Fall 2009

The growth of the quantum hyperbolic invariants of the figure eight knot

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THE GROWTH OF THE QUANTUM HYPERBOLIC INVARIANTS OF THE FIGURE EIGHT KNOT

by

Heather Michelle Mollé

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy degree
in Mathematics in
the Graduate College of
The University of Iowa

December 2009

Thesis Supervisor: Professor Charles Frohman

ABSTRACT

Baseilhac and Benedetti have created a quantum hyperbolic knot invariant similar to the colored Jones polynomial. Their invariant is based on the polyhedral decomposition of the knot complement into ideal tetrahedra. The edges of the tetrahedra are assigned cross ratios based on their interior angles. Additionally, these edges are decorated with charges and flattenings which can be determined by assigning weights to the longitude and meridian of the boundary torus of a neighborhood of the knot. Baseilhac and Benedetti then use a summation of matrix dilogarithms to get their invariants. This thesis investigates these invariants for the figure eight knot. In fact, it will be shown that,

$$\lim_{N\to\infty} \frac{\log |H_N(4_1)|}{N} \leq \frac{1}{2\pi} Vol(S^3 \setminus 4_1)$$

where $H_N(4_1)$ are the quantum hyperbolic invariants of the figure eight knot.

Abstract Approved: ______________________________
Thesis Supervisor

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Title and Department

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Date

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the December 2009 graduation.

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I would like to thank everyone who has supported me over the last six years.

ABSTRACT

Baseilhac and Benedetti have created a quantum hyperbolic knot invariant similar to the colored Jones polynomial. Their invariant is based on the polyhedral decomposition of the knot complement into ideal tetrahedra. The edges of the tetrahedra are assigned cross ratios based on their interior angles. Additionally, these edges are decorated with charges and flattenings which can be determined by assigning weights to the longitude and meridian of the boundary torus of a neighborhood of the knot. Baseilhac and Benedetti then use a summation of matrix dilogarithms to get their invariants. This thesis investigates these invariants for the figure eight knot. In fact, it will be shown that,

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INTRODUCTION

1.1 Motivation

A topological knot is similar to a knot one might tie in a piece of string. However, after tying the knot, the two ends of the string are glued together. This way the knot cannot be untied without breaking the string. Yet it can be tangled up so that it is difficult to tell what, if any, knot is present. Knot theorists study knot invariants in order to determine whether a certain knot diagram contains a knot. One such invariant is the Jones polynomial $J(K)$. The Jones polynomial assigns a polynomial to a knot diagram by resolving the crossings. If two knot diagrams represent the same knot, they will have the same Jones polynomial. However, two different knots may have the same Jones polynomial. A more complex version of the Jones polynomial is the colored Jones polynomial $J_N(K)$. To compute the colored Jones polynomial, strands of the knot are first "colored" by integers between 1 and N. Another knot invariant is the volume of a knot, or rather, the volume of the complement of the knot. No matter how tangled up a knot may get, the volume of its complement in hyperbolic space remains the same. It is believed that the colored Jones polynomial is related to the volume of the complement of the knot.

The Volume Conjecture states that for any knot K,

$$\lim_{N\to\infty} \frac{\log |J_N(K)|_{e^{\frac{i\pi}{N}}}|}{N} = \frac{1}{2\pi} Vol(S^3 \setminus K)$$

This conjecture has already been proven for several specific examples including the figure eight knot, the 5_2 knot, and torus knots. The Volume Conjecture is interesting in its own right as it relates an algebraic quantity with a geometric one. It is of even greater interest though as it has been proven [6] that if the Volume Conjecture is true, then a knot diagram is unknotted if and only if its Vassiliev invariants are trivial.

Baseilhac and Benedetti have created a quantum hyperbolic knot invariant similar to the colored Jones polynomial [2, 3]. Their invariant is based on the polyhedral decomposition of the knot complement into ideal tetrahedra. The edges of the tetrahedra are assigned cross ratios based on their interior angles. Additionally, these edges are decorated with charges and flattenings which can be determined by assigning weights to the longitude and meridian of the boundary torus of a neighborhood of the knot. Baseilhac and Benedetti then use a summation of matrix dilogarithms to get their invariants. This thesis investigates these invariants for the figure eight knot. In fact, it will be shown that,

$$\lim_{N\to\infty} \frac{\log |H_N(4_1)|}{N} \leq \frac{1}{2\pi} Vol(S^3 \setminus 4_1)$$

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1.2 Overview

Chapter 2 provides definitions and background information that will be necessary for the remainder of this thesis. Included are a brief introduction to hyperbolic geometry, a discussion on möbius transformations, and a thorough treatment of the Lobachevsky function. In particular, an expression for the volume of an ideal tetrahedron involving the Lobachevsky function is derived.

Chapter 3 discusses the decomposition of a knot complement into ideal tetrahedra. The tetrahedra are labeled according to a branching, and values are assigned to the edges of the tetrahedra according to gluing consistency conditions. The figure eight knot is presented as an example.

In Chapter 4, the quantum hyperbolic invariants are defined. The invariants are calculated for the figure eight knot.

Chapter 5 explores the growth of the quantum hyperbolic invariants of the figure eight knot. Using simple analysis and the Lobachevsky function, an upper bound for the growth of these invariants is found.

Chapter 6 provides some ideas for future research. Since only an upper bound was found in this thesis, naturally, the next step would be to look for a lower bound. A few techniques for finding a lower bound will be discussed. Additionally, the quantum hyperbolic invariants of Baseilhac and Benedetti can be computed not only for knot complements, but for other hyperbolic manifolds as well. Two manifolds of interest are the sister manifold of the complement of the figure eight knot and the n-sheeted covers of the figure eight knot.

CHAPTER 2
DEFINITIONS AND BACKGROUND

2.1 Hyperbolic geometry

The biggest difference between Euclidean geometry and hyperbolic geometry is the parallel postulate. In Euclidean geometry, the parallel postulate states:

Postulate 2.1. *Given a line l and a point p not on l, there is exactly one line through p which does not intersect l.*

However, in hyperbolic geometry, this postulate reads:

Postulate 2.2. *Given a line l and a point p not on l, there are at least two lines through p which do not intersect l.*

This is illustrated in the upper half-plane model for hyperbolic geometry $\mathbb{H}^2$. In this model, hyperbolic space is represented by all points above the x-axis. *Lines* are vertical lines as well as half-circles perpendicular to the x-axis. In Figure 2.1, the *lines* m and n both pass through point p and are both considered parallel to the *line* l. One advantage of this particular model is that it is conformal; that is, it preserves angles.

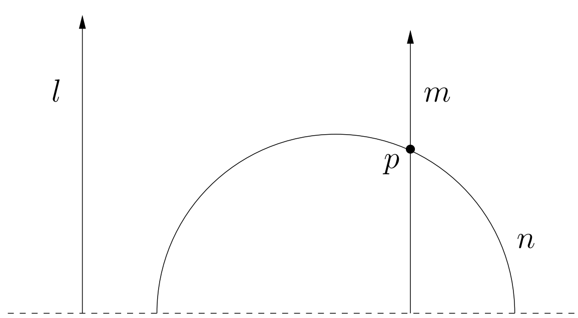


Figure 2.1: Three *lines* in the upper half-plane model

The upper half-plane model of two-dimensional hyperbolic space can be extended to a three-dimensional model. The upper half-space model of hyperbolic space $\mathbb{H}^3$ contains all points above the xy-plane. *Lines* are still either vertical lines or half-circles perpendicular to the x-axis. *Planes* are vertical planes as well as half-spheres perpendicular to the xy-plane.

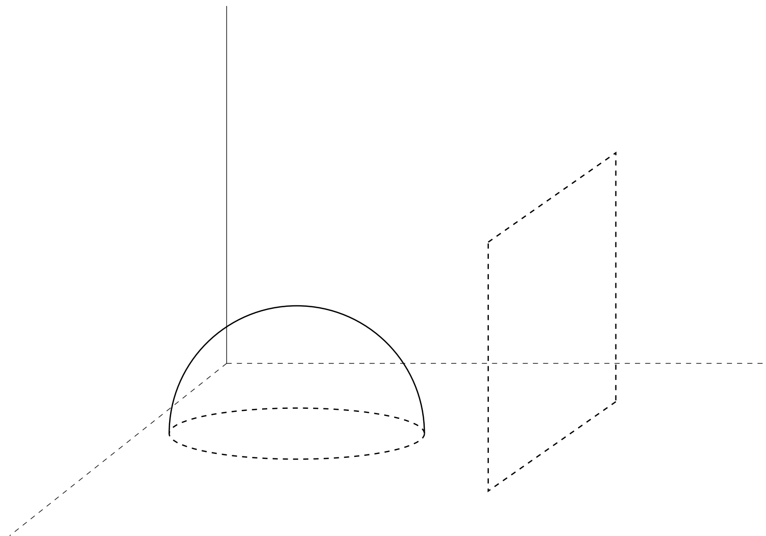


Figure 2.2: Two *planes* in the upper half-space model

Definition 2.3. An ideal tetrahedron is a tetrahedron with its four vertices at infinity.

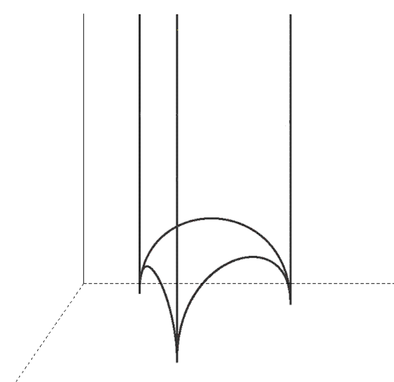


Figure 2.3: An ideal tetrahedron in $\mathbb{H}^3$

In the upper half-space model, an ideal tetrahedron can be constructed by placing three vertices on the xy-plane and the fourth vertex at the infinity of the z-axis as shown in Figure 2.3.

Theorem 2.4. *The dihedral angles of opposite edges of an ideal tetrahedron are equal.*

Proof. Consider an ideal tetrahedron with dihedral angles α, β, and γ around one vertex. Let α' be the dihedral angle opposite α, β' be the dihedral angle opposite β, and γ' be the dihedral angle opposite γ. Since the upper half-space model is conformal, around each vertex the dihedral angles sum to π. Hence

$$\alpha + \beta + \gamma = \pi$$
$$\alpha + \beta' + \gamma' = \pi$$
$$\alpha' + \beta' + \gamma = \pi$$
$$\alpha' + \beta + \gamma' = \pi$$

Combining the first two equations and last two equations gives

$$2\alpha + (\beta + \beta') + (\gamma + \gamma') = 2\pi \text{ and } 2\alpha' + (\beta + \beta') + (\gamma + \gamma') = 2\pi.$$

Thus $\alpha = \alpha'$. Similarly, $\beta = \beta'$ and $\gamma = \gamma'$ □

2.2 Möbius transformations

For this section, it will be necessary to consider $\mathbb{H}^3$ as a subset of $\mathbb{C}^2$. Thus all points in $\mathbb{H}^3$ are also points in $\mathbb{C}^2$.

Definition 2.5. A möbius transformation is a rational function of the form

$$f(z) = \frac{az + b}{cz + d}$$

where $ad - bc \neq 0$.

Theorem 2.6. *All isometries of the upper half-space model are möbius transformations.*

Proof. See [4]. □

Definition 2.7. The cross ratio of four distinct points is given by

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

If z_n is ∞ for $n \in 1, 2, 3, 4$, divide both the numerator and denominator by z_n and take $\frac{1}{\infty} = 0$ [4].

Theorem 2.8. *Möbius transformations preserve cross ratios.*

Proof. Let $f(z) = \frac{az+b}{cz+d}$ be a möbius transformation, and let z_1, z_2, z_3, and z_4, be four distinct points. Consider the cross ratio

$$\begin{aligned}
(f((z_1), f(z_2); f(z_3), f(z_4)) &= \frac{(f(z_1) - f(z_3))(f(z_2) - f(z_4))}{(f(z_1) - f(z_4))(f(z_2) - f(z_3))} \\
&= \frac{(\frac{az_1+b}{cz_1+d} - \frac{az_3+b}{cz_3+d})(\frac{az_2+b}{cz_2+d} - \frac{az_4+b}{cz_4+d})}{(\frac{az_1+b}{cz_1+d} - \frac{az_4+b}{cz_4+d})(\frac{az_2+b}{cz_2+d} - \frac{az_3+b}{cz_3+d})} \\
&= \frac{(\frac{(az_1+b)(cz_3+d)-(az_3+b)(cz_1+d)}{(cz_1+d)(cz_3+d)})(\frac{(az_2+b)(cz_4+d)-(az_4+b)(cz_2+d)}{(cz_2+d)(cz_4+d)})}{(\frac{(az_1+b)(cz_4+d)-(az_4+b)(cz_1+d)}{(cz_1+d)(cz_4+d)})(\frac{(az_2+b)(cz_3+d)-(az_3+b)(cz_2+d)}{(cz_2+d)(cz_3+d)})} \\
&= \frac{\frac{((az_1+b)(cz_3+d)-(az_3+b)(cz_1+d))((az_2+b)(cz_4+d)-(az_4+b)(cz_2+d))}{(cz_1+d)(cz_3+d)(cz_2+d)(cz_4+d)}}{\frac{((az_1+b)(cz_4+d)-(az_4+b)(cz_1+d))((az_2+b)(cz_3+d)-(az_3+b)(cz_2+d))}{(cz_1+d)(cz_4+d)(cz_2+d)(cz_3+d)}} \\
&= \frac{(az_1 + b)(cz_3 + d) - (az_3 + b)(cz_1 + d)}{(az_1 + b)(cz_4 + d) - (az_4 + b)(cz_1 + d)} \\
&\quad * \frac{(az_2 + b)(cz_4 + d) - (az_4 + b)(cz_2 + d)}{(az_2 + b)(cz_3 + d) - (az_3 + b)(cz_2 + d)} \\
&= \frac{(bcz_3 + adz_1 - adz_3 - bcz_1)(bcz_4 + adz_2 - adz_4 - bcz_2)}{(bcz_4 + adz_1 - adz_4 - bcz_1)(bcz_3 + adz_2 - adz_3 - bcz_2)} \\
&= \frac{(ad - bc)^2(z_1 - z_3)(z_2 - z_4)}{(ad - bc)^2(z_1 - z_4)(z_2 - z_3)} \\
&= \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \\
&= (z_1, z_2; z_3, z_4).
\end{aligned}$$

□

Theorem 2.9. *A möbius transformation is completely determined by specifying where any three points are sent.*

Proof. Let z_1, z_2, and z_3 be three distinct points in $\mathbb{H}^3 \subseteq \mathbb{C}^2$. Let f be a möbius transformation which sends z_1 to w_1, z_2 to w_2, and z_3 to w_3. Let z be any point in $\mathbb{H}^3 \subseteq \mathbb{C}^2$. Then $\frac{(z_1-z_3)(z_2-z)}{(z_1-z)(z_2-z_3)} = \frac{(w_1-w_3)(w_2-f(z))}{(w_1-f(z))(w_2-w_3)}$ by Theorem 2.8. Thus $f(z) = \frac{(z_1-z)(z_2-z_3)(w_1-w_3)w_2-(z_1-z_3)(z_2-z)(w_2-w_3)w_1}{(z_1-z)(z_2-z_3)(w_1-w_3)-(z_1-z_3)(z_2-z)(w_2-w_3)}$. □

Consider the four vertices of an ideal tetrahedron. These points can be considered elements of $\mathbb{C} \cup \{\infty\}$, since they either lie on the xy-plane or are the infinity of the z-axis. So a möbius transformation could be used to send three of the vertices of an ideal tetrahedron to the points 0, 1, and ∞. The fourth vertex would be sent to a point z. Taking the cross ratio of these vertices yields $(z, 1; 0, \infty) = z$. Thus each ideal tetrahedron can be uniquely described by the cross ratio of its vertices. Of course, this depends on an ordering of the vertices.

Consider an ideal tetrahedron with vertices at points a, b, c, and d. The möbius transformation $f_1(z) = \frac{(b-c)z-a(b-c)}{(b-a)z-c(b-a)}$ sends a to 0, b to 1, and c to ∞. The fourth vertex d will be sent to $\frac{(d-a)(b-c)}{(d-c)(b-a)} = (d, b; a, c) = w$. Taking the intersection of the tetrahedron under the möbius transformation f_1 with a plane parallel to the xy-plane results in a Euclidean triangle with vertices at 0, 1, and w in $\mathbb{C}$. The interior angle of this triangle at 0 is $\arg(w)$. Thus the edge of the tetrahedron between vertex a and vertex c is assigned the cross ratio w as the dihedral angle at this edge is $\arg(w)$. The möbius transformation $f_2(z) = \frac{(d-c)z-b(d-c)}{(d-b)z-c(d-b)}$ sends b to 0, d to 1, and c to ∞. The fourth vertex a will be sent to $\frac{(a-b)(d-c)}{(a-c)(d-b)} = (a, d; b, c) = \frac{1}{1-w}$. Taking the intersection of the tetrahedron under the möbius transformation f_2 with a plane parallel to the xy-plane results in a Euclidean triangle with vertices at 0, 1, and $\frac{1}{1-w}$ in $\mathbb{C}$. The interior angle of this triangle at 0 is $\arg(\frac{1}{1-w})$. Thus the edge of the tetrahedron between vertex b and vertex c is assigned the cross ratio $\frac{1}{1-w}$ as the dihedral angle between this edge is $\arg(\frac{1}{1-w})$. Finally, the möbius transformation $f_3(z) = \frac{(a-c)z-d(a-c)}{(a-d)z-c(a-d)}$ sends d to 0, a to 1, and c to ∞. The fourth vertex b will be sent to $\frac{(b-d)(a-c)}{(b-c)(a-d)} = (b, a; d, c) = \frac{w-1}{w}$. Taking the intersection of the tetrahedron under the möbius transformation f_3 with a plane parallel to the xy-plane results in a Euclidean triangle with vertices at 0, 1, and $\frac{w-1}{w}$ in $\mathbb{C}$. The interior angle of this triangle at 0 is $\arg(\frac{w-1}{w})$. Thus the edge of the tetrahedron between vertex d and vertex c is assigned the cross ratio $\frac{w-1}{w}$ as the dihedral angle between this edge is $\arg(\frac{w-1}{w})$. Since the dihedral angles of opposite edges are equal, Theorem 2.4, opposite edges are assigned the same cross ratios. However, these can also be determined using möbius transformations as above.

2.3 The Lobachevsky function

This section directly follows chapter 7 of Thurston's notes [8].

Definition 2.10. For an angle θ, the Lobachevsky function $\Lambda(\theta)$ is given by

$$\Lambda(\theta) = -\int_0^\theta \log|2\sin u|du$$

.

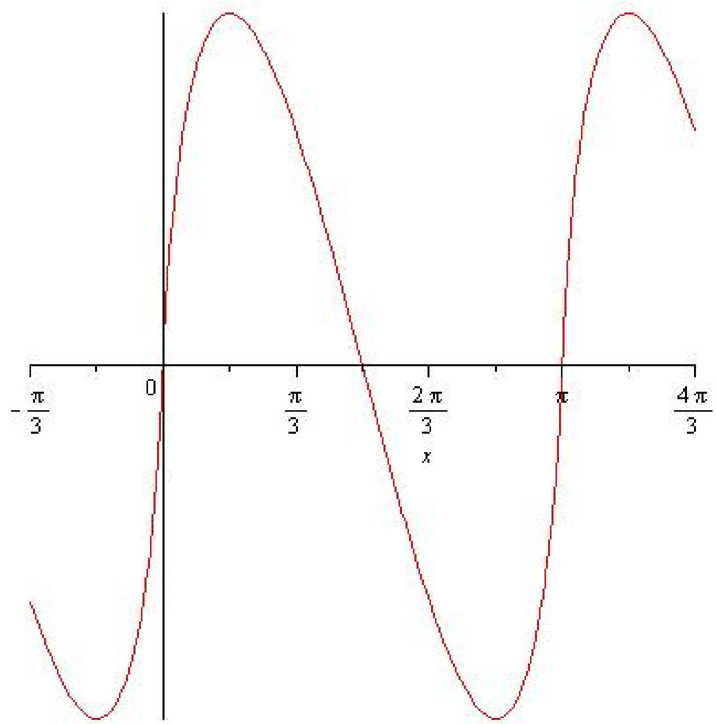


Figure 2.4: The Lobachevsky function

Theorem 2.11. *The Lobachevsky function achieves a maximum at* $\theta = \frac{\pi}{6}$.

Proof. The maximum can be found by setting the derivative of $\Lambda(\theta)$ equal to zero and solving for θ.

$$\begin{aligned}
d/d\theta(\Lambda(\theta)) &= 0 \\
d/d\theta(-\int_0^\theta \log|2\sin u|du) &= 0 \\
-\log|2\sin\theta| &= 0 \\
e^0 &= 2\sin\theta \\
\frac{1}{2} &= \sin\theta \\
\theta &= \frac{\pi}{6}
\end{aligned}$$

□

Lemma 2.12. *The Lobachevsky function has a uniformly convergent Fourier series expansion*

$$\Lambda(\theta) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(2n\theta)}{n^2}$$

when $0 \le \theta \le \pi$.

Proof. Recall that $\log(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}$. Let $|z| \le 1$, and define $\Psi(z) = -\int_0^z \frac{1}{w}\log(1-w)dw$. Then

$$\begin{aligned}
\Psi(z) &= -\int_0^z \frac{1}{w}\log(1-w)dw \\
&= -\int_0^z \frac{1}{w}(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{((1-w)-1)^n}{n})dw \\
&= -\int_0^z \frac{1}{w}(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n w^n}{n})dw \\
&= -\int_0^z \sum_{n=1}^{\infty}(-1)^{2n+1}\frac{w^{n-1}}{n}dw \\
&= \int_0^z \sum_{n=1}^{\infty}\frac{w^{n-1}}{n}dw \\
&= \sum_{n=1}^{\infty}\int_0^z \frac{w^{n-1}}{n}dw \\
&= \sum_{n=1}^{\infty}\frac{z^n}{n^2}
\end{aligned}$$

Now consider $\frac{1}{w}\log(1-w)dw$ when $w = e^{2i\phi}$ and $dw = 2ie^{2i\phi}d\phi$ for $0 \le \phi \le \pi$.

$$\begin{aligned}
\frac{1}{w}\log(1-w)dw &= \frac{1}{e^{2i\phi}}\log(1-e^{2i\phi})2ie^{2i\phi}d\phi \\
&= 2i\log(-e^{i\phi}(-e^{-i\phi}+e^{i\phi}))d\phi \\
&= 2i\log(-2ie^{i\phi}\frac{e^{i\phi}-e^{-i\phi}}{2i})d\phi \\
&= 2i\log(-2ie^{i\phi}\sin\phi)d\phi \\
&= 2i(\log(-i)+\log(e^{i\phi})+\log(2\sin\phi))d\phi \\
&= 2i(\frac{-i\pi}{2}+i\phi+\log(2\sin\phi))d\phi \\
&= (\pi-2\phi+2i\log(2\sin\phi))d\phi
\end{aligned}$$

Then

$$\begin{aligned}
\Psi(e^{2i\theta})-\Psi(1) &= -\int_0^{e^{2i\theta}}\frac{1}{w}\log(1-w)dw+\int_0^1\frac{1}{w}\log(1-w)dw \\
&= -\int_0^{e^{2i\theta}}\frac{1}{w}\log(1-w)dw-\int_1^0\frac{1}{w}\log(1-w)dw \\
&= -\int_1^{e^{2i\theta}}\frac{1}{w}\log(1-w)dw \\
&= -\int_0^{\theta}(\pi-2\phi+2i\log(2\sin\phi))d\phi \\
&= -(\pi\phi-\phi^2)|_0^\theta-2i\int_0^\theta\log(2\sin\phi)d\phi \\
&= -\pi\theta+\theta^2+2i\Lambda(\theta)
\end{aligned}$$

Taking the imaginary parts of both sides yields

$$\begin{aligned}
Im[-\pi\theta+\theta^2+2i\Lambda(\theta)] &= Im[\Psi(e^{2i\theta})-\Psi(1)] \\
2\Lambda(\theta) &= Im[\sum_{n=1}^{\infty}\frac{e^{2i\theta n}}{n^2}-\sum_{n=1}^{\infty}\frac{1^n}{n^2}] \\
2\Lambda(\theta) &= Im[\sum_{n=1}^{\infty}\frac{\cos(2n\theta)+i\sin(2n\theta)}{n^2}-\sum_{n=1}^{\infty}\frac{1}{n^2}] \\
2\Lambda(\theta) &= \sum_{n=1}^{\infty}\frac{\sin(2n\theta)}{n^2} \\
\Lambda(\theta) &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(2n\theta)}{n^2}
\end{aligned}$$

□

Lemma 2.13. *The function $\Lambda(\theta)$ is periodic of period π.*

Proof. Using Lemma 2.12, $\Lambda(0) = \Lambda(\pi) = 0$. Furthermore, $\frac{d}{d\theta}\Lambda(\theta) = -\log|2\sin\theta|$ which is periodic with period π. Hence, $\Lambda(\theta)$ is periodic of period π. $\square$

Lemma 2.14. *The function $\Lambda(\theta)$ is an odd function.*

Proof. Consider $\Lambda(-\theta) = -\int_0^{-\theta} \log|2\sin u|du$. Substituting $v = -u$ results in $\int_0^{-\theta} \log|2\sin v|dv = -\Lambda(\theta)$. $\square$

Lemma 2.15. *For any integer $n \neq 0$, $\Lambda(n\theta) = \sum\limits_{j \mod n} n\Lambda(\theta + \frac{j\pi}{n})$.*

Proof. Let $z = e^{2iu}$ in the identity $z^n - 1 = \prod\limits_{k=0}^{n-1}(z - e^{\frac{-2\pi ik}{n}})$. This yields $2\sin(nu) = \prod\limits_{k=0}^{n-1} 2\sin(u + \frac{k\pi}{n})$. Taking the logarithm of both sides and the multiplying by n gives $-n\log(2\sin(nu)) = -n\sum\limits_{k=0}^{n-1}\log(2\sin(u + \frac{k\pi}{n})$. Finally, integrating from 0 to θ gives the desired result. $\square$

The Lobachevsky function can be used to compute the volume of an ideal tetrahedron. Let $S_{\alpha,\beta,\gamma}$ be a tetrahedron with one vertex at infinity and three right dihedral angles as shown in Figure 2.5. Note that $\beta = \frac{\pi}{2} - \alpha$.

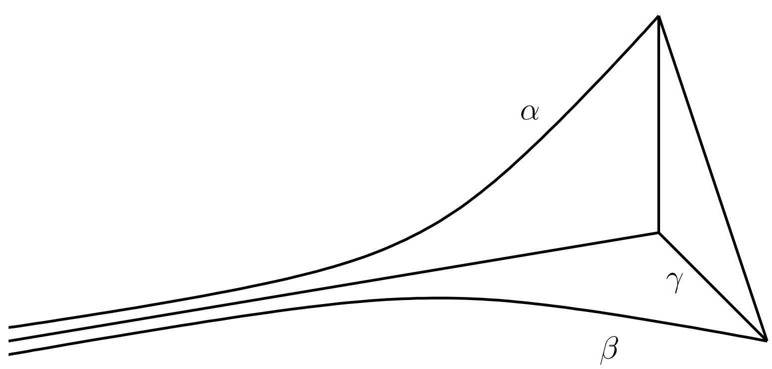


Figure 2.5: A tetrahedron with one ideal vertex

Lemma 2.16. *The volume of $S_{\alpha,\frac{\pi}{2}-\alpha,\gamma}$ equals $\frac{1}{4}[\Lambda(\alpha+\gamma) + \Lambda(\alpha-\gamma) + 2\Lambda(\frac{\pi}{2} - \alpha)]$.*

Proof. Consider $S_{\alpha,\frac{\pi}{2}-\alpha,\gamma}$ in the upper half-space model as shown in Figure 2.6. The infinite vertex is at ∞, and the other vertices lie on the unit hemisphere. Note that one of the vertices on the unit sphere projects to the origin of the xy-plane.

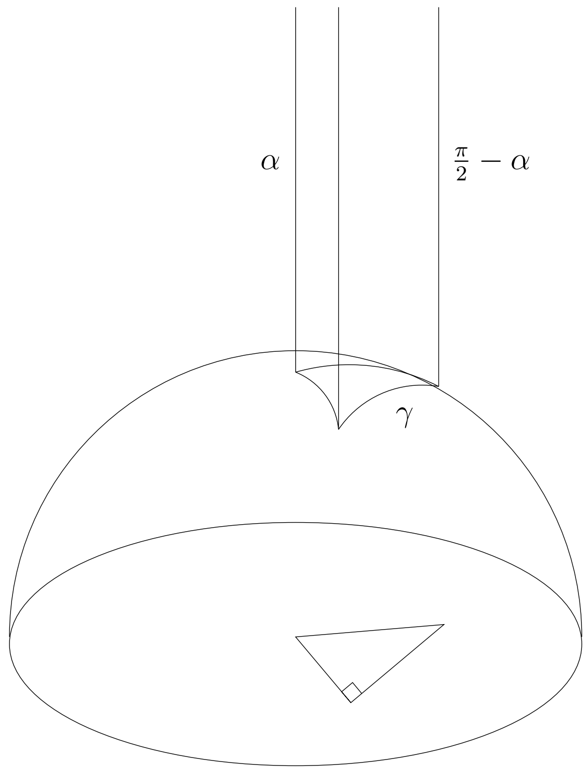


Figure 2.6: A tetrahedron with three vertices on the unit sphere and one at ∞

Recall that the volume element for the hyperbolic metric is $dV = \frac{dxdydz}{z^3}$. Thus the volume of this tetrahedron is computed as follows

$$\begin{aligned} Vol(S_{\alpha,\frac{\pi}{2}-\alpha,\gamma}) &= \int_0^{\cos\gamma}\int_0^{x\tan\alpha}\int_{\sqrt{1-x^2-y^2}}^{\infty}\frac{1}{z^3}dzdydx \\ &= \int_0^{\cos\gamma}\int_0^{x\tan\alpha}\frac{1}{2(1-x^2-y^2)}dydx. \end{aligned}$$

Making the substitution $x = \cos\theta$ gives

$$\begin{aligned}
Vol(S_{\alpha,\frac{\pi}{2}-\alpha,\gamma}) &= \int_{\frac{\pi}{2}}^{\gamma}\int_0^{\cos\theta\tan\alpha} \frac{1}{2(1-\cos^2(\theta)-y^2)}dy\sin\theta d\theta \\
&= -\int_{\frac{\pi}{2}}^{\gamma}\frac{1}{4}\log\frac{2\sin(\theta+\alpha)}{2\sin(\theta-\alpha)}d\theta \\
&= \frac{1}{4}[-\int_{\frac{\pi}{2}}^{0}\log(2\sin(\theta+\alpha)) - \int_0^{\gamma}\log(2\sin(\theta+\alpha)) \\
&\quad + \int_{\frac{\pi}{2}}^{0}\log(2\sin(\theta-\alpha)) + \int_0^{\gamma}\log(2\sin(\theta+\alpha))] \\
&= \frac{1}{4}[-\Lambda(\frac{\pi}{2}+\alpha) + \Lambda(\gamma+\alpha) + \Lambda(\frac{\pi}{2}-\alpha) - \Lambda(\gamma-\alpha)].
\end{aligned}$$

Finally, Lemmas 2.13 and 2.14 give $\frac{1}{4}[\Lambda(\alpha+\gamma)+\Lambda(\alpha-\gamma)+2\Lambda(\frac{\pi}{2}-\alpha)]$. $\square$

Notice that if a second vertex is placed at infinity, then $\alpha = \gamma$ and the tetrahedron has volume $\frac{1}{4}[\Lambda(2\alpha)+2\Lambda(\frac{\pi}{2}-\alpha)]$. By Lemmas 2.13, 2.14, and 2.15, this reduces to $\frac{1}{2}\Lambda(\alpha)$.

Now let $\Sigma_{\alpha,\beta,\gamma}$ be an ideal tetrahedron with dihedral angles α, β, and γ.

Theorem 2.17. *The volume of* $\Sigma_{\alpha,\beta,\gamma}$ *equals* $\Lambda(\alpha)+\Lambda(\beta)+\Lambda(\gamma)$.

Proof. Observe that an ideal tetrahedron can be placed in the upper half-space model so that one vertex is at ∞ and the other three vertices lie on the unit circle of the xy-plane.

Case 1 Suppose the tetrahedron projects onto the xy-plane with the center of the unit circle inside the projection triangle as shown in Figure 2.7.

This leads to the system of equations

$$\alpha+\beta+\gamma=\pi$$

$$A+B+C+D+E+F=2\pi$$

$$G+H=\alpha$$

$$I+J=\beta$$

$$K+L=\gamma$$

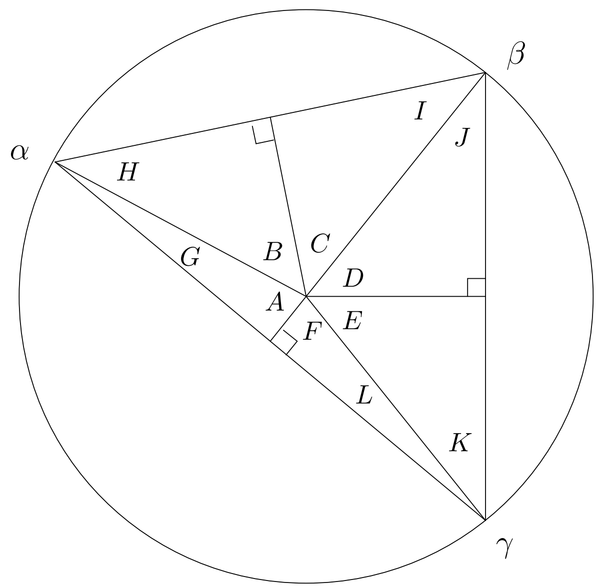


Figure 2.7: One possible projection of an ideal tetrahedron to the xy-plane

$$
\begin{aligned}
A + G &= \frac{\pi}{2} & A &= F \\
B + H &= \frac{\pi}{2} & B &= C \\
C + I &= \frac{\pi}{2} & D &= E \\
D + J &= \frac{\pi}{2} & G &= L \\
E + K &= \frac{\pi}{2} & H &= I \\
F + L &= \frac{\pi}{2} & J &= K
\end{aligned}
$$

which solves to give

$$A = \beta,\ B = \gamma,\ C = \gamma,\ D = \alpha,\ E = \alpha,\ F = \beta,$$

$$G = \tfrac{\pi}{2} - \beta,\ H = \tfrac{\pi}{2} - \gamma,\ I = \tfrac{\pi}{2} - \gamma,\ J = \tfrac{\pi}{2} - \alpha,\ K = \tfrac{\pi}{2} - \alpha, \text{ and } L = \tfrac{\pi}{2} - \beta.$$

Hence $Vol(\Sigma_{\alpha,\beta,\gamma}) = 2Vol(S_{\alpha,\frac{\pi}{2}-\alpha,\alpha}) + 2Vol(S_{\beta,\frac{\pi}{2}-\beta,\beta}) + 2Vol(S_{\gamma,\frac{\pi}{2}-\gamma,\gamma}) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$.

Case 2 Suppose the tetrahedron projects onto the xy-plane with the center of the unit circle outside the projection triangle as shown in Figure 2.8.

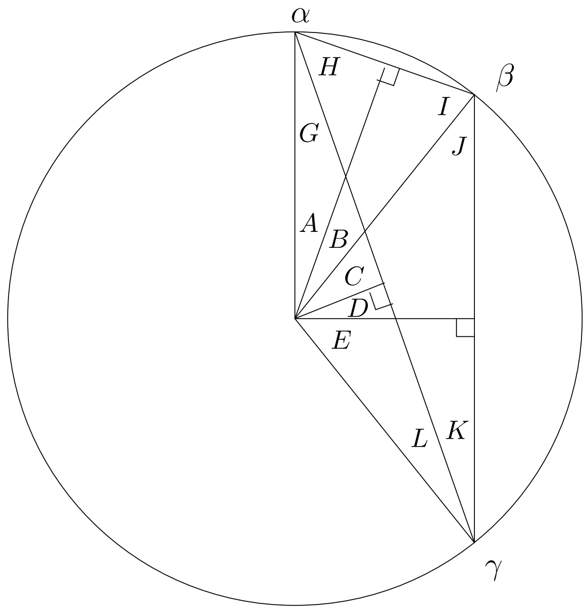


Figure 2.8: The other possible projection of an ideal tetrahedron to the xy-plane

This leads to the system of equations

$$\alpha + \beta + \gamma = \pi$$

$$G + H = \alpha$$

$$I + J = \beta$$

$$K + L = \gamma$$

$$\begin{aligned} A + B + C + G &= \frac{\pi}{2} & A + B + C &= D + E \\ A + G + H &= \frac{\pi}{2} & A &= B \\ B + I &= \frac{\pi}{2} & C + D &= E \\ C + D + J &= \frac{\pi}{2} & G &= L \\ E + K + L &= \frac{\pi}{2} & G + H &= I \\ D + E + L &= \frac{\pi}{2} & J &= K + L \end{aligned}$$

which solves to give

$$A = \gamma,\ B = \gamma,\ C + D = \alpha,\ E = \alpha,\ A + B + C = \pi - \beta,\ D + E = \pi - \beta,$$

$G = \frac{\pi}{2} - (\pi - \beta)$, $H + G = \frac{\pi}{2} - \gamma$, $I = \frac{\pi}{2} - \gamma$, $J = \frac{\pi}{2} - \alpha$, $K + L = \frac{\pi}{2} - (\pi - \alpha)$, and

$$L = \frac{\pi}{2} - (\pi - \beta).$$

So $Vol(\Sigma_{\alpha,\beta,\gamma}) = 2Vol(S_{\alpha,\frac{\pi}{2}-\alpha,\alpha}) - 2Vol(S_{(\pi-\beta),\frac{\pi}{2}-(\pi-\beta),\beta}) + 2Vol(S_{\gamma,\frac{\pi}{2}-\gamma,\gamma})$. By Lemma 2.16, $Vol(S_{\pi-\beta,\frac{\pi}{2}-(\pi-\beta),\beta}) = \frac{1}{4}[\Lambda(\beta+(\pi-\beta)) - \Lambda(\beta-(\pi-\beta)) + 2\Lambda(\frac{\pi}{2}-(\pi-\beta))] = \frac{1}{4}[-\Lambda(2\beta) + 2\Lambda(\beta-\frac{\pi}{2})]$. By Lemma 2.15, $-\Lambda(2\beta) + 2\Lambda(\beta - \frac{\pi}{2}) = -2\Lambda(\beta) - 2\Lambda(\beta+\frac{\pi}{2}) + 2\Lambda(\beta-\frac{\pi}{2}) = -2\Lambda(\beta)$. Hence, $Vol(\Sigma_{\alpha,\beta,\gamma}) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$. $\square$

CHAPTER 3
THE COMPLEMENT OF A HYPERBOLIC KNOT

This chapter will follow the ideas and the notation of Baseilhac and Benedetti [2, 3].

3.1 The decomposition of a knot complement

Consider a hyperbolic knot embedded in S^3. It may be useful to think of S^3 as the one point compactification of $\mathbb{R}^3$.

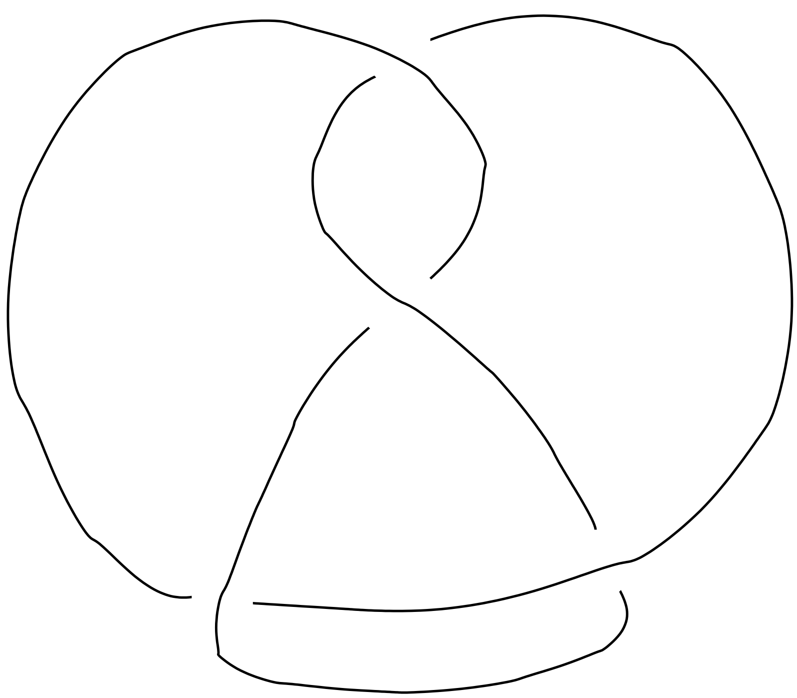


Figure 3.1: A knot diagram of the figure eight knot

The knot diagram lies on a plane dividing $\mathbb{R}^3$ in two. Imagine the complement of the knot as being composed of polyhedra which are glued together along that plane. Within the plane, the knot diagram divides the plane into several regions. Regions which are surrounded by three or more crossings will be the n-gonal faces of the polyhedra. The edges of these n-gons are represented by edges near the crossings while the vertices of these edges are smushed out along the knot. Regions which are surrounded by only two crossings are bigons. The bigons are labeled with an arrow. The edges which surround the bigons will be glued together along these arrows.

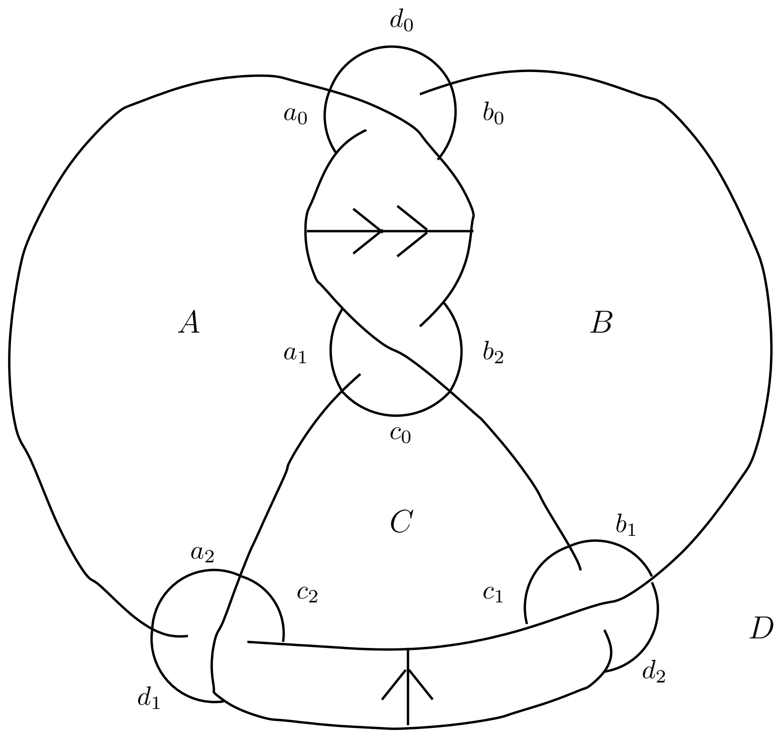
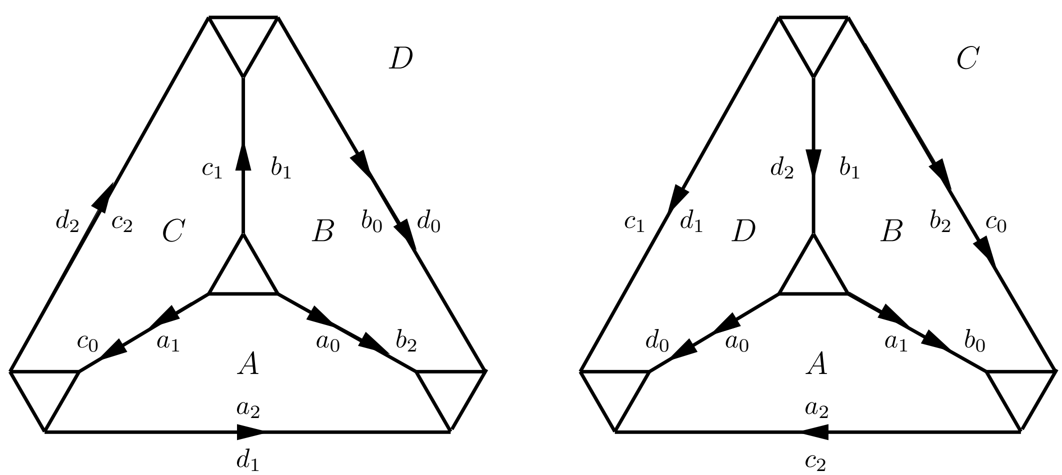


Figure 3.2: The knot diagram divides the plane into several regions

To construct the polyhedron above the plane, slide each labeled edge along an over crossing until it meets up with another labeled edge. Together, these two edges will form one edge of the polyhedron. For example, in Figure 3.2, edge a_0 and b_2 match up to form one edge of the tetrahedron. Additionally this edge will be given an arrow. Slide either a_0 or b_2 along the knot until it meets up with one of the arrows from the bigons. In this case, a_0 meets up with a double arrow which points from a_1 to a_2. Likewise, b_2 meets up with a double arrow which points from b_1 to b_0.

To construct the polyhedron below the plane, slide each labeled edge along an under crossing until it meets up with another labeled edge. Together, these two edges will form one edge of the polyhedron. For example, in Figure 3.2, edge a_0 and d_0 match up to form one edge of the tetrahedron. Additionally this edge will be given an arrow. Slide either a_0 or d_0 along the knot until it meets up with one of the arrows from the bigons. In this case, a_0 meets up with a double arrow which points from a_1 to a_2. Likewise, d_0 meets up with a double arrow which points from d_2 to d_1.

Since the vertices of the polyhedra lie on the knot, the complement of a hyperbolic knot is actually composed of two truncated polyhedra. To reproduce the complement of the hyperbolic knot from the truncated polyhedra, glue each face of one polyhedron to its corresponding face on the other polyhedron. Then glue together the edges which are marked with the same arrow type.



the tetrahedron above the plane the tetrahedron below the plane

Figure 3.3: The two tetrahedra of the figure eight knot complement

In the case of the figure eight knot, the complement decomposes into two tetrahedra. For other hyperbolic knots, the complement will decompose into two polyhedra which then require a triangulation into tetrahedra. An appropriate triangulation is one which induces a branching on each tetrahedron. That is each vertex can be labeled 0, 1, 2, or 3 based on how many arrows point towards that vertex. The vertex which has three arrows pointing away from it will be labeled 0, the vertex with two arrows pointing away from it and one arrow pointing towards it will be labeled 1, the vertex with one arrow pointing away from it and two arrows pointing towards it will be labeled 2, and the vertex which has three arrows pointing towards it will be labeled 3. Arrows must always point from a vertex labeled by a smaller number to a vertex labeled by a larger number. Not all triangulations will induce a branching.

The tetrahedra are assigned an orientation based on the branching. Via a right hand rule, with the thumb pointing in the direction of vertex 3 and the fingers curling around the remaining vertices, the tetrahedron is given a positive orientation, $*b = +1$, if the fingers curl in the order $0-1-2$ and a negative orientation, $*b = -1$, if the fingers curl in the order $0 - 2 - 1$.

The edges of the tetrahedra will be labeled according to the vertices. The edge between vertex 0 and vertex 1 will be e_0, the edge between vertex 1 and vertex 2 will be e_1, and the edge between vertex 0 and vertex 2 will be e_2. The edges opposite e_0, e_1, and e_2 will be labeled e_0', e_1', and e_2', respectively. However, all values will be chosen so that opposite edges are equal.

The four faces of the tetrahedra will be labeled as well. The face opposite vertex 0 is labeled j, the face opposite vertex 1 is labeled l, the face opposite vertex 2 is labeled i, and the face opposite vertex 3 is labeled k.

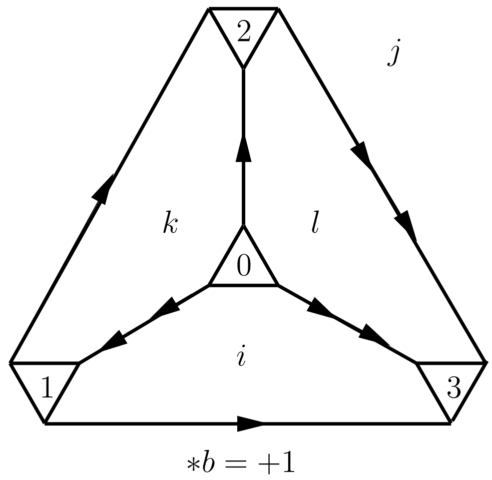


Figure 3.4: The branching for the figure eight knot complement

3.2 The homology of the figure eight knot

The triangles that result from truncating the tetrahedra glue together to form the boundary of a neighborhood of the knot. The boundary of the neighborhood will be a torus; however, the longitude and meridian will not be the standard longitude and meridian because the torus is knotted. To determine the longitude and meridian of the boundary torus, it will be necessary to look at the homology of the knot.

Consider the complement of the figure eight knot. Imagine the two tetrahedra are connected by strings. An α string connects the A face of the bottom tetrahedron to the A face of the top tetrahedron. Similarly, a β string connects the B face of the

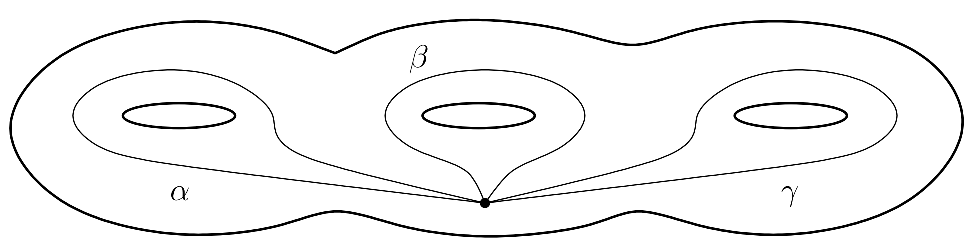


Figure 3.5: A genus three handle body with generators α, β, and γ

bottom tetrahedron to the B face of the top tetrahedron, a γ string connects the C face of the bottom tetrahedron to the C face of the top tetrahedron, and a δ string connects the D face of the bottom tetrahedron to the D face of the top tetrahedron. Gluing the tetrahedra together along the D faces removes the δ string, but turns the other three strings into loops. Continue gluing the faces together. This results in a genus three handlebody with α, β, and γ as shown in Figure 3.5. Gluing the single

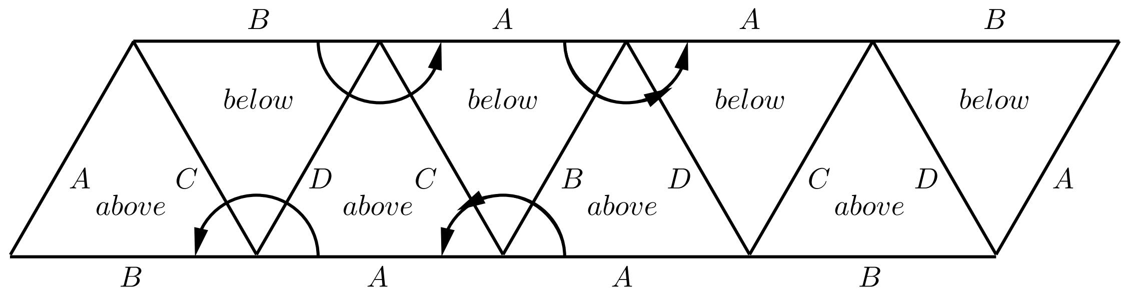


Figure 3.6: The boundary torus of the figure eight knot

and double arrowed edges together corresponds to adding two 2-handles along the curves $\gamma\beta^{-1}\gamma^{-1}\alpha$ and $\beta^{-1}\gamma\alpha^{-1}\beta\alpha$, respectively. Thus the fundamental group of the complement of the figure eight knot is

$$\pi_1(S^3 \setminus 4_1) = \left\langle \alpha, \beta, \gamma | \gamma\beta^{-1}\gamma^{-1}\alpha, \beta^{-1}\gamma\alpha^{-1}\beta\alpha \right\rangle$$

Abelianizing this group gives the first homology of the complement of the figure eight knot

$$H_1(S^3 \setminus 4_1) = \langle \alpha, \beta, \gamma | \alpha - \beta = 0, \gamma = 0 \rangle = \langle \alpha \rangle$$

The meridinal curve of the boundary torus must correspond to the generator of this group while the longitude must be null homologous to the group. Notice that the standard meridian corresponds to α, the generator of the first homology. The standard choice of longitude, however, is not null homologous. It corresponds to the word

$$-\gamma - \gamma + \beta + \gamma + \alpha = \beta + \alpha$$

Adding two additional twists gives the word

$$-\gamma - \gamma + \beta + \gamma - \beta - \beta + \alpha = 0$$

Thus the meridian and longitude are as shown in Figure 3.7. The double arrowed curve represents the meridian while the single arrowed curve represents the longitude.

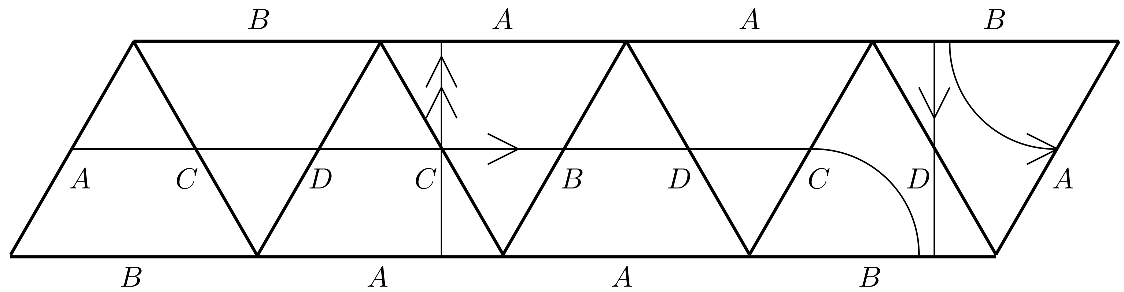


Figure 3.7: The longitude and meridian of the boundary torus

3.3 Cross ratios, charges, and flattenings

Each edge of the tetrahedra is assigned a value determined by the cross ratio of its vertices. The edge e_n will be assigned the cross ratio w_n, for $n \in 0, 1, 2$. Recall that an ideal tetrahedron can be uniquely described by the cross ratio of its vertices up to an ordering of the vertices. Thus

$$w_0 = w,\ w_1 = \tfrac{1}{1-w},\ \text{and}\ w_2 = \tfrac{w-1}{w}.$$

and opposite edges will have the same cross ratio. This imposes the condition that around any vertex, the cross ratios will multiply to -1.

$$w_0 w_1 w_2 = w\frac{1}{1-w}\frac{w-1}{w} = -1$$

which corresponds with the condition that around any ideal vertex, the dihedral angles sum to 2π. Additionally, the cross ratios must satisfy consistency conditions from gluing the tetrahedra together. The product of the cross ratios, where cross ratios from tetrahedra with $*b = -1$ will be given negative exponents, around any edge must be 1. Thus for every arrow type there will be another relation. For the complement of the figure eight knot, the consistency conditions are

$$(w_0^+)^2(w_1^+)(w_1^-)^{-1}(w_2^-)^{-2} = 1$$

from the single arrows, and

$$(w_1^+)(w_2^+)^2(w_0^-)^{-2}(w_1^-)^{-1} = 1$$

from the double arrows. Finally, to ensure the complete hyperbolic structure, the derivative of the holonomy for the meridian and longitude must equal 1 giving two additional relations. For the complement of the figure eight knot, these are

$$w_0^- w_0^+ = 1 \text{ and } w_0^+ w_1^- w_1^+ w_2^- w_0^+ w_1^- (w_2^+)^{-1} w_0^- w_0^+ (w_1^-)^{-1} = 1.$$

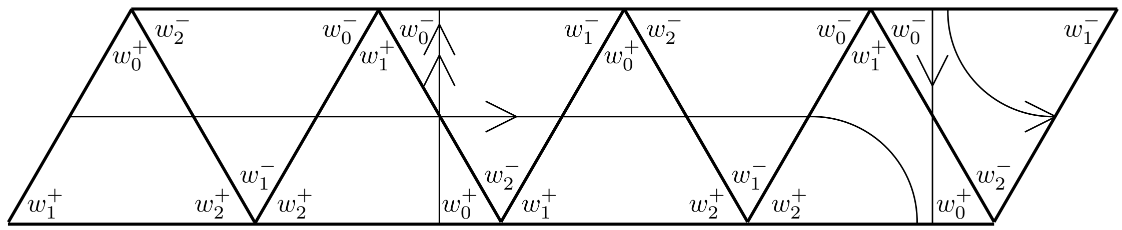


Figure 3.8: The cross ratios on the boundary torus

Solving this system of equations gives cross ratios

$$w_0^+ = w_1^+ = w_2^+ = e^{\frac{i\pi}{3}} \text{ and } w_0^- = w_1^- = w_2^- = e^{\frac{-i\pi}{3}}$$

for the complement of the figure eight knot. The triple (w_0, w_1, w_2) is considered nondegenerate if the imaginary parts of the cross ratios are not 0. In this case, the tetrahedron is assigned an orientation $*w$ which is 1 if the imaginary parts are positive and -1 if the imaginary parts are negative. For the complement of the figure eight knot $*w = *b$ for both tetrahedra.

Each edge of the tetrahedra is also assigned a charge. The edge e_n will be assigned the charge c_n, for $n \in 0, 1, 2$. The sum of the charges around any vertex must add up to 1.

$$c_0 + c_1 + c_2 = 1$$

Thus, as in Theorem 2.4, the charges of opposite edges will be equal. Additionally, the charges must satisfy consistency conditions from gluing the two tetrahedra together. The sum of the charges around any edge must be 2. Thus for every arrow there will be another relation. For the complement of the figure eight knot the consistency conditions are

$$2c_0^+ + c_1^+ + c_1^- + 2c_2^- = 2$$

from the single arrows, and

$$c_1^+ + 2c_2^+ + 2c_0^- + c_1^- = 2$$

from the double arrows. Finally, the charges depend on values given to the meridian and longitude of the boundary torus via the homomorphism $H_1(\partial M) \to \mathbb{Z}$ giving two additional relations. For the complement of the figure eight knot, these are

$$-c_0^+ + c_0^- = \mu \text{ and } c_0^+ - c_1^- + c_1^+ - c_2^- + c_0^+ - c_1^- - c_2^+ - c_0^- + c_0^+ + c_1^- = \lambda.$$

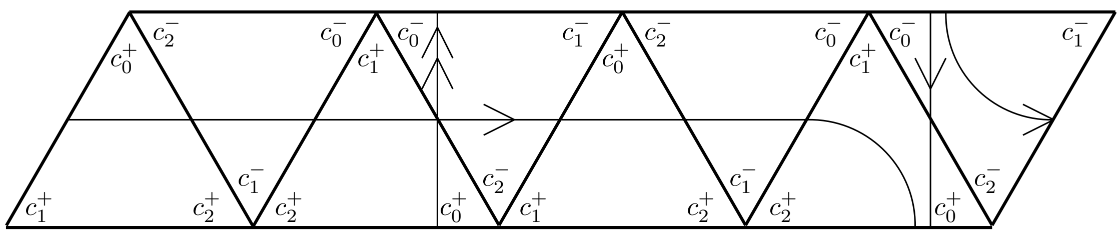


Figure 3.9: The charges on the boundary torus

Solving this system of equations gives charges

$$\begin{aligned}
c_0^+ &= c & c_0^- &= c + \mu \\
c_1^+ &= 1 - 2c + \frac{1}{2}\lambda & c_1^- &= 1 - 2c + \frac{1}{2}\lambda - 2\mu \\
c_2^+ &= c - \frac{1}{2}\lambda & c_2^- &= c - \frac{1}{2}\lambda + \mu
\end{aligned}$$

for the complement of the figure eight knot.

Lastly, each edge of the tetrahedra is assigned a flattening. The edge e_n will be assigned the flattening f_n, for $n \in 0, 1, 2$. Consider the log-branch $l = \log(w) + i\pi f$. The sum of the log-branches around any vertex must add up to 0.

$$\log(w_0) + i\pi f_0 + \log(w_1) + i\pi f_1 + \log(w_2) + i\pi f_3 = 0$$

which leads to the condition

$$f_0 + f_1 + f_2 = -(\arg(w_0) + \arg(w_1) + \arg(w_2))$$

Thus, as in Theorem 2.4, the flattenings of opposite edges will be equal. In the case of the figure eight knot, these relations are

$$f_0^+ + f_1^+ + f_2^+ = -1 \text{ and } f_0^- + f_1^- + f_2^- = 1.$$

Additionally, the flattenings must satisfy consistency conditions from gluing the two tetrahedra together. The sum of the log-branches around an edge must be 0, where the log-branch from a tetrahedron with $*b = +1$ is positive and the log-branch from a tetrahedron with $*b = -1$ is negative. For the figure eight knot, these consistency conditions are

$$2(\log(w_0^+) + i\pi f_0^+) + \log(w_1^+) + i\pi f_1^+ - (\log(w_1^-) + i\pi f_1^-) - 2(\log(w_2^-) + i\pi f_2^-) = 0$$

from the single arrows, and

$$\log(w_1^+) + i\pi f_1^+ + 2(\log(w_2^+) + i\pi f_2^+) - 2(\log(w_0^-) + i\pi f_0^-) - (\log(w_1^-) + i\pi f_1^-) = 0$$

from the double arrows. These conditions reduce to

$$2f_0^+ + f_1^+ - f_1^- - 2f_2^- = -2$$

and

$$f_1^+ + 2f_2^+ - 2f_0^- - f_1^- = -2$$

Finally, the flattenings depend on values given to the meridian and longitude of

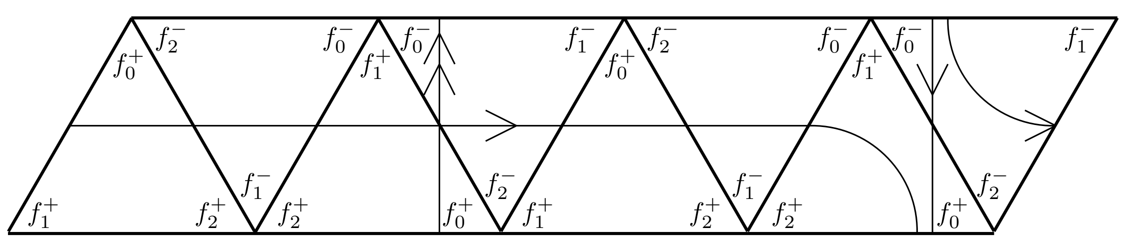


Figure 3.10: The flattenings on the boundary torus

the boundary torus via the homomorphism $H_1(\partial M) \to \mathbb{Z}$. In this case, the trivial homomorphism is used. Thus

$$-f_0^+ - f_0^- = 0 \text{ and } f_0^+ + f_1^- + f_1^+ + f_2^- + f_0^+ + f_1^- - f_2^+ + f_0^- + f_0^+ - f_1^- = 0.$$

Solving this system of equations gives flattenings

$$
\begin{aligned}
f_0^+ &= f & \qquad f_0^- &= -f \\
f_1^+ &= -1 - 2f & \qquad f_1^- &= 1 + 2f \\
f_2^+ &= f & \qquad f_2^- &= -f
\end{aligned}
$$

for the complement of the figure eight knot.

CHAPTER 4
QUANTUM HYPERBOLIC INVARIANTS

This chapter defines the quantum hyperbolic invariants as discussed by Baseilhac and Benedetti [2, 3].

4.1 Defining the invariants

Let ξ be the primitive Nth root of unity $\xi = e^{\frac{2i\pi}{N}}$ and define the Nth root of x as $x^{\frac{1}{N}} = e^{\frac{\log x}{N}} = e^{\frac{\log|x| + i \arg x}{N}}$ for nonzero x and $x^{\frac{1}{N}} = 0$ when $x = 0$. Let $\delta(n) = 1$ if $n \equiv 0 \mod N$ and $\delta(n) = 0$ otherwise. Also let $[x] = \frac{1-x^N}{N(1-x)}$. Define $g(x) = \prod_{j=0}^{N-1}(1 - x\xi^{-j})^{\frac{j}{N}}$ and set $h(x) = \frac{g(x)}{g(1)}$.

Definition 4.1. Let (Δ, b, w, c, f) be an ideal tetrahedron with branching b giving orientation, $*b$, cross ratios w_0, w_1, and w_2, charges c_0, c_1, and c_2, and flattenings f_0, f_1, and f_2. For $j \in 0, 1, 2$, let w'_j be the Nth root of w_j

$$w'_j = e^{\frac{log(w_j) + i\pi(N+1)(f_j - *bc_j)}{N}}$$

Definition 4.2. For any $u', v' \in \mathbb{C}$ such that $(u')^N + (v')^N = 1$ and any $n \in \mathbb{N}$, let

$$\omega(u', v'|n) = \prod_{j=1}^{n} \frac{v'}{1 - u'\xi^j}$$

and set $\omega(u', v'|0) = 1$.

Proposition 4.3. *For any $u', v' \in \mathbb{C}$ such that $(u')^N + (v')^N = 1$, $\omega(u', v'|n)$ is periodic of period N.*

Proof. Observe that

$$\begin{aligned}
\omega(u', v'|N) &= \prod_{j=1}^{N} \frac{v'}{1 - u'\xi^j} \\
&= \frac{(v')^N}{\prod_{j=1}^{N} 1 - u'\xi^j} \\
&= \frac{(v')^N}{1 - (u')^N} \\
&= 1
\end{aligned}$$

So $\omega(u', v'|0) = \omega(u', v'|N)$. Also, for any nonzero $n \in \mathbb{N}$,

$$\begin{aligned}
\omega(u', v'|n+N) &= \prod_{j=1}^{N} \frac{v'}{1-u'\xi^j} \prod_{j=N+1}^{N+n} \frac{v'}{1-u'\xi^j} \\
&= \prod_{j=N+1}^{N+n} \frac{v'}{1-u'\xi^j} \\
&= \prod_{j=1}^{n} \frac{v'}{1-u'\xi^{N+j}} \\
&= \prod_{j=1}^{n} \frac{v'}{1-u'\xi^N\xi^j} \\
&= \prod_{j=1}^{n} \frac{v'}{1-u'\xi^j} \\
&= \omega(u', v'|n)
\end{aligned}$$

Hence $\omega(u', v'|n)$ is periodic with period N. □

Using Proposition 4.3, $\omega(u', v'|n)$ can be extended to all $n \in \mathbb{Z}$.

Proposition 4.4. *For any $u', v' \in \mathbb{C}$ such that $(u')^N + (v')^N = 1$,*

$$(\omega((u')^*, (v')^*|n)^* = (\omega(\frac{u'}{\xi}, v'|N-n))^{-1}.$$

Proof. Consider

$$\begin{aligned}
(\omega((u')^*, (v')^*|n)^* &= (\prod_{k=1}^{n} \frac{(v')^*}{1-(u')^*\xi^k})^* \\
&= \prod_{k=1}^{n} \frac{(v')^{**}}{1-(u')^{**}(\xi^k)^*} \\
&= \prod_{k=1}^{n} \frac{v'}{1-u'\xi^{-k}}
\end{aligned}$$

Since $\prod_{k=1}^{N} v' = (v')^N$, it is the case that $\prod_{k=1}^{n} v' = (v')^N \prod_{k=n+1}^{N} \frac{1}{v'}$, and since $\prod_{k=1}^{N} \frac{1}{1-u'\xi^{-k}} = \frac{1}{1-(u')^N}$, it is the case that $\prod_{k=1}^{n} \frac{1}{1-u'\xi^{-k}} = \frac{1}{1-(u')^N} \prod_{k=n+1}^{N} (1-u'\xi^{-k})$. So

$$\begin{aligned}
\prod_{k=1}^{n} \frac{v'}{1-u'\xi^{-k}} &= \frac{(v')^N}{1-(u')^N} \prod_{k=n+1}^{N} \frac{1-u'\xi^{-k}}{v'} \\
&= \prod_{k=n+1}^{N} \frac{1-u'\xi^{-k}}{v'}
\end{aligned}$$

Now let $j = N - k + 1$. Then

$$\begin{aligned}\prod_{k=n+1}^{N} \frac{1 - u'\xi^{-k}}{v'} &= \prod_{j=1} N - n\frac{1 - u'\xi^{j-1-N}}{v'} \\ &= \prod_{j=1}^{N-n} \frac{1 - u'\xi^{j-1}\xi^{-N}}{v'} \\ &= \prod_{j=1}^{N-n} \frac{1 - (\frac{u'}{\xi})\xi^{j}}{v'} \\ &= (\omega(\frac{u'}{\xi}, v'|N-n))^{-1}\end{aligned}$$

Hence $(\omega((u')^*, (v')^*|n)^* = (\omega(\frac{u'}{\xi}, v'|N-n))^{-1}$. $\square$

Definition 4.5. Define the matrix dilogarithm of level N for $N \geq 3$ to be

$$R_N(\Delta, b, w, c, f) = ((w_0')^{-c_1}(w_1')^{c_0})^{\frac{N-1}{2}} L_N^{*b}(w_0', \frac{1}{w_1'})$$

where

$$L_N^{+1}(u', v')^{i\,j}_{k\,l} = h(u')\xi^{kj + \frac{N+1}{2}k^2}\omega(u', v'|i-k)\delta(i+j-l)$$

and

$$L_N^{-1}(u', v')^{k\,l}_{i\,j} = \frac{[u']}{h(u')}\xi^{-kj - \frac{N+1}{2}k^2}\frac{\delta(i+j-l)}{\omega(\frac{u'}{\xi}, v'|i-k)}$$

Note that an index will be assigned to each of the faces i, j, k, and l. Any pair of faces that is glued together will always receive the same index.

Definition 4.6. The quantum hyperbolic invariant of a knot complement K is given by the quantum hyperbolic state sums

$$H_N(K) = \sum_{face\ labelings} \prod_{n=1}^{t} R_N(\Delta_n, b_n, w_n, c_n, f_n)$$

where t is the number of tetrahedra in the decomposition.

4.2 The invariants of the figure eight knot

The complement of the figure eight knot is composed of two tetrahedra Δ^+, the tetrahedron with $*b = +1$, and Δ^-, the tetrahedron with $*b = -1$, which are glued together in the following way: face i^+ is glued to face j^-, face j^+ is glued to face i^-, face k^+ is glued to face l^-, and face l^+ is glued to face k^-. Thus these faces must be given the same index. Hence the quantum hyperbolic invariants of the figure eight knot are

$$\begin{aligned}
H_N(4_1) &= \sum_{\alpha,\beta,\gamma,\delta=0}^{N-1} R_N(\Delta^+, b^+, w^+, c^+, f^+)_{\gamma\ \delta}^{\alpha\ \beta} R_N(\Delta^-, b^-, w^-, c^-, f^-)_{\beta\ \alpha}^{\delta\ \gamma} \\
&= \sum_{\alpha,\beta,\gamma,\delta=0}^{N-1} ((w_0^{+\prime})^{-c_1^+}(w_1^{+\prime})^{c_0^+})^{\frac{N-1}{2}} h(w_0^{+\prime}) \xi^{\gamma\beta+\frac{N+1}{2}\gamma^2} \omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|\alpha-\gamma) \\
&\quad * \delta(\alpha+\beta-\delta)((w_0^{-\prime})^{-c_1^-}(w_1^{-\prime})^{c_0^-})^{\frac{N-1}{2}} \frac{[w_0^{-\prime}]}{h(w_0^{-\prime})} \xi^{-\delta\alpha-\frac{N+1}{2}\delta^2} \frac{\delta(\beta+\alpha-\gamma)}{\omega(\frac{w_0^{-\prime}}{\xi}, \frac{1}{w_1^{-\prime}}|\beta-\delta)} \\
&= ((w_0^{+\prime})^{-c_1^+}(w_1^{+\prime})^{c_0^+})^{\frac{N-1}{2}} ((w_0^{-\prime})^{-c_1^-}(w_1^{-\prime})^{c_0^-})^{\frac{N-1}{2}} [w_0^{-\prime}] \frac{g(w_0^{+\prime})}{g(w_0^{-\prime})} \\
&\quad * \sum_{\alpha,\beta,\gamma,\delta=0}^{N-1} \xi^{\gamma\beta+\frac{N+1}{2}\gamma^2-\delta\alpha-\frac{N+1}{2}\delta^2} \frac{\omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|\alpha-\gamma)}{\omega(\frac{w_0^{-\prime}}{\xi}, \frac{1}{w_1^{-\prime}}|\beta-\delta)} \delta(\alpha+\beta-\delta)\delta(\beta+\alpha-\gamma)
\end{aligned}$$

In order for a term in the summation to be nonzero, $\alpha+\beta-\delta \equiv 0 \mod N$ and $\alpha+\beta-\gamma \equiv 0 \mod N$. Since $0 \le \alpha,\beta,\gamma,\delta \le N-1$, either $\delta=\gamma=\alpha+\beta$ or $\delta=\gamma=\alpha+\beta-N$.

If $\delta=\gamma=\alpha+\beta$, then

$$\begin{aligned}
\gamma\beta+\frac{N+1}{2}\gamma^2-\delta\alpha-\frac{N+1}{2}\delta^2 &= (\alpha+\beta)\beta+\frac{N+1}{2}(\alpha+\beta)^2 \\
&\quad -(\alpha+\beta)\alpha-\frac{N+1}{2}(\alpha+\beta)^2 \\
&= \beta^2-\alpha^2
\end{aligned}$$

Thus $\xi^{\gamma\beta+\frac{N+1}{2}\gamma^2-\delta\alpha-\frac{N+1}{2}\delta^2} = \xi^{\beta^2-\alpha^2}$. Also because $\alpha-\gamma=\alpha-(\alpha+\beta)=-\beta$, it is the case that $\omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|\alpha-\gamma) = \omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|-\beta) = \omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|N-\beta)$. Similarly, $\omega(\frac{w_0^{-\prime}}{\xi}, \frac{1}{w_1^{-\prime}}|\beta-\delta) = \omega(\frac{w_0^{-\prime}}{\xi}, \frac{1}{w_1^{-\prime}}|N-\alpha)$.

If $\delta=\gamma=\alpha+\beta-N$, then

$$\begin{aligned}
\gamma\beta+\frac{N+1}{2}\gamma^2-\delta\alpha-\frac{N+1}{2}\delta^2 &= (\alpha+\beta-N)\beta+\frac{N+1}{2}(\alpha+\beta-N)^2 \\
&\quad -(\alpha+\beta-N)\alpha-\frac{N+1}{2}(\alpha+\beta-N)^2 \\
&= \beta^2-\beta N-\alpha^2+\alpha N \\
&= \beta^2-N(\beta-\alpha)-\alpha^2
\end{aligned}$$

Since ξ is an Nth root of unity and $\beta-\alpha \in \mathbb{Z}$, $\xi^{-N(\beta-\alpha)}=1$. Thus $\xi^{\gamma\beta+\frac{N+1}{2}\gamma^2-\delta\alpha-\frac{N+1}{2}\delta^2}$ $= \xi^{\beta^2-N(\beta-\alpha)-\alpha^2} = \xi^{\beta^2-\alpha^2}$. Also because $\alpha-\gamma=\alpha-(\alpha+\beta-N)=N-\beta$, it is the case that $\omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|\alpha-\gamma) = \omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|N-\beta)$. Similarly, $\omega(\frac{w_0^{-\prime}}{\xi}, \frac{1}{w_1^{-\prime}}|\beta-\delta) = \omega(\frac{w_0^{-\prime}}{\xi}, \frac{1}{w_1^{-\prime}}|N-\alpha)$.

Hence,

$$\begin{aligned} H_N(4_1) &= ((w_0^{+\prime})^{-c_1^+}(w_1^{+\prime})^{c_0^+}(w_0^{-\prime})^{-c_1^-}(w_1^{-\prime})^{c_0^-})^{\frac{N-1}{2}}[w_0^{-\prime}]\frac{g(w_0^{+\prime})}{g(w_0^{-\prime})} \\ &\quad * \sum_{\alpha,\beta,=0}^{N-1} \xi^{\beta^2-\alpha^2}\omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|N-\beta)(\omega(\frac{w_0^{-\prime}}{\xi}, \frac{1}{w_1^{-\prime}}|N-\alpha))^{-1} \\ &= ((w_0^{+\prime})^{-c_1^+}(w_1^{+\prime})^{c_0^+}(w_0^{-\prime})^{-c_1^-}(w_1^{-\prime})^{c_0^-})^{\frac{N-1}{2}}[w_0^{-\prime}]\frac{g(w_0^{+\prime})}{g(w_0^{-\prime})} \\ &\quad * \sum_{\alpha=0}^{N-1}\sum_{\beta=0}^{N-1} \xi^{\beta^2-\alpha^2}\omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|N-\beta)(\omega(\frac{w_0^{-\prime}}{\xi}, \frac{1}{w_1^{-\prime}}|N-\alpha))^{-1} \\ &= ((w_0^{+\prime})^{-c_1^+}(w_1^{+\prime})^{c_0^+}(w_0^{-\prime})^{-c_1^-}(w_1^{-\prime})^{c_0^-})^{\frac{N-1}{2}}[w_0^{-\prime}]\frac{g(w_0^{+\prime})}{g(w_0^{-\prime})} \\ &\quad * \sum_{\alpha=0}^{N-1} \xi^{-\alpha^2}(\omega(\frac{w_0^{-\prime}}{\xi}, \frac{1}{w_1^{-\prime}}|N-\alpha))^{-1}\sum_{\beta=0}^{N-1}\xi^{\beta^2}\omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|N-\beta) \end{aligned}$$

as shown in [3].

Let $x \in [0, N-1]$, and consider the sum $\sum_{\beta=0}^{N-1} \xi^{\beta^2}\omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|N-\beta)$. The xth term is

$$\xi^{x^2}\omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|N-x) = \xi^{x^2}\prod_{j=1}^{N-x}\frac{\frac{1}{w_1^{+\prime}}}{1-w_0^{+\prime}\xi^j}$$

and the $(N-x)$th term is

$$\begin{aligned} \xi^{(N-x)^2}\omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|N-(N-x)) &= \xi^{(N-x)^2}\prod_{j=1}^{x}\frac{\frac{1}{w_1^{+\prime}}}{1-w_0^{+\prime}\xi^j} \\ &= \xi^{N^2-2N\beta+\beta^2}\prod_{j=1}^{x}\frac{\frac{1}{w_1^{+\prime}}}{1-w_0^{+\prime}\xi^j} \\ &= \xi^{\beta^2}\prod_{j=1}^{x}\frac{\frac{1}{w_1^{+\prime}}}{1-w_0^{+\prime}\xi^j} \end{aligned}$$

Now consider the sum $\sum_{\beta=0}^{N-1} \xi^{\beta^2}\omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|\beta)$. The xth term is

$$\xi^{x^2}\omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|x) = \xi^{x^2}\prod_{j=1}^{x}\frac{\frac{1}{w_1^{+\prime}}}{1-w_0^{+\prime}\xi^j}$$

while the $N-x$th term is

$$\begin{aligned}\xi^{(N-x)^2}\omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|N-x) &= \xi^{(N-x)^2}\prod_{j=1}^{N-x}\frac{\frac{1}{w_1^{+\prime}}}{1-w_0^{+\prime}\xi^j} \\ &= \xi^{N^2-2N\beta+\beta^2}\prod_{j=1}^{N-x}\frac{\frac{1}{w_1^{+\prime}}}{1-w_0^{+\prime}\xi^j} \\ &= \xi^{\beta^2}\prod_{j=1}^{N-x}\frac{\frac{1}{w_1^{+\prime}}}{1-w_0^{+\prime}\xi^j}\end{aligned}$$

Since x was chosen arbitrarily,

$$\sum_{\beta=0}^{N-1}\xi^{\beta^2}\omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|N-\beta) = \sum_{\beta=0}^{N-1}\xi^{\beta^2}\omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|\beta)$$

Also, by Proposition 4.4,

$$(\omega(\frac{w_0^{-\prime}}{\xi}, \frac{1}{w_1^{-\prime}}|N-\alpha))^{-1} = (\omega((w_0^{-\prime})^*, (\frac{1}{w_1^{-\prime}})^*|\alpha))^*$$

Thus the quantum hyperbolic invariants of the figure eight knot are given by

$$\begin{aligned}H_N(4_1) &= ((w_0^{+\prime})^{-c_1^+}(w_1^{+\prime})^{c_0^+}(w_0^{-\prime})^{-c_1^-}(w_1^{-\prime})^{c_0^-})^{\frac{N-1}{2}}[w_0^{-\prime}]\frac{g(w_0^{+\prime})}{g(w_0^{-\prime})} \\ &\quad * \sum_{\alpha=0}^{N-1}\xi^{-\alpha^2}(\omega((w_0^{-\prime})^*, (\frac{1}{w_1^{-\prime}})^*|\alpha))^*\sum_{\beta=0}^{N-1}\xi^{\beta^2}\omega(w_0^{+\prime}, \frac{1}{w_1^{+\prime}}|\beta)\end{aligned}$$

CHAPTER 5
THE GROWTH OF THE INVARIANTS

This chapter investigates the quantum hyperbolic invariants of the figure eight knot as $N \to \infty$. In doing so, the following theorem will be proved.

Theorem 5.1. *For the figure eight knot, the quantum hyperbolic invariants of Baseilhac and Benedetti grow at a rate no greater than $\frac{1}{2\pi}$ times the volume of the complement of the knot. That is*

$$\lim_{N\to\infty} \frac{\log|H_N(4_1)|}{N} \leq \frac{1}{2\pi} Vol(S^3 \setminus 4_1)$$

The quantum hyperbolic invariants on the figure eight knot can be broken into three pieces.

$$\begin{aligned}
\lim_{N\to\infty} \frac{\log|H_N(4_1)|}{N} &= \lim_{N\to\infty} \frac{1}{N} \log|((w_0^{+\prime})^{-c_1^+}(w_1^{+\prime})^{c_0^+}(w_0^{-\prime})^{-c_1^-}(w_1^{-\prime})^{c_0^-})^{\frac{N-1}{2}} \\
&\quad *[w_0^{-\prime}]\frac{g(w_0^{+\prime})}{g(w_0^{-\prime})}\sum_{\alpha=0}^{N-1}\xi^{-\alpha^2}(\omega((w_0^{-\prime})^*,(\frac{1}{w_1^{-\prime}})^*|\alpha))^* \\
&\quad *\sum_{\beta=0}^{N-1}\xi^{\beta^2}\omega(w_0^{+\prime},\frac{1}{w_1^{+\prime}}|))| \\
&= \lim_{N\to\infty} \frac{1}{N} \log\left|((w_0^{+\prime})^{-c_1^+}(w_1^{+\prime})^{c_0^+}(w_0^{-\prime})^{-c_1^-}(w_1^{-\prime})^{c_0^-})^{\frac{N-1}{2}}[w_0^{-\prime}]\frac{g(w_0^{+\prime})}{g(w_0^{-\prime})}\right| \\
&\quad + \lim_{N\to\infty} \frac{1}{N} \log\left|\sum_{\alpha=0}^{N-1}\xi^{-\alpha^2}(\omega((w_0^{-\prime})^*,(\frac{1}{w_1^{-\prime}})^*|\alpha))^*\right| \\
&\quad + \lim_{N\to\infty} \frac{1}{N} \log\left|\sum_{\beta=0}^{N-1}\xi^{\beta^2}\omega(w_0^{+\prime},\frac{1}{w_1^{+\prime}}|\beta)\right|
\end{aligned}$$

5.1 The Riemann sum

First consider

$$\begin{aligned}
S_\beta &= \sum_{\beta=0}^{N-1}\xi^{\beta^2}\omega(w_0^{+\prime},\frac{1}{w_1^{+\prime}}|\beta) \\
&= \sum_{\beta=0}^{N-1} e^{\frac{2\pi i\beta^2}{N}} \prod_{k=1}^{\beta} \frac{e^{-\frac{1}{N}(\log(w_1^+)+i\pi(N+1)(f_1^+-c_1^+))}}{1-e^{-\frac{1}{N}(\log(w_0^+)+i\pi(N+1)(f_0^+-c_0^+))}e^{\frac{2\pi ik}{N}}}
\end{aligned}$$

Recall that for the figure eight knot,

$$w_0^+ = w_1^+ = w_2^+ = e^{\frac{\pi i}{3}}$$

$$f_0^+ = f,\ f_1^+ = -1-2f, \text{ and } f_2^+ = f$$

$$c_0^+ = c,\ c_1^+ = 1-2c+\tfrac{1}{2}\lambda, \text{ and } c_2^+ = c-\tfrac{1}{2}\lambda.$$

So

$$\begin{aligned}
S_\beta &= \sum_{\beta=0}^{N-1} e^{\frac{2\pi i\beta^2}{N}} \prod_{k=1}^{\beta} \frac{e^{-\frac{1}{N}(\frac{i\pi}{3}+i\pi(N+1)(-1-2f-1+2c-\frac{1}{2}\lambda))}}{1-e^{\frac{1}{N}(\frac{i\pi}{3}+i\pi(N+1)(f-c))}e^{\frac{2\pi ik}{N}}} \\
&= \sum_{\beta=0}^{N-1} e^{\frac{2\pi i\beta^2}{N}} e^{-\frac{\beta}{N}(\frac{i\pi}{3}+i\pi(N+1)(-2-2f+2c-\frac{1}{2}\lambda))} \prod_{k=1}^{\beta} \frac{1}{1-e^{\frac{1}{N}(\frac{i\pi}{3}+i\pi(N+1)(f-c))}e^{\frac{2\pi ik}{N}}} \\
&= \sum_{\beta=0}^{N-1} e^{\frac{2\pi i\beta^2}{N}-\frac{\beta}{N}(\frac{i\pi}{3}+i\pi(N+1)(-2-2f+2c-\frac{1}{2}\lambda))} \prod_{k=1}^{\beta} \frac{1}{1-e^{\frac{i\pi(1+3(N+1)(f-c)+6k)}{3N}}} \\
&= \sum_{\beta=0}^{N-1} e^{\frac{2\pi i\beta^2}{N}-\frac{\beta}{N}(\frac{i\pi}{3}+i\pi(N+1)(-2-2f+2c-\frac{1}{2}\lambda))} \\
&\quad * \prod_{k=1}^{\beta} \frac{e^{-\frac{i\pi(1+3(N+1)(f-c)+6k)}{6N}}}{e^{-\frac{i\pi(1+3(N+1)(f-c)+6k)}{6N}}-e^{\frac{i\pi(1+3(N+1)(f-c)+6k)}{6N}}} \\
&= \sum_{\beta=0}^{N-1} e^{\frac{2\pi i\beta^2}{N}-\frac{\beta}{N}(\frac{i\pi}{3}+i\pi(N+1)(-2-2f+2c-\frac{1}{2}\lambda))-\frac{i\pi(\beta+3\beta(N+1)(f-c)+6\frac{\beta(\beta+1)}{2})}{6N}} \\
&\quad * \prod_{k=1}^{\beta} \frac{-1}{i} \frac{1}{2\sin(\frac{\pi(1+3(N+1)(f-c)+6k)}{6N})} \\
&= \sum_{\beta=0}^{N-1} e^{\frac{2\pi i\beta^2}{N}-\frac{\beta}{N}(\frac{i\pi}{3}+i\pi(N+1)(-2-2f+2c-\frac{1}{2}\lambda))-\frac{i\pi(\beta+3\beta(N+1)(f-c)+3\beta(\beta+1))}{6N}+\frac{i\pi\beta}{2}} \\
&\quad * \prod_{k=1}^{\beta} \frac{1}{2\sin(\frac{\pi(1+3(N+1)(f-c)+6k)}{6N})} \\
&= \sum_{\beta=0}^{N-1} e^{\frac{i\pi}{2N}(3\beta^2+\beta+3\beta(f-c)+\beta\lambda+5N\beta+3N\beta(f-c)+N\beta\lambda)} \\
&\quad * \prod_{k=1}^{\beta} \frac{1}{2\sin(\frac{\pi(1+3(N+1)(f-c)+6k)}{6N})}
\end{aligned}$$

Thus S_β is a summation of vectors in $\mathbb{C}$ with magnitude $\prod_{k=1}^{\beta} \frac{1}{2\sin(\frac{\pi(1+3(N+1)(f-c)+6k)}{6N})}$ and direction $e^{\frac{i\pi}{2N}(3\beta^2+\beta+3\beta(f-c)+\beta\lambda+5N\beta+3N\beta(f-c)+N\beta\lambda)}$. The goal will be to show that $\prod_{k=1}^{\beta} \frac{1}{2\sin(\frac{\pi(1+3(N+1)(f-c)+6k)}{6N})} \approx e^{\frac{N}{\pi}\Lambda(\frac{\beta\pi}{N})}$.

Using the addition formula for sine, notice that

$$\begin{aligned}
\sin(\frac{\pi(1+3(N+1)(f-c)+6k)}{6N}) &= \sin(\frac{\pi(1+3N(f-c)+3(f-c)+6k)}{6N}) \\
&= \sin(\frac{\pi(1+3(f-c)+6k)}{6N} + \frac{3\pi N(f-c)}{6N}) \\
&= \sin(\frac{\pi(1+3(f-c)+6k)}{6N})\cos(\frac{\pi(f-c)}{2}) \\
&\quad + \cos(\frac{\pi(1+3(f-c)+6k)}{6N})\sin(\frac{\pi(f-c)}{2})
\end{aligned}$$

Since $(f-c) \in \mathbb{Z}$,

$$\cos(\frac{\pi(f-c)}{2}) = \begin{cases} 1 & \text{if } (f-c) \equiv 0 \mod 4 \\ 0 & \text{if } (f-c) \equiv 1 \mod 4 \\ -1 & \text{if } (f-c) \equiv 2 \mod 4 \\ 0 & \text{if } (f-c) \equiv 3 \mod 4 \end{cases}$$

and

$$\sin(\frac{\pi(f-c)}{2}) = \begin{cases} 0 & \text{if } (f-c) \equiv 0 \mod 4 \\ 1 & \text{if } (f-c) \equiv 1 \mod 4 \\ 0 & \text{if } (f-c) \equiv 2 \mod 4 \\ -1 & \text{if } (f-c) \equiv 3 \mod 4 \end{cases}$$

To simplify calculations, assume $f - c \equiv 0 \mod 4$. Then

$$\sin(\frac{\pi(1+3(N+1)(f-c)+6k)}{6N}) = \sin(\frac{\pi(1+3(f-c)+6k)}{6N})$$

Furthermore, using the addition formula for sine again results in

$$\begin{aligned}
\sin(\frac{\pi(1+3(f-c)+6k)}{6N}) &= \sin(\frac{k\pi}{N})\cos(\frac{\pi(1+3(f-c))}{6N}) \\
&\quad + \cos(\frac{k\pi}{N})\sin(\frac{\pi(1+3(f-c))}{6N})
\end{aligned}$$

Since $-1 \leq \cos(\frac{k\pi}{N}) \leq 1$ and $\lim_{N\to\infty} \sin(\frac{\pi(1+3(f-c))}{6N}) = 0$, the second term disappears. Also $\lim_{N\to\infty} \cos(\frac{\pi(1+3(f-c))}{6N}) = 1$. So, for large enough N,

$$\sin(\frac{\pi(1+3(f-c)+6k)}{6N}) \approx \sin(\frac{k\pi}{N})$$

Observe that

$$\begin{aligned}
\prod_{k=1}^{\beta} \frac{1}{2\sin(\frac{k\pi}{N})} &= e^{\log(\prod_{k=1}^{\beta} \frac{1}{2\sin(\frac{k\pi}{N})})} \\
&= e^{-\sum_{k=1}^{\beta} \log(2\sin(\frac{k\pi}{N}))}
\end{aligned}$$

where $\sum_{k=1}^{\beta} \log(2\sin(\frac{k\pi}{N}))$ can be thought of as the Riemann sum of $\log(2\sin(\frac{x\pi}{N}))$ on the interval $[0, \beta]$ with $\Delta x = 1$. Multiplying by $\frac{\pi}{N}$ gives the Riemann sum of $\log(2\sin(x))$ on the interval $[0, \frac{\beta\pi}{N}]$ with $\Delta x = \frac{\pi}{N}$. Now as $N \to \infty$, $\Delta x \to 0$ and the Riemann sum approximates the integral $\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du$. So

$$\begin{aligned} e^{-\sum_{k=1}^{\beta} \log(2\sin(\frac{k\pi}{N}))} &= e^{-\frac{N}{\pi}\sum_{k=1}^{\beta} \frac{\pi}{N}\log(2\sin(\frac{k\pi}{N}))} \\ &= e^{-\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du + \frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du - \frac{N}{\pi}\sum_{k=1}^{\beta} \frac{\pi}{N}\log(2\sin(\frac{k\pi}{N}))} \\ &= e^{\frac{N}{\pi}\Lambda(\frac{\beta\pi}{N}) + \frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du - \frac{N}{\pi}\sum_{k=1}^{\beta} \frac{\pi}{N}\log(2\sin(\frac{k\pi}{N}))} \end{aligned}$$

where

$$\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du - \frac{N}{\pi}\sum_{k=1}^{\beta} \frac{\pi}{N}\log(2\sin(\frac{k\pi}{N}))$$

is the error between the integral and the Riemann sum approximation.

5.2 The error term

Consider the case when $\beta \leq \frac{N}{2}$. Although $\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du$ converges, $\log(2\sin u)$ has a singularity at $u = 0$. Thus, in order to evaluate the error term, it will be necessary to borrow a trick from numerical analysis [5]. While $\lim_{u\to 0} 2\sin u = 0$, $\lim_{u\to 0} \frac{2\sin u}{2u} = 1$. So $\log(\frac{2\sin u}{2u})$ is bounded as $u \to 0$. Furthermore, the integral of $\log(2u)$ is bounded at 0. Hence replacing

$$\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du$$

with

$$\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du - \int_0^{\frac{\beta\pi}{N}} \log(2u)du + \int_0^{\frac{\beta\pi}{N}} \log(2u)du$$

resolves the singularity problem. The same is done for the summation. That is,

$$\sum_{k=1}^{\beta} \frac{\pi}{N}\log(2\sin(\frac{k\pi}{N}))$$

is replaced by

$$\sum_{k=1}^{\beta} \frac{\pi}{N}\log(2\sin(\frac{k\pi}{N})) - \sum_{k=1}^{\beta} \frac{\pi}{N}\log(2(\frac{k\pi}{N})) + \sum_{k=1}^{\beta} \frac{\pi}{N}\log(2(\frac{k\pi}{N}))$$

So now the error term is

$$\frac{N}{\pi}\left[\int_0^{\frac{\beta\pi}{N}} \log(\frac{\sin u}{u})du - \sum_{k=1}^{\beta} \frac{\pi}{N}\log(\frac{\sin(\frac{k\pi}{N})}{\frac{k\pi}{N}}) + \int_0^{\frac{\beta\pi}{N}} \log(2u)du - \sum_{k=1}^{\beta} \frac{\pi}{N}\log(2\frac{k\pi}{N})\right]$$

Observe that

$$\begin{aligned} \frac{d}{du}(\log(\frac{\sin u}{u})) &= \frac{u\cos u - \sin u}{u\sin u} \\ \frac{d^2}{du^2}(\log(\frac{\sin u}{u})) &= \frac{\sin^2 u - u^2}{u^2\sin^2 u} \end{aligned}$$

When $0 \leq u \leq \frac{\pi}{2}$, $0 \leq \sin u \leq u$. So $\sin^2 u \leq u^2$. Thus $\frac{d^2}{du^2}(\log(\frac{\sin u}{u})) \leq 0$. This

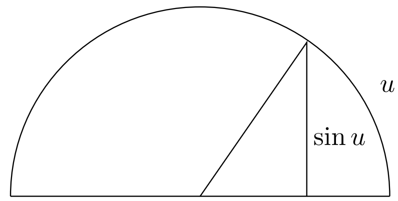


Figure 5.1: If $0 \le u \le \frac{\pi}{2}$, then $\sin u \le u$

means that $\frac{d}{du}(\log(\frac{\sin u}{u}))$, the slope of $\log(\frac{\sin u}{u})$, is monotone decreasing. So the maximum slope will occur when $u = 0$, and the minimum slope will occur when $u = \frac{\pi}{2}$. Using L'Hospital's rule,

$$\begin{aligned}\lim_{u\to 0} \frac{u\cos u - \sin u}{u \sin u} &= \lim_{u\to 0} \frac{-u\sin u}{u\cos u + \sin u}\\ &= \lim_{u\to 0}\frac{-u\cos u - \sin u}{-u\sin u + 2\cos u}\\ &= 0\end{aligned}$$

Also

$$\begin{aligned}\lim_{u\to \frac{\pi}{2}} \frac{u\cos u - \sin u}{u \sin u} &= \lim_{u\to \frac{\pi}{2}} \frac{\frac{\pi}{2}\cos\frac{\pi}{2} - \sin\frac{\pi}{2}}{\frac{\pi}{2}\sin\frac{\pi}{2}}\\ &= -\frac{2}{\pi}\end{aligned}$$

Thus $|\frac{d}{du}(\log(\frac{\sin u}{u}))| \le \frac{2}{\pi}$.

Consider the kth rectangle of the Riemann sum $\sum_{k=1}^{\beta} \frac{\pi}{N}\log(\frac{\sin(\frac{k\pi}{N})}{\frac{k\pi}{N}})$. For some $c \in [\frac{(k-1)\pi}{N}, \frac{k\pi}{N}]$, this rectangle contributes $\frac{\pi}{N}\log(\frac{\sin(c)}{c})$ to the sum. Because $|\frac{d}{du}(\log(\frac{\sin u}{u}))| \le \frac{2}{\pi}$, for any $\frac{(k-1)\pi}{N} \le x \le c$,

$$\log(\frac{\sin c}{c}) - \frac{2}{\pi}(c-x) \le \log(\frac{\sin x}{x}) \le \log(\frac{\sin c}{c}) + \frac{2}{\pi}(c-x)$$

Similarly, for any $c \le x \le \frac{k\pi}{N}$,

$$\log(\frac{\sin c}{c}) - \frac{2}{\pi}(x-c) \le \log(\frac{\sin x}{x}) \le \log(\frac{\sin c}{c}) + \frac{2}{\pi}(x-c)$$

So the largest error possible between the integral $\int_0^{\frac{\beta\pi}{N}} \log(\frac{\sin u}{u})du$ and the Riemann sum $\sum_{k=1}^{\beta} \frac{\pi}{N}\log(\frac{\sin(\frac{k\pi}{N})}{\frac{k\pi}{N}})$ on the interval $[\frac{(k-1)\pi}{N}, \frac{k\pi}{N}]$ is shown in grey in Figure 5.2. This area can be easily calculated. Let $l_1 = c - \frac{(k-1)\pi}{N}$ and $l_2 = \frac{k\pi}{N} - c$. Then

$$\begin{aligned} A &= \frac{1}{2}l_1(\frac{2}{\pi}l_1) + \frac{1}{2}l_2(\frac{2}{\pi}l_2) \\ &= \frac{1}{\pi}(l_1^2 + l_2^2) \\ &\leq \frac{1}{\pi}(l_1 + l_2)^2 \\ &= \frac{1}{\pi}(\frac{\pi}{N})^2 \\ &= \frac{\pi}{N^2} \end{aligned}$$

Since there are β such intervals,

$$|\int_0^{\frac{\beta\pi}{N}} \log(\frac{\sin u}{u})du - \sum_{k=1}^{\beta} \frac{\pi}{N} \log(\frac{\sin(\frac{k\pi}{N})}{\frac{k\pi}{N}})| \leq \frac{\beta\pi}{N^2}$$

Furthermore, because $\frac{d}{du}(\log(\frac{\sin u}{u})) \leq 0$ when $0 \leq u \leq \frac{\pi}{2}$ and the Riemann sum $\sum_{k=1}^{\beta} \frac{\pi}{N} \log(\frac{\sin(\frac{k\pi}{N})}{\frac{k\pi}{N}})$ follows the right endpoint rule, the sum is an underestimate of the integral. Hence

$$\int_0^{\frac{\beta\pi}{N}} \log(\frac{\sin u}{u})du - \sum_{k=1}^{\beta} \frac{\pi}{N} \log(\frac{\sin(\frac{k\pi}{N})}{\frac{k\pi}{N}}) \leq \frac{\beta\pi}{N^2}$$

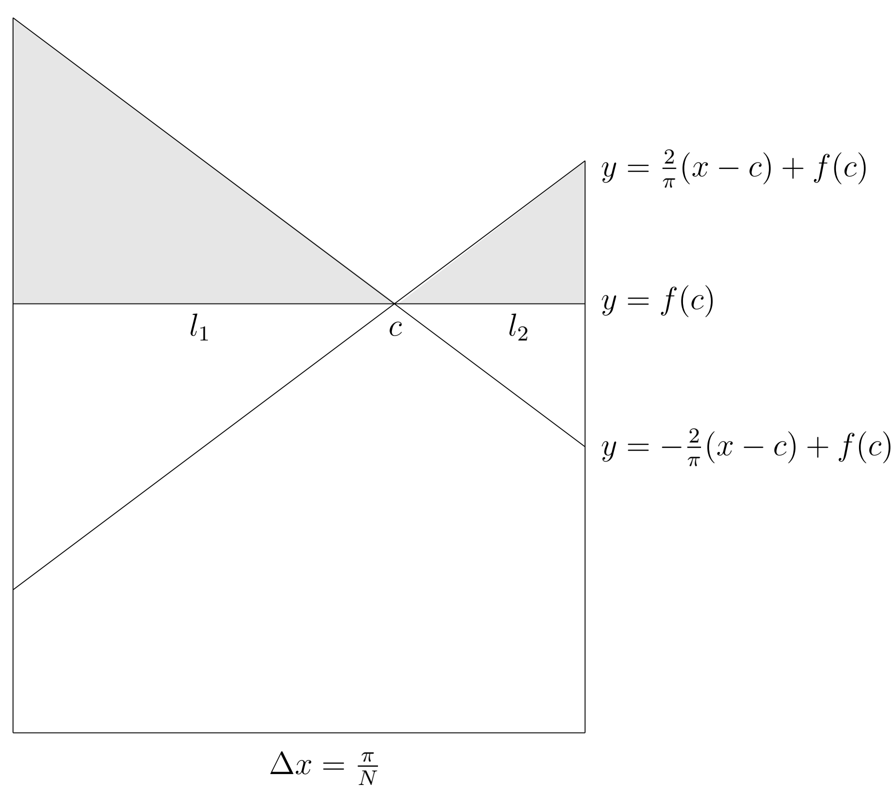


Figure 5.2: One rectangle of the Riemann sum

Observe that $\int_0^{\frac{\beta\pi}{N}} \log(2x)dx$ can be integrated analytically using integration by parts where $u = \log(2x)$ and $dv = dx$. Thus the remaining error can be computed directly as follows.

$$\begin{aligned}
\int_0^{\frac{\beta\pi}{N}} \log(2x)dx - \sum_{k=1}^{\beta} \frac{\pi}{N}\log(2\frac{k\pi}{N}) &= \lim_{t\to 0}\int_t^{\frac{\beta\pi}{N}} \log(2x)dx - \sum_{k=1}^{\beta}\frac{\pi}{N}\log(2\frac{k\pi}{N}) \\
&= \lim_{t\to 0} x\log(2x)|_t^{\frac{\beta\pi}{N}} - \lim_{t\to 0}\int_t^{\frac{\beta\pi}{N}} dx \\
&\quad -\sum_{k=1}^{\beta}\frac{\pi}{N}\log(2\frac{k\pi}{N}) \\
&= \frac{\beta\pi}{N}\log(2\frac{\beta\pi}{N}) - \lim_{t\to 0} t\log(2t) - \frac{\beta\pi}{N} + \lim_{t\to 0} t \\
&\quad -\sum_{k=1}^{\beta}\frac{\pi}{N}\log(2\frac{k\pi}{N}) \\
&= \frac{\pi}{N}(\beta\log(2\frac{\beta\pi}{N}) - \beta - \sum_{k=1}^{\beta}\log(2\frac{k\pi}{N})) \\
&= \frac{\pi}{N}\sum_{k=1}^{\beta}(\log(2\frac{\beta\pi}{N}) - 1 - \log(2\frac{k\pi}{N})) \\
&= \frac{\pi}{N}\sum_{k=1}^{\beta}(\log(\frac{\beta}{k}) - 1)
\end{aligned}$$

Therefore, when $0 \leq \beta \leq \frac{N}{2}$,the total error between the integral and the Riemann sum is given by

$$\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du - \frac{N}{\pi}\sum_{k=1}^{\beta}\frac{\pi}{N}\log(2\sin(\frac{k\pi}{N})) \leq \frac{\beta}{N} + \sum_{k=1}^{\beta}(\log\frac{\beta}{k} - 1)$$

5.3 The limit

Now recall that S_β is part of the limit

$$\begin{aligned}
\lim_{N\to\infty}\frac{1}{N}\log|S_\beta| &= \lim_{N\to\infty}\frac{1}{N}\log|\sum_{\beta=0}^{N-1}\xi^{\beta^2}\omega(w_0^{+\prime},\frac{1}{w_1^{+\prime}}|\beta)| \\
&= \lim_{N\to\infty}|\sum_{\beta=0}^{N-1}e^{\frac{i\pi}{2N}(3\beta^2+\beta+3\beta(f-c)+\beta\lambda+5N\beta+3N\beta(f-c)+N\beta\lambda)} \\
&\quad *\prod_{k=1}^{\beta}\frac{1}{2\sin(\frac{\pi(1+3(N+1)(f-c)+6k)}{6N})}| \\
&\leq \lim_{N\to\infty}|\sum_{\beta=0}^{N-1}\prod_{k=1}^{\beta}\frac{1}{2\sin(\frac{\pi(1+3(N+1)(f-c)+6k)}{6N})}| \\
&= \lim_{N\to\infty}\frac{1}{N}\log|\sum_{\beta=0}^{\frac{N}{2}}e^{-\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}}\log(2\sin u)du+\frac{\beta}{N}+\sum\limits_{k=1}\beta(\log\frac{\beta}{k}-1)} \\
&\quad +\sum_{\beta=\frac{N}{2}}^{N-1}e^{-\sum\limits_{k=1}^{\beta}\log(2sin(\frac{k\pi}{N}))}|
\end{aligned}$$

Consider $-\sum\limits_{k=1}^{\beta}\log(2\sin(\frac{k\pi}{N}))$ when $\frac{N}{2}\leq\beta\leq N$. When $\frac{N}{2}\leq\beta\leq\frac{5N}{6}$, $-\log(2\sin(\frac{k\pi}{N}))\leq 0$. So if $-\sum\limits_{k=1}^{\frac{N}{2}}\log(2\sin(\frac{k\pi}{N}))\leq 0$ then $-\sum\limits_{k=1}^{\beta}\log(2\sin(\frac{k\pi}{N}))\leq 0$ for $\frac{N}{2}\leq\beta\leq\frac{5N}{6}$. Also $-\log(2\sin(\frac{k\pi}{N}))\geq 0$ when $\frac{5N}{6}\leq\beta\leq N-1$. So if $-\sum\limits_{k=1}^{N-1}\log(2\sin(\frac{k\pi}{N}))\leq 0$ then $-\sum\limits_{k=1}^{\beta}\log(2\sin(\frac{k\pi}{N}))\leq 0$ for $\frac{5N}{6}\leq\beta\leq N-1$.

Recall that $x^N-1=\prod\limits_{k=0}^{N-1}(x-\xi^k)$. So

$$\frac{x^N-1}{x-1}=x^{N-1}+x^{N-2}+...+x+1=\prod_{k=1}^{N-1}(x-\xi^k)$$

Letting $x = 1$ gives

$$\begin{aligned}
N &= \prod_{k=1}^{N-1}(1 - e^{\frac{2k\pi}{N}}) \\
&= \prod_{k=1}^{N-1} 2ie^{\frac{ki\pi}{N}} \frac{e^{-\frac{ki\pi}{N}} - e^{\frac{ki\pi}{N}}}{2i} \\
&= \prod_{k=1}^{N-1} 2ie^{\frac{ki\pi}{N}} \sin(-\frac{k\pi}{N}) \\
&= (-i)^{N-1} e^{\frac{N(N-1)i\pi}{2N}} \prod_{k=1}^{N-1} 2\sin(\frac{k\pi}{N}) \\
&= e^{\frac{3(N-1)i\pi}{2}} e^{\frac{(N-1)i\pi}{2}} \prod_{k=1}^{N-1} 2\sin(\frac{k\pi}{N}) \\
&= \prod_{k=1}^{N-1} 2\sin(\frac{k\pi}{N})
\end{aligned}$$

Thus

$$-\sum_{k=1}^{N-1} \log(2\sin(\frac{k\pi}{N})) = -\log\left(\prod_{k=1}^{N-1} 2\sin(\frac{k\pi}{N})\right) = -\log(N) \leq 0$$

So $-\sum_{k=1}^{\beta} \log(2\sin(\frac{k\pi}{N})) \leq 0$ for $\frac{5N}{6} \leq \beta \leq N-1$.

Because $\sin\theta$ is symmetric about $\theta = \frac{\pi}{2}$, $\sin(\pi - \theta) = \sin\theta$. So $\sin(\frac{k\pi}{N}) = \sin(\pi - \frac{k\pi}{N}) = \sin(\frac{(N-k)\pi}{N})$. Then, for odd N, $\prod_{k=1}^{\frac{N-1}{2}} 2\sin(\frac{k\pi}{N}) = \prod_{k=\frac{N+1}{2}}^{N-1} 2\sin(\frac{k\pi}{N})$. Hence $\sqrt{N} = \prod_{k=1}^{\frac{N-1}{2}} 2\sin(\frac{k\pi}{N})$. Thus

$$-\sum_{k=1}^{\frac{N-1}{2}} \log(2\sin(\frac{k\pi}{N})) = -\log\left(\prod_{k=1}^{\frac{N-1}{2}} 2\sin(\frac{k\pi}{N})\right) = -\log(\sqrt{N}) \leq 0$$

So $-\sum_{k=1}^{\beta} \log(2\sin(\frac{k\pi}{N}) \leq 0$ for $\frac{N}{2} \leq \beta \leq \frac{5N}{6}$.

Therefore $-\sum_{k=1}^{\beta} \log(2\sin(\frac{k\pi}{N}) \leq 0$ when $\frac{N}{2} \leq \beta \leq N-1$. So $e^{-\sum_{k=1}^{\beta} \log(2\sin(\frac{k\pi}{N})} \leq 1$ and $\sum_{\beta=\frac{N}{2}}^{N-1} e^{-\sum_{k=1}^{\beta} \log(2\sin(\frac{k\pi}{N})} \leq \frac{N}{2}$. Because

$$\sum_{\beta=0}^{\frac{N}{2}} e^{-\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du + \frac{\beta}{N} + \sum_{k=1}^{\beta}(\log\frac{\beta}{k} - 1)} \geq 1$$

it is the case that

$$\sum_{\beta=0}^{\frac{N}{2}} e^{-\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du+\frac{\beta}{N}+\sum\limits_{k=1}^{\beta}(\log\frac{\beta}{k}-1)}+\frac{N}{2} \leq (\sum_{\beta=0}^{\frac{N}{2}} e^{-\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du+\frac{\beta}{N}+\sum\limits_{k=1}^{\beta}(\log\frac{\beta}{k}-1)})\frac{N}{2}$$

Hence

$$\begin{aligned}
\lim_{N\to\infty}\frac{1}{N}\log|S_\beta| &\leq \lim_{N\to\infty}\frac{1}{N}\log|\sum_{\beta=0}^{\frac{N}{2}} e^{-\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du+\frac{\beta}{N}+\sum\limits_{k=1}^{\beta}(\log\frac{\beta}{k}-1)} \\
&\quad +\sum_{\beta=\frac{N}{2}}^{N-1} e^{-\sum\limits_{k=1}^{\beta}\log(2\sin(\frac{k\pi}{N}))}| \\
&\leq \lim_{N\to\infty}\frac{1}{N}\log|\sum_{\beta=0}^{\frac{N}{2}} e^{-\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du+\frac{\beta}{N}+\sum\limits_{k=1}^{\beta}(\log\frac{\beta}{k}-1)}+\frac{N}{2}| \\
&\leq \lim_{N\to\infty}\frac{1}{N}\log|(\sum_{\beta=0}^{\frac{N}{2}} e^{-\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du+\frac{\beta}{N}+\sum\limits_{k=1}^{\beta}(\log\frac{\beta}{k}-1)})\frac{N}{2}| \\
&= \lim_{N\to\infty}\frac{1}{N}\log|\sum_{\beta=0}^{\frac{N}{2}} e^{-\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du+\frac{\beta}{N}+\sum\limits_{k=1}^{\beta}(\log\frac{\beta}{k}-1)}| \\
&\quad +\lim_{N\to\infty}\frac{1}{N}\log|\frac{N}{2}| \\
&= \lim_{N\to\infty}\frac{1}{N}\log|\sum_{\beta=0}^{\frac{N}{2}} e^{-\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du+\frac{\beta}{N}+\sum\limits_{k=1}^{\beta}(\log\frac{\beta}{k}-1)}|
\end{aligned}$$

Now consider the error term $\frac{\beta}{N}+\sum\limits_{k=1}^{\beta}(\log\frac{\beta}{k}-1)$. If $0\leq\beta\leq\frac{N}{2}$, then $\frac{\beta}{N}\leq\frac{1}{2}$. So $\sum\limits_{k=1}^{\beta}(\log\frac{\beta}{k}-1)\leq\sum\limits_{k=1}^{\frac{N}{2}}(\log\frac{N}{2k}-1)$ for any $0\leq\beta\leq\frac{N}{2}$. Furthermore, observe that

$$\begin{aligned}
\sum_{k=1}^{\frac{N}{2}}(\log(\frac{N}{2k})-1) &= \log(\prod_{k=1}^{\frac{N}{2}}\frac{N}{2k})-\sum_{k=1}^{\frac{N}{2}}1 \\
&= \log(\frac{N^{\frac{N}{2}}}{2^{\frac{N}{2}}(\frac{N}{2})!})-\frac{N}{2}
\end{aligned}$$

According to Stirling's approximation, $n!\approx\sqrt{2n\pi}(\frac{n}{e})^n$ for large n [1]. Substituting

$\frac{N}{2}$ for n gives $(\frac{N}{2})! \approx \sqrt{N\pi}(\frac{N}{2e})^{\frac{N}{2}}$. So for large enough N,

$$\begin{aligned}
\log(\frac{N^{\frac{N}{2}}}{2^{\frac{N}{2}}(\frac{N}{2})!}) - \frac{N}{2} &\approx \log(\frac{N^{\frac{N}{2}}}{2^{\frac{N}{2}}(\sqrt{N\pi}(\frac{N}{2e}))^{\frac{2}{N}}}) - \frac{N}{2} \\
&= \log(\frac{e^{\frac{N}{2}}}{\sqrt{N\pi}}) - \frac{N}{2} \\
&= \log(e^{\frac{N}{2}}) - \log(\sqrt{N\pi}) - \frac{N}{2} \\
&= -\frac{1}{2}\log(N\pi) \\
&\leq 0
\end{aligned}$$

Thus

$$\lim_{N\to\infty} \tfrac{1}{N} \log |\sum_{\beta=0}^{\frac{N}{2}} e^{-\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du + \frac{\beta}{N} + \sum\limits_{k=1} \beta(\log\frac{\beta}{k}-1)}|$$

$$\leq \lim_{N\to\infty} \tfrac{1}{N} \log |\sum_{\beta=0}^{\frac{N}{2}} e^{-\frac{N}{\pi}\int_0^{\frac{\pi}{6}} \log(2\sin u)du + \frac{1}{2}}|$$

Furthermore, by Theorem 2.11, $-\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du \leq -\frac{N}{\pi}\int_0^{\frac{\pi}{6}} \log(2\sin u)du$.

Therefore

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N} \log |\sum_{\beta=0}^{\frac{N}{2}} e^{-\frac{N}{\pi}\int_0^{\frac{\pi}{6}} \log(2\sin u)du + \frac{1}{2}}| &= \lim_{N\to\infty} \frac{1}{N} \log |\frac{N}{2} e^{-\frac{N}{\pi}\int_0^{\frac{\pi}{6}} \log(2\sin u)du + \frac{1}{2}}| \\
&= \lim_{N\to\infty} \frac{1}{N} \log |\frac{N}{2}| \\
&\quad + \lim_{N\to\infty} \frac{1}{N} \log |e^{-\frac{N}{\pi}\int_0^{\frac{\pi}{6}} \log(2\sin u)du + \frac{1}{2}}| \\
&= \lim_{N\to\infty} \frac{1}{N}(-\frac{N}{\pi}\int_0^{\frac{\pi}{6}} \log(2\sin u)du + \frac{1}{2}) \\
&= -\frac{1}{\pi}\int_0^{\frac{\pi}{6}} \log(2\sin u)du + \lim_{N\to\infty} \frac{1}{2N} \\
&= \frac{1}{\pi}\Lambda(\frac{\pi}{6})
\end{aligned}$$

5.4 When $f - c$ is not equivalent to $0 \mod 4$

Thus far, it has been assumed that $f - c \equiv 0 \mod 4$. This assumption was made to simplify calculations; however, the results hold when $f - c \not\equiv 0 \mod 4$.

Recall that

$$\begin{aligned}\sin(\frac{\pi(1+3(N+1)(f-c)+6k)}{6N}) &= \sin(\frac{\pi(1+3N(f-c)+3(f-c)+6k)}{6N}) \\ &= \sin(\frac{\pi(1+3(f-c)+6k)}{6N}+\frac{3\pi N(f-c)}{6N}) \\ &= \sin(\frac{\pi(1+3(f-c)+6k)}{6N})\cos(\frac{\pi(f-c)}{2}) \\ &\quad +\cos(\frac{\pi(1+3(f-c)+6k)}{6N})\sin(\frac{\pi(f-c)}{2})\end{aligned}$$

Since $(f-c)\in\mathbb{Z}$,

$$\cos(\frac{\pi(f-c)}{2}) = \begin{cases} 1 & \text{if } (f-c)\equiv 0 \mod 4 \\ 0 & \text{if } (f-c)\equiv 1 \mod 4 \\ -1 & \text{if } (f-c)\equiv 2 \mod 4 \\ 0 & \text{if } (f-c)\equiv 3 \mod 4 \end{cases}$$

and

$$\sin(\frac{\pi(f-c)}{2}) = \begin{cases} 0 & \text{if } (f-c)\equiv 0 \mod 4 \\ 1 & \text{if } (f-c)\equiv 1 \mod 4 \\ 0 & \text{if } (f-c)\equiv 2 \mod 4 \\ -1 & \text{if } (f-c)\equiv 3 \mod 4 \end{cases}$$

Now suppose $f-c\equiv 2 \mod 4$. Then

$$\sin(\frac{\pi(1+3(N+1)(f-c)+6k)}{6N}) = -\sin(\frac{\pi(1+3(f-c)+6k)}{6N})$$

Furthermore,using the addition formula for sine again results in

$$\begin{aligned}-\sin(\frac{\pi(1+3(f-c)+6k)}{6N}) &= -\sin(\frac{k\pi}{N})\cos(\frac{\pi(1+3(f-c))}{6N}) \\ &\quad -\cos(\frac{k\pi}{N})\sin(\frac{\pi(1+3(f-c))}{6N})\end{aligned}$$

Since $-1\le\cos(\frac{k\pi}{N})\le 1$ and $\lim\limits_{N\to\infty}\sin(\frac{\pi(1+3(f-c))}{6N})=0$, the second term disappears. Also $\lim\limits_{N\to\infty}\cos(\frac{\pi(1+3(f-c))}{6N})=1$. So, for large enough N,

$$-\sin(\frac{\pi(1+3(f-c)+6k)}{6N}) \approx -\sin(\frac{k\pi}{N})$$

Observe that

$$\begin{aligned}\prod_{k=1}^{\beta}\frac{1}{2(-\sin(\frac{k\pi}{N}))} &= -\prod_{k=1}^{\beta}\frac{1}{2\sin(\frac{k\pi}{N})} \\ &= -e^{\log(\prod_{k=1}^{\beta}\frac{1}{2\sin(\frac{k\pi}{N})})} \\ &= -e^{-\sum_{k=1}^{\beta}\log(2\sin(\frac{k\pi}{N}))}\end{aligned}$$

where $\sum\limits_{k=1}^{\beta}\log(2\sin(\frac{k\pi}{N}))$ can be thought of as the Riemann sum of $\log(2\sin(\frac{x\pi}{N}))$

on the interval $[0,\beta]$ with $\Delta x = 1$. Multiplying by $\frac{\pi}{N}$ gives the Riemann sum of $\log(2\sin(x))$ on the interval $[0,\frac{\beta\pi}{N}]$ with $\Delta x = \frac{\pi}{N}$. Now as $N \to \infty$, $\Delta x \to 0$ and the Riemann sum approximates the integral $\int_0^{\frac{\beta\pi}{N}} \log(2\sin u)du$. So

$$\begin{aligned}
-e^{-\sum_{k=1}^{\beta}\log(2\sin(\frac{k\pi}{N}))} &= -e^{-\frac{N}{\pi}\sum_{k=1}^{\beta}\frac{\pi}{N}\log(2\sin(\frac{k\pi}{N}))} \\
&= -e^{-\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}}\log(2\sin u)du+\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}}\log(2\sin u)du-\frac{N}{\pi}\sum_{k=1}^{\beta}\frac{\pi}{N}\log(2\sin(\frac{k\pi}{N}))} \\
&= -e^{\frac{N}{\pi}\Lambda(\frac{\beta\pi}{N})+\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}}\log(2\sin u)du-\frac{N}{\pi}\sum_{k=1}^{\beta}\frac{\pi}{N}\log(2\sin(\frac{k\pi}{N}))}
\end{aligned}$$

where

$$\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}}\log(2\sin u)du - \frac{N}{\pi}\sum_{k=1}^{\beta}\frac{\pi}{N}\log(2\sin(\frac{k\pi}{N}))$$

is the error between the integral and the Riemann sum approximation. Furthermore, observe that

$$\begin{aligned}
|S_\beta| &= \left|\sum_{\beta=0}^{N-1} e^{\frac{i\pi}{2N}(3\beta^2+\beta+3\beta(f-c)+\beta\lambda+5N\beta+3N\beta(f-c)+N\beta\lambda)}\prod_{k=1}^{\beta}\frac{1}{2\sin(\frac{\pi(1+3(N+1)(f-c)+6k)}{6N})}\right| \\
&= \left|\sum_{\beta=0}^{N-1} e^{p(\beta)}(-e^{\frac{N}{\pi}\Lambda(\frac{\beta\pi}{N})+\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}}\log(2\sin u)du-\frac{N}{\pi}\sum_{k=1}^{\beta}\frac{\pi}{N}\log(2\sin(\frac{k\pi}{N}))})\right| \\
&= \left|-\sum_{\beta=0}^{N-1} e^{p(\beta)}e^{\frac{N}{\pi}\Lambda(\frac{\beta\pi}{N})+\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}}\log(2\sin u)du-\frac{N}{\pi}\sum_{k=1}^{\beta}\frac{\pi}{N}\log(2\sin(\frac{k\pi}{N}))}\right| \\
&= \left|\sum_{\beta=0}^{N-1} e^{p(\beta)}e^{\frac{N}{\pi}\Lambda(\frac{\beta\pi}{N})+\frac{N}{\pi}\int_0^{\frac{\beta\pi}{N}}\log(2\sin u)du-\frac{N}{\pi}\sum_{k=1}^{\beta}\frac{\pi}{N}\log(2\sin(\frac{k\pi}{N}))}\right|
\end{aligned}$$

where $p(\beta) = \frac{i\pi}{2N}(3\beta^2+\beta+3\beta(f-c)+\beta\lambda+5N\beta+3N\beta(f-c)+N\beta\lambda)$ as before. Hence this case reduces to the case when $f-c \equiv 0 \mod 4$.

5.5 The other Riemann sum

Now consider the remaining summation

$$\begin{aligned}
S_\alpha &= \sum_{\alpha=0}^{N-1}\xi^{-\alpha^2}\omega((w_0^{-\prime})^*,(\frac{1}{w_1^{-\prime}})^*|\alpha) \\
&= \sum_{\alpha=0}^{N-1} e^{\frac{-2\pi i\alpha^2}{N}}(\prod_{k=1}^{\alpha}\frac{(e^{-\frac{1}{N}(\log(w_1^-)+i\pi(N+1)(f_1^-+c_1^-))})^*}{1-(e^{\frac{1}{N}(\log(w_0^-)+i\pi(N+1)(f_0^-+c_0^-))})^* e^{\frac{2\pi ik}{N}}})^* \\
&= \sum_{\alpha=0}^{N-1} e^{\frac{-2\pi i\alpha^2}{N}}\prod_{k=1}^{\alpha}\frac{e^{-\frac{1}{N}(\log(w_1^-)+i\pi(N+1)(f_1^-+c_1^-))}}{1-e^{\frac{1}{N}(\log(w_0^-)+i\pi(N+1)(f_0^-+c_0^-))}e^{\frac{-2\pi ik}{N}}}
\end{aligned}$$

Recall that for the figure eight knot,

$$w_0^- = w_1^- = w_2^- = e^{-\frac{\pi i}{3}}$$

$$f_0^- = -f,\ f_1^- = 1+2f, \text{ and } f_2^- = -f$$

$$c_0^- = c+\mu,\ c_1^- = 1-2c+\tfrac{1}{2}\lambda-2\mu, \text{ and } c_2^- = c-\tfrac{1}{2}\lambda+\mu.$$

So

$$\begin{aligned}
S_\alpha &= \sum_{\alpha=0}^{N-1} e^{\frac{-2\pi i\alpha^2}{N}} \prod_{k=1}^{\alpha} \frac{e^{-\frac{1}{N}(\frac{-\pi i}{3}+i\pi(N+1)(1+2f+1-2c+\frac{1}{2}\lambda-2\mu))}}{1-e^{\frac{1}{N}(\frac{-\pi i}{3}+i\pi(N+1)(-f+c+\mu))}e^{\frac{-2\pi ik}{N}}} \\
&= \sum_{\alpha=0}^{N-1} e^{\frac{-2\pi i\alpha^2}{N}} e^{-\frac{\alpha}{N}(\frac{-\pi i}{3}+i\pi(N+1)(2+2f-2c+\frac{1}{2}\lambda-2\mu))} \prod_{k=1}^{\alpha} \frac{1}{1-e^{\frac{1}{N}(\frac{-\pi i}{3}+i\pi(N+1)(-f+c+\mu))}e^{\frac{-2\pi ik}{N}}} \\
&= \sum_{\alpha=0}^{N-1} e^{\frac{-2\pi i\alpha^2}{N}-\frac{\alpha}{N}(\frac{-\pi i}{3}+i\pi(N+1)(2+2f-2c+\frac{1}{2}\lambda-2\mu))} \prod_{k=1}^{\alpha} \frac{1}{1-e^{-\frac{\pi i(1-3(N+1)(-f+c+\mu)+6k)}{3N}}} \\
&= \sum_{\alpha=0}^{N-1} e^{\frac{-2\pi i\alpha^2}{N}-\frac{\alpha}{N}(\frac{-\pi i}{3}+i\pi(N+1)(2+2f-2c+\frac{1}{2}\lambda-2\mu))} \\
&\quad * \prod_{k=1}^{\beta} \frac{e^{\frac{\pi i(1-3(N+1)(-f+c+\mu)+6k)}{6N}}}{e^{\frac{\pi i(1-3(N+1)(-f+c+\mu)+6k)}{6N}}-e^{-\frac{\pi i(1-3(N+1)(-f+c+\mu)+6k)}{6N}}} \\
&= \sum_{\alpha=0}^{N-1} e^{\frac{-2\pi i\alpha^2}{N}-\frac{\alpha}{N}(\frac{-\pi i}{3}+i\pi(N+1)(2+2f-2c+\frac{1}{2}\lambda-2\mu))+\frac{\pi i(\alpha-3\alpha(N+1)(-f+c+\mu)+6\frac{\alpha(\alpha+1)}{2})}{6N}} \\
&\quad * \prod_{k=1}^{\alpha} \frac{1}{i}\frac{1}{2\sin(\frac{\pi(1-3(N+1)(-f+c+\mu)+6k)}{6N})} \\
&= \sum_{\alpha=0}^{N-1} e^{\frac{-2\pi i\alpha^2}{N}-\frac{\alpha}{N}(\frac{-\pi i}{3}+i\pi(N+1)(2+2f-2c+\frac{1}{2}\lambda-2\mu))+\frac{\pi i(\alpha-3\alpha(N+1)(-f+c+\mu)+3\alpha(\alpha+1))}{6N}-\frac{i\pi\alpha}{2}} \\
&\quad * \prod_{k=1}^{\alpha} \frac{1}{2\sin(\frac{\pi(1-3(N+1)(-f+c+\mu)+6k)}{6N})} \\
&= \sum_{\alpha=0}^{N-1} e^{\frac{i\pi}{2N}(-3\alpha^2-2\alpha-3\alpha(f-c-\mu)-\alpha\lambda-5N\alpha-3N\alpha(f-c-\mu)-N\alpha\lambda)} \\
&\quad * \prod_{k=1}^{\alpha} \frac{1}{2\sin(\frac{\pi(1+3(N+1)(f-c-\mu)+6k)}{6N})}
\end{aligned}$$

Thus S_α is a summation of vectors in $\mathbb{C}$ with magnitude $\prod_{k=1}^{\alpha} \frac{1}{2\sin(\frac{\pi(1+3(N+1)(f-c-\mu)+6k)}{6N})}$ and direction $e^{\frac{i\pi}{2N}(-3\alpha^2-2\alpha-3\alpha(f-c-\mu)-\alpha\lambda-5N\alpha-3N\alpha(f-c-\mu)-N\alpha\lambda)}$. As with the S_β summation, the goal will be to show that $\prod_{k=1}^{\alpha} \frac{1}{2\sin(\frac{\pi(1+3(N+1)(f-c-\mu)+6k)}{6N})} \approx e^{\frac{N}{\pi}\Lambda(\frac{\alpha\pi}{N})}$.

Using the addition formula for sine, note that

$\sin(\frac{\pi(1+3(N+1)(f-c-\mu)+6k)}{6N})$

$$= \sin(\tfrac{\pi(1+3N(f-c-\mu)+3(f-c-\mu)+6k)}{6N})$$
$$= \sin(\tfrac{\pi(1+3(f-c-\mu)+6k)}{6N} + \tfrac{3\pi N(f-c-\mu)}{6N})$$
$$= \sin(\tfrac{\pi(1+3(f-c-\mu)+6k)}{6N})\cos(\tfrac{\pi(f-c-\mu)}{2}) + \cos(\tfrac{\pi(1+3(f-c-\mu)+6k)}{6N})\sin(\tfrac{\pi(f-c-\mu)}{2})$$

Since $(f-c-\mu) \in \mathbb{Z}$,

$$\cos(\frac{\pi(f-c-\mu)}{2}) = \begin{cases} 1 & \text{if } (f-c-\mu) \equiv 0 \mod 4 \\ 0 & \text{if } (f-c-\mu) \equiv 1 \mod 4 \\ -1 & \text{if } (f-c-\mu) \equiv 2 \mod 4 \\ 0 & \text{if } (f-c-\mu) \equiv 3 \mod 4 \end{cases}$$

and

$$\sin(\frac{\pi(f-c-\mu)}{2}) = \begin{cases} 0 & \text{if } (f-c-\mu) \equiv 0 \mod 4 \\ 1 & \text{if } (f-c-\mu) \equiv 1 \mod 4 \\ 0 & \text{if } (f-c-\mu) \equiv 2 \mod 4 \\ -1 & \text{if } (f-c-\mu) \equiv 3 \mod 4 \end{cases}$$

To simplify calculations, assume $(f-c-\mu) \equiv 0 \mod 4$. Then

$$\sin(\frac{\pi(1+3(N+1)(f-c-\mu)+6k)}{6N}) = \sin(\frac{\pi(1+3(f-c-\mu)+6k)}{6N})$$

Furthermore,

$$\begin{aligned} \sin(\frac{\pi(1+3(f-c-\mu)+6k)}{6N}) &= \sin(\frac{k\pi}{N})\cos(\frac{\pi(1+3(f-c-\mu))}{6N}) \\ &\quad + \cos(\frac{k\pi}{N})\sin(\frac{\pi(1+3(f-c-\mu))}{6N}) \end{aligned}$$

Since $-1 \leq \cos(\frac{k\pi}{N}) \leq 1$ and $\lim_{N\to\infty} \sin(\frac{\pi(1+3(N+1)(f-c-\mu))}{6N}) = 0$, the second term disappears. Also $\lim_{N\to\infty} \cos(\frac{\pi(1+3(N+1)(f-c-\mu))}{6N}) = 1$. So, for large enough N,

$$\sin(\frac{\pi(1+3(f-c-\mu)+6k)}{6N}) \approx \sin(\frac{k\pi}{N})$$

Observe that

$$\begin{aligned} \prod_{k=1}^{\alpha} \frac{1}{2\sin(\frac{k\pi}{N})} &= e^{\log(\prod_{k=1}^{\alpha} \frac{1}{2\sin(\frac{k\pi}{N})})} \\ &= e^{-\sum_{k=1}^{\alpha} \log(2\sin(\frac{k\pi}{N}))} \end{aligned}$$

where $\sum_{k=1}^{\alpha} \log(2\sin(\frac{k\pi}{N}))$ can be thought of as the Riemann sum of $\log(2\sin(\frac{x\pi}{N}))$ on the interval $[0,\alpha]$ with $\Delta x = 1$. Multiplying by $\frac{\pi}{N}$ gives the Riemann sum of $\log(2\sin(x))$ on the interval $[0, \frac{\alpha\pi}{N}]$ with $\Delta x = \frac{\pi}{N}$. Now as $N \to \infty$, $\Delta x \to 0$ and the Riemann sum approximates the integral $\int_0^{\frac{\alpha\pi}{N}} \log(2\sin u)du$. So

$$\begin{aligned} e^{-\sum_{k=1}^{\alpha} \log(2\sin(\frac{k\pi}{N}))} &= e^{-\frac{N}{\pi}\sum_{k=1}^{\alpha} \frac{\pi}{N}\log(2\sin(\frac{k\pi}{N}))} \\ &= e^{-\frac{N}{\pi}\int_0^{\frac{\alpha\pi}{N}} \log(2\sin u)du + \frac{N}{\pi}\int_0^{\frac{\alpha\pi}{N}} \log(2\sin u)du - \frac{N}{\pi}\sum_{k=1}^{\alpha} \frac{\pi}{N}\log(2\sin(\frac{k\pi}{N}))} \\ &= e^{\frac{N}{\pi}\Lambda(\frac{\alpha\pi}{N}) + \frac{N}{\pi}\int_0^{\frac{\alpha\pi}{N}} \log(2\sin u)du - \frac{N}{\pi}\sum_{k=1}^{\alpha} \frac{\pi}{N}\log(2\sin(\frac{k\pi}{N}))} \end{aligned}$$

where

$$\frac{N}{\pi}\int_0^{\frac{\alpha\pi}{N}} \log(2\sin u)du - \frac{N}{\pi}\sum_{k=1}^{\alpha} \frac{\pi}{N}\log(2\sin(\frac{k\pi}{N}))$$

is the error between the integral and the Riemann sum approximation. As before,

$$\frac{N}{\pi}\int_0^{\frac{\alpha\pi}{N}} \log(2\sin u)du - \frac{N}{\pi}\sum_{k=1}^{\alpha} \frac{\pi}{N}\log(2\sin(\frac{k\pi}{N})) \leq \frac{\alpha}{N} + \sum_{k=1}^{\alpha}(\log\frac{\alpha}{k} - 1)$$

when $0 \leq \alpha \leq \frac{N}{2}$.

Now

$$\begin{aligned} \lim_{N\to\infty}\frac{1}{N}\log|S_\alpha| &= \lim_{N\to\infty}\frac{1}{N}\log|\sum_{\alpha=0}^{N-1} \xi^{-\alpha^2}(\omega((w_0^{-\prime})^*, (\frac{1}{w_1^{-\prime}})^*|\alpha))^*| \\ &= \lim_{N\to\infty}\frac{1}{N}\log|\sum_{\alpha=0}^{\frac{N}{2}} e^{\frac{i\pi}{2N}(-3\alpha^2-2\alpha-3\alpha(f-c-\mu)-\alpha\lambda-5N\alpha-3N\alpha(f-c-\mu)-N\alpha\lambda)} \\ &\quad * \prod_{k=1}^{\alpha}\frac{1}{2\sin(\frac{\pi(1+3(N+1)(f-c-\mu)+6k)}{6N})}| \\ &\leq \lim_{N\to\infty}\frac{1}{N}\log|\sum_{\alpha=0}^{\frac{N}{2}}\prod_{k=1}^{\alpha}\frac{1}{2\sin(\frac{\pi(1+3(N+1)(f-c-\mu)+6k)}{6N})}| \\ &= \lim_{N\to\infty}\frac{1}{N}\log|\sum_{\alpha=0}^{\frac{N}{2}} e^{-\frac{N}{\pi}\int_0^{\frac{\alpha\pi}{N}}\log(2\sin u)du + \frac{\alpha}{N} + \sum_{k=1}\alpha(\log\frac{\alpha}{k}-1)} \\ &\quad + \sum_{\alpha=\frac{N}{2}}^{N-1} e^{-\sum_{k=1}^{\alpha}\log(2sin(\frac{k\pi}{N}))}| \\ &\leq \frac{1}{\pi}\Lambda(\frac{\pi}{6}) \end{aligned}$$

In the cases when $(f - c - \mu) \not\equiv 0 \mod 4$, the result follows similarly to when $f - c \not\equiv 0 \mod 4$ in S_β.

5.6 The remaining term

There is one more piece to consider

$$\lim_{N\to\infty}\frac{1}{N}\log|((w_0^{+\prime})^{-c_1^+}(w_1^{+\prime})^{c_0^+}(w_0^{-\prime})^{-c_1^-}(w_1^{-\prime})^{c_0^-})^{\frac{N-1}{2}}[w_0^{-\prime}]\frac{g(w_0^{+\prime})}{g(w_0^{-\prime})}|$$

which can be broken into three pieces

$$\lim_{N\to\infty}\frac{\log|((w_0^{+\prime})^{-c_1^+}(w_1^{+\prime})^{c_0^+}(w_0^{-\prime})^{-c_1^-}(w_1^{-\prime})^{c_0^-})^{\frac{N-1}{2}}|}{N}+\lim_{N\to\infty}\frac{\log|[w_0^{-\prime}]|}{N}+\lim_{N\to\infty}\frac{\log|\frac{g(w_0^{+\prime})}{g(w_0^{-\prime})}|}{N}$$

First note that

$$\begin{aligned}
(w_0^{+\prime}) &= e^{-\frac{1}{N}(\frac{i\pi}{3}+i\pi(N+1)(f-c))}\\
(w_1^{+\prime}) &= e^{\frac{1}{N}(\frac{i\pi}{3}+i\pi(N+1)(-2-2f+2c-\frac{1}{2}\lambda))}\\
(w_0^{-\prime}) &= e^{-\frac{1}{N}(-\frac{i\pi}{3}+i\pi(N+1)(-f+c+\mu))}\\
(w_1^{-\prime}) &= e^{\frac{1}{N}(-\frac{i\pi}{3}+i\pi(N+1)(2+2f-2c+\frac{1}{2}\lambda-2\mu))}
\end{aligned}$$

So

$$\begin{aligned}
(w_0^{+\prime})^{-c_1^+} &= e^{-\frac{1}{N}(\frac{i\pi}{3}+i\pi(N+1)(f-c))(1-2c+\frac{1}{2}\lambda)}\\
&= \exp(-\frac{i\pi}{N}(\frac{1}{3}+(N+1)f-(N+1)c-\frac{2c}{3}-2(N+1)fc+2(N+1)c^2\\
&\quad +\frac{\lambda}{6}+\frac{1}{2}(N+1)f\lambda+\frac{1}{2}(N+1)c\lambda))\\
(w_1^{+\prime})^{c_0^+} &= e^{\frac{1}{N}(\frac{i\pi}{3}+i\pi(N+1)(-2-2f+2c-\frac{1}{2}\lambda))c}\\
&= \exp(\frac{i\pi}{N}(\frac{c}{3}-2(N+1)c-2(N+1)fc+2(N+1)c^2-\frac{1}{2}(N+1)c\lambda))\\
(w_0^{-\prime})^{-c_1^-} &= e^{-\frac{1}{N}(-\frac{i\pi}{3}+i\pi(N+1)(-f+c+\mu))(1-2c+\frac{1}{2}\lambda-2\mu)}\\
&= \exp(-\frac{i\pi}{N}(-\frac{1}{3}-(N+1)f+(N+1)c+(N+1)\mu+\frac{2c}{3}+2(N+1)fc\\
&\quad -2(N+1)c^2-4(N+1)c\mu-\frac{\lambda}{6}-\frac{1}{2}(N+1)f\lambda-\frac{1}{2}(N+1)c\lambda\\
&\quad -\frac{1}{2}(N+1)\lambda\mu+\frac{2\mu}{3}+2(N+1)f\mu-2(N+1)\mu^2))\\
(w_1^{-\prime})^{c_0^-} &= e^{\frac{1}{N}(-\frac{i\pi}{3}+i\pi(N+1)(2+2f-2c+\frac{1}{2}\lambda-2\mu))(c+\mu)}\\
&= \exp(\frac{i\pi}{N}(-\frac{c}{3}+2(N+1)c+2(N+1)fc-2(N+1)c^2+\frac{1}{2}(N+1)c\lambda\\
&\quad -4(N+1)c\mu-\frac{\mu}{3}+2(N+1)\mu+2(N+1)f\mu+\frac{1}{2}(N+1)\lambda\mu\\
&\quad -2(N+1)\mu^2))
\end{aligned}$$

Then

$$((w_0^{+\prime})^{-c_1^+}(w_1^{+\prime})^{c_0^+}(w_0^{-\prime})^{-c_1^-}(w_1^{-\prime})^{c_0^-})^{\frac{N-1}{2}}=(e^{\frac{i\pi}{N}(-\mu+(N+1)\mu)})^{\frac{N-1}{2}}=e^{\frac{i\pi(N-1)\mu}{2}}$$

Since N is odd, $N-1$ is even. Also recall that $\mu\in 2\mathbb{Z}$. Hence $\frac{(N-1)\mu}{2}$ is even. So

$e^{\frac{i\pi(N-1)\mu}{2}} = 1$. Thus
$$\lim_{N\to\infty} \frac{\log|((w_0^{+\prime})^{-c_1^+}(w_1^{+\prime})^{c_0^+}(w_0^{-\prime})^{-c_1^-}(w_1^{-\prime})^{c_0^-})^{\frac{N-1}{2}}|}{N} = 0$$

Next consider
$$[w_0^{-\prime}] = \frac{1-(w_0^{-\prime})^N}{N(1-w_0^{-\prime})} = \frac{1-e^{-\frac{i\pi}{3}+i\pi(N+1)(-f+c+\mu)}}{N(1-e^{\frac{1}{N}(-\frac{i\pi}{3}+i\pi(N+1)(-f+c+\mu))})}$$
Since $N+1$ is even and f, c, $\mu \in \mathbb{Z}$, it is the case that
$$e^{-\frac{i\pi}{3}+i\pi(N+1)(-f+c+\mu)} = e^{-\frac{i\pi}{3}}$$

So
$$[w_0^{-\prime}] = \frac{1-e^{-\frac{i\pi}{3}}}{N(1-e^{\frac{1}{N}(-\frac{i\pi}{3}+i\pi(N+1)(-f+c+\mu))})} = \frac{1-e^{-\frac{i\pi}{3}}}{N(1-e^{-\frac{i\pi}{3N}}e^{i\pi\frac{N+1}{N}(-f+c+\mu)})}$$

If $(-f+c+\mu)$ is even, then $\lim_{N\to\infty} e^{i\pi\frac{N+1}{N}(-f+c+\mu)} = 1$. This would mean that $\lim_{N\to\infty} 1-e^{-\frac{1\pi}{3N}}e^{i\pi\frac{N+1}{N}(-f+c+\mu)} = 0$. Using L'Hospital's rule,

$\lim_{N\to\infty} N(1-e^{-\frac{i\pi}{3N}}e^{i\pi\frac{N+1}{N}(-f+c+\mu)})$
$$\begin{aligned}
&= \lim_{N\to\infty} \frac{1-e^{-\frac{i\pi}{3N}}e^{i\pi\frac{N+1}{N}(-f+c+\mu)}}{\frac{1}{N}} \\
&= \lim_{N\to\infty} \frac{-\frac{1}{N^2}(\frac{i\pi}{3}-i\pi(-f+c+\mu))e^{\frac{-i\pi}{3N}}ei\pi\frac{N+1}{N}(-f+c+\mu)}{-\frac{1}{N^2}} \\
&= \lim_{N\to\infty} (\frac{i\pi}{3}-i\pi(-f+c+\mu))e^{\frac{-i\pi}{3N}}ei\pi\frac{N+1}{N}(-f+c+\mu) \\
&= \frac{i\pi}{3}-i\pi(-f+c+\mu)
\end{aligned}$$

Hence
$$\lim_{N\to\infty} \frac{\log|[w_0^{-\prime}]|}{N} = \lim_{N\to\infty} \frac{\log|\frac{1-e^{-\frac{i\pi}{3}}}{\frac{i\pi}{3}-i\pi(-f+c+\mu)}|}{N} = 0$$

If, on the other hand, $(-f+c+\mu)$ is odd, then $\lim_{N\to\infty} e^{i\pi\frac{N+1}{N}(-f+c+\mu)} = -1$. So
$$\lim_{N\to\infty} \frac{1-e^{-\frac{i\pi}{3}}}{N(1-e^{-\frac{i\pi}{3N}}e^{i\pi\frac{N+1}{N}(-f+c+\mu)})} = 0$$
and thus
$$\lim_{N\to\infty} \log|\frac{1-e^{-\frac{i\pi}{3}}}{N(1-e^{-\frac{i\pi}{3N}}e^{i\pi\frac{N+1}{N}(-f+c+\mu)})}| = -\infty$$
Using L'Hospital's rule
$$\begin{aligned}
\lim_{N\to\infty} \frac{\log|\frac{1-e^{-\frac{i\pi}{3}}}{N(1-e^{-\frac{i\pi}{3N}}e^{i\pi\frac{N+1}{N}(-f+c+\mu)})}|}{N} &= \frac{(1+\frac{i\pi}{3N}-\frac{i\pi(-f+c+\mu)}{N})e^{-\frac{i\pi}{3N}}e^{i\pi\frac{N+1}{N}(-f+c+\mu)}-1}{N(1-e^{-\frac{i\pi}{3N}}e^{i\pi\frac{N+1}{N}(-f+c+\mu)})} \\
&= \lim_{N\to\infty} \frac{-2}{2N} \\
&= 0
\end{aligned}$$

Finally consider

$$\begin{aligned}
\left|\frac{g(w_0^{+\prime})}{g(w_0^{-\prime})}\right| &= \left|\frac{\prod_{j=0}^{N-1}(1-w_0^{+\prime}\xi^{-j})^{\frac{1}{N}}}{\prod_{k=0}^{N-1}(1-w_0^{-\prime}\xi^{-k})^{\frac{1}{N}}}\right| \\
&= \frac{\left|\prod_{j=0}^{N-1}(1-e^{\frac{1}{N}(\frac{i\pi}{3}+i\pi(N+1)(f-c))}e^{-\frac{2i\pi j}{N}})^{\frac{1}{N}}\right|}{\left|\prod_{k=0}^{N-1}(1-e^{\frac{1}{N}(-\frac{i\pi}{3}+i\pi(N+1)(-f+c+\mu))}e^{-\frac{2i\pi k}{N}})^{\frac{1}{N}}\right|} \\
&= \frac{\left|(\prod_{j=0}^{N-1}(1-e^{\frac{1}{N}(\frac{i\pi}{3}+i\pi(N+1)(f-c))}e^{-\frac{2i\pi j}{N}}))^{\frac{1}{N}}\right|}{\left|(\prod_{k=0}^{N-1}(1-e^{\frac{1}{N}(-\frac{i\pi}{3}+i\pi(N+1)(-f+c+\mu))}e^{-\frac{2i\pi k}{N}}))^{\frac{1}{N}}\right|} \\
&= \frac{\left|(1-e^{\frac{i\pi}{3}+i\pi(N+1)(f-c)})^{\frac{1}{N}}\right|}{\left|(1-e^{-\frac{i\pi}{3}+i\pi(N+1)(-f+c+\mu)})^{\frac{1}{N}}\right|}
\end{aligned}$$

Now

$$\begin{aligned}
1-e^{\frac{i\pi}{3}+i\pi(N+1)(f-c)} &= 2ie^{\frac{\frac{i\pi}{3}+i\pi(N+1)(f-c)}{2}}\frac{e^{-\frac{\frac{i\pi}{3}+i\pi(N+1)(f-c)}{2}}-e^{\frac{\frac{i\pi}{3}+i\pi(N+1)(f-c)}{2}}}{2i} \\
&= 2ie^{\frac{\frac{i\pi}{3}+i\pi(N+1)(f-c)}{2}}\sin(-\frac{\frac{\pi}{3}+\pi(N+1)(f-c)}{2})
\end{aligned}$$

and

$$\begin{aligned}
1-e^{-\frac{i\pi}{3}+i\pi(N+1)(-f+c+\mu)} &= 2ie^{\frac{-\frac{i\pi}{3}+i\pi(N+1)(-f+c+\mu)}{2}} \\
&\quad *\frac{e^{-\frac{-\frac{i\pi}{3}+i\pi(N+1)(-f+c+\mu)}{2}}-e^{\frac{-\frac{i\pi}{3}+i\pi(N+1)(-f+c+\mu)}{2}}}{2i} \\
&= 2ie^{\frac{-\frac{i\pi}{3}+i\pi(N+1)(-f+c+\mu)}{2}} \\
&\quad *\sin(-\frac{-\frac{\pi}{3}+\pi(N+1)(-f+c+\mu)}{2})
\end{aligned}$$

Also recall that $x^{\frac{1}{N}} = e^{\frac{\log|x| + i\arg x}{N}}$. So $\left|x^{\frac{1}{N}}\right| = \left|e^{\frac{\log|x|}{N}} e^{\frac{1 \arg x}{N}}\right| = e^{\frac{\log|x|}{N}}$. Thus

$$\begin{aligned}
\left|\frac{g(w_0^{+\prime})}{g(w_0^{-\prime})}\right| &= \frac{\left|(1 - e^{\frac{i\pi}{3} + i\pi(N+1)(f-c)})^{\frac{1}{N}}\right|}{\left|(1 - e^{-\frac{i\pi}{3} + i\pi(N+1)(-f+c+\mu)})^{\frac{1}{N}}\right|} \\
&= \frac{\left|e^{\frac{\log|2ie^{\frac{\frac{i\pi}{3} + i\pi(N+1)(f-c)}{2}} \sin(-\frac{\frac{\pi}{3} + \pi(N+1)(f-c)}{2})|}{N}}\right|}{\left|e^{\frac{\log|2ie^{\frac{-\frac{i\pi}{3} + i\pi(N+1)(-f+c+\mu)}{2}} \sin(-\frac{-\frac{\pi}{3} + \pi(N+1)(-f+c+\mu)}{2})|}{N}}\right|} \\
&= \frac{\left|e^{\frac{\log|2i|}{N}} e^{\frac{\log|e^{\frac{\frac{i\pi}{3} + i\pi(N+1)(f-c)}{2}}|}{N}} e^{\frac{\log|\sin(-\frac{\frac{\pi}{3} + \pi(N+1)(f-c)}{2})|}{N}}\right|}{\left|e^{\frac{\log|2i|}{N}} e^{\frac{\log|e^{\frac{-\frac{i\pi}{3} + i\pi(N+1)(-f+c+\mu)}{2}}|}{N}} e^{\frac{\log|\sin(-\frac{-\frac{\pi}{3} + \pi(N+1)(-f+c+\mu)}{2})|}{N}}\right|} \\
&= \frac{\left|e^{\frac{\log|\sin(-\frac{\frac{\pi}{3} + \pi(N+1)(f-c)}{2})|}{N}}\right|}{\left|e^{\frac{\log|\sin(-\frac{-\frac{\pi}{3} + \pi(N+1)(-f+c+\mu)}{2})|}{N}}\right|}
\end{aligned}$$

since $|e^{\frac{\frac{i\pi}{3} + i\pi(N+1)(f-c)}{2}}| = |e^{\frac{-\frac{i\pi}{3} + i\pi(N+1)(-f+c+\mu)}{2}}| = 1$. Because $N+1$ is even, $\frac{N+1}{2} \in \mathbb{Z}$. So

$$\begin{aligned}
|\sin(-\frac{\frac{\pi}{3} + \pi(N+1)(f-c)}{2})| &= |\sin(\frac{\frac{\pi}{3} + \pi(N+1)(f-c)}{2})| \\
&= |\sin(\frac{\pi}{6})\cos(\frac{\pi(N+1)(f-c)}{2}) \\
&\quad + \sin(\frac{\pi(N+1)(f-c)}{2})\cos(\frac{\pi}{6})| \\
&= \sin(\frac{\pi}{6})
\end{aligned}$$

and

$$\begin{aligned}
|\sin(-\frac{-\frac{\pi}{3} + \pi(N+1)(-f+c+\mu)}{2})| &= |\sin(\frac{-\frac{\pi}{3} + \pi(N+1)(-f+c+\mu)}{2})| \\
&= |\sin(\frac{\pi}{6})\cos(\frac{\pi(N+1)(-f+c+\mu)}{2}) \\
&\quad + \sin(\frac{\pi(N+1)(-f+c+\mu)}{2})\cos(\frac{\pi}{6})| \\
&= \sin(\frac{\pi}{6})
\end{aligned}$$

Therefore

$$\lim_{N\to\infty}\frac{\log\left|\frac{g(w_0^{+\prime})}{g(w_0^{-\prime})}\right|}{N}=\lim_{N\to\infty}\frac{\log\left|\frac{e^{\frac{\log(\sin(\frac{\pi}{6}))}{N}}}{e^{\frac{\log(\sin(\frac{\pi}{6}))}{N}}}\right|}{N}=0$$

5.7 The total growth

Finally, combining all three pieces gives the result

$$\begin{aligned}\lim_{N\to\infty}\frac{\log|H_N(4_1)|}{N} &= \lim_{N\to\infty}\frac{1}{N}\log|((w_0^{+\prime})^{-c_1^+}(w_1^{+\prime})^{c_0^+}(w_0^{-\prime})^{-c_1^-}(w_1^{-\prime})^{c_0^-})^{\frac{N-1}{2}}[w_0^{-\prime}]\frac{g(w_0^{+\prime})}{g(w_0^{-\prime})}| \\ &\quad+\lim_{N\to\infty}\frac{1}{N}\log|\sum_{\alpha=0}^{N-1}\xi^{-\alpha^2}(\omega((w_0^{-\prime})^*,(\frac{1}{w_1^{-\prime}})^*|\alpha))^*| \\ &\quad+\lim_{N\to\infty}\frac{1}{N}\log|\sum_{\beta=0}^{N-1}\xi^{\beta^2}\omega(w_0^{+\prime},\frac{1}{w_1^{+\prime}}|\beta)| \\ &\leq 0+\frac{1}{\pi}\Lambda(\frac{\pi}{6})+\frac{1}{\pi}\Lambda(\frac{\pi}{6})\end{aligned}$$

By Lemma 2.15, $\Lambda(\frac{\pi}{6})=\frac{3}{2}\Lambda(\frac{\pi}{3})$. Recall that the complement of the figure eight knot is composed of two ideal tetrahedra with dihedral angles $\frac{\pi}{3}$. Hence, by Theorem 2.17, $Vol(S^3\setminus 4_1)=6\Lambda(\frac{\pi}{3})$. Thus proves Theorem 5.1.

CHAPTER 6
FUTURE RESEARCH

6.1 Finding a lower bound

Since only an upper bound was found in Chapter 5, it is natural to wonder about a lower bound. As evidenced in [7], the Baseilhac and Benedetti quantum hyperbolic invariants grow exponentially for certain values of λ and oscillate for other values of λ. The behavior of the invariants is based on the direction vectors of S_β and S_α, which were ignored in finding the upper bound.

Recall that S_β is a summation of vectors with magnitude $\prod_{k=1}^{\beta} \frac{1}{2\sin(\frac{\pi(1+3(N+1)(f-c)+6k)}{6N})}$ pointing in the direction $e^{\frac{i\pi}{2N}(3\beta^2+\beta+3\beta(f-c)+\beta\lambda+5N\beta+3N\beta(f-c)+N\beta\lambda)}$. To find an upper bound for S_β, the direction of the vectors could be ignored. To find a nonzero lower bound for S_β, the direction must be considered. For some values of λ, the vectors will oscillate rapidly causing cancellation. However when $\frac{i\pi}{2N}(3\beta^2+\beta+3\beta(f-c)+\beta\lambda+5N\beta+3N\beta(f-c)+N\beta\lambda)$ is changing at a rate of $2ni\pi$ for some $n \in \mathbb{Z}$ the vectors will point in the same direction and their magnitudes will add together. If this occurs when the magnitudes of the vectors are the greatest, then $S_\beta \approx e^{\frac{N}{\pi}\Lambda(\frac{\beta\pi}{N})}$. Observe that

$$\begin{aligned}
\frac{d}{d\beta}\frac{i\pi}{2N}(3\beta^2+\beta+3\beta(f-c)+\beta\lambda+5N\beta+3N\beta(f-c)+N\beta\lambda) &= 2ni\pi \\
\frac{i\pi}{2N}(6\beta+1+3(f-c)+\lambda+5N+3N(f-c)+N\lambda) &= 2ni\pi \\
6\beta+1+3(f-c)+\lambda+5N+3N(f-c)+N\lambda &= 4nN \\
-4nN+1+3(f-c)+\lambda+5N+3N(f-c)+N\lambda &= -6\beta \\
4nN-1-3(f-c)-\lambda-5N-3N(f-c)-N\lambda &= 6\beta \\
\frac{N(4n-5-3(f-c)-\lambda)-1-3(f-c)-\lambda}{6} &= \beta
\end{aligned}$$

So the vectors point in roughly the same direction near $\beta = \frac{N(4n-5-3(f-c)-\lambda)-1-3(f-c)-\lambda}{6}$. As $N \to \infty$, $\frac{N(4n-5-3(f-c)-\lambda)-1-3(f-c)-\lambda}{6} \to \frac{N(4n-5-3(f-c)-\lambda)}{6}$. When $4n-5-3(f-$

$c) - \lambda = 1$, this is exactly where $\Lambda(\frac{\beta\pi}{N})$ achieves a maximum.

Consider the case when $f - c \equiv 0 \mod 4$. Then $f - c = 4a$ for some $a \in \mathbb{Z}$. Also recall that $\lambda \in 2\mathbb{Z}$; so suppose $\lambda \equiv 0 \mod 4$. Then $\lambda = 4b$ for some $b \in \mathbb{Z}$. Thus $4n - 5 - 3(f - c) - \lambda = 1 \Leftrightarrow 4n - 12a - 4b = 6 \Leftrightarrow n - 3a - b = \frac{3}{2}$. Since n, a, and b are integers, this cannot be true. So suppose instead that $\lambda \equiv 2 \mod 4$. Then $\lambda = 2 + 4b$ for some $b \in \mathbb{Z}$. Thus $4n - 5 - 3(f - c) - \lambda = 1 \Leftrightarrow 4n - 12a - (2 + 4b) = 6 \Leftrightarrow 4n - 12a - 4b = 8 \Leftrightarrow n - 3a - b = 2$. Since n, a, and b are integers, this is possible when $n = 2 + \frac{3(f-c)}{4} + \frac{\lambda-2}{4}$. Indeed this supports the findings in [7]. The invariants grow exponentially when $\lambda \equiv 2 \mod 4$, but not when $\lambda \equiv 0 \mod 4$. The vectors which point in the same half-plane as the vector $\exp(\frac{i\pi}{12N}(\frac{1}{2}(N-1-3(f-c)-\lambda)^2 + (N-1-3(f-c)-\lambda) + 3(N-1-3(f-c)-\lambda)(f-c) + (N-1-3(f-c)-\lambda)\lambda + 5N(N-1-3(f-c)-\lambda) + 3N(N-1-3(f-c)-\lambda)(f-c) + N(N-1-3(f-c)-\lambda)\lambda))$ will contribute the most to S_β. Investigating these vectors is the next step in this project.

Similarly, S_α is a summation of vectors with magnitude $\prod_{k=1}^{\alpha} \frac{1}{2\sin(\frac{\pi(1+3(N+1)(f-c-\mu)+6k)}{6N})}$ pointing in the direction $e^{\frac{i\pi}{2N}(-3\alpha^2-2\alpha-3\alpha(f-c-\mu)-\alpha\lambda-5N\alpha-3N\alpha(f-c-\mu)-N\alpha\lambda)}$. As with S_β, for some values of λ, the vectors will oscillate rapidly causing cancellation. However when $\frac{i\pi}{2N}(-3\alpha^2 - 2\alpha - 3\alpha(f - c - \mu) - \alpha\lambda - 5N\alpha - 3N\alpha(f - c - \mu) - N\alpha\lambda)$ is changing at a rate of $2ni\pi$ for some $n \in \mathbb{Z}$ the vectors will point in the same direction and their magnitudes will add together. If this occurs when the magnitudes of the vectors are the greatest, then $S_\alpha \approx e^{\frac{N}{\pi}\Lambda(\frac{\alpha\pi}{N})}$. Observe that

$$\begin{aligned}
\frac{d}{d\alpha}\frac{i\pi}{2N}(-3\alpha^2 - 2\alpha - 3\alpha(f-c-\mu) - \alpha\lambda - 5N\alpha - 3N\alpha(f-c-\mu) - N\alpha\lambda) &= 2ni\pi \\
\frac{i\pi}{2N}(-6\alpha - 2 - 3(f-c-\mu) - \lambda - 5N - 3N(f-c-\mu) - N\lambda) &= 2ni\pi \\
-6\alpha - 2 - 3(f-c-\mu) - \lambda - 5N - 3N(f-c-\mu) - N\lambda &= 4nN \\
-4nN - 2 - 3(f-c-\mu) - \lambda - 5N - 3N(f-c-\mu) - N\lambda &= 6\alpha \\
\frac{N(-4n - 5 - 3(f-c-\mu) - \lambda) - 2 - 3(f-c-\mu) - \lambda}{6} &= \alpha
\end{aligned}$$

So the vectors point in the same direction near $\alpha = \frac{N(-4n-5-3(f-c-\mu)-\lambda)-2-3(f-c-\mu)-\lambda}{6}$.

As $N \to \infty$, $\frac{N(-4n-5-3(f-c-\mu)-\lambda)-2-3(f-c-\mu)-\lambda}{6} \to \frac{N(-4n-5-3(f-c-\mu)-\lambda)}{6}$. When $-4n-5-3(f-c-\mu)-\lambda = 1$, this is exactly where $\Lambda(\frac{\alpha\pi}{N})$ achieves a maximum.

Consider the case when $f-c-\mu \equiv 0 \mod 4$. Then $f-c-\mu = 4a$ for some $a \in \mathbb{Z}$. Also recall that $\lambda \in 2\mathbb{Z}$; so suppose $\lambda \equiv 0 \mod 4$. Then $\lambda = 4b$ for some $b \in \mathbb{Z}$. Thus $-4n-5-3(f-c-\mu)-\lambda = 1 \Leftrightarrow -4n-12a-4b = 6 \Leftrightarrow -n-3a-b = \frac{3}{2}$. Since n, a, and b are integers, this cannot be true. So suppose instead that $\lambda \equiv 2 \mod 4$. Then $\lambda = 2+4b$ for some $b \in \mathbb{Z}$. Then $-4n-5-3(f-c-\mu)-\lambda = 1 \Leftrightarrow -4n-12a-(2+4b) = 6 \Leftrightarrow -4n-12a-4b = 8 \Leftrightarrow -n-3a-b = 2$. Since n, a, and b are integers, this is possible when $n = -2 - \frac{3(f-c-\mu)}{4} - \frac{\lambda-2}{4}$. Again this supports the findings in [7]. The invariants grow exponentially when $\lambda \equiv 2 \mod 4$, but not when $\lambda \equiv 0 \mod 4$. The vectors which point in the same half-plane as the vector $\exp(\frac{i\pi}{12N}(-\frac{1}{2}(N-2-3(f-c-\mu)-\lambda)^2 - 2(N-2-3(f-c-\mu)-\lambda) - 3(N-2-3(f-c-\mu)-\lambda)(f-c-\mu) - (N-2-3(f-c-\mu)-\lambda)\lambda - 5N(N-2-3(f-c-\mu)-\lambda) - 3N(N-2-3(f-c-\mu)-\lambda)(f-c-\mu) - N(N-2-3(f-c-\mu)-\lambda)\lambda))$ will contribute the most to S_α. It is expected that these vectors will be similar to those in the S_β case.

6.2 The sister manifold of the figure eight knot

One advantage of the Baseilhac and Benedetti invariant is that it is not limited to knot complements. It can be computed for any hyperbolic manifold. The sister manifold of the figure eight knot is a hyperbolic manifold which is also composed of two ideal tetrahedra with dihedral angles $\frac{\pi}{3}$. Thus this manifold will have the same volume as the complement of the figure eight knot. Because of its similarities with the complement of the figure eight knot, computing the quantum hyperbolic invariants for the sister manifold should mirror the computations in this thesis. It would be of great interest to see if the invariants of the sister manifold also grew at the same rate. If they did, the result would support the conjecture that

$$\lim_{N\to\infty} \frac{\log |H_N(M)|}{N} = \frac{1}{2\pi} Vol(M)$$

where M is a hyperbolic manifold. If they did not, the result would suggest that the Baseilhac and Benedetti invariants act differently on knot complements than they do for other manifolds.

6.3 The n-sheeted covers of the figure eight knot

Because of the complexity of the Baseilhac and Benedetti invariants, they are difficult to compute for most hyperbolic knots. The n-sheeted covers of the figure eight knot provides another example of a fairly simple manifold. The computations should be similar to those of the figure eight knot. Again, a second result would support the volume conjecture above.

REFERENCES

[1] Robert G. Bartle. *The Elements of Real Analysis*. John Wiley & Sons, Inc., 1975.

[2] Stéphane Baseilhac and Riccardo Benedetti. Classical and quantum dilogarithmic invariants of flat $\mathrm{PSL}(2,\mathbb{C})$-bundles over 3-manifolds. *Geometry & Topology*, 9:493–569, 2005.

[3] Stéphane Baseilhac and Riccardo Benedetti. Quantum hyperbolic geometry. *Algebraic & Geometric Topology*, 7:845–917, 2007.

[4] Werner Fenchel. *Elementary Geometry in Hyperbolic Space*. Walter de Gruyter & Co., 1989.

[5] Gerald Flynn. Numerical methods for improper integrals. *The College Mathematics Journal*, 26:284–291, 1995.

[6] Hitoshi Murakami and Jun Murakami. The colored jones polynomials and the simplicial volume of a knot. *Acta Mathematica*, 186:85–104, 2001.

[7] Matthew Stoeckel. *Computing Quantum Hyperbolic Invariants*. PhD thesis, The University of Iowa, 2007.

[8] William Thurston. *The Geometry and Topology of Three-Manifolds*. http://www.msri.org/publications/books/gt3m, 1980.