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# The structure of gluons in point form quantum chromodynamics

Kevin Christoher Murphy  
*University of Iowa*

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THE STRUCTURE OF GLUONS IN POINT FORM QUANTUM  
CHROMODYNAMICS

by

Kevin Christoher Murphy

An Abstract

Of a thesis submitted in partial fulfillment of the  
requirements for the Doctor of Philosophy  
degree in Applied Mathematical and Computational Sciences  
in the Graduate College of  
The University of Iowa

December 2009

Thesis Supervisor: Professor William H. Klink

## ABSTRACT

This dissertation investigates part of the strong nuclear force in point form QCD. The quark sector is neglected to focus on gluons and their self-interactions. The structure of gluons is investigated by building up a field theory for massless particles. Single gluon states are defined, and a condition is implemented to make the wave function inner product positive definite. The transformation between gluon and classical gluon fields generates a differentiation inner product, and the kernels allow for transition between momentum and position space. Then, multiparticle gluon states are introduced as symmetric tensor products of gluon Hilbert spaces generated by creation and annihilation operators. In order to assure that the resulting Fock space inner product is positive definite, an annihilator condition is needed and gauge transformations are introduced. The four momentum operator consists of the stress-energy tensor integrated over the forward hyperboloid. The free gluon four momentum operator introduced via the Lagrangian and stress-energy tensor is shown to be equivalent to that generated by gluon irreducible representations when acting on the physical Fock space.

Next the vacuum problem is discussed, where the vacuum state is the state that is annihilated by the the four momentum operator and is invariant under Lorentz and color transformations. To find such a state, the vacuum problem is simplified by considering a one degree of freedom model. The Hamiltonian for such a model, the one dimensional energy operator, is solved under a variety of different ansatzes.

It is shown that the Hamiltonian has a continuous eigenvalue spectrum, and that the vacuum can be constructed in a way that eliminates the interaction term of the Hamiltonian. This one dimensional vacuum model is adapted to the full problem where it is shown that such a result cannot be replicated.

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Graduate College  
The University of Iowa  
Iowa City, Iowa

CERTIFICATE OF APPROVAL

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PH.D. THESIS

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This is to certify that the Ph.D. thesis of

Kevin Christoher Murphy

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Applied Mathematical and Computational Sciences at the December 2009 graduation.

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# CHAPTER 1

## INTRODUCTION

### 1.1 Introduction

#### 1.1.1 Background

One of the major goals in nuclear physics is to understand the nature of the strong nuclear force. In traditional nuclear physics, protons and neutrons, known collectively as nucleons, are the fundamental building blocks out of which nuclei are built. Historically, nuclear forces were generated phenomenologically and the resulting Hamiltonians were used to calculate the bound and scattering states of nuclei [10].

In a modern understanding of the fundamental forces of nature, forces are produced by the exchange of mesons between fermions. In a first attempt to understand the strong nuclear force in this way, the fermions were taken to be nucleons, spin  $\frac{1}{2}$  particles with mass, while the mesons were taken to be the three pions, having spin 0, and charge 0,  $+e$ , and  $-e$  [8]. Nuclear forces are then produced by the exchange of pions between nucleons. Moreover, the quantum field theory (QFT) that underlies such a notion of forces is also able to explain particle production, wherein the collision between nucleons, or between pions and nucleons, produce new particles by converting energy into mass. One of the main drawbacks with such a theory is that the coupling strength between pions and nucleons is large, meaning that perturbative methods are not very useful in calculating cross sections for scattering and production reactions. Nevertheless, using this QFT to explore the nature of the strong nuclear

force is still actively pursued.

A more serious difficulty with such a theory is that it is known experimentally that nucleons and pions are not fundamental, but are themselves comprised of more fundamental entities called quarks. There are a number of different experiments that point to this conclusion, starting with the anomalous magnetic moments of protons and neutrons [10]. That is, in contrast to the electron, which is thought to be fundamental and has a magnetic moment predicted by the Dirac equation, the magnetic moments of the proton and neutron deviate significantly from the electron magnetic moment. Additionally, there is a spectrum of excited states of the nucleons that would indicate that they have a composite nature. The most significant experiments, however, involve the scattering of protons off of protons, the spectrum of which indicates the existence of entities that make up the proton [7].

To account for these experimental facts, a quantum field theory in which the fermions are quarks and the mediating mesons are gluons provides the theoretical basis for understanding the strong nuclear force. Such a field theory, called quantum chromodynamics (QCD), has a number of peculiar features [13]. Perhaps the most prominent is that the forces between quarks, produced by the mediating gluons, increase as the distance between the quarks increase. This results in the quarks being confined and unable to exist as free particles. Conversely, when the quarks are close together, they behave almost as free particles.

### 1.1.2 Gluons and Quantum Chromodynamics

The main objects of study in this dissertation are the gluons, uncharged massless particles of spin 1 which carry an internal symmetry called color. The internal symmetry group is  $SU(3)$  [7], so that the fundamental representation carries three “colors”, red, green and blue, named because the three can combine to make white or “colorless”. Quarks transform as the fundamental representation and hence come in three “colors”. Gluons transform as the adjoint representation of  $SU(3)$ , so that there are eight gluons which consist of color-anticolor pairings. Color as an internal symmetry was originally introduced to solve a problem involving the permutation symmetry of hadronic wave functions [4] (hadron is the general term given to experimentally observed strongly interacting particles). Color was then incorporated as a gauge symmetry to generate QCD. In this dissertation, we will take the color symmetry as given and focus on its role in the structure of gluons.

One of the most striking features of gluons is that they can self-interact. That is, though the gluons mediate the forces between quarks, they can also interact among themselves. This raises a host of questions [2, 6]. Is it possible to express the self-interaction as a potential between gluons? If so, are there bound states of gluons? Here both the experimental and theoretical situation is quite clouded. There are claims and counterclaims for the existence of so called “glueballs”, alleged experimental bound states of gluons [2]. There are also (usually non-relativistic) potential models which compute glueball spectra [6]. Chapter 3 will discuss some simple models of the gluon sector of QCD in which the infinite degree of freedom nature of QCD is

reduced to one degree of freedom, where it will be shown that the self-interaction can be transformed away. In Chapter 4, the full infinite degree of freedom structure is restored and insights gained from Chapter 3 are used to address the gluon structure in its full complexity.

Hadrons are all color singlets or “colorless”; they are bound states of quarks and gluons that transform as the identity representation of color  $SU(3)$ . In contrast, quarks and gluons, which do not exist as free particles, transform as the fundamental (3 dimensional) and adjoint (8 dimensional) representations of color  $SU(3)$  respectively. Hadronic wave functions are tensor products of quark, antiquark and gluon wave functions, made in such a way that they transform as the identity representation of color  $SU(3)$ . In particular, glueballs, if they exist, must be made from tensor products of multiparticle gluon states that transform as the identity representation of color  $SU(3)$ . In Chapter 2, the structure of gluons will be studied in detail and the many particle nature of gluons analyzed using gluon creation and annihilation operators. Products of gluon fields, which are invariant under color  $SU(3)$  and made from the gluon creation and annihilation operators, will play an important role in the study of gluons in Chapter 4.

## 1.2 Point Form Quantum Field Theory

### 1.2.1 The Poincaré Group

A main objective of a relativistic many body quantum theory is to find a realization of the Poincaré algebra on some generalized Fock space. Once this has



been accomplished, the vacuum and one particle states can be computed, as well as scattering data and other particle correlations.

The Poincaré group is the set of all isometries of Minkowski spacetime. It is a ten dimensional group consisting of six Lorentz elements (boosts and rotations) and four spacetime translations. Its irreducible representations (or irreps) generate one particle Hilbert spaces for massive and massless particles. In Chapter 2, the one particle representations of gluons will be generated by the massless spin 1 representations of the Poincaré group.

The Lie algebra of the Poincaré group consists of Lorentz generators, written  $M^{\mu\nu}$ , and spacetime generators,  $P^\mu$ , called the four momentum operator. They satisfy the commutation relations

$$\begin{aligned}
 [P^\mu, P^\nu] &= 0 \\
 [M^{\mu\nu}, P^\rho] &= \eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu \\
 [M^{\mu\nu}, M^{\rho\sigma}] &= \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\rho} M^{\mu\sigma} + \eta^{\nu\sigma} M^{\mu\rho}
 \end{aligned} \tag{1.1}$$

where  $\eta$  is the Minkowski metric defined as

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The general four vector product is written

$$p \cdot q = p^\alpha \eta_{\alpha\beta} q^\beta$$

which is summed over the indices as in the standard Einstein summation notation.

A relativistic quantum theory is one in which the operators  $P^\mu$  and  $M^{\mu\nu}$  act on some Hilbert (Fock) space and satisfy the Poincaré commutation relations in Eq. 1.1.

### 1.2.2 Possible Approaches

There are several ways of generating operators that satisfy the Poincaré commutation relations. The traditional way is via a Lagrangian which is given in terms of free fields [8]. In turn, the Lagrangian generates the stress-energy tensor,  $T^{\mu\nu}$ , and the integral over some spacelike surface,  $\sigma$ , then gives the four momentum operator

$$P^\mu = \int d\sigma_\nu T^{\mu\nu}$$

If the stress-energy tensor is conserved,  $\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0$ , then the four momentum operator  $P^\mu$  will be conserved as well.

In the early days of relativistic QFT, Schwinger, Tomonaga, and others [9, 11] proved that any spacelike surface could be used to obtain the Poincaré generators. In this dissertation we will take the spacelike surface to be the forward hyperboloid in Minkowski space. This has the consequence that interactions occur only in the four momentum, and the Lorentz generators are purely kinematic (contain no interactions) [1]. Other choices for spacelike surfaces lead to interactions occurring in different sets of Poincaré generators. The most common choice, a time constant surface (used prominently in textbooks, for example [13]), leads to what is called instant form QFT, while the choice of a light front in Minkowski space leads to what is called front

form QFT [3].

The main advantage of the point form, wherein the spacelike surface is the forward hyperboloid, is that Lorentz transformations can be treated globally and quantities can be written that explicitly exhibit global Lorentz transformation properties. In that case the Poincaré commutation relations can be integrated to

$$\begin{aligned} [P^\mu, P^\nu] &= 0 \\ U_\Lambda P^\mu U_\Lambda^{-1} &= (\Lambda^{-1})^\mu{}_\nu P^\nu \end{aligned} \tag{1.2}$$

where  $U_\Lambda$  is the unitary operator representing the Lorentz transformation  $\Lambda$ . The goal of a point form QFT is to find the four momentum operator satisfying these “point form” equations, with the interpretation that the energy ( $P^{\mu=0}$ ) and momentum ( $P^{\mu=1,2,3}$ ) commute so that energy and momentum can be simultaneously measured, even though both energy and momentum operators contain interactions. In this dissertation, the four momentum operator will be that of self-interacting gluons only; a complete point form QCD would include the four momentum operator for quarks as well. The form of the four momentum for gluons is discussed in detail in Chapter 2.

An alternate way to build up a many particle quantum theory is to start with the irreducible representations of the Poincaré group [12]. These irreps act on Hilbert spaces, which are the underlying one particle spaces of the theory. A many particle theory can then be generated by introducing creation and annihilation operators that have the same transformation properties as the one particle operators. In particular, in the point form, the unitary operators representing Lorentz transformations arise

from the one particle irreps.

In Chapter 2, the irreducible representations of gluons will be induced from a finite dimensional non-unitary representation of  $E(2)$ , the Euclidean group in two dimensions, which is the little group for massless particles [5]. Since the representation of  $E(2)$  is non-unitary, a special inner product is required for Lorentz invariance. This inner product, however, is not positive definite. Chapter 2 also shows how to restrict the polarization degrees of freedom to make the inner product positive definite while at the same time showing how to introduce gauge transformations into the theory.

### 1.2.3 Extension to Many Body Theory

A many body gluon structure is obtained with gluon creation and annihilation operators. In order to assure that the resulting Fock space inner product stays positive definite, an annihilator operator condition is introduced. This operator condition also serves to define the physical Fock space, and the resulting free four momentum operator is shown to be gauge invariant. From gluon creation and annihilation operators, free gluon fields are introduced. The most important theorem of Chapter 2 is that the free gluon four momentum operator introduced via the Lagrangian and stress-energy tensor is the same as that generated from gluon irreps (the details are provided in Appendix B). A similar theorem was previously proven for scalar charged mesons and spin  $\frac{1}{2}$  fermions [1], but here we show it also holds for massless particles.

Given the gluon fields, self-interacting gluons are obtained with the help of the field tensor. The four momentum operator is shown to be the sum of free, cubic

and quartic gluon fields. This leads to the generalized eigenvalue problem

$$P^\mu |\Psi_p\rangle = p^\mu |\Psi_p\rangle \quad (1.3)$$

where  $P^\mu$  is the total four momentum operator including self-interactions, and  $p^\mu$  is the eigenvalue for the generalized eigenvector  $|\Psi_p\rangle$ .

The vacuum problem for gluons is then given by

$$\begin{aligned} P^\mu |\Omega\rangle &= 0 \\ U_\Lambda |\Omega\rangle &= |\Omega\rangle \\ U_h |\Omega\rangle &= |\Omega\rangle \end{aligned}$$

with  $h \in SU(3)$ , and  $|\Omega\rangle$  is the vacuum state, a state which has zero energy and momentum and is invariant under Lorentz and color symmetry.

How to obtain  $|\Omega\rangle$  is the subject of Chapter 3. The gluon degrees of freedom are suppressed to one degree of freedom and the eigenvalue problem analyzed with this greatly simplified model. The main result of Chapter 3 is to show that if the vacuum state is written as  $|\Omega\rangle = e^S |0\rangle$ , where  $|0\rangle$  is the Fock vacuum, then choosing  $S$  as a product of three gluon fields effectively transforms away the self-interactions of the four momentum operator. In Chapter 4, the full gluon vacuum problem is considered in light of this result, and the method is duplicated. It is determined that despite transforming away the interactions for the one degree of freedom problem in Chapter 3, the corresponding form of  $S$  for the full problem is unable to eliminate the self-interactions.

Chapter 5 summarizes the results and indicates how the earlier chapters can be

used to investigate the full vacuum problem for gluons. Additionally, some properties of glueballs are introduced for consideration of future work.

## CHAPTER 2 THE STRUCTURE OF GLUONS

The goal of this chapter is to investigate the structure of gluons in the context of point form quantum field theory.

### 2.1 Gluons and Their Properties

Gluons are massless, spin 1 bosons carrying the color charge. Because gluons are massless, they only have two spin states, despite being spin 1 bosons; conventionally the spin projector is chosen to be the helicity, which is the spin projection in the direction of the momentum of the particle. A simple way to understand the nature of gluons is by comparing it to the photon (which is discussed in [5]).

#### 2.1.1 The Little Group $E(2)$

As in the photon case, the structure of the gluon stems from representations of the Poincaré group for massless particles, for which the little group is the two dimensional Euclidean group,  $E(2)$ .

**Definition 2.1.**  $E(2)$  consists of all continuous transformations on the Euclidean space  $\mathbb{R}^2$  which leave the length of all vectors invariant.

A general transformation sends  $\mathbf{x} \rightarrow \mathbf{x}' = R(\phi)\mathbf{x} + \mathbf{b}$  where  $R(\phi)$  is a rotation matrix through an angle  $\phi$  and  $\mathbf{b}$  is a translation.

If  $\Lambda \in SO(1, 3)$ , the proper Lorentz group, then  $E(2)$  can also be defined as the subgroup of the proper Lorentz group leaving a standard four-vector invariant:

$$E(2) := \{\Lambda \in SO(1, 3) | \Lambda k^{st} = k^{st}\} \quad (2.1)$$

where  $k^{st} := (1, 0, 0, 1)$ . Eq. 2.1 is essentially a four-dimensional non-unitary representation of  $E(2)$  for which the representation of  $e_2 \in E(2)$  is written as  $\Lambda(e_2)$ , to indicate the Lorentz transformation representing the group element  $e_2$ .

To verify this is a representation of  $E(2)$ , the elements of the Euclidean group in this representation can be written explicitly as

$$\Lambda(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi & 0 \\ 0 & \sin \phi & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.2)$$

$$\Lambda(\mathbf{b}) = \begin{pmatrix} 1 + \frac{|\mathbf{b}|^2}{2} & b_x & b_y & -\frac{|\mathbf{b}|^2}{2} \\ b_x & 1 & 0 & -b_x \\ -b_y & 0 & 1 & b_y \\ \frac{|\mathbf{b}|^2}{2} & b_x & -b_y & 1 - \frac{|\mathbf{b}|^2}{2} \end{pmatrix} \quad (2.3)$$

where it is trivial to show that  $\Lambda(\phi), \Lambda(\mathbf{b}) \in E(2)$  as defined in Eq. 2.1.

**Definition 2.2.** A helicity boost,  $B_H(k)$ , is a Lorentz transformation satisfying  $k = B_H(k)k^{st}$ .

This assures that

$$k \cdot k = k^\alpha \eta_{\alpha\beta} k^\beta = k^{st\alpha} \eta_{\alpha\beta} k^{st\beta} = 0$$



as required for massless gluons.

Any Lorentz transformation can be written as a boost times a Euclidean group element, where  $B_H(k)$  is a coset representative of  $SO(1,3)$  with respect to  $E(2)$ . The helicity boost choice for massless particles is

$$\begin{aligned} B_H(k) &= R(\hat{k})\Lambda_z(|\mathbf{k}|) \\ &= \begin{pmatrix} \cosh \chi & 0 & 0 & \sinh \chi \\ \sinh \chi \hat{k}_1 & \hat{k}_1 & \hat{k}_2 & \cosh \chi \hat{k}_1 \end{pmatrix} \end{aligned}$$

where  $R(\hat{k})$  is a rotation matrix taking  $\hat{z}$  to the unit vector  $\hat{k}$ ,  $\Lambda_z(|\mathbf{k}|)$  is a Lorentz transformation about the  $z$ -axis with  $|\mathbf{k}| = e^\chi$ ,  $\hat{k}_1 = (\cos \phi \cos \theta, \sin \phi \cos \theta, -\sin \theta)$ , and  $\hat{k}_2 = (-\sin \phi, \cos \phi, 0)$ .

### 2.1.2 Gluon Basis States and Their Properties

A standard gluon basis state is defined in terms of the standard four-vector,  $k^{st}$ , spin projection  $\rho$  and color  $a$  such that it transforms as a representation of  $E(2)$ ,

$$U_{e_2}|k^{st}, \rho, a \rangle = \sum_{\rho'} |k^{st}, \rho', a \rangle \Lambda_{\rho' \rho}(e_2), \forall e_2 \in E(2)$$

**Definition 2.3.** The one particle gluon state is given by boosting the standard gluon basis state,

$$|k, \rho, a \rangle := U_{B(k)}|k^{st}, \rho, a \rangle$$

From this definition the transformation properties of a one particle gluon state under Lorentz and spacetime transformations are then determined. This will allow for conversion to or from the standard vector via a unitary Lorentz transformation.

$$\begin{aligned}
U_\Lambda |k, \rho, a \rangle &= U_\Lambda U_{B(k)} |k^{st}, \rho, a \rangle \\
&= \sum_{\rho'} |\Lambda k, \rho', a \rangle \Lambda_{\rho' \rho}(e_W) \\
U_b |k, \rho, a \rangle &= e^{ik \cdot b} |k, \rho, a \rangle \\
U_h |k, \rho, a \rangle &= \sum_{a'} |k, \rho, a' \rangle D_{aa'}(h)
\end{aligned} \tag{2.4}$$

where  $b$  is a spacetime translation and  $h \in SU(3)$ .  $D(h)$  are the adjoint representation matrices of color  $SU(3)$ .  $\Lambda(e_W)$  is the massless analogue of a Wigner rotation defined by

$$\Lambda(e_W) := B^{-1}(\Lambda k) \Lambda B(k) \tag{2.5}$$

Notice that  $\Lambda(e_W) \in E(2)$ , since

$$\begin{aligned}
\Lambda(e_W) k^{st} &= B^{-1}(\Lambda k) \Lambda B(k) k^{st} \\
&= B^{-1}(\Lambda k) \Lambda k \\
&= k^{st}
\end{aligned}$$

### 2.1.3 Single Particle Wave Function

**Definition 2.4.** A general gluon state can be written in terms of gluon wave functions and basis states,

$$|\phi \rangle := \sum_{\rho, a} \int \frac{d^3 k}{2k_0} \phi(k, \rho, a) |k, \rho, a \rangle$$

where  $\phi(k, \rho, a)$  is a single particle gluon wave function. Here  $\frac{d^3 k}{2k_0}$  is the Lorentz invariant measure where  $k_0 := |\mathbf{k}|$ .

An immediate consequence of this definition is that the transformation properties of gluon wave functions are inherited from those of the states. Specifically, the action of the Lorentz transformation in Eq. 2.4 on states is transferred to the wave function, resulting in

$$(U_\Lambda \phi)(k, \rho, a) = \sum_{\rho'} \Lambda_{\rho' \rho}(e_W(k, \Lambda^{-1})) \phi(\Lambda^{-1}k, \rho', a)$$

using the notation  $\Lambda(e_W(k, \Lambda^{-1})) = B^{-1}(k)\Lambda B(\Lambda^{-1}k)$  instead of  $e_W$  as defined in Eq. 2.5.

The usual Hilbert space of square integrable gluon wave functions is not Lorentz invariant. Modify the gluon inner product to be

$$(\phi, \psi) := - \sum_a \int \frac{d^3k}{2k_0} \phi^*(k, \rho, a) \eta^{\rho\rho} \psi(k, \rho, a) \quad (2.6)$$

Then  $\|U_\Lambda \phi\|^2 = \|\phi\|^2$ . However, as defined, this inner product is not positive definite as can be seen by setting  $\phi(k, \rho, a) = 0$ , for  $\rho = 1, 2, 3$  with  $\phi(k, 0, a) \neq 0$ . In such a case,  $\|\phi\|^2 < 0$ . Therefore, constraints must be imposed on  $\phi$  for the inner product to be positive definite.

**Lemma 2.1.** *The wave function condition,*

$$k^{st\rho} \phi(k, \rho, a) = 0 \quad (2.7)$$

*which sets  $\phi(k, 0, a) = \phi(k, 3, a)$ , makes the wave function inner product positive definite.*

Proof.

$$\begin{aligned}
(\phi, \phi) &= \sum_a \int \frac{d^3k}{2k_0} [-\phi^*(k, 0, a)\phi(k, 0, a) + \phi^*(k, 1, a)\phi(k, 1, a) + \\
&\quad \phi^*(k, 2, a)\phi(k, 2, a) + \phi^*(k, 3, a)\phi(k, 3, a)] \\
&= \sum_a \int \frac{d^3k}{2k_0} [\phi^2(k, 1, a) + \phi^2(k, 2, a)] \\
&\geq 0
\end{aligned}$$

**Corollary 2.2.** *Eq. 2.7 is Lorentz invariant.*

Proof (of Corollary).

$$\begin{aligned}
U_\Lambda k^{st\rho} \phi(k, \rho, a) &= k^{st\rho} \Lambda_{\rho'\rho}(e_W(k, \Lambda^{-1}))\phi(\Lambda^{-1}k, \rho', a) \\
&= k^{st\rho'} \phi(\Lambda^{-1}k, \rho', a) \\
&= 0
\end{aligned}$$

The one gluon Hilbert Space is then  $\mathcal{H}_g = \{(\phi, \phi) | \phi(k, 0, a) = \phi(k, 3, a)\}$ , where the equality of the zero and third components of the spin means there are only two independent gluon spin/helicity states.

#### 2.1.4 Classical Gluon Fields

**Definition 2.5.** Classical free gluon fields are four vector fields over spacetime that carry color; they are defined as

$$G_a^\mu(x) = \int \frac{d^3k}{(2\pi)^{3/2}2k_0} B^{\mu\rho}(k)\phi(k, \rho, a)e^{-ik\cdot x} \quad (2.8)$$

with a norm extracted from a differentiation inner product

$$(\psi, \chi)_\sigma = -i \int_\sigma d\sigma^\mu(x) [\psi^*(x) \frac{\partial \chi(x)}{\partial x^\mu} - \chi(x) \frac{\partial \psi^*(x)}{\partial x^\mu}]$$

where  $\sigma$  is a space-like hypersurface of Minkowski spacetime [8].

For point form QFT the natural hypersurface is the forward hyperboloid  $\sigma_\tau$  :

$x_\mu x^\mu = \tau^2$ , with measure

$$d\sigma^\mu(x) = 2d^4x \delta(x \cdot x - \tau^2) \theta(x_0) x^\mu$$

Then the norm for a classical free gluon field,

$$\begin{aligned} (G, G) &= -i \sum_a \int 2d^4x \delta(x \cdot x - \tau^2) \theta(x_0) x^\alpha (G_a^{\mu*}(x) \frac{\partial G_{\mu a}(x)}{\partial x^\alpha} - \frac{\partial G_a^{\mu*}(x)}{\partial x^\alpha} G_{\mu a}(x)) \\ &= \sum_a \int \frac{d^3k}{(2\pi)^{3/2} 2k_0} \frac{d^3k'}{(2\pi)^{3/2} 2k'_0} B^{\mu\rho}(k) B_\mu^{\rho'}(k') \phi^*(k, \rho, a) \phi(k', \rho', a) \\ &\quad \int 2d^4x \delta(x \cdot x - \tau^2) \theta(x_0) x^\alpha (k' + k)_\alpha e^{ix \cdot (k' - k)} \end{aligned}$$

The spatial integral is the Lorentz invariant distribution  $W(p, q)$  where  $p = k' + k$  and  $q = k' - k$ . As shown in the Appendix of Ref. [1],  $W(p, q) = (2\pi)^3 p_0 \delta^3(\mathbf{q})$

so that

$$\begin{aligned} (G, G) &= - \sum_a \int \frac{d^3k}{(2\pi)^{3/2} 2k_0} \frac{d^3k'}{(2\pi)^{3/2} 2k'_0} B^{\mu\rho}(k) B_\mu^{\rho'}(k') \phi^*(k, \rho, a) \phi(k', \rho', a) W(k' + k, k' - k) \\ &= - \sum_a \int \frac{d^3k}{2k_0} \frac{d^3k'}{2k'_0} B^{\mu\rho}(k) B_\mu^{\rho'}(k') \phi^*(k, \rho, a) \phi(k', \rho', a) (k_0 + k'_0) \delta^3(\mathbf{k} - \mathbf{k}') \\ &= - \sum_a \int \frac{d^3k}{2k_0} \phi^*(k, \rho, a) \eta^{\rho\rho'} \phi(k, \rho', a) \\ &= (\phi, \phi) \end{aligned}$$

where  $B^{\mu\rho}(k) B_\mu^{\rho'}(k) = \eta^{\rho\rho'}$  has been used.

Similarly,  $\phi(k, \rho, a)$  can be written in terms of gluon fields by introducing the kernel

$$K_\rho^\mu(k, x) = B_\rho^\mu(k)e^{ik \cdot x}$$

and the other direction can also be shown. Therefore, the inner product of free gluon fields over the forward hyperboloid is equivalent to that of the gluon momentum space wave functions.

## 2.2 Multiparticle Gluon States

Because gluons are bosons, multiparticle gluon states are symmetric tensor products of gluon Hilbert spaces. The infinite direct sum of all symmetrized tensor products forms the gluon Fock space. Multiparticle gluon states are naturally generated by creation and annihilation operators.

### 2.2.1 Creation and Annihilation Operators

Let  $g^\dagger(k, \rho, a)$  be the creation operator for a gluon with momentum  $k$ , spin  $\rho$  and color  $a$ . The adjoint  $g(k, \rho, a)$  is the annihilation operator.

**Definition 2.6.** Creation and annihilation operators act on the Fock vacuum,  $|0\rangle$ , the state of no gluons, as follows

$$g(k, \rho, a)|0\rangle = 0, \forall k, \rho, a$$

$$g^\dagger(k, \rho, a)|0\rangle = |k, \rho, a\rangle$$

and satisfy boson commutation relations

$$[g(k, \rho, a), g^\dagger(k', \rho', a')] = -\eta_{\rho\rho'} k_0 \delta^3(\mathbf{k} - \mathbf{k}') \delta_{aa'}$$

With these operators it is possible to generate multiparticle gluon states, as elements in the gluon Fock space. The following transformation properties are inherited from the one particle gluon properties in Eq. 2.4:

$$\begin{aligned} U_\Lambda g(k, \rho, a) U_\Lambda^{-1} &= \sum_{\rho'} g(\Lambda k, \rho', a) \Lambda_{\rho' \rho}(e_W) \\ U_h g(k, \rho, a) U_h^{-1} &= \sum_{a'} g(k, \rho, a') D_{a' a}(h) \end{aligned} \quad (2.9)$$

The free four momentum operator,  $P_{fr}^\mu$ , can be written

$$P_{fr}^\mu = - \sum_a \int \frac{d^3 k}{k_0} k^\mu g^\dagger(k, \rho, a) \eta^{\rho\rho} g(k, \rho, a) \quad (2.10)$$

which gives the free four momentum of a multiparticle gluon state.

As with the one particle Hilbert space, the Fock space inner product will not be positive definite unless the  $0^{th}$  and  $3^{rd}$  components of multiparticle wave functions are equal.

**Theorem 2.3.** *The annihilator condition that guarantees that the Fock space inner product is positive definite is*

$$\sum_a k^{st\rho} g(k, \rho, a) |\phi \rangle = 0 \quad (2.11)$$

Proof.

Consider the two gluon state,

$$\begin{aligned} |\phi_2 \rangle &= \sum_{\rho_1, \rho_2, a_1, a_2} \frac{1}{(2\pi)^3} \int \frac{d^3 k_1}{2k_{01}} \frac{d^3 k_2}{2k_{02}} \phi_2(k_1, \rho_1, a_1; k_2, \rho_2, a_2) |k_1, \rho_1, a_1; k_2, \rho_2, a_2 \rangle \\ &= \sum_{\rho_1, \rho_2, a_1, a_2} \frac{1}{(2\pi)^3} \int \frac{d^3 k_1}{2k_{01}} \frac{d^3 k_2}{2k_{02}} \phi_2(k_1, \rho_1, a_1; k_2, \rho_2, a_2) |k_1, \rho_1, a_1 \rangle \otimes |k_2, \rho_2, a_2 \rangle \end{aligned}$$

Then applying Eq. 2.11,

$$\begin{aligned}
\sum_a k^{st\rho} g(k, \rho, a) |\phi_2\rangle &= \sum_a \sum_{\rho_1, \rho_2, a_1, a_2} k^{st\rho} g(k, \rho, a) \frac{1}{(2\pi)^3} \int \frac{d^3 k_1}{2k_{01}} \frac{d^3 k_2}{2k_{02}} \phi_2(k_1, \rho_1, a_1; k_2, \rho_2, a_2) |k_1, \rho_1, a_1\rangle \otimes |k_2, \rho_2, a_2\rangle \\
&= \sum_a \sum_{\rho_1, \rho_2, a_1, a_2} k^{st\rho} \frac{1}{(2\pi)^3} \int \frac{d^3 k_1}{2k_{01}} \frac{d^3 k_2}{2k_{02}} \phi_2(k_1, \rho_1, a_1; k_2, \rho_2, a_2) g(k, \rho, a) g^\dagger(k_1, \rho_1, a_1) g^\dagger(k_2, \rho_2, a_2) |0\rangle \\
&= \sum_a \sum_{\rho_1, \rho_2, a_1, a_2} k^{st\rho} \frac{1}{(2\pi)^3} \int \frac{d^3 k_1}{2k_{01}} \frac{d^3 k_2}{2k_{02}} \phi_2(k_1, \rho_1, a_1; k_2, \rho_2, a_2) [-\eta_{\rho\rho_1} k_{01} \delta^3(\mathbf{k} - \mathbf{k}_1) \delta_{aa_1} g^\dagger(k_2, \rho_2, a_2) \\
&\quad + g^\dagger(k_1, \rho_1, a_1) g(k, \rho, a) g^\dagger(k_2, \rho_2, a_2)] |0\rangle \\
&= \sum_a \sum_{\rho_1, \rho_2, a_1, a_2} k^{st\rho} \frac{1}{(2\pi)^3} \int \frac{d^3 k_1}{2k_{01}} \frac{d^3 k_2}{2k_{02}} \phi_2(k_1, \rho_1, a_1; k_2, \rho_2, a_2) [-\eta_{\rho\rho_1} k_{01} \delta^3(\mathbf{k} - \mathbf{k}_1) \delta_{aa_1} g^\dagger(k_2, \rho_2, a_2) \\
&\quad - \eta_{\rho\rho_2} k_{02} \delta^3(\mathbf{k} - \mathbf{k}_2) \delta_{aa_2} g^\dagger(k_1, \rho_1, a_1)] |0\rangle \\
&= -\sum_a \sum_{\rho_2, a_2} k^{st\rho} \frac{1}{(2\pi)^3} \int \frac{d^3 k_2}{2k_{02}} \eta_{\rho\rho_1} \phi_2(k, \rho_1, a; k_2, \rho_2, a_2) |k_2, \rho_2, a_2\rangle \\
&\quad - \sum_a \sum_{\rho_1, a_1} k^{st\rho} \frac{1}{(2\pi)^3} \int \frac{d^3 k_1}{2k_{01}} \eta_{\rho\rho_2} \phi_2(k_1, \rho_1, a_1; k, \rho_2, a) |k_1, \rho_1, a_1\rangle \\
&= -\sum_a \sum_{\rho_2, a_2} \frac{1}{(2\pi)^3} \int \frac{d^3 k_2}{2k_{02}} [\phi_2(k, 0, a; k_2, \rho_2, a_2) - \phi_2(k, 3, a; k_2, \rho_2, a_2)] |k_2, \rho_2, a_2\rangle \\
&\quad - \sum_a \sum_{\rho_1, a_1} \frac{1}{(2\pi)^3} \int \frac{d^3 k_1}{2k_{01}} [\phi_2(k_1, \rho_1, a_1; k, 0, a) - \phi_2(k_1, \rho_1, a_1; k, 3, a)] |k_1, \rho_1, a_1\rangle \\
&= 0
\end{aligned}$$

Recall that the wave function condition from Eq. 2.7 was introduced to assure that the inner product be positive definite. For the two particle state, the above calculation shows that there are now two conditions that need to be satisfied,

$$\begin{aligned}
\phi_2(k, 0, a; k', \rho', a') &= \phi_2(k, 3, a; k', \rho', a') \\
&\text{and} \tag{2.12}
\end{aligned}$$

$$\phi_2(k', \rho', a'; k, 0, a) = \phi_2(k', \rho', a'; k, 3, a)$$

It is clear how this generalizes for a many gluon state.



For a generic  $\phi_n$ , this leaves

$$\begin{aligned}
(\phi_n, \phi_n) &= (-1)^n \sum_{a_i, \rho_i} \int \prod_i \frac{d^3 k_i}{2k_{0i}} |\phi_n(k_1, \rho_1, a_1 \dots k_n, \rho_n, a_n)|^2 \eta_{\rho_1 \rho_1} \dots \eta_{\rho_n \rho_n} \\
&= \sum_{a_i} \sum_{\rho_i=1,2} \int \frac{d^3 k}{2k_0} |\phi_n(k_1, \rho_1, a_1 \dots k_n, \rho_n, a_n)|^2 \\
&\geq 0
\end{aligned}$$

where all of the  $0^{th}$  components have cancelled with the  $3^{rd}$  components by the generalized conditions of Eq. 2.12.

**Corollary 2.4.** *Eq. 2.11 is Lorentz invariant.*

Proof (of Corollary).

$$\begin{aligned}
U_\Lambda k^{st\rho} g(k, \rho, a) |\phi \rangle &= k^{st\rho} U_\Lambda g(k, \rho, a) U_\Lambda^{-1} U_\Lambda |\phi \rangle \\
&= k^{st\rho} g(\Lambda k, \rho', a) \Lambda_{\rho' \rho}(e_W) U_\Lambda |\phi \rangle \\
&= k^{st\rho'} g(\Lambda k, \rho', a) U_\Lambda |\phi \rangle \\
&= 0
\end{aligned}$$

With the result of Thm 2.3, it seems that the  $0^{th}$  and  $3^{rd}$  components of a multiparticle gluon wave function play no role in the structure of gluons. That, however, is not the case. Under Lorentz transformations the  $0^{th}$  and  $3^{rd}$  components of a multiparticle wave function change, although in such a way that the components remain equal (Cor. 2.4). If the  $0^{th}$  and  $3^{rd}$  components are simply suppressed, the gluon wave functions do not have the correct Lorentz transformation properties. For example, Weinberg starts with 2 component gluons and then shows that they do

not have good Lorentz transformation properties [13]. By retaining the 0<sup>th</sup> and 3<sup>rd</sup> components, the gluons are guaranteed to have the correct Lorentz transformation properties.

A more important consequence of retaining the 0<sup>th</sup> and 3<sup>rd</sup> components has to do with gauge transformations and gauge invariance. A gauge transformation is an automorphism on the algebra of gluon creation and annihilation operators that preserves their commutation relations:

$$\begin{aligned} g(k, \rho, a) &\rightarrow g'(k, \rho, a) = g(k, \rho, a) + k_\rho^{st} f(k) I \\ g^\dagger(k, \rho, a) &\rightarrow g'^\dagger(k, \rho, a) = g^\dagger(k, \rho, a) \\ [g'(k, \rho, a), g'^\dagger(k', \rho', a')] &= [g(k, \rho, a), g^\dagger(k', \rho', a')] \end{aligned}$$

Here  $f(k)$  is a Lorentz invariant distribution and  $I$  is the identity operator. It is then straightforward to show that the wave function condition in Eq. 2.7 is invariant under a gauge transformation.

### 2.2.2 Gluon Fields Using Creation and Annihilation Operators

**Definition 2.7.** Define the gluon field at the space-time point zero

$$G_a^\mu(0) := - \sum_\rho \int \frac{d^3k}{(2\pi)^{3/2} 2k_0} B^{\mu\rho}(k) (g(k, \rho, a) + g^\dagger(k, \rho, a))$$

The free gluon field for a general  $x$  can then be determined by

$$\begin{aligned} G_a^\mu(x) &:= e^{iP_{fr}\cdot x} G_a^\mu(0) e^{-iP_{fr}\cdot x} \\ &= - \sum_\rho \int \frac{d^3k}{(2\pi)^{3/2} 2k_0} B^{\mu\rho}(k) (e^{-ik\cdot x} g(k, \rho, a) + e^{ik\cdot x} g^\dagger(k, \rho, a)) \end{aligned}$$

with  $P_{fr}$  defined as in 2.10. The second equality uses

$$\begin{aligned} e^{\imath P_{fr} \cdot x} g^\dagger(k, \rho, a) e^{-\imath P_{fr} \cdot x} &= e^{\imath k \cdot x} g^\dagger(k, \rho, a) \\ e^{\imath P_{fr} \cdot x} g(k, \rho, a) e^{-\imath P_{fr} \cdot x} &= e^{-\imath k \cdot x} g(k, \rho, a) \end{aligned}$$

which are found by applying

$$\begin{aligned} [g^\dagger(k', \rho', a'), -\imath P_{fr}^\mu(k) \cdot x] &= \imath \left( \sum_a \int \frac{d^3 k}{k_0} k^\mu x_\mu g^\dagger(k, \rho, a) \eta^{\rho\rho} [g^\dagger(k', \rho', a'), g(k, \rho, a)] \right) \\ &= \imath k \cdot x g^\dagger(k, \rho, a) \end{aligned}$$

$$[g(k', \rho', a'), -\imath P_{fr}^\mu(k) \cdot x] = -\imath k \cdot x g(k, \rho, a)$$

to the Taylor expansion for the exponential.

As a result, the free gluon field is now given in terms of creation and annihilation operators as

$$G_a^\mu(x) = - \int \frac{d^3 k}{(2\pi)^{3/2} 2k_0} B^{\mu\rho}(k) (e^{-\imath k \cdot x} g(k, \rho, a) + e^{\imath k \cdot x} g^\dagger(k, \rho, a))$$

and satisfies  $\square^2 G_a^\mu(x) = 0$ . The spatial derivative is

$$\frac{\partial G_a^\mu}{\partial x_\nu} = \imath \int \frac{d^3 k}{(2\pi)^{3/2} 2k_0} k^\nu B^{\mu\rho}(k) (e^{-\imath k \cdot x} g(k, \rho, a) - e^{\imath k \cdot x} g^\dagger(k, \rho, a))$$

Moreover a gauge transformation on free fields now takes the form

$$G_a^\mu(x) \rightarrow G_a'^\mu(x) = G_a^\mu(x) + \frac{\partial \tilde{f}(x, a)}{\partial x_\mu} I$$

where

$$\tilde{f}(x, a) = \int \frac{d^3 k}{(2\pi)^{3/2} 2k_0} e^{-\imath k \cdot x} f(k, a)$$

From the gluon field point of view a gauge transformation is an element of a map group, mapping a point in Minkowski space to an element of the internal symmetry group  $SU(3)$ . With the notion of a gauge transformation, the condition on the physical gluon Fock space can be written as

$$\frac{\partial G_a^\mu(x)}{\partial x^\mu} |\phi\rangle = 0$$

and is gauge invariant [5].

### 2.2.2.1 Gluon Field Commutation Relations

A free gluon field is local, meaning

$$[G_a^\mu(x), G_b^\nu(y)] = 0, (x - y)^2 < 0 \quad (2.13)$$

It will also be useful to calculate

$$\left[ \frac{\partial G_a^\mu(x)}{\partial x_\beta}, G_b^\nu(y) \right] = \eta^{\mu\nu} \delta_{ab} \frac{\partial \Delta(x-y)}{\partial x_\beta} I \quad (2.14)$$

These properties are verified in Appendix A. The commutator in Eq. 2.14 is a multiple of the identity, with the multiple a derivative of the Pauli-Jordan function,  $\Delta(x - y) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k_0} (e^{ik \cdot (x-y)} - e^{-ik \cdot (x-y)})$ . When extended to a product of fields one has

$$\left[ \frac{\partial G_a^\mu(x)}{\partial x_\beta}, \prod_i G_{b_i}^{\nu_i}(y_i) \right] = \sum_i \eta^{\mu\nu_i} \delta_{ab_i} \frac{\partial \Delta(x-y_i)}{\partial x_\beta} \prod_{j \neq i} G_{b_j}^{\nu_j}(y_j)$$

and furthermore

$$\left[ \frac{\partial G_a^\mu(x)}{\partial x_\beta}, e^{\prod_i G_{b_i}^{\nu_i}(y_i)} \right] = e^{\prod_k G_{b_k}^{\nu_k}(y_k)} \sum_i \eta^{\mu\nu_i} \delta_{ab_i} \frac{\partial \Delta(x-y_i)}{\partial x_\beta} \prod_{j \neq i} G_{b_j}^{\nu_j}(y_j)$$

The goal of the later chapters is to write the physical vacuum in terms of gluon fields, so these commutation relations will be needed when investigating the vacuum problem.

### 2.3 The Gluon Four Momentum

As in quantum electrodynamics, the gluon field tensor is built up from gluon fields and derivatives in such a way to be gauge invariant. From any standard book in QCD [7] the field tensor is given by

$$F_a^{\mu\nu}(x) = \frac{\partial G_a^\nu}{\partial x_\mu} - \frac{\partial G_a^\mu}{\partial x_\nu} + \alpha c_{abc} G_b^\mu(x) G_c^\nu(x)$$

where  $\alpha$  is the bare coupling constant and  $c_{abc}$  are the structure constants for the color charge. Notice that the field tensor is antisymmetric in its four-vector indices,  $F_a^{\mu\nu} = -F_a^{\nu\mu}$ , because the structure constants are antisymmetric in all three indices.

The stress-energy tensor in terms of the field tensor is given by

$$T^{\mu\nu}(x) = \sum_a F_a^{\alpha\beta}(x) [\eta_{\alpha'}^\mu \eta_{\alpha}^{\nu'} \eta_{\beta\beta'} + \eta_{\alpha'}^\nu \eta_{\alpha}^{\mu'} \eta_{\beta\beta'} - \frac{1}{2} \eta^{\mu\nu} \eta_{\alpha\alpha'} \eta_{\beta\beta'}] F_a^{\alpha'\beta'}(x)$$

from which it follows that the gluon four momentum operator is

$$P^\mu = \int dx_\nu T^{\mu\nu}(x)$$

where the integration is over the forward hyperboloid. Further,  $P^\mu$  satisfies the point form equations given in Eq. 1.2.

The four momentum operator consists of a product of field tensors with differing coefficients dependent on the metric. It is useful to break the four momentum into parts,

$$P^\mu = P_{KE}^\mu + P_{tri}^\mu + P_{quar}^\mu$$

The parts are separated by breaking the field tensor into a term dependent on  $\alpha$  and one not. The quartic part inherits its name from the product of the two  $\alpha$  dependent pieces of the field tensor, the trilinear from the two pairings of one  $\alpha$  dependent with an independent piece, and the kinetic energy part from the two  $\alpha$  independent pieces.

The general structure for the resulting parts are

$$\begin{aligned} P_{KE}^\mu &= - \sum \int dx^\mu \frac{d^3k}{k_0} (B^{\nu\rho_1}(k_1)k_1^\mu - B^{\sigma\rho_1}(k_1)k_1^\nu)(B_\nu^{\rho_2}(k_2)k_2^\mu - B_\sigma^{\rho_2}(k_2)k_2^\nu) \\ &\quad (e^{-ik_1 \cdot x} g(k_1, \rho_1, a) - e^{ik_1 \cdot x} g^\dagger(k_1, \rho_1, a)) \\ &\quad (e^{-ik_2 \cdot x} g(k_2, \rho_2, a) - e^{ik_2 \cdot x} g^\dagger(k_2, \rho_2, a)) \end{aligned}$$

$$\begin{aligned} P_{tri}^\mu &= i\alpha \sum c_{abc} \int dx^\mu dk_1 dk_2 dk_3 \\ &\quad (B^{\nu\rho_1}(k_1)k_1^\mu - B^{\sigma\rho_1}(k_1)k_1^\nu) B_\sigma^{\rho_2}(k_2) B_\nu^{\rho_3}(k_3) \\ &\quad (e^{-ik_1 \cdot x} g(k_1, \rho_1, a) - e^{ik_1 \cdot x} g^\dagger(k_1, \rho_1, a)) \\ &\quad (e^{-ik_2 \cdot x} g(k_2, \rho_2, b) + e^{ik_2 \cdot x} g^\dagger(k_2, \rho_2, b)) \\ &\quad (e^{-ik_3 \cdot x} g(k_3, \rho_3, c) + e^{ik_3 \cdot x} g^\dagger(k_3, \rho_3, c)) \end{aligned}$$

$$\begin{aligned}
P_{quar}^\mu &= \alpha^2 \sum \int dx^\mu dk_1 dk_2 dk_3 dk_4 c_{abc} c_{ab'c'} \\
& B^{\sigma\rho_1}(k_1) B^{\nu\rho_2}(k_2) B^{\rho_3}(k_3) B^{\rho_4}(k_4) \\
& (e^{-ik_1 \cdot x} g(k_1, \rho_1, b) + e^{ik_1 \cdot x} g^\dagger(k_1, \rho_1, b)) \\
& (e^{-ik_2 \cdot x} g(k_2, \rho_2, c) + e^{ik_2 \cdot x} g^\dagger(k_2, \rho_2, c)) \\
& (e^{-ik_3 \cdot x} g(k_3, \rho_3, b') + e^{ik_3 \cdot x} g^\dagger(k_3, \rho_3, b')) \\
& (e^{-ik_4 \cdot x} g(k_4, \rho_4, c') + e^{ik_4 \cdot x} g^\dagger(k_4, \rho_4, c'))
\end{aligned}$$

This leads to the key theorem of this chapter.

**Theorem 2.5.** *On the physical Fock space,  $P_{KE}^\mu$  is equivalent to  $P_{fr}^\mu$ ,*

$$P_{fr}^\mu = - \sum_a \int \frac{d^3k}{2k_0} k^\mu \eta^{\rho\rho} (g^\dagger(k, \rho, a) g(k, \rho, a) + g(k, \rho, a) g^\dagger(k, \rho, a))$$

The proof of this result is lengthy and given in detail in Appendix B.

Recall that for point form QFT the dynamics resides in the four momentum operator. With the four momentum defined in terms of creation and annihilation operators, the physical vacuum can be investigated as the state for which the following equations hold,

$$P^\mu |\Omega\rangle = 0$$

$$U_\Lambda |\Omega\rangle = |\Omega\rangle$$

$$U_h |\Omega\rangle = |\Omega\rangle$$

meaning that the vacuum,  $|\Omega\rangle$ , is a state having zero energy and momentum and is invariant under Lorentz and color symmetry.

The next two chapters center on how to choose  $|\Omega\rangle$  using free gluon fields in such a way to satisfy these equations.



### CHAPTER 3 THE ONE DEGREE OF FREEDOM PROBLEM

Given the gluon four momentum operator,  $P^\mu$ , the first problem to be addressed is the vacuum problem,  $P^\mu|\Omega\rangle = 0$ , where

$$P^\mu = \sum_a \int dx_\nu F_a^{\alpha\beta}(x) [\eta_{\alpha'}^\mu \eta_{\alpha}^\nu \eta_{\beta\beta'} + \eta_{\alpha'}^\nu \eta_{\alpha}^\mu \eta_{\beta\beta'} - \frac{1}{2} \eta^{\mu\nu} \eta_{\alpha\alpha'} \eta_{\beta\beta'}] F_a^{\alpha'\beta'}(x) \quad (3.1)$$

and

$$\begin{aligned} F_a^{\mu\nu}(x) &= \frac{\partial G_a^\nu}{\partial x^\mu} - \frac{\partial G_a^\mu}{\partial x^\nu} + \alpha c_{abc} G_b^\mu(x) G_c^\nu(x) \\ G_a^\mu(x) &= \int dk B^{\mu\rho}(k) (e^{-ik \cdot x} g(k, \rho, a) + e^{ik \cdot x} g^\dagger(k, \rho, a)) \\ \frac{\partial G_a^\mu(x)}{\partial x^\nu} &= -i \int dk k^\nu B^{\mu\rho}(k) (e^{-ik \cdot x} g(k, \rho, a) - e^{ik \cdot x} g^\dagger(k, \rho, a)) \end{aligned}$$

with  $|\Omega\rangle$  the state to be found. How to solve such a generalized eigenvalue problem isn't obvious, so to investigate the structure of the vacuum without the complications of the infinite degrees of freedom, the problem will initially be truncated to one degree of freedom. Then the four momentum operator is replaced by a 1-D Hamiltonian operator,  $H$ . The resulting vacuum problem becomes finding an  $|\Omega\rangle$  that satisfies  $H|\Omega\rangle = 0$ .

The 1-D Hamiltonian should have the same basic structure as the four momentum with the following simplifications

$$\begin{aligned} g^\dagger(k, \rho, a) &\rightarrow g^\dagger \\ G_a^\mu(x) &\rightarrow g + g^\dagger \\ \frac{\partial G_a^\mu}{\partial x^\nu} &\rightarrow i(g - g^\dagger) \\ [g, g^\dagger] &= 1 \end{aligned}$$

which results in

$$H = [\imath(g - g^\dagger) + \alpha(g + g^\dagger)^2]^2 \quad (3.2)$$

or

$$\begin{aligned} H &= H_{KE} + H_{tri} + H_{quar} \\ &= -(g - g^\dagger)^2 + \imath\alpha[(g - g^\dagger)(g + g^\dagger)^2 + (g + g^\dagger)^2(g - g^\dagger)] + \alpha^2(g + g^\dagger)^4 \end{aligned}$$

when broken into components.

It is worth noting that  $F_a^{\mu\nu}$  is antisymmetric in its indices, but the simplification to one degree of freedom no longer allows for this antisymmetry. As a result, for the one degree case,  $\frac{\partial G_a^\nu}{\partial x^\mu} - \frac{\partial G_a^\mu}{\partial x^\nu} \rightarrow \imath(g - g^\dagger)$  since that is the overall structure of creation and annihilation operators in the full problem.

### 3.1 A Discrete Spectrum for $H$

#### 3.1.1 The Anharmonic Oscillator

The Hamiltonian problem under consideration is a 1-D eigenvalue problem that has been rewritten in terms of creation and annihilation operators, similar to the study of the 1-D harmonic oscillator Hamiltonian problem where  $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$ . Investigation of the harmonic oscillator Hamiltonian involves constructing raising and lowering operators  $a$  and  $a^\dagger$ , where the position  $x \propto (a + a^\dagger)$ , and the momentum  $p \propto \imath(a - a^\dagger)$ . The similarity of our Hamiltonian,  $H = [\imath(g - g^\dagger) + \alpha(g + g^\dagger)^2]^2 \sim [p + \alpha x^2]^2$  to the anharmonic oscillator,  $p^2 + x^2 + \alpha_3x^3 + \alpha_4x^4$ , suggests that the Hamiltonian from Eq. 3.2 might have a discrete eigenvalue spectrum.

As a preliminary ansatz, the vacuum will be written as a polynomial of creation operators acting on the Fock vacuum. Since the Hamiltonian,  $H$ , is assumed to have

a discrete spectrum, the goal is to calculate the value of  $\alpha$  which makes the lowest eigenvalue zero. If such a procedure could be generalized, it would be possible to “fine tune” the value of the strong interaction coupling constant.

### 3.1.2 Bargmann Space Realization

A concrete realization of the creation and annihilation operators is given by

$$\begin{aligned} g^\dagger &\rightarrow z \\ g &\rightarrow \partial_z \end{aligned}$$

which then transforms  $H|\Omega\rangle = \lambda|\Omega\rangle$  into a differential equation. This identification is possible since  $[\partial_z, z] = 1$ , where  $z$  and  $\partial_z$  are adjoints on a Bargmann space for holomorphic functions. The inner product for this space is

$$(F, G) = F^*(\partial_z)G(z)|_{z=0} \quad (3.3)$$

In this formulation,  $|\Omega\rangle$  is a holomorphic function of  $z$ , where  $|0\rangle$  is a constant, since  $g|0\rangle = 0 \Rightarrow \partial_z|0\rangle = 0$ .

Substituting in for  $H$  and using the commutation relation to attain a normal ordering gives

$$\begin{aligned} H &= z\partial_z + i\alpha[(\partial_z - z)(\partial_z + z)^2 + (\partial_z + z)^2(\partial_z - z)] + \alpha^2(\partial_z + z)^4 \\ &= z\partial_z + 2i\alpha[\partial_z^3 + z\partial_z^2 + (1 - z^2)\partial_z - (z + z^3)] + \\ &\quad \alpha^2[\partial_z^4 + 4z\partial_z^3 + 6(1 + z^2)\partial_z^2 + (12z + 4z^3)\partial_z + (3 + 6z^2 + z^4)] \end{aligned} \quad (3.4)$$

Note that, for the full problem,

$$\begin{aligned}
P_{KE}^\mu &= \sum_a \int dx_\nu \left( \frac{\partial G_a^\beta}{\partial x^\alpha} - \frac{\partial G_a^\alpha}{\partial x^\beta} \right) [\eta_{\alpha'}^\mu \eta_{\alpha'}^\nu \eta_{\beta\beta'} + \eta_{\alpha'}^\nu \eta_{\alpha'}^\mu \eta_{\beta\beta'} - \frac{1}{2} \eta^{\mu\nu} \eta_{\alpha\alpha'} \eta_{\beta\beta'}] \left( \frac{\partial G_a^{\beta'}}{\partial x^{\alpha'}} - \frac{\partial G_a^{\alpha'}}{\partial x^{\beta'}} \right) \\
&= - \sum_a \int \frac{d^3 k}{k_0} k^\mu \eta^{\rho\rho} (g^\dagger(k, \rho, a) g(k, \rho, a) + g(k, \rho, a) g^\dagger(k, \rho, a))
\end{aligned}$$

which is proven in Appendix B. In our 1-D realization, the kinetic energy is chosen

$H_{KE} = z\partial_z$ , although  $z\partial_z \neq -(\partial_z - z)^2$  in the one degree of freedom case.

### 3.1.2.1 The Polynomial Approach

The simplest ansatz for the vacuum state is  $|\Omega\rangle = S(z)|0\rangle$  where  $S(z)$  is a polynomial in  $z$ . An orthonormal basis for the inner product defined in Eq. 3.3 is determined by noticing

$$\begin{aligned}
(z^n, z^m) &= \partial_z^n z^m |_{z=0} \\
&= \begin{cases} 0 & n \neq m \\ n! & n = m \end{cases}
\end{aligned}$$

Therefore an orthonormal basis has the form  $\frac{z^n}{\sqrt{n!}}$ . Thus  $S(z) = \sum_{i=0}^{\infty} f_i \frac{z^i}{\sqrt{i!}}$ . Since  $H$  has up to four derivatives, a truncation of  $S(z)$  should be at least fourth order,  $S(z) = f_0 + f_1 z + f_2 \frac{z^2}{\sqrt{2}} + f_3 \frac{z^3}{\sqrt{6}} + f_4 \frac{z^4}{2\sqrt{6}}$ . Calculating  $H|\Omega\rangle$  using Eq. 3.4 for this vacuum representation results in a polynomial in terms of  $z$ .

The eigenvalue equations for  $H|\Omega\rangle = \lambda|\Omega\rangle$  can be extracted by utilizing the orthonormality of the basis states. Taking the inner product,

$$\begin{aligned}
\left( \frac{z^n}{\sqrt{n!}}, HS(z) \right) &= \left( \frac{z^n}{\sqrt{n!}}, \lambda S(z) \right) \\
&= \lambda f_n
\end{aligned}$$

gives the following five equations

$$\begin{aligned}
3\alpha^2 f_0 + 2i\alpha f_1 + 6\sqrt{2}\alpha^2 f_2 + 2\sqrt{6}i\alpha f_3 + 2\sqrt{6}\alpha^2 f_4 &= \lambda f_0 \\
-2i\alpha f_0 + (1 + 15\alpha^2)f_1 + 4\sqrt{2}i\alpha f_2 + 10\sqrt{6}\alpha^2 f_3 + 4\sqrt{6}i\alpha f_4 &= \lambda f_1 \\
6\sqrt{2}\alpha^2 f_0 - 4\sqrt{2}i\alpha f_1 + (2 + 39\alpha^2)f_2 + 6\sqrt{3}i\alpha f_3 + 28\sqrt{3}\alpha^2 f_4 &= \lambda f_2 \quad (3.5) \\
-2\sqrt{6}i\alpha f_0 + 10\sqrt{6}\alpha^2 f_1 - 6\sqrt{3}i\alpha f_2 + (3 + 75\alpha^2)f_3 + 16i\alpha f_4 &= \lambda f_3 \\
2\sqrt{6}\alpha^2 f_0 - 4\sqrt{6}i\alpha f_1 + 28\sqrt{3}\alpha^2 f_2 - 16i\alpha f_3 + (4 + 123\alpha^2)f_4 &= \lambda f_4
\end{aligned}$$

The goal is to find the value of the coupling constant,  $\alpha$ , which gives a minimum eigenvalue,  $\lambda_1 = 0$ . Writing Eq. 3.5 in matrix form gives,

$$\begin{pmatrix}
3\alpha^2 & 2i\alpha & 6\sqrt{2}\alpha^2 & 2\sqrt{6}i\alpha & 2\sqrt{6}\alpha^2 \\
-2i\alpha & 1 + 15\alpha^2 & 4\sqrt{2}i\alpha & 10\sqrt{6}\alpha^2 & 4\sqrt{6}i\alpha \\
6\sqrt{2}\alpha^2 & -4\sqrt{2}i\alpha & 2 + 39\alpha^2 & 6\sqrt{3}i\alpha & 28\sqrt{3}\alpha^2 \\
-2\sqrt{6}i\alpha & 10\sqrt{6}\alpha^2 & -6\sqrt{3}i\alpha & 3 + 75\alpha^2 & 16i\alpha \\
2\sqrt{6}\alpha^2 & -4\sqrt{6}i\alpha & 28\sqrt{3}\alpha^2 & -16i\alpha & 4 + 123\alpha^2
\end{pmatrix}
\begin{pmatrix}
f_0 \\
f_1 \\
f_2 \\
f_3 \\
f_4
\end{pmatrix}
= \lambda
\begin{pmatrix}
f_0 \\
f_1 \\
f_2 \\
f_3 \\
f_4
\end{pmatrix} \quad (3.6)$$

where the matrix representation for the Hamiltonian (labeled  $H_4$  for this truncation) is Hermitian, since  $H$  is a Hermitian operator.

Setting the characteristic polynomial of  $H_4$  to zero gives the value(s) of  $\alpha$  corresponding to the state having lowest eigenvalue of 0. Due to the choice of  $H_{KE} = z\partial_z$ ,  $\alpha = 0$  will automatically be a solution. The characteristic polynomial for  $H_4$  is symmetric about  $\alpha = 0$  and has only one positive root at  $\alpha \approx 1.27941$ .

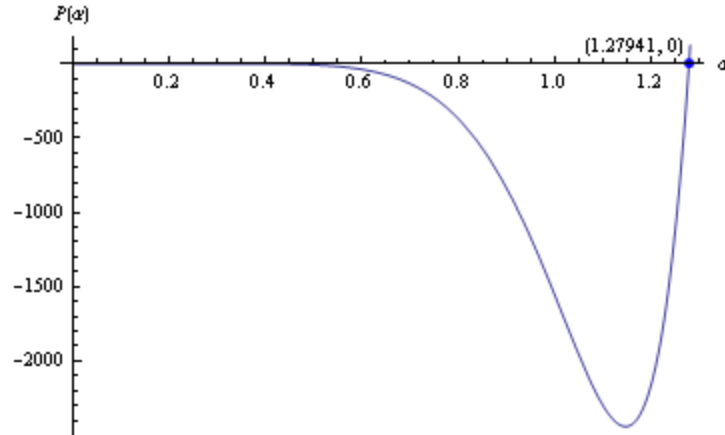


Figure 3.1: Characteristic polynomial plot for  $H_4$

The same method can be applied for higher order polynomials. Due to the structure of the matrix equations, the Hamiltonians are nested,  $H_i \subset H_{i+1}$ , with new information only appearing in the additional row/column resulting from the additional coefficient,  $f_{i+1}$ . All the characteristic polynomial plots pass through the origin. Additionally, the kinetic energy term appears to pull the characteristic polynomial down, while the trilinear and quartic terms pull  $P(\alpha)$  up until  $\alpha$  becomes large enough for the higher order terms to dominate as in Fig. 3.1. Thus, in this simple model it is possible to “fine tune” the coupling constant. The following table summarizes the nonzero alpha value calculations for truncations from fourth order up to tenth.

Matrix Truncation	$\alpha$ value
$H_4$	1.27941
$H_5$	1.44213
$H_6$	1.91875
$H_7$	2.10005
$H_8$	2.60845
$H_9$	2.8079
$H_{10}$	3.3482

Table 3.1:  $\alpha$  results for selected truncations

### 3.1.2.2 The Exponential Approach

The attractiveness of the polynomial vacuum representation is the ease of the calculation. However, as seen in Table 3.1, it is not clear whether increasing the order of the polynomial converges to the exact solution. Rather than continue increasing the length of a finite order polynomial, consider a vacuum of the form

$$|\Omega \rangle = e^{\sum_{i=0}^{\infty} f_i z^i} = e^{S(z)}$$

This has the advantage that a finite polynomial,  $S(z)$ , will still result in what is equivalent to an overall infinite polynomial for  $|\Omega \rangle$ , but the linear structure from the previous case is sacrificed making it impossible to solve the problem as simply as before.

The major obstacle comes from the quartic component of the Hamiltonian.

$H_{quar}e^{S(z)} = (\partial_z + z)^4 e^{S(z)}$  will result in four additional equations for each increase in degree of  $S(z)$ , an increase of only one unknown.

$$\begin{aligned}\partial_z^4(e^{f_i z^i}) &= O((z^{i-1})^4) \\ \partial_z^4(e^{f_{i+1} z^{i+1}}) &= O((z^i)^4)\end{aligned}$$

The ansatz for the vacuum state should minimize the effect of the quartic component. Thus consider  $|\Omega\rangle = e^{S(\partial_z + z)}|0\rangle$ , with  $[H_{quar}, e^{S(\partial_z + z)}] = 0$ . By allowing the quartic term to commute with the vacuum operator, the problem is transformed from

$$H e^{S(\partial_z + z)}|0\rangle = \lambda e^{S(\partial_z + z)}|0\rangle$$

to

$$(\tilde{H} - \lambda)|0\rangle = 0$$

where  $\tilde{H} = e^{-S(\partial_z + z)} H e^{S(\partial_z + z)}$ .  $\tilde{H}_{quar} = H_{quar}$  is a direct consequence of the choice of the vacuum structure.

The following commutation relation will be useful to determine the other pieces of  $\tilde{H}$

$$\begin{aligned}[\partial_z - z, \partial_z + z] &= 2 \\ [\partial_z - z, (\partial_z + z)^2] &= 4(\partial_z + z) \\ [\partial_z - z, (\partial_z + z)^3] &= 6(\partial_z + z)^2 \\ &\vdots \\ [\partial_z - z, S(\partial_z + z)] &= 2\dot{S}(\partial_z + z)\end{aligned}\tag{3.7}$$



or uncoupled

$$[z, S(\partial_z + z)] = -\dot{S}(\partial_z + z)$$

$$[\partial_z, S(\partial_z + z)] = \dot{S}(\partial_z + z)$$

Recall

$$H = H_{KE} + H_{Tri} + H_{Quar}$$

or

$$H = z\partial_z + \iota\alpha[(\partial_z - z)(\partial_z + z)^2 + (\partial_z + z)^2(\partial_z - z)] + \alpha^2(\partial_z + z)^4$$

First write

$$(e^{-S(\partial_z+z)} H e^{S(\partial_z+z)} - \lambda)|0 \rangle = (\tilde{H} - \lambda)|0 \rangle = 0$$

where solving for  $\tilde{H}$  is a matter of commuting the exponential through the Hamiltonian operator. Solving for the components gives

$$\begin{aligned}
\tilde{H}_{KE}|0\rangle &= e^{-S(\partial_z+z)} z \partial_z e^{S(\partial_z+z)} |0\rangle \\
&= e^{-S(\partial_z+z)} z (\dot{S} e^{S(\partial_z+z)} + e^{S(\partial_z+z)} \partial_z) |0\rangle \\
&= e^{-S(\partial_z+z)} z (\dot{S} e^{S(\partial_z+z)}) |0\rangle \\
&= e^{-S(\partial_z+z)} (-\ddot{S} e^{S(\partial_z+z)} + \dot{S}^2 e^{S(\partial_z+z)} + \dot{S} e^{S(\partial_z+z)} z) |0\rangle \\
&= e^{-S(\partial_z+z)} e^{S(\partial_z+z)} (-\ddot{S} - \dot{S}^2 + \dot{S} z) |0\rangle \\
&= (-\ddot{S} - \dot{S}^2 + \dot{S} z) |0\rangle
\end{aligned}$$

$$\begin{aligned}
\tilde{H}_{Tri}|0\rangle &= i\alpha e^{-S(\partial_z+z)} [(\partial_z - z)(\partial_z + z)^2 + (\partial_z + z)^2(\partial_z - z)] e^{S(\partial_z+z)} |0\rangle \\
&= i\alpha e^{-S(\partial_z+z)} [(\partial_z - z)(e^{S(\partial_z+z)}(\partial_z + z)^2) + (\partial_z + z)^2 e^{S(\partial_z+z)}(2\dot{S} + (\partial_z - z))] |0\rangle \\
&= i\alpha e^{-S(\partial_z+z)} [(2\dot{S} e^{S(\partial_z+z)}(\partial_z + z)^2 + e^{S(\partial_z+z)} 4(\partial_z + z) \\
&\quad + e^{S(\partial_z+z)}(\partial_z + z)^2(\partial_z - z)) + e^{S(\partial_z+z)}(\partial_z + z)^2(2\dot{S} - z)] |0\rangle \\
&= i\alpha [4\dot{S} \cdot (\partial_z + z)^2 + 4(\partial_z + z) + (\partial_z + z)^2(\partial_z - z) - (\partial_z + z)^2 z] |0\rangle \\
&= i\alpha [4\dot{S} \cdot (\partial_z + z)^2 + 4z - 2(\partial_z + z)^2 z] |0\rangle
\end{aligned}$$

$$\begin{aligned}
\tilde{H}_{Quar}|0\rangle &= e^{-S(\partial_z+z)} (\alpha^2 (\partial_z + z)^4) e^{S(\partial_z+z)} |0\rangle \\
&= \alpha^2 (\partial_z + z)^4 |0\rangle
\end{aligned}$$

Condensing the terms back together results in

$$\begin{aligned}
\tilde{H}|0\rangle &= [-\ddot{S} - \dot{S}^2 + \dot{S}z + \imath\alpha(4\dot{S} \cdot (\partial_z + z)^2 + 4z - 2(\partial_z + z)^2z) + \alpha^2(\partial_z + z)^4]|0\rangle \\
&= [-\ddot{S} - \dot{S}^2 + \dot{S}z + \imath\alpha(4\dot{S} \cdot (1 + z^2) - 2z - 2z^3) + \alpha^2(3 + 6z^2 + z^4)]|0\rangle
\end{aligned}$$

Since  $\tilde{H}$  depends on  $\dot{S}$  and  $\ddot{S}$ , choose  $\dot{S} = f_0 + f_1(\partial_z + z) + f_2(\partial_z + z)^2$ . Notice that the  $\dot{S}^2$  term is responsible for the loss of linearity. The resulting set of equations are

$$\begin{aligned}
3\alpha^2 + 4\imath\alpha f_0 - f_0^2 - f_1^2 + 12\imath\alpha f_2 - 2f_0f_2 - 3f_2^2 &= \lambda \\
(-2\imath\alpha + f_0 + 12\imath\alpha f_1 - 2f_0f_1 + f_2 - 6f_1f_2)z &= 0 \\
(6\alpha^2 + 4\imath\alpha f_0 + f_1 - f_1^2 + 24\imath\alpha f_2 - 2f_0f_2 - 6f_2^2)z^2 &= 0 \\
(-2\imath\alpha + 4\imath\alpha f_1 + f_2 - 2f_1f_2)z^3 &= 0 \\
(\alpha^2 + 4\imath\alpha f_2 - f_2^2)z^4 &= 0
\end{aligned}$$

Unlike the polynomial case, where we had an equation for each coefficient of  $S(z)$ , here there are five equations but only four unknowns ( $f_0, f_1, f_2$ , and  $\alpha$ ). A truncation must be made, but it is not clear which equation to truncate. Truncating the highest term and solving for  $\lambda = 0$  results in  $\alpha = 0, \frac{1}{6\sqrt{2}} \approx 0.117581$ . Truncating the  $z^3$  term instead gives  $\alpha = 0, \sqrt{\frac{5\sqrt{\frac{11}{3}}-7}{576}} \approx 0.0668522$ . All other truncations have no nontrivial ( $\alpha = 0$ ) solutions.

To test the stability of the exponential model, additional terms in  $\dot{S}$  are needed, but each increase in one term of  $\dot{S}$  results in two additional equations due to the  $\dot{S}^2$  term, meaning more arbitrary truncations will be necessary.

### 3.1.3 Bound States

If the spectrum of  $H$  is discrete, the vacuum eigenvalue problem is only the first eigenvalue problem leading to bound states, which in this simplified model gives the analogue of a glueball spectrum.

For the full problem it is impossible to have a color singlet with only one gluon. Therefore, any bound state calculation will require at least two gluon creation operators, or  $z^2|0\rangle$ . Calculating properties for glueballs requires consideration of

$$He^{S(\partial_z+z)}z^2|0\rangle = \lambda e^{S(\partial_z+z)}z^2|0\rangle$$

which results in two additional equations with one new variable  $\lambda$ , where  $\lambda$  is no longer zero. It is important to note that the coefficients found for the polynomial  $S(z)$  in the vacuum case will not be the same as those in the bound state problem, although the method will be similar. Instead of solving for  $\alpha$  when  $\lambda = 0$ ,  $\lambda$  will be solved by using the  $\alpha$  found from the vacuum case, which will give the “glueball” eigenvalue.

## 3.2 $H$ has a Continuous Spectrum

In the previous section it was assumed that  $H$  has a discrete spectrum, and the consequences of this assumption were explored. We will now show that despite the similarity to the anharmonic oscillator, the spectrum of  $H$  is actually continuous. Further, the main insight of the previous section, that the vacuum state be written as an exponential in gluon fields, will be used to prove this result.

Instead of considering the vacuum problem in pieces,  $H = H_{KE} + H_{tri} + H_{quar}$ , recall from Eq. 3.2 that the 1-D Hamiltonian is

$$H = [\imath(g - g^\dagger) + \alpha(g + g^\dagger)^2]^2$$

Writing  $|\Omega\rangle = e^{S(g+g^\dagger)}|0\rangle$ , we need to solve

$$\begin{aligned} [\imath(g - g^\dagger) + \alpha(g + g^\dagger)^2]^2 e^{S(g+g^\dagger)}|0\rangle &= \lambda e^{S(g+g^\dagger)}|0\rangle \\ e^{-S(g+g^\dagger)}[\imath(g - g^\dagger) + \alpha(g + g^\dagger)^2]^2 e^{S(g+g^\dagger)}|0\rangle &= \lambda|0\rangle \\ e^{-S(g+g^\dagger)}[\imath(g - g^\dagger) + \alpha(g + g^\dagger)^2]e^{S(g+g^\dagger)}e^{-S(g+g^\dagger)}[\imath(g - g^\dagger) + \alpha(g + g^\dagger)^2]e^{S(g+g^\dagger)}|0\rangle &= \lambda|0\rangle \\ (e^{-S(g+g^\dagger)}[\imath(g - g^\dagger) + \alpha(g + g^\dagger)^2]e^{S(g+g^\dagger)})^2|0\rangle &= \lambda|0\rangle \end{aligned}$$

where the ground state eigenvalue  $\lambda = 0$ . The pertinent operator commutation relations adapted from Eq. 3.7 are

$$\begin{aligned} [g, g^\dagger] &= 1 \\ [g - g^\dagger, S(g + g^\dagger)] &= 2\dot{S}(g + g^\dagger) \end{aligned}$$

We can use these commutation relations to get

$$\begin{aligned} e^{-S(g+g^\dagger)}[\imath(g - g^\dagger) + \alpha(g + g^\dagger)^2]e^{S(g+g^\dagger)} &= [2\imath\dot{S}(g + g^\dagger) + \imath(g - g^\dagger) + \alpha(g + g^\dagger)^2] \\ (e^{-S(g+g^\dagger)}[\imath(g - g^\dagger) + \alpha(g + g^\dagger)^2]e^{S(g+g^\dagger)})^2|0\rangle &= [2\imath\dot{S}(g + g^\dagger) + \imath(g - g^\dagger) + \alpha(g + g^\dagger)^2]^2|0\rangle \end{aligned}$$

If  $S$  is chosen so that  $\dot{S} = \frac{\imath\alpha}{2}(g + g^\dagger)^2$ , the quadratic terms are cancelled, leaving the free kinetic energy which has a continuous spectrum!

$$\begin{aligned}\tilde{H}|0\rangle &= -(g - g^\dagger)^2|0\rangle \\ &= H_{KE}|0\rangle\end{aligned}$$

Therefore  $H$  has a continuous spectrum and the vacuum state  $|\Omega\rangle$  converts the Hamiltonian with trilinear and quartic interactions to one with only kinetic energy. The gluon interaction terms have been transformed away.

**Theorem 3.1.** *The spectrum of  $H$  is continuous and agrees with  $H_{KE}$ .*

The question is whether such a procedure also works for the full infinite degree of freedom problem.

## CHAPTER 4 THE INFINITE DEGREE OF FREEDOM PROBLEM

### 4.1 The Infinite Degree of Freedom Problem

The major result from the one degree of freedom problem in Chapter 3, was that setting  $\dot{S} = \frac{i\alpha}{2}(g + g^\dagger)^2$ , where  $|\Omega\rangle = e^{S(g+g^\dagger)}|0\rangle$  resulted in the cancellation of the interaction term of the Hamiltonian. To apply a similar approach to the full four momentum problem, properties of gluons derived in Chapter 2 will be needed.

Recall that the gluon four momentum operator is

$$P^\mu = \sum_a \int dx_\nu F_a^{\alpha\beta}(x) [\eta_{\alpha'}^\mu \eta_{\alpha'}^\nu \eta_{\beta\beta'} + \eta_{\alpha'}^\nu \eta_{\alpha'}^\mu \eta_{\beta\beta'} - \frac{1}{2} \eta^{\mu\nu} \eta_{\alpha\alpha'} \eta_{\beta\beta'}] F_a^{\alpha'\beta'}(x)$$

where

$$\begin{aligned} F_a^{\mu\nu}(x) &= \frac{\partial G_a^\nu}{\partial x^\mu} - \frac{\partial G_a^\mu}{\partial x^\nu} + \alpha c_{a,b,c} G_b^\mu(x) G_c^\nu(x) \\ G_a^\mu(x) &= \int dk B^{\mu\rho}(k) (e^{-ik \cdot x} g(k, \rho, a) + e^{ik \cdot x} g^\dagger(k, \rho, a)) \\ \frac{\partial G_a^\mu}{\partial x_\nu} &= -i \int dk k^\nu B^{\mu\rho}(k) (e^{-ik \cdot x} g(k, \rho, a) - e^{ik \cdot x} g^\dagger(k, \rho, a)) \end{aligned}$$

Also from Chapter 2,

$$\left[ \frac{\partial G_a^\mu(x)}{\partial x_\beta}, \prod_i G_{b_i}^{\nu_i}(y_i) \right] = \sum_i \eta^{\mu\nu_i} \delta_{ab_i} \frac{\partial \Delta(x-y_i)}{\partial x_\beta} \prod_{j \neq i} G_{b_j}^{\nu_j}(y_j)$$

which we will need for calculations on the vacuum, which we will be writing as an exponential of a product of gluon fields.

### 4.1.1 The Structure of $S$

If  $S$  is chosen to be a product of three gluon fields, calculating

$$e^{-S} \left( \frac{\partial G_a^\alpha}{\partial x_\beta} - \frac{\partial G_a^\beta}{\partial x_\alpha} \right) e^S = \frac{\partial G_a^\alpha}{\partial x_\beta} - \frac{\partial G_a^\beta}{\partial x_\alpha} + \dots G^\alpha G^\beta$$

should make it possible to choose the coefficients of  $S$  in such a way to cancel off the interaction term of the field tensor,  $\alpha c_{abc} G_b^\mu(x) G_c^\nu(x)$ .

A general form for  $S$  is given by

$$S = \int dx_1 dx_2 dx_3 c_{a_1 a_2 a_3} f_{\mu_1 \mu_2 \mu_3}(x_1, x_2, x_3) G_{a_1}^{\mu_1}(x_1) G_{a_2}^{\mu_2}(x_2) G_{a_3}^{\mu_3}(x_3)$$

Since the vacuum state is written  $|\Omega\rangle = e^S |0\rangle$ ,  $S$  must be invariant under Lorentz and color transformations.  $S$  is a function of three triplets of indices  $(a_1, a_2, a_3)$ ,  $(\mu_1, \mu_2, \mu_3)$ , and  $(x_1, x_2, x_3)$ . The gluon fields are local so they commute with one another for  $x - y$  spacelike, meaning that a permutation of  $(a_i, \mu_i, x_i) \rightarrow (a_j, \mu_j, x_j)$  must leave  $S$  unchanged. To cancel the structure constant,  $c_{a_1 a_2 a_3}$ , from the field tensor,  $f$  is chosen to be proportional to  $c_{a_1 a_2 a_3}$  which is antisymmetric in the color indices,  $(a_1, a_2, a_3)$ . As a result,  $f_{\mu_1 \mu_2 \mu_3}(x_1, x_2, x_3)$  must be antisymmetric under permutations of  $(\mu_i, x_i) \rightarrow (\mu_j, x_j)$  where  $i \neq j$ .



The field tensor can be rewritten into two similar terms,

$$\begin{aligned}
F_a^{\mu\nu}(x) &= \frac{\partial G_a^\nu}{\partial x_\mu} - \frac{\partial G_a^\mu}{\partial x_\nu} + \alpha c_{abc} G_b^\mu(x) G_c^\nu(x) \\
&= \frac{\partial G_a^\nu}{\partial x_\mu} - \frac{\partial G_a^\mu}{\partial x_\nu} + \frac{\alpha}{2} (c_{abc} G_b^\mu(x) G_c^\nu(x) + c_{abc} G_c^\nu(x) G_b^\mu(x)) \\
&= \frac{\partial G_a^\nu}{\partial x_\mu} - \frac{\partial G_a^\mu}{\partial x_\nu} + \frac{\alpha}{2} (c_{abc} G_b^\mu(x) G_c^\nu(x) + c_{acb} G_b^\nu(x) G_c^\mu(x)) \\
&= \frac{\partial G_a^\nu}{\partial x_\mu} - \frac{\partial G_a^\mu}{\partial x_\nu} + \frac{\alpha}{2} (c_{abc} G_b^\mu(x) G_c^\nu(x) - c_{abc} G_b^\nu(x) G_c^\mu(x)) \\
&= \left( \frac{\partial G_a^\nu}{\partial x_\mu} - \frac{\alpha}{2} c_{abc} G_b^\nu(x) G_c^\mu(x) \right) - \left( \frac{\partial G_a^\mu}{\partial x_\nu} - \frac{\alpha}{2} c_{abc} G_b^\mu(x) G_c^\nu(x) \right)
\end{aligned}$$

so that if coefficients of  $S$  are chosen in such a way that

$$e^{-S} \left( \frac{\partial G_a^\alpha}{\partial x_\beta} - \frac{\alpha}{2} c_{aa_2 a_3} G_{a_2}^\alpha(x) G_{a_3}^\beta(x) \right) e^S = \frac{\partial G_a^\alpha}{\partial x_\beta}$$

then

$$e^{-S} F_a^{\mu\nu}(x) e^S = \frac{\partial G_a^\nu}{\partial x_\mu} - \frac{\partial G_a^\mu}{\partial x_\nu}$$

which transforms away the self-interaction and converts the gluon four momentum operator into the free four momentum operator,  $P^\mu |\Omega \rangle = P_{fr}^\mu |\Omega \rangle$ .

From Chapter 3 we know to choose  $S$  in such a way to commute with the free gluon fields, thereby commuting with the interaction term of the field tensor.

Consider the noncommuting term,

$$\begin{aligned}
e^{-S} \frac{\partial G_a^\alpha(x)}{\partial x_\beta} e^S &= \frac{\partial G_a^\alpha(x)}{\partial x_\beta} + \int dx_1 dx_2 dx_3 c_{a_1 a_2 a_3} f_{\mu_1 \mu_2 \mu_3}(x_1, x_2, x_3) \\
&\quad \left[ \eta^{\alpha \mu_1} \delta_{a a_1} \frac{\partial \Delta(x - x_1)}{\partial x_\beta} \frac{1}{2} (G_{a_2}^{\mu_2}(x_2) G_{a_3}^{\mu_3}(x_3) + G_{a_3}^{\mu_3}(x_3) G_{a_2}^{\mu_2}(x_2)) \right. \\
&\quad + \eta^{\alpha \mu_2} \delta_{a a_2} \frac{\partial \Delta(x - x_2)}{\partial x_\beta} \frac{1}{2} (G_{a_1}^{\mu_1}(x_1) G_{a_3}^{\mu_3}(x_3) + G_{a_3}^{\mu_3}(x_3) G_{a_1}^{\mu_1}(x_1)) \\
&\quad \left. + \eta^{\alpha \mu_3} \delta_{a a_3} \frac{\partial \Delta(x - x_3)}{\partial x_\beta} \frac{1}{2} (G_{a_1}^{\mu_1}(x_1) G_{a_2}^{\mu_2}(x_2) + G_{a_2}^{\mu_2}(x_2) G_{a_1}^{\mu_1}(x_1)) \right]
\end{aligned}$$

which accounts for the fact that the gluon fields commute.

Re-indexing and contracting gives

$$\begin{aligned}
e^{-S} \frac{\partial G_a^\alpha(x)}{\partial x_\beta} e^S - \frac{\partial G_a^\alpha(x)}{\partial x_\beta} &= \frac{1}{2} \int dx_1 dx_2 dx_3 \frac{\partial \Delta(x-x_1)}{\partial x_\beta} \eta^{\alpha\alpha} \\
& [c_{aa_2 a_3} f_{\alpha\mu_2\mu_3}(x_1, x_2, x_3) G_{a_2}^{\mu_2}(x_2) G_{a_3}^{\mu_3}(x_3) \\
& + c_{aa_3 a_2} f_{\alpha\mu_3\mu_2}(x_1, x_3, x_2) G_{a_3}^{\mu_3}(x_3) G_{a_2}^{\mu_2}(x_2) \\
& + c_{a_2 a a_3} f_{\mu_2\alpha\mu_3}(x_2, x_1, x_3) G_{a_2}^{\mu_2}(x_2) G_{a_3}^{\mu_3}(x_3) \\
& + c_{a_3 a a_2} f_{\mu_3\alpha\mu_2}(x_3, x_1, x_2) G_{a_3}^{\mu_3}(x_3) G_{a_2}^{\mu_2}(x_2) \\
& + c_{a_3 a_2 a} f_{\mu_3\mu_2\alpha}(x_3, x_2, x_1) G_{a_3}^{\mu_3}(x_3) G_{a_2}^{\mu_2}(x_2) \\
& + c_{a_2 a_3 a} f_{\mu_2\mu_3\alpha}(x_2, x_3, x_1) G_{a_2}^{\mu_2}(x_2) G_{a_3}^{\mu_3}(x_3)] \\
& = \frac{1}{2} \int dx_1 dx_2 dx_3 \frac{\partial \Delta(x-x_1)}{\partial x_\beta} c_{aa_2 a_3} [f_{\alpha\mu_2\mu_3}(x_1, x_2, x_3) \\
& - f_{\alpha\mu_3\mu_2}(x_1, x_3, x_2) - f_{\mu_2\alpha\mu_3}(x_2, x_1, x_3) \\
& + f_{\mu_3\alpha\mu_2}(x_3, x_1, x_2) - f_{\mu_3\mu_2\alpha}(x_3, x_2, x_1) \\
& + f_{\mu_2\mu_3\alpha}(x_2, x_3, x_1)] \eta^{\alpha\alpha} G_{a_2}^{\mu_2}(x_2) G_{a_3}^{\mu_3}(x_3)
\end{aligned}$$

The last equality results from the antisymmetry of the structure constants for color  $SU(3)$ . To cancel the field tensor interaction term, the  $f$ 's must be chosen so

that

$$\begin{aligned}
\frac{\alpha}{2}c_{aa_2a_3}G_{a_2}^\alpha(x)G_{a_3}^\beta(x) &= \frac{1}{2}\int dx_1dx_2dx_3\frac{\partial\Delta(x-x_1)}{\partial x_\beta}c_{aa_2a_3}[f_{\alpha\mu_2\mu_3}(x_1,x_2,x_3) \\
&\quad -f_{\alpha\mu_3\mu_2}(x_1,x_3,x_2) - f_{\mu_2\alpha\mu_3}(x_2,x_1,x_3) \\
&\quad +f_{\mu_3\alpha\mu_2}(x_3,x_1,x_2) - f_{\mu_3\mu_2\alpha}(x_3,x_2,x_1) \\
&\quad +f_{\mu_2\mu_3\alpha}(x_2,x_3,x_1)]\eta^{\alpha\alpha}G_{a_2}^{\mu_2}(x_2)G_{a_3}^{\mu_3}(x_3)
\end{aligned}$$

The interaction term of the field tensor can be rewritten

$$\frac{\alpha}{2}c_{aa_2a_3}G_{a_2}^\alpha(x)G_{a_3}^\beta(x) = \frac{\alpha}{2}c_{aa_2a_3}\int dx_2dx_3\delta(x-x_2)\delta(x-x_3)\eta_{\mu_2}^\alpha\eta_{\mu_3}^\beta G_{a_2}^{\mu_2}(x_2)G_{a_3}^{\mu_3}(x_3)$$

so that the equation for the  $f$ 's simplifies to

$$\begin{aligned}
\int dx_1\frac{\partial\Delta(x-x_1)}{\partial x_\beta}[f_{\alpha\mu_2\mu_3}(x_1,x_2,x_3) - f_{\alpha\mu_3\mu_2}(x_1,x_3,x_2) - f_{\mu_2\alpha\mu_3}(x_2,x_1,x_3) + f_{\mu_3\alpha\mu_2}(x_3,x_1,x_2) \\
-f_{\mu_3\mu_2\alpha}(x_3,x_2,x_1) + f_{\mu_2\mu_3\alpha}(x_2,x_3,x_1)]\eta^{\alpha\alpha} - \frac{\alpha}{2}\delta(x-x_2)\delta(x-x_3)\eta_{\mu_2}^\alpha\eta_{\mu_3}^\beta = 0
\end{aligned} \tag{4.1}$$

## 4.2 Determining the $f$ 's

Consider setting  $f_{\mu_1\mu_2\mu_3}(x_1,x_2,x_3) = f_1(\mu_1,\mu_2,\mu_3)f_2(x_1,x_2,x_3)$ . If  $f_1$  is antisymmetric in its indices and  $f_2$  is symmetric in its indices, then  $f$  is antisymmetric overall as needed. However, applying the symmetry of  $f_2$  by permuting  $(x_i,x_j,x_k) \rightarrow (x_1,x_2,x_3)$  results in the cancellation all of the  $f$  terms from Eq. 4.1, meaning that the equality cannot hold. Similarly, if  $f_1$  is symmetric and  $f_2$  antisymmetric, applying the symmetry of  $f_1$  by permuting  $(\mu_i,\mu_j,\mu_k) \rightarrow (\alpha,\alpha,\beta)$  and

noticing that  $x_2 = x_3 = x$  in order for the equality in Eq. 4.1 to hold, again all of the terms cancel and the equality cannot hold.

The only other way to satisfy Eq. 4.1 when separating  $f$  into a product of separate indices is to require  $f_1$  and  $f_2$  to have a mixed symmetry where the individual functions are neither symmetric nor antisymmetric but have an overall antisymmetry under permutations  $(\mu_i, x_i) \rightarrow (\mu_j, x_j)$  for  $i \neq j$ . Mixed symmetry is discussed further in Appendix C.

#### 4.2.1 Mixed Symmetry

The mixed representation is a two dimensional representation with projection operators from Appendix C:

$$\begin{aligned} P_+ &= \sum_g D_{11}^{-1}(g)U(g) \\ P_- &= \sum_g D_{22}^{-1}(g)U(g) \end{aligned} \tag{4.2}$$

where  $D(g)$  is a  $2 \times 2$  irrep (shown in Eq. C.1) and  $U(g)$  is the permutation operator for the group element  $g \in S_3$ .

The most general way of writing the antisymmetric

$$f_{\mu_1\mu_2\mu_3}(x_1, x_2, x_3) = f_1(\mu_1, \mu_2, \mu_3)f_2(x_1, x_2, x_3)$$

is in terms of Clebsch-Gordan Coefficients (or CGK) as

$$\begin{aligned}
f_{\mu_1\mu_2\mu_3}^A(x_1, x_2, x_3) &= C_{++}^{AMM} P_+ f_1(\mu_1, \mu_2, \mu_3) P_+ f_2(x_1, x_2, x_3) + \\
&C_{+-}^{AMM} P_+ f_1(\mu_1, \mu_2, \mu_3) P_- f_2(x_1, x_2, x_3) + \\
&C_{-+}^{AMM} P_- f_1(\mu_1, \mu_2, \mu_3) P_+ f_2(x_1, x_2, x_3) + \\
&C_{--}^{AMM} P_- f_1(\mu_1, \mu_2, \mu_3) P_- f_2(x_1, x_2, x_3)
\end{aligned}$$

where  $A$  is for antisymmetric,  $M$  is for the mixed representation and  $+$ ,  $-$  represent the projection operators.

Since  $f$  is antisymmetric under interchange of both its indices  $(\mu_i, x_i) \rightarrow (\mu_j, x_j)$ , where  $i \neq j$ , permuting the indices will change the sign of  $f$ , for example

$$U_{(12)} f_{\mu_1\mu_2\mu_3}^A(x_1, x_2, x_3) = -f_{\mu_1\mu_2\mu_3}^A(x_1, x_2, x_3)$$

It is shown in Appendix C that  $U_{(12)} P_+ = P_+$  and  $U_{(12)} P_- = -P_-$ . Therefore

$$\begin{aligned}
U_{(12)} f_{\mu_1\mu_2\mu_3}^A(x_1, x_2, x_3) &= C_{++}^{AMM} P_+ f_1(\mu_1, \mu_2, \mu_3) P_+ f_2(x_1, x_2, x_3) - \\
&C_{+-}^{AMM} P_+ f_1(\mu_1, \mu_2, \mu_3) P_- f_2(x_1, x_2, x_3) - \\
&C_{-+}^{AMM} P_- f_1(\mu_1, \mu_2, \mu_3) P_+ f_2(x_1, x_2, x_3) + \\
&C_{--}^{AMM} P_- f_1(\mu_1, \mu_2, \mu_3) P_- f_2(x_1, x_2, x_3) \\
&= -f_{\mu_1\mu_2\mu_3}^A(x_1, x_2, x_3)
\end{aligned}$$

or

$$C_{++}^{AMM} P_+ f_1(\mu_1, \mu_2, \mu_3) P_+ f_2(x_1, x_2, x_3) + C_{--}^{AMM} P_- f_1(\mu_1, \mu_2, \mu_3) P_- f_2(x_1, x_2, x_3) = 0$$

which requires  $C_{++}^{AMM} = C_{--}^{AMM} = 0$ .

This leaves

$$f_{\mu_1\mu_2\mu_3}^A(x_1, x_2, x_3) = C_{+-}^{AMM} P_+ f_1(\mu_1, \mu_2, \mu_3) P_- f_2(x_1, x_2, x_3) + C_{-+}^{AMM} P_- f_1(\mu_1, \mu_2, \mu_3) P_+ f_2(x_1, x_2, x_3)$$

Applying the projection operators from Eq. 4.2 to  $f_1(\mu_1, \mu_2, \mu_3)$  gives

$$\begin{aligned} P_+ f_1(\mu_1, \mu_2, \mu_3) &= f_1(\mu_1, \mu_2, \mu_3) + f_1(\mu_2, \mu_1, \mu_3) - \frac{1}{2} f_1(\mu_3, \mu_2, \mu_1) - \\ &\quad \frac{1}{2} f_1(\mu_1, \mu_3, \mu_2) - \frac{1}{2} f_1(\mu_3, \mu_1, \mu_2) - \frac{1}{2} f_1(\mu_2, \mu_3, \mu_1) \\ P_- f_1(\mu_1, \mu_2, \mu_3) &= f_1(\mu_1, \mu_2, \mu_3) - f_1(\mu_2, \mu_1, \mu_3) + \frac{1}{2} f_1(\mu_3, \mu_2, \mu_1) + \\ &\quad \frac{1}{2} f_1(\mu_1, \mu_3, \mu_2) - \frac{1}{2} f_1(\mu_3, \mu_1, \mu_2) - \frac{1}{2} f_1(\mu_2, \mu_3, \mu_1) \end{aligned}$$

In order for the equality in Eq. 4.1 to hold,  $\mu_1 = \mu_2 = \alpha$ , and  $\mu_3 = \beta$  giving

$$\begin{aligned} P_+ f_1(\mu_1, \mu_2, \mu_3) &= 2f_1(\alpha, \alpha, \beta) - f_1(\beta, \alpha, \alpha) - f_1(\alpha, \beta, \alpha) \\ P_- f_1(\mu_1, \mu_2, \mu_3) &= 0 \end{aligned}$$

The fact that  $P_- f = 0$ , means that even in the mixed symmetry, there is no way to write  $f$  as a product of functions of the  $x$  indices and the  $\mu$  indices.

#### 4.2.2 The Kernel

Although  $f$  cannot be factored any further, there is the possibility that an  $f$  exists to transform away the gluon interaction. To further investigate, it is necessary to study the structure of the kernel  $\frac{\partial \Delta(x-y)}{\partial x_\beta}$  and in particular see if it has an inverse.

Recall from the calculation in Appendix A, that

$$\begin{aligned}\frac{\partial \Delta(x-y)}{\partial x_\beta} &= \frac{1}{(2\pi)^3} \int \frac{d^3 k}{2k_0} \imath k^\beta (e^{\imath k \cdot (x-y)} + e^{-\imath k \cdot (x-y)}) \\ &= \frac{1}{(2\pi)^3} \int \frac{d^3 k}{k_0} \imath k^\beta \cos(k \cdot (x-y))\end{aligned}$$

meaning that the kernel is symmetric under interchange of  $x - y$ .

#### 4.2.2.1 Does the Kernel Have an Inverse?

In Chapter 2 the Lorentz invariant distribution was introduced, namely

$$W(p, q) = 2 \int d^4 x \delta(x \cdot x - \tau^2) \Theta(x_0) x_\nu p^\nu e^{-\imath q \cdot x}$$

where  $p = k + k'$ , and  $q = k - k'$ . For  $\frac{\partial \Delta(x-y)}{\partial x_\beta}$ , there is an integral over a different Lorentz invariant measure,  $\frac{d^3 k}{2k_0}$ , but an otherwise identical structure. Renaming  $p = x + y$  and  $q = x - y$  and substituting the measure transforms to

$$W(p, q) = \int \frac{d^3 k}{2k_0} p_\beta k^\beta e^{-\imath q \cdot k}$$

and

$$\int \frac{d^3 k}{2k_0} \imath k^\beta e^{-\imath k \cdot q} = \frac{p^\beta}{p^2} (2\pi)^3 p_0 \delta^3(\mathbf{q}) + \frac{q^\beta}{q^2} W(q, q)$$

as in Appendix 2. Similarly

$$\int \frac{d^3 k}{2k_0} \imath k^\beta e^{\imath k \cdot q} = \frac{p^\beta}{p^2} (2\pi)^3 p_0 \delta^3(\mathbf{q}) + \frac{q^\beta}{q^2} W(q, -q)$$

Putting these pieces together gives

$$\frac{\partial \Delta(x-y)}{\partial x_\beta} = \frac{p^\beta}{p^2} (2\pi)^3 2p_0 \delta^3(\mathbf{q}) + \frac{q^\beta}{q^2} \widetilde{W}(q)$$

where  $\widetilde{W}(q) = \int \frac{d^3k}{k_0} i q \cdot k (e^{-iq \cdot k} + e^{iq \cdot k})$  which is odd in  $q$  or antisymmetric under interchange of  $x - y$ . This leaves the kernel in the form  $M = D + N$  where the first part,  $D$ , is diagonal which makes it plausible to assume that an inverse exists.

#### 4.2.2.2 Finding the $f$ 's Using the Inverse Kernel

We want to solve for the function  $f$  that satisfies

$$\int dx_1 \frac{\partial \Delta(x-x_1)}{\partial x_\beta} [f_{\alpha\mu_2\mu_3}(x_1, x_2, x_3) - f_{\alpha\mu_3\mu_2}(x_1, x_3, x_2) - f_{\mu_2\alpha\mu_3}(x_2, x_1, x_3) + f_{\mu_3\alpha\mu_2}(x_3, x_1, x_2) - f_{\mu_3\mu_2\alpha}(x_3, x_2, x_1) + f_{\mu_2\mu_3\alpha}(x_2, x_3, x_1)] \eta^{\alpha\alpha} - \frac{\alpha}{2} \delta(x-x_2) \delta(x-x_3) \eta_{\mu_2}^\alpha \eta_{\mu_3}^\beta = 0$$

Let us assume that the kernel is invertible. In order for the above equation to be satisfied, the  $\beta$  term must match up with the  $\mu_3$  term. Focusing on the first  $f$  term and writing  $f$  in terms of the inverse kernel,  $K^{-1}$ , leaves

$$\int dx_1 \frac{\partial \Delta(x-x_1)}{\partial x_\beta} f_{\alpha\mu_2\mu_3}(x_1, x_2, x_3) \eta^{\alpha\alpha} = \int dx_1 \frac{\partial \Delta(x-x_1)}{\partial x_\beta} K_{\mu_3}^{-1}(x_1 - x_3) \widetilde{f}(x_2, \alpha, \mu_2)$$

Matching to the interaction term of the field tensor gives

$$\frac{\alpha}{2} (x+x_3)_0 \delta(x-x_3) \delta(x-x_2) \eta_{\mu_2}^\alpha \eta_{\mu_3}^\beta = \int dx_1 \frac{\partial \Delta(x-x_1)}{\partial x_\beta} K_{\mu_3}^{-1}(x_1 - x_3) \widetilde{f}(x_2, \alpha, \mu_2)$$

so that  $\widetilde{f}(x_2, \alpha, \mu_2) \propto \delta(x-x_2) \eta_{\mu_2}^\alpha$ . Finally, this gives

$$\int dx_1 \frac{\partial \Delta(x-x_1)}{\partial x_\beta} K_{\mu_3}^{-1}(x_1 - x_3) = (x+x_3)_0 \delta(x-x_3) \eta_{\mu_3}^\beta \quad (4.3)$$

For  $\mu_3 = \beta$ , Eq. 4.3 shows that  $f$  can be chosen in such a way to transform away the gluon self interaction. It is only possible to satisfy the condition in Eq. 4.3 with the inverse kernel, so if no inverse exists the gluon self-interactions cannot be eliminated by this method either.



Lastly, when  $\mu_3 \neq \beta$  the interaction term is zero since the metric is diagonal, but there is no reason to expect

$$\int dx_1 \frac{\partial \Delta(x - x_1)}{\partial x_\beta} K_{\mu_3}^{-1}(x_1 - x_3) = 0 \quad (4.4)$$

Without calculating the inverse kernel explicitly, it is impossible to guarantee Eq. 4.4 cannot hold, but it is reasonable to assume that such a condition will not be satisfied.

Based on this reasoning it is the conclusion of this chapter that in the full degree of freedom problem, the results of the one degree of freedom case cannot be replicated. Therefore the gluon four momentum self-interactions cannot be transformed away such that  $P^\mu |\Omega \rangle = P_{fr}^\mu |\Omega \rangle$ .

## CHAPTER 5 CONCLUSIONS AND FUTURE WORK

### 5.1 Summary and Conclusions

The goal of this dissertation has been to investigate the structure of gluons in point form QCD. Chapter 1 discussed historical approaches to studying the strong nuclear force, where it was determined that in order to understand the strong force, it is necessary to first understand gluons and their self-interactions. Chapter 2 builds up a field theory for gluons starting with representations of the Poincaré group for massless particles, using the little group  $E(2)$ . Helicity boosts were then defined to boost from a standard gluon basis state to a single particle gluon state. Then single particle gluon wave functions were defined where the Hilbert space arose from restricting the polarization degrees of freedom to assure that the wave function inner product be positive definite. Next, classical gluon fields were defined which allowed for a transformation from position space to momentum space. Then, multiparticle gluon states were introduced as symmetric tensor products of gluon Hilbert spaces generated by creation and annihilation operators. An annihilator condition was included to assure that the resulting Fock space inner product remained positive definite. The four momentum operator, where all dynamics are located in the point form, was defined in terms of free gluon fields via the stress-energy tensor, which was integrated over the forward hyperboloid. Gauge transformations were introduced for the gluon fields to ensure that the four momentum was gauge invariant. The major result

from Chapter 2 was that the free gluon four momentum operator introduced via the Lagrangian and stress-energy tensor was shown to be equivalent to that generated by gluon irreps when acting on the physical Fock space.

After the four momentum was constructed in Chapter 2, it was implemented in Chapter 3 for the simplest case, the one degree of freedom problem. Some results for the coupling constant,  $\alpha$ , were generated corresponding to the ground state eigenvalue,  $\lambda = 0$  when writing the vacuum as a polynomial of gluon creation operators. Increasing the degree of the polynomial resulted in the  $\alpha$  values climbing and did not appear to stabilize. A better ansatz for the vacuum was to express  $|\Omega\rangle$  as an exponential of gluon creation and annihilation operators (the 1-D equivalent of free gluon fields). This formulation allowed for  $|\Omega\rangle$  to be chosen in such a way to eliminate the dependence on  $\alpha$ , transforming the Hamiltonian problem into just the kinetic energy component,  $H|\Omega\rangle = H_{KE}|\Omega\rangle$ .

Chapter 4 tested this structure for the full four momentum operator, by adjusting the 1-D vacuum to satisfy the infinite degree of freedom case. The full vacuum was now written  $|\Omega\rangle = e^S|0\rangle$ , where

$$S = \int dx_1 dx_2 dx_3 c_{a_1 a_2 a_3} f_{\mu_1 \mu_2 \mu_3}(x_1, x_2, x_3) G_{a_1}^{\mu_1}(x_1) G_{a_2}^{\mu_2}(x_2) G_{a_3}^{\mu_3}(x_3)$$

Chapter 4 involved solving for  $f_{\mu_1 \mu_2 \mu_3}(x_1, x_2, x_3)$  that would eliminate the  $\alpha$  dependence from the field tensor, transforming away the gluon self-interactions. It was shown that unlike in the 1-D case, it is doubtful that a formulation for  $f_{\mu_1 \mu_2 \mu_3}(x_1, x_2, x_3)$  exists allowing  $P^\mu|\Omega\rangle = P_{fr}^\mu|\Omega\rangle$ . If such a result were possible and gluon self-interactions were eliminated, gluon bound states or glueballs could not exist. How-

ever, it is the conclusion of this dissertation that although the gluon self-interactions can be eliminated in the one degree of freedom problem, a similar approach does not yield similar results in the full degree of freedom case.

## 5.2 Open Questions and Future Work

One of the main goals of this dissertation is to lay the groundwork for studying gluon eigenvalue problems in the context of point form QCD. The first question to be answered is whether or not an inverse to the kernel,  $\frac{\partial \Delta(x-y)}{\partial x_\beta}$ , exists. If an inverse for one component of the kernel can be determined, say  $K_0^{-1}$ , it will be possible to determine whether or not the gluon self-interactions can actually be transformed away entirely. Although an argument was given that these interactions cannot be eliminated, it is still a possibility until either an inverse is calculated or it is shown that no inverse can exist.

The framework has been provided for further investigation into the vacuum and bound state problems for gluons using the full degree of freedom four momentum operator,  $P^\mu$ . One possible approach would be to adapt the models from the discrete spectrum Hamiltonian in Chapter 3 to the full degree of freedom case. Starting with an ansatz of a vacuum consisting of a polynomial in terms of gluon fields,  $|\Omega \rangle = F|0 \rangle$ , here

$$F = f_0 I + \sum \int dk_1 dk_2 f_2((k_1+k_2)^2) B^{\mu\rho_1}(k_1) B_\mu^{\rho_2}(k_2) \delta_{a_1 a_2} g^\dagger(k_1, \rho_1, a_1) g^\dagger(k_2, \rho_2, a_2) + \dots$$

where  $f_2((k_1 + k_2)^2)$  is a Lorentz invariant function. It is possible to factor out the infinite Lorentz volume leaving a set of recursive equations which have no infinities.

Similarly, one can consider an exponential vacuum ansatz where the quartic term commutes with the vacuum, although the self-interactions cannot be removed entirely. As in Chapter 3, the next step would be to consider the bound state problem where gluon bound states can be investigated. The simplest glueballs are bound states of two gluons held together via their self-interactions and written

$$|k_1, \rho_1, a_1 \rangle \otimes |k_2, \rho_2, a_2 \rangle = g^\dagger(k_1, \rho_1, a_1) g^\dagger(k_2, \rho_2, a_2) |0 \rangle$$

such that  $\mathbf{k}_1 + \mathbf{k}_2 = 0$ . Then a velocity state can be defined

$$|v, \mathbf{k} \rangle := U_{B(v)} |k_1, \rho_1, a_1; k_2, -\rho_1, a_2 \rangle \delta_{a_1 a_2}$$

where  $\mathbf{k} = \mathbf{k}_1$  and  $v$  is the four velocity of the glueball state. A unitary Lorentz transformation on this velocity state gives

$$\begin{aligned} U_\Lambda |v, \mathbf{k} \rangle &= U_\Lambda U_{B(v)} |k_1, \rho_1, a_1; k_2, -\rho_1, a_1 \rangle \\ &= U_{B(\Lambda v)} U_{R_W} |k_1, \rho_1, a_1; k_2, -\rho_1, a_1 \rangle \\ &= U_{B(\Lambda v)} |R_W k_1, \rho_1, a_1; R_W k_2, -\rho_1, a_1 \rangle \\ &= |\Lambda v, R_W \mathbf{k} \rangle \end{aligned}$$

where  $R_W$  is a rotation.

Again a set of bound state equations in powers of gluon creation operators results generated by

$$P^\mu |v, \mathbf{k} \rangle = M v^\mu |v, \mathbf{k} \rangle$$

where  $M$  is the mass of the glueball.

## APPENDIX A GLUON FIELD COMMUTATION RELATIONS

Within this appendix are proofs and calculations that were omitted from Chapter 2. These calculations pertain to commutation relations of free gluon fields which are written in terms of creation and annihilation operators. Recall the creation and annihilation operator commutator is

$$[g(k, \rho, a), g^\dagger(k', \rho', a')] = -\eta_{\rho\rho'} k_0 \delta^3(\mathbf{k} - \mathbf{k}') \delta_{aa'}$$

The following is the calculation for Eq. 2.13,

$$[G_a^\mu(x), G_b^\nu(y)] = 0, (x - y)^2 < 0$$

which means that gluon fields are local.

$$\begin{aligned}
[G_a^\mu(x), G_b^\nu(y)] &= \int \frac{d^3k}{(2\pi)^{3/2}2k_0} \frac{d^3k'}{(2\pi)^{3/2}2k'_0} B^{\mu\rho}(k) B^{\nu\rho'}(k') \\
&\quad [e^{-ik\cdot x} g(k, \rho, a) + e^{ik\cdot x} g^\dagger(k, \rho, a), e^{-ik'\cdot y} g(k', \rho', b) + e^{ik'\cdot y} g^\dagger(k', \rho', b)] \\
&= \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k_0} \frac{d^3k'}{2k'_0} B^{\mu\rho}(k) B^{\nu\rho'}(k') \\
&\quad (e^{-ik\cdot x} e^{ik'\cdot y} [g(k, \rho, a), g^\dagger(k', \rho', b)] + e^{ik\cdot x} e^{-ik'\cdot y} [g^\dagger(k, \rho, a), g(k', \rho', b)]) \\
&= \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k_0} \frac{d^3k'}{2k'_0} B^{\mu\rho}(k) B^{\nu\rho'}(k') (-e^{-ik\cdot x} e^{ik'\cdot y} \eta_{\rho\rho'} k_0 \delta^3(\mathbf{k} - \mathbf{k}') \delta_{ab} \\
&\quad + e^{ik\cdot x} e^{-ik'\cdot y} \eta_{\rho\rho'} k_0 \delta^3(\mathbf{k} - \mathbf{k}') \delta_{ab}) \\
&= \frac{\delta_{ab}}{(2\pi)^3} \int \frac{d^3k}{2k_0} B^{\mu\rho}(k) B^{\nu\rho'}(k) \eta_{\rho\rho'} (-e^{ik\cdot(y-x)} + e^{ik\cdot(x-y)}) \\
&= \eta^{\mu\nu} \frac{\delta_{ab}}{(2\pi)^3} \int \frac{d^3k}{2k_0} (e^{ik\cdot(x-y)} - e^{-ik\cdot(x-y)}) \\
&= \eta^{\mu\nu} \delta_{ab} \Delta(x-y) \\
&= 0, \forall (x-y)^2 < 0
\end{aligned}$$

The last equality holds because when  $x - y$  is spacelike, a continuous Lorentz transformation can take  $(x - y) \rightarrow -(x - y)$ . [7]

A similar calculation can be done for Eq. 2.14,

$$[\frac{\partial G_a^\mu(x)}{\partial x_\beta}, G_b^\nu(y)] = \eta^{\mu\nu} \delta_{ab} \frac{\partial \Delta(x-y)}{\partial x_\beta} I$$

$$\begin{aligned}
\left[\frac{\partial G_a^\mu(x)}{\partial x_\beta}, G_b^\nu(y)\right] &= \frac{-i}{(2\pi)^3} \int \frac{d^3k}{2k_0} \frac{d^3k'}{2k'_0} k'^\beta B^{\mu\rho}(k) B^{\nu\rho'}(k') [e^{-ik\cdot x} g(k, \rho, a) - e^{ik\cdot x} g^\dagger(k, \rho, a), \\
&\quad e^{-ik'\cdot y} g(k', \rho', b) + e^{ik'\cdot y} g^\dagger(k', \rho', b)] \\
&= \frac{-i}{(2\pi)^3} \int \frac{d^3k}{2k_0} \frac{d^3k'}{2k'_0} k'^\beta B^{\mu\rho}(k) B^{\nu\rho'}(k') (e^{-ik\cdot x} e^{ik'\cdot y} [g(k, \rho, a), g^\dagger(k', \rho', b)] \\
&\quad - e^{ik\cdot x} e^{-ik'\cdot y} [g^\dagger(k, \rho, a), g(k', \rho', b)]) \\
&= \frac{i}{(2\pi)^3} \int \frac{d^3k}{2k_0} \frac{d^3k'}{2k'_0} k'^\beta B^{\mu\rho}(k) B^{\nu\rho'}(k') (e^{-ik\cdot x} e^{ik'\cdot y} \eta_{\rho\rho'} k_0 \delta^3(\mathbf{k} - \mathbf{k}') \delta_{ab} \\
&\quad + e^{ik\cdot x} e^{-ik'\cdot y} \eta_{\rho\rho'} k_0 \delta^3(\mathbf{k} - \mathbf{k}') \delta_{ab}) \\
&= \eta^{\mu\nu} \delta_{ab} \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k_0} ik^\beta (e^{ik\cdot(x-y)} + e^{-ik\cdot(x-y)}) \\
&= \eta^{\mu\nu} \delta_{ab} \frac{\partial}{\partial x_\beta} \left( \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k_0} (e^{ik\cdot(x-y)} - e^{-ik\cdot(x-y)}) \right) \\
&= \eta^{\mu\nu} \delta_{ab} \frac{\partial \Delta(x-y)}{\partial x_\beta} I
\end{aligned}$$

The fact that this commutator is a multiple of the identity is essential to the ansatz of writing the physical vacuum as a product of free fields.



## APPENDIX B DERIVATION OF THE KINETIC COMPONENT OF THE FOUR MOMENTUM

The full four momentum operator,  $P^\mu$  results from integrating the stress-energy tensor,  $T^{\mu\nu}$ , over the forward hyperboloid

$$P^\mu = \int d^4x \delta(x \cdot x - \tau^2) \Theta(x_0) x_\nu T^{\mu\nu}$$

where  $\Theta(x_0)$  is the Heaviside function and

$$T^{\mu\nu}(x) = \sum_a F_a^{\alpha\beta}(x) \eta_{\beta\beta'} [\eta_{\alpha'}^\mu \eta_\alpha^\nu + \eta_{\alpha'}^\nu \eta_\alpha^\mu - \frac{1}{2} \eta^{\mu\nu} \eta_{\alpha\alpha'}] F_a^{\alpha'\beta'}(x)$$

$T^{\mu\nu}$  is defined in terms of the field tensor,  $F^{\mu\nu}$ , which is defined in terms of the free gluon fields  $G_a^\mu(x)$ ,

$$F_a^{\mu\nu}(x) = \frac{\partial G_a^\nu}{\partial x_\mu} - \frac{\partial G_a^\mu}{\partial x_\nu} + \alpha c_{abc} G_b^\mu(x) G_c^\nu(x)$$

where  $c_{abc}$  are the color structure constants and  $\alpha$  is the strong bare coupling constant. Recall that the free fields are defined in terms of the fundamental creation and annihilation operators,

$$\begin{aligned} G_a^\mu(x) &= \int \frac{d^3k}{(2\pi)^{3/2} 2k_0} B^{\mu\rho}(k) (e^{-ik \cdot x} g(k, \rho, a) + e^{ik \cdot x} g^\dagger(k, \rho, a)) \\ \frac{\partial G_a^\mu}{\partial x_\nu} &= -i \int \frac{d^3k}{(2\pi)^{3/2} 2k_0} k^\nu B^{\mu\rho}(k) (e^{-ik \cdot x} g(k, \rho, a) - e^{ik \cdot x} g^\dagger(k, \rho, a)) \end{aligned}$$

as discussed in Chapter 2.

The four momentum consists of a product of two field tensors. It is useful to break the four momentum into parts,

$$P^\mu = P_{KE}^\mu + P_{tri}^\mu + P_{quar}^\mu$$

based on this product. The three parts are distinguished by their dependence on  $\alpha$ . The quartic piece depends on  $\alpha^2$  and inherits its name from the product of four free fields, the trilinear consists of the two ways of garnering an  $\alpha$  dependence, while the kinetic energy part is independent of  $\alpha$ . It is not apparent that  $P_{KE}^\mu$  defined in this way is equivalent to the free four momentum operator

$$P_{fr}^\mu = - \sum_a \int \frac{d^3k}{2k_0} k^\mu \eta^{\rho\rho} (g^\dagger(k, \rho, a)g(k, \rho, a) + g(k, \rho, a)g^\dagger(k, \rho, a))$$

Proving this equivalence is the purpose of this Appendix.

For the following calculations, set the strong bare coupling constant,  $\alpha$ , to zero to isolate the free terms. Then

$$\begin{aligned} F_a^{\alpha\beta} &= \frac{\partial G_a^\beta}{\partial x_\alpha} - \frac{\partial G_a^\alpha}{\partial x_\beta} \\ &= -i \int \frac{d^3k}{(2\pi)^{3/2} 2k_0} (k^\alpha B^{\beta\rho}(k) - k^\beta B^{\alpha\rho}(k)) (e^{-ik \cdot x} g(k, \rho, a) - e^{ik \cdot x} g^\dagger(k, \rho, a)) \\ P^\mu &= \sum_a 2 \int d^4x \delta(x \cdot x - \tau^2) \Theta(x_0) x_\nu F_a^{\alpha\beta} F_a^{\alpha'\beta'} \eta_{\beta\beta'} (\eta_\alpha^\mu \eta_{\alpha'}^\nu + \eta_\alpha^\nu \eta_{\alpha'}^\mu - \frac{1}{2} \eta^{\mu\nu} \eta_{\alpha\alpha'}) \end{aligned}$$

The product of the two field tensors results in four terms which will be treated individually according to their creation/annihilation pairs and labeled accordingly;

$$P_{gg}^\mu, P_{g^\dagger g^\dagger}^\mu, P_{g^\dagger g}^\mu, P_{g g^\dagger}^\mu.$$

First consider

$$\begin{aligned}
P_{gg}^\mu &= \sum_a 2 \int d^4x \delta(x \cdot x - \tau^2) \Theta(x_0) x_\nu \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k_0} \frac{d^3k'}{2k'_0} e^{-i(k+k') \cdot x} \\
&\quad [(k^\mu B^{\alpha\rho}(k) - k^\alpha B^{\mu\rho}(k))(k'^\nu B'_\alpha{}^{\rho'}(k') - k'_\alpha B^{\nu\rho'}(k')) + \\
&\quad (k^\nu B^{\alpha\rho}(k) - k^\alpha B^{\nu\rho}(k))(k'^\mu B'_\alpha{}^{\rho'}(k') - k'_\alpha B^{\mu\rho'}(k')) - \\
&\quad \frac{1}{2} \eta^{\mu\nu} (k^\delta B^{\alpha\rho}(k) - k^\alpha B^{\delta\rho}(k))(k'_\delta B'_\alpha{}^{\rho'}(k') - k'_\alpha B'_\delta{}^{\rho'}(k'))] \\
&\quad g(k, \rho, a) g(k', \rho', a)
\end{aligned} \tag{B.1}$$

Notice that for the other three terms of the four momentum, the only differences are the overall sign, the term in the exponential, and the creation/annihilation operator pairing.

Define a Lorentz invariant distribution as

$$W(p, q) = 2 \int d^4x \delta(x \cdot x - \tau^2) \Theta(x_0) x_\nu p^\nu e^{-iq \cdot x} \tag{B.2}$$

where  $p = k + k'$ , and  $q = k - k'$ . As a result, the spatial integral in Eq. B.1 becomes

$$2 \int d^4x \delta(x \cdot x - \tau^2) \Theta(x_0) x_\nu e^{-i(k+k') \cdot x} = \frac{p_\nu}{p^2} W(p, p)$$

which is shown in the Appendix of Ref. [1]. Also recall that  $k^2 = k'^2 = 0$  since gluons are massless particles, meaning  $p \cdot p = 2k \cdot k'$  and  $q \cdot q = -2k \cdot k'$ .

The common boost terms from Eq. B.1 can now be factored out to give

$$\begin{aligned}
P_{gg}^\mu &= \sum_a \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k_0} \frac{d^3k'}{2k'_0} \left( \frac{k_\nu + k'_\nu}{2k \cdot k'} W(p, p) \right) B^{\alpha\rho}(k) B^{\alpha'\rho'}(k') \\
&\quad [(k^\mu k'^\nu \eta_{\alpha\alpha'} - k^\mu k'_\alpha \delta_{\alpha'}^\nu - k'^\nu k_{\alpha'} \delta_\alpha^\mu + k \cdot k' \delta_{\alpha'}^\nu \delta_\alpha^\mu) + \\
&\quad (k^\nu k'^\mu \eta_{\alpha\alpha'} - k^\nu k'_\alpha \delta_{\alpha'}^\mu - k'^\mu k_{\alpha'} \delta_\alpha^\nu + k \cdot k' \delta_\alpha^\nu \delta_{\alpha'}^\mu) \\
&\quad - \eta^{\mu\nu} k \cdot k' \eta_{\alpha\alpha'} + \eta^{\mu\nu} k_{\alpha'} k'_\alpha] g(k, \rho, a) g(k', \rho', a) \\
&= \sum_a \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k_0} \frac{d^3k'}{2k'_0} \left( \frac{W(p, p)}{2k \cdot k'} \right) B^{\alpha\rho}(k) B^{\alpha'\rho'}(k') \\
&\quad [(k^\mu k \cdot k' \eta_{\alpha\alpha'} - k^\mu k'_\alpha (k_{\alpha'} + k'_{\alpha'}) - k \cdot k' k_{\alpha'} \delta_\alpha^\mu + k \cdot k' (k_{\alpha'} + k'_{\alpha'}) \delta_\alpha^\mu) + \\
&\quad (k'^\mu k \cdot k' \eta_{\alpha\alpha'} - k \cdot k' k'_\alpha \delta_{\alpha'}^\mu - k'^\mu k_{\alpha'} (k_\alpha + k'_\alpha) + k \cdot k' (k_\alpha + k'_\alpha) \delta_{\alpha'}^\mu) \\
&\quad - (k^\mu + k'^\mu) k \cdot k' \eta_{\alpha\alpha'} + (k^\mu + k'^\mu) k_{\alpha'} k'_\alpha] g(k, \rho, a) g(k', \rho', a) \\
&= \sum_a \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k_0} \frac{d^3k'}{2k'_0} \left( \frac{w(P, P)}{2k \cdot k'} \right) B^{\alpha\rho}(k) B^{\alpha'\rho'}(k') \\
&\quad [-k^\mu k'_\alpha k'_{\alpha'} + k \cdot k' k'_{\alpha'} \delta_\alpha^\mu - k'^\mu k_{\alpha'} k_\alpha + k \cdot k' k_\alpha \delta_{\alpha'}^\mu] g(k, \rho, a) g(k', \rho', a)
\end{aligned}$$

Each of the remaining terms has the form  $B^{\alpha\rho}(k) k_\alpha g(k, \rho, a) = k^{st\rho} g(k, \rho, a)$

which is zero from the wave function condition

$$\sum_a k^{st\rho} g(k, \rho, a) |\phi\rangle = 0 \quad (\text{B.3})$$

leaving  $P_{gg}^\mu = 0$  when acting on the physical subspace.

In calculating the remaining three terms of  $P^\mu$ , the major difference is the exponential term. For  $P_{g^\dagger g^\dagger}^\mu$ ,  $e^{-z(k+k') \cdot x} \rightarrow e^{z(k+k') \cdot x}$ , with  $W(p, p) \rightarrow W(p, -p)$  the only change. Although the wave function condition in Eq. B.3 no longer applies, we can require an operator condition

$$\sum_a k^{st\rho} g^\dagger(k, \rho, a) = 0 \quad (\text{B.4})$$

Each of the four  $P_{g^\dagger g^\dagger}^\mu$  terms has the form  $B^{\alpha\rho}(k)k_\alpha g^\dagger(k, \rho, a) = k^{st\rho} g^\dagger(k, \rho, a) = 0$ , leading to the same conclusion,  $P_{g^\dagger g^\dagger}^\mu = 0$  on the physical subspace.

For  $P_{gg^\dagger}^\mu$ ,  $e^{-i(k+k')\cdot x} \rightarrow e^{-i(k-k')\cdot x}$ , changing the spatial integral to

$$\begin{aligned} \int 2d^4x \delta(x \cdot x - \tau^2) \Theta(x_0) x_\nu e^{-i(k-k')\cdot x} &= \frac{p_\nu}{p^2} (2\pi)^3 p_0 \delta^3(\mathbf{q}) + \frac{q_\nu}{q^2} W(q, q) \\ &= \frac{k_\nu + k'_\nu}{2k \cdot k'} (2\pi)^3 (k_0 + k'_0) \delta^3(\mathbf{k} - \mathbf{k}') + \frac{k_\nu - k'_\nu}{-2k \cdot k'} W(q, q) \end{aligned}$$

as calculated in the Appendix of Ref. [1]. However, this result is only valid for particles with mass, because otherwise  $k \cdot k' \rightarrow 0$  as  $k \rightarrow k'$  causing the first term to blow up.

The second term avoids this complication and carrying out the same calculations for  $q_\nu$  instead of  $p_\nu$  gives

$$\begin{aligned} P_{gg^\dagger}^\mu &= \tilde{P}_{gg^\dagger}^\mu + \sum_a \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k_0} \frac{d^3k'}{2k'_0} \left( \frac{W(q, q)}{-2k \cdot k'} \right) B^{\alpha\rho}(k) B^{\alpha'\rho'}(k') \\ &\quad [k^\mu k'_\alpha k'_{\alpha'} - k \cdot k' k'_\alpha \delta_\alpha^\mu - k'^\mu k_{\alpha'} k_\alpha + k \cdot k' k_\alpha \delta_{\alpha'}^\mu] \\ &\quad g(k, \rho, a) g^\dagger(k', \rho', a') \end{aligned}$$

where the latter two parts are eliminated by the wave function condition, and the former two by the operator condition, leaving  $P_{gg^\dagger}^\mu = \tilde{P}_{gg^\dagger}^\mu$

In order to calculate the first piece, it will be useful to define

$$k := \lim_{\epsilon \rightarrow 0} \begin{pmatrix} \sqrt{\mathbf{k}^2 + \epsilon^2} \\ \mathbf{k} \end{pmatrix}$$

so that  $k \cdot k = \lim_{\epsilon \rightarrow 0} \epsilon^2 = 0$ .

$P_{gg^\dagger}^\mu$  is now

$$\begin{aligned}
P_{gg^\dagger}^\mu &= - \sum_a \int \frac{d^3k}{2k_0} \frac{d^3k'}{2k'_0} \left( \frac{k_0 + k'_0}{2k \cdot k'} \delta^3(\mathbf{k}-\mathbf{k}') \right) (k_\nu + k'_\nu) B^{\alpha\rho}(k) B^{\alpha'\rho'}(k') \\
&\quad [(k^\mu k'^\nu \eta_{\alpha\alpha'} - k^\mu k'_\alpha \delta_{\alpha'}^\nu - k'^\nu k_{\alpha'} \delta_\alpha^\mu + k \cdot k' \delta_{\alpha'}^\nu \delta_\alpha^\mu) + \\
&\quad (k^\nu k'^\mu \eta_{\alpha\alpha'} - k^\nu k'_\alpha \delta_{\alpha'}^\mu - k'^\mu k_{\alpha'} \delta_\alpha^\nu + k \cdot k' \delta_\alpha^\nu \delta_{\alpha'}^\mu) \\
&\quad - \eta^{\mu\nu} k \cdot k' \eta_{\alpha\alpha'} + \eta^{\mu\nu} k_{\alpha'} k'_\alpha] g(k, \rho, a) g^\dagger(k', \rho', a) \\
&= - \lim_{\epsilon \rightarrow 0} \sum_a \int \frac{d^3k}{4k_0 \epsilon^2} B^{\alpha\rho}(k) B^{\alpha'\rho'}(k) (2k_\nu) \\
&\quad [(k^\mu k^\nu \eta_{\alpha\alpha'} - k^\mu k_\alpha \delta_{\alpha'}^\nu - k^\nu k_{\alpha'} \delta_\alpha^\mu + \epsilon^2 \delta_{\alpha'}^\nu \delta_\alpha^\mu) + \\
&\quad (k^\nu k^\mu \eta_{\alpha\alpha'} - k^\nu k_\alpha \delta_{\alpha'}^\mu - k^\mu k_{\alpha'} \delta_\alpha^\nu + \epsilon^2 \delta_\alpha^\nu \delta_{\alpha'}^\mu) \\
&\quad - \eta^{\mu\nu} \epsilon^2 \eta_{\alpha\alpha'} + \eta^{\mu\nu} k_{\alpha'} k_\alpha] g(k, \rho, a) g^\dagger(k, \rho', a) \\
&= - \lim_{\epsilon \rightarrow 0} \sum_a \int \frac{d^3k}{2k_0 \epsilon^2} B^{\alpha\rho}(k) B^{\alpha'\rho'}(k) \\
&\quad [\epsilon^2 k^\mu \eta_{\alpha\alpha'} - k^\mu k_\alpha k_{\alpha'} - \epsilon^2 k_{\alpha'} \delta_\alpha^\mu + \epsilon^2 k_{\alpha'} \delta_\alpha^\mu + \\
&\quad \epsilon^2 k^\mu \eta_{\alpha\alpha'} - \epsilon^2 k_\alpha \delta_{\alpha'}^\mu - k^\mu k_{\alpha'} k_\alpha + \epsilon^2 k_\alpha \delta_{\alpha'}^\mu \\
&\quad - k^\mu \epsilon^2 \eta_{\alpha\alpha'} + k^\mu k_{\alpha'} k_\alpha] g(k, \rho, a) g^\dagger(k, \rho', a) \tag{B.5}
\end{aligned}$$

Seven of the ten terms from Eq. B.5 are eliminated by Eqs. B.3 and B.4.

This leaves

$$\begin{aligned}
P_{gg^\dagger}^\mu &= -\lim_{\epsilon \rightarrow 0} \sum_a \int \frac{d^3k}{2k_0\epsilon^2} B^{\alpha\rho}(k) B^{\alpha'\rho'}(k) \\
&\quad [\epsilon^2 k^\mu \eta_{\alpha\alpha'} + \epsilon^2 k^\mu \eta_{\alpha\alpha'} - k^\mu \epsilon^2 \eta_{\alpha\alpha'}] g(k, \rho, a) g^\dagger(k, \rho', a) \\
&= -\sum_a \int \frac{d^3k}{2k_0} B^{\alpha\rho}(k) B^{\alpha'\rho'}(k) \eta_{\alpha\alpha'} k^\mu g(k, \rho, a) g^\dagger(k, \rho', a) \\
&= -\sum_a \int \frac{d^3k}{2k_0} k^\mu \eta^{\rho\rho'} g(k, \rho, a) g^\dagger(k, \rho', a) \\
&= -\sum_a \int \frac{d^3k}{2k_0} k^\mu \eta^{\rho\rho} g(k, \rho, a) g^\dagger(k, \rho, a)
\end{aligned}$$

An identical argument can be made for  $P_{g^\dagger g}^\mu$  leaving

$$P_{g^\dagger g}^\mu = -\sum_a \int \frac{d^3k}{2k_0} k^\mu \eta^{\rho\rho} g^\dagger(k, \rho, a) g(k, \rho, a)$$

and

$$P_{KE}^\mu = -\sum_a \int \frac{d^3k}{2k_0} k^\mu \eta^{\rho\rho} (g^\dagger(k, \rho, a) g(k, \rho, a) + g(k, \rho, a) g^\dagger(k, \rho, a))$$

as desired.

## APPENDIX C MIXED SYMMETRY

The group  $S_3$  is the set of all rotations of three elements, usually represented by its six group elements written,  $I$ ,  $(12)$ ,  $(13)$ ,  $(23)$ ,  $(123)$ ,  $(321)$ . In this notation  $(12)$  represents switching the first two elements.  $(12)$  can be represented as a  $3 \times 3$  matrix by  $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ , since the first and second elements of a three vector are swapped and the third is left unchanged. This matrix representation, however, is reducible.

The six irreducible representations of  $S_3$  can be calculated by block diagonalizing the three dimensional representations into irreducible representations. The eigenvalues of the three dimensional representation of  $(12)$  are 1, 1, and  $-1$ . Diagonalizing  $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ , and using the diagonalization matrices on the remaining five representations block diagonalizes the three dimensional representations into a 1 and



the following  $2 \times 2$  irreps,

$$\begin{aligned}
D(I) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
D((12)) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
D((13)) &= \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \\
D((23)) &= \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \\
D((123)) &= \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} \\
D((321)) &= \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}
\end{aligned} \tag{C.1}$$

**Definition C.1.** The generalized projection operator for the mixed representation is

$P_{ij} := \sum_g D_{ij}^{-1}(g)U(g)$  where  $U(g)$  is the permutation operator for the group element  $g \in S_3$ .

To show that these are generalized projection operators, two properties must hold.

1.  $P_{ij}$  must transform irreducibly for the mixed representation or  $U(g)P_{ij} = \sum_k P_{kj}D_{ik}(g)$ .
2.  $P_{ij}P_{kl} = \delta_{jk}P_{il}$ .

These properties are shown in most group theory books, for example Ref. [12].

**Definition C.2.**  $P_{ii}$  are the projection operators since they satisfy the property that

$$P_{ii}P_{jj} = \begin{cases} P_{ii} & i = j \\ 0 & i \neq j \end{cases}$$

The mixed representation is a two dimensional representation and the two projection operators will be renamed  $P_{11} = P_+$  and  $P_{22} = P_-$ . A consequence of the choice to diagonalize  $D((12))$  is that  $U_{(12)}P_+ = P_+$  and  $U_{(12)}P_- = -P_-$ .

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