2007

Curvature arbitrage

Yang Ho Choi
University of Iowa

CURVATURE ARBITRAGE

by

Yang Ho Choi

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Applied Mathematical and Computational Sciences
in the Graduate College of
The University of Iowa

July 2007

Thesis Supervisor: Professor Palle Jorgensen

ABSTRACT

The Black-Scholes model is one of the most important concepts in modern financial theory. It was developed in 1973 by Fisher Black, Robert Merton and Myron Scholes and is still widely used today, and regarded as one of the best ways of determining fair prices of options. In the classical Black-Scholes model for the market, it consists of an essentially riskless bond and a single risky asset. So far there is a number of straightforward extensions of the Black-Scholes analysis. Here we consider more complex products where each component in a portfolio entails several variables with constraints. This leads to elegant models based on multivariable stochastic integration, and describing several securities simultaneously. We derive a general asymptotic solution in a short time interval using the heat kernel expansion on a Riemannian metric. We then use our formula to predict the better price of options on multiple underlying assets. Especially, we apply our method to the case known as the one of two-color rainbow options, outperformance option, i.e., the special case of the model with two underlying assets. This asymptotic solution is important, as it explains hidden effects in a class of financial models.

Abstract Approved: ______________________________
Thesis Supervisor

____
Title and Department

____
Date

CURVATURE ARBITRAGE

by

Yang Ho Choi

A thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Applied Mathematical and Computational Sciences
in the Graduate College of
The University of Iowa

July 2007

Thesis Supervisor: Professor Palle Jorgensen

Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Yang Ho Choi

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Applied Mathematical and Computational Sciences at the July 2007 graduation.

Thesis Committee:

Palle Jorgensen, Thesis Supervisor

Yi Li

Oguz Durumeric

Walter Seaman

Yannick Meurice

ACKNOWLEDGEMENTS

I would like to express my gratitude to Prof. Li, Prof. Durumeric, Prof. Seaman and Prof. Meurice for serving committee members of my Ph.D. I also like to thank all my friends who have treated me with care.

I am deeply indebted to my supervisor Prof. Jorgensen and former supervisor Prof. Branson. I could not have imagined having better advisors and mentors, and without their help, stimulating suggestions, encouragement and patience, I would never reach this stage. It has truly been an honor and I feel very blessed to have them as my advisors.

I am greatly grateful to my family, Jinho Choi, Aeran Choi, Younghan Kim, Youmi and Minwoo for their support. I would express heartfelt thanks to my father-in-law, Kwansik Moon, mother-in-law, Moonhee Park and sister-in-law Soyoung for their endless love and care. I would like to give my special thanks to my wife, Hyeyoung and baby girl, Elaine whose patience and love enabled me to complete this work. And last but not least, I would like to thank my father, Dongwoon Choi and mother, Hyangja Ahn in heaven.

Words cannot express my gratitude for all these people.

ABSTRACT

The Black-Scholes model is one of the most important concepts in modern financial theory. It was developed in 1973 by Fisher Black, Robert Merton and Myron Scholes and is still widely used today, and regarded as one of the best ways of determining fair prices of options. In the classical Black-Scholes model for the market, it consists of an essentially riskless bond and a single risky asset. So far there is a number of straightforward extensions of the Black-Scholes analysis. Here we consider more complex products where each component in a portfolio entails several variables with constraints. This leads to elegant models based on multivariable stochastic integration, and describing several securities simultaneously. We derive a general asymptotic solution in a short time interval using the heat kernel expansion on a Riemannian metric. We then use our formula to predict the better price of options on multiple underlying assets. Especially, we apply our method to the case known as the one of two-color rainbow options, outperformance option, i.e., the special case of the model with two underlying assets. This asymptotic solution is important, as it explains hidden effects in a class of financial models.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1
INTRODUCTION

1.1 Overview

This thesis combines ideas from stochastic differential equations and from geometry in the study of a multidimensional version of the Black-Scholes equations from mathematical economics. The Black-Scholes model lies on the interface of mathematics and economics, and it has both theoretical and practical significance. Indeed, it offers a beautiful mathematical formula for the theoretical value of financial strategies such as "European put" and "call" stock options. They may be derived from the assumptions built into the model, and the solution of the equation. The fundamental insight of Black and Scholes is that the call stock option is implicitly priced if the stock is traded.

Our approach is motivated by the need to introduce geometric invariants in the study of the multi-dimensional case, as the variable configurations in particular models naturally form non-linear manifold structures.

As is well known, the original Black-Scholes equation is a stochastic partial differential equation (PDE) which models financial instruments of the varying price over time

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$$

The equation is named after its inventors, the two economists Fischer Black and Myron Scholes. In its original form and presentation, it is fair to say that mathematical rigor was lacking somewhat.

Here are the basic assumptions for the fundamental Black-Scholes analysis [18]:

- The price of the underlying assets follows a certain Brownian motion.
- The risk-free interest rate r and the asset volatility σ are known functions of time over the life of the option.
- There are no transaction costs associated with hedging a portfolio.
- There are no arbitrage possibilities.
- Trading of the underlying asset can take place continuously.
- There are no dividends.
- Short selling is permitted and the assets are divisible.

We will develop the multidimensional version of the Black-Scholes formula in much the same way as one develops the original Black-Scholes model. In order to calculate explicitly the multidimensional Black-Scholes formula with a certain terminal condition, we need to see the Itô stochastic calculus and the Feynman-Kac representation formula which connects between stochastic differential equation and certain second order parabolic partial differential equations. In addition, we can derive this solution by transferring the multidimensional Black-Scholes to the heat equation in Chapter 5. After we consider this simplest model in multiple asset case, we will propose more elegant one which we call curvature arbitrage model. We then apply our method to the case known as the famous rainbow option.

1.2 Outline

Chapter 2 contains terminology from the world of finance and much of the background necessary for the rest of the thesis. We see financial derivatives whose value depends on the values of basic underlying variables. Especially, we focus on a stock option which is a derivative whose value is dependent on the price of a stock. Also, we give a brief review of geometric invariants.

In Chapter 3 we introduce the multiple stock models where the terms $S = (S_i)$, $i = 1, \cdots, n$, (for asset) and W (the random or Brownian component) are vector valued. This means that the corresponding probabilistic covariance will be matrix valued. We discuss the case of uncorrelated Brownian terms, as well as the correlated case.

In Chapter 4 we introduce the Itô formula and the multifactor Girsanov Theorem for change measure to construct the martingale measure and the multidimensional Feynman-Kac stochastic representation formula which connects between a stochastic differential equation and the multidimensional Black- Scholes formula.

In Chapter 5 we introduce value $V(S_1, \cdots, S_n, t)$ of a vector-valued Black-Scholes model as a function of time as well as the maturity $P = P(S_1, \cdots, S_n, T)$, and we show that the corresponding equation for dV is a stochastic partial differential equation (4.2) involving the first and second order partial derivatives of V with respect to the variables S_i by the delta hedging method. We then derive a Black-Scholes equation in a finite time interval $[0, T]$ and for the terminal condition $V = P$ for $t = T$, i.e., the time T when the value matches maturity in two ways. Not surprisingly, the solution coming from (4.4),(5.4) can be expressed in terms of the multivariate normal distribution.

In Chapter 6 we review some standard heat equation formulas on curved manifolds, which are then applied to the Black-Scholes equation for the investment model called "the rainbow option." Furthermore, this section contains our new heat kernel asymptotics method with off-diagonal terms in $\mathbf{R}^2$. The new heat kernel asymptotics predict a better option price. We will use it on the 2-color outperformance option in Chapter 7.

This thesis has threefold; first we give a systematic approach to the multi-dimensional Black-Scholes equation; secondly, in order to understand the intrinsic nonlinearities in the multi-dimensional case, we introduce two fundamental curvature invariants from geometry; and thirdly, we study time asymptotics of the multi-dimensional Black-Scholes equation with the use of heat equation asymptotics.

Several geometric methods for pricing financial derivatives have recently been developed. In addition, we are able to see the exact solutions in the SABR model and special affine and quadratic two-factor term structure models from these methods [7] and [13]. In this paper, we will focus on deriving more precise general heat kernel asymptotics in 2 dimensions and applying it to one of the 2-color rainbow options. In a future work, we will find the time-dependent heat kernel asymptotics and use it in the real finance markets. Some good references on Black-Scholes, options theory, stochastic differential equations, and stochastic integration are [8] and [12].

Finally, the summary and future study are introduced in Chapter 8.

CHAPTER 2
TERMINOLOGY AND BACKGROUND

2.1 Finance

First we have to see the idea behind stocks or known as shares. A company that needs money for building a new factory or making a new product, can sell shares to investors at the open market. Then the company is owned by its shareholders; if the company makes a profit, part of it may be paid out to shareholders as a dividend of each share. Shares thus have a value which represent how well the company is going to do in the future. This value is quantified by the price at which shares are bought and sold on stock exchanges. We have a collection of markets on which assets of various kinds are bought and sold. As markets have become more sophisticated, more complex contracts have been introduced. Known as financial derivatives, they can give investors a great range of opportunities to design their investment needs. There are many types of financial instruments that are grouped under the term derivatives, but options, forwards and futures are among the most common. Our central purpose is to determine how much one should be willing to pay for a derivative security.

Options on stocks were first traded on an organized exchange in 1973. Since then there has been a dramatic growth in options markets. There are two basic types of options. A *call option* gives the holder the right to buy the underlying asset by a certain date for a certain price. A *put option* gives the holder the right to sell the underlying asset by a certain date for a certain price. The price in the contract is known as the *exercise price* or *strike price*; the date in the contract is known as the *expiration date* or *maturity.* For the holder of the option, this contract is a right and not an obligation. The other party to the contract known as writer, does have a potential obligation: he must sell or buy the asset if the holder exercises call or put option at the time of maturity. Before we see more details about options, we introduce the single most important example: a call option.

Example 2.1. A trader wants to buy one European call option contract on IBM stock with a strike price of $ 100, which means the trader purchases the right to buy an IBM share for $ 100. Suppose that the current stock price is $98, the expiration date of the option is in three months. Because the option is European, the trader can exercise only on the expiration date. If the stock price on this date is less than $ 100, the trader will clearly choose not to exercise. In these circumstances the trader loses the certain option price. If the stock price is above $ 100 on the expiration date, the option will be exercised. For example, suppose that the stock price is $ 110. By exercising the option, the trader is able to buy an IBM stock for $100. If the trader sells the stock immediately at the open market, he makes a profit of $10, ignoring transaction costs and the initial cost of the option.

Remark. One contract of options is usually an agreement to buy or sell 100 shares. For the simplicity, we use 1 share of European call option at this example.

The above example is especially about the *European call option.* As we can see the prefix *European* means that the option can only be exercised at exactly the date of expiration. There also exist *American options*, which give the holder the right to exercise the option at any time before the date of expiration.

Apart from options, a *forward contract* is an agreement between two parties whereby one contracts to buy a specified asset from the other for a specified price, known as the *forward price*, on a specified date in the future, the *delivery date* or *maturity date*. A *future contract* is almost the same as a forward contract, but futures are usually traded on an exchange which specifies certain standard features of the contract such as delivery date and contract size.

It should be emphasized that an option gives the holder the right to do something. The holder does not have to exercise this right. This fact distinguishes options from forwards and futures, where the holder is obligated to buy or sell the underlying asset. Note that whereas it costs nothing to enter into a forward or futures contract, there is a cost to acquiring an option.

2.2 Geometric Invariants

In this section, we give a brief review of the geometric invariants used in the paper. We can find more details from [16] and [19].

The notion of a differentiable manifold is necessary for extending the methods of differential calculus to spaces more general than $\mathbf{R}^n$. An example of a manifold is a regular surface in $\mathbf{R}^3$. A subset $S \subset \mathbf{R}^3$ is called a regular surface if, for every point $p \in S$, there exist a neighborhood V of p in $\mathbf{R}^3$ and a mapping $\mathbf{x} : U \subset \mathbf{R}^2 \to V \cap S$ of an open set $U \subset \mathbf{R}^2$ onto $V \cap S$, such that:

- $\mathbf{x}$ is a differentiable homeomorphism;

- The differential $(d\mathbf{x})_q : \mathbf{R}^2 \to \mathbf{R}^3$ is injective for all $q \in U$.

The mapping $\mathbf{x}$ is called a parametrization of S at p. The most important consequence of this definition is that if $\mathbf{x}_\alpha : U_\alpha \to S$ and $\mathbf{x}_\beta : U_\beta \to S$ are two parametrizations such that $\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha : \mathbf{x}_\alpha^{-1}(W) \to \mathbf{R}^2$ and $\mathbf{x}_\alpha^{-1} \circ \mathbf{x}_\beta^{-1} : \mathbf{x}_\beta^{-1}(W) \to \mathbf{R}^2$ are differentiable.

Thus, a regular surface is intuitively a union of open sets of $\mathbf{R}^2$, organized in such a way that when two such open sets intersect the change from one to the other can be made in a differentiable manner. From this property, we define a differential manifold for an arbitrary dimension n.

Definition 2.1. A differentiable manifold of dimension n is a set M and a family of injective mappings $\mathbf{x}_\alpha : U_\alpha \subset \mathbf{R}^n \to M$ of open sets U_α of $\mathbf{R}^n$ into M such that:

(1) $\cup_\alpha \mathbf{x}_\alpha(U_\alpha) = M$.

(2) for any pair α, β, with $\mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\beta(U_\beta) = W \neq \varnothing$, the sets $\mathbf{x}_\alpha^{-1}(W)$ and $\mathbf{x}_\beta^{-1}(W)$ are open sets in $\mathbf{R}^n$ and the mappings $\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$ are differentiable (Figure 2.1).

(3) the family $\{U_\alpha, \mathbf{x}_\alpha\}$ is maximal relative to the conditions (1) and (2).

The pair $(U_\alpha, \mathbf{x}_\alpha)$ with $p \in \mathbf{x}_\alpha(U_\alpha)$ is called a parametrization (or system of coordinates) of M at p; $\mathbf{x}_\alpha(U_\alpha)$ is then called a coordinate neighborhood at p. A family $\{U_\alpha, \mathbf{x}_\alpha\}$ satisfying (1) and (2) is called a differentiable structure on M.

Given a surface $S \subset \mathbf{R}^3$, we have a natural way of measuring the lengths of vectors tangent to S, namely: the inner product $< v, w >$ of two vectors tangent to S at a point p of S is simply the inner product of these vectors in $\mathbf{R}^3$. The definition

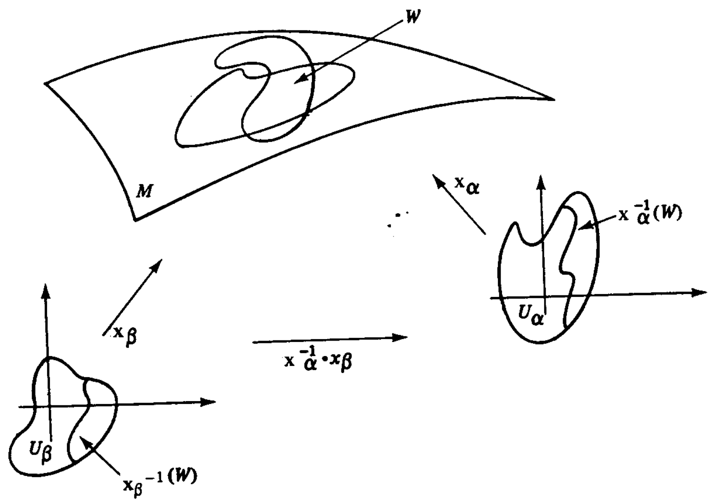

Figure 2.1: Differentiable manifolds.

of $<,>$ permits us to measure the lengths of curves in S, as well as the angle between two curves, and all the other metric ideas used in geometry. Now we will define a Riemannian metric or Reimannian structure on a differentiable manifold.

Definition 2.2. A Riemannian metric on a differentiable manifold M is a correspondence which associates to each point p of M an inner product $<,>_p$ on the tangent space T_pM, which varies differentiably in the following sense: If $\mathbf{x} : U \subset \mathbf{R}^n \to M$ is a system of coordinates around p, with $\mathbf{x}(x_1, x_2, \ldots, x_n) = q \in \mathbf{x}(U)$ and $\frac{\partial}{\partial x_i}(q) = d\mathbf{x}_q(0, \ldots, 1, \ldots, 0)$, then $< \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) >_q = g_{ij}(x_1, \ldots, x_n)$ is a differentiable function on U.

It is obvious this definition does not depend on the choice of coordinate system. The function g_{ij} is called the local representation of the Riemannian metric or the

g_{ij} of the metric in the coordinate system $\mathbf{x} : U \subset \mathbf{R}^n \rightarrow M$. A differentiable manifold with a given Reimannian metric will be called a Riemannian manifold. Once a local coordinate system x^i is chosen, the metric tensor appears as a matrix. g_{ij} is conventionally used for the components of the metric tensor. The following example is almost trivial one.

Example 2.2. $M = \mathbf{R}^n$ with $\frac{\partial}{\partial x_i} := \partial_i = e_i = (0, \ldots, 1, \ldots, 0)$. The metric is given by $< e_i, e_j >= \delta_{ij}$ [1] , which is the Dirac measure. $\mathbf{R}^n$ is called Euclidean space of dimension n and the Riemannian geometry of this space is metric Euclidean geometry.

We are now ready to see the introduction of the Levi-Civita parallelism which is a fundamental tool in the development of differential geometry. Here is a basic idea. Let $S \subset \mathbf{R}^3$ be a surface and let $c : I \rightarrow S$ be a parametrized curve in S, with $V : I \rightarrow \mathbf{R}^3$ a vector field along c tangent to S. The vector $\frac{dV}{dt}(t)$, $t \in I$, does not in general belong to the tangent plane of S, $T_{c(t)}S$. The concept of differentiating a vector field is not therefore an intrinsic geometric notion on S. To remedy this state of affairs we consider, instead of the usual derivative $\frac{dV}{dt}(t)$, the orthogonal projection of $\frac{dV}{dt}(t)$ on $T_{c(t)}S$. this orthogonally projected vector we call the covariant derivative and denote it by $\frac{DV}{dt}(t)$. The covariant derivative of V is the derivative of V as seen from the viewpoint of S.

We will define a notion of derivation of vector fields with certain properties which is called an affine connection. By using the Levi-Civita connection, a choice of

[1]In this section, δ_{ij} stands for the Kronecker delta. However, the notation δ will be used for the Dirac delta-function in the section 5.2.

a Riemannian metric on a manifold M uniquely determines a certain affine connection on M.

Let $\mathcal{VF}(M)$ by the set of all vector fields of class C^∞ on M and by $\mathcal{D}(M)$ the ring of real-valued functions of class C^∞ defined on M.

Definition 2.3. An affine connection ∇ on a differentiable manifold M is a mapping

$$\nabla : \mathcal{VF}(M) \times \mathcal{VF}(M) \to \mathcal{VF}(M)$$

which is denoted by $\nabla : (X, Y) \to \nabla_X Y$ and which satisfies the following properties:

(1) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$.

(2) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$.

(3) $\nabla_X (fY) = f\nabla_X Y + X(f)Y$,

in which $X, Y, Z \in \mathcal{VF}(M)$ and $f, g \in \mathcal{D}(M)$.

This definition is not as clear as that of Riemannian structure. The following lemma should clarify the situation a little bit.

Lemma 2.1. *Let M be a differentiable manifold with an affine connection ∇. There exists a unique correspondence which associates to a vector field V along the differentiable curve $c : I \to M$ another vector field $\frac{DV}{dt}$ along c, called the covariant derivative of V along c, such that:*

(a) $\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$.

(b) $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$, *where* W *is a vector field along* c *and* f *is a differentiable function of* I.

(c) *If* V *is induced by a vector field* $Y \in \mathcal{VF}(M)$, *i.e.,* $V(t) = Y(c(t))$, *then* $\frac{DV}{dt} = \nabla_{\frac{dc}{dt}} Y$.

In other words, ∇ has the connection one-form ω such as $\nabla_i = \partial_i + \omega_i$. This ω_i is uniquely determined by coordinate system.

Before we see the Levi-Civita connection, we need to see two more definitions.

Definition 2.4. Let M be a differentiable manifold with an affine connection ∇ and a Riemannian metric $<,>$. A connection is said to be compatible with the metric $<,>$, when for any smooth curve c and any pair of parallel vector fields P and P' along c, we have $< P, P' >$=constant.

Definition 2.5. An affine connection ∇ on a smooth manifold M is said to be symmetric when

$$\nabla_X Y - \nabla_Y X = [X, Y] = XY - YX$$

for all $X, Y \in \mathcal{VF}(M)$.

We are now able to state the fundamental theorem, the Levi-Civita connection.

Theorem 2.2. *(Levi-Civita) Given a Riemannian manifold* M, *there exists a unique affine connection* ∇ *on* M *satisfying the conditions:*

(a) ∇ *is symmetric.*

(b) ∇ is compatible with the Riemannian metric.

Let us define more terms in a coordinate system $(U, \mathbf{x})$. The functions Γ_{ij}^k defined on U by $\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k$, are called the coefficients of the connection ∇ on U or the Christoffel symbols of the connection. It also satisfies the condition

$$\sum_l \Gamma_{ij}^l g_{lk} = \frac{1}{2}\left\{\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}\right\},$$

where $g_{ij} =< X_i, X_j >$. The followings are the notion of curvature in a Riemannian manifold.

Definition 2.6. The curvature R of a Riemannian manifold M is a correspondence that associates to every pair $X, Y \in \mathcal{VF}(M)$ a mapping $R(X,Y) : \mathcal{VF}(M) \to \mathcal{VF}(M)$ given by

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, \quad Z \in \mathcal{VF}(M),$$

where $\nabla$is the Riemannian connection of M.

As we will see it at chapter 6, the followings are more convenient forms. The Riemannian curvature R can be defined by

$$[\nabla_i, \nabla_j] X^k = \nabla_j \nabla_i X^k - \nabla_i \nabla_j X^k = R_{\ell ij}^k X^\ell$$

for a vector field X. In addition, the Ricci tensor r and scalar curvature K are defined by

$$r_{ij} = R_{ikj}^k, \qquad K = g^{ij} r_{ij}.$$

We will use these geometric invariants at chapter 6.

CHAPTER 3
MULTIPLE STOCK MODELS

3.1 Modeling Using a Correlated Standard Brownian Motion

In the same way as the book [18] show a simple model for a single asset price. We can obtain multiple stock models. The corresponding return on the asset i, $dS_i(t)/S_i(t)$ consists of two parts. One is $\mu_i(t)dt$, where $\mu_i(t)$ is a measure of the rate of growth of the asset price i. The second part is represented by a random sample drawn from the standard normal distribution, $\sigma_i(t)dW_i(t)$. Here $\sigma_i(t)$ is a function called the volatility of the asset i, which measures the standard deviation of the returns and $dW_i(t)$ is the standard Brownian motion. Also, we use $\rho_{ij}(t)$ for the correlation between two standard Brownian motions $dW_i(t)$ and $dW_j(t)$. In the simplest model, $\mu_i(t)$ and $\sigma_i(t)$ are assumed to be constants. The corresponding return is

$$\frac{dS_i(t)}{S_i(t)} = \mu_i(t)dt + \sigma_i(t)dW_i(t), \quad \text{for} \quad t \geq 0, \quad i = 1, \cdots, n, \tag{3.1}$$

where $dW_i(t)$ is a standard Brownian motion with the properties

(a) $E(dW_i(t)) = 0$,

(b) $Var(dW_i(t)) = dt$,

(c) $Cov(dW_i(t), dW_j(t)) = \rho_{ij}(t)dt$.

3.2 Modeling Using an Uncorrelated Standard Brownian Motion

Equivalently, we take a set of n independent Brownian motions and drive the asset prices by linear combinations of these. This results in an uncorrelated standard Brownian model.

$$\frac{dS_i(t)}{S_i(t)} = \mu_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dW_j(t), \quad \text{for} \quad t \geq 0, \quad i = 1, \cdots, n. \tag{3.2}$$

We can express this model as vectors and matrices. The equation (3.2) becomes

$$d\overrightarrow{S}(t) = \overrightarrow{\mu}(t)dt + [\sigma(t)]d\overrightarrow{W}(t), \tag{3.3}$$

where,

$$\overrightarrow{S}(t) = \begin{pmatrix} S_1(t) \\ \vdots \\ S_n(t) \end{pmatrix}, \overrightarrow{\mu}(t) = \begin{pmatrix} \mu_1(t)S_1(t) \\ \vdots \\ \mu_n(t)S_n(t) \end{pmatrix}, \overrightarrow{W}(t) = \begin{pmatrix} W_1(t) \\ \vdots \\ W_n(t) \end{pmatrix},$$

$$[\sigma(t)] = \begin{pmatrix} \sigma_{11}(t)S_1(t) & \cdots & \sigma_{1n}(t)S_1(t) \\ \vdots & & \vdots \\ \sigma_{n1}(t)S_n(t) & \cdots & \sigma_{nn}(t)S_n(t) \end{pmatrix} = [S(t)][\sigma_0(t)]$$

with

$$[S(t)] = \begin{pmatrix} S_1(t) & & 0 \\ & \ddots & \\ 0 & & S_n(t) \end{pmatrix}, [\sigma_0(t)] = \begin{pmatrix} \sigma_{11}(t) & \cdots & \sigma_{1n}(t) \\ \vdots & & \vdots \\ \sigma_{n1}(t) & \cdots & \sigma_{nn}(t). \end{pmatrix}$$

Now we show that the models in (3.1) and (3.2) are exactly the same.

Lemma 3.1. *Two models in (3.1) and (3.2) are equivalent.*

Proof. Let the stochastic process $Z(t) = [\sigma_0(t)]W(t)$. This allows us to rewrite (3.2) more compactly as

$$\frac{dS_i(t)}{S_i(t)} = \mu_i(t)dt + dZ_i(t), \quad \text{for} \quad t \geq 0, \quad i = 1, \cdots, n,$$

because $[\sigma_0]$ is a constant matrix. It is easily verified that $Z(t)$ is an n-dimensional correlated non-standard Brownian motion. In particular, for $i = 1, \cdots, n$, we have

$$\begin{aligned} Var[Z_i(t)] =& Var[\sigma_{i1}(t)Z_1(t) + \cdots + \sigma_{in}(t)Z_n(t)] \\ =& (\sigma_{i1}^2(t) + \cdots + \sigma_{in}^2(t))t. \end{aligned}$$

For this reason, we let

$$\sigma_i^2(t) := \sigma_{i1}^2(t) + \cdots + \sigma_{in}^2(t).$$

We call this the instantaneous variance of $Z_i(t)$. Similarly, since

$$\begin{aligned} Cov[Z_i(t), Z_j(t)] =& Cov[\sigma_{i1}(t)Z_1(t) + \cdots + \sigma_{in}(t)Z_n(t), \sigma_{j1}(t)Z_1(t) + \cdots + \sigma_{jn}(t)Z_n(t)] \\ =& (\sigma_{i1}(t)\sigma_{j1}(t) + \cdots + \sigma_{in}(t)\sigma_{jn}(t))t, \end{aligned}$$

we call

$$C_{ij}(t) := \sigma_{i1}(t)\sigma_{j1}(t) + \cdots + \sigma_{in}(t)\sigma_{jn}(t) \tag{3.4}$$

the instantaneous covariance between $Z_i(t)$ and $Z_j(t)$, and

$$C(t) := \begin{pmatrix} C_{11}(t) & \cdots & C_{1n}(t) \\ \vdots & & \vdots \\ C_{n1}(t) & \cdots & C_{nn}(t) \end{pmatrix} = [\sigma_0(t)][\sigma_0(t)]^T$$

the covariance matrix of $Z(t)$. Note that different volatility matrices $[\sigma_0(t)]$ might result in the same covariance matrix $C(t)$. That is, different volatility matrices $[\sigma_0(t)]$

might lead to the same stock price model. Thus, estimating the parameters of the model specifies the covariance matrix $C(t)$, but does not specify the volatility matrix $[\sigma_0(t)]$ uniquely.

Finally, the correlation between $Z_i(t)$ and $Z_j(t)$ is defined as

$$\begin{aligned}\rho_{ij}(t) :=& \frac{C_{ij}(t)}{\sqrt{\sigma_i^2(t)\sigma_j^2(t)}} \\ =& \frac{C_{ij}(t)}{\sigma_i(t)\sigma_j(t)}.\end{aligned} \tag{3.5}$$

Hence,

$$C_{ij}(t) = \sigma_i(t)\sigma_j(t)\rho_{ij}(t). \tag{3.6}$$

Therefore, (3.1) is equivalent to (3.2), by (3.5) or (3.6). □

CHAPTER 4
STOCHASTIC CALCULUS

4.1 Multifactor Girsanov Theorem

Let us start with the multiple asset model in (3.2). The most basic tool will be an n-factor Itô formula. Assume some option whose value $V(S_1, \cdots, S_n, t)$ depends only on $S_1, \cdots, S_n, t$. It is not necessary to specify what V is at this time. If we apply Taylor's Theorem, we can obtain

$$dV = \frac{\partial V}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial V}{\partial S_i}dS_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 V}{\partial S_i \partial S_j}dS_i dS_j + \cdots . \tag{4.1}$$

Since we use an uncorrelated standard Brownian motion, we have the multiplication

Table 4.1: Multiplication table

	$dW_i(t)$	$dW_j(t)$	dt
$dW_i(t)$	dt	0	0
$dW_j(t)$	0	dt	0
dt	0	0	0

table, and this gives $dS_i dS_j = \sum_{k=1}^{n} \sigma_{ik}\sigma_{jk} S_i(t) S_j(t) dt$. The same multiplication table tells us that $dS_i(t)dS_j(t)dS_k(t)$ is $o(dt)$[1] [10] and [21]. Then we can get the famous multifactor Itô formula.

Lemma 4.1. *Let*

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t), \quad \textit{for} \quad i = 1, \cdots, n,$$

be an uncorrelated standard Brownian motion. Then the option value $V(S_1, \cdots, S_n, t)$ *will be satisfied by*

$$dV = \sum_{i=1}^{n} \frac{\partial V}{\partial S_i} dS_i + \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{n} C_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} \right) dt, \tag{4.2}$$

where $C_{ij}(t)$ *is defined by the equation (3.6).*

Proof. From the multiplication table (Table 4.1), we obtain the following results easily.

$$dS_i dS_j = \sum_{k=1}^{n} \sigma_{ik} \sigma_{jk} S_i S_j dt,$$

and other terms which is higher than degree of 2 will be eliminated under the assumption. □

Pricing and hedging in the multiple stock model will follow the same pattern in the original model. First, we find an equivalent probability measure under the condition all of the discounted stock prices are martingales. We then use a multifactor version of the Martingale Representation Theorem to construct a replicating

---

[1] If $F(t)/G(t) \to 0$ as $t \to 0$, we write

$$F(t) = o(G(t)) \quad \text{as} \quad t \to 0;$$

portfolio. Constructing the martingale measure, we need to see some definitions and a multifactor version of the Girsanov Theorem [11] and [22].

When we start with a random variable, we must specify a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a set, the sample space, $\mathcal{F}$ is a collection of subsets of Ω, events, and $\mathbb{P}$ specifies the probability of each event $A \in \mathcal{F}$. The collection $\mathcal{F}$ is a σ-field, that is, $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under the operations of countable union and taking complements. The probability $\mathbb{P}$ must satisfy the usual axioms of probability:

- $0 \leq \mathbb{P}[A] \leq 1$, for all $A \in \mathcal{F}$,
- $\mathbb{P}[\Omega] = 1$,
- $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B]$ for any disjoint $A, B \in \mathcal{F}$,
- if $A_n \in \mathcal{F}$ for all $n \in \mathbf{N}$ and $A_1 \subseteq A_2 \subseteq \cdots$ then $\mathbb{P}[A_n] \to \mathbb{P}[\cup_n A_n]$ as $n \to \infty$.

Definition 4.1. A real-valued random variable, X, is a real-valued function on Ω that is $\mathcal{F}$-measurable. In the case of a discrete random variable this means

$$\{\omega \in \Omega : X(\omega) = x\} \in \mathcal{F},$$

so that $\mathbb{P}$ assigns a probability to the event $\{X = x\}$. For a general real-valued random variable we require that

$$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F},$$

so that we can define the distribution function, $F(x) = \mathbb{P}[X \leq x]$.

To specify a stochastic process, we need to have a special σ-field. The collection $\{\mathcal{F}_n\}$ with $\mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subseteq \cdots \subseteq \mathcal{F}$ is called a filtration and the quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ is called a filtered probability space.

Definition 4.2. A real-valued stochastic process is just a sequence of real-valued function, $\{X_n\}$, on Ω. We say that it is adapted to the filtration $\{\mathcal{F}_n\}$ if X_n is $\mathcal{F}_n$-measurable for each n.

Definition 4.3. Suppose that $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ is filtered probability space. The sequence of random variables $\{X_n\}$ is a martingale with respect to $\mathbb{P}$ and $\{\mathcal{F}_n\}$ if

$$\mathbb{E}[|X_n|] < \infty,$$

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n,$$

for all n.

Definition 4.4. Given a filtration $\{\mathcal{F}_n\}$, the process $\{A_n\}$ is $\{\mathcal{F}_n\}$- previsible if A_n is $\mathcal{F}_{n-1}$-measurable for all $n \geq 1$.

Theorem 4.2. *(Multifactor Girsanov Theorem) Let $\{W_i(t)\}$ be independent Brownian motions under the measure $\mathbb{P}$ generating the filtration $\{\mathcal{F}(t)\}$ and let $\{\theta_i(t)\}$ be $\{\mathcal{F}(t)\}$-previsible processes such that*

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \sum_{i=1}^{n} \theta_i^2(s)ds\right)\right] < \infty. \tag{4.3}$$

Define

$$L(t) = \exp\left(-\sum_{i=1}^{n}\left(\int_0^T \theta_i(s)dW_i(s) + \frac{1}{2}\int_0^T \theta_i^2(s)ds\right)\right)$$

and let $\mathbb{P}^{(L)}$ *be the probability measure defined by*

$$\frac{d\mathbb{P}^{(L)}}{d\mathbb{P}}|_{\mathcal{F}(t)} = L(t).$$

Then under $\mathbb{P}^{(L)}$ *the processes* $\{X_i(t)\}$*, defined by*

$$X_i(t) = W_i(t) + \int_0^t \theta_i(s)ds$$

are all martingales.

Proof. The proof is similar in the one-factor model [5]. Let $L(t) = \prod_{i=1}^n L_i(t)$ where

$$L_i(t) = \exp\left(-\int_0^t \theta_i(s)dW_i(s) - \frac{1}{2}\int_0^t \theta_i^2(s)ds\right).$$

That $\{L(t)\}$ defines a martingale follows from (4.3) and the independence of the Brownian motions $\{W_i(t)\}$, $i = 1, \cdots, n$.

To check that $\{X_i(t)\}$ is a $\mathbb{P}^{(L)}$-martingale we find the stochastic differential equation satisfied by $\{X_i(t)L(t)\}$. Since

$$dL_i(t) = -\theta_i(t)L_i(t)dW_i(t),$$

repeated application of our product rule gives

$$dL(t) = -L(t)\sum_{i=1}^n \theta_i(t)dW_i(t).$$

Moreover,

$$dX_i(t) = dW_i(t) + \theta_i(t)dt,$$

and so another application of our product rule gives

$$\begin{aligned} d(X_i(t)L(t)) =& X_i(t)dL(t) + L(t)dW_i(t) + L(t)\theta_i(t)dt - L(t)\theta_i(t)dt \\ =& -X_i(t)L(t)\sum_{i=1}^n \theta_i(t)dW_i(t) + L(t)dW_i(t). \end{aligned}$$

Combined with the boundedness condition (4.3), this proves that $\{X_i(t)L(t)\}$ is $\mathbb{P}$-martingale and hence $\{X_i(t)\}$ is a $\mathbb{P}^{(L)}$-martingale. $\mathbb{P}^{(L)}$ is equivalent to $\mathbb{P}$ so $\{X_i(t)\}$ has quadratic variation $[X_i(t)] = t$ with $\mathbb{P}^{(L)}$-probability one and once again Lévy's characterization of Brownian motion confirms that $\{X_i(t)\}$ is a $\mathbb{P}^{(L)}$-Brownian motion as required. $\square$

Remark. Since we want to distinguish which probability measure is used, we will use $\mathbb{P}$ for a probability measure, usually the market measure and $\mathbb{Q}$ a martingale measure equivalent to the market measure. In addition, the notation $\mathbb{E}^{\mathbb{Q}}$ means the expectation under the measure $\mathbb{Q}$.

As promised by the multifactor Girsanov theorem, we find a measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, under which the discounted stock price processes $\{\widetilde{S}_i(t)\}$, defined by $\widetilde{S}_i(t) = e^{-rt}S_i(t)$, are all martingales [24]. We need to identify the appropriate drifts $\{\theta_i(t)\}$.

The discounted stock price $\{\widetilde{S}_i(t)\}$ is governed by the stochastic differential equation

$$\begin{aligned} d\widetilde{S}_i(t) =& \widetilde{S}_i(t)(\mu_i(t) - r)dt + \widetilde{S}_i(t)\sum_{j=1}^{n} \sigma_{ij}(t)dW_j(t) \\ =& \widetilde{S}_i(t)\left(\mu_i(t) - r - \sum_{j=1}^{n} \theta_j(t)\sigma_{ij}(t)\right)dt + \widetilde{S}_i(t)\sum_{j=1}^{n} \sigma_{ij}(t)dX_j(t), \end{aligned}$$

where as in Theorem 4.3

$$dX_j(t) = \theta_j(t)dt + dW_j(t).$$

The discounted stock price processes will be local martingales under $\mathbb{Q} = \mathbb{P}^{(L)}$ if we can make all the drift terms vanish. That is, if we can find $\{\theta_j(t)\}$ such that

$$\mu_i(t) - r - \sum_{j=1}^{n} \theta_j(t)\sigma_{ij}(t) = 0 \qquad \text{for all} i = 1, \cdots, n.$$

Dropping the dependence on t in our notation and writing

$$\mu = (\mu_1, \cdots, \mu_n), \quad \theta = (\theta_1, \cdots, \theta_n), \quad \mathbf{1} = (1, \cdots, 1) \ \text{ and } \ \sigma = (\sigma_{ij}),$$

this becomes

$$\mu - r\mathbf{1} = \theta\sigma.$$

A solution exists if the matrix σ is invertible.

In order to guarantee that the discounted price processes are martingales, not just local martingales, once again we impose a Novikov condition:

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_0^t \frac{1}{2}\sum_{j=1}^{n} \sigma_{ij}^2(t)dt\right)\right] < \infty \ \text{ for each } \ i.$$

At this point, the value of a claim $C(t) \in \mathcal{F}(t)$ at time $t < T$ is its discounted expected value under the measure $\mathbb{Q}$. To prove this we show that there is a self-financing replicating portfolio and this we infer from a multifactor version of the Martingale Representation Theorem.

Theorem 4.3. *Let $W_i(t)$ be independent $\mathbb{P}$-Brownian motions generating the filtration $\{\mathcal{F}(t)\}$. Let $M_i(t)$ be given by*

$$dM_i(t) = \sum_{j=1}^{n} \sigma_{ij}(t)dW_j(t),$$

where

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \sum_{j=1}^{n} \sigma_{ij}^2(t)dt\right)\right] < \infty.$$

Suppose further that the volatility matrix $(\sigma_{ij}(t))$ *is non-singular (with probability one). Then if* $\{N(t)\}]$ *is any one-dimensional* $(\mathbb{P}, \{\mathcal{F}(t)\})$*-martingale there exists an* n*- dimensional* $\{\mathcal{F}(t)\}$*-previsible process* $\{\phi(t)\} = \{\phi_1(t), \cdots, \phi_n(t)\}$ *such that*

$$N(t) = N(0) + \sum_{j=1}^{n} \int_0^t \phi_j(s) dM_j(s).$$

A proof of this result is found in [20]. We now verify that the value of a claim in the multifactor world is its discounted expected value under the martingale measure $\mathbb{Q}$.

Let $C(T) \in \mathcal{F}(T)$ be a claim at time T and let $\mathbb{Q}$ be the martingale measure obtained above. We write

$$M(t) = \mathbb{E}^{\mathbb{Q}}[B^{-1}(T)C(T)|\mathcal{F}(t)].$$

Since, by assumption, the matrix $\sigma = (\sigma_{ij})$ is invertible, the n-factor Martingale Representation Theorem tells us that there is an $\{\mathcal{F}(t)\}$-previsible process $\{\phi_1(t), \cdots, \phi_n(t)\}$ such that

$$M(t) = M(0) + \sum_{j=1}^{n} \int_0^t \phi_j(s) d\widetilde{S}_j(s).$$

Our hedging strategy will be to hold $\phi_i(t)$ units of the ith stock at time t for each $i = 1, \cdots, n$, and to hold $\psi(t)$ units of bond where

$$\psi(t) = M(t) - \sum_{j=1}^{n} \phi_j(t) \widetilde{S}_j(t).$$

The value of the portfolio is then $V(t) = B(t)M(t)$, which at time T is exactly the value of the claim, and the portfolio is self-financing in that

$$dV(t) = \sum_{j=1}^{n} \phi_i(t) dS_j(t) + \psi(t) dB(t).$$

In the absence of arbitrage the value of the derivative at time t is

$$V(t) = B(t)\mathbb{E}^{\mathbb{Q}}[B^{-1}(T)C(T)|\mathcal{F}(t)] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[C(T)|\mathcal{F}(t)].$$

4.2 Multidimensional Feynman-Kac Stochastic representation

We will use a delta-hedging argument to obtain the multidimensional Black-Scholes formula at Chapter 5. In this section, the partial differential equation can also be obtained directly from the expectation price and a multidimensional version of the Feynman-Kac stochastic representation.

Theorem 4.4. *Let $\sigma(t,x) = (\sigma_{ij}(t,x))$ be a real symmetric $n \times n$ matrix, $\Phi : \mathbf{R}^n \to \mathbf{R}$ and $\mu_i : [0,\infty) \times \mathbf{R}^n \to \mathbf{R}$, $i = 1, \cdots, n$, be real-valued functions and r be a constant. We suppose that the functions $F(t,x)$, defined for $(t,x) \in [0,\infty) \times \mathbf{R}^n$, solves the boundary value problem*

$$\partial_t F(t,x) + \sum_{i=1}^{n} \mu_i(t,x)\partial_i F(t,x) + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}\partial_i\partial_j F(t,x) - rF(t,x) = 0,$$

$$F(T,x) = \Phi(x).$$

Assume further that for each $i = 1, \cdots, n$, the process $\{X_i(t)\}$ solves the stochastic differential equation

$$dX_i(t) = \mu_i(t, X(t))dt + \sum_{j=1}^{n} \sigma_{ij}(t, X(t))dW_j(t),$$

where $X(t) = \{X_1(t), \cdots, X_n(t)\}$. Finally, suppose that

$$\int_0^T \mathbb{E}\left[\sum_{j=1}^{n} (\sigma_{ij}(s, X(s))\partial_i F(s, X(s)))^2\right] ds < \infty, \quad i = 1, \cdots, n.$$

Then

$$F(t,x) = e^{-r(T-t)}\mathbb{E}[\Phi(X(T))|X(t) = x].$$

Lemma 4.5. *Let* $S(t) = \{S_1(t), \cdots, S_n(t)\}$ *be as above and* $C(T) = \Phi(S(T))$ *be a claim at time* T*. Then the price of the claim at time* $t < T$*,*

$$V(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\Phi(S(T))|\mathcal{F}(t)] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\Phi(S(T))|S(t) = x] := F(t,x) \quad (4.4)$$

satisfies

$$\partial_t F(t,x) + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}(t,x)x_i x_j \partial_i \partial_j F(t,x) + r\sum_{i=1}^{n} x_i \partial_i F(t,x) - rF(t,x) = 0,$$

$$F(T,x) = \Phi(x).$$

Proof. The process $\{S(t)\}$ is governed by

$$dS_i(t) = rS_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t, S(t))S_i(t)dX_j(t),$$

where $\{X_j(t)\}, j = 1, \cdots, n$, are $\mathbb{Q}$-Brownian motions, so the result follows from an application of Theorem 4.4. $\square$

CHAPTER 5
THE MULTIDIMENSIONAL BLACK-SCHOLES FORMULA

5.1 The Black-Scholes analysis

In this section, we will describe the multidimensional Black-Scholes formula under the assumptions from the Section 1.1. Here, we will apply an n-factor Itô formula 4.2 in the stock model (3.2). Then we can get the multidimensional Black-Scholes partial differential equation by the delta hedging method.

Remark. Notice that the matrix $C = [C_{ij}]$ is positive definite, since for all $\eta \in \mathbf{R}^n$,

$$\eta^T C\eta = \eta^T[\sigma_0][\sigma_0^T]\eta = |\sigma_0^T\eta|^2 \geq 0.$$

Now we construct a portfolio consisting of one option and numbers $-\triangle_i$ of the underlying asset S_i. These numbers are as yet specified. The value of this portfolio is

$$\Pi = V(S_1, \cdots, S_n, t) - \sum_{i=1}^{n} \triangle_i S_i.$$

The jump in the value of this portfolio in one time-step is

$$d\Pi = r\Pi dt = r(V - \sum_{i=1}^{n} \triangle_i S_i)dt, \tag{5.1}$$

with the risk-free rate r. Also, we find that Π follows the random walk

$$\begin{aligned} d\Pi =& dV - \sum_{i=1}^{n} \triangle_i dS_i \\ =& \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}S_iS_j\frac{\partial^2 V}{\partial S_i \partial S_j}\right)dt + \sum_{i=1}^{n}\frac{\partial V}{\partial S_i}dS_i - \sum_{i=1}^{n}\triangle_i dS_i. \end{aligned} \tag{5.2}$$

Here, we can eliminate the random component in this random walk by choosing $\triangle_i = \frac{\partial V}{\partial S_i}$. From (5.1) and (5.2), we obtain the multidimensional Black-Scholes formula

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{n} r S_i \frac{\partial V}{\partial S_i} - rV = 0. \tag{5.3}$$

5.2 PDE derivation

As we see, Equation (5.3) is not the heat equation, but we can transform it into the heat equation by performing a change of variables and using a property of the volatility matrix C. For a specific terminal condition, we are able to find an expression for the value by solving the Theorem 5.1.

Theorem 5.1. *The explicit solution of multidimensional Black-Scholes formula with a certain terminal condition*

$$\begin{cases} V(S_1, \cdots, S_n, T) = P(S_1, \cdots, S_n), \\ 0 \le S_i < \infty, 0 \le t \le T. \end{cases}$$

is

$$\begin{aligned} V(S,t) = &\left[\frac{1}{2\pi(T-t)}\right]^{\frac{n}{2}} \frac{\exp(-r(T-t))}{|\det C|^{\frac{1}{2}}} \\ &\int_0^\infty \cdots \int_0^\infty \frac{P(\eta_1, \cdots, \eta_n)}{\eta_1 \cdots \eta_n} \exp\left[-\frac{\overrightarrow{\alpha}^T C^{-1} \overrightarrow{\alpha}}{2(T-t)}\right] d\eta_1 \cdots d\eta_n, \end{aligned} \tag{5.4}$$

with $\overrightarrow{\alpha}^T = (\alpha_1 \cdots \alpha_n)$, *where* $\alpha_i = \ln \frac{S_i}{\eta_i} + \left(r - \frac{C_{ii}}{2}\right)(T-t)$.

Proof. Let $s_i = \ln S_i$. We obtain the backward Kolmogorov equation

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij} \frac{\partial^2 V}{\partial s_i \partial s_j} + \sum_{i=1}^{n} \left(r - \frac{C_{ii}}{2}\right) \frac{\partial V}{\partial s_i} - rV = 0 \\ \Leftrightarrow &\frac{\partial V}{\partial t} + \frac{1}{2} D_s{}^T C D_s V + \overrightarrow{b}^T D_s V - rV = 0, \end{aligned} \tag{5.5}$$

with

$$\overrightarrow{b} = \begin{pmatrix} r - \frac{C_{11}}{2} \\ \vdots \\ r - q_n - \frac{C_{nn}}{2} \end{pmatrix}, D_s{}^1 = \begin{pmatrix} \frac{\partial}{\partial s_1} \\ \vdots \\ \frac{\partial}{\partial s_n} \end{pmatrix}.$$

Because C is positive definite, we can find the orthonormal matrix B such that

$$BCB^T = M = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

where $\lambda_1, \cdots, \lambda_n$ are eigenvalues of C. Furthermore, we can find the eigenvector

$$\overrightarrow{\xi_i} = \begin{pmatrix} \xi_{i1} \\ \vdots \\ \xi_{in} \end{pmatrix},$$

for each corresponding eigenvalue λ_i. So the orthonormal matrix B is given by

$$\begin{pmatrix} \overrightarrow{\xi_i}^T \\ \vdots \\ \overrightarrow{\xi_n}^T \end{pmatrix} = \begin{pmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \vdots & & & \vdots \\ \xi_{n1} & \xi_{n2} & \cdots & \xi_{nn} \end{pmatrix}.$$

If we use the change of variables $\overrightarrow{z} = B\overrightarrow{s}$ and $D_s = B^T D_z$, then Equation (5.5) becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2} D_z{}^T (BCB^T) D_z V + (B\overrightarrow{b})^T D_z V - rV = 0, \tag{5.6}$$

[1]Here, D_s means the gradient as usual. It can be replaced by ∇_s. However, we adopt this notation because the notation ∇ will be used for covariant derivatives, not for partial derivatives. Similarly, the notation D_z on the next page is the gradient of z.

with the terminal condition $V(z_1, \cdots, z_n, T) = P(z_1, \cdots, z_n)$. Now we are able to get the fundamental solution for the equation (5.6). This means that the terminal condition is assumed to be the Dirac delta-function.

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{i=1}^{n} \lambda_i \dfrac{\partial^2 V}{\partial z_i^2} + \sum_{i=1}^{n} (\overrightarrow{\xi_i}^T \overrightarrow{b}) \dfrac{\partial V}{\partial z_i} - rV = 0, \\ \\ V(z_1, \cdots, z_n, T) = \delta(z_1 - z_1^0, \cdots, z_n - z_n^0), \end{cases} \tag{5.7}$$

where δ is a dirac measure which has a property $\delta(z_1, \cdots, z_n) = \delta(z_1) \cdots \delta(z_n)$. The equation (5.7) is a heat equation. We need to get the heat equation by using a change of variable $V = \exp\big(\overrightarrow{\omega}^T \overrightarrow{z} + \beta(T-t)\big)W$, for some constant vector $\overrightarrow{\omega}^T = (\omega_1 \cdots \omega_n)$ and constant β to be found. Then differentiation gives

$$\frac{\partial W}{\partial t} + \sum_{i=1}^{n} [\lambda_i \omega_i + (\overrightarrow{\xi_i}^T \overrightarrow{b})] \frac{\partial W}{\partial z_i} + \frac{1}{2} \sum_{i=1}^{n} \lambda_i \frac{\partial^2 W}{\partial z_i^2} - [r + \beta - \frac{1}{2}(\overrightarrow{\omega}^T M \overrightarrow{\omega}) - \overrightarrow{b}^T B^T \overrightarrow{\omega}]W = 0.$$

We obtain an equation with no $\frac{\partial W}{\partial z_i}$ term by choosing $\omega_i = -\frac{1}{\lambda_i}(\overrightarrow{\xi_i}^T \overrightarrow{b})$, that is $\overrightarrow{\omega} = -M^{-1} B \overrightarrow{b}$, while the choice

$$\begin{aligned} \beta &= -r + \frac{1}{2}(\overrightarrow{\omega}^T M \overrightarrow{\omega}) + \overrightarrow{b}^T B^T \overrightarrow{\omega} \\ &= -r + \frac{1}{2}(B\overrightarrow{b})^T M^{-1} M M^{-1} (B\overrightarrow{b}) - \overrightarrow{b}^T B^T M^{-1} B \overrightarrow{b} \\ &= -r - \frac{1}{2}(B\overrightarrow{b})^T M^{-1} (B\overrightarrow{b}) \end{aligned}$$

eliminates the W term as well. We then have

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{i=1}^{n} \lambda_i \dfrac{\partial^2 W}{\partial z_i^2} = 0, \\ \\ W(z_1, \cdots, z_n, T) = \exp(-\overrightarrow{\omega}^T \overrightarrow{z_0}) \delta(z_1 - z_1^0, \cdots, z_n - z_n^0), \end{cases} \tag{5.8}$$

where $\overrightarrow{z_0}^T = (z_1^0 \cdots z_n^0)$. The fundamental solution of the above equation is

$$
\begin{aligned}
W(z,t;z_0) &= \exp\left(-\overrightarrow{\omega}^T\overrightarrow{z_0}\right)\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\lambda_i(T-t)}}\exp\left(-\frac{(z_i-z_i^0)^2}{2\lambda_i(T-t)}\right)\\
&= \exp\left(-\overrightarrow{\omega}^T\overrightarrow{z_0}\right)\left[\frac{1}{2\pi(T-t)}\right]^{\frac{n}{2}}(\det M^{-1})^{\frac{1}{2}}\exp\left(-\frac{(\overrightarrow{z}-\overrightarrow{z_0})^T M^{-1}(\overrightarrow{z}-\overrightarrow{z_0})}{2(T-t)}\right).
\end{aligned}
$$

From the relation between the variables s and z, we can find the fundamental solution Γ of the equation (5.6).

$$
\begin{aligned}
\Gamma(s,t;s_0) = &\left[\frac{1}{2\pi(T-t)}\right]^{\frac{n}{2}}\frac{\exp(-r(T-t))}{|\det C|^{\frac{1}{2}}}\exp\left\{-\frac{(B(\overrightarrow{s}-\overrightarrow{s_0}))^T M^{-1}(B(\overrightarrow{s}-\overrightarrow{s_0}))}{2(T-t)}\right.\\
&\left.-\frac{1}{2}(B\overrightarrow{b})^T M^{-1}(B\overrightarrow{b})(T-t)-(M^{-1}B\overrightarrow{b})^T B(\overrightarrow{s}-\overrightarrow{s_0})\right\}.
\end{aligned}
$$

Since $B^TM^{-1}B = B^{-1}M^{-1}(B^{-1})^T = (B^TMB)^{-1} = C^{-1}$,

$$
\begin{aligned}
\Gamma(s,t;s_0) = &\left[\frac{1}{2\pi(T-t)}\right]^{\frac{n}{2}}\frac{\exp(-r(T-t))}{|\det C|^{\frac{1}{2}}}\\
&\exp\left\{-\frac{1}{2(T-t)}[(\overrightarrow{s}-\overrightarrow{s_0}+\overrightarrow{b}(T-t))^T C^{-1}(\overrightarrow{s}-\overrightarrow{s_0}+\overrightarrow{b}(T-t))]\right\}.
\end{aligned}
$$

Consequently the convolution

$$
V(s,t) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Gamma(s,t;\zeta)P(e^{\zeta_1},\cdots,e^{\zeta_n})d\zeta_1\cdots d\zeta_n
$$

is the solution of the initial value problem at Theorem 5.1. If we put the original variables back into the above, this is the solution as we desire. □

Finally, we can obtain the explicit solution for the multidimensional Black-Scholes formula. However, this model still has many unrealistic assumptions, such as constant volatilities. To solve the multidimensional problem we will introduce the heat kernel asymptotics which are developed in differential geometry. From the heat kernel asymptotics, we can get a better approximation because we will no longer be making the constant volatility assumption. We will compute the heat kernel asymptotics in n-dimensional case in general. Moreover, we will derive the heat kernel asymptotics in the two-dimensional case in Section 6.2.

CHAPTER 6
USE OF CURVATURE AND HEAT EQUATION ASYMPTOTICS

6.1 The Heat Kernel Asymptotics in $\mathbf{R}^n$

In this section, the equation in Theorem 5.1 is a parabolic partial differential equation (PDE) which can be related to the canonical heat equation on a Riemannian manifold without boundary (see [6] and [15]). The heat kernel is generally given in terms of the Riemann tensor and its covariant derivatives. But, if the heat kernel asymptotics on a Riemannian metric are chosen by the needed order, the Riemann tensor is replaced by the scalar curvature K and its covariant derivatives. See Section 2.2 for more details about the metric g, the affine connection ∇, the scalar curvature K and bundle endomorphism E. Here, the bundle endomorphism E is calculated by the equation (6.1), the metric inverse g^{ij} and K are directly related to the covariance matrix C because metric needs to be $\frac{1}{2}C_{ij}$ and the scalar curvature is defined as the trace of Ricci curvature with respect to the metric.

Let $\tau = T - t$ and let $D = -\frac{1}{2}\sum_{i,j=1}^{n} C_{ij}\partial_i\partial_j - \sum_{i=1}^{n}\left(r - \frac{C_{ii}}{2}\right)\partial_i + r$ be the differential operator which appears in (5.5). Then D is a second-order elliptic Laplace type operator. We want to move toward a version of this that is good in any coordinate system x^i:

$$D = -g^{ij}\nabla_i\nabla_j - E. \tag{6.1}$$

Here, we have used the Einstein summation convention which is a notational convention useful when dealing with coordinate equations or formulas. According to this convention, when an index variable appears twice in a single term, it implies that we are summing over all of its possible values. We will use the Einstein convention throughout this section. Clearly the metric inverse is $g^{ij} = \frac{1}{2}C_{ij}$ in the s_i coordinates. The ∇ is made from the Levi-Civita connection ∇^{LC} associated with g, and a certain line bundle connection $\nabla^{\mathcal{L}}$ which will be chosen exactly to make sure there is no order 1 term in the PDE. That is,

$$\nabla = \nabla^{LC} \otimes 1 + 1 \otimes \nabla^{\mathcal{L}}$$

After this choice is made uniquely, E is a line bundle endomorphism. Then for the equation

$$V_\tau + (-g^{ij}\nabla_i\nabla_j - E)V = 0, \tag{6.2}$$

the heat kernel asymptotics are given in terms of g and its inverse, ∇, the Riemann curvature R of ∇^{LC}, and the curvature ω of the line bundle connection.

If we go back to see the original problem (5.5), the connection 1-form ω_i is uniquely determined to be

$$\omega_i = \frac{1}{2}\left\{g_{ij}\left(r - \frac{C_{jj}}{2}\right) + {\Gamma_{ji}}^{j}\right\},$$

with indices raised and lowered using the metric. We introduced the Christoffel symbol ${\Gamma_{ji}}^{j}$ in the Section 2.2. The bundle endomorphism is

$$E = -r - \{\partial^j\omega_j - {\Gamma^j}_j{}^k\omega_k + \omega^j\omega_j\}. \tag{6.3}$$

Then the small-time asymptotics of the heat kernel at the origin y are

$$H(\tau, x, y) \sim (4\pi\tau)^{-\frac{n}{2}} \exp\left(-\frac{\sigma(x,y)^2}{4\tau}\right) U(\tau, x, y), \qquad \tau \to 0.$$

Here $\sigma = \sigma(x, y)$ is the geodesic distance and $U(\tau, x, y) = \sum_{i=0}^{\infty} \tau^i U_i(x, y)$.

If the initial condition is $V(0, x)$, the solution for the equation (6.2) is

$$V(\tau, x) = \int H_0(\tau, x, y)\{U_0(x, y) + \tau U_1(x, y)\}V(0, y)dy,$$

with the flat heat kernel $H_0(\tau, x, y)$.

This theorem was developed by Pleijel-Minakshisundaram-De Witt-Gilkey [17] and [6]. In many references, only the diagonal values $U(\tau, x, x)$ of the heat kernel are considered; these of course produce the largest analytic effect. In fact, the two-point functions are

$$\begin{aligned} U_0(x, x) &= 1, \\ U_1(x, x) &= E + \frac{1}{6}K, \cdots . \end{aligned} \tag{6.4}$$

However, I claim that there is information to be gained from off-diagonal values $U(\tau, x, y)$ in the parabolic region where $|x - y|^2$ and τ are comparable, and that this information has implications for the multi-asset model. This off-diagonal information is carried by higher-order asymptotics in the multivariate quantity $x - y$. See [1] for more details on off-diagonal values. $U_i \in C^\infty$ are uniquely defined by the system in the paper [3].

$$\begin{cases} 2\langle d\sigma, dU_0\rangle - (\triangle\sigma + 2n)U_0 = 0, \\ 2\langle d\sigma, dU_i\rangle - (\triangle\sigma + 2n - 4i)U_i = -4DU_{i-1}, \\ U_0(y, y) = 1, \quad U_i \text{ bounded as } x \to y, \quad i > 0, \end{cases} \tag{6.5}$$

where $\langle \cdot, \cdot \rangle$ is the inner product. Since

$$
\begin{aligned}
\triangle\sigma &= -\frac{1}{\sqrt{g}}\partial_i\left(g^{ij}\left(\partial_j(\mathring{g}_{kl}x^k x^l)\right)\sqrt{g}\right) \\
&= -\frac{1}{\sqrt{g}}\partial_i\left(\delta^i{}_k 2x^k\sqrt{g}\right) \\
&= -2n - \frac{1}{\sqrt{g}}2x^i\partial_i\sqrt{g} \\
&= -2n - \frac{1}{\sqrt{g}}2X\sqrt{g},
\end{aligned}
$$

with g the determinant of the metric (g_{ij}) and $\mathring{g}_{ij} = \delta_{ij}$ the constant matrix encoding the entries of g_{ij} at the origin of normal coordinates, we can obtain

$$
\begin{aligned}
\triangle\sigma + 2n &= -\frac{2}{\sqrt{g}}X\sqrt{g} \\
&= -2X(\ln\sqrt{g}) \\
&= -X(\ln g),
\end{aligned}
$$

and $\langle d\sigma, dU_0\rangle = 2XU_0$. From the equation in (6.5),

$$4XU_0 + \{X(\ln g)\}U_0 = 0.$$

Therefore,

$$U_0(x, y) = g^{-\frac{1}{4}} \tag{6.6}$$

because $4XU_0 + \{X(\ln g)\}U_0 = 4X(g^{-\frac{1}{4}}U_0)$. Now we need to calculate all expansions for U_i, U_{i-1}, σ and g in terms of x. First, we consider a covariant expansion for g_{ij} in [15]. The method used is one developed by Herglotz and later modified by Günther in the Appendix. The expansion for g_{ij} to 4th order is

$$
\begin{aligned}
g_{ij} =& \mathring{g}_{ij} - \frac{1}{3}\mathring{R}_{i\alpha_1 j\alpha_2}x^{\alpha_1}x^{\alpha_2} - \frac{1}{6}\nabla_{\alpha_3}\mathring{R}_{i\alpha_1 j\alpha_2}x^{\alpha_1}x^{\alpha_2}x^{\alpha_3} \\
&- \frac{1}{20}\left(\nabla_{\alpha_4}\nabla_{\alpha_3}\mathring{R}_{i\alpha_1 j\alpha_2} - \frac{8}{9}\mathring{R}_{i\alpha_1 j\alpha_2}\mathring{R}^{j}_{\alpha_3 j\alpha_4}\right)x^{\alpha_1}x^{\alpha_2}x^{\alpha_3}x^{\alpha_4} + O(x^5).
\end{aligned}
$$

From the above setup, we can find off-diagonal values in general. However, we try to find off-diagonal values in 2 dimensions because we will apply it to 2-color rainbow options.

6.2 The Heat Kernel Asymptotics in $\mathbf{R}^2$

Theorem 6.1. *The heat kernel asymptotics in* $\mathbf{R}^2$ *are*

$$
H(\tau, x, y) \sim (4\pi\tau)^{-1}\exp\left(-\frac{\sigma(x,y)^2}{4\tau}\right)\left\{U_0(x,y) + \tau U_1(x,y)\right\},
$$

where

$$
\begin{aligned}
U_0(x,y) &= 1 + \frac{1}{24}K\mathring{g}_{ij}(x^i - y^i)(x^j - y^j) + \frac{1}{48}\nabla_{\alpha_3}K\mathring{g}_{ij}(x^i - y^i)(x^j - y^j)(x^{\alpha_3} - y^{\alpha_3}), \\
U_1(x,y) &= E + \frac{1}{6}K + \frac{1}{12}\nabla_i K(x^i - y^i),
\end{aligned}
$$

with E is given at (6.3).

Proof. In 2 dimensions, the condition (6.5) becomes

$$
\begin{cases}
2\langle d\sigma, dU_0\rangle - (\triangle\sigma + 4)U_0 = 0, \\
2\langle d\sigma, dU_i\rangle - (\triangle\sigma + 4 - 4i)U_i = -4DU_{i-1}, \\
U_0(y,y) = 1, \quad U_i \text{ bounded as } x \to y, \quad i > 0.
\end{cases}
\tag{6.7}
$$

In addition, we can use

$$R_{i\alpha_1 j\alpha_2} = \frac{1}{2}K(g_{ij}g_{\alpha_1\alpha_2} - g_{i\alpha_2}g_{\alpha_1 j}),$$
$$\nabla_{\alpha_n}\cdots\nabla_{\alpha_3}R_{i\alpha_1 j\alpha_2} = \frac{1}{2}(\nabla_{\alpha_n}\cdots\nabla_{\alpha_3}K)(g_{ij}g_{\alpha_1\alpha_2} - g_{i\alpha_2}g_{\alpha_1 j}),$$

where K is the scalar curvature. Thus,

$$\nabla_{\alpha_n}\cdots\nabla_{\alpha_3}R_{i\alpha_1 j\alpha_2}x^{\alpha_1}\cdots x^{\alpha_n} = \frac{1}{2}(\nabla_{\alpha_n}\cdots\nabla_{\alpha_3}K)(g_{ij}\sigma^2 - x_i x_j)x^{\alpha_3}\cdots x^{\alpha_n},$$

From the expansion of g_{ij}, we can get an expansion for g. It is

$$\begin{aligned} g =& g_{11}g_{22} - {g_{12}}^2 \\ =& 1 - \frac{1}{6}K\mathring{g}_{ij}x^i x^j - \frac{1}{12}\nabla_{\alpha_3}K\mathring{g}_{ij}x^i x^j x^{\alpha_3} \\ &- \frac{1}{40}\nabla_{\alpha_4}\nabla_{\alpha_3}K\mathring{g}_{ij}x^i x^j x^{\alpha_3}x^{\alpha_4} + \frac{1}{90}K^2((x^1)^4 + (x^2)^4) \\ &- \frac{1}{180}\nabla_{\alpha_5}\nabla_{\alpha_4}\nabla_{\alpha_3}K\mathring{g}_{ij}x^i x^j x^{\alpha_3}x^{\alpha_4}x^{\alpha_5} + \frac{1}{90}K\nabla_{\alpha_3}K((x^1)^4 + (x^2)^4)x^{\alpha_3} \\ &- \frac{1}{1008}\nabla_{\alpha_6}\nabla_{\alpha_5}\nabla_{\alpha_4}\nabla_{\alpha_3}K\mathring{g}_{ij}x^i x^j x^{\alpha_3}x^{\alpha_4}x^{\alpha_5}x^{\alpha_6} \\ &+ \frac{17}{5040}K(\nabla_{\alpha_4}\nabla_{\alpha_3}K)((x^1)^4 + (x^2)^4)x^{\alpha_3}x^{\alpha_4} \\ &- \frac{1}{2520}K^3((x^1)^6 + (x^2)^6) - \frac{1}{540}((x^1)^6 + (x^2)^6) + O(x^7). \end{aligned}$$

From the expansion of g and (6.6), U_0 can be extended in terms of x.

$$\begin{aligned} U_0(x,y) =& g^{-\frac{1}{4}} \\ =& 1 + \frac{1}{24}K\mathring{g}_{ij}x^i x^j + \frac{1}{48}\nabla_{\alpha_3}K\mathring{g}_{ij}x^i x^j x^{\alpha_3} \\ &+ \frac{1}{160}\nabla_{\alpha_4}\nabla_{\alpha_3}K\mathring{g}_{ij}x^i x^j x^{\alpha_3}x^{\alpha_4} + \frac{1}{640}K^2(\mathring{g}_{ij}x^i x^j)^2 + \frac{1}{180}K^2(x^1)^2(x^2)^2 + O(x^5). \end{aligned}$$

For the expansions for $U_i's$, we need to get the expansions for $\ln g$, $X(\ln g)$, $\sqrt{g}$ and

$\frac{1}{\sqrt{g}}$. These expansions come from the expansion of g.

$$\begin{aligned}\ln g =& -\frac{1}{6}K\mathring{g}_{ij}x^ix^j - \frac{1}{12}\nabla_{\alpha_3}K\mathring{g}_{ij}x^ix^jx^{\alpha_3} - \frac{1}{40}\nabla_{\alpha_4}\nabla_{\alpha_3}K\mathring{g}_{ij}x^ix^jx^{\alpha_3}x^{\alpha_4}\\ &- \frac{1}{360}K^2(\mathring{g}_{ij}x^ix^j)^2 - \frac{1}{45}K^2(x^1)^2(x^2)^2 + O(x^5),\end{aligned}$$

$$\begin{aligned}X(\ln g) =& -\frac{1}{3}K\mathring{g}_{ij}x^ix^j - \frac{1}{4}\nabla_{\alpha_3}K\mathring{g}_{ij}x^ix^jx^{\alpha_3} - \frac{1}{10}\nabla_{\alpha_4}\nabla_{\alpha_3}K\mathring{g}_{ij}x^ix^jx^{\alpha_3}x^{\alpha_4}\\ &- \frac{1}{90}K^2(\mathring{g}_{ij}x^ix^j)^2 - \frac{4}{45}K^2(x^1)^2(x^2)^2 + O(x^5),\end{aligned}$$

$$\begin{aligned}\sqrt{g} =& 1 - \frac{1}{12}K\mathring{g}_{ij}x^ix^j - \frac{1}{24}\nabla_{\alpha_3}K\mathring{g}_{ij}x^ix^jx^{\alpha_3} - \frac{1}{80}\nabla_{\alpha_4}\nabla_{\alpha_3}K\mathring{g}_{ij}x^ix^jx^{\alpha_3}x^{\alpha_4}\\ &+ \frac{1}{480}K^2((x^1)^4 + (x^2)^4) - \frac{1}{144}K^2(x^1)^2(x^2)^2 + O(x^5),\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\sqrt{g}} =& 1 + \frac{1}{12}K\mathring{g}_{ij}x^ix^j + \frac{1}{24}\nabla_{\alpha_3}K\mathring{g}_{ij}x^ix^jx^{\alpha_3} + \frac{1}{80}\nabla_{\alpha_4}\nabla_{\alpha_3}K\mathring{g}_{ij}x^ix^jx^{\alpha_3}x^{\alpha_4}\\ &+ \frac{7}{1440}K^2((x^1)^4 + (x^2)^4) + \frac{1}{48}K^2(x^1)^2(x^2)^2 + O(x^5).\end{aligned}$$

Now we can try to obtain U_1. From the equation (6.7),

$$4X(U_1) + (X(\ln g) + 4)U_1 = 4g^{ij}\nabla_i\nabla_jU_0 + 4EU_0. \tag{6.8}$$

The U_i have their own asymptotic tails. In fact, in normal coordinates x^i with origin y, we can say $U_1 = \overset{(0)}{U_1} + (\overset{(1)}{U_1})_ix^i + (\overset{(2)}{U_1})_{ij}x^ix^j + \cdots$. At the constant level,

$$4\overset{(0)}{U_1} = \frac{2}{3}K + 4E,$$

so that

$$\overset{(0)}{U_1} = \frac{1}{6}K + E.$$

Similarly, we can get the (x^i) and $(x^i x^j)$ levels. Therefore,

$$
\begin{aligned}
U_1 = &\left(\frac{1}{6}K + E\right) + \frac{1}{12}\nabla_i K x^i \\
&+ \left(\frac{13}{1080}K^2 + \frac{1}{240}(7\nabla_1\nabla_1 K + \nabla_2\nabla_2 K) + \frac{1}{216}K^2 + \frac{1}{12}KE_{11}\right)(x^1)^2 \\
&+ 2\left(\frac{1}{40}\nabla_1\nabla_2 K + \frac{1}{18}KE_{12}\right)x^1x^2 \\
&+ \left(\frac{13}{1080}K^2 + \frac{1}{240}(7\nabla_2\nabla_2 K + \nabla_1\nabla_1 K) + \frac{1}{216}K^2 + \frac{1}{12}KE_{22}\right)(x^2)^2 + O(x^3),
\end{aligned}
$$

where E_{ij} is the (i,j)-th entry of the matrix E. Systematically, we can obtain U_2. It is

$$U_2 = \frac{1}{30}K^2 + \frac{1}{30}\mathring{g}_{ij}\nabla_i\nabla_j K + \frac{1}{12}K(E_{11} + E_{22}) + \frac{1}{12}KE + \frac{1}{2}E^2.$$

So far, we can obtain the new heat kernel asymptotics. From the above process, we are able to calculate more U_i's. However, we will say the asymptotics are defined at Theorem 6.1 because $dx^2 \to d\tau$ as $d\tau \to 0$. In other words, the order $d\tau$ terms play a significant part and any other choice for the order of dx would not lead to interesting results. For this reason, we expand U_0 and U_1 up to the third power of x. $\square$

CHAPTER 7
RAINBOW OPTIONS

There are three groups of the multifactor options: rainbow, quanto and basket options [9]. In this section, we will find a price of rainbow options. The value of a rainbow option is determined by the performance of two or more underlying assets. There are three different payoffs for two-color rainbow options.

1. Better-of/worse-of rainbow option: $\max(S_1, S_2)/\min(S_1, S_2)$,

2. Outerperformance option: $\max(S_2 - S_1, 0)$,

3. Max/min call option

$$\max(\max(S_1 - X_1, S_2 - X_2), 0)/\max(\min(S_1 - X_1, S_2 - X_2), 0),$$

where X_i are the strike price of the asset i, $i = 1, 2$.

In particular, we will find the price of an outperformance option[1]. After we get the price of the outperformance option, we can derive the price of others.

7.1 Two-color outperformance option

As we can see, the outperformance option is based on the difference in the performance of two assets. For example, an investor holding a position in the S&P 100 could protect against the S&P 100 underperforming the Dow Jones Industrial Average by purchasing a rainbow outperformance option that would pay out according to the

[1]Outperformance options are sometimes referred to as "spread" options. The payoff structure of spread option is $\max(S_2 - S_1 - X, 0)$, where X is the strike price of this option.

difference in returns of the two indexes. If the Dow Jones Industrial Average increased by 12% and the S&P 100 increased by 5%, the holder would receive a return based on the 7% spread between the returns. But if the Dow Jones Industrial Average underperformed the S&P 100, the payout is zero.

Here is the partial differential equation and the terminal condition for this option.

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{2} C_{ij} S_i S_j \dfrac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{2} r S_i \dfrac{\partial V}{\partial S_i} - rV = 0, \\ \\ V(S_1, S_2, T) = \max(S_2 - S_1, 0). \end{cases} \tag{7.1}$$

We will price it by using the heat kernel asymptotics. Let $x_i = \ln S_i$, $\tau = T - t$. From these substitutions, we obtain

$$\begin{cases} \dfrac{\partial V}{\partial \tau} + \left(-\dfrac{1}{2}\displaystyle\sum_{i,j=1}^{2} C_{ij} \dfrac{\partial^2 V}{\partial x_i \partial x_j} - \sum_{i=1}^{2} \left(r - \dfrac{C_{ii}}{2} \right) \dfrac{\partial V}{\partial x_i} + rV \right) = 0, \\ \\ V(x_1, x_2, 0) = \max(e^{x_2} - e^{x_1}, 0). \end{cases} \tag{7.2}$$

We choose the connection 1-form $\omega_i = \frac{1}{2}\left\{ g_{ij}\left(r - \frac{C_{jj}}{2} \right) + \Gamma_{ji}{}^{j} \right\}$ and the line bundle connection $E = -r - \{\partial^j \omega_j - \Gamma^j{}_j{}^k \omega_k + \omega^j \omega_j\}$. Equation (7.2) then becomes

$$\begin{cases} \dfrac{\partial V}{\partial \tau} + \left(-\dfrac{1}{2}\displaystyle\sum_{i,j=1}^{2} C_{ij} \nabla_i \nabla_j V - EV \right) = 0, \\ \\ V(x_1, x_2, 0) = \max(e^{x_2} - e^{x_1}, 0). \end{cases} \tag{7.3}$$

From the heat kernel asymptotics in (6.4), the solution for (7.3) will be written by

$$\begin{aligned} V(x_1,x_2,\tau) =& \frac{1}{4\pi\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{\sigma^2}{4\tau}\right)\left(1+\tau\left(E+\frac{1}{6}K\right)\right)V(y_1,y_2,0)dy_1dy_2 \\ =& \frac{1}{4\pi\tau}\left(1+\tau\left(E+\frac{1}{6}K\right)\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{\sigma^2}{4\tau}\right)V(y_1,y_2,0)dy_1dy_2, \end{aligned} \tag{7.4}$$

where $\sigma^2 = (x_1-y_1)^2+(x_2-y_2)^2$. From (9) and (10) in Appendix, the option price $V(S_1,S_2,t)$ is

$$e^{(T-t)}\left(1+(T-t)\left(E+\frac{1}{6}K\right)\right)\left\{S_2N\left(\alpha(S_1,S_2)\right)-S_1N\left(\alpha(S_1,S_2)-2\sqrt{T-t}\right)\right\},$$

where $N(\cdot)$ is the cumulative distribution function for the normal distribution and

$$\alpha(S_1,S_2) = \frac{1}{2\sqrt{T-t}}\left\{\ln\left(\frac{S_2}{S_1}\right)+2(T-t)\right\}.$$

Now we will apply off-diagonal values which have more precise terms. From the off-diagonal values at the end of Section 6.2, we can price the option much more accurately.

$$V(x_1,x_2,\tau) = \frac{1}{4\pi\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{\sigma^2}{4\tau}\right)H(\tau,x,y)V(y_1,y_2,0)dy_1dy_2,$$

where $H(\tau,x,y) = U_0(x,y)+\tau U_1(x,y)$, with

$$\begin{aligned} U_0(x,y) &= 1+\frac{1}{24}K\delta_{ij}(x_i-y_i)(x_j-y_j)+\frac{1}{48}\nabla_k K\delta_{ij}(x_i-y_i)(x_j-y_j)(x_k-y_k), \\ U_1(x,y) &= E+\frac{1}{6}K+\frac{1}{12}\nabla_i K(x_i-y_i). \end{aligned}$$

In this case, $\mathring{g}_{ij} = \delta_{ij}$ because of the construction of normal coordinates.

If we plug off-diagonal values and rewrite it, we can obtain the solution more explicitly.

$$
\begin{aligned}
&V(x_1,x_2,\tau) = \frac{1}{4\pi\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{\sigma^2}{4\tau}\right)H(\tau,x,y)V(y_1,y_2,0)dy_1dy_2\\
&=\frac{1}{4\pi\tau}\left[\left(1+\tau\left(E+\frac{1}{6}K\right)\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{\sigma^2}{4\tau}\right)V(y_1,y_2,0)dy_1dy_2\right.\\
&\quad+\left(\frac{1}{12}\nabla_i K\tau\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{\sigma^2}{4\tau}\right)(x_i-y_i)V(y_1,y_2,0)dy_1dy_2\\
&\quad+\left(\frac{1}{24}K\delta_{ij}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{\sigma^2}{4\tau}\right)(x_i-y_i)(x_j-y_j)V(y_1,y_2,0)dy_1dy_2\\
&\quad\left.+\left(\frac{1}{48}\nabla_k K\delta_{ij}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{\sigma^2}{4\tau}\right)(x_i-y_i)(x_j-y_j)(x_k-y_k)V(y_1,y_2,0)dy_1dy_2\right].
\end{aligned}
$$

For convenience, let the flat heat kernel

$$
\frac{1}{4\pi\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{\sigma^2}{4\tau}\right)V(y_1,y_2,0)dy_1dy_2 := F((x_1,x_2),\tau).
$$

From (11), (12) and (13) in Appendix,

$$
\begin{aligned}
&V(x_1,x_2,\tau)\\
&=\left(1+\tau\left(E+\frac{1}{3}K\right)\right)F-\frac{1}{2}\tau^2\left(\nabla_1 K\frac{\partial F}{\partial x_1}+\nabla_2 K\frac{\partial F}{\partial x_2}\right)+\frac{1}{6}K\tau^2\left(\frac{\partial^2 F}{\partial {x_1}^2}+\frac{\partial^2 F}{\partial {x_2}^2}\right)\\
&\quad-\frac{1}{6}\tau^3\left\{\nabla_1 K\left(\frac{\partial^3 F}{\partial {x_1}^3}+\frac{\partial^3 F}{\partial x_1\partial {x_2}^2}\right)+\nabla_2 K\left(\frac{\partial^3 F}{\partial {x_2}^3}+\frac{\partial^3 F}{\partial {x_1}^2\partial x_2}\right)\right\}.
\end{aligned}
$$

Finally, we can get the explicit solution using off-diagonal heat asymptotics.

$$
\begin{aligned}
&V(S_1,S_2,t)\\
&=A_2e^{(T-t)}S_2N(\alpha)-A_1e^{(T-t)}S_1N(\alpha-2\sqrt{T-t})\\
&\quad+B_2e^{(T-t)}S_2\exp\left\{-\frac{\alpha^2}{2}\right\}-B_1e^{(T-t)}S_1\exp\left\{-\frac{(\alpha-2\sqrt{T-t})^2}{2}\right\}\\
&\quad-C_2e^{(T-t)}\alpha S_2\exp\left\{-\frac{\alpha^2}{2}\right\}+C_1e^{(T-t)}(\alpha-2\sqrt{T-t})S_1\exp\left\{-\frac{(\alpha-2\sqrt{T-t})^2}{2}\right\}\\
&\quad+D\left[e^{(T-t)}\alpha^2e^{x_2}\exp\left\{-\frac{\alpha^2}{2}\right\}-e^{(T-t)}(\alpha-2\sqrt{T-t})^2S_1\exp\left\{-\frac{(\alpha-2\sqrt{T-t})^2}{2}\right\}\right],
\end{aligned}
$$

where

$$A_i = 1 + \left(E + \frac{1}{3}K\right)(T-t) + \left(\frac{1}{6}K - \frac{1}{2}\nabla_i K\right)(T-t)^2 - \frac{1}{6}\nabla_i K(T-t)^3,$$

$$B_2 = \frac{1}{24\sqrt{2\pi(T-t)}}\left[(5\nabla_1 K - 5\nabla_2 K + 4K)(T-t)^2 + 2\left(\nabla_1 K - 3\nabla_2 K\right)(T-t)^3\right],$$

$$B_1 = \frac{1}{24\sqrt{2\pi(T-t)}}\left[(5\nabla_1 K - 5\nabla_2 K - 4K)(T-t)^2 + 2\left(3\nabla_1 K - \nabla_2 K\right)(T-t)^3\right],$$

$$C_2 = \frac{1}{12\sqrt{2\pi}}\left\{K(T-t) + (\nabla_1 K + \nabla_2 K)(T-t)^2\right\},$$

$$C_1 = \frac{1}{12\sqrt{2\pi}}\left\{K(T-t) + (-2\nabla_1 K + \nabla_2 K)(T-t)^2\right\},$$

$$D = \frac{1}{24\sqrt{2\pi(T-t)}}(\nabla_1 K - \nabla_2 K)(T-t)^2,$$

for $i = 1, 2$.

From the historical data, we are able to catch the trend of the volatility. Then we directly compute the scalar curvature K, the bundle endomorphism E from the function of the volatility using the relation defined at the beginning of Section 5. However, we make a short version of the above answer if we use an estimated volatility function. In a financial market, we can find out volatilities are increasing near the time of the maturity. So we can say that the scalar curvature K can be written in terms of $-\epsilon$ for any positive ϵ. Moreover, we can choose $|\nabla K| \ll \epsilon$ because the volatility does not fluctuate in any unusual way. For simplicity, we can say $\nabla_1 K = \nabla_2 K = \epsilon'$

with $|\epsilon'| \ll \epsilon$. If we use this simple setup, we can obtain

$$
\begin{aligned}
&V(S_1, S_2, t)\\
=&Ae^{(T-t)}\left[S_2 N(\alpha) - S_1 N(\alpha - 2\sqrt{T-t})\right]\\
&- Be^{(T-t)}\left[S_2 \exp\left\{-\frac{\alpha^2}{2}\right\} - S_1 \exp\left\{-\frac{(\alpha - 2\sqrt{T-t})^2}{2}\right\}\right]\\
&+ Ce^{(T-t)}\alpha S_2 \exp\left\{-\frac{\alpha^2}{2}\right\} - De^{(T-t)}(\alpha - 2\sqrt{T-t}) S_1 \exp\left\{-\frac{(\alpha - 2\sqrt{T-t})^2}{2}\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
A &= 1 + \left(E - \frac{1}{3}\epsilon\right)(T-t) - \frac{1}{6}(\epsilon - 3\epsilon')(T-t)^2 - \frac{1}{6}\epsilon'(T-t)^3,\\
B &= \frac{1}{6\sqrt{2\pi(T-t)}}\left\{\epsilon(T-t)^2 + \epsilon'(T-t)^3\right\},\\
C &= \frac{1}{12\sqrt{2\pi}}\left\{\epsilon(T-t) - 2\epsilon'(T-t)^2\right\},\\
D &= C - \frac{1}{12\sqrt{2\pi}}\epsilon'(T-t)^2.
\end{aligned}
$$

7.2 Other options in two-color rainbow

If we adjust the terminal conditions of other options, then we can price them using the solution from the previous subsection.

1. Better-of option: $\max(S_1, S_2) = S_1 + \max(S_2 - S_1, 0)$,

2. Worse-of option: $\min(S_1, S_2) = -\max(-S_1, -S_2)$,

3. Max/min option is similar to that of better-of/worse-of options when the strike prices are set to zero.

CHAPTER 8
CONCLUSION

8.1 Summary

By a simple substitution, the multidimensional Black-Scholes formula turns into a backward Kolmogorov equation. Moving toward a covariant version of this which is good in any coordinate system, we find a general asymptotic solution in a short time interval using the heat kernel expansion on a Riemannian metric. The asymptotic resolution of the heat kernel for a short time is an important problem in theoretical physics and mathematics.

A Fourier transform approach is only capable of yielding a solution based on the flat heat kernel or 0th order asymptotics. Certain observed market effects suggest a curved heat kernel, which may be approximated by taking higher heat kernel asymptotics as in [17]. The heat kernel asymptotics are smooth functions and depend on geometric invariants such as the scalar curvature.

In many references, for examples [6] and [17], only the diagonal values $H(t,x,x)$ of the heat kernel are considered; these of course produce the largest analytic effect. I claim, however, that there is information to be gained from off-diagonal values $H(t,x,y)$ in the parabolic region where $|x-y|^2$ and t are comparable, and that this information has implications for the multi-asset model. This off-diagonal information is carried by higher-order aymptotics in the multivariate quantity $x-y$. I have succeeded in obtaining some of this additional information in terms of the scalar curvature and the affine connection in the two underlying asset case by direct calculation. By using my result as a model, we obtain better, more precise solutions. Furthermore, we can now explain hidden effects in a financial model.

8.2 Future Study

First, I will work with one of my colleagues who works in a financial institution at Japan and show that my idea works better than other models. Although it takes more time to calculate extra terms, we are quite sure to have a better result. Second, my idea can be generalized to the n asset based model. We need this generalized model to get a better understanding of financial markets which always evolve themselves. In addition, investers can hedge their risk more efficiently because they can diversify them. Third, I strongly believe that my model can be applied in other contexts such as the derivation for a Libor market model and a credit risk model.

APPENDIX

In this appendix, we will want to review the Hergoltz and Günther formulas more carefully and then learn the details to get the price of the outperformance option. Two formulas are proven already, but we will prove them with a different method. A different lower and upper index and new notations will be used to prove the two formulas.

Let $\mathring{g}_{\mu\nu}$ be the constant matrix encoding the entries of $g_{\mu\nu}$ at the origin of the normal coordinate system. We have

$$(g_{\mu\nu} - \mathring{g}_{\mu\nu})x^\nu = 0.$$

Let X be the vector field with $X^\alpha = x^\alpha$. Then

$$\begin{aligned} 2x^\alpha \Gamma_{\alpha\mu\nu} &= x^\alpha\{\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\alpha\nu} - \partial_\nu g_{\alpha\mu}\} \\ &= Xg_{\mu\nu} + \partial_\mu(x^\alpha g_{\alpha\nu}) - \delta_\mu{}^\alpha g_{\alpha\nu} - \partial_\nu(x^\alpha g_{\alpha\mu}) + \delta_\nu{}^\alpha g_{\alpha\mu} \\ &= Xg_{\mu\nu} + \partial_\mu(x^\alpha g_{\alpha\nu}) - \partial_\nu(x^\alpha g_{\alpha\mu}) \\ &= Xg_{\mu\nu} + \delta_\mu{}^\alpha \mathring{g}_{\alpha\nu} - \delta_\nu{}^\alpha \mathring{g}_{\alpha\mu} \\ &= Xg_{\mu\nu} =: \gamma_{\mu\nu}. \end{aligned} \tag{1}$$

The geodesic condition $\nabla_T T = 0$, in terms of $X = rT$, reads

$$\nabla_X X = \nabla_{rT}(rT) = r\nabla_T(rT) = r^2\nabla_T T + r(Tr)T = X, \tag{2}$$

because $\nabla_T T = 0$, $Tr = 1$. The idea now is to compute $X^\nu[\nabla_\mu, \nabla_\nu]X^\sigma$ in different ways. On the one hand,

$$X^\nu[\nabla_\mu, \nabla_\nu]X^\sigma = X^\nu R^\sigma{}_{\rho\mu\nu}X^\rho = x^\nu R^\sigma{}_{\rho\mu\nu}x^\rho =: k^\sigma{}_\mu.$$

On the other hand,

$$\begin{aligned} X^\nu \nabla_\mu \nabla_\nu X^\sigma &= \nabla_\mu (X^\nu \nabla_\nu X^\sigma) - (\nabla_\mu X^\nu) \nabla_\nu X^\sigma \\ &= \nabla_\mu (\nabla_X X)^\sigma - (\partial_\mu X^\nu + \Gamma_{\mu\rho}{}^\nu X^\rho) \nabla_\nu X^\sigma \\ &= \nabla_\mu X^\sigma - \nabla_\mu X^\sigma - \frac{1}{2}\gamma_\mu{}^\nu \nabla_\nu X^\sigma = -\frac{1}{2}\gamma_\mu{}^\nu \nabla_\nu X^\sigma, \end{aligned} \tag{3}$$

using (1) and (2). Furthermore,

$$\begin{aligned} -X^\nu \nabla_\nu \nabla_\mu X^\sigma &= -X^\nu \{-\partial_\nu \nabla_\mu X^\sigma - \Gamma^\rho_{\nu\mu} \nabla_\rho X^\sigma + \Gamma_{\nu\rho}{}^\sigma \nabla_\mu X^\rho\} \\ &= -X(\partial_\mu X^\sigma + \Gamma_{\mu\rho}{}^\sigma X^\rho) + \frac{1}{2}\gamma_\mu{}^\rho \nabla_\rho X^\sigma - \frac{1}{2}\gamma_\rho{}^\sigma \nabla_\mu X^\rho. \end{aligned} \tag{4}$$

Adding (3) and (4), we get

$$\begin{aligned} X^\nu [\nabla_\mu, \nabla_\nu] X^\sigma &= -X(\partial_\mu X^\sigma + \Gamma_{\mu\rho}{}^\sigma X^\rho) - \frac{1}{2}\gamma_\rho{}^\sigma \nabla_\mu X^\rho \\ &= -X(\delta_\mu{}^\sigma + \frac{1}{2}\gamma_\mu{}^\sigma) - \frac{1}{2}(\partial_\mu X^\rho + \Gamma_{\mu\alpha}{}^\rho X^\alpha) \\ &= -\frac{1}{2}X\gamma_\mu{}^\sigma - \frac{1}{2}\gamma_\rho{}^\sigma (\delta_\mu{}^\sigma + \frac{1}{2}\gamma_\mu{}^\rho) \\ &= -\frac{1}{2}(X+1)(\gamma_{\mu\nu} g^{\mu\nu}) - \frac{1}{4}\gamma_{\mu\nu} g^{\nu\rho} \gamma_{\rho\alpha} g^{\alpha\sigma} \\ &= -\frac{1}{2}g^{\nu\alpha}(X+1)Xg_{\mu\nu} - \frac{1}{2}(Xg_{\mu\nu})Xg^{\nu\sigma} - \frac{1}{4}(Xg_{\mu\nu})g^{\nu\rho}(Xg_{\rho\alpha})g^{\alpha\sigma}. \end{aligned} \tag{5}$$

The middle term of the last expression in (5) of expands via

$$Xg^{\nu\sigma} = -g^{\nu\alpha}(Xg_{\alpha\beta})g^{\beta\sigma}.$$

This makes (5) into

$$X^\nu [\nabla_\mu, \nabla_\nu] X^\sigma = -\frac{1}{2}g^{\nu\sigma}(X+1)Xg_{\mu\nu} + \frac{1}{4}(Xg_{\mu\nu})g^{\nu\rho}(Xg_{\rho\alpha})g^{\alpha\sigma}.$$

Viewing g^{-1}, g, Xg, XXg, etc. as matrices, this says

$$2k^\sigma{}_\mu = g^{-1}(-(X+1)Xg + \frac{1}{2}g^{-1}(Xg)g^{-1}Xg)^\sigma{}_\mu,$$

or

$$2k_{\tau\mu} = (-(X+1)Xg + \frac{1}{2}(Xg)g^{-1}Xg)_{\tau\mu},$$

or finally

$$2k = -(X+1)Xg + \frac{1}{2}(Xg)g^{-1}Xg.$$

This is the Herglotz formula. We want to use this inductively to get some kind of calculation of the $X^k g$. If $m \geq 2$ and $X^l g$ is known for $l < m$, then

$$X^m g = X^{m-2}X^2 g = X^{m-2}\left(-2k - Xg + \frac{1}{2}g^{-1}(Xg)g^{-1}Xg\right).$$

Everything is thus written in terms of previously understood quantities, except $X^{m-2}k$.

This is the content of the Günther formula. Let

$$k^{(p)}_{\tau\mu} = (\nabla_{\alpha_p}\cdots\nabla_{\alpha_1}R_{\tau\rho\mu\nu})x^\rho x^\nu x^{\alpha_1}\cdots x^{\alpha_p}.$$

Note that $k^{(0)} = k$. The Günther formula is

$$k^{(p)} = Xk^{(p-1)} - (p+1)k^{(p-1)} - \frac{1}{2}((Xg)g^{-1}k^{(p-1)} + k^{(p-1)}g^{-1}Xg).$$

Indeed, setting $h := k^{(p)}$ and $H := k^{(p+1)}$, we have

$$\begin{aligned}
(\nabla_X h)_{\tau\mu} &= X^\alpha \nabla_\alpha h_{\tau\mu} \\
&= X^\alpha(\partial_\alpha h_{\tau\mu} - \Gamma_{\alpha\tau}{}^\rho - \Gamma_{\alpha\mu}{}^\rho h_{\tau\rho}) \\
&= Xh_{\tau\mu} - \frac{1}{2}(\gamma_\tau{}^\rho h_{\rho\mu} + \gamma_\mu{}^\rho h_{\tau\rho}) \\
&= Xh_{\tau\mu} - \frac{1}{2}(\gamma_{\tau\beta}g^{\beta\rho}h_{\rho\mu} + h_{\tau\rho}g^{\rho\beta}\gamma_{\beta\mu}),
\end{aligned}$$

so that

$$\nabla_X h = Xh - \frac{1}{2}((Xg)g^{-1}h + hg^{-1}Xg).$$

Furthermore, using $\nabla_X X = X$, we get

$$\begin{aligned} H_{\tau\mu} &= X^{\alpha_{p+1}}(\nabla_{\alpha_{p+1}}\nabla_{\alpha_p})\cdots\nabla_{\alpha_1}R_{\tau\rho\mu\nu}X^\rho X^\nu X^{\alpha_1}\cdots X^{\alpha_p} \\ &= (\nabla_X h)_{\tau\mu} - (\nabla_{\alpha_p})\cdots\nabla_{\alpha_1}R_{\tau\rho\mu\nu})\nabla_X(X^\rho X^\nu X^{\alpha_1}\cdots X^{\alpha_p}) \\ &= (\nabla_X h)_{\tau\mu} - (\nabla_{\alpha_p})\cdots\nabla_{\alpha_1}R_{\tau\rho\mu\nu})(p+2)X^\rho X^\nu X^{\alpha_1}\cdots X^{\alpha_p}, \end{aligned}$$

so that

$$H = (\nabla_X - (p+2))h = (X - (p+2))h - \frac{1}{2}((Xg)g^{-1}h + hg^{-1}Xg),$$

as desired.

We now introduce the notation $f[\ell]$ for the order ℓ part of the normal expansion (if any) of f. For example,

$$\begin{aligned} g =& g[0] + g[2] + \cdots, \\ Xg =& 2g[2] + 3g[3] + \cdots, \\ k^{(p)} =& k^{(p)}[p+2] + \cdots. \end{aligned} \tag{6}$$

Note that

$$k^{(p)}_{\tau\mu}[p+2] = (\nabla_{\alpha_p}\cdots\nabla_{\alpha_1}\mathring{R}_{\tau\rho\mu\nu})x^\rho x^\nu x^{\alpha_1}\cdots x^{\alpha_p},$$

where $\mathring{R}$ denotes evaluation at the origin of the normal coordinate system. Generalizing the second equation of (6), we have

$$(Xf)[\ell] = \ell f[\ell].$$

The Herglotz formula says that

$$2k[\ell] = -(\ell+1)\ell g[\ell] + \frac{1}{2}\sum_{r+s+t=\ell}(rg[r])g^{-1}[s](tg[t]). \tag{7}$$

Since $rg[r]$ is only non-vanishing for $r \geq 2$, this expresses $g[\ell]$ in terms of $k[\ell]$, plus $g[r]$ for $r \leq \ell - 4$, plus $g^{-1}[s]$ for $s \leq \ell - 4$. Since knowledge of g to a certain order gives a knowledge of g^{-1} to that order, the dependence is only on $k[\ell]$ and the $g[r]$ for $r \leq \ell - 2$.

The Günther formula says that

$$\begin{aligned} k^{(p)}[\ell] &= (\ell - p - 1)k^{(p-1)}[\ell] \\ &\quad - \frac{1}{2} \sum_{r+s+t=\ell} \{(rg[r])g^{-1}[s]k^{(p-1)}[t] + k^{(p-1)}[t]g^{-1}[s](rg[r])\}. \end{aligned} \tag{8}$$

In particular,

$$k^{(p)}[p+2] = k^{(p-1)}[p+2].$$

It is important to note that both (7) and (8) should be viewed as equations for the first terms on their respective right sides. That is, (7) solves for $g[\ell]$, and (8) takes a given $k^{(q)}[\ell]$ and replaces it with terms in which q is increased, or ℓ is decreased. The idea of this is that $k^{(p)}[p+2]$ is well understood, so we try to reduce the “gap” $\ell - q \geq 2$ in $k^{(q)}[\ell]$ until it is just 2. In the process of reducing the gap, terms from the power series of g appear, but in the overall process of inductively computing $g[\ell]$, only $g[s]$ with $s \leq \ell - 2$ appear. In fact in (8), the g terms appearing have homogeneity at most $\lambda - p - 1 = \text{gap} - 2$ (where the gap is that of the first term on the right). Since the initial gap (from (7)) is ℓ, the recursion works.

From the viewpoint of solving the recursion in this way, the fundamental formulas read:

$$g[\ell] = \frac{1}{(\ell+1)\ell} \left\{ -2k[\ell] + \frac{1}{2} \sum_{r+s+t=\ell} (rg[r])g^{-1}[s](tg[t]) \right\}$$

and

$$k^{(p-1)}[\ell] = \frac{1}{\ell - p - 1}\left\{k^{(p)}[\ell] + \frac{1}{2}\sum_{r+s+t=\ell}\{(rg[r])g^{-1}[s]k^{(p-1)}[t] + k^{(p-1)}[t]g^{-1}[s](rg[r])\}\right\}.$$

Let us perform some of this recursion:

$$g[2] = -\frac{1}{3}k[2] = -\frac{1}{3}\mathring{R}_{\bullet\rho\bullet\nu}x^\rho x^\nu$$

Furthermore,

$$g[3] = -\frac{1}{6}k[3] = -\frac{1}{6}k^{(1)}[3] = -\frac{1}{6}\nabla_\alpha\mathring{R}_{\bullet\rho\bullet\nu}x^\alpha x^\rho x^\nu.$$

These are terminal formulas because they involve only $k^{(p)}[\ell]$ with $\ell - p = 2$; this is reflected in the fact that the curvature quantities on the far right are underlined (so evaluated at the origin).

Next,

$$\begin{aligned} g[4] &= \frac{1}{20}\{-2k[4] + 2g[2]g^{-1}[0]g[2]\}, \\ k[4] &= \frac{1}{2}\{k^{(1)}[4] + (g[2]g^{-1}[0]k[2] + k[2]g^{-1}[0]g[2])\} \\ &= \frac{1}{2}\{k^{(1)}[4] - \frac{2}{3}k[2]g^{-1}[0]k[2]\}, \\ k^{(1)}[4] &= k^{(2)}[4], \end{aligned}$$

so

$$g[4] = -\frac{1}{20}k^{(2)}[4] + \frac{2}{45}k[2]g^{-1}[0]k[2] = (-\frac{1}{20}\nabla_\beta\nabla_\alpha R_{\bullet\rho\bullet\nu} + \frac{2}{45}R_{\bullet\rho\lambda\nu}R^\lambda{}_{\alpha\bullet\beta})x^\alpha x^\beta x^\rho x^\nu.$$

The above method gives us a more efficient way to prove the two formulas.

The following calculation is details for pricing a rainbow option. For calculating (7.4), it may seem like a long way to travel from the original formation.

$$\begin{aligned}
&\frac{1}{4\pi\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{\sigma^2}{4\tau}\right)V(y_1,y_2,0)dy_1dy_2\\
=&\frac{1}{4\pi\tau}\int_{-\infty}^{\infty}\left[\int_{y_1}^{\infty}\exp\left(-\frac{\sigma^2}{4\tau}\right)(e^{y_2}-e^{y_1})dy_2\right]dy_1\\
=&\frac{1}{4\pi\tau}\int_{-\infty}^{\infty}\left[\exp\left\{-\frac{(x_1-y_1)^2}{4\tau}\right\}\int_{y_1}^{\infty}\exp\left\{-\frac{(x_2-y_2)^2}{4\tau}+y_2\right\}dy_2\right]dy_1\\
&-\frac{1}{4\pi\tau}\int_{-\infty}^{\infty}\left[\exp\left\{-\frac{(x_1-y_1)^2}{4\tau}+y_1\right\}\int_{y_1}^{\infty}\exp\left\{-\frac{(x_2-y_2)^2}{4\tau}\right\}dy_2\right]dy_1\\
=&I_2-I_1,
\end{aligned}$$

say. We evaluate I_2 by completing the square in the exponent to get a standard integral:

$$\begin{aligned}
I_2=&\frac{\sqrt{2\tau}}{4\pi\tau}e^{x_2+\tau}\int_{-\infty}^{\infty}\left[\int_{f(y_1)}^{\infty}\exp\left\{-\frac{(x_1-y_1)^2}{4\tau}\right\}e^{-\frac{n^2}{2}}dn\right]dy_1\\
=&\frac{\sqrt{2\tau}}{4\pi\tau}e^{x_2+\tau}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{g(n)}\exp\left\{-\frac{(x_1-y_1)^2}{4\tau}\right\}e^{-\frac{n^2}{2}}dy_1\right]dn\\
=&\frac{1}{2\pi}e^{x_2+\tau}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{n+\sqrt{2}\alpha(x_1,x_2)}e^{-\frac{m^2+n^2}{2}}dm\right]dn,
\end{aligned}$$

with

$$\begin{aligned}
f(y_1)&=\frac{y_1-(x_2+2\tau)}{\sqrt{2\tau}},\\
g(n)&=\sqrt{2\tau}n+(x_2+2\tau),\\
\alpha(x_1,x_2)&=\frac{x_2-x_1+2\tau}{2\sqrt{\tau}}.
\end{aligned}$$

From the Leibniz's Rule, we can simplify the double integral further. Since

$$
\begin{aligned}
&\frac{\partial}{\partial \alpha}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{n+\sqrt{2}\alpha(x_1,x_2)} e^{-\frac{m^2+n^2}{2}}dm\right]dn \\
&=\sqrt{2}\int_{-\infty}^{\infty}\exp\left\{-\frac{(n+\sqrt{2}\alpha)^2+n^2}{2}\right\}dn \\
&=\sqrt{2}\exp\left\{-\frac{\alpha^2}{2}\right\}\int_{-\infty}^{\infty}\exp\left\{-\left(n+\frac{\sqrt{2}}{2}\alpha\right)^2\right\}dn \\
&=\sqrt{2\pi}\exp\left\{-\frac{\alpha^2}{2}\right\},
\end{aligned}
$$

the double integral can be expressed in terms of the accumulative normal distribution $2\pi N(\alpha)$. Therefore,

$$
I_2 = e^{x_2+\tau}N(\alpha). \tag{9}
$$

The calculation of I_1 is identical to that of I_2, except that some constant terms are different.

$$
I_1 = e^{x_1+\tau}N(\alpha(x_1,x_2)-2\sqrt{\tau}) \tag{10}
$$

In addition, we will consider F and find the partial derivatives with respect to the variable x_i. We have

$$
\frac{\partial F}{\partial x_i} = \left(-\frac{1}{2\tau}\right)\frac{1}{4\pi\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_i-y_i)\exp\left(-\frac{\sigma^2}{4\tau}\right)V(y_1,y_2,0)dy_1dy_2.
$$

In this way, we obtain the following useful formula:

$$
\frac{1}{4\pi\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_i-y_i)\exp\left(-\frac{\sigma^2}{4\tau}\right)V(y_1,y_2,0)dy_1dy_2 = -2\tau\frac{\partial F}{\partial x_i}. \tag{11}
$$

To obtain the remaining terms, the calculation is almost identical.

$$
\begin{aligned}
&\frac{1}{4\pi\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_i-y_i)(x_j-y_j)\exp\left(-\frac{\sigma^2}{4\tau}\right)V(y_1,y_2,0)dy_1dy_2 \\
&=4\tau^2\frac{\partial^2 F}{\partial x_i\partial x_j}+2\tau F\cdot I(i=j),
\end{aligned} \tag{12}
$$

and

$$\begin{aligned}\frac{1}{4\pi\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_i-y_i)(x_j-y_j)(x_k-y_k)\exp\left(-\frac{\sigma^2}{4\tau}\right)V(y_1,y_2,0)dy_1dy_2 \\ =-8\tau^3\frac{\partial^3 F}{\partial x_i\partial x_j\partial x_k}-4\tau^2\frac{\partial F}{\partial x_k}-8\tau^2\frac{\partial F}{\partial x_k}\cdot I(i=j=k),\end{aligned} \tag{13}$$

where $I(\cdot)$ is the indicator function.

REFERENCES

[1] I. G. Avramidi. The covariant technique for calculation of one-loop effective action. *Nuclear Phys.*, 1991.

[2] M. W. Baxter and A. J. O. Rennie. *Financial Calculus: An introduction to derivative pricing*. Cambridge Univ. Press, UK, 1996.

[3] T.P. Branson and B. Orsted. Conformal indices of riemannian manifolds. *Compositio Math.*, pages 261–293, 1986.

[4] M. Broadie and J. Detemple. The valuation of american options on multiple assets. *Math. Finance*, 1997.

[5] A. Etheridge. *A Course in Financial Calculus*. Cambridge Univ. Press, UK, 2002.

[6] P. B. Gilkey. *Invariance theory, the heat equation, and the Atiyah-Singer index theorem.* EMIS server http://emis.library.cornell.edu/monographs/gilkey/index.html.

[7] P. Henry-Labordére. A general asymptotic implied volatility for stochastic volatility models. *ccsd-00004632*, 2005.

[8] J. Hull. *Introduction to Futures and Options Markets*. Simon & Schuster, New York, 1991.

[9] L. Jiang. *Mathematical Modeling and Methods of Option Pricing*. Higher Education, Beijing, China, 2003.

[10] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*. Springer-Verlag, New York, 1991.

[11] I. Karatzas and S. E. Shreve. *Methods of Mathematical Finace*. Springer-Verlag, New York, 1998.

[12] H. H. Kuo. *Introduction to Stochastic Integration*. Universitext. Springer, Berlin Heidelberg, 2006.

[13] P. Laurence and T. H. Wang. Closed form solutions for quadratic and inverse quadratic term structure models. *Int. J. Theor. Appl. Finance*, (8):1059–1083, 2005.

[14] T. J. Lyons. Uncertain volatility and the risk-free synthesis of derivatives. *Appl Math Finance*, pages 117–133, 1995.

[15] R. G. McLenaghan. An explicit determination of the empty space-times on which the wave equation satisfies huygens' principle. *Proc. Camb. Philol. Soc.*, pages 139–155, 1969.

[16] R. S. Millman and G. D. Parker. *Elements of differential Geometry*. Prentice-Hall Inc., New Jersey, 1977.

[17] S. Minakshisundaram and Å. Pleijel. Some properties of the eigenfunctions of the laplace-operator on riemannian manifolds. *Canad. J. Math.*, pages 242–256, 1949.

[18] S. Howison P. Wilmott and J. Dewynne. *The Mathematics of Financial Derivatives*. Cambridge University Press, UK, 1995.

[19] M. perdigão Do Carmo. *Riemannian Geometry*. Birkhäuser, Boston, 1992.

[20] P. Protter. *Stochastic Integration and Differential Equations*. Springer-Verlag, Heidelberg, 1990.

[21] S. E. Shreve. *Stochastic Calculus for Finance I*. Springer-Verlag, New York, 2004.

[22] S. E. Shreve. *Stochastic Calculus for Finance II*. Springer-Verlag, New York, 2004.

[23] C. Smithson. Multifactor options. *Risk*, (43), 1997.

[24] R. Zagst. *The Mathematics of Financial Derivatives*. Springer-Verlag, Berlin Heidelberg New York, 2002.