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# Cellularity and Jones basic construction

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# CELLULARITY AND THE JONES BASIC CONSTRUCTION

by

John Eric Graber

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy degree
in Mathematics in
the Graduate College of
The University of Iowa

July 2009

Thesis Supervisor: Professor Frederick M. Goodman

## ABSTRACT

This thesis establishes a framework for cellularity of algebras related to the Jones basic construction. The framework allows a uniform proof of cellularity of Brauer algebras, BMW algebras, walled Brauer algebras, partition algebras, and others. In this setting, the cellular bases are labeled by paths on certain branching diagrams rather than by tangles. Moreover, for this class of algebras, the cellular structures are compatible with restriction and induction of modules.

Abstract Approved: _____
Thesis Supervisor

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Title and Department

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Date

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John Eric Graber

A thesis submitted in partial fulfillment of the
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Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

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PH.D. THESIS

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This is to certify that the Ph.D. thesis of

John Eric Graber

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the July 2009 graduation.

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To my family

## ABSTRACT

This thesis establishes a framework for cellularity of algebras related to the Jones basic construction. The framework allows a uniform proof of cellularity of Brauer algebras, BMW algebras, walled Brauer algebras, partition algebras, and others. In this setting, the cellular bases are labeled by paths on certain branching diagrams rather than by tangles. Moreover, for this class of algebras, the cellular structures are compatible with restriction and induction of modules.

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# CHAPTER 1
# INTRODUCTION

## 1.1 Motivation

Cellularity is a concept due to Graham and Lehrer [22] that is useful for studying non–semisimple specializations of certain algebras such as Hecke algebras, $q$–Schur algebras, etc. A number of important examples of cellular algebras, including the Hecke algebras of type $A$ and the Birman–Wenzl–Murakami (BMW) algebras, actually occur in towers $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ with coherent cellular structures. Coherence means that the cellular structures are well–behaved with respect to induction and restriction.

This thesis establishes a framework for proving cellularity of towers of algebras $(A_n)_{n\geq 0}$ that are obtained by repeated Jones basic constructions from a coherent tower of cellular algebras $(Q_n)_{n\geq 0}$.

Examples that fit the framework include: Temperley-Lieb algebras, Brauer algebras, walled Brauer algebras, Birman–Wenzl–Murakami (BMW) algebras, cyclotomic BMW algebras, partition algebras, and contour algebras. Thus, the framework gives a uniform proof of cellularity for all of these algebras.

Once the general result is proved (Theorem 3.2.1), it is generally very easy to check that an example fits the framework, and thus that the tower $(A_n)_{n\geq 0}$ in each example is a coherent tower of cellular algebras. This information is, for the most part, already in the literature, or completely elementary. The application of this method to the cyclotomic BMW algebras depends on a very recent result of Mathas regarding induced modules of cyclotomic Hecke algebras [42].

## 1.2 Related work

For most of the examples, cellularity has been established previously (but coherence of the cellular structures is a new result). Many of the existing proofs

of cellularity for these algebras follow the pattern made explicit by Xi in his paper on cellularity of the partition algebras [57]. The cellular bases obtained are pieced together from cellular bases of the (quotient) algebras $Q_k$ and bases of certain $R$–modules $V_k$ of tangles or diagrams, where $R$ is the ground ring for $A_n$. It is not evident that these "tangle bases" yield coherent cellular structures. By contrast, the cellular bases produced in this thesis are indexed by paths on the branching diagram (Bratteli diagram) for the generic semisimple representation theory of the tower $(A_n)_{n\geq 0}$ over a field, and coherence is built into the construction. For example, for the BMW algebras, the cellular basis of the $n$–th algebra is indexed by up–down tableaux of length $n$. In this regard, the results below generalize and simplify the work of Enyang on the BMW algebra [15] (although the information is less detailed than that obtained by Enyang).

It will be obvious that the approach is influenced by the work of König and Xi on cellularity and Morita equivalence [36] as well as by the work of Cox et. al. on "towers of recollement" [9]. In fact, the idea behind the approach is roughly the following: Each algebra $A_n$ in the examples contains an idempotent $e_{n-1}$ with the properties that $e_{n-1}A_ne_{n-1} \cong A_{n-2}$ and $A_n/(A_ne_{n-1}A_n) \cong Q_n$, where $Q_n$ is a cellular algebra. Now, $I_n = A_ne_{n-1}A_n$ and $A_{n-2} \cong e_{n-1}A_ne_{n-1}$ are (in some sense) Morita equivalent, and according to König and Xi, Morita equivalence should preserve cellularity. Assuming inductively that $A_{n-2}$ is cellular, it should follow that $I_n$ is cellular. It should then be possible to obtain a cellular basis of $A_n$ by gluing together a cellular bases of $I_n$ and $Q_n$. However, there are technical difficulties to realizing this sketch. Consequently, the framework does not directly apply any results from König and Xi, but the arguments nevertheless reflect their ideas.

Following Cox et. al. [9], the approach below employs the interaction between induction and restriction functors relating $A_{n-1}$–mod and $A_n$–mod, on the

one hand, and localization and globalization functions relating $A_n$–mod and $A_{n-2}$–mod, on the other hand. (Write $e = e_{n-1} \in A_n$. The localization functor $F : A_n\text{–mod} \to eA_ne\text{–mod} \cong A_{n-2}\text{–mod}$ is $F : M \mapsto eM$. The globalization function $G : A_{n-2}\text{–mod} \cong eA_ne\text{–mod} \to A_n\text{–mod}$ is $G : N \mapsto A_ne \otimes_{eA_ne} N$.)

The framework below and that of Cox et. al. dovetail nicely; in fact, the main result (Theorem 3.2.1) says that if $(A_n)$, $(Q_n)$ are two sequences of algebras satisfying the framework axioms, then $(A_n)$ satisfies a cellular version of the axioms for towers of recollement; see [8] for a discussion of cellularity and towers of recollement.

The generic ground ring plays an important role for the examples. For each of the examples $(A_n)_{n\geq 0}$, there is a generic ground ring $R$ such that any specialization $A_n^S$ to a ground ring $S$ is obtained as $A_n^S = A_n^R \otimes_R S$. Moreover, $R$ is an integral domain, and if $F$ denotes the field of fractions of $R$, then the algebras $(A_n^F)_{n\geq 0}$ are split semisimple with a known representation theory and branching diagram. It suffices to prove that the sequence of algebras defined over the generic ground ring $R$ is a coherent cellular tower, and use the structure of the algebras defined over $F$ as a tool to accomplish this.

Several other general frameworks have been proposed for cellularity which also successfully encompass most of the examples; see [36, 23, 53].

# CHAPTER 2
# PRELIMINARIES

## 2.1 Algebras with involution

Let $R$ be a commutative ring with identity. In the following, assume $A$ is an $R$–algebra with an involution $i$ (that is, an $R$–linear algebra anti–automorphism of $A$ with $i^2 = \mathrm{id}$).

If $M$ is a left $A$–module, we define a right $A$–module $i(M)$ as follows. As a set, $i(M)$ is a copy of $M$, with elements marked with the symbol $i$, $i(M) = \{i(m) : m \in M\}$. The $R$–module structure of $i(M)$ is given by $i(m_1) + i(m_2) = i(m_1 + m_2)$, and $ri(m) = i(rm)$. Finally, the right $A$–module structure is defined by $i(m)a = i((i(a)m)$. If $\alpha : M \to N$ is a homomorphism of left $A$–modules, define $i(\alpha) : i(M) \to i(N)$ by $i(\alpha)(i(m)) = i(\alpha(m))$. Then $i : A\text{–mod} \to \text{mod–}A$ is a functor. For any fixed $M$, $i : M \to i(M)$ given by $m \mapsto i(m)$ is, by definition, an isomorphism of $R$–modules.

If $\Delta$ is a left ideal in $A$, we have two possible meanings for $i : \Delta \to i(\Delta)$, namely the restriction to $\Delta$ of the involution $i$, whose image is a right ideal in $A$, or the application of the functor $i$. However, there is no problem with this, as the right $A$–module obtained by applying the functor $i$ can be identified with the right ideal $i(\Delta)$.

The same construction gives a map from right $A$–modules to left $A$–modules. Moreover, if $A$ and $B$ are $R$–algebras with involutions $i_A$ and $i_B$, and $M$ is an $A$–$B$–bimodule, then $i(M)$, defined as above as an $R$–module has the structure of a $B$–$A$–bimodule with $b\, i(m) a = i(i_A(a) m\, i_B(b))$. Note that $i \circ i(M)$ is naturally isomorphic to $M$, so $i$ is an equivalence between the categories of $A$–$B$–bimodules and the category of $B$–$A$–bimodules.

**Lemma 2.1.1.** *Suppose $A$, $B$, and $C$ are $R$–algebras with involutions $i_A$, $i_B$, and*

*$i_C$. Let ${}_BP_A$ and ${}_AQ_C$ be bimodules. Then*

$$i(P \otimes_A Q) \cong i(Q) \otimes_A i(P),$$

*as $C$–$B$–bimodules.*

*Proof.* It is straightforward to check that there is a well defined $R$–linear isomorphism $f_0 : P \otimes_A Q \to i(Q) \otimes_A i(P)$ such that $f_0(p \otimes q) = i(q) \otimes i(p)$. Then $f = f_0 \circ i^{-1} : i(P \otimes_A Q) \to i(Q) \otimes_A i(P)$ is an $R$–linear isomorphism. Finally, one can check that $f$ is a $C$–$B$–bimodule map. □

**Remark 2.1.2.** Note that if we identify $i(P \otimes_A Q)$ with $i(Q) \otimes_A i(P)$ via $f$, then we have the formula $i(p \otimes q) = i(q) \otimes i(p)$. In particular, let $M$ be a $B$–$A$–bimodule, and identify $i \circ i(M)$ with $M$, and $i(M \otimes_A i(M))$ with $i \circ i(M) \otimes_A i(M) = M \otimes_A i(M)$. Then we have the formula $i(x \otimes i(y)) = y \otimes i(x)$. We will use these identifications throughout the paper.

## 2.2 Cellularity

We recall the definition of *cellularity* from [22]; see also [41]. The version of the definition given here is slightly weaker than the original definition in [22]; and justified below.

**Definition 2.2.1.** Let $R$ be an integral domain and $A$ a unital $R$–algebra. A *cell datum* for $A$ consists of an algebra involution $i$ of $A$; a partially ordered set $(\Lambda, \geq)$ and for each $\lambda \in \Lambda$ a set $\mathcal{T}(\lambda)$; and a subset $\mathcal{C} = \{c^\lambda_{s,t} : \lambda \in \Lambda \text{ and } s, t \in \mathcal{T}(\lambda)\} \subseteq A$; with the following properties:

1. $\mathcal{C}$ is an $R$–basis of $A$.

2. For each $\lambda \in \Lambda$, let $\breve{A}^\lambda$ be the span of the $c^\mu_{s,t}$ with $\mu > \lambda$. Given $\lambda \in \Lambda$, $s \in \mathcal{T}(\lambda)$, and $a \in A$, there exist coefficients $r^s_v(a) \in R$ such that for all $t \in \mathcal{T}(\lambda)$:

$$ac^\lambda_{s,t} \equiv \sum_v r^s_v(a) c^\lambda_{v,t} \mod \breve{A}^\lambda.$$

3. $i(c^\lambda_{s,t}) \equiv c^\lambda_{t,s} \mod \breve{A}^\lambda$ for all $\lambda \in \Lambda$ and, $s, t \in \mathcal{T}(\lambda)$.

$A$ is said to be a *cellular algebra* if it has a cell datum.

For brevity, we will write that $(\mathcal{C}, \Lambda)$ is a cellular basis of $A$.

**Remark 2.2.2.**

1. The original definition in [22] requires that $i(c^\lambda_{s,t}) = c^\lambda_{t,s}$ for all $\lambda, s, t$. However, one can check that the basic consequences of the definition ([22], pages 7-13) remain valid with our weaker axiom.

2. In case $2 \in R$ is invertible, one can check that our definition is equivalent to the original.

We recall some basic structures related to cellularity, see [22]. Given $\lambda \in \Lambda$. Let $A^\lambda$ denote the span of the $c^\mu_{s,t}$ with $\mu \geq \lambda$. It follows that both $A^\lambda$ and $\breve{A}^\lambda$ (defined above) are $i$–invariant two sided ideals of $A$. If $t \in \mathcal{T}(\lambda)$, define $C^\lambda_t$ to be the $R$-submodule of $A^\lambda/\breve{A}^\lambda$ with basis $\{c^\lambda_{s,t} + \breve{A}^\lambda : s \in \mathcal{T}(\lambda)\}$. Then $C^\lambda_t$ is a left $A$-module by Definition 2.2.1 (2). Furthermore, the action of $A$ on $C^\lambda_t$ is independent of $t$, i.e $C^\lambda_u \cong C^\lambda_t$ for any $u, t \in \mathcal{T}(\lambda)$. The *left cell module* $\Delta^\lambda$ is defined as follows: as an $R$–module, $\Delta^\lambda$ is free with basis $\{c^\lambda_s : s \in \mathcal{T}(\lambda)\}$; for each $a \in A$, the action of $a$ on $\Delta^\lambda$ is defined by $ac^\lambda_s = \sum_v r^s_v(a) c^\lambda_v$ where $r^s_v(a)$ is as in Definition 2.2.1 (2). Then $\Delta^\lambda \cong C^\lambda_t$, for any $t \in \mathcal{T}(\lambda)$. For all $s, t \in \mathcal{T}(\lambda)$, we have a canonical $A - A$–bimodule isomorphism $\alpha : A^\lambda/\breve{A}^\lambda \to \Delta^\lambda \otimes_R i(\Delta^\lambda)$ defined by $\alpha(c^\lambda_{s,t} + \breve{A}^\lambda) = c^\lambda_s \otimes_R i(c^\lambda_t)$. Moreover, we have $i \circ \alpha = \alpha \circ i$, using Remark 2.1.2 and point (3) of Definition 2.2.1.

**Definition 2.2.3.** Suppose $A$ is a unital $R$–algebra with involution $i$, and $J$ is a $i$–invariant ideal; then we have an induced algebra involution $i$ on $A/J$. Let us say that $J$ is a *cellular ideal* in $A$ if it satisfies the axioms for a cellular algebra (except for being unital) with cellular basis

$$\{c^\lambda_{s,t} : \lambda \in \Lambda_J \text{ and } s, t \in \mathcal{T}(\lambda)\} \subseteq J$$

and we have, as in point (2) of the definition of cellularity,

$$ac^{\lambda}_{s,t} \equiv \sum_{v} r^{s}_{v}(a) c^{\lambda}_{v,t} \mod \breve{J}^{\lambda}$$

not only for $a \in J$ but also for $a \in A$.

**Remark 2.2.4.** (On extensions of cellular algebras.) If $J$ is a cellular ideal in $A$, and $H = A/J$ is cellular (with respect to the involution induced from the involution on $A$), then $A$ is cellular. In fact, let $(\Lambda_J, \geq)$ be the partially ordered set in the cell datum for $J$ and $\mathcal{C}_J$ the cellular basis. Let $(\Lambda_H, \geq)$ be the partially ordered set in the cell datum for $H$ and $\{\bar{h}^{\mu}_{u,v}\}$ the cellular basis. Then $A$ has a cell datum with partially ordered set $\Lambda = \Lambda_J \cup \Lambda_H$, with partial order agreeing with the original partial orders on $\Lambda_J$ and on $\Lambda_H$ and with $\lambda > \mu$ if $\lambda \in \Lambda_J$ and $\mu \in \Lambda_H$. A cellular basis of $A$ is $\mathcal{C}_J \cup \{h^{\mu}_{s,t}\}$, where $h^{\mu}_{s,t}$ is any lift of $\bar{h}^{\mu}_{s,t}$.

With the original definition of [22], the assertions of this remark would be valid only if the ideal $J$ has an $i$–invariant $R$–module complement in $A$. The ease of handling extensions is our motivation for using the weaker definition of cellularity.

## 2.3 Basis–free formulations of cellularity

König and Xi have given a basis–free definition of cellularity [36]. Definition 2.3.1 is a slight weakening of their definition, which corresponds exactly to the weaker form of Graham–Lehrer cellularity

**Definition 2.3.1** (König and Xi)**.** Let $R$ be an integral domain and $A$ a unital $R$-algebra with involution $i$. An $i$–invariant two sided ideal $J$ in $A$ is called a *split ideal* if, and only if, there exists a left ideal $\Delta$ of $A$ contained in $J$, with $\Delta$ finitely generated and free over $R$, and there is an isomorphism of $A$–$A$–bimodules $\alpha : J \to \Delta \otimes_R i(\Delta)$ making the following diagram commute:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_R i(\Delta) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ J & \xrightarrow{\alpha} & \Delta \otimes_R i(\Delta). \end{array}$$

A finite chain of $i$–invariant two sided ideals

$$0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$$

is called a *cell chain* if for each $j$ $(1 \leq j \leq n)$, the quotient $J_j/J_{j-1}$ is a split ideal of $A/J_{j-1}$ (with respect to the involution induced by $i$ on $A/J$).

**Remark 2.3.2.**

1. König and Xi call a split ideal a "cell ideal." We changed the terminology to avoid confusion with other concepts.

2. The definition of a cell chain differs from the one given by Konig and Xi in that we dropped the requirement that $J_{j-1}$ have an $i$-invariant $R$-module complement in $J_j$.

**Lemma 2.3.3.** *Let $R$ be an integral domain and let $A$ be a unital $R$–algebra with involution $i$. An ideal $J$ of $A$ is split if, and only if, there exists a left $A$–module $M$ that is finitely generated and free as an $R$–module, and there exists an isomorphism of $A$–$A$–bimodules $\gamma : J \to M \otimes_R i(M)$ making the following diagram commute:*

$$\begin{array}{ccc} J & \xrightarrow{\gamma} & M \otimes_R i(M) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ J & \xrightarrow{\gamma} & M \otimes_R i(M). \end{array}$$

*Proof.* If $J$ is split, it clearly satisfies the condition of the lemma. Conversely, suppose the condition of the lemma is satisfied. Fix some element $b_0$ of the basis of $M$ over $R$ and define a left $A$–module map $\beta : M \to A$ by $\beta(m) = \gamma^{-1}(m \otimes b_0)$. Then $\beta$ is an isomorphism of $M$ onto a left ideal $\Delta$ of $A$ contained in $J$.

Now we have $\beta \otimes i(\beta) : M \otimes_R i(M) \to \Delta \otimes_R i(\Delta)$ is an isomorphism satisfying $(\beta \otimes i(\beta)) \circ i = i \circ (\beta \otimes i(\beta))$. It follows that $\alpha = (\beta \otimes i(\beta)) \circ \gamma : J \to \Delta \otimes_R i(\Delta)$ is an isomorphism of $A$–$A$–bimodules satisfying the requirement for a split ideal, namely, $\alpha \circ i = i \circ \alpha$. □

**Lemma 2.3.4** (König and Xi)**.** *Let $A$ be an $R$–algebra with involution. $A$ is cellular if, and only if, $A$ has a finite cell chain.*

*Proof.* We sketch the proof from [35], p. 372.

Suppose $A$ has a cell datum with partially ordered set $(\Lambda, \geq)$ and cell basis $\{c^{\lambda}_{s,t}\}$. Write $\Lambda$ as a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_1$ is maximal in $\Lambda$, and, for $1 \leq j < n$, $\lambda_{j+1}$ is maximal in $\Lambda \setminus \{\lambda_1, \ldots, \lambda_j\}$. Then for each $j \geq 1$, $\Gamma_j = \{\lambda_1, \ldots, \lambda_j\}$ is an order ideal in $\Lambda$. Set $\Gamma_0 = \emptyset$. Define $A(\Gamma_j)$ to be the $R$-submodule of $A$ spanned by the basis elements $c^{\lambda}_{u,v}$, with $\lambda \in \Gamma_j$. Then $A(\Gamma_j)$ is an $i$–invariant two sided ideal in $A$, and

$$0 = A(\Gamma_0) \subset A(\Gamma_1) \subset \cdots \subset A(\Gamma_n) = A.$$

Moreover (see [22], p. 6),

$$A(\Gamma_j)/A(\Gamma_{j-1}) \cong A^{\lambda_j}/\breve{A}^{\lambda_j} \cong \Delta^{\lambda_j} \otimes_R i(\Delta^{\lambda_j}),$$

and the isomorphism $\alpha : A(\Gamma_j)/A(\Gamma_{j-1}) \to \Delta^{\lambda_j} \otimes_R i(\Delta^{\lambda_j})$ satisfies $\alpha \circ i = i \circ \alpha$. Thus $(A(\Gamma_j))_{1 \leq j \leq n}$ is a cell chain.

Conversely, suppose $(J_j)_{0 \leq j \leq n}$ is a cell chain in $A$. Then for each $j \geq 1$, we have an $A$–module $\Delta_j$ that is finitely generated and free as an $R$–module, and an isomorphism of $A$–$A$–bimodules $\alpha_j : J_j/J_{j-1} \to \Delta_j \otimes_R i(\Delta_j)$ satisfying $i \circ \alpha_j = \alpha_j \circ i$. Let $\{b^j_s : s \in \mathcal{T}(j)\}$ be an $R$–basis of $\Delta_j$ and let $c^j_{s,t}$ be any lift in $J_j$ of $\alpha_j^{-1}(b^j_s \otimes i(b^j_t))$. Now take $\Lambda'$ to be $\Lambda$ with the order $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. Let $\mathcal{C} = \{c^{\lambda_j}_{s,t} : 1 \leq j \leq n;\ s, t \in \mathcal{T}(j)\}$. Then $(\mathcal{C}, \Lambda')$ is a cellular basis of $A$. $\square$

Note that if we follow the procedure of the proof, starting with a cell datum on $A$ with partially ordered set $(\Lambda, \geq)$, then the only information that we retain about $\Lambda$ is that $\lambda_{j+1}$ is maximal in $\Lambda \setminus \Gamma_j$; we cannot recover the partial order on $\Lambda$ from this. Moreover, if we continue to produce a cellular basis $\{c^j_{s,t}\}$ from the cell chain $(A(\Gamma_j))_{0 \leq j \leq n}$, the result will not necessarily have the properties of a cellular basis with respect to the original partially ordered set $(\Lambda, \geq)$.

In order to prove the main results, Definition 2.3.5 provides a different basis–free formulation of cellularity that allows us to pass back and forth between the formulation of Definition 2.2.1 and the basis–free formulation without losing information about the partially ordered set.

**Definition 2.3.5.** Let $A$ be an $R$–algebra with involution $i$. Let $(\Lambda, \geq)$ be a finite partially ordered set. For $\lambda \in \Lambda$, let $\Gamma_{\geq\lambda}$ denote the order ideal $\{\mu : \mu \geq \lambda\}$ and $\Gamma_{>\lambda}$ the order ideal $\{\mu : \mu > \lambda\}$.

A $\Lambda$–*cell net* is a map from the set of order ideals of $\Lambda$ to the set of $i$–invariant two sided ideals of $A$, $\Gamma \mapsto A_\Gamma$, with the following properties:

1. $A_\emptyset = \{0\}$. If $\Gamma_1 \subseteq \Gamma_2$, then $A_{\Gamma_1} \subseteq A_{\Gamma_2}$.

2. For $\lambda \in \Lambda$, write $A_{\geq\lambda} = A_{\Gamma_{\geq\lambda}}$ and $A_{>\lambda} = A_{\Gamma_{>\lambda}}$. Then
$$A = \operatorname{span}\{A_{\geq\mu} : \mu \in \Lambda\},$$
and for all $\lambda \in \Lambda$,
$$A_{>\lambda} = \operatorname{span}\{A_{\geq\mu} : \mu > \lambda\}.$$

3. For each $\lambda \in \Lambda$, there is an $A$–module $M^\lambda$, finitely generated and free as an $R$–module, such that whenever $\Gamma \subseteq \Gamma'$ are order ideals of $\Lambda$, with $\Gamma' \setminus \Gamma = \{\lambda\}$, then there exists an isomorphism of $A$–$A$–bimodules
$$\alpha : A_{\Gamma'}/A_\Gamma \to M^\lambda \otimes_R i(M^\lambda),$$
satisfying $i \circ \alpha = \alpha \circ i$.

**Lemma 2.3.6.** *Let $A$ be an $R$–algebra with involution, and let $(\Lambda, \geq)$ be a finite partially ordered set. Then $A$ has a cell datum with partially ordered set $\Lambda$ if, and only if, $A$ has a $\Lambda$–cell net.*

*Proof.* Suppose that $A$ has a cell datum with partially ordered set $\Lambda$ and cell basis $\{c^\lambda_{s,t}\}$. For each order ideal $\Gamma$ of $\Lambda$, let $A(\Gamma)$ denote the span of those $c^\lambda_{s,t}$ with $\lambda \in \Gamma$. Then $\Gamma \mapsto A(\Gamma)$ is a $\Lambda$–cell net.

Conversely, suppose that $A$ has a $\Lambda$–cell net, $\Gamma \mapsto A_\Gamma$. For each $\lambda \in \Lambda$, we have an isomorphism of $A$–$A$–bimodules $\alpha_\lambda : A_{\geq\lambda}/A_{>\lambda} \to M^\lambda \otimes_R i(M^\lambda)$. Let $\{b^\lambda_s : s \in \mathcal{T}(\lambda)\}$ be an $R$–basis of $M^\lambda$ and let $c^\lambda_{s,t}$ be any lift of $\alpha_\lambda^{-1}(b^\lambda_s \otimes i(b^\lambda_t))$ to $A_{\geq\lambda}$. We claim that
$$\mathcal{C} = \{c^\lambda_{s,t} : \lambda \in \Lambda; s, t \in \mathcal{T}(\lambda)\}$$

is an $R$–basis of $A$.

Let $A^\lambda$ be the span of those $c^\mu_{s,t}$ with $\mu \geq \lambda$ and $\breve{A}^\lambda$ the span of those $c^\mu_{s,t}$ with $\mu > \lambda$. If $\mu \geq \lambda$, then for all $s, t \in \mathcal{T}(\mu)$, $c^\mu_{s,t} \in A_{\geq\mu} \subseteq A_{\geq\lambda}$, using point (1) of Definition 2.3.5. Hence $A^\lambda \subseteq A_{\geq\lambda}$. Similarly, $\breve{A}^\lambda \subseteq A_{>\lambda}$.

We claim that

$$\text{for all } \lambda \in \Lambda, \quad A_{\geq\lambda} = A^\lambda. \tag{2.3.1}$$

This is clear if $\lambda$ is a maximal element of $\Lambda$. (Note that $A_{>\lambda} = A_\emptyset = \{0\}$.) Now suppose that $\lambda$ is not maximal and that for all $\mu > \lambda$, $A_{\geq\mu} = A^\mu$. Then

$$A_{>\lambda} = \operatorname{span}\{A_{\geq\mu} : \mu > \lambda\} = \operatorname{span}\{A^\mu : \mu > \lambda\} = \breve{A}^\lambda,$$

where the first equality comes from (2) of Definition 2.3.5 and the second from the induction hypothesis. By definition of $\{c^\lambda_{s,t}\}$, we have

$$A_{\geq\lambda} = \operatorname{span}\{c^\lambda_{s,t}\} + A_{>\lambda} = \operatorname{span}\{c^\lambda_{s,t}\} + \breve{A}^\lambda = A^\lambda.$$

Assertion (2.3.1) now follows by induction. Point (2) of Definition 2.3.5 and (2.3.1) imply that $A_{>\lambda} = \breve{A}^\lambda$ for all $\lambda \in \Lambda$, and that $A = \operatorname{span}(\mathcal{C})$.

We now proceed to establish linear independence of $\mathcal{C}$. Write $\Lambda$ as a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_K)$ with $\lambda_1$ maximal and $\lambda_{j+1}$ maximal in $\Lambda \setminus \{\lambda_1, \ldots, \lambda_j\}$ for $1 \leq j < K$. Put $\Gamma_j = \{\lambda_1, \ldots, \lambda_j\}$ for $j \geq 1$ and $\Gamma_0 = \emptyset$. Then $(\Gamma_j)_{0\leq j\leq K}$ is a maximal chain of order ideals. Since $\Gamma_j \setminus \Gamma_{j-1} = \{\lambda_j\}$, we have an isomorphism $\gamma_j : A_{\Gamma_j}/A_{\Gamma_{j-1}} \to M^{\lambda_j} \otimes_R i(M^{\lambda_j})$ with $i \circ \gamma_j = \gamma_j \circ i$. Thus $(A_{\Gamma_j})_{0\leq j\leq K}$ is a cell chain in $A$. So by the proof of Lemma 2.3.4, $A$ has a cellular basis

$$\mathcal{B} = \{b^\lambda_{s,t} : \lambda \in \Lambda;\ s, t, \in \mathcal{T}(\lambda)\},$$

but with respect to the "wrong" partial order on $\Lambda$. Since $\mathcal{C}$ is a spanning set of the same cardinality as the basis $\mathcal{B}$, it follows that $\mathcal{C}$ is linearly independent over $R$, and thus an $R$–basis of $A$.

Because $A_{>\lambda} = \breve{A}^\lambda$ for all $\lambda \in \Lambda$, it is now easy to see that properties (2) and (3) of Definition 2.2.1 are satisfied by $\mathcal{C}$. $\square$

## 2.4 Coherent towers of cellular algebras

**Definition 2.4.1.** Let $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ be an increasing sequence of cellular algebras over an integral domain $R$. Let $\Lambda_n$ denote the partially ordered set in the cell datum for $H_n$. We say that $(H_n)_{n\geq 0}$ is a *coherent tower of cellular algebras* if the following conditions are satisfied:

1. The involutions are consistent; that is, the involution on $H_{n+1}$, restricted to $H_n$, agrees with the involution on $H_n$.

2. For each $n \geq 0$ and for each $\lambda \in \Lambda_n$, the induced module $\mathrm{Ind}_{H_n}^{H_{n+1}}(\Delta^\lambda)$ has a filtration by cell modules of $H_{n+1}$. That is, there is a filtration
$$\mathrm{Ind}_{H_n}^{H_{n+1}}(\Delta^\lambda) = M_t \supseteq M_{t-1} \supseteq \cdots \supseteq M_0 = (0)$$
such that for each $j \geq 1$, there is a $\mu_j \in \Lambda_{n+1}$ with $M_j/M_{j-1} \cong \Delta^{\mu_j}$.

3. For each $n \geq 0$ and for each $\mu \in \Lambda_{n+1}$, the restriction $\mathrm{Res}_{H_n}^{H_{n+1}}(\Delta^\mu)$ has a filtration by cell modules of $H_n$. That is, there is a filtration
$$\mathrm{Res}_{H_n}^{H_{n+1}}(\Delta^\mu) = N_s \supseteq N_{s-1} \supseteq \cdots \supseteq N_0 = (0)$$
such that for each $i \geq 1$, there is a $\lambda_i \in \Lambda_n$ with $N_j/N_{j-1} \cong \Delta^{\lambda_i}$.

The modification of the definition for a *finite* tower of cellular algebras is obvious.

Define a filtration as in (2) and (3) to be a *cell filtration*. The examples will also have *uniqueness of the multiplicities* of the cell modules appearing as subquotients of the cell filtrations, and *Frobenius reciprocity* connecting the multiplicities in the two types of filtrations. Definition 2.4.1 did not include uniqueness of multiplicities and Frobenius reciprocity as requirements, as they will follow from additional assumptions imposed later; see Lemma 2.7.4.

**Example 2.4.2.** *The tower of Hecke algebras of type $A$ is a coherent tower of cellular algebras.* Let $R$ be an integral domain and $q$ an invertible element of $R$. Let $H_n(R,q)$ denote the Hecke algebra of type $A$ generated by elements $T_1, \ldots, T_{n-1}$

satisfying the braid relations and the quadratic relations $(T_j - q)(T_j + 1) = 0$ for $1 \leq j \leq n-1$. When $q = 1$, $H_n(R, q)$ is the group algebra $R\mathfrak{S}_n$ of the symmetric group $\mathfrak{S}_n$. As is well known, $H_n(R, q)$ has a basis $T_w$ ($w \in \mathfrak{S}_n$) given by $T_w = T_{j_1} \dots T_{j_\ell}$ for any reduced expression $w = s_{j_1} \dots s_{j_\ell}$. The map defined by $i(T_w) = T_{w^{-1}}$ is an algebra involution. The map defined by $(T_w)^\# = (-q)^{\ell(w)}(T_{w^{-1}})^{-1}$ is an algebra automorphism. The assignment $T_w \mapsto T_w$ is an embedding of $H_n(R, q)$ into $H_{n+1}(R, q)$. The algebra involutions are consistent on $(H_n)_{n \geq 0}$.

Dipper and James [10, 11] studied the representation theory of the Hecke algebras, defining Specht modules $S^\lambda$ which generalize Specht modules for symmetric groups. They showed that induced modules of Specht modules have a filtration by Specht modules [10]. Jost [32] showed that restrictions of Specht modules have Specht filtrations.

Murphy [46] showed that the Hecke algebras are cellular (before the formalization of the notion of cellularity in [22]). Murphy shows that his cell modules $\Delta^\lambda$ satisfy $\Delta^\lambda \cong (S^{\lambda'})^\#$, where $\lambda'$ is the transpose of $\lambda$ and the superscript $\#$ means that the module is twisted by the automorphism $\#$. Thus it follows from the results of Dipper, James, and Jost cited above that restricted modules and induced modules of Murphy's cell modules have cell filtrations.

## 2.5 Branching diagrams

A general source for the material in this section is [18].

A finite dimensional split semisimple algebra over a field $F$ is one which is isomorphic to a finite direct sum of full matrix algebras over $F$.

Suppose $A \subseteq B$ are finite dimensional split semisimple algebras over $F$ (with the same identity element). Let $A(i)$, $i \in I$, be the minimal ideals of $A$ and $B(j)$, $j \in J$, the minimal ideals of $B$. We associate a $J \times I$ *inclusion matrix* $\Omega$ to the inclusion $A \subseteq B$, as follows. Let $W_j$ be a simple $B(j)$–module. Then $W_j$ becomes an $A$–module via the inclusion, and $\Omega(j, i)$ is the multiplicity of a simple $A_i$–module

in the decomposition of $W_j$ as an $A$–module. An equivalent characterization of the inclusion matrix is the following. Let $q_i$ be a minimal idempotent in $A(i)$ and let $z_j$ be the identity of $B(j)$ (a minimal central idempotent in $B$). Then $q_i z_j$ is the sum of $\Omega(j,i)$ minimal idempotents in $B(j)$.

It is convenient to encode an inclusion matrix by a bipartite graph, called the *branching diagram*; the branching diagram has vertices labeled by $I$ arranged on one horizontal line, vertices labeled by $J$ arranged along a second (higher) horizontal line, and $\Omega(j,i)$ edges connecting $j \in J$ to $i \in I$.

If $A_1 \subseteq A_2 \subseteq A_3 \cdots$ is a (finite or infinite) sequence of inclusions of finite dimensional split semisimple algebras over $F$, then the branching diagram for the sequence is obtained by stacking the branching diagrams for each inclusion, with the upper vertices of the diagram for $A_i \subseteq A_{i+1}$ being identified with the lower vertices of the diagram for $A_{i+1} \subseteq A_{i+2}$.

For the purposes of this thesis, it suffices to restrict to the case that $A_0 \cong F$. In most of the examples, the entries in each inclusion matrix are all 0 or 1; thus in the branching diagram there are no multiple edges between vertices.

**Definition 2.5.1.** An (infinite) abstract branching diagram $\mathfrak{B}$ is an infinite graph with vertex set $V = \coprod_{i \geq 0} V_i$, with the following properties

1. $V_0$ is a singleton and $V_i$ is finite for all $i$.

2. Two vertices $v \in V_i$ and $w \in V_j$ are adjacent only if $|i-j| = 1$. Multiple edges are allowed between adjacent vertices.

3. If $i \geq 1$ and $v \in V_i$, then $v$ is adjacent to at least one vertex in $V_{i-1}$ and to at least one vertex in $V_{i+1}$.

The definition can be modified in the obvious way for a *finite* abstract branching diagram. When treating the walled Brauer algebra in Section 5.6, weaken the definition by dropping the requirement that $V_0$ is a singleton.

The branching diagram for a sequence of finite dimensional split semisimple algebras (with the restrictions mentioned above) is an abstract branching diagram, and conversely, given an abstract branching diagram $\mathfrak{B}$, one can construct a sequence of finite dimensional split semisimple algebras (over any given field) whose branching diagram is (isomorphic to) $\mathfrak{B}$.

Let $\mathfrak{B}$ be an abstract branching diagram with vertex set $V = \coprod_{i\geq 0} V_i$. By convention, denote the unique element of $V_0$ by $\emptyset$. Picture $\mathfrak{B}$ with the elements of $V_i$ arranged on the horizontal line $y = -i$ in the plane, and call $V_i$ the $i$–th *row* of vertices in $\mathfrak{B}$. If $v \in V_i$ and $w \in V_{i+1}$ are adjacent, write $v \nearrow w$. The subgraph of $\mathfrak{B}$ consisting of $V_i$ and $V_{i+1}$ and the edges connecting them is called the $i$–th *level* of $\mathfrak{B}$.

Now suppose that $\mathfrak{B}_0$ is an abstract branching diagram with vertex set $V^{(0)} = \coprod_{i\geq 0} V_i^{(0)}$. Construct a new abstract branching diagram $\mathfrak{B}$ as follows: The vertex set of $\mathfrak{B}$ is $V = \coprod_{k\geq 0} V_k$, where

$$V_k = \coprod_{\substack{i\leq k \\ k-i \text{ even}}} V_i^{(0)} \times \{k\}.$$

Thus the $k$–th row of vertices of $\mathfrak{B}$ consists of copies of rows $k$, $k-2$, $k-4$, ... of vertices of $\mathfrak{B}_0$. Now if $(\lambda, k) \in V_k$ and $(\mu, k+1) \in V_{k+1}$, there exist $i \leq k$ with $k-i$ even such that $\lambda \in V_i^{(0)}$, and $j \leq k+1$ with $k+1-j$ even such that $\mu \in V_j^{(0)}$. Let $(\lambda, k) \nearrow (\mu, k+1)$ if, and only if, $|i-j| = 1$ and $\lambda$ and $\mu$ are adjacent in $\mathfrak{B}_0$. The number of edges connecting $(\lambda, k)$ and $(\mu, k+1)$ is the same as the number of edges connecting $\lambda$ and $\mu$ in $\mathfrak{B}_0$.

The first few levels of $\mathfrak{B}$ is picture schematically in Figure 2.1, where each diagonal line represents all the edges connecting vertices in $V_i^{(0)}$ with vertices in $V_{i\pm 1}^{(0)}$. Note that the $k$–th level of $\mathfrak{B}$ is a folded copy of the first $k$ levels of $\mathfrak{B}_0$. We call $\mathfrak{B}$ the *branching diagram obtained by reflections from* $\mathfrak{B}_0$.

**Example 2.5.2.** Take $\mathfrak{B}_0$ to be Young's lattice. Thus $V_k^{(0)}$ consists of Young

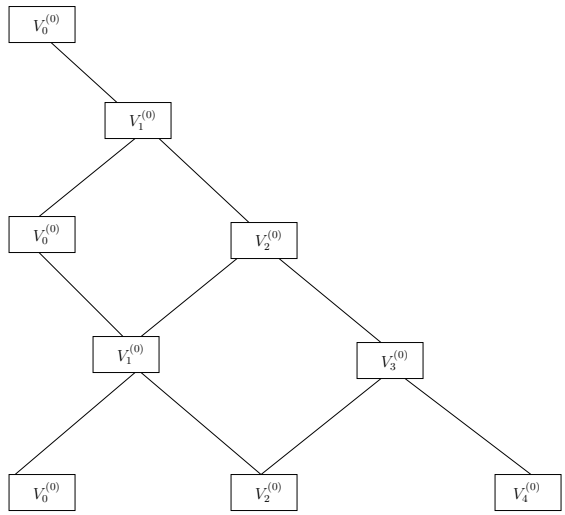



Figure 2.1: Branching diagram obtained by reflections

diagrams of size $k$, and $\lambda \nearrow \mu$ in $\mathfrak{B}_0$ if $\mu$ is obtained from $\lambda$ by adding one box. Then the $k$–th row of vertices in the abstract branching diagram $\mathfrak{B}$ obtained from $\mathfrak{B}_0$ by reflections consists of all pairs $(\lambda, k)$, where $\lambda$ is a Young diagram of size $i \leq k$, with $k - i$ even. Moreover, $(\lambda, k) \nearrow (\mu, k+1)$ in $\mathfrak{B}$ if, and only if, $\mu$ is obtained from $\lambda$ either by adding one box or by removing one box.

## 2.6 The Jones basic construction

This thesis could be written without ever mentioning the Jones basic construction. Nevertheless, the basic construction plays an essential role behind the scenes.

The Jones basic construction was introduced [29] in the theory of von Neumann algebras and is crucial in the analysis of von Neumann subfactors. Translated to the context of finite dimensional split semisimple algebras over a field, the basic construction was a fundamental ingredient in Wenzl's analysis of the generic structure of the Brauer algebras and the BMW algebras [51, 6, 52] .

The basic construction for finite dimensional split semisimple algebras can be described as follows (see [18]): let $A \subseteq B$ be finite dimensional split semisimple

algebras over field $F$. The basic construction for the pair $A \subseteq B$ is the algebra $\mathrm{End}(B_A)$. This algebra is also split semisimple and the inclusion matrix for the pair $B \subseteq \mathrm{End}(B_A)$ is a transpose of that for the pair $A \subseteq B$. Suppose now that $B$ has a faithful $F-$ valued trace $\varepsilon$ with faithful restriction to $A$. Here faithful means that the bilinear form $(x, y) \mapsto \varepsilon(xy)$ is non–degenerate. In this case there is a unique trace preserving conditional expectation $\varepsilon_A : B \to A$, i.e. a unital $A$–$A$–bimodule map satisfying $\varepsilon \circ \varepsilon_A = \varepsilon$. Identify $B$ with its image in $\mathrm{End}_F(B)$ under the left regular representation. The basic construction $\mathrm{End}(B_A)$ is equal to $B\varepsilon_A B = \{\sum_{i=1}^n b_i' \varepsilon_a b_i'' : n \geq 1, b_i', b_i'' \in B\}$. Moreover, $B\varepsilon_A B \cong B \otimes_A B$, where the latter is given the algebra structure determined by $(b_1 \otimes b_2)(b_3 \otimes b_4) = b_1 \otimes \varepsilon_A(b_2 b_3) b_4$. Note that we have three realizations for the basic construction,

$$\mathrm{End}(B_A) \cong B\varepsilon_A B \cong B \otimes_A B,$$

any of which could serve as a potential definition of the basic construction in a more general setting.

Suppose in addition that we are given an algebra $C$ with $B \subseteq C$ and that $C$ contains an idempotent $e$ such that $exe = \varepsilon_A(x)e$ for $x \in B$, and $x \mapsto xe$ is injective from $B$ to $Be \subseteq C$. Note that $BeB$ is a possibly non–unital subalgebra of $C$. By [51], Theorem 1.3, $BeB \cong B\varepsilon_A B \cong \mathrm{End}(B_A)$, and, in particular, $BeB$ is unital and semisimple.

Wenzl used these ideas to show the generic semisimplicity of the Brauer algebras, see Chapter 5.2 for the definition of the Brauer algebras. Consider the Brauer algebras $B_n = B_n(F, \delta)$ over $F = \mathbb{C}$ or $F = \mathbb{Q}(\delta)$, in the first case with parameter $\delta$ a non-integer complex number, and in the second case with parameter $\delta$ an indeterminant over $\mathbb{Q}$. The Brauer algebras have a canonical $F$–valued trace $\varepsilon$ and conditional expectations $\varepsilon_n : B_n \to B_{n-1}$ preserving the trace. Each Brauer algebra $B_n$ contains an essential idempotent $e_{n-1}$ with $e_{n-1}^2 = \delta e_{n-1}$ and $e_{n-1} x e_{n-1} = \delta \varepsilon_{n-1}(x) e_{n-1}$ for $x \in B_{n-1}$. Moreover, $x \mapsto x e_{n-1}$ is injective from $B_{n-1}$ to $B_n$ and one has

$B_n/B_ne_{n-1}B_n \cong F\mathfrak{S}_n$, which is semisimple, since $F$ has characteristic 0. Let $f_{n-1} = \delta^{-1}e_{n-1}$; then $f_{n-1}$ is an idempotent with $f_{n-1}xf_{n-1} = \varepsilon_{n-1}(x)f_{n-1}$ for $x \in B_{n-1}$. We have $B_0 \cong B_1 \cong F$.

Suppose it is known for some $n$ that $B_k$ is split semisimple and that the trace $\varepsilon$ is faithful on $B_k$ for $k \leq n$. By Wenzl's observation applied to $B_{n-1} \subseteq B_n \subseteq B_{n+1}$ and the idempotent $f_n \in B_{n+1}$, we have $B_ne_nB_n = B_nf_nB_n \cong B_n\varepsilon_nB_n \cong \mathrm{End}((B_n)_{B_{n-1}})$. But it is elementary to check that $B_ne_nB_n = B_{n+1}e_nB_{n+1}$. Thus we have that the ideal $B_{n+1}e_nB_{n+1} \subseteq B_{n+1}$ is split semisimple, and the quotient of $B_{n+1}$ by this ideal ($\cong F\mathfrak{S}_{n+1}$) is also split semisimple, so $B_{n+1}$ is split semisimple. To continue the inductive argument, it is necessary to verify that the trace $\varepsilon$ is faithful on $B_{n+1}$. Wenzl uses a Lie theory argument for this.

This thesis develops a cellular analog of this argument. To illustrate this, continue to use the example of the Brauer algebras. Cellularity is a property that is preserved under specializations, so it suffices to consider the Brauer algebras over the generic ring $R = \mathbb{Z}[\delta]$. Let $F$ denote the field of fractions of $R$, $F = \mathbb{Q}(\delta)$. Write $B_n$ for $B_n(R, \delta)$ and $B_n^F$ for $B_n(F, \delta)$. By Wenzl's theorem, $B_n^F$ is split semisimple. We have $B_0 \cong B_1 \cong R$.

Suppose it is known for some $n$ that $B_k$ is cellular for $k \leq n$. We want to show that $B_{n+1}e_nB_{n+1} = B_ne_nB_n$ is a cellular ideal in $B_{n+1}$. It will then follow that $B_{n+1}$ is cellular, because the quotient $B_{n+1}/B_{n+1}e_nB_{n+1} \cong R\mathfrak{S}_{n+1}$ is cellular. Let $\Lambda_{n-1}$ denote the partially ordered set in the cell datum for $B_{n-1}$. For each order ideal $\Gamma$ of $\Lambda_{n-1}$, write $J(\Gamma)$ for the span in $B_{n-1}$ of all $c_{s,t}^{\lambda}$ with $\lambda \in \Gamma$. The crucial point is to show that $\Gamma \mapsto B_ne_nJ(\Gamma)B_n = B_{n+1}e_nJ(\Gamma)B_{n+1}$ is a $\Lambda_{n-1}$–cell net in $B_{n+1}e_nB_{n+1}$. Along the way to doing this, we show that

$$J'(\Gamma) := B_n \otimes_{B_{n-1}} J(\Gamma) \otimes_{B_{n-1}} B_n \cong B_ne_nJ(\Gamma)B_n \tag{2.6.1}$$

via $b' \otimes x \otimes b'' \mapsto b'e_nxb''$; consequently, if $\Gamma_1 \subseteq \Gamma_2$, then $J'(\Gamma_1)$ imbeds in $J'(\Gamma_2)$.

In particular,

$$B_n \otimes_{B_{n-1}} B_n \cong B_n e_n B_n = B_{n+1} e_n B_{n+1}, \tag{2.6.2}$$

and $J'(\Gamma)$ imbeds as an ideal in the (non–unital) algebra $B_n \otimes_{B_{n-1}} B_n$. Essentially, what we show is that $B_{n+1}e_nB_{n+1} = B_ne_nB_n$ is isomorphic to the basic construction $B_n \otimes_{B_{n-1}} B_n$, and that $\Gamma \mapsto J'(\Gamma)$ is a $\Lambda_{n-1}$–cell net in $B_n \otimes_{B_{n-1}} B_n$.

Note that $B_n$ is *not* a projective $B_{n-1}$–module, but the isomorphisms (2.6.1) and the embeddings $J'(\Gamma_1) \hookrightarrow J'(\Gamma_2)$ reflect the projectivity of $B_n^F$ over $B_{n-1}^F$

## 2.7 Extension of the ground ring

Let $R$ be an integral domain and let $F$ denote the field of fractions of $R$. Consider a coherent tower $(H_n)_{n\geq 0}$ of cellular algebras over $R$ such that for all $n$, the $F$–algebra $H_n^F := H_n \otimes_R F$ is (split) semisimple. In this situation, the filtrations of induced and restricted modules by cell modules have uniqueness of multiplicities, and Frobenius reciprocity connecting these multiplicities.

For any algebra $A$ over $R$, write $A^F$ for the $F$–algebra $A \otimes_R F$. Moreover, for a left (or right) $A$–module $M$, write $M^F$ for the left (or right) $A^F$ module $M \otimes_R F$.

**Lemma 2.7.1.** *Let $R$ be an integral domain and $F$ its field of fractions. Let $A$ and $B$ be $R$-algebras. For modules $M_A$ and ${}_AN$, we have*

$$M \otimes_A N \otimes_R F \cong M^F \otimes_{A^F} N^F \tag{2.7.1}$$

*as $F$-vector spaces. The isomorphism*

$$M \otimes_A N \otimes_R F \to M^F \otimes_{A^F} N^F$$

*is determined by $(x \otimes_A y \otimes_R f) \mapsto (x \otimes_R \mathbf{1}_F) \otimes_{A^F} (y \otimes_R f)$. If ${}_AN_B$ is a bimodule, then the isomorphism in (2.7.1) is an isomorphism of right $B^F$–modules, and similarly, if ${}_BM_A$ is a bimodule, then the isomorphism is an isomorphism of left $B^F$–modules.*

*Proof.* Note that

$$\begin{aligned} M \otimes_A (N \otimes_R F) &\cong M \otimes_A A^F \otimes_{A^F} (N \otimes_R F) \\ &= (M \otimes_A A \otimes_R F) \otimes_{A^F} (N \otimes_R F) \\ &\cong (M \otimes_R F) \otimes_{A^F} (N \otimes_R F) \\ &= M^F \otimes_{A^F} N^F. \end{aligned}$$

If we track a simple tensor through these equalities and isomorphisms, we see that

$$\begin{aligned} x \otimes_A y \otimes_R f &\mapsto x \otimes_A \mathbf{1}_{A^F} \otimes_{A^F} (y \otimes_R f) \\ &= x \otimes_A \mathbf{1}_A \otimes_R \mathbf{1}_F \otimes_{A^F} (y \otimes_R f) \mapsto (x \otimes_R \mathbf{1}_F) \otimes_{A^F} (y \otimes_R f). \end{aligned}$$

The final statement follows from this. □

**Lemma 2.7.2.** *Let $R$ be an integral domain and $F$ its field of fractions. If $M$ is a free $R$–module, then the map $M \to M \otimes_R F$ determined by $x \mapsto x \otimes 1_F$ is injective.*

*Proof.* It follows from [27], Propositions 3.2 and 3.3 that the map $x \mapsto x \otimes 1$ takes an $R$–basis of $M$ to an $F$–basis of $M \otimes_R F$. In particular, the map is injective. □

**Lemma 2.7.3.** *Let $R$ be an integral domain and $F$ its field of fractions. Let $N_1 \subseteq N_2$ be $R$–modules with $N_2$ free. Let $\iota : N_1 \to N_2$ denote the injection. Then $\iota \otimes \mathrm{id}_F : N_1 \otimes_R F \to N_2 \otimes_R F$ is injective.*

*Proof.* Any element of $N_1 \otimes_R F$ can be written as $y = (1/q)(x \otimes 1_F)$, with $q \in R^\times$ and $x \in N_1$. Then $\iota \otimes \mathrm{id}_F(y) = (1/q)(\iota(x) \otimes 1_F) = (1/q)\, \gamma \circ \iota(x)$, where $\gamma : N_2 \to N_2 \otimes_R F$ is determined by $z \mapsto z \otimes 1_F$. Because $N_2$ is a free $R$–module, $\gamma$ is injective, by Lemma 2.7.2, and it follows that $\iota \otimes \mathrm{id}_F$ is injective. □

**Lemma 2.7.4.** *Let $R$ be an integral domain with field of fractions $F$. Suppose that $(H_n)_{n \geq 0}$ is a coherent tower of cellular algebras over $R$ and that $H_n^F$ is split semisimple for all $n$. Let $\Lambda_n$ denote the partially ordered set in the cell datum for $H_n$. Then*

1. *$\{(\Delta^\lambda)^F : \lambda \in \Lambda_n\}$ is a complete family of simple $H_n^F$–modules.*

2. *Let* $[\omega(\mu,\lambda)]_{\mu\in\Lambda_{n+1},\,\lambda\in\Lambda_n}$ *denote the inclusion matrix for* $H_n^F \subseteq H_{n+1}^F$. *Then for any* $\lambda \in \Lambda_n$ *and* $\mu \in \Lambda_{n+1}$, *and any cell filtration of* $\mathrm{Res}_{H_n}^{H_{n+1}}(\Delta^\mu)$, *the number of subquotients of the filtration isomorphic to* $\Delta^\lambda$ *is* $\omega(\mu,\lambda)$.

3. *Likewise, for any* $\lambda \in \Lambda_n$ *and* $\mu \in \Lambda_{n+1}$, *and any cell filtration of* $\mathrm{Ind}_{H_n}^{H_{n+1}}(\Delta^\lambda)$, *the number of subquotients of the filtration isomorphic to* $\Delta^\mu$ *is* $\omega(\mu,\lambda)$.

*Proof.* For point (1), $(\Delta^\lambda)^F$ is a cell module for $H_n^F$, and, for a semisimple cellular algebra, the cell modules are precisely the simple modules.

We have

$$(\mathrm{Res}_{H_n}^{H_{n+1}}(\Delta^\mu))^F = \mathrm{Res}_{H_n^F}^{H_{n+1}^F}((\Delta^\mu)^F) \cong \bigoplus_{\lambda\in\Lambda_n} \omega(\mu,\lambda)(\Delta^\lambda)^F, \tag{2.7.2}$$

by the definition of the inclusion matrix. On the other hand, if

$$\mathrm{Res}_{H_n}^{H_{n+1}}(\Delta^\mu) = N_s \supseteq N_{s-1} \supseteq \cdots \supseteq N_0 = (0)$$

is a cell filtration, with $N_j/N_{j-1} \cong \Delta^{\lambda_j}$, then

$$(\mathrm{Res}_{H_n}^{H_{n+1}}(\Delta^\mu))^F = N_s^F \supseteq N_{s-1}^F \supseteq \cdots \supseteq N_0^F = (0),$$

by Lemma 2.7.3, because all the modules $N_j$ are free as $R$–modules. Moreover, $N_j^F/N_{j-1}^F \cong (N_j/N_{j-1})^F \cong (\Delta^{\lambda_j})^F$ by right exactness of tensor products. Since $H_n^F$ modules are semisimple,

$$(\mathrm{Res}_{H_n}^{H_{n+1}}(\Delta^\mu))^F \cong \bigoplus_{j=1}^{s} (\Delta^{\lambda_j})^F. \tag{2.7.3}$$

Comparing (2.7.2) and (2.7.3) and taking into account that $\Delta^\lambda \mapsto (\Delta^\lambda)^F$ is injective, we obtain conclusion (2).

Likewise,

$$(\mathrm{Ind}_{H_n}^{H_{n+1}}(\Delta^\lambda))^F = H_{n+1} \otimes_{H_n} \Delta^\lambda \otimes_R F \cong H_{n+1}^F \otimes_{H_n^F} (\Delta^\lambda)^F,$$

by Lemma 2.7.1. But

$$H_{n+1}^F \otimes_{H_n^F} (\Delta^\lambda)^F = \mathrm{Ind}_{H_n^F}^{H_{n+1}^F}((\Delta^\lambda)^F) \cong \bigoplus_{\mu\in\Lambda_{n+1}} \omega(\mu,\lambda)(\Delta^\mu)^F,$$

using (2.7.2) and Frobenius reciprocity. The rest of the argument for point (3) is similar to that for point (2). □

**Lemma 2.7.5.** *Adopt the assumptions and notation of Lemma 2.7.4. Assume in*

*addition that the branching diagram $\mathfrak{B}$ for $(H_n^F)_{n\geq 0}$ has no multiple edges and that $H_0^F = F$. It follows that each $H_n$ has a cell datum (perhaps different from the one initially given) with the same partially ordered set $\Lambda_n$ but with $\mathcal{T}(\lambda)$ equal to the set of paths on $\mathfrak{B}$ from $\emptyset$ to $\lambda$.*

*Proof.* Referring to the proof of Lemma 2.3.6, it suffices to show that, for each $n$ and for each $\lambda \in \Lambda_n$, the cell module $\Delta^\lambda$ has an $R$–basis indexed by the set $\mathcal{P}(\lambda)$ of paths $\mathfrak{B}$ from $\emptyset$ to $\lambda$. But this says only that the rank of $\Delta^\lambda$ over $R$ is $|\mathcal{P}(\lambda)|$, and this is true because $\mathrm{rank}_R(\Delta^\lambda) = \dim_F(\Delta^\lambda \otimes_R F) = |\mathcal{P}(\lambda)|$. See also the following remark. $\square$

**Remark 2.7.6.** In principle, in the situation of Lemma 2.7.5, we can recursively build bases of cell modules, using the cell filtrations of restrictions. Suppose we have bases of $\Delta^\lambda$ for all $\lambda \in \Lambda_n$ for some $n$. Let $\mu \in \Lambda_{n+1}$. Then $\Delta^\mu$, regarded as an $H_n$–module, has a filtration by cell modules of $H_n$,

$$\Delta^\mu = N_s \supseteq N_{s-1} \supseteq \cdots \supseteq N_0 = (0),$$

with $N_j/N_{j-1} \cong \Delta^{\lambda_j}$; and $\lambda \in \Lambda_n$ appears (exactly once) in the list of $\lambda_j$, if, and only if, $\lambda \nearrow \mu$. Now we inductively build bases of the $N_j$ to obtain a basis of $N_s = \Delta^\mu$. The isomorphism $N_1 \cong \Delta^{\lambda_1}$ provides a basis of $N_1$. For $j \geq 2$, if we have a basis of $N_{j-1}$, then that basis together with any lift of a basis of $N_j/N_{j-1} \cong \Delta^{\lambda_j}$ gives a basis of $N_j$.

# CHAPTER 3
# A FRAMEWORK FOR CELLULARITY

This chapter describes the framework for cellularity of algebras related to the Jones basic construction.

## 3.1 Framework axioms

Let $R$ be an integral domain with field of fractions $F$. Consider two sequences of $R$–algebras

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots, \quad \text{and} \quad Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \cdots.$$

Assume the following axioms:

1. $(Q_n)_{n\geq 0}$ is a coherent tower of cellular algebras.
2. There is an algebra involution $i$ on $\cup_n A_n$ such that $i(A_n) = A_n$.
3. $A_0 = Q_0 = R$, and $A_1 = Q_1$ (as algebras with involution).
4. For all $n$, $A_n^F := A_n \otimes_R F$ (and hence also $Q_n^F := Q_n \otimes_R F$) is split semisimple.
5. For $n \geq 2$, $A_n$ contains an essential idempotent $e_{n-1}$ such that $i(e_{n-1}) = e_{n-1}$ and $A_n/(A_n e_{n-1} A_n) \cong Q_n$, as algebras with involution.
6. For $n \geq 2$, $e_{n-1}$ commutes with $A_{n-2}$ and $e_{n-1} A_{n-1}^F e_{n-1} = A_{n-2}^F e_{n-1}$.
7. For $n \geq 2$, $A_n e_{n-1} = A_{n-1} e_{n-1}$, and the map $x \mapsto x e_{n-1}$ is injective from $A_{n-1}$ to $A_{n-1} e_{n-1}$.
8. For $n \geq 2$, $e_{n-1} \in A_{n+1} e_n A_{n+1}$.

**Remark 3.1.1.**

1. Let $\Lambda_n^{(0)}$ denote the partially ordered set in the cell datum for $Q_n$. It follows from axioms (1) and (4) and Lemma 2.7.4 that $\Lambda_n^{(0)}$ can be identified with the $n$–th row of vertices of the branching diagram for $(Q_n^F)_{n\geq 0}$.

2. Applying the involution in axiom (7), $e_{n-1}A_{n-1} = e_{n-1}A_n$, and the map $x \mapsto e_{n-1}x$ is injective from $A_{n-1}$ to $e_{n-1}A_{n-1}$. It follows from axioms (6) and (7) that $e_{n-1}A_n^F e_{n-1} = A_{n-2}^F e_{n-1}$.

3. From axiom (6), for every $x \in A_{n-1}$, there is a $y \in A_{n-2}$ such that $e_{n-1}xe_{n-1} = ye_{n-1}$; but by axiom (7), $y$ is uniquely determined, so there exists a map $\mathrm{cl}_{n-1} : A_{n-1} \to A_{n-2}$ with $e_{n-1}xe_{n-1} = \mathrm{cl}_{n-1}(x)e_{n-1}$. It is easy to check that $\mathrm{cl}_{n-1}$ is an $A_{n-2}$–$A_{n-2}$–bimodule map, but it is not unital in general; if $e_{n-1}^2 = \delta e_{n-1}$, then $\mathrm{cl}_{n-1}(\mathbf{1}) = \delta\mathbf{1}$. If $\delta$ is invertible in $R$, then $\varepsilon_{n-1} = (1/\delta)\mathrm{cl}_{n-1}$ is a conditional expectation, i.e., a unital $A_{n-2}$–$A_{n-2}$–bimodule map.

## 3.2 The main theorem

**Theorem 3.2.1.** *Let $R$ be an integral domain with field of fractions $F$. Let $(Q_k)_{k\geq 0}$ and $(A_k)_{k\geq 0}$ be two towers of $R$–algebras satisfying the framework axioms of Section 3.1. Then*

1. *$(A_k)_{k\geq 0}$ is a coherent tower of cellular algebras.*

2. *For all $k$, the partially ordered set in the cell datum for $A_k$ can be realized as*
$$\Lambda_k = \coprod_{\substack{i\leq k \\ k-i \text{ even}}} \Lambda_i^{(0)} \times \{k\},$$
*with the following partial order: Let $\lambda \in \Lambda_i^{(0)}$ and $\mu \in \Lambda_j^{(0)}$, with $i$, $j$, and $k$ all of the same parity. Then $(\lambda, k) > (\mu, k)$ if, and only if, $i < j$, or $i = j$ and $\lambda > \mu$ in $\Lambda_i^{(0)}$.*

3. *Suppose $k \geq 2$ and $(\lambda, k) \in \Lambda_i^{(0)} \times \{k\} \subseteq \Lambda_k$. Let $\Delta^{(\lambda,k)}$ be the corresponding cell module. If $i < k$, then $A_k e_{k-1} A_k\ \Delta^{(\lambda,k)} = \Delta^{(\lambda,k)}$, while if $i = k$ then $A_k e_{k-1} A_k\ \Delta^{(\lambda,k)} = 0$.*

4. *The branching diagram $\mathfrak{B}$ for $(A_k^F)_{k\geq 0}$ is that obtained by reflections from the branching diagram $\mathfrak{B}_0$ for $(Q_k^F)_{n\geq 0}$.*

**Remark 3.2.2.** In most of the examples, the branching diagrams have no multiple edges. In this case, for all $k$ and for all $(\lambda, k) \in \Lambda_k$, the index set $\mathcal{T}((\lambda, k))$ in the cell datum for $A_k$ can be taken to be the set of paths on $\mathfrak{B}$ from $\emptyset$ to $(\lambda, k)$. This follows from (1) and (4), using Lemma 2.7.5.

# CHAPTER 4
# PROOF OF THE MAIN THEOREM

This chapter contains the proof Theorem 3.2.1. The strategy is to prove the following statement by induction on $n$:

**Claim:** *For all $n \geq 0$, the statements (1) –(4) of Theorem 3.2.1 hold for the finite tower $(A_k)_{0\leq k\leq n}$.*

Of course, by statement (4) for the finite tower, the branching diagram for the finite tower $(A_k^F)_{0\leq k\leq n}$ is that obtained by reflections from the branching diagram of the finite tower $(Q_k^F)_{0\leq k\leq n}$.

The claim holds trivially for $n = 0$ and $n = 1$. Assume that the claim holds for some $n \geq 1$ and prove that it also holds for $n + 1$.

## 4.1 $A_{n+1}$ is cellular

This section contains the proof that $A_{n+1}$ is a cellular algebra.

Since $A_{n+1}/A_{n+1}e_nA_{n+1} \cong Q_{n+1}$ is cellular, to prove that $A_{n+1}$ is cellular, it suffices to show that $A_{n+1}e_nA_{n+1}$ is a cellular ideal in $A_{n+1}$; see Remark 2.2.4.

Recall that $\Lambda_k$ denotes the partially ordered set in the cell datum for $A_k$ for each $k$, $0 \leq k \leq n$. Denote the elements of the cellular basis of $A_k$ by $c^\lambda_{u,v}$ for $\lambda \in \Lambda_k$ and $u, v \in \mathcal{T}(\lambda)$.

For each order ideal $\Gamma$ of $\Lambda_{n-1}$, recall that $A_{n-1}(\Gamma)$ is the span in $A_{n-1}$ of all $c^\lambda_{s,t}$ with $\lambda \in \Gamma$. $A_{n-1}(\Gamma)$ is an $i$–invariant two sided ideal of $A_{n-1}$. In the following, we will write $J(\Gamma) = A_{n-1}(\Gamma)$ and

$$\hat{J}(\Gamma) = A_ne_nJ(\Gamma)A_n = A_{n+1}e_nJ(\Gamma)A_{n+1},$$

which is a two sided ideal in $A_{n+1}$. Our goal is to show that $\Gamma \mapsto \hat{J}(\Gamma)$ is a $\Lambda_{n-1}$–cell net in $A_{n+1}e_nA_{n+1}$.

**Lemma 4.1.1.** *Let $R$ be an integral domain and $F$ its field of fractions. Suppose*

*that $A$ and $B$ are $R$-algebras. Let $P_A$, ${}_AM_A$ and ${}_AQ$ be modules. Then*

$$P \otimes_A M \otimes_A Q \otimes_R F \cong P^F \otimes_{A^F} M^F \otimes_{A^F} Q^F$$

*as $F$-vector spaces. The isomorphism*

$$P \otimes_A M \otimes_A Q \otimes_R F \to P^F \otimes_{A^F} M^F \otimes_{A^F} Q^F$$

*is determined by*

$$x \otimes_A y \otimes_A z \otimes_R f \mapsto (x \otimes_R \mathbf{1}_F) \otimes_{A^F} (y \otimes_R \mathbf{1}_F) \otimes_{A^F} (z \otimes_R f).$$

*If ${}_BP_A$ and ${}_AQ_B$ are bimodules, then the isomorphism is an isomorphism of $B^F$–$B^F$–bimodules.*

*Proof.* By Lemma 2.7.1,

$$(P \otimes_A M) \otimes_A Q \otimes_R F \cong (P \otimes_A M)^F \otimes_{A^F} Q^F. \tag{4.1.1}$$

Applying Lemma 2.7.1 again, we have that

$$(P \otimes_A M)^F \cong P^F \otimes_{A^F} M^F \tag{4.1.2}$$

as right $A^F$–modules. Combining the two isomorphisms we have

$$P \otimes_A M \otimes_A Q \otimes_R F \cong P^F \otimes_{A^F} M^F \otimes_{A^F} Q^F. \tag{4.1.3}$$

If we track a simple tensor through these isomorphisms, we see that

$$\begin{aligned} x \otimes_A y \otimes_A z \otimes_R f &\mapsto (x \otimes_A y \otimes_R \mathbf{1}_F) \otimes_{A^F} (z \otimes_R f) \\ &\mapsto (x \otimes_R \mathbf{1}_F) \otimes_{A^F} (y \otimes_R \mathbf{1}_F) \otimes_{A^F} (z \otimes_R f). \end{aligned}$$

If ${}_BP_A$ and ${}_AQ_B$ are bimodules, then the isomorphism in (4.1.1) is an isomorphism of $B^F$–$B^F$–bimodules, and the isomorphism in (4.1.2) is an isomorphism of $B^F$–$A^F$–bimodules. Hence the final isomorphism (4.1.3) is an isomorphism of $B^F$–$B^F$–bimodules. □

**Lemma 4.1.2.** *Let $K$ be a field and $A$ a semisimple $K$-algebra. Suppose that $I \subseteq A$ is a two-sided ideal and $M_A$, ${}_AN$ are modules. Then the homomorphism $M \otimes_A I \otimes_A N \to M \otimes_A N$ defined by $x \otimes y \otimes z \mapsto x \otimes yz$ is injective.*

*Proof.* The semisimplicity of $A$ implies that all $A$-modules are projective. Thus

$N \otimes_A -$ and $- \otimes_A M$ are exact, and

$$N \otimes_A I \otimes_A M \to N \otimes_A A \otimes_A M \cong N \otimes_A M$$

is injective. □

**Proposition 4.1.3.** *For all order ideals $\Gamma$ of $\Lambda_{n-1}$:*

1. *The map*

$$\Phi_\Gamma : \ A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n \to A_n e_n J(\Gamma) A_n$$

*determined by*

$$\Phi_\Gamma(a_1 e_n \otimes x \otimes e_n a_2) = a_1 e_n x a_2$$

*is an isomorphism of $A_{n+1}$–$A_{n+1}$–bimodules.*

2. *$A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n$ is a free $R$–module.*

3. *Let $\Gamma'$ be another order ideal containing $\Gamma$, such that $\Gamma' \setminus \Gamma$ is a singleton. Let $\iota$ denote the injection $J(\Gamma) \ \to J(\Gamma')$. Then*

$$\beta_{\Gamma,\Gamma'} := \mathrm{id} \otimes \iota \otimes \mathrm{id} : A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n \to$$
$$A_n e_n \otimes_{A_{n-1}} J(\Gamma') \otimes_{A_{n-1}} e_n A_n$$

*is injective.*

The following two lemmas will prove Proposition 4.1.3.

**Lemma 4.1.4.** *Let $\Gamma \subseteq \Gamma'$ be two order ideals in $\Lambda_{n-1}$ such that $\Gamma' \setminus \Gamma$ is a singleton. Suppose that $\Phi_\Gamma$ is an isomorphism and that $A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n$ is a free $R$–module. Then $\beta_{\Gamma,\Gamma'}$ is injective and $A_n e_n \otimes_{A_{n-1}} J(\Gamma') \otimes_{A_{n-1}} e_n A_n$ is a free $R$–module.*

*Proof.* Let $\{\lambda\} = \Gamma' \setminus \Gamma$. Since $\Phi_\Gamma$ is assumed injective, it follows from considering the commutative diagram below that $\beta_{\Gamma,\Gamma'}$ is also injective:

$$\begin{array}{ccc} A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n & \xrightarrow{\Phi_\Gamma} & A_n e_n J(\Gamma) A_n \\ \Big\downarrow \beta_{\Gamma,\Gamma'} & & \Big\downarrow \\ A_n e_n \otimes_{A_{n-1}} J(\Gamma') \otimes_{A_{n-1}} e_n A_n & \xrightarrow{\Phi_{\Gamma'}} & A_n e_n J(\Gamma') A_n \end{array}$$

By the right exactness of tensor products, we have

$$(A_n e_n \otimes_{A_{n-1}} J(\Gamma') \otimes_{A_{n-1}} e_n A_n)/\beta_{\Gamma,\Gamma'}(A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n)$$

$$\cong A_n e_n \otimes_{A_{n-1}} (J(\Gamma')/J(\Gamma)) \otimes_{A_{n-1}} e_n A_n \tag{4.1.4}$$

$$\cong A_n e_n \otimes_{A_{n-1}} \Delta^\lambda \otimes_R i(\Delta^\lambda) \otimes_{A_{n-1}} e_n A_n$$

Consider $A_n e_n = A_{n+1} e_n$ as an $A_{n+1}$–$A_{n-1}$–bimodule. One can easily check that $i(A_n e_n) \cong e_n A_n$ as $A_{n-1}$–$A_{n+1}$–bimodules. Therefore,

$$i(\Delta^\lambda) \otimes_{A_{n-1}} e_n A_n \cong i(\Delta^\lambda) \otimes_{A_{n-1}} i(A_n e_n) \cong i(A_n e_n \otimes_{A_{n-1}} \Delta^\lambda), \tag{4.1.5}$$

using Lemma 2.1.1. By framework axioms (6) and (7), $A_n e_n \cong A_n$ as $A_n$–$A_{n-1}$–bimodules. Hence,

$$A_n e_n \otimes_{A_{n-1}} \Delta^\lambda \cong A_n \otimes_{A_{n-1}} \Delta^\lambda = \mathrm{Ind}_{A_{n-1}}^{A_n}(\Delta^\lambda), \tag{4.1.6}$$

as $A_n$ modules. Combining (4.1.4), (4.1.5), and (4.1.6), we have

$$\begin{aligned} &(A_n e_n \otimes_{A_{n-1}} J(\Gamma') \otimes_{A_n} e_n A_n)/\beta_{\Gamma,\Gamma'}(A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n) \\ &\cong \mathrm{Ind}_{A_{n-1}}^{A_n}(\Delta^\lambda) \otimes_R i(\mathrm{Ind}_{A_{n-1}}^{A_n}(\Delta^\lambda)), \end{aligned} \tag{4.1.7}$$

as $A_n$–$A_n$–bimodules.

By the induction assumption on $n$, $\mathrm{Ind}_{A_{n-1}}^{A_n}(\Delta^\lambda)$ has a filtration with subquotients isomorphic to cell modules for $A_n$, and in particular $\mathrm{Ind}_{A_{n-1}}^{A_n}(\Delta^\lambda)$ is a free $R$–module. By (4.1.7),

$$(A_n e_n \otimes_{A_{n-1}} J(\Gamma') \otimes_{A_{n-1}} e_n A_n)/\beta_{\Gamma,\Gamma'}(A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n)$$

is a free $R$–module. Since $A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n$ is free by hypothesis, and $\beta_{\Gamma,\Gamma'}$ is injective,

$$\beta_{\Gamma,\Gamma'}(A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n)$$

is a free $R$–module. Hence

$$A_n e_n \otimes_{A_{n-1}} J(\Gamma') \otimes_{A_{n-1}} e_n A_n$$

is also a free $R$–module. □

**Lemma 4.1.5.** *Let $\Gamma$ be an order ideal in $\Lambda_{n-1}$. If $A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n$ is a free $R$–module, then $\Phi_\Gamma$ is an isomorphism.*

*Proof.* $\Phi_\Gamma$ is surjective, so it suffices to prove $\Phi_\Gamma$ is injective. Define

$$\alpha_1 \ : \ A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n \to A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n \otimes_R F$$

and

$$\alpha_2 \ : \ A_n e_n J(\Gamma) A_n \to A_n e_n J(\Gamma) A_n \otimes_R F$$

by $x \mapsto x \otimes \mathbf{1}_F$. Since $A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n$ is a free $R$–module, by assumption, $\alpha_1$ is injective, according to Lemma 2.7.2. Let

$$\tau : A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n \otimes_R F \to A_n^F e_n \otimes_{A_{n-1}^F} J(\Gamma)^F \otimes_{A_{n-1}^F} e_n A_n^F$$

be the isomorphism from Lemma 4.1.1. (Write $e_n$ for $e_n \otimes \mathbf{1}_F$.) Let

$$\Phi_\Gamma^F : A_n^F e_n \otimes_{A_{n-1}^F} J(\Gamma)^F \otimes_{A_{n-1}^F} e_n A_n^F \to A_n^F e_n J(\Gamma)^F A_n^F$$

be defined by $x e_n \otimes a \otimes e_n y \mapsto x e_n a y$.

Consider the following diagram

$$\begin{array}{ccc}
A_n^F e_n \otimes_{A_{n-1}^F} J(\Gamma)^F \otimes_{A_{n-1}^F} e_n A_n^F & \xrightarrow{\Phi_\Gamma^F} & A_n^F e_n J(\Gamma)^F A_n^F \\
\tau \uparrow & & \| \\
A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n \otimes_R F & \xrightarrow{\Phi_\Gamma \otimes id_F} & A_n e_n J(\Gamma) A_n \otimes_R F \\
\alpha_1 \uparrow & & \uparrow \alpha_2 \\
A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n & \xrightarrow{\Phi_\Gamma} & A_n e_n J(\Gamma) A_n.
\end{array}$$

It is straightforward to check that $\Phi_\Gamma^F \circ \tau \circ \alpha_1 = \alpha_2 \circ \Phi_\Gamma$. Thus, to prove that $\Phi_\Gamma$ is injective, it suffices to show that $\Phi_\Gamma^F$ is injective.

Define

$$\beta \ : \ A_n^F e_n \otimes_{A_{n-1}^F} J(\Gamma)^F \otimes_{A_{n-1}^F} e_n A_n^F \to A_n^F e_n \otimes_{A_{n-1}^F} e_n A_n^F$$

by $\beta(x \otimes y \otimes z) = x \otimes yz$. Observe that $\beta$ is injective by Lemma 4.1.2. Define

$$\phi^F : A_n^F e_n \otimes_{A_{n-1}^F} e_n A_n^F \to A_n^F e_n A_n^F$$

by $\phi^F(x e_n \otimes e_n y) = x e_n y$. Observe that $\phi^F \circ \beta = \Phi_\Gamma^F$, so to prove that $\Phi_\Gamma^F$ is injective, it suffices to show that $\phi^F$ is injective.

Since $A_{n+1}^F$ is split semisimple, the ideal $A_{n+1}^F e_n A_{n+1}^F = A_n^F e_n A_n^F$ is a unital algebra in its own right, and Morita equivalent to $e_n A_{n+1}^F e_n = e_n A_n^F e_n \cong A_{n-1}^F$. In fact, let

$$\psi^F : e_n A_n \otimes_{A_n^F e_n A_n^F} A_n^F e_n \to e_n A_n^F e_n$$

be given by $e_n x \otimes y e_n \mapsto (1/\delta) e_n x y e_n$, where $e_n^2 = \delta e_n$. Then

$$(e_n A_n^F e_n, A_n^F e_n A_n^F, A_n^F e_n, e_n A_n^F, \psi^F, \phi^F)$$

is a Morita context, in the sense of [27], Section 3.12, with surjective bimodule maps $\psi^F$ and $\phi^F$. It follows from Morita theory, for example [27], Morita Theorem I, page 167, that $\psi^F$ and $\phi^F$ are isomorphisms. □

*Proof of Proposition 4.1.3*: Let $\Gamma$ be an order ideal of $\Lambda_{n-1}$. There exists a chain of order ideals

$$\emptyset = \Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq \Gamma_s = \Gamma,$$

such that the difference between any two successive order ideals is a singleton. Write $\beta_j$ for $\beta_{\Gamma_j, \Gamma_{j+1}}$, for $0 \leq j < s$.

Proceed by induction to prove that for $0 \leq j \leq s$, $\Phi_{\Gamma_j}$ is an isomorphism and $A_n e_n \otimes_{A_{n-1}} J(\Gamma_j) \otimes_{A_{n-1}} e_n A_n$ is a free $R$–module; and that for $0 \leq j < s$, $\beta_j$ is injective. For $j = 0$, these statements are trivial since $J(\emptyset) = 0$.

Fix $j$ $(0 \leq j < s)$ and suppose that $A_n e_n \otimes_{A_{n-1}} J(\Gamma_j) \otimes_{A_{n-1}} e_n A_n$ is a free $R$–module, that $\Phi_{\Gamma_j}$ is an isomorphism. Then it follows from Lemma 4.1.4 that $A_n e_n \otimes_{A_{n-1}} J(\Gamma_{j+1}) \otimes_{A_{n-1}} e_n A_n$ is a free $R$–module. Next, it follows from Lemma 4.1.5 that $\Phi_{\Gamma_{j+1}}$ is an isomorphism.

Thus $A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n$ is a free $R$–module and $\Phi_\Gamma$ is an isomorphism. Applying Lemma 4.1.4 again gives statement (3) of the Proposition. □

Continue to work with the following assumptions: $R$ is an integral domain with field of fractions $F$. $(Q_k)_{k \geq 0}$ and $(A_k)_{k \geq 0}$ are two towers of $R$–algebras satisfying the framework axioms of Section 3.1. The following induction assumption is in force: For some fixed $n \geq 1$, the conclusions (1) –(4) of Theorem 3.2.1 hold for the finite tower $(A_k)_{0 \leq k \leq n}$. Use the notation of the discussion preceding Lemma 4.1.1.

The following is a corollary of Proposition 4.1.3.

**Corollary 4.1.6.** *$A_n e_n \otimes_{A_{n-1}} e_n A_n \cong A_n e_n A_n$, as $A_{n+1}$–$A_{n+1}$ bimodules, with the isomorphism determined by $x e_n \otimes e_n y \mapsto x e_n y$.*

*Proof.* In Proposition 4.1.3, take $\Gamma = \Lambda_{n-1}$, so $J(\Gamma) = A_{n-1}$. $\square$

**Proposition 4.1.7.**

1. *$\Gamma \mapsto \hat{J}(\Gamma)$ is a $\Lambda_{n-1}$–cell net in $A_n e_n A_n$.*

2. *$A_n e_n A_n$ is a cellular ideal in $A_{n+1}$.*

3. *$A_{n+1}$ is a cellular algebra. The partially ordered set in the cell datum for $A_{n+1}$ can be realized as $\Lambda_{n+1} = \Lambda_{n-1} \cup \Lambda_{n+1}^{(0)}$, where $\Lambda_{n+1}^{(0)}$ is the partially ordered set in the cell datum for $Q_{n+1}$; moreover the partial order on $\Lambda_{n+1}$ agrees with the original partial orders on $\Lambda_{n-1}$ and $\Lambda_{n+1}^{(0)}$, and satisfies $\lambda > \mu$ if $\lambda \in \Lambda_{n-1}$ and $\mu \in \Lambda_{n+1}^{(0)}$.*

4. *Let $\lambda \in \Lambda_{n-1}$, and let $\Delta^\lambda$ denote the corresponding cell module of $A_{n-1}$. The cell module of $A_{n+1}$ corresponding to $\lambda$ is isomorphic to $A_n e_n \otimes_{A_{n-1}} \Delta^\lambda$.*

*Proof.* It is evident that $\hat{J}(\emptyset) = \{0\}$, and that $\Gamma_1 \subseteq \Gamma_2$ implies $\hat{J}(\Gamma_1) \subseteq \hat{J}(\Gamma_2)$. Note that $J(\Gamma_{\geq\lambda}) = A_{n-1}^\lambda$, so $\hat{J}(\Gamma_{\geq\lambda}) = A_n e_n A_{n-1}^\lambda A_n$. Similarly, $\hat{J}(\Gamma_{>\lambda}) = A_n e_n \breve{A}_{n-1}^\lambda A_n$. It follows that $A_n e_n A_n = \operatorname{span}\{\hat{J}(\Gamma_{\geq\lambda}) : \lambda \in \Lambda_{n-1}\}$ and that for all $\lambda \in \Lambda_{n-1}$, $\hat{J}(\Gamma_{>\lambda}) = \operatorname{span}\{\hat{J}(\Gamma_{\geq\mu}) : \mu > \lambda\}$. We have shown that $\Gamma \mapsto \hat{J}(\Gamma)$ satisfies conditions (1) and (2) of Definition 2.3.5.

Next we show that $\Gamma \mapsto \hat{J}(\Gamma)$ satisfies condition (3) of Definition 2.3.5. Let $\Gamma \subseteq \Gamma'$ be two order ideals of $\Lambda_{n-1}$, with $\Gamma' \backslash \Gamma = \{\lambda\}$. From the proof of Proposition 4.1.3, we already have $\hat{J}(\Gamma')/\hat{J}(\Gamma) \cong M^\lambda \otimes_R i(M^\lambda)$, with $M^\lambda = A_n e_n \otimes_{A_{n-1}} \Delta^\lambda$. Let $\chi : \hat{J}(\Gamma')/\hat{J}(\Gamma) \to M^\lambda \otimes_R i(M^\lambda)$ denote the isomorphism. We have to check that $\chi \circ i = i \circ \chi$. The isomorphism $\Phi_\Gamma$ of Proposition 4.1.3 satisfies $i \circ \Phi_\Gamma = \Phi_\Gamma \circ i$. Moreover,

$$\beta_{\Gamma,\Gamma'}(A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n) \subseteq A_n e_n \otimes_{A_{n-1}} J(\Gamma') \otimes_{A_{n-1}} e_n A_n$$

and $\hat{J}(\Gamma) \subseteq \hat{J}(\Gamma')$ are $i$–invariant, so the induced isomorphism

$$\tilde{\Phi}_\Gamma : A_n e_n \otimes_{A_{n-1}} J(\Gamma') \otimes_{A_{n-1}} e_n A_n / \beta_{\Gamma,\Gamma'}(A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n)$$

$$\to \hat{J}(\Gamma')/\hat{J}(\Gamma)$$

satisfies $i \circ \tilde{\Phi}_\Gamma = \tilde{\Phi}_\Gamma \circ i$. Next, the map

$$\pi : A_n e_n \otimes_{A_{n-1}} J(\Gamma') \otimes_{A_{n-1}} e_n A_n \to A_n e_n \otimes_{A_{n-1}} J(\Gamma')/J(\Gamma) \otimes_{A_{n-1}} e_n A_n$$

satisfies $i \circ \pi = \pi \circ i$, so the induced isomorphism

$$\tilde{\pi} : A_n e_n \otimes_{A_{n-1}} J(\Gamma') \otimes_{A_{n-1}} e_n A_n / \beta_{\Gamma,\Gamma'}(A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n)$$

$$\to A_n e_n \otimes_{A_{n-1}} J(\Gamma')/J(\Gamma) \otimes_{A_{n-1}} e_n A_n$$

satisfies $i \circ \tilde{\pi} = \tilde{\pi} \circ i$. Finally, we have an isomorphism $\alpha : J(\Gamma')/J(\Gamma) \to \Delta^\lambda \otimes_R i(\Delta^\lambda)$ satisfying $i \circ \alpha = \alpha \circ i$, so the map

$$\bar{\alpha} = \mathrm{id} \otimes \alpha \otimes \mathrm{id} : A_n e_n \otimes_{A_{n-1}} J(\Gamma')/J(\Gamma) \otimes_{A_{n-1}} e_n A_n \to$$

$$A_n e_n \otimes_{A_{n-1}} \Delta^\lambda \otimes_R i(\Delta^\lambda) \otimes_{A_{n-1}} e_n A_n$$

satisfies $i \circ \bar{\alpha} = \bar{\alpha} \circ i$. The map $\chi$ is $\bar{\alpha} \circ \tilde{\pi} \circ \tilde{\Phi}_\Gamma^{-1}$, so we have $i \circ \chi = \chi \circ i$.

This completes the proof that $\Gamma \mapsto \hat{J}(\Gamma)$ is a $\Lambda_{n-1}$–cell net in $A_n e_n A_n$. By Lemma 2.3.6, $A_n e_n A_n$ has a cell datum with partially ordered set equal to $\Lambda_{n-1}$. Moreover, since the isomorphisms $\hat{J}(\Gamma')/\hat{J}(\Gamma) \cong M^\lambda \otimes_R i(M^\lambda)$ are actually isomorphisms of $A_{n+1}$–$A_{n+1}$–bimodules, the cellular basis $\tilde{\mathcal{C}}$ of $A_n e_n A_n$ satisfies the property (2) of Definition 2.2.1 not only for $a \in A_n e_n A_n$ but also for $a \in A_{n+1}$; that is $A_n e_n A_n$ is a cellular ideal in $A_{n+1}$.

Statement (3) of the Lemma follows from applying Remark 2.2.4. Statement (4) follows from the isomorphism $\hat{J}(\Gamma')/\hat{J}(\Gamma) \cong M^\lambda \otimes_R i(M^\lambda)$. □

**Corollary 4.1.8.** *The description of the partially ordered set given in Theorem 3.2.1, point (2), is valid for $k = n + 1$.*

*Proof.* Combining point (3) of Proposition 4.1.7 with the induction assumption (specifically the description of $\Lambda_{n-1}$ as the union of copies of $\Lambda_{n-1}^{(0)}$, $\Lambda_{n-3}^{(0)}$, etc.), we see that $\Lambda_{n+1}$ is the union of copies of $\Lambda_{n+1}^{(0)}$, $\Lambda_{n-1}^{(0)}$, $\Lambda_{n-3}^{(0)}$, etc., with the following partial order: the partial order agrees with the original partial order on each $\Lambda_i^{(0)}$, and $\lambda > \mu$ if $\lambda \in \Lambda_i^{(0)}$, $\mu \in \Lambda_j^{(0)}$, and $i < j$. □

For the remainder of Chapter 4, denote elements of $\Lambda_k$ $(0 \leq k \leq n+1)$ by ordered pairs $(\lambda, k)$, where it is understood that $\lambda \in \Lambda_i^{(0)}$ for some $i \leq k$ with $k - i$ even.

**Corollary 4.1.9.** *Point (3) of Theorem 3.2.1 holds for $k = n+1$.*

*Proof.* The cell modules of $A_{n+1}$ are of two types: There are the cell modules $\Delta^{(\lambda,n+1)}$ with $\lambda \in \Lambda_{n+1}^{(0)}$, which are actually cell modules of $A_{n+1}/(A_n e_n A_n) \cong Q_{n+1}$. These satisfy

$$A_n e_n A_n \ \Delta^{(\lambda,n+1)} = 0.$$

On the other hand, there are the cell modules of the cellular ideal $A_n e_n A_n$, namely $\Delta^{(\lambda,n+1)} = A_n e_n \otimes_{A_{n-1}} \Delta^{(\lambda,n-1)}$, with $\lambda \in \Lambda_i^{(0)}$ for some $i < n+1$ with $n+1-i$ even. These satisfy

$$A_n e_n A_n \ \Delta^{(\lambda,n+1)} = (A_n e_n A_n)(A_n e_n) \otimes_{A_{n-1}} \Delta^{(\lambda,n-1)}.$$

But $(A_n e_n A_n)(A_n e_n) = A_n A_{n-1} e_n = A_n e_n$, using framework axiom (6), so we have

$$A_n e_n A_n \ \Delta^{(\lambda,n+1)} = \Delta^{(\lambda,n+1)}.$$

□

## 4.2 Cell filtrations of Res and Ind

Next we show that the restriction of a cell module from $A_{n+1}$ to $A_n$, and the induction of a cell module from $A_n$ to $A_{n+1}$, have cell filtrations.

**Proposition 4.2.1.** *Let $(\lambda, n+1) \in \Lambda_{n+1}$, and let $\Delta = \Delta^{(\lambda,n+1)}$ be the corresponding cell module of $A_{n+1}$. Then the restriction of $\Delta$ to $A_n$ has a cell filtration.*

*Proof.* Write $\mathrm{Res}(\Delta)$ for the restriction to $A_n$.

If $A_{n+1} e_n A_{n+1} \ \Delta = 0$, then $\Delta$ is a $Q_{n+1}$–module; moreover, by framework axiom (8) from Section 3.1, $A_n e_{n-1} A_n \ \mathrm{Res}(\Delta) = 0$ as well, so $\mathrm{Res}(\Delta)$ is a $Q_n$–module. Then it follows from the assumption of coherence of $(Q_k)_{k \geq 0}$ that $\mathrm{Res}(\Delta)$ has a cell filtration as an $Q_n$–module, hence as an $A_n$–module.

If $A_{n+1}e_nA_{n+1}\ \Delta = \Delta$, then $\lambda \in \Lambda_i^{(0)}$ for some $i < n$, and

$$\Delta \cong A_n e_n \otimes_{A_{n-1}} \Delta^{(\lambda,n-1)}.$$

Since $A_n e_n \cong A_n$ as $A_n$–$A_{n-1}$ bimodules, $\mathrm{Res}(\Delta) \cong \mathrm{Ind}_{A_{n-1}}^{A_n}(\Delta^{(\lambda,n-1)})$, which has a cell filtration by the induction assumption. □

**Lemma 4.2.2.** *Let $R$ be an integral domain with field of fractions $F$. Let $A$ be a unital $R$–algebra, $P$ a right $A$–module, and $N_1 \subseteq N_2$ left $A$–modules, such that*

1. *$A^F = A \otimes_R F$ is semisimple, and*
2. *$N_2$ and $P \otimes_A N_1$ are free $R$–modules.*

*Let $\iota : N_1 \to N_2$ denote the injection. Then*

$$\mathrm{id}_P \otimes \iota : P \otimes_A N_1 \to P \otimes_A N_2$$

*is injective.*

*Proof.* First, $\iota \otimes \mathrm{id}_F : N_1 \otimes_R F \to N_2 \otimes_R F$ is injective by Lemma 2.7.3. Write $\beta = \mathrm{id}_P \otimes \iota$, and let

$$\beta^F = \mathrm{id}_{P^F} \otimes (\iota \otimes \mathrm{id}_F) : P^F \otimes_{A^F} N_1^F \to P^F \otimes_{A^F} N_2^F.$$

Since $A^F$ is semisimple, $P^F$ is projective; hence $\beta^F$ is injective.

Consider the following diagram:

$$\begin{tikzcd}
P^F \otimes_{A^F} N_1^F \arrow[r, "\beta^F"] & P^F \otimes_{A^F} N_2^F \\
P \otimes_A N_1 \otimes_R F \arrow[u, "\tau_1"] \arrow[r, "\beta\otimes id_F"] & P \otimes_A N_2 \otimes_R F \arrow[u, "\tau_2"'] \\
P \otimes_A N_1 \arrow[u, "\alpha_1"] \arrow[r, "\beta"] & P \otimes_A N_2, \arrow[u, "\alpha_2"']
\end{tikzcd}$$

where $\alpha_i$ is determined by $x \mapsto x \otimes 1_F$ and $\tau_i$ is the isomorphism of Lemma 2.7.1 ($i = 1, 2$). Note that $\alpha_1$ is injective by Lemma 2.7.2, since $P \otimes_A N_1$ is assumed to be free over $R$. One can check that $\beta^F \circ \tau_1 \circ \alpha_1 = \tau_2 \circ \alpha_2 \circ \beta$. It follows that $\beta$ is injective. □

**Lemma 4.2.3.** *Let $M$ be an $A_{n-1}$ module with a cell filtration:*

$$(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M,$$

*with $M_j/M_{j-1} \cong \Delta^{(\lambda_j, n-1)}$ for $1 \leq j \leq t$. Then for $1 \leq j \leq t$,*

1. *$A_n e_n \otimes_{A_{n-1}} M_j$ is a free $R$–module,*

2. *$A_n e_n \otimes_{A_{n-1}} M_{j-1}$ imbeds in $A_n e_n \otimes_{A_{n-1}} M_j$, and*

3. *$(A_n e_n \otimes_{A_{n-1}} M_j)/(A_n e_n \otimes_{A_{n-1}} M_{j-1}) \cong A_n e_n \otimes_{A_{n-1}} \Delta^{(\lambda_j, n-1)}$.*

*Thus, the $A_{n+1}$–module $A_n e_n \otimes_{A_{n-1}} M$ has a cell filtration with subquotients $\Delta^{(\lambda_j, n+1)} = A_n e_n \otimes_{A_{n-1}} \Delta^{(\lambda_j, n-1)}$ $(1 \leq j \leq t)$.*

*Proof.* We have $M_1 \cong \Delta^{(\lambda_1, n-1)}$, so $A_n e_n \otimes_{A_{n-1}} M_1$ is a free $R$–module. Fix $j \geq 2$ and suppose that $A_n e_n \otimes_{A_{n-1}} M_{j-1}$ is a free $R$–module. Let $\iota : M_{j-1} \to M_j$ denote the injection and let

$$\beta = \mathrm{id}_{A_n e_n} \otimes \iota : A_n e_n \otimes_{A_{n-1}} M_{j-1} \to A_n e_n \otimes_{A_{n-1}} M_j.$$

Then $\beta$ is injective by an application of Lemma 4.2.2, with $A = A_{n-1}$, $P = A_n e_n$, $N_1 = M_{j-1}$, and $N_2 = M_j$. The quotient

$$(A_n e_n \otimes_{A_{n-1}} M_j, )/\beta(A_n e_n \otimes_{A_{n-1}} M_{j-1})$$

is free over $R$, because

$$\begin{aligned}(A_n e_n \otimes_{A_{n-1}} M_j)/\beta(A_n e_n \otimes_{A_{n-1}} M_{j-1}) \\ \cong A_n e_n \otimes_{A_{n-1}} (M_j/M_{j-1}) \\ \cong A_n e_n \otimes_{A_{n-1}} \Delta^{(\lambda_j, n-1)}.\end{aligned}$$

Consequently, $A_n e_n \otimes_{A_{n-1}} M_j$ is free over $R$. All the assertions of the lemma now follow by induction on $j$. □

**Lemma 4.2.4.** *Let $M$ be an $A_n$–module, and let $\mathrm{Res}(M)$ denote the restriction of $M$ to $A_{n-1}$. We have*

$$A_n e_n A_n \otimes_{A_n} M \cong A_n e_n \otimes_{A_{n-1}} \mathrm{Res}(M),$$

*as $A_{n+1}$ modules.*

*Proof.* By Corollary 4.1.6, we have $A_n e_n A_n \cong A_n e_n \otimes_{A_{n-1}} e_n A_n \cong A_n e_n \otimes_{A_{n-1}} A_n$ as $A_{n+1}$–$A_n$ bimodules. Thus

$$A_n e_n A_n \otimes_{A_n} M \cong A_n e_n \otimes_{A_{n-1}} A_n \otimes_{A_n} M \cong A_n e_n \otimes_{A_{n-1}} \mathrm{Res}(M).$$

□

**Proposition 4.2.5.** *Let $(\mu, n) \in \Lambda_n$ and let $\Delta^{(\mu,n)}$ be the corresponding cell module of $A_n$.*

1. *$A_n e_n A_n \otimes_{A_n} \Delta^{(\mu,n)}$ has cell filtration (as an $A_{n+1}$–module). In particular, $A_n e_n A_n \otimes_{A_n} \Delta^{(\mu,n)}$ is free as an $R$–module.*

2. *$A_n e_n A_n \otimes_{A_n} \Delta^{(\mu,n)}$ imbeds in $\mathrm{Ind}_{A_n}^{A_{n+1}}(\Delta^{(\mu,n)})$, and*
$$\mathrm{Ind}_{A_n}^{A_{n+1}}(\Delta^{(\mu,n)})/(A_n e_n A_n \otimes_{A_n} \Delta^{(\mu,n)}) \cong Q_{n+1} \otimes_{A_n} \Delta^{(\mu,n)}.$$

3. *$Q_{n+1} \otimes_{A_n} \Delta^{(\mu,n)}$ has cell filtration (as a $Q_{n+1}$–module, hence as an $A_{n+1}$–module).*

4. *$\mathrm{Ind}_{A_n}^{A_{n+1}}(\Delta^{(\mu,n)})$ has a cell filtration.*

*Proof.* For point (1), let $\mathrm{Res}(\Delta^{(\mu,n)})$ denote the restriction to $A_{n-1}$. By Lemma 4.2.4, we have $A_n e_n A_n \otimes_{A_n} \Delta^{(\mu,n)} \cong A_n e_n \otimes_{A_{n-1}} \mathrm{Res}(\Delta^{(\mu,n)})$, as $A_{n+1}$ modules. By the induction assumption stated at the beginning of Section 4, $\mathrm{Res}(\Delta^{(\mu,n)})$ has cell filtration,

$$(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = \mathrm{Res}(\Delta^{(\mu,n)}),$$

with $M_j/M_{j-1} \cong \Delta^{(\lambda_j, n-1)}$ for some $(\lambda_j, n-1) \in \Lambda_{n-1}$. By Lemma 4.2.3, $A_n e_n \otimes_{A_{n-1}} \mathrm{Res}(\Delta^{(\mu,n)})$ has a cell filtration with subquotients $\Delta^{(\lambda_j, n+1)} = A_n e_n \otimes_{A_{n-1}} \Delta^{(\lambda_j, n-1)}$.

Point (2) follows from Lemma 4.2.2 (with left and right modules interchanged), taking $A = A_n$, $P = \Delta^{(\mu,n)}$, $N_1 = A_n e_n A_n$, and $N_2 = A_{n+1}$. Note that $A_{n+1}$ is a free $R$–module by Proposition 4.1.7, and $A_n e_n A_n \otimes_{A_n} \Delta^{(\mu,n)}$ is a free $R$–module by point (1). The statement regarding the quotient follows from the right exactness of tensor products.

For $n = 1$, $A_1 = Q_1$, and $\Delta^{(\mu,n)}$ is an $Q_1$–cell module; statement (3) follows from the assumption of coherence of $(Q_k)_{k\geq 0}$. If $n \geq 2$, then by the induction assumption, either $A_n e_{n-1} A_n \ \Delta^{(\mu,n)} = \Delta^{(\mu,n)}$, or $A_n e_{n-1} A_n \ \Delta^{(\mu,n)} = (0)$. In the former case,

$$\begin{aligned} Q_{n+1} \otimes_{A_n} \Delta^{(\mu,n)} &= Q_{n+1} \otimes_{A_n} A_n e_{n-1} A_n \ \Delta^{(\mu,n)} \\ &= Q_{n+1} A_n e_{n-1} A_n \otimes_{A_n} \Delta^{(\mu,n)} = 0, \end{aligned}$$

because $e_{n-1} \in A_{n+1} e_n A_{n+1}$, by the framework axiom (8). In the latter case, $A_n e_{n-1} A_n$ annihilates both $Q_{n+1}$ and $\Delta^{(\mu,n)}$, so both are $A_n/(A_n e_{n-1} A_n) \cong Q_n$–modules. Thus $Q_{n+1} \otimes_{A_n} \Delta^{(\mu,n)} = Q_{n+1} \otimes_{Q_n} \Delta^{(\mu,n)}$, which has a $Q_{n+1}$–cell filtration by the assumption of coherence of $(Q_k)_{k\geq 0}$. This proves point (3).

Finally, we have an exact sequence

$$0 \to A_n e_n A_n \otimes_{A_n} \Delta^{(\mu,n)} \to \mathrm{Ind}_{A_n}^{A_{n+1}}(\Delta^{(\mu,n)}) \to Q_{n+1} \otimes_{A_n} \Delta^{(\mu,n)} \to 0,$$

where both $A_n e_n A_n \otimes_{A_n} \Delta^{(\mu,n)}$ and $Q_{n+1} \otimes_{A_n} \Delta^{(\mu,n)}$ have $A_{n+1}$–cell filtrations. Hence $\mathrm{Ind}_{A_n}^{A_{n+1}}(\Delta^{(\mu,n)})$ has an $A_{n+1}$–cell filtration. □

**Corollary 4.2.6.** *The finite tower $(A_k)_{0\leq k\leq n+1}$ is a coherent tower of cellular algebras.*

*Proof.* Combine the induction hypothesis, Proposition 4.1.7, Proposition 4.2.1, and Proposition 4.2.5. □

**Corollary 4.2.7.** *The branching diagram for the finite tower $(A_k^F)_{0\leq k\leq n+1}$ is that obtained by reflections from the branching diagram of the finite tower $(Q_k^F)_{0\leq k\leq n+1}$.*

*Proof.* From the induction hypothesis, we already know that the branching diagram for $(A_k^F)_{0\leq k\leq n}$ is obtained by reflections from the branching diagram of the finite tower $(Q_k^F)_{0\leq k\leq n}$. So we have only to consider the branching diagram for $A_{n-1}^F \subseteq A_n^F \subseteq A_{n+1}^F$; specifically, we need to show that if $\lambda \in \Lambda_i^{(0)}$ with $i < n+1$ and $n+1-i$ even, and $(\mu, n) \in \Lambda_n$ is arbitrary, then

$$(\mu, n) \nearrow (\lambda, n+1) \quad \text{if, and only if} \quad (\lambda, n-1) \nearrow (\mu, n),$$

in the branching diagram for $A^F_{n-1} \subseteq A^F_n \subseteq A^F_{n+1}$, and the number of edges connecting $(\mu, n)$ and $(\lambda, n+1)$ is the same as the number of edges connecting $(\lambda, n-1)$ and $(\mu, n)$. But this follows from Lemma 2.7.4 and the proof of either Proposition 4.2.1, or Proposition 4.2.5, point (1). $\square$

*Conclusion of the proof of Theorem 3.2.1.* Under the assumption that statements (1)–(4) of the theorem are valid for the finite tower $(A_k)_{0 \leq k \leq n}$, for some fixed $n$, we had to show that they are also valid for the tower $(A_k)_{0 \leq k \leq n+1}$. This was verified in Corollary 4.2.6, Corollary 4.1.8, Corollary 4.1.9, and Corollary 4.2.7.

# CHAPTER 5
# EXAMPLES

## 5.1 Preliminaries on tangle diagrams

Several of the examples involve *tangle diagrams* in the rectangle $\mathcal{R} = [0,1] \times [0,1]$. Fix points $a_i \in [0,1]$, $i \geq 1$, with $0 < a_1 < a_2 < \cdots$. Write $\boldsymbol{i} = (a_i, 1)$ and $\overline{\boldsymbol{i}} = (a_i, 0)$.

Recall that a *knot diagram* means a collection of piecewise smooth closed curves in the plane which may have intersections and self-intersections, but only simple transverse intersections. At each intersection or crossing, one of the two strands (curves) which intersect is indicated as crossing over the other.

An $(n,n)$–*tangle diagram* is a piece of a knot diagram in $\mathcal{R}$ consisting of exactly $n$ topological intervals and possibly some number of closed curves, such that: (1) the endpoints of the intervals are the points $\mathbf{1}, \ldots \boldsymbol{n}, \overline{\mathbf{1}}, \ldots, \overline{\boldsymbol{n}}$, and these are the only points of intersection of the family of curves with the boundary of the rectangle, and (2) each interval intersects the boundary of the rectangle transversally.

An $(n,n)$–*Brauer diagram* is a "tangle" diagram containing no closed curves, in which information about over and under crossings is ignored. Two Brauer diagrams are identified if the pairs of boundary points joined by curves is the same in the two diagrams. By convention, there is a unique $(0,0)$–Brauer diagram, the empty diagram with no curves. For $n \geq 1$, the number of $(n,n)$–Brauer diagrams is $(2n-1)!! = (2n-1)(2n-3)\cdots(3)(1)$.

A *Temperley–Lieb* diagram is a Brauer diagram without crossings. For $n \geq 0$, the number of $(n,n)$–Temperley–Lieb diagrams is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

For any of these types of diagrams, we call $P = \{\mathbf{1}, \ldots, \boldsymbol{n}, \overline{\mathbf{1}}, \ldots, \overline{\boldsymbol{n}}\}$ the set of *vertices* of the diagram, $P^+ = \{\mathbf{1}, \ldots, \boldsymbol{n}\}$ the set of *top vertices*, and $P^- = \{\overline{\mathbf{1}}, \ldots, \overline{\boldsymbol{n}}\}$ the set of *bottom vertices*. A curve or *strand* in the diagram is called a

*vertical* or *through* strand if it connects a top vertex and a bottom vertex, and a *horizontal* strand if it connects two top vertices or two bottom vertices.

## 5.2 The Brauer algebras

### 5.2.1 Definition of the Brauer algebras

Let $S$ be a commutative ring with identity, with a distinguished element $\delta$. The Brauer algebra $B_n(S, \delta)$ is the free $S$–module with basis the set of $(n, n)$–Brauer diagrams, and with multiplication defined as follows. The product of two Brauer diagrams is defined to be a certain multiple of another Brauer diagram. Namely, given two Brauer diagrams $a, b$, first "stack" $b$ over $a$; the result is a planar tangle that may contain some number of closed curves. Let $r$ denote the number of closed curves, and let $c$ be the Brauer diagram obtained by removing all the closed curves. Then $ab = \delta^r c$.

**Definition 5.2.1.** For $n \geq 1$, the *Brauer algebra* $B_n(S, \delta)$ over $S$ with parameter $\delta$ is the free $S$-module with basis the set of $(n, n)$-Brauer diagrams, with the bilinear product determined by the multiplication of Brauer diagrams. In particular, $B_0(S, \delta) = S$.

Note that the Brauer diagrams with only vertical strands are in bijection with permutations of $\{1, \ldots, n\}$, and that the multiplication of two such diagrams coincides with the multiplication of permutations. Thus the Brauer algebra contains the group algebra $S\mathfrak{S}_n$ of the permutation group $\mathfrak{S}_n$. The identity element of the Brauer algebra is the diagram corresponding to the trivial permutation.

### 5.2.2 Brief history of the Brauer algebras

The Brauer algebras were introduced by Brauer [7] as a device for studying the invariant theory of orthogonal and symplectic groups. Wenzl [51] observed that generically, the sequence of Brauer algebras (over a field) is obtained by repeated

Jones basic constructions from the symmetric group algebras; he used this to show that $B_n(k,\delta)$ is semisimple, when $k$ is a field of characteristic zero and $\delta$ is not an integer. Graham and Lehrer [22] showed that the Brauer algebras are cellular, and classified the simple modules of $B_n(k,\delta)$ when $k$ is a field and $\delta$ is arbitrary. Another illuminating proof of cellularity of the Brauer algebras was given by König and Xi [37]. Enyang's two proofs of cellularity for Birman–Wenzl algebras [14, 15] also apply to the Brauer algebras.

### 5.2.3 Some properties of the Brauer algebras

For $n \geq 1$, let $\iota$ denote the map from $(n,n)$–Brauer diagrams to $(n+1,n+1)$–Brauer diagrams that adds an additional strand to a diagram, connecting $\boldsymbol{n+1}$ to $\overline{\boldsymbol{n+1}}$.

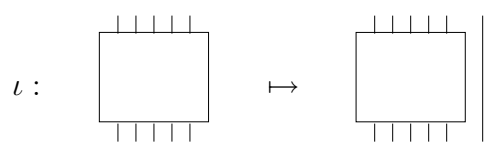



Figure 5.1: Inclusion map

The linear extension of $\iota$ to $B_n(S,\delta)$ is an injective unital homomorphism into $B_{n+1}(S,\delta)$. Using $\iota$, we identify $B_n(S,\delta)$ with its image in $B_{n+1}(S,\delta)$.

For $n \geq 1$ define a map cl from $(n,n)$–Brauer diagrams into $B_{n-1}(S,\delta)$ as follows. First "partially close" a given $(n,n)$–Brauer diagram by adding an additional smooth curve connecting $\boldsymbol{n}$ to $\overline{\boldsymbol{n}}$, see Figure 5.2. In case the resulting "tangle" contains a closed curve (which happens precisely when the original diagram already had a strand connecting $\boldsymbol{n}$ to $\overline{\boldsymbol{n}}$), remove this loop and replace it with a factor of $\delta$. The linear extension of cl to $B_n(S,\delta)$ is a (non-unital) $B_{n-1}$–$B_{n-1}$ bimodule map,

and $\mathrm{cl} \circ \iota(x) = \delta\ x$ for $x \in B_n(S, \delta)$.

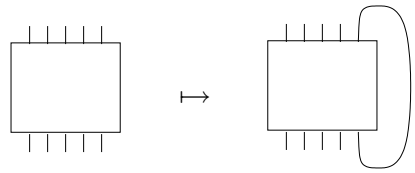


Figure 5.2: A geometric representation of cl

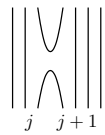
If $\delta$ is invertible in $S$, we can define $\varepsilon_n = (1/\delta)\mathrm{cl}$, which is a conditional expectation, that is, a unital $B_{n-1}$–$B_{n-1}$ bimodule map. We have $\varepsilon_{n+1} \circ \iota(x) = x$ for $x \in B_n(S, \delta)$. The map $\varepsilon = \varepsilon_1 \circ \cdots \circ \varepsilon_n : B_n \to B_0 \cong S$ is a normalized trace; that is, $\varepsilon(\mathbf{1}) = 1$ and $\varepsilon(ab) = \varepsilon(ba)$ for all $a, b$. The value of $\varepsilon$ on a Brauer diagram $d$ is obtained as follows: first close all the strands of $d$ by introducing new curves joining $\boldsymbol{j}$ to $\overline{\boldsymbol{j}}$ for all $j$; let $c$ be the number of components (closed loops) in the resulting $(0, 0)$–tangle; then $\varepsilon(d) = \delta^{c-n}$ if $d \in B_n$. The trace and condition expectation play an essential role in Wenzl's treatment of the structure of the Brauer algebra over $\mathbb{Q}(\boldsymbol{\delta})$ [51], and thus implicitly in our verification of the framework axioms in Proposition 5.7.7.

The involution $i$ on $(n, n)$–Brauer diagrams which reflects a diagram in the axis $y = 1/2$ extends linearly to an algebra involution of $B_n(S, \delta)$. We have $\iota \circ i = i \circ \iota$ and $\mathrm{cl} \circ i = i \circ \mathrm{cl}$.

The products $ab$ and $ba$ of two Brauer diagrams have at most as many through strands as $a$. Consequently, the span of diagrams with at most $r$ through strands ($r \leq n$ and $n - r$ even) is a two–sided ideal $J_r$ in $B_n(S, \delta)$. $J_r$ is $i$–invariant.

Let $e_j$ and $s_j$ denote the $(n, n)$–Brauer diagrams, see Figure 5.3. Note that $e_j^2 = \delta e_j$, so $e_j$ is an essential idempotent if $\delta \neq 0$, and nilpotent if $\delta = 0$. We have $i(e_j) = e_j$ and $i(s_j) = s_j$. It is easy to see that $e_1, \ldots, e_{n-1}$ and $s_1, \ldots, s_{n-1}$ generate

$B_n(S,\delta)$ as an algebra.

$e_j =$ 


$s_j =$


Figure 5.3: Generators of Brauer algebas

Let $r \leq n$ with $n-r$ even, and let $f_r = e_{r+1}e_{r+3}\cdots e_{n-1}$. Any Brauer diagram with exactly $r$ through strands can be factored as $\pi_1 f_r \pi_2$, where $\pi_i$ are permutation diagrams. Consequently, $J_r$ is generated by $f_r$. In particular the ideal $J = J_{n-2}$ spanned by diagrams with fewer than $n$ through strands is generated by $e_{n-1}$. We have $B_n(S,\delta)/J \cong S\mathfrak{S}_n$, as algebras with involutions.

**Lemma 5.2.2.** *Write $B_n$ for $B_n(S,\delta)$.*

1. *For $n \geq 3$, $e_{n-1}B_{n-1}e_{n-1} = B_{n-2}e_{n-1}$.*

2. *$e_1B_1e_1 = \delta B_0 e_1$*

3. *For $n \geq 2$, $e_{n-1}$ commutes with $B_{n-2}$.*

*Proof.* For $n \geq 3$, if $x$ is an $(n-1,n-1)$–Brauer diagram, then $e_{n-1}xe_{n-1} \in B_{n-2}\, e_{n-1}$. Thus, $e_{n-1}B_{n-1}\, e_{n-1} \subseteq B_{n-2}\, e_{n-1}$. On the other hand, for $x \in B_{n-2}$, we have $e_{n-1}xe_{n-2}e_{n-1} = xe_{n-1}$. Hence, $e_{n-1}B_{n-1}e_{n-1} \supseteq B_{n-2}\, e_{n-1}$. This proves (1). Points (2) and (3) are obvious. □

**Lemma 5.2.3.** *Write $B_n$ for $B_n(S,\delta)$. For $n \geq 2$, $B_n\, e_{n-1} = B_{n-1}\, e_{n-1}$. Moreover, $x \mapsto xe_{n-1}$ is injective from $B_{n-1}$ to $B_n$.*

*Proof.* By [51], Proposition 2.1, any $(n,n)$–Brauer diagram is either already in $B_{n-1}$, or can be written in the form $a\chi_{n-1}b$, with $a,b \in B_{n-1}$ and $\chi_{n-1} \in \{e_{n-1}, s_{n-1}\}$. Applying this again to $b$, either $b \in B_{n-2}$, or $b$ can be factored as $b_1\chi_{n-2}b_2$, with

$b_i \in B_{n-2}$ and $\chi_{n-2} \in \{e_{n-2}, s_{n-2}\}$. Since $e_{n-1}^2 = \delta e_{n-1}$ and $s_{n-1}e_{n-1} = e_{n-1}$, it follows that if $b \in B_{n-2}$, then $a\chi_{n-1}be_{n-1} = ab\chi_{n-1}e_{n-1} \in B_{n-1}e_{n-1}$. If $b = b_1\chi_{n-2}b_2$, then $a\chi_{n-1}be_{n-1} = ab_1\chi_{n-1}\chi_{n-2}e_{n-1}b_2$. Now we can apply the following identities: $e_{n-1}\chi_{n-2}e_{n-1} = e_{n-1}$ for $\chi_{n-2} \in \{e_{n-2}, s_{n-2}\}$, $s_{n-1}e_{n-2}e_{n-1} = s_{n-2}e_{n-1}$, and $s_{n-1}s_{n-2}e_{n-1} = e_{n-2}e_{n-1}$ to conclude that $a\chi_{n-1}be_{n-1} \in B_{n-1}e_{n-1}$. This shows that $B_ne_{n-1} = B_{n-1}e_{n-1}$.

For $x \in B_{n-1}$, we have $\mathrm{cl}(xe_{n-1}) = x$, so the map $x \mapsto xe_{n-1}$ is injective from $B_{n-1}$ to $B_{n-1}e_{n-1}$. □

### 5.2.4 Verification of framework axioms for $B_n(S, \delta)$

Take $R = \mathbb{Z}[\boldsymbol{\delta}]$, where $\boldsymbol{\delta}$ is an indeterminant. Then $R$ is the universal ground ring for the Brauer algebras; for any commutative ring $S$ with distinguished element $\delta$, we have $B_n(S, \delta) \cong B_n(R, \boldsymbol{\delta}) \otimes_R S$. Let $F = \mathbb{Q}(\boldsymbol{\delta})$ denote the field of fractions of $R$. Write $B_n = B_n(R, \boldsymbol{\delta})$.

**Proposition 5.2.4.** *The two sequences of $R$–algebras $(B_n)_{n\geq 0}$ and $(R\mathfrak{S}_n)_{n\geq 0}$ satisfy the framework axioms of Section 3.1.*

*Proof.* According to Example 2.4.2, $(R\mathfrak{S}_n)_{n\geq 0}$ is a coherent tower of cellular algebras, so axiom (1) holds. Framework axioms (2) and (3) are evident. $B_n^F$ is split semisimple by [51], Theorem 3.2, so axiom (4) holds.

We take $e_{n-1} \in B_n$ to be the element defined in the previous section. Let us verify the axioms (5)–(8) involving $e_{n-1}$. As observed above, $e_{n-1}$ is $i$–invariant, $J = B_ne_{n-1}B_n$ is the ideal spanned by diagrams with fewer than $n$ through strands, and $B_n/J \cong R\mathfrak{S}_n$ as algebras with involution. This verifies axiom (5). Axiom (6) follows from Lemma 5.2.2 and axiom (7) from Lemma 5.2.3. Axiom (8) holds because $e_{n-1}e_ne_{n-1} = e_{n-1}$. □

**Corollary 5.2.5.** *For any commutative ring $S$ and for any $\delta \in S$, the sequence*

*of Brauer algebras* $(B_n(S,\delta))_{n\geq 0}$ *is a coherent tower of cellular algebras.* $B_n(S,\delta)$ *has cell modules indexed by all Young diagrams of size* $n$, $n-2$, $n-4,\ldots$. *The cell module labeled by a Young diagram* $\lambda$ *has a basis labeled by up–down tableaux of length* $n$ *and shape* $\lambda$.

## 5.3 The Jones–Temperley–Lieb algebras

### 5.3.1 Definition of the Jones–Temperley–Lieb algebras

Let $S$ be a commutative ring with identity, with distinguished element $\delta$. The Jones–Temperley–Lieb algebra $T_n(S,\delta)$ is the unital $S$–algebra with generators $e_1,\ldots,e_{n-1}$ satisfying the relation:

1. $e_j^2 = \delta e_j$,
2. $e_j e_{j\pm 1} e_j = e_j$,
3. $e_j e_k = e_k e_j$, if $|j-k|\geq 2$,

whenever all indices involved are in the range from 1 to $n-1$.

### 5.3.2 Diagramatic realization of $T_n(S,\delta)$

The $S$–span $\tilde{T}_n(S,\delta)$ of Temperley–Lieb diagrams is a subalgebra of the Brauer algebra. There exists an algebra map $\varphi$ from $T_n(S,\delta)$ to $\tilde{T}_n(S,\delta)$, determined by $e_j \mapsto e_j$ for $1\leq j\leq n-1$. Kauffman shows ([33], Theorem 4.3) that the map is an isomorphism. In fact, to show that $\varphi$ is surjective, it suffices to show that any Temperley–Lieb diagram can be written as a product of $e_j$'s. Kauffman indicates by example how this is to be done, and it is not difficult to invent a measure of complexity of Temperley–Lieb diagrams and to show this formally, by induction on complexity. For injectivity, Jones shows ([29], p. 14) that $T_n(S,\delta)$ is spanned by a family $\mathbb{B}$ of $\frac{1}{n+1}\binom{2n}{n}$ reduced words in the $e_j$'s. Since $\varphi$ is surjective and $\tilde{T}_n(S,\delta)$ is a free $S$–module of rank $\frac{1}{n+1}\binom{2n}{n}$, it follows easily that $\mathbb{B}$ is a basis and $\varphi$ is an

isomorphism. Because of this, we will no longer distinguish between $T_n(S,\delta)$ and $\tilde{T}_n(S,\delta)$.

### 5.3.3 Brief history of $T_n(S,\delta)$

The Jones-Temperely-Lieb algebras were introduced by Jones in his study of subfactors [29] and then employed by him to define the Jones link invariant [30]. The name derives from the appearance of specific representations of the algebras in statistical mechanics that had been found some years earlier. By now, there is a huge literature related to these algebras because of their multiple roles in subfactor theory, invariants of links and 3-manifolds, statistical mechanics and quantum field theory. The Jones–Tempereley–Lieb algebras were shown to be cellular in [22]. Several other proofs of cellularity are known, for example [53, 23].

### 5.3.4 Some properties of $T_n(S,\delta)$

The Brauer algebra maps $\iota$, cl, $\varepsilon_n$ (when $\delta$ is invertible), and $i$ restrict to maps of the Jones–Temperley–Lieb algebras having similar properties. For example, $i$ is an algebra involution on each $T_n(S,\delta)$ and $i \circ \iota = \iota \circ i$.

The span of Temperley–Lieb diagrams having at least one horizontal strand is an ideal $J$ in $T_n(S,\delta)$, and $T_n(S,\delta)/J \cong S$. The proof of surjectivity of $\varphi$ sketched above shows that any Temperley–Lieb diagram with at least one horizontal edge can be written as a non-trivial product of $e_j$'s; so $J$ is equal to the ideal generated by all of the $e_j$'s. However, the identities $e_j e_{j+1} e_j = e_j$ imply that $J$ is the ideal generated by $e_{n-1}$.

### 5.3.5 Verification of the framework axioms for $T_n(S, \delta)$

Take $R = \mathbb{Z}[\boldsymbol{\delta}]$, where $\boldsymbol{\delta}$ is an indeterminant. Then $R$ is the universal ground ring for the Jones–Temperley–Lieb algebras; for any integral domain $S$ with distinguished element $\delta$, we have $T_n(S, \delta) \cong T_n(R, \boldsymbol{\delta}) \otimes_R S$. Let $F = \mathbb{Q}(\boldsymbol{\delta})$ denote the field of fractions of $R$. Write $T_n = T_n(R, \boldsymbol{\delta})$.

**Proposition 5.3.1.** *The two sequences of $R$–algebras $(T_n)_{n\geq 0}$ and $(R)_{n\geq 0}$ satisfy the framework axioms of Section 3.1.*

*Proof.* Axioms (1), (2), and (3) are obvious. For semisimplicity of $T_n^F$, see [18], Theorem 2.8.5. This gives axiom (4).

We checked axiom (5) in the previous section. The proof for axiom (6) is the same as for the Brauer algebras.

According to [29], Lemma 4.1.2, any $(n, n)$–Temperley–Lieb diagram is either already in $T_{n-1}$, or can be written in the form $ae_{n-1}b$, with $a, b \in T_{n-1}$. Given this, the verification of axiom (7) is the same as for the Brauer algebras; only use the identity $e_{n-1}e_{n-2}e_{n-1} = e_{n-1}$ in place of several similar identities for the Brauer algebras.

As for the Brauer algebras, axiom (8) follows from the identity $e_{n-1}e_ne_{n-1} = e_{n-1}$. □

**Corollary 5.3.2.** *For any ring $S$ and $\delta \in S$, the sequence of Jones–Temperley–Lieb algebras $(T_n(S, \delta))_{n\geq 0}$ is a coherent tower of cellular algebras. The cell modules of $T_n(S, \delta)$ can be labeled by Young diagrams with one or two rows and size $n$, and the basis of the cell module labeled by $\lambda$ has a basis labeled by standard tableaux of shape $\lambda$.*

*Proof.* We only have to remark that the vertices on the $n$-th row of the branching diagram for $(T_k^F)_{k\geq 0}$ (see [18], Lemma 2.8.4) can be labeled by Young diagrams of size $n$ with no more than 2 rows, and the paths on the branching diagram by

standard tableaux. (Alternatively, the vertices on the $n$-th row of the branching diagram can be labeled by Young diagrams with one row and size $n, n-2, n-4, \ldots,$ and the paths on the branching diagram by up–down tableaux.) $\square$

## 5.4 The Birman–Wenzl–Murakami (BMW) algebras

### 5.4.1 Definition of the BMW algebras

The BMW algebras were first introduced by Birman and Wenzl [6] and independently by Murakami [45] as abstract algebras defined by generators and relations. The version of the presentation given here follows [44].

**Definition 5.4.1.** Let $S$ be a commutative unital ring with invertible elements $\rho$ and $q$ and an element $\delta$ satisfying $\rho^{-1} - \rho = (q^{-1} - q)(\delta - 1)$. The *Birman–Wenzl–Murakami algebra* $W_n(S; \rho, q, \delta)$ is the unital $S$–algebra with generators $g_i^{\pm 1}$ and $e_i$ $(1 \leq i \leq n-1)$ and relations:

1. (Inverses) $g_i g_i^{-1} = g_i^{-1} g_i = 1$.
2. (Essential idempotent relation) $e_i^2 = \delta e_i$.
3. (Braid relations) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ and $g_i g_j = g_j g_i$ if $|i-j| \geq 2$.
4. (Commutation relations) $g_i e_j = e_j g_i$ and $e_i e_j = e_j e_i$ if $|i-j| \geq 2$.
5. (Tangle relations) $e_i e_{i\pm 1} e_i = e_i$, $g_i g_{i\pm 1} e_i = e_{i\pm 1} e_i$, and $e_i g_{i\pm 1} g_i = e_i e_{i\pm 1}$.
6. (Kauffman skein relation) $g_i - g_i^{-1} = z(e_i - 1)$.
7. (Untwisting relations) $g_i e_i = e_i g_i = \rho^{-1} e_i$, and $e_i g_{i\pm 1} e_i = \rho e_i$.

### 5.4.2 Geometric realization of the BMW algebras

A geometric realization of the BMW algebra is as the algebra of framed $(n, n)$–tangles in the disc cross the interval, modulo certain skein relations. It is more

convenient, at least for our purposes, to describe this geometric version in terms of tangle diagrams.

First, tangle diagrams can be multiplied by stacking, as for Brauer or Temperley–Lieb diagrams (but closed loops are allowed, and there is no reduction by removing closed loops after stacking). Recall that our convention is that the product $ab$ of tangle diagrams is given by stacking $b$ over $a$. This makes $(n, n)$–tangle diagrams into a monoid, the identity being the tangle diagram in which each top vertex $\boldsymbol{j}$ is connected to the bottom vertex $\overline{\boldsymbol{j}}$ by a vertical line segment, when $n \geq 1$. (The identity for the monoid of $(0, 0)$–tangle diagrams is the empty tangle.)

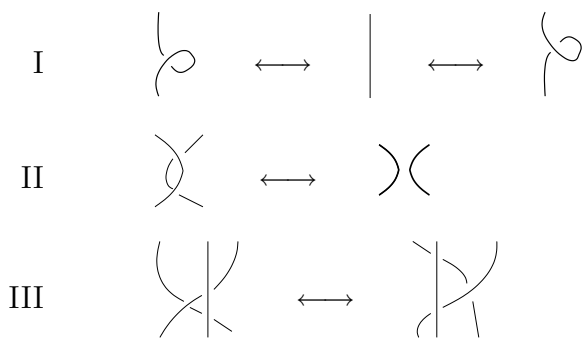
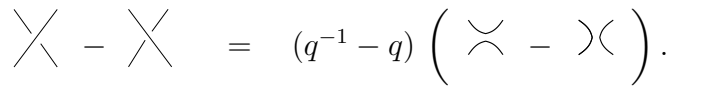


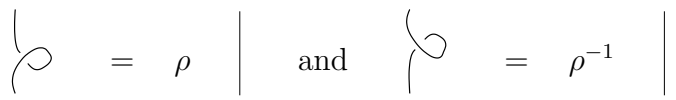

Figure 5.4: Reidemeister moves

Two tangle diagrams are said to be *regularly isotopic* if they are related by a sequence of Reidemeister moves of types II and III, followed by an isotopy of $\mathcal{R}$ fixing the boundary. (Reidemeister moves of type I are not allowed.) See the figure above for the Reidemeister moves.

Stacking of tangle diagrams respects regular isotopy; thus one obtains a monoid structure on the regular isotopy classes of $(n, n)$–tangle diagrams. Let us denote this monoid by $\mathcal{U}_n$. Let $S$ be a ring with elements $\rho$, $q$ and $\delta$ as in the definition of the BMW algebras. The *Kauffman tangle algebra* $\mathrm{KT}_n(S; \rho, q, \delta)$ is the monoid

algebra $S\,\mathcal{U}_n$ modulo the following skein relations:

1. Crossing relation: 


2. Untwisting relation: 


3. Free loop relation: $T \cup \bigcirc = \delta\, T$, where $T \cup \bigcirc$ means the union of a tangle diagram $T$ and a closed loop having no crossings with $T$.

   Let $e_i$ and $g_j$ denote the following $(n, n)$–tangle diagrams:

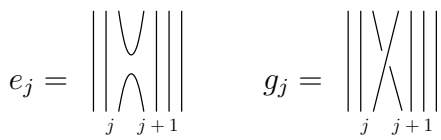



Figure 5.5: Generators of BMW algebras

Morton and Wassermann [44] showed that the assignments $e_j \mapsto e_j$ and $g_j \mapsto g_j$ determine an isomorphism from $W_n(S; \rho, q, \delta)$ to $\mathrm{KT}_n(S; \rho, q, \delta)$. Given this, we will no longer distinguish between the BMW algebras and the Kauffman tangle algebras.

### 5.4.3 Brief history of the BMW algebras

The origin of the BMW algebras was in knot theory. Kauffman defined [33] an invariant of regular isotopy for links in $S^3$, determined by skein relations. Birman and Wenzl [6] and Murakami [45] then defined the BMW algebras in order to give an algebraic setting for the Kauffman invariant. The BMW algebras were implicitly modeled on algebras of tangles. The definition of the Kauffman tangle algebra was made explicit by Morton and Traczyk [43], who also showed that $\mathrm{KT}_n(S; \rho, q, \delta)$ is free as an $S$–module of rank $(2n-1)!!$. Morton and Wassermann [44] showed that the BMW algebras and Kauffman tangle algebras are isomorphic.

Xi showed [58] that the tangle basis of Morton and Traczyk is a cellular basis. Enyang has exhibited two cellular bases of BMW algebras; the first [14] is a tangle type basis, and the second [15] is a basis indexed by up–down tableaux, which demonstrates the coherence of the cellular structures on $(W_n)_{n\geq 0}$.

### 5.4.4 Some properties of the BMW algebras

In this subsection, write $W_n$ for $W_n(S;\rho,q,\delta)$.

The BMW algebras have an algebra involution $i$ uniquely determined by $i(e_j) = e_j$ and $i(g_j) = g_j$ for all $j$. The action of $i$ on tangle diagrams is by the rotation through the axis $y = 1/2$. (It is by rotation rather than reflection, since the reflection would take $g_j \mapsto g_j^{-1}$.)

For $n \geq 0$, there is a unique homomorphism $\iota$ from $W_n$ to $W_{n+1}$ determined by $e_i \mapsto e_i$ and $g_i \mapsto g_i$ for $1 \leq i \leq n-1$. On the level of tangle diagrams, the map is given by adding a new vertical strand connecting $\boldsymbol{n+1}$ and $\overline{\boldsymbol{n+1}}$, as for the Brauer algebras.

For $n \geq 1$, a map cl from $(n,n)$–tangle diagrams to $(n-1,n-1)$–tangle diagrams can be defined as for Brauer diagrams. The linear extension of this map respects regular isotopy and the Kauffman skein relations, so determines a linear map from $W_n$ to $W_{n-1}$. We have $i \circ \mathrm{cl} = \mathrm{cl} \circ i$ and $\mathrm{cl} \circ \iota(x) = \delta\, x$. Moreover, for $x \in W_n$, $x = \mathrm{cl}(\iota(x)e_n)$, so it follows that $\iota : W_n \to W_{n+1}$ is injective. The involution $i$ and inclusion $\iota$ satisfy $i \circ \iota = \iota \circ i$. Using $\iota$, identify $W_n$ as a subalgebra of $W_{n+1}$.

If $\delta$ is invertible in $S$, define $\varepsilon_n = (1/\delta)\mathrm{cl}$, which is a conditional expectation, that is, an unital $W_{n-1}$–$W_{n-1}$ bimodule map. We have $\varepsilon_{n+1} \circ \iota(x) = x$ for $x \in W_n$.

The ideal $J$ in $W_n$ generated by $e_{n-1}$ contains $e_j$ for all $j$ because of the relations $e_j e_{j+1} e_j = e_j$. It follows from the BMW relations that $W_n/J$ is isomorphic to the Hecke algebra $H_n(S;q^2)$ with the quadratic relation $g_j - g_j^{-1} = q - q^{-1}$, or $(g_j - q)(g_j + q^{-1}) = 0$.

**Lemma 5.4.2.**

1. *For* $n \geq 3$, $e_{n-1}W_{n-1}e_{n-1} = W_{n-2}e_{n-1}$.

2. $e_1 W_1 e_1 = \delta\, W_0\, e_1$

3. *For* $n \geq 2$, $e_{n-1}$ *commutes with* $W_{n-2}$.

*Proof.* The proof is the same as that of Lemma 5.2.2 for the Brauer algebras, using the tangle realization of the BMW algebras. □

**Lemma 5.4.3.** *For* $n \geq 2$, $W_n\, e_{n-1} = W_{n-1}\, e_{n-1}$. *Moreover,* $x \mapsto xe_{n-1}$ *is injective from* $W_{n-1}$ *to* $W_{n-1}e_{n-1}$.

*Proof.* According to [6], Lemma 3.1, any $(n,n)$–tangle is already in $W_{n-1}$, or it can be written as a linear combination of elements $a\chi_{n-1}b$, with $a, b \in W_{n-1}$ and $\chi_{n-1} \in \{e_{n-1}, g_{n-1}\}$. Given this, the proof of the lemma is the same as the proof of Lemma 5.2.3 for the Brauer algebras, using the tangle relations and untwisting relations of Definition 5.4.1 in place of similar identities for the Brauer algebras. □

### 5.4.5 Verification of the framework axioms for $W_n$

The generic or universal ground ring for the BMW algebras is

$$R = \mathbb{Z}[\boldsymbol{\rho}^{\pm 1}, \boldsymbol{q}^{\pm 1}, \boldsymbol{\delta}]/\langle \boldsymbol{\rho}^{-1} - \boldsymbol{\rho} = (\boldsymbol{q}^{-1} - \boldsymbol{q})(\boldsymbol{\delta} - 1)\rangle,$$

where $\boldsymbol{\rho}$, $\boldsymbol{q}$, and $\boldsymbol{\delta}$ are indeterminants over $\mathbb{Z}$. Suppose that $S$ is an appropriate ground ring for the BMW algebras; that is, $S$ is a commutative unital ring with invertible elements $\rho$ and $q$ and an element $\delta$ satisfying $\rho^{-1} - \rho = (q^{-1} - q)(\delta - 1)$. Then $W_n(S; \rho, q, \delta) \cong W_n(R; \boldsymbol{\rho}, \boldsymbol{q}, \boldsymbol{\delta}) \otimes_R S$.

$R$ is an integral domain whose field of fractions is $F \cong \mathbb{Q}(\boldsymbol{\rho}, \boldsymbol{q})$ (with $\boldsymbol{\delta} = (\boldsymbol{\rho}^{-1} - \boldsymbol{\rho})/(\boldsymbol{q}^{-1} - \boldsymbol{q}) + 1$ in $F$). We write $W_n$ for $W_n(R; \boldsymbol{\rho}, \boldsymbol{q}, \boldsymbol{\delta})$ and $H_n$ for $H_n(R; \boldsymbol{q}^2)$ in this subsection.

**Proposition 5.4.4.** *The two sequences of algebras $(W_n)_{n\geq 0}$ and $(H_n)_{n\geq 0}$ satisfy the framework axioms of Section 3.1.*

*Proof.* According to example 2.4.2 , $(H_n)_{n\geq 0}$ is a coherent tower of cellular algebras, so axiom (1) holds. Axioms (2) and (3) are evident. $W_n^F$ is semisimple by [6], Theorem 3.7, or [52], Theorem 3.5. Thus axiom (4) holds.

We observed above that $W_n/W_n e_{n-1} W_n \cong H_n$; it is easy to check that the isomorphism respects the involutions. Thus axiom (5) holds. Axiom (6) follows from Lemma 5.4.2 and axiom (7) from Lemma 5.4.3. Finally, axiom (8) holds again because of the relation $e_{n-1}e_n e_{n-1} = e_{n-1}$. □

**Corollary 5.4.5.** *Let $S$ be any ground ring for the BMW algebras, with parameters $\rho$, $q$, and $\delta$. The sequence of BMW algebras $(W_n)_{n\geq 0}$ is a coherent tower of cellular algebras. $W_n$ has cell modules indexed by all Young diagrams of size $n$, $n-2$, $n-4, \ldots$. The cell module labeled by a Young diagram $\lambda$ has a basis labeled by up–down tableaux of length $n$ and shape $\lambda$.*

## 5.5 The cyclotomic BMW algebras

### 5.5.1 Definition of the cyclotomic BMW algebras

In general, our notation will follow [21]. In order to simplify statements, we establish the following convention.

**Definition 5.5.1.** Fix an integer $r \geq 1$. A *ground ring* $S$ is a commutative unital ring with parameters $\rho$, $q$, $\delta_j$ $(j \geq 0)$, and $u_1, \ldots, u_r$, with $\rho$, $q$, and $u_1, \ldots, u_r$ invertible, and with $\rho^{-1} - \rho = (q^{-1} - q)(\delta_0 - 1)$.

**Definition 5.5.2.** Let $S$ be a ground ring with parameters $\rho$, $q$, $\delta_j$ $(j \geq 0)$, and $u_1, \ldots, u_r$. The *cyclotomic BMW algebra* $W_{n,S,r}(u_1, \ldots, u_r)$ is the unital $S$–algebra with generators $y_1^{\pm 1}$, $g_i^{\pm 1}$ and $e_i$ $(1 \leq i \leq n-1)$ and relations:

(1) (Inverses) $g_i g_i^{-1} = g_i^{-1} g_i = 1$ and $y_1 y_1^{-1} = y_1^{-1} y_1 = 1$.

(2) (Idempotent relation) $e_i^2 = \delta_0 e_i$.

(3) (Affine braid relations)

(a) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ and $g_i g_j = g_j g_i$ if $|i-j| \geq 2$.

(b) $y_1 g_1 y_1 g_1 = g_1 y_1 g_1 y_1$ and $y_1 g_j = g_j y_1$ if $j \geq 2$.

(4) (Commutation relations)

(a) $g_i e_j = e_j g_i$ and $e_i e_j = e_j e_i$ if $|i-j| \geq 2$.

(b) $y_1 e_j = e_j y_1$ if $j \geq 2$.

(5) (Affine tangle relations)

(a) $e_i e_{i\pm 1} e_i = e_i$,

(b) $g_i g_{i\pm 1} e_i = e_{i\pm 1} e_i$ and $e_i g_{i\pm 1} g_i = e_i e_{i\pm 1}$.

(c) For $j \geq 1$, $e_1 y_1^j e_1 = \delta_j e_1$.

(6) (Kauffman skein relation) $g_i - g_i^{-1} = (q - q^{-1})(1 - e_i)$.

(7) (Untwisting relations) $g_i e_i = e_i g_i = \rho^{-1} e_i$ and $e_i g_{i\pm 1} e_i = \rho e_i$.

(8) (Unwrapping relation) $e_1 y_1 g_1 y_1 = \rho e_1 = y_1 g_1 y_1 e_1$.

(9) (Cyclotomic relation) $(y_1 - u_1)(y_1 - u_2) \cdots (y_1 - u_r) = 0$.

Thus, a cyclotomic BMW algebra is the quotient of the affine BMW algebra [19], by the cyclotomic relation $(y_1 - u_1)(y_1 - u_2) \cdots (y_1 - u_r) = 0$.

### 5.5.2 Geometric realization

We recall from [19] that the affine BMW algebra is isomorphic to the affine Kauffman tangle algebra, which is an algebra of "affine tangle diagrams," modulo Kauffman skein relations. An affine $(n, n)$–tangle diagram is just an ordinary $(n +$

$1, n+1)$–tangle diagram with a fixed vertical strand connecting $\mathbf{1}$ and $\overline{\mathbf{1}}$, as in the following figure.

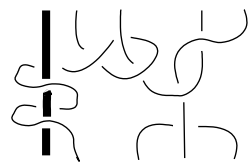


Figure 5.6: An affine $(n, n)$-tangle diagram

The affine Kauffman tangle algebra is generated by the following affine tangle diagrams:

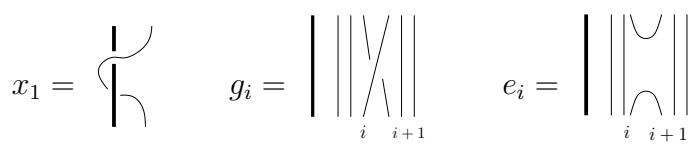



Figure 5.7: Generators of the affine Kauffman tangle algebra

One can also define a cyclotomic Kauffman tangle algebra as the quotient of the affine Kauffman tangle algebra by a cyclotomic skein relation, which is a "local" version of the cyclotomic relation of Definition 5.5.2 (9). See [20] for the precise definition. It is shown in [21] and in [56] that the cyclotomic BMW algebra is isomorphic to the cyclotomic Kauffman tangle algebra with the same parameters, assuming admissibility conditions on the ground ring (see Section 5.5.4).

### 5.5.3 Brief history of cyclotomic BMW algebras

Affine and cyclotomic BMW algebras were introduced by Häring–Oldenberg [26] and have recently been studied by three groups of mathematicians: Goodman and Hauschild Mosley [19, 20, 21, 17], Rui, Xu, and Si [49, 48], and Wilcox and Yu [54, 55, 56, 59]. Under various admissibility assumptions on the ground ring (see Section 5.5.4) all three groups have shown that the algebra $W_{n,S,r}$ is free over $S$ of rank $r^n(2n-1)!!$ and in fact is cellular. The cellular bases produced by all three groups are essentially tangle bases, i.e., cyclotomic analogues of the basis of Morton, Traczyk, and Wassermann for the ordinary BMW algebras. Goodman & Hauschild Mosley and Wilcox & Yu have shown that the algebras can be realized as algebras of tangles, when the ground ring is admissible. Rui et. al. have achieved additional representation theoretic results. Further background on cyclotomic BMW algebras, motivation for the study of these algebras, relations to other mathematical topics (quantum groups, knot theory), and further literature citations can be found in [20] and in the other papers cited above.

### 5.5.4 Admissibility conditions on the ground ring.

The cyclotomic BMW algebras can be defined over arbitrary ground rings. However, it is necessary to impose conditions on the parameters in order to get a satisfactory theory.

One can see by a simple computation why one has to expect conditions on the parameters. First, one can show that there are elements $\delta_{-j}$ in the ground ring $S$ for $j \geq 1$ such that $e_1 y_1^{-j} e_1 = \delta_{-j}\, e_1$; moreover, $\delta_{-j}$ is a polynomial in $\rho^{-1}$, $q - q^{-1}$, and $\delta_0, \delta_1, \ldots, \delta_j$; see [21], Lemma 2.5. If one now multiplies the cyclotomic relation, Definition 5.5.2 (9), by $y_1^a$ and pre– and postmultiplies by $e_1$, one gets $(\sum_{k=0}^{r} a_k \delta_{k+a}) e_1 = 0$, for $a \in \mathbb{Z}$, where the $a_k$ are signed elementary symmetric polynomials in $u_1, \ldots, u_r$. Therefore, either $e_1$ is a torsion element over $S$, or the

following *weak admissibility conditions* hold:

$$\sum_{k=0}^{r} a_k \delta_{k+a} = 0, \quad \text{for } a \in \mathbb{Z}.$$

If $S$ is a field and the weak admissibility conditions do not hold, then $e_1 = 0$; it follows that all the $e_i$ are zero, and the algebra reduces to the cyclotomic Hecke algebra over $S$ with parameters $q^2$ and $u_1, \ldots, u_r$.

The weak admissibility conditions are complicated and apparently not strong enough to give satisfactory results on the representation theory of the algebras. Therefore, one wishes to find conditions that are both simpler and stronger. Two apparently different conditions have been proposed, one by Wilcox and Yu [54], and another by Rui and Xu [49]. It has been shown in [16] that the two conditions are equivalent in the case of greatest interest, when $S$ is an integral domain with $q - q^{-1} \neq 0$. We consider only this case from now on.

**Definition 5.5.3.** Let $S$ be an integral ground ring with parameters $\rho$, $q$, $\delta_j$ ($j \geq 0$) and $u_1, \ldots, u_r$, with $q - q^{-1} \neq 0$. One says that $S$ is *admissible* (or that the parameters are *admissible*) if the weak admissibility conditions hold and $\{e, y_1 e, \ldots, y_1^{r-1} e\} \subseteq W_{2,S,r}$ is linearly independent over $S$.

It is shown in [54] that admissibility is equivalent to finitely many (explicit) polynomial relations on the parameters. Moreover, these relations give $\rho$ and $(q - q^{-1})\delta_j$ as Laurent polynomials in the remaining parameters $q, u_1, \ldots, u_r$; see [54] and [21] for details.

### 5.5.5 Morphisms of ground rings and a generic ring

We consider what are the appropriate morphisms between ground rings for cyclotomic BMW algebras. The obvious notion would be that of a ring homomorphism taking parameters to parameters; that is, if $S$ is a ground ring with parameters $\rho$, $q$, etc., and $S'$ another ground ring with parameters $\rho'$, $q'$, etc., then a morphism $\varphi : S \to S'$ would be required to map $\rho \mapsto \rho'$, $q \mapsto q'$, etc.

However, it is better to require less, for the following reason: The parameter $q$ enters into the cyclotomic BMW relations only in the expression $q^{-1} - q$, and the transformation $q \mapsto -q^{-1}$ leaves this expression invariant. Moreover, the transformation $g_i \mapsto -g_i$, $\rho \mapsto -\rho$, $q \mapsto -q$ (with all other generators and parameters unchanged) leaves the cyclotomic BMW relations unchanged.

Taking this into account, we arrive at the following notion:

**Definition 5.5.4.** Let $S$ be a ground ring with parameters $\rho$, $q$, $\delta_j$ $(j \geq 0)$, and $u_1, \ldots, u_r$. Let $S'$ be another ground ring with parameters $\rho'$, $q'$, etc.

A unital ring homomorphism $\varphi : S \to S'$ is a *morphism of ground rings* if it maps

$$\begin{cases} \rho \mapsto \rho', \text{ and} \\ q \mapsto q' \text{ or } q \mapsto -q'^{-1}, \end{cases}$$

or

$$\begin{cases} \rho \mapsto -\rho', \text{ and} \\ q \mapsto -q' \text{ or } q \mapsto q'^{-1}, \end{cases}$$

and strictly preserves all other parameters.

Suppose there is a morphism of ground rings $\psi : S \to S'$. Then $\psi$ extends to a homomorphism from $W_{n,S,r}$ to $W_{n,S',r}$. Moreover, $W_{n,S,r} \otimes_S S' \cong W_{n,S',r}$ as $S'$–algebras. These statements are discussed in [21], Section 2.4.

Let $S$ be a ground ring with admissible parameters $\rho$, $q$, $\delta_j$ $(j \geq 0)$, and $u_1, \ldots, u_r$. Then

$$\rho, -q^{-1}, \delta_j \ (j \geq 0), \text{ and } u_1, \ldots, u_r$$

and

$$-\rho, -q, \delta_j \ (j \geq 0), \text{ and } u_1, \ldots, u_r$$

are also sets of admissible parameters. Suppose that $S$ is an integral ground ring with admissible parameters, with $q - q^{-1} \neq 0$, and that $S'$ is another integral ground ring; if $\varphi : S \to S'$ is a morphism of ground rings such that $\varphi(q - q^{-1}) \neq 0$, then $S'$ is also admissible.

It is easy to show (see [21], Theorem 3.19) that there is a universal integral

admissible ground ring $R$, with parameters $\boldsymbol{\rho}$, $\boldsymbol{q}$, $\boldsymbol{\delta}_j$ ($j \geq 0$), and $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$, with the following properties:

1. The parameters $\boldsymbol{q}$, $\boldsymbol{u}_1$, $\ldots$, $\boldsymbol{u}_r$ of $R$ are algebraically independent over $\mathbb{Z}$.

2. $R$ is generated as a ring by $\boldsymbol{q}^{\pm 1}$, $\boldsymbol{\rho}^{\pm 1}$, $\boldsymbol{\delta}_0$, $\boldsymbol{\delta}_1$, $\ldots \boldsymbol{\delta}_{r-1}$, and $\boldsymbol{u}_1^{\pm 1}, \ldots, \boldsymbol{u}_r^{\pm 1}$.

3. Whenever $S$ is an integral ground ring with admissible parameters, with $q - q^{-1} \neq 0$, there exists a morphism of ground rings from $R$ to $S$; thus $W_{n,S,r} \cong W_{n,R,r} \otimes_R S$.

4. The field of fractions of $R$ is $\mathbb{Q}(\boldsymbol{q}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_r)$.

### 5.5.6 Some properties of cyclotomic BMW algebras.

We restrict attention to the case of an integral admissible ground ring $S$ with $q - q^{-1} \neq 0$. Identify the cyclotomic BMW algebra with the cyclotomic Kauffman tangle algebra via the isomorphism mentioned in Subsection 5.5.2. Write $W_n$ for $W_{n,S,r}(u_1, \ldots, u_r)$.

The cyclotomic BMW algebras have an algebra involution $i$ uniquely determined by $i(e_j) = e_j$ and $i(g_j) = g_j$ for all $j$, and $i(y_1) = y_1$. The action of $i$ on affine tangle diagrams is by the rotation through the axis $y = 1/2$.

For $n \geq 0$, there is a unique homomorphism $\iota$ from $W_n$ to $W_{n+1}$ determined by $e_i \mapsto e_i$ and $g_i \mapsto g_i$ for $1 \leq i \leq n-1$, and $y_1 \mapsto y_1$. On the level of affine tangle diagrams, the map is given by adding a new vertical strand connecting $\boldsymbol{n+1}$ and $\overline{\boldsymbol{n+1}}$, as for the Brauer algebras.

For $n \geq 1$, a map cl from affine $(n,n)$–tangle diagrams to affine $(n-1, n-1)$–tangle diagrams can be defined as for Brauer diagrams and ordinary tangle diagrams. The linear extension of this map respects regular isotopy and all the skein relations defining the cyclotomic Kauffman tangle algebras, so determines a linear map from $W_n$ to $W_{n-1}$. (See [19], Section 2.7, and [20], Section

3.3 for details.) We have $i \circ \mathrm{cl} = \mathrm{cl} \circ i$ and $\mathrm{cl} \circ \iota(x) = \delta\, x$. Moreover, for $x \in W_n$, $x = \mathrm{cl}(\iota(x)e_n)$, so it follows that $\iota : W_n \to W_{n+1}$ is injective. The involution $i$ and inclusion $\iota$ satisfy $i \circ \iota = \iota \circ i$. Using $\iota$, identify $W_n$ as a subalgebra of $W_{n+1}$.

If $\delta$ is invertible in $S$, define $\varepsilon_n = (1/\delta)\mathrm{cl}$, which is a conditional expectation, that is, an unital $W_{n-1}$–$W_{n-1}$ bimodule map. We have $\varepsilon_{n+1} \circ \iota(x) = x$ for $x \in W_n$.

**Lemma 5.5.5.**

1. *For $n \geq 3$, $e_{n-1}W_{n-1}e_{n-1} = W_{n-2}e_{n-1}$.*

2. $e_1 W_1 e_1 = \delta\, W_0\, e_1$

3. *For $n \geq 2$, $e_{n-1}$ commutes with $W_{n-2}$.*

*Proof.* The proof is the same as that of Lemma 5.2.2 for the Brauer algebras, using the tangle realization of the cyclotomic BMW algebras. □

**Lemma 5.5.6.** *For $n \geq 2$, $W_n\, e_{n-1} = W_{n-1}\, e_{n-1}$. Moreover, $x \mapsto xe_{n-1}$ is injective from $W_{n-1}$ to $W_{n-1}e_{n-1}$.*

*Proof.* Regarding the first statement, the analogous fact for affine BMW algebras is established in [19], Proposition 3.20, and the result for the cyclotomic BMW algebras follows from this. The proof of the second statement is the same as for the ordinary BMW algebras or the Brauer algebras. □

### 5.5.7 The cyclotomic Hecke algebra

Recall the definition of the affine and cyclotomic Hecke algebras, see [1].

**Definition 5.5.7.** Let $S$ be a commutative unital ring with an invertible element $q$. The *affine Hecke algebra* $\widehat{H}_{n,S}(q^2)$ over $S$ is the $S$–algebra with generators $t_1, g_1, \ldots, g_{n-1}$, with relations:

1. The generators $g_i$ are invertible, satisfy the braid relations, and $g_i - g_i^{-1} = (q - q^{-1})$.

2. The generator $t_1$ is invertible, $t_1g_1t_1g_1 = g_1t_1g_1t_1$ and $t_1$ commutes with $g_j$ for $j \geq 2$.

Let $u_1, \ldots, u_r$ be additional elements in $S$. The *cyclotomic Hecke algebra* $H_{n,S,r}(q^2; u_1, \ldots, u_r)$ is the quotient of the affine Hecke algebra $\widehat{H}_{n,S}(q^2)$ by the polynomial relation $(t_1 - u_1) \cdots (t_1 - u_r) = 0$.

Since the generator $t_1$ can be rescaled by an arbitrary invertible element of $S$, only the ratios of the parameters $u_i$ have invariant significance in the definition of the cyclotomic Hecke algebra.

Now let $S$ be a ground ring with parameters $\rho$, $q$, $\delta_j$, and $u_1, \ldots, u_r$. For each $n$, let $I_n$ be the two sided ideal in $W_{n,S,r}$ generated by $e_{n-1}$. Because of the relations $e_je_{j\pm1}e_j = e_j$, the ideal $I_n$ is generated by any $e_i$ $(1 \leq i \leq n-1)$ or by all of them. It is easy to check that the quotient of $W_{n,S,r}$ by $I_n$ is isomorphic to the cyclotomic Hecke algebra $H_{n,S,r}(q^2; u_1, \ldots, u_r)$.

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ be an $r$–tuple of Young diagrams. The total size of $\boldsymbol{\lambda}$ is $|\boldsymbol{\lambda}| = \sum_i |\lambda^{(i)}|$. If $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are $r$–tuples of Young diagrams of total size $f-1$ and $f$ respectively, we write $\boldsymbol{\mu} \subset \boldsymbol{\lambda}$ if $\boldsymbol{\mu}$ is obtained from $\boldsymbol{\lambda}$ by removing one box from one component of $\boldsymbol{\lambda}$.

**Theorem 5.5.8** ([1]). *Let $F$ be a field. The cyclotomic Hecke algebra $H_{n,F,r}(q^2; u_1, \ldots, u_r)$ is split semisimple for all $n$ as long as $q$ is not a proper root of unity and $u_i/u_j$ is not a power of $q$ for all $i \neq j$. In this case, the simple components of $H_{n,F,r}(q; u_1, \ldots, u_r)$ are labeled by $r$–tuples of Young diagrams of total size $n$, and a simple $H_{n,F,r}$ module $V_{\boldsymbol{\lambda}}$ decomposes as a $H_{n-1,F,r}$ module as the direct sum of all $V_{\boldsymbol{\mu}}$ with $\boldsymbol{\mu} \subset \boldsymbol{\lambda}$.*

Let us call the branching diagram for the cyclotomic Hecke algebras, as described in the theorem, the *$r$–Young lattice*. Note that, as for the usual Young's lattice, the $r$–Young lattice has no multiple edges.

**Theorem 5.5.9** (Ariki, Koike, Dipper, James, Mathas). *The sequence of cyclotomic*

*Hecke algebras* $(H_{n,S,r}(q^2; u_1, \ldots, u_r))_{n\geq 0}$ *is a coherent tower of cellular algebras.*

*Proof.* Write $H_n$ for $H_{n,S,r}(q^2; u_1, \ldots, u_r)$. Ariki and Koike showed that the cyclotomic Hecke algebras are free as $S$ modules [2], which implies that $H_n$ imbeds naturally in $H_{n+1}$. Moreover, the algebras $H_n$ have involutions that are consistent with the inclusions. Dipper, James and Mathas [12] constructed a cellular basis of the cyclotomic Hecke algebras, generalizing the Murphy basis of ordinary Hecke algebras. Ariki and Mathas showed [3], Proposition 1.9, that restrictions of cell modules from $H_{n+1}$ to $H_n$ have cell filtrations. Finally, Mathas has shown [42] that the module obtained from inducing a cell module from $H_n$ to $H_{n+1}$ has a cell filtration. □

### 5.5.8 Verification of the framework axioms for $W_{n,R,r}$

Let $R$ be the generic admissible integral ground ring, with parameters $\boldsymbol{\rho}$, $\boldsymbol{q}$, $\boldsymbol{\delta}_j$ $(j \geq 0)$, and $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$, as introduced at the end of Section 5.5.5. In this section, write $W_n$ for $W_{n,R,r}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r)$ and $H_n$ for $H_{n,R,r}(\boldsymbol{q}^2; \boldsymbol{u}_1, \ldots, \boldsymbol{u}_r)$. Recall that the field of fractions of $R$ is $F = \mathbb{Q}(\boldsymbol{q}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_r)$.

**Proposition 5.5.10.** *The two sequences of algebras* $(W_n)_{n\geq 0}$ *and* $(H_n)_{n\geq 0}$ *satisfy the framework axioms of Section 3.1.*

*Proof.* According to Proposition 5.5.9, $(H_n)_{n\geq 0}$ is a coherent tower of cellular algebras, so axiom (1) holds. Axioms (2) and (3) are evident. $W_n^F$ is semisimple by [21], Theorem 4.8, or [48]. Thus axiom (4) holds.

We observed above that $W_n/W_n e_{n-1} W_n \cong H_n$; it is easy to check that the isomorphism respects the involutions. Thus axiom (5) holds. Axiom (6) follows from Lemma 5.5.5 and axiom (7) from Lemma 5.5.6. Finally, axiom (8) holds because of the relation $e_{n-1} e_n e_{n-1} = e_{n-1}$. □

**Corollary 5.5.11.** *Let* $S$ *be any admissible integral ground ring with* $q - q^{-1} \neq 0$.

*The sequence of cyclotomic BMW algebras* $(W_{n,S,r})_{n\geq 0}$ *is a coherent tower of cellular algebras.* $W_{n,S,r}$ *has cell modules indexed by all* $r$*–tuples of Young diagrams of total size* $n$, $n-2$, $n-4, \ldots$. *The cell module labeled by an* $r$*–tuple of Young diagrams* $\boldsymbol{\lambda}$ *has a basis labeled by up–down tableaux of length* $n$ *and shape* $\boldsymbol{\lambda}$.

**Remark 5.5.12.** The degenerate cyclotomic BMW algebras (cyclotomic Nazarov Wenzl algebras) studied in [4] can be treated in the same fashion.

## 5.6 The walled Brauer algebras

### 5.6.1 Definition of the walled Brauer algebras

Let $S$ be a commutative ring with identity, with a distinguished element $\delta$. The walled (or rational) Brauer algebra $B_{r,s}(S,\delta)$ is a unital subalgebra of the Brauer algebra $B_{r+s}(S,\delta)$ spanned by certain Brauer diagrams. Divide the $r+s$ top vertices into a left cluster consisting of the leftmost $r$ vertices and a right cluster consisting of the remaining $s$ vertices, and similarly for the bottom vertices. The walled Brauer diagrams are those in which no vertical strand connects a left vertex and a right vertex, and every horizontal strand connects a left vertex and a right vertex. (If we draw a vertical line–the wall–separating left and right vertices, then vertical strands are forbidden to cross the wall, and horizontal strands are required to cross the wall.) One can easily check that the span of walled Brauer diagrams is a unital subalgebra of $B_{r+s}(S,\delta)$.

### 5.6.2 Brief history of the walled Brauer algebras

The walled Brauer algebras were introduced by Turaev [50] and by Koike [34], and studied by Benkart et. al. [5] and by Nikitin [47]. The walled Brauer algebras arise in connection with the invariant theory of the general linear group acting on mixed tensors. Cellularity of walled Brauer algebras was proved by Green and Martin [23] and by Cox et. al. [8]; the latter authors show that walled Brauer

algebras can be arranged into coherent cellular towers.

### 5.6.3 Some properties of the walled Brauer algebras

The walled Brauer algebra $B_{r,s}$ is invariant under the involution $i$ of the Brauer algebra $B_{r+s}$. Moreover, the inclusion map $\iota : B_{r+s} \to B_{r+s+1}$ maps $B_{r,s}$ to $B_{r,s+1}$, and the closure map $\mathrm{cl} : B_{r+s} \to B_{r+s-1}$ maps $B_{r,s}$ to $B_{r,s-1}$, when $s \geq 1$. If $\delta$ is invertible, $\varepsilon_{r,s} = (1/\delta)\,\mathrm{cl} : B_{r,s} \to B_{r,s-1}$ is a conditional expectation, and, of course, the trace $\varepsilon$ on $B_{r+s}$ restricts to a trace on $B_{r,s}$.

The Brauer algebras have an involutive inner automorphism $\rho$ which maps each Brauer diagram to its reflection in the vertical line $x = 1/2$. (We might as well take the vertical line to coincide with our wall.) It is clear that $\rho$ restricts to an isomorphism from $B_{r,s}$ to $B_{s,r}$. Given this, we can define "left versions" of $\iota$, cl and $\varepsilon_{r,s}$ by $\iota' = \rho \circ \iota \circ \rho : B_{r,s} \to B_{r+1,s}$, $\mathrm{cl}' = \rho \circ \mathrm{cl} \circ \rho : B_{r,s} \to B_{r-1,s}$, and $\varepsilon' = \rho \circ \varepsilon \circ \rho : B_{r,s} \to B_{r-1,s}$. Note that $\iota'$ adds a vertical strand on the left, and $\mathrm{cl}'$ partially closes diagrams on the left.

Let $e_{a,b}$ be the Brauer diagram with horizontal strands connecting $\boldsymbol{a}$ to $\boldsymbol{b}$ and $\overline{\boldsymbol{a}}$ to $\overline{\boldsymbol{b}}$ and vertical strands connecting $\boldsymbol{j}$ to $\overline{\boldsymbol{j}}$ for all $j \neq a, b$. One can easily check the following properties:

**Lemma 5.6.1.**

1. $e_{a,b}^2 = \delta e_{a,b}$.

2. $e_{a,b}\, e_{a,b\pm 1}\, e_{a,b} = e_{a,b}$ *and* $e_{a,b}\, e_{a\pm 1,b}\, e_{a,b} = e_{a,b}$.

3. *For* $e_{a,b} \in B_{r,s}$, $\iota(e_{a,b}) = e_{a,b}$ *and* $\iota'(e_{a,b}) = e_{a+1,b+1}$.

4. *For* $x \in B_{r,s+1}$, *we have* $e_{1,r+s+2}\, \iota'(x)\, e_{1,r+s+2} = \iota' \circ \iota \circ \mathrm{cl}(x)\, e_{1,r+s+2}$.

5. *For* $x \in B_{r+1,s}$, *we have* $e_{1,r+s+2}\, \iota(x)\, e_{1,r+s+2} = \iota' \circ \iota \circ \mathrm{cl}'(x)\, e_{1,r+s+2}$.

6. $e_{1,r+s+2}$ *commutes with* $\iota' \circ \iota(x)$ *for all* $x \in B_{r,s}$.

The following statement is also easy to check:

**Lemma 5.6.2.** *The ideal $J$ in $B_{r,s}(S,\delta)$ generated by $e_{1,r+s}$ is the ideal spanned by diagrams with fewer than $r+s$ through strands, and $B_{r,s}(S,\delta)/J \cong S(\mathfrak{S}_r \times \mathfrak{S}_s)$.*

**Lemma 5.6.3.**

1. $B_{r,s+1}\, e_{1,r+s+1} = \iota(B_{r,s})\, e_{1,r+s+1}$.

2. $B_{r+1,s}\, e_{1,r+s+1} = \iota'(B_{r,s})\, e_{1,r+s+1}$.

*Proof.* To prove part (1), we have to show that if $d$ is a diagram in $B_{r,s+1}$, then there is a diagram $d' \in \iota(B_{r,s})$ such that $d\, e_{1,r+s+1} = d'\, e_{1,r+s+1}$. We can suppose that $d$ is not already in $\iota(B_{r,s})$; therefore, the vertex $\overline{\boldsymbol{r+s+1}}$ in $d$ is connected to some vertex $v$ other than $\mathbf{1}$ and $\boldsymbol{r+s+1}$. There are two cases to consider.

The first is that the vertices $\mathbf{1}$ and $\boldsymbol{r+s+1}$ are not connected to each other in $d$; let $a$ and $b$ be the vertices connected to $\mathbf{1}$ and $\boldsymbol{r+s+1}$. Now let $d'$ be the diagram in which $a$ and $b$ are connected to each other; $\boldsymbol{r+s+1}$ is connected to $\overline{\boldsymbol{r+s+1}}$; $\mathbf{1}$ is connected to $v$; and all other strands are as in $d$. Then we have $d\, e_{1,r+s+1} = d'\, e_{1,r+s+1}$. The case that the vertices $\mathbf{1}$ and $\boldsymbol{r+s+1}$ are connected to each other is similar and will be omitted.

Part (2) is proved by applying the map $\rho$ to both sides of the equality in part (1) and then interchanging the roles of $r$ and $s$. □

A unital trace $\varepsilon$ on an $S$–algebra $A$ is *non-degenerate* if for every non–zero $x \in A$ there exists a $y \in A$ such that $\varepsilon(xy) \neq 0$.

**Lemma 5.6.4.**

1. *The trace $\varepsilon$ on $B_n(\mathbb{Q}(\boldsymbol{\delta}),\boldsymbol{\delta})$ is non-degenerate, for any $n$.*

2. *The trace $\varepsilon$ on $B_{r,s}(\mathbb{Q}(\boldsymbol{\delta}),\boldsymbol{\delta})$ is non-degenerate, for any $r,s$.*

*Sketch of proof.* The argument for part (1) is from [43]. It suffices to show that the determinant of the Gram matrix $\varepsilon(dd')_{d,d'}$, where $d,d'$ run over the list of all Brauer

diagrams (in some order), is non-zero. Recall that $\varepsilon(dd')$ is $\boldsymbol{q}^{c(dd')-n}$, where $c(dd')$ is the number of components in the tangle obtained by closing all the strands of $dd'$. One can check that $c(d\,i(d)) = n$ and $c(dd') < n$ for all diagrams other than $i(d)$. Therefore, each row and column of the Gram matrix has exactly one entry equal to 1 and all other entries have the form $\boldsymbol{q}^{-k}$ for some $k > 0$.

The argument for part (2) is identical. □

### 5.6.4 Verification of the framework axioms for $B_{r,s}$

To fit the walled Brauer algebras to the framework, reduce the double sequence of algebras to a single sequence. Adopt the following scheme, as in [47], or [8]: Fix some integer $t \geq 0$. For any $S$ and $\delta \in S$, consider the sequence of walled Brauer algebras $A_n = A_n(S, \delta)$, where $A_{2k}(S, \delta) = B_{k,k+t}(S, \delta)$, and $A_{2k+1}(S, \delta) = B_{k,k+t+1}(S, \delta)$, with the inclusions

$$A_{2k} \xrightarrow{\iota} A_{2k+1} \xrightarrow{\iota'} A_{2k+2}.$$

Let $f_{2k-1} = e_{1,2k+t} \in A_{2k}$ and $f_{2k} = e_{1,2k+t+1} \in A_{2k+1}$. Identify $A_n$ as a subalgebra of $A_{n+1}$ via these embeddings. With these conventions, Lemma 5.6.1, points (2) and (3) give $f_n f_{n\pm1} f_n = f_n$. Moreover, if we write $\mathrm{cl}_n = \mathrm{cl}$ when $n$ is even and $\mathrm{cl}_n = \mathrm{cl}'$ when $n$ is odd, then $f_{n-1} x f_{n-1} = \mathrm{cl}_{n-1}(x) f_{n-1}$ for $x \in A_{n-1}$, by Lemma 5.6.1, points (4) and (5). Point (6) of the Lemma says that $f_{n-1}$ commutes with $A_{n-2}$.

If $J$ is the ideal in $A_n$ generated by $f_{n-1}$, then $A_{2k}/J \cong S(\mathfrak{S}_k \times \mathfrak{S}_{k+t})$, and $A_{2k+1}/J \cong S(\mathfrak{S}_k \times \mathfrak{S}_{k+t+1})$. So set $Q_{2k}(S) = S(\mathfrak{S}_k \times \mathfrak{S}_{k+t})$ and $Q_{2k+1}(S) = S(\mathfrak{S}_k \times \mathfrak{S}_{k+t+1})$, with the natural embeddings.

Since $A_0 = B_{0,t} \cong S\mathfrak{S}_t$, and $A_1 = B_{0,t+1} \cong S\mathfrak{S}_{t+1}$, we cannot hope to satisfy framework axiom (3). However, replace axiom (3) with the weaker

(3′) $A_0 \cong Q_0$, and $A_1 \cong Q_1$.

Drop the usual convention (see Definition 2.5.1) regarding branching diagrams that

the 0–th row of the branching diagram has a single vertex. The conclusions will have to be modified, but not severely.

Take $R = \mathbb{Z}[\boldsymbol{\delta}]$ and $\delta = \boldsymbol{\delta}$. $R$ is the generic ground ring for walled Brauer algebras; if $S$ is any commutative unital ring with parameter $\delta$, then $B_{r,s}(S, q) = B_{r,s}(R, \boldsymbol{q}) \otimes_R S$. Let $F = \mathbb{Q}(\boldsymbol{\delta})$. In the remainder of this section, write $A_n = A_n(R, \boldsymbol{\delta})$ and $Q_n = Q_n(R)$. (Recall that $Q_n(R) = R(\mathfrak{S}_k \times \mathfrak{S}_{k+t})$ if $n = 2k$, and $Q_n(R) = R(\mathfrak{S}_k \times \mathfrak{S}_{k+t+1})$ if $n = 2k + 1$.)

**Lemma 5.6.5.** *The walled Brauer algebra $B_{r,s}(\mathbb{Q}(\boldsymbol{\delta}), \boldsymbol{\delta})$ is split semisimple.*

*Sketch of proof.* It suffices to show that (for any $t$) the algebras in the sequence $A_n$ are split semisimple. This was proved by Nikitin in [47], following Wenzl's method for the Brauer algebra in [51]. Nikitin's proof involves obtaining the weights of the trace $\varepsilon$, but little detail is given. For the purposes of this thesis, bypass this issue, and use Lemma 5.6.4 instead. Then the method of proof of Theorem 3.2 from [51] applies. □

**Proposition 5.6.6.** *The two sequence of $R$–algebras $(A_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ satisfy the framework axioms of Section 3.1, with axiom (3) replaced by (3′), specified above, and with the elements $f_n$ taking the role of the elements $e_n$ in the list of framework axioms.*

*Proof.* The sequence $(Q_n)_{n\geq 0}$ is clearly a coherent tower of cellular algebras, so axiom (1) holds. Axiom (2) is evident, and we have remarked about substituting axiom (3′) for axiom (3). $A_n^F$ is split semisimple by Lemma 5.6.5. Thus axiom (4) holds.

We have $f_{n-1}$ is an essential idempotent with $i(f_{n-1}) = f_{n-1}$. We have $A_n/(A_n f_{n-1} A_n) \cong Q_n$ by Lemma 5.6.2, which gives axiom (5).

We have seen that $f_{n-1}$ commutes with $A_{n-2}$ and $f_{n-1}A_{n-1}f_{n-1} \subseteq A_{n-2}f_{n-1}$. Moreover, if $x \in A_{n-2}$, then $f_{n-1}xf_{n-1} = \boldsymbol{\delta}xf_{n-1}$, so $f_{n-1}A_{n-1}f_{n-1} \supseteq \boldsymbol{\delta}A_{n-2}f_{n-1}$. Therefore, $f_{n-1}A_{n-1}^F f_{n-1} = A_{n-2}^F f_{n-1}$, so axiom (6) holds.

Axiom (7) results from Lemma 5.6.3, and axiom (8) from $f_{n-1}f_nf_{n-1} = f_{n-1}$.

□

**Remark 5.6.7.** The branching diagram for the sequence $(Q_n^F)$ is the following: Each row has vertices labeled by pairs of Young diagrams; on an even row $2k$, the the first Young diagram in a pair has $k$ boxes and the second $k+t$ boxes; on an odd row $2k+1$, the first Young diagram has $k$ boxes and the second $k+t+1$ boxes; finally, there is an edge between pairs of Young diagrams in successive rows that differ by exactly one box.

**Corollary 5.6.8.** *Let $S$ be any commutative unital ring with parameter $\delta$.*

1. *The walled Brauer algebras $B_{r,s}(S,\delta)$ are cellular algebras.*

2. *The family is coherent in the sense that the restriction of a cell module from $B_{r,s}(S,\delta)$ to $B_{r-1,s}(S,\delta)$ or to $B_{r,s-1}(S,\delta)$ and induction of a cell module from $B_{r,s}(S,\delta)$ to $B_{r+1,s}(S,\delta)$ or to $B_{r,s+1}(S,\delta)$ have filtrations by cell modules.*

3. *The cell modules of $B_{r,s}(S,\delta)$ are labeled by pairs of Young diagrams $(\lambda^{(1)},\lambda^{(2)})$, where $|\lambda^{(2)}| - |\lambda^{(1)}| = s-r$ and $|\lambda^{(2)}| + |\lambda^{(1)}| \leq s+r$.*

A basis for any cell module for $B_{r,s}$ can be labeled by paths on a certain branching diagram. Suppose without loss of generality that $t = s-r \geq 0$. Let $(A_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ be the two sequences of algebras defined above, depending on $t$, so in particular, $B_{r,s} = A_{2r}$. Let $\mathfrak{B}_0$ be the branching diagram for $(Q_n^F)_{n\geq 0}$, which was described above, and let $\mathfrak{B}$ be that obtained by reflections from $\mathfrak{B}_0$. On the 0–th row, $\mathfrak{B}$ has vertices labeled by all pairs $(\emptyset,\lambda)$, where $\lambda$ is a Young diagram of size $t$. Finally, augment $\mathfrak{B}$ with a copy of Young's lattice up to the $(t-1)$–st level, with vertices labeled by pairs $(0,\mu)$ with $0 \leq |\mu| \leq t-1$. The pairs of Young diagrams labeling the cell modules of $B_{r,s}$ are located on the $r+s$–th row of the augmented branching diagram, and a basis of any cell module can be labeled by paths on the augmented branching diagram from $(\emptyset,\emptyset)$ to the pair in question.

Note that several of the results of Section 3 of [8] follow from the application of our method to the walled Brauer algebras.

## 5.7 Partition algebras

### 5.7.1 Definition of the partition algebras

Let $n$ be an integer, $n \geq 1$. Let $[\boldsymbol{n}] = \{\mathbf{1}, \ldots, \boldsymbol{n}\}$ and $[\overline{\boldsymbol{n}}] = \{\overline{\mathbf{1}}, \ldots, \overline{\boldsymbol{n}}\}$ be disjoint sets of size $n$, and let $X_n$ be the family of all set partitions of $[\boldsymbol{n}] \cup [\overline{\boldsymbol{n}}]$.

Represent an element $x$ of $X_n$ by any graph with vertex set equal to $[\boldsymbol{n}] \cup [\overline{\boldsymbol{n}}]$ whose connected components are the blocks or classes of the partition $x$. Picture such a graph as a diagram in the rectangle $\mathcal{R}$, with the vertices in $[\boldsymbol{n}]$ arranged on the top edge and those in $[\overline{\boldsymbol{n}}]$ arranged on the bottom edge of $\mathcal{R}$, as in the tangle diagrams discussed in section 5.1.

Let $S$ be any commutative ring with identity, with a distinguished element $\delta$. Define a product on $X_n$ as follows: Let $x$ and $y$ be elements of $X_n$. Realize $y$ as a set partition of $[\boldsymbol{n}] \cup [\boldsymbol{n}']$ (with $[\boldsymbol{n}']$ the set of bottom vertices). Realize $x$ as a set partition of $[\boldsymbol{n}'] \cup [\overline{\boldsymbol{n}}]$ (with $[\boldsymbol{n}']$ the set of top vertices). Let $E_x$ and $E_y$ be the corresponding equivalence relations, regarded as equivalence relations on $[\boldsymbol{n}] \cup [\boldsymbol{n}'] \cup [\overline{\boldsymbol{n}}]$. Let $E$ be the smallest equivalence relation on $[\boldsymbol{n}] \cup [\boldsymbol{n}'] \cup [\overline{\boldsymbol{n}}]$ containing $E_x \cup E_y$. Let $r$ be the number of equivalence classes of $E$ contained in $[\boldsymbol{n}']$. Let $E_{xy}$ be the equivalence relation obtained by restricting $E$ to $[\boldsymbol{n}] \cup [\overline{\boldsymbol{n}}]$, and let $z$ be the corresponding set partition of $[\boldsymbol{n}] \cup [\overline{\boldsymbol{n}}]$. Then $xy$ is defined to be $\delta^r z$.

Here is an example of two set partitions represented by graphs and their product.

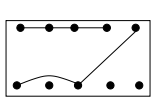
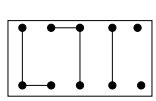
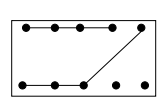
$y =$  $x =$  $xy = \delta$ 

Figure 5.8: Multiplication in partition algebras

Let $A_{2n}(S,\delta)$ be the free $S$ module with basis $X_n$. Give $A_{2n}(S,\delta)$ the bilinear product extending the product defined on $X_n$. One can check the multiplication is associative. Note that $A_0(S,\delta) \cong S$. For $n \geq 1$, the multiplicative identity of $A_{2n}(S,\delta)$ is the partition with blocks $\{\boldsymbol{i}, \bar{\boldsymbol{i}}\}$ for $1 \leq i \leq n$.

For $n \geq 1$, Let $X'_n \subset X_n$ be the family of set partitions with $\boldsymbol{n}$ and $\overline{\boldsymbol{n}}$ in the same block. The $S$–span of $X'_n$ is a unital subalgebra of $A_{2n}(S,\delta)$, which we denote by $A_{2n-1}(S,\delta)$.

The algebras $A_k(S,\delta)$ for $k \geq 0$ are called the *partition algebras*.

Note that the set partitions $x \in X_n$ each of whose blocks has size 2 can be identified with Brauer diagrams on $2n$ vertices and the product of such diagrams in the Brauer algebra $B_n(S,\delta)$ agrees with the product in $A_{2n}$. Thus $B_n(S,\delta)$ can be identified with a unital subalgebra of $A_{2n}(S,\delta)$.

### 5.7.2 Brief history of the partition algebras

The partition algebras $A_{2n}$ were introduced independently by Martin [38, 39] and Jones [31]. Partition algebras arise as centralizer algebras for the symmetric group $\mathfrak{S}_k$ acting as a subgroup of $\mathrm{GL}(k,\mathbb{C})$ on tensor powers of $\mathbb{C}^k$ [31, 40]. The algebras $A_{2n+1}$ have been used as an auxiliary device for studying the partition algebras, by Martin and others. Halverson and Ram [25] emphasized putting the even and odd algebras on an equal footing, which reveals the role played by the basic construction. Cellularity of the partition algebras was proved in [57, 13, 53]. For further literature citations, see the review article [25].

### 5.7.3 Some properties of the partition algebras

Fix a ground ring $S$ and $\delta \in S$. In this subsection, write $A_k$ for $A_k(S,\delta)$.

For $n \geq 1$, $A_{2n-1}$ is defined as a subalgebra of $A_{2n}$. The map $\iota : X_n \to X'_{n+1}$ which adds the additional block $\{\boldsymbol{n+1}, \overline{\boldsymbol{n+1}}\}$ to $x \in X_n$ is an imbedding; the

linear extension of $\iota$ to $A_{2n}$ is a unital algebra monomorphism into $A_{2n+1}$. Using $\iota$, we identify $A_{2n}$ with its image in $A_{2n+1}$.

For $n \geq 1$, let $p_{2n-1} \in A_{2n}$ be the set partition of $[\boldsymbol{n}] \cup [\overline{\boldsymbol{n}}]$ with blocks $\{\boldsymbol{n}\}$, $\{\overline{\boldsymbol{n}}\}$, and $\{\boldsymbol{i}, \overline{\boldsymbol{i}}\}$ for $1 \leq i \leq n-1$, . The element $p_{2n-1}$ satisfies $p_{2n-1}^2 = \delta\, p_{2n-1}$. Let $p_{2n} \in A_{2n+1}$ be the set partition of $[\boldsymbol{n+1}] \cup [\overline{\boldsymbol{n+1}}]$ with blocks $\{\boldsymbol{n}, \boldsymbol{n+1}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{n+1}}\}$ and $\{\boldsymbol{i}, \overline{\boldsymbol{i}}\}$ for $1 \leq i \leq n-1$. Then $p_{2n}$ is an idempotent.

Here are graphs representing the $p_k$ for $k$ even and odd:

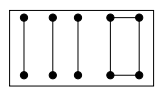
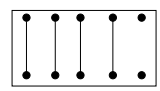
$p_8 =$  $p_9 =$ 

Figure 5.9: Idempotents in partition algebras

One can check that

$$p_k p_{k\pm1} p_k = p_k \quad \text{for all } k. \tag{5.7.1}$$

Define an involution $i$ on $X_n$ by interchanging $\boldsymbol{j}$ with $\overline{\boldsymbol{j}}$ for each $j$. The map $i$ reflects a graph $d(x)$ representing $x \in X_n$ in the line $y = 1/2$. The linear extension of $i$ to $A_n$ is an algebra involution. Note that $X'_n$ and $A_{2n-1}$ are invariant under $i$. The embeddings of $A_k$ in $A_{k+1}$ commute with the involutions. The elements $p_k$ are invariant under $i$.

Define a map $\mathrm{cl} : X_n \to X'_n$ by merging the blocks containing $\boldsymbol{n}$ and $\overline{\boldsymbol{n}}$, and define $\mathrm{cl} : A_{2n} \to A_{2n-1}$ as the linear extension of the map $\mathrm{cl} : X_n \to X'_n$.

Define a map $\mathrm{cl} : X'_n \to A_{2n-2}$ as follows: For $x \in X'_n$, if $\{\boldsymbol{n}, \overline{\boldsymbol{n}}\}$ is a block of $x$, then $\mathrm{cl}(x) = \delta\, x'$, where $x' \in X_{n-1}$ is obtained by removing the block $\{\boldsymbol{n}, \overline{\boldsymbol{n}}\}$. Otherwise, $\mathrm{cl}(x) \in X_{n-1}$ is obtained by intersecting each block of $x$ with $[\boldsymbol{n-1}] \cup [\overline{\boldsymbol{n-1}}]$. Define $\mathrm{cl} : A_{2n-1} \to A_{2n-2}$ as the linear extension of the map $\mathrm{cl} : X'_n \to A_{2n-2}$.

One can check that for all $k$, $\mathrm{cl} : A_k \to A_{k-1}$ is a non–unital $A_{k-1}$–$A_{k-1}$

bimodule map. Moreover, $\mathrm{tr} = \mathrm{cl} \circ \mathrm{cl} \circ \cdots \circ \mathrm{cl} : A_k \to A_0 \cong S$ is a non–unital trace. The trace tr can be computed as follows: given $x \in X_n$, let $d(x)$ be any graph representing $x$ and let $d'(x)$ be the graph augmented by drawing edges between each pair of vertices $\{\boldsymbol{j}, \overline{\boldsymbol{j}}\}$; then $\mathrm{tr}(x) = \delta^r$, where $r$ is the number of components of $d'(x)$.

The maps cl commute with the algebra involutions $i$, and $\mathrm{tr}(a) = \mathrm{tr}(i(a))$. Moreover,

$$p_k x p_k = \mathrm{cl}(x) p_k \quad \text{for all } x \in A_k,\ k \geq 1. \tag{5.7.2}$$

If $\delta$ is invertible, define $\varepsilon_{2n} : A_{2n} \to A_{2n-1}$ by $\varepsilon_{2n} = \mathrm{cl}$, and $\varepsilon_{2n-1} : A_{2n-1} \to A_{2n-2}$ by $\varepsilon_{2n-1} = \delta^{-1}\,\mathrm{cl}$. Then the maps $\varepsilon_k$ are unital conditional expectations, and the map $\varepsilon = \varepsilon_1 \circ \cdots \varepsilon_k : A_k \to A_0 \cong S$ is a unital trace.

Let $x \in X_n$. Call a block of $x$ a *through block* if the block has non–empty intersection with both $[\boldsymbol{n}]$ and $[\overline{\boldsymbol{n}}]$. The number of through blocks of $x$ is called the propagating number of $x$, denoted $\mathrm{pn}(x)$. Clearly, $\mathrm{pn}(x) \leq n$ for all $x \in X_n$. The only $x \in X_n$ with propagating number equal to $n$ are Brauer diagrams with only vertical strands, i.e. permutation diagrams.

If $x, y \in X_n$ and $xy = \delta^r z$, then $\mathrm{pn}(z) \leq \min\{\mathrm{pn}(x), \mathrm{pn}(y)\}$. Hence the span of the set of $x \in X_n$ with $\mathrm{pn}(x) < n$ is an ideal $J_{2n} \subset A_{2n}$. Moreover, $J_{2n-1} := J_{2n} \cap A_{2n-1}$ is the span of $x \in X'_n$ with $\mathrm{pn}(x) < n$.

**Lemma 5.7.1.** *For $n \geq 1$, $A_{2n}/J_{2n} \cong S\mathfrak{S}_n$, and $A_{2n-1}/J_{2n-1} \cong S\mathfrak{S}_{n-1}$, as algebras with involution.*

*Proof.* The span of permutation diagrams is a linear complement to $J_{2n}$, and is an $i$–invariant subalgebra of $A_{2n}$ isomorphic to $S\mathfrak{S}_n$; hence, $A_{2n}/J_{2n} \cong S\mathfrak{S}_n$. The span of permutation diagrams $\pi$ with $\pi(n) = n$ is a linear complement to $J_{2n-1}$ in $A_{2n-1}$; hence $A_{2n-1}/J_{2n-1} \cong S\mathfrak{S}_{n-1}$. □

**Lemma 5.7.2.** *For $k \geq 2$, $J_k = A_{k-1} p_{k-1} A_{k-1}$.*

*Proof.* It is straightforward to check that if $x \in X_n$ has propagating number strictly less than $n$, then $x$ can be factored as $x = x' p_{2n-1} x''$, with $x', x'' \in X'_n$. Likewise, if $n \geq 2$ and $x \in X'_n$ has propagating number strictly less than $n$, then $x$ can be factored as $x = x' p_{2n-2} x''$ with $x', x'' \in X'_{n-1}$. □

**Lemma 5.7.3.**

1. *For $k \geq 3$, $p_{k-1} A_{k-1} p_{k-1} = A_{k-2} p_{k-1}$.*

2. *$p_1 A_1 p_1 = \delta\, A_0\, p_1$.*

3. *For $k \geq 2$, $p_{k-1}$ commutes with $A_{k-2}$.*

*Proof.* Let $x \in A_{2n}$ with $n \geq 1$. Then $p_{2n-1} x p_{2n-1}$ is contained in the span of $y \in X_n$ such that $\{\boldsymbol{n}\}$ and $\{\overline{\boldsymbol{n}}\}$ are blocks of $y$, and any such $y$ can be written as $y = z p_{2n-1}$, where $z \in A_{2n-2}$.

Now consider $x \in A_{2n+1}$ with $n \geq 1$. Then $p_{2n} x p_{2n}$ is contained in the span of $y \in X'_{n+1}$ such that $\{\boldsymbol{n}, \boldsymbol{n+1}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{n+1}}\}$ is contained in one block of $y$. Any such $y$ can be written as $y = z p_{2n}$ where $z \in A_{2n-1}$.

This shows that $p_{k-1} A_k p_{k-1} \subseteq A_{k-2} p_{k-1}$ for all $k \geq 3$. On the other hand, if $x \in A_{k-2}$ then $x p_{k-1} = x p_{k-1} p_{k-2} p_{k-1} = p_{k-1} x p_{k-2} p_{k-1} \in p_{k-1} A_k p_{k-1}$, so $p_{k-1} A_k p_{k-1} \supseteq A_{k-2} p_{k-1}$. This proves (1).

Points (2) and (3) are easy to check. □

**Lemma 5.7.4.** *For $k \geq 2$, $A_k p_{k-1} = A_{k-1} p_{k-1}$. Moreover, $x \mapsto x e_{k-1}$ is injective from $A_{k-1}$ to $A_k$.*

*Proof.* For $k = 2$, we have $A_2 p_1 = S p_1 = A_1 p_1$. For $k \geq 3$, we have

$$\begin{aligned} A_k p_{k-1} &= A_k p_{k-1} p_{k-2} p_{k-1} \\ &\subseteq J_k p_{k-1} = A_{k-1} p_{k-1} A_{k-1} p_{k-1} \\ &\subseteq A_{k-1} A_{k-2} p_{k-1} = A_{k-1} p_{k-1}. \end{aligned}$$

Checking $k$ odd and even separately, one can check that $x = \mathrm{cl}(x p_{k-1})$ for $k \geq 2$ and $x \in A_{k-1}$. □

**Lemma 5.7.5.** *The trace $\varepsilon$ on $A_k(\mathbb{Q}(\boldsymbol{\delta}), \boldsymbol{\delta})$ is non–degenerate.*

*Proof.* For any set partition $x \in X_n$, let $r(x)$ be the number of blocks of $x$. Let $E_x$ be the equivalence relation on $[\boldsymbol{n}] \cup [\overline{\boldsymbol{n}}]$ whose equivalence classes are the blocks of $x$.

For any $x, y \in X_n$, define an integer $r(x, y)$ as follows: Let $E(x, y)$ be the smallest equivalence relation on $[\boldsymbol{n}] \cup [\overline{\boldsymbol{n}}]$ containing $E_x \cup E_{i(y)}$ and let $r(x, y)$ be the number of equivalence classes of $E(x, y)$. Clearly, $r(x, y) \leq \min\{r(x), r(y)\}$. Moreover, if $r(x) = r(y)$, then $r(x, y) < r(x)$ unless $y = i(x)$, and $r(x, i(x)) = r(x)$.

It is not hard to see that $\mathrm{tr}(xy) = \boldsymbol{\delta}^{r(x,y)}$, so $\varepsilon(x, y) = \boldsymbol{\delta}^{r(x,y)-n}$. It follows that the Gram determinant $\det(\varepsilon(xy))_{x,y}$ is a Laurent polynomial that has a unique term of highest degree namely $\pm \prod_x \varepsilon(x\, i(x))$. In particular the Gram determinant is non–zero. This shows that the trace on $A_{2n}(\mathbb{Q}(\boldsymbol{\delta}), \boldsymbol{\delta})$ is non–degenerate, and the same method shows that the restriction of the trace to $A_{2n-1}(\mathbb{Q}(\boldsymbol{\delta}), \boldsymbol{\delta})$ is non–degenerate. □

**Lemma 5.7.6.** *$A_k(\mathbb{Q}(\boldsymbol{\delta}), \boldsymbol{\delta})$ is split semisimple. The branching diagram for $(A_k(\mathbb{Q}(\boldsymbol{\delta}), \boldsymbol{\delta}))_{k \geq 0}$ has vertices on levels $2n$ and $2n + 1$ labeled by all Young diagrams of size $j$, $0 \leq j \leq n$. There is an edge connecting $\lambda$ on level $2n$ and $\mu$ on level $2n \pm 1$ if, and only if, $\lambda = \mu$ or $\mu$ is obtained by removing one box from $\lambda$.*

*Proof.* This is proved by Martin [38]. It can also be proved using the method of Wenzl from [51], using Lemma 5.7.5. □

### 5.7.4 Verification of framework axioms for $A_k(R, \boldsymbol{\delta})$

Take $R = \mathbb{Z}[\boldsymbol{\delta}]$, where $\boldsymbol{\delta}$ is an indeterminant. Then $R$ is the universal ground ring for the partition algebras; for any commutative ring $S$ with distinguished element $\delta$, $A_k(S, \delta) \cong A_k(R, \boldsymbol{\delta}) \otimes_R S$. Let $F = \mathbb{Q}(\boldsymbol{\delta})$ denote the field of fractions of $R$. Write $A_k = A_k(R, \boldsymbol{\delta})$. Define $Q_{2n} = Q_{2n+1} = R\mathfrak{S}_n$.

**Proposition 5.7.7.** *The two sequences of $R$–algebras $(A_k)_{k\geq 0}$ and $(Q_k)_{k\geq 0}$ satisfy the framework axioms of Section 3.1.*

*Proof.* According to Example 2.4.2, $(Q_k)_{k\geq 0}$ is a coherent tower of cellular algebras, so axiom (1) holds. Framework axioms (2) and (3) are evident. $A_k^F$ is split semisimple by Lemma 5.7.6. This verifies axiom (4).

Take $p_{k-1} \in A_k$ to be the element defined in the previous subsection. Then $p_{k-1}$ is an $i$–invariant essential idempotent with $J_k = A_k p_{k-1} A_k$. Thus $A_k/J_k \cong Q_k$ as algebras with involution by Lemma 5.7.1. This verifies axiom (5).

Axiom (6) follows from Lemma 5.7.3, and axiom (7) from Lemma 5.7.4. Axiom (8) holds because $p_{n-1}p_n p_{n-1} = p_{n-1}$. □

**Corollary 5.7.8.** *For any commutative ring $S$ and for any $\delta \in S$, the sequence of partition algebras $(A_n(S,\delta))_{n\geq 0}$ is a coherent tower of cellular algebras. $A_n(S,\delta)$ has cell modules indexed by all Young diagrams of size $j$, $0 \leq j \leq n$. The cell module labeled by a Young diagram $\lambda$ has a basis labeled by paths on the branching diagram for $(A_k(\mathbb{Q}(\boldsymbol{\delta}),\boldsymbol{\delta}))_{k\geq 0}$, described in Lemma 5.7.6.*

## 5.8 Contour algebras

We define generalizations of the *contour algebras* of Cox et. al. [9], which in turn include several sorts of diagram algebras. The algebras are obtained as a sort of wreath product of the Jones–Temperley–Lieb algebras with some other algebra $A$ with involution; varying $A$ gives a wide variety of examples.

### 5.8.1 Definition of contour algebras

Let $S$ be a commutative ring with distinguished element $\delta$. Let $A$ be an $S$–algebra with involution $i$ and with a unital $S$–valued trace $\varepsilon$. We first define the $A$–Temperley–Lieb algebras $T_n(A)$ and then the contour algebras $C_n^d(A)$ as subalgebras of $T_n(A)$. In case we need to emphasize the ground ring $S$ and parameter $\delta$, write

$C_n^d(A, S, \delta)$.

An $A$–Temperley–Lieb diagram is a Temperley–Lieb (TL) diagram with strands labeled by elements of $A$. For convenience, adopt the convention that an unlabeled strand is the same as a strand labeled with the identity of $A$.

We will define the product of two $A$–Temperley–Lieb diagrams. Note that ordinary TL diagrams have an inherent orientation. Label the top vertices of a TL diagram by $\mathbf{1}, \ldots, \boldsymbol{n}$ and the bottom vertices by $\overline{\mathbf{1}}, \ldots, \overline{\boldsymbol{n}}$. Place a small arrow pointing down at each odd numbered vertex (top or bottom) and a small arrow pointing up at each even numbered vertex. Then because of the planarity of TL diagrams, each strand of a TL diagram must connect one arrow pointing into the rectangle $\mathcal{R}$ of the diagram with one arrow pointing out of $\mathcal{R}$; the strand can be thought of as oriented from the inward pointing arrow to the outward pointing arrow. When two TL diagrams are multiplied by stacking, the orientation of composed strands agrees.

Now consider two $A$–Temperley–Lieb diagrams $X$ and $Y$. To form the product $XY$, stack $Y$ over $X$ as for tangles, forming a composite diagram $X \circ Y$. Label each non–closed composite strand with the product of the labels of its component strands from $X$ and $Y$, taken in the order of their occurrence as the strand is traversed according to its orientation. For each closed strand $s$ in $X \circ Y$, let $\varepsilon(s)$ be the trace of the product of the labels of its component strands; the product is unique up to cyclic permutation of the factors, so the trace is uniquely determined. Let $r$ be the number of closed strands and let $Z$ be the labeled diagram obtained by removing all the closed strands. Then $XY = \delta^r(\prod_s \varepsilon(s))\, Z$.

As an $S$–module, $T_n(A)$ is $A^{\otimes n} \otimes T_n(S, \delta) = \bigoplus_x (A^{\otimes n} \otimes x)$, where the sum is over ordinary Temperley–Lieb diagrams $x$. Identify a simple tensor $a_1 \otimes \cdots \otimes a_n \otimes x$ with a labeling of $x$ with the labels $a_1, \ldots, a_n$. We have to specify how to place the labels. Fix an ordering of the vertices, for example $\mathbf{1} < \cdots < \boldsymbol{n} < \overline{\mathbf{1}} < \cdots \overline{\boldsymbol{n}}$,

and then order the strands of $x$ according to the order of the initial vertex of each (oriented) strand. The simple tensor $a_1 \otimes \cdots \otimes a_n \otimes x$ is identified with the diagram with underlying TL diagram $x$, with the $j$–th strand of $x$ labeled by $a_j$ for each $j$.

Fix TL diagrams $x$ and $y$. The product of $A$–Temperley–Lieb diagrams with underlying TL diagrams $x$ and $y$, defined above, determines a multilinear map $A^{2n} \to A^{\otimes n} \otimes xy$, and hence a bilinear map $(A^{\otimes n} \otimes x) \times (A^{\otimes n} \otimes y) \to A^{\otimes n} \otimes xy$. This product extends to a bilinear product on $T_n(A)$, which one can check to be associative.

Next we define an involution on $T_n(A)$. Define $i$ on an $A$–labeled TL diagram by flipping the diagram over the line $y = 1/2$ and applying the involution in $A$ to the label of each strand. For a fixed TL diagram $x$, this gives a multilinear map from $A^n$ to $A^{\otimes n} \otimes i(x)$, and hence a linear map from $A^{\otimes n} \otimes x$ to $A^{\otimes n} \otimes i(x)$. Now $i$ extends to a linear map on $T_n(A)$. One can check that $i$ is an algebra involution.

This completes the definition of the $A$–Temperley–Lieb algebra, as an algebra with involution.

Next we define the $A$–contour algebras. Assign a depth to each strand in an ordinary TL diagram $x$, as follows: Draw a curve from a point on a given strand $s$ to the western boundary of $\mathcal{R}$, having only transverse intersections with any strands of $x$. The depth of $s$ is the minimum, over all such curves $\gamma$, of the number of points of intersection of $\gamma$ with the strands of $x$ (including $s$). The depth of an $A$–labeled TL diagram is the maximum depth of the strands with non–identity labels.

Fix $d \leq n$. As an $S$–module $C_n^d(A)$ is the span of those $A$–labeled TL diagrams of depth no greater than $d$. It is easy to check as in [9] Lemma 2.1 that $C_n^d(A)$ is an $i$–invariant subalgebra of $T_n(A)$.

For $a \in A$ and $1 \leq j \leq n$ let $a^{(j)}$ be the identity TL diagram in $T_n(A)$ with the $j$–th strand labeled with $a$ (and the other strands unlabeled). Then $a^{(j)}$ and $b^{(k)}$ commute if $j \neq k$. Also $a^{(j)}$ commutes with $e_k$ unless $j \in \{k, k+1\}$ and

$e_k a^{(k)} = e_k a^{(k+1)}$, and, likewise, $a^{(k)} e_k = a^{(k+1)} e_k$. Note that $a \mapsto a^{(k)}$ is an algebra homomorphism if $k$ is odd, but an algebra anti-homomorphism if $k$ is even.

**Lemma 5.8.1.** *$C_n^d(A)$ is generated as an algebra by $e_1, \ldots, e_{n-1}$ and by $\{a^{(k)} : 1 \leq k \leq d\}$.*

*Sketch:* It is enough to show that if $x$ is a Temperley–Lieb diagram and $X = xa^{(k)}$ has depth $r$, then $X$ can be rewritten as a product of $a^{(r)}$ and TL diagrams. First one can check that $X$ can be written as $x_1\, x_2 a^{(k')}\, x_3$ where the $x_i$ are TL diagrams, $x_2$ is a product of commuting $e_i$'s, and the depth of $x_2\, a^{(k')}$ is $r$. Finally, it suffices to show that $x_2\, a^{(k')}$ can be written as a product of TL diagrams with $a^{(r)}$. We give an example that captures the idea: $e_1 e_3 a^{(6)}$ has depth 2. We have

$$\begin{aligned} e_1 e_3 a^{(6)} &= (e_1 e_3)(e_2 e_4)(e_1 e_3) a^{(6)} \\ &= (e_1 e_3)(e_2 e_4)(e_3 e_5)(e_2 e_4)(e_1 e_3) a^{(6)} \\ &= (e_1 e_3)(e_2 e_4)(e_3 e_5) a^{(2)} (e_2 e_4)(e_1 e_3), \end{aligned}$$

by repeated use of the relations listed before the statement of the lemma.

### 5.8.2 Brief history of contour algebras

The contour algebras introduced by Cox et. al. [9] are the special case with $A$ the group algebra of the cyclic group $\mathbb{Z}_m$. On the other hand, the $A$–Temperley–Lieb algebras $T_n(A)$ have been considered in [28], Example 2.2. The contour subalgebras of $T_n(A)$ were discussed in [24].

### 5.8.3 Some properties of $T_n(A)$ and $C_n^d(A)$

We deal with the contour algebras and the $A$–Temperley–Lieb algebras together; regard $T_n(A)$ as $C_n^\infty(A)$.

Define maps $\iota : C_n^d(A) \to C_{n+1}^d(A)$ as for other classes of diagram or tangle algebras, and likewise maps $\mathrm{cl} : C_n^d(A) \to C_{n-1}^d(A)$; if closing the rightmost strand of an $A$–Temperley–Lieb diagram produces a closed loop, remove the loop and multiply the resulting diagram by $\delta$ times the trace of the product of labels along

the loop. The map $\iota$ is injective, since $x = \mathrm{cl}(\iota(x)e_n)$ for $x \in C_n^d(A)$. The maps $\iota$ and cl commute with the involutions.

If $\delta$ is invertible in $S$, define $\varepsilon_n = (1/\delta)\mathrm{cl} : C_n^d(A) \to C_{n-1}^d(A)$, which is a unital conditional expectation. We have $\varepsilon_{n+1} \circ \iota(x) = x$ for $x \in C_n^d(A)$. The map $\varepsilon = \varepsilon_1 \circ \cdots \circ \varepsilon_n : C_n^d(A) \to C_0^d(A) \cong S$ is a normalized trace. The value of $\varepsilon$ on an $A$-Temperley–Lieb diagram $X$ with $n$ strands is obtained as follows: first close all the strands of $X$ by introducing new curves joining $\boldsymbol{j}$ to $\overline{\boldsymbol{j}}$ for all $j$; let $r$ be the number of closed loops in the resulting diagram; then $\varepsilon(X) = \delta^{r-n} \prod_s \varepsilon(s)$, where the product is over the collection of closed loops $s$, and $\varepsilon(s)$ denotes the trace in $A$ of the product of labels along the loop $s$.

The span $J$ of $A$–Temperley–Lieb diagrams of depth $\leq d$ and with at least one horizontal strand is an ideal in $C_n^d(A)$. By Lemma 5.8.1, any $A$–Temperley–Lieb diagram with depth $\leq d$ can be written as a word in the $e_i$'s and in elements $a^{(k)}$ with $k \leq d$; the diagram is in $J$ if, and only if, some $e_i$ appears in the word. Thus $J$ is the ideal generated by the $e_i$'s. Because of the relations $e_i e_{i\pm1} e_i = e_i$, $J$ is generated by $e_{n-1}$. The quotient $C_n^d(A)/J$ is isomorphic (as algebras with involution) to the subalgebra generated by the $a^{(k)}$ with $k \leq d$, and thus to $A^{\otimes d}$ if $n \geq d$ and $A^{\otimes n}$ if $n < d$.

**Lemma 5.8.2.**

1. *For $n \geq 3$, $e_{n-1}\, C_{n-1}^d(A)\, e_{n-1} = C_{n-2}^d(A)\, e_{n-1}$.*

2. *$e_1\, C_1^d(A)\, e_1 = \delta\, S\, e_1$*

3. *For $n \geq 2$, $e_{n-1}$ commutes with $C_{n-2}^d(A)$.*

*Proof.* The proof is the same as that of Lemma 5.2.2 for the Brauer algebras. □

**Lemma 5.8.3.** *For $n \geq 2$, $C_n^d(A)\, e_{n-1} = C_{n-1}^d(A)\, e_{n-1}$. Moreover, $x \mapsto xe_{n-1}$ is injective from $C_{n-1}^d(A)$ to $C_{n-1}^d(A)\, e_{n-1}$.*

*Proof.* Any $A$–TL diagram in $C_n^d(A)$ is either already in $C_{n-1}^d(A)$, or it can be written as $\alpha\chi\beta$, with $\alpha, \beta \in C_{n-1}^d(A)$, and $\chi \in \{e_{n-1}, a^{(n)}\}$ if $n \leq d$, or $\chi = e_{n-1}$ if $n > d$.

The remainder of the proof is the same as the proof of Lemma 5.2.3 for the Brauer algebras, using the identities: $a^{(n)}xe_{n-1} = xa^{(n-1)}e_{n-1}$, and $e_{n-1}xe_{n-1} = \mathrm{cl}(x)e_{n-1}$ for $x \in C_{n-1}^d(A)$. □

### 5.8.4 Hypotheses on the algebra $A$

We will suppose that the algebra $A$ has a generic version defined over an integral domain $R_0$. Let $F_0$ be the field of fractions of $R_0$. We suppose that $A = A(R_0)$ satisfies the following hypotheses:

1. $A = A(R_0)$ is cellular.
2. $A(F_0) = A(R_0) \otimes_{R_0} F_0$ is split semisimple.
3. The trace $\varepsilon$ on $A(R_0)$ is non–degenerate.

Take $R = R_0[\boldsymbol{\delta}]$, where $\boldsymbol{\delta}$ is an indeterminant, and let $F = F_0(\boldsymbol{\delta})$ denote the field of fractions of $R$. We will show that $(C_n^d(A, R, \boldsymbol{\delta}))_{n\geq 0}$ is a coherent tower of cellular algebras.

### 5.8.5 Special instances

The cellular algebra $A$ in Subsection 5.8.4 can be taken to be the generic version of any of the diagram or tangle algebras treated in this thesis. $A$ could be taken to be a generic Hecke algebra or cyclotomic Hecke algebra, or the group ring of a symmetric group over $R_0 = \mathbb{Z}$.

The contour algebras of Cox et. al. [9] are recovered by taking $R_0 = \mathbb{Z}[\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_{m-1}]$ and $A$ the group algebra of $\mathbb{Z}_m$ over $R_0$. The trace on $A$ is determined by $\varepsilon([k]) = \boldsymbol{\delta}_k$ for $[k] \neq [0]$ and $\varepsilon([0]) = 1$. The parameter $\delta_0$ in [9]

becomes identified with our $\delta$.

### 5.8.6 Verification of the framework axioms for $C_n^d(A)$

Adopt the hypotheses and notation of Subsection 5.8.4.

**Lemma 5.8.4.** *The trace $\varepsilon$ on $C_n^d(A, F, \boldsymbol{\delta})$ is non–degenerate.*

*Proof.* Take any basis $\mathbb{A}$ of $A$ over $F_0$ with $\mathbf{1} \in \mathbb{A}$. As a basis $\mathbb{B}$ of $C_n^d(A)$ over $F$ we take all $n$–strand TL diagrams decorated up to depth $d$ with elements of $\mathbb{A}$. Consider the modified Gram determinant $\det[\varepsilon(Xi(Y))]_{X,Y\in\mathbb{B}}$. If $X$ and $Y$ have different underlying TL diagrams, then $\varepsilon(Xi(Y)) \in \delta^{-1}F_0$.

Next consider matrix entries $\varepsilon(Xi(Y))$ where $X$ and $Y$ have the same underlying TL diagram, say $x$. Suppose $x$ has $\ell$ strands at depth $d$ or less and these strands are decorated by basis elements $a_1, \ldots, a_\ell$ in $X$, respectively $b_1, \ldots, b_\ell$ in $Y$. Then $\varepsilon(Xi(Y)) = \prod_{j=1}^{\ell} \varepsilon(a_j i(b_j))$. The determinant of the square submatrix of $[\varepsilon(Xi(Y))]$ consisting of those entries for which $X$ and $Y$ both have underlying TL diagram $x$ is therefore $D^\ell$, where $D$ is the determinant of $[\varepsilon(ai(b))]_{a,b\in\mathbb{A}}$. It follows that $\det[\varepsilon(Xi(Y))]_{X,Y\in\mathbb{B}}$ is equal to a power of $D$ modulo $\delta^{-1}R_0$, and is therefore non–zero. □

Consider

$$Q_n = C_n^d(A)/J \cong \begin{cases} A^{\otimes n} & \text{if } n < d \\ A^{\otimes d} & \text{if } n \geq d. \end{cases}$$

By the assumptions in Subsection 5.8.4, $Q_n(R)$ is cellular and $Q_n(F)$ is split semisimple. Moreover, it is easy to see that $(Q_n)_{n\geq 0}$ is a coherent tower of cellular algebras.

**Lemma 5.8.5.** *$C_n^d(A, F, \boldsymbol{\delta})$ is split semisimple for all $n$.*

*Proof.* The method of Wenzl from [51] applies, using the non–degeneracy of the trace and the split semisimplicity of $Q_n(F)$ for all $n$. □

**Proposition 5.8.6.** *The pair of sequences $(C_n^d(A, R, \boldsymbol{\delta}))_{n\geq 0}$ and $(Q_n(R))_{n\geq 0}$ satisfy the framework axioms of Section 3.1. Hence, $(C_n^d(A, R, \boldsymbol{\delta}))_{n\geq 0}$ is a coherent tower*

*of cellular algebras.*

*Proof.* We observed above that $(Q_k)_{k\geq 0}$ is a coherent tower of cellular algebras, so axiom (1) holds. Framework axioms (2) and (3) are evident. Framework axiom (4) follows from Lemma 5.8.5.

The elements $e_k$ are $i$–invariant essential idempotents. With $J = C_k^d(A)e_{k-1}C_k^d(A)$, we have $C_k^d(A)/J \cong Q_k$ as algebras with involution. This verifies axiom (5). Axiom (6) follows from Lemma 5.8.2, and axiom (7) from Lemma 5.8.3. Axiom (8) holds because $e_{n-1}e_ne_{n-1} = e_{n-1}$. $\square$

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