2010

# Source localization from received signal strength under lognormal shadowing

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# SOURCE LOCALIZATION FROM RECEIVED SIGNAL STRENGTH UNDER LOGNORMAL SHADOWING

by

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A thesis submitted in partial fulfillment of the
requirements for the Master of Science degree
in Electrical and Computer Engineering
in the Graduate College of
The University of Iowa

May 2010

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CERTIFICATE OF APPROVAL

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MASTER'S THESIS

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This is to certify that the Master's thesis of

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has been approved by the Examining Committee for the thesis requirement for the Master of Science degree in Electrical and Computer Engineering at the May 2010 graduation.

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To my parents

# ACKNOWLEDGMENTS

I would like to convey my deep gratitude to my thesis advisor, Prof. Soura Dasgupta, for his guidance and valuable advice during the course of my thesis project. His style of teaching marked by clarity and simplicity made my work a delight and his influence an inspiration. I truly appreciate his patience in mentoring me through my frequent phases of indecision. I am grateful to my committee members, Dr. Raghu Madumbai and Prof. Mark Andersland for their comments and valuable time. I sincerely thank Prof. Sudhakar.M.Reddy for giving me guidance and counsel.

I wish to thank my best friend here in Iowa, Elham, for gifting me wonderful times during my years as a graduate student. It is a pleasure to express my acknowledgement to Cathy and Dina for helping me through the graduate paperwork and for always being ready to lend an ear whenever I barged into the department office.

I thank my parents, Maheswari Devi and Sankara Reddy, and my sister Sree Navya, for their unflagging faith and immense confidence in me, which has been crucial in all my achievements. Their unflinching encouragement and support has seen me through the most difficult times of my life. I am indebted endlessly to my father, for inculcating in me the desire to learn and for encouraging me to enjoy whatever I undertake. Most importantly I would like to thank my partner in life, Pavan, for unstinted encouragement through my times of procrastination and for being a wonderful friend.

## ABSTRACT

This thesis considers statistical issues in source localization from the received signal strength (RSS) measurements at sensor locations, under the practical assumption of log-normal shadowing. Distance information of source from sensor locations can be estimated from RSS measurements and many algorithms directly use powers of distances to localize the source, even though distance measurements are not directly available. The first part of the thesis considers the statistical analysis of distance estimation from RSS measurments. We show that the underlying problem is inefficient and there is only one unbiased estimator for this problem and its mean square error (MSE) grows exponentially with noise power. Later, we provide the linear minimum mean square error (MMSE) estimator whose bias and MSE are bounded in noise power. The second part of the thesis establishes an isomorphism between estimates of differences between squares of distances and the source location. This is used to completely characterize the class of unbiased estimates of the source location and to show that their MSEs grow exponentially with noise powers. Later, we propose an estimate based on the linear MMSE estimate of distances that has error variance and bias that is bounded in the noise variance.

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# CHAPTER 1
# INTRODUCTION

The increase in the emerging applications of localization in sensor networks has stimulated significant research activity in this area, [1]. In source localization a group of sensors at known positions jointly estimate the unknown location of a source using some relative position information of the source. In sensor localization a sensor estimates its own location using some information related to its relative position to a set of sensors at known locations called anchors. This thesis concerns only issues in source localization. More specifically, this thesis considers localization from received signal strength (RSS) measurements under log-normal shadowing, terms that will be defined in the sequel. Our goal is to consider statistical estimators and study their error variance and bias.

## 1.1 Background and Applications

In sensor networks, acquiring location information has become vital in a number of emerging applications. In order to process a signal in sensor networks, sensors must locate the origin of the signal source and localization issues become crucial when there is an uncertainty about the location of the source. For example, a sensor network deployed from an aircraft to monitor wildlife in a remote forest must provide precise location information about each animal, just as a sensor network installed to combat bioterrorism must detect as well as locate the source of a potential threat. Similarly the tasks of routing, tracking and efficiently using resources in large sensor networks can be facilitated by estimating the location of source or node, [2]-[4]. Manual configuration of sensor nodes might be difficult in large scale sensor networks, particularly if the nodes move frequently, [2]. Hence, algorithms have to be developed to estimate the location of nodes.

An overview of the various applications of wireless location technology and location based services is presented in [7]. With the emergence of wireless networks and mobile devices, it has become important to provide location based services to emergency and security applications, and to commercial applications [7]. One example of emergency applications is providing timely and accurate location information of the mobile phone from which an E911 call is made. Other examples include advanced public safety applications by tracking and monitoring assets, fleet management to enhance transportation safety and ensure efficient utilization of resources, and location based wireless access to enhance network security. Expanding mobile markets will span a multibillion dollar market for services based on wireless location technology, [7]. Examples of such mobile marketing services include location specific advertising that takes into account the location of the customer's wireless device, location sensitive billing and offering various services and plans based on caller location to attract the customers.

The increase in the use of personal electronic devices and wireless networking has led to significant research activity in the area of pervasive(wireless) computing, [2]-[4]. In this area, location information is commonly considered to be an important information as it enables a wide range of applications. Some example applications include services which have to identify and select resources based on their proximity to them, e.g. selecting the nearest printer in the building with matching capabilities for the end user, notifying the end user about events happening in the vicinity.

As discussed earlier, in large scale sensor networks and in the scenario where sensor nodes move frequently, manual configuration of sensor nodes may not be feasible. The existing global positioning systems (GPS) technology does not represent a feasible solution in indoor environments where a clean line of sight (LOS) is not available, [6]. It also requires expensive hardware of unwieldy size and large power consumption, [7]. Most of the other existing location finding algorithms exploit relative position information obtained from physical measurements like time difference of arrival (TDOA), angle of arrival (AOA) and received signal strength (RSS). TDOA techniques typically require a synchronous network and also demand accurate time delay measurements, [6]. The AOA measurements can be estimated by steering antennae in the direction of the arriving signal, [6] and typically require expensive antenna arrays at each sensor node. RSS measurements are relatively less expensive and easy to obtain and are therefore a suitable choice of information for localization. RSS at a sensor is indirectly related to distance of a source from that sensor and in the absence of noise it directly provides distance. In this thesis, we are only concerned about localization algorithms based on RSS. We investigate issues concerning both location and distance estimation from RSS measurements.

## 1.2 System Model

Suppose $\{x_1, x_2....x_N\}$ $(N > 2)$ are known locations of the sensors which are placed non-collinearly in a 2-dimensional space. Suppose a source located at position $y$ emits a signal that has strength $A$ at a unit distance from the source and the signal strength at a sensor located at $x_i$ is $s_i$. Define the distance between the source and the sensor located at $x_i$ as

$$\|x_i - y\| = d_i \quad \text{for } i \in \{1, 2, ...N\} \tag{1.1}$$

Assume here and in the rest of the thesis that all vector norms are 2-norms. Then with $\beta$ as the path loss coefficient, in the absence of noise one has:

$$s_i = \frac{A}{d_i^\beta} \tag{1.2}$$

In the noise free case, $s_i$ directly provides the distance $d_i$, given the knowledge of $A$ and $\beta$. From here on, we will assume the knowledge of these two parameters.

A key difficulty with RSS measurements is obtaining an accurate estimate of $\beta$. In an uncluttered environment it has a value two. In a cluttered environment its value can be unpredictable. One way of modeling such uncertainty is through the assumption of log-normal shadowing, where (1.2) must be replaced by

$$\ln s_i = \ln A - \beta \ln d_i + w_i \tag{1.3}$$

the noise $w_i$ being independent and identically distributed (iid) Gaussian random variables obeying: $w_i \sim N(0, \sigma^2)$. We assume the knowledge of noise variance $\sigma^2$ throughout the thesis. The main problem of this thesis is estimating the unknown location $y$ of the source from the noisy RSS measurements $\{s_1, s_2, \ldots, s_N\}$ under (1.3) obtained at the sensors located at $\{x_1, x_2 .... x_N\}$, given the knowledge of $A$, $\beta$ and $\sigma^2$.

In the absence of noise, with one distance we can determine the location of the source to within a circle in the case of two-dimensional localization, and to within a sphere in the case of three-dimensional localization. With two distances, the location of source can be determined to within a flip ambiguity in two dimensions. This flip ambiguity may be resolved if some a priori information about the location of source is available. In general, we require distances from at least three non-collinearly placed sensors to determine the position of the source in two-dimensions. In three dimensions, one generally requires distances from at least four non-coplanar sensors.

## 1.3 Previous Approaches

There are several papers in the literature which present localization algorithms based on the assumption that distance measurements are available, [7]-[11]. In general, distances of sources from sensors are not directly available but need to be estimated using TDOA or RSS measurements. Linear algorithms based on the distance estimates are proposed in [6] and [8]. It is argued in [11] that linear algorithms may deliver highly inaccurate estimates even with small noise levels in the available distance measurements. In [9]-[11], nonlinear estimation approaches are adopted and these algorithms are based on the minimization of formulated cost functions. These algorithms generally suffer from slow convergence and locally attractive false minima. But in most of these algorithms, the statistical analysis of the proposed method is rarely considered and none of these algorithms are optimal under the practical assumption of log-normal shadowing.

Several papers such as [18] and [19] present lower bounds for the localization problem based on RSS measurements. In [19], barakin bounds are presented to study the performance of the location estimation algorithms based on the given RSS measurements. In [18], the accuracy and performance of the proposed algorithm for this problem is studied by providing the Cramer-Rao bound, the concentration eclipse and the circular error probability. Both the papers assume a Gaussian noise model, which is not practical for most practical environments.

## 1.4 Contributions

The first main contribution of this thesis is the study of distance estimation from RSS measurements under log-normal shadowing with particular focus on bias and variance. We derive the Cramer-Rao lower bound (CRLB) for the underlying estimation problem to study the performance of estimation algorithms. The primary result proves that there is a unique unbiased estimate for this problem and its mean square error (MSE) grows exponentially with the noise power. We derive the linear minimum mean square error (MMSE) estimate and show that its bias and MSE are bounded in noise power unlike the unbiased estimate and maximum likelihood (ML) estimate whose MSE grow exponentially and the Cramer-Rao bound which increases linearly with noise power. In fact, MSE of the linear MMSE estimate is upper bounded by the square of the distance that we are trying to estimate.

The second main contribution of this thesis is related to source location estimation from RSS measurements affected by log-normal shadowing. The next result completely characterizes the class of unbiased estimators for the underlying estimation problem. The result is in some respects a negative result as we show that MSE of all the unbiased estimators for this problem grows exponentially with noise power. We then propose a biased estimate based on the Linear MMSE estimate of distances and show that both the bias and MSE of this estimate are bounded in noise power

## 1.5 Outline of the thesis

Chapter 2 considers estimation of powers of distances from RSS measurements affected by log-normal shadowing. In section 2.2 we provide the CRLB for this problem and argue that the underlying estimation problem is inefficient, i.e. no unbiased estimator meets the CRLB. We also provide the ML estimate and show that both its bias and MSE grow exponentially with $\sigma^2$. In section 2.3 we investigate the class of unbiased estimators and show that there is a unique unbiased estimator and its MSE also grows exponentially with $\sigma^2$. Section 2.4 provides the linear MMSE estimator and the comparison of its MSE with that of the unique unbiased estimator, ML estimator and the CRLB.

Chapter 3 systematically analyzes the main estimation problem, estimating the unknown source location from noisy RSS measurements. In section 3.2 we derive the CRLB to analyze the performance of unbiased estimators. We show that there exists no unbiased estimator which meets the CRLB, , i.e. prove that localization from RSS under log-normal shadowing is inefficient. Section 3.3 considers the direct implication of distance estimation results to source localization. We then characterize the complete class of unbiased estimators for this problem and provide the MSEs associated with them. In section 3.4 we propose another class of biased estimators and argue that these estimates perform better than unbiased estimators as far as MSE is concerned. Finally conclusion of the thesis and some interesting problems concerning future research are presented in chapter 4.

# CHAPTER 2
# DISTANCE ESTIMATION FROM RSS

## 2.1 Introduction

In this chapter, we consider the topic of distance estimation from received signal strength (RSS) measurements affected by log-normal shadowing with focus on statistical properties of bias and error variance. The first main result in this chapter proves that there is a unique unbiased estimate of $d_i^2$ and its error variance increases exponentially with noise power. In the second main result we present a Linear MMSE estimator whose bias and error variance are bounded by the correct distance and its square, respectively.

In section 2.2 we derive the Cramer-Rao Lower Bound (CRLB) for this problem of distance estimation from RSS. It is argued that the problem is insufficient in that there is no unbiased estimator that meets the CRLB. We show that both the bias and the mean square error of the maximum-likelihood estimator grow exponentially with noise power. This motivates us further to consider the statistical analysis of this problem. In this regard, we first consider the nature of unbiased estimators. In section 2.3 we show that there is a unique unbiased estimator for this problem using the techniques on complete sufficient statistics of exponential family of distributions presented in [20]. We show that error variance of this estimator grows exponentially with $\sigma^2$. In section 2.4 we provide the Linear Minimum Mean Square Error (MMSE) estimator whose bias is bounded by the distance that is being estimated and mean square error by the distance square irrespective of the noise variance.

## 2.2 Preliminaries

In this section, we consider the statistical analysis of the estimation of distance $d$ from RSS measurement $s$ under the assumed noise model of log-normal shadowing given by (1.3).

For notational convenience, we define:

$$\alpha = \frac{2}{\beta}, z = \left(\frac{A}{s_l}\right)^{\alpha} \tag{2.1}$$

and

$$p = d^2 \tag{2.2}$$

Then (1.3) can be written as

$$z = pe^{-\alpha w} \tag{2.3}$$

Then the underlying estimation problem is to estimate $p$ from the observation of $z$ satisfying (2.3), given the knowledge of $\alpha$ and $\sigma^2$. The estimator that we obtain must work for all $p > 0$, $\alpha$ and $\sigma^2$.

We now consider the derivation of CRLB for this estimation problem. Taking the logarithm of (2.3), we obtain

$$\ln z = \ln p - \alpha w \tag{2.4}$$

Define $l = lnz$ and observe that

$$l \sim N(lnp, \alpha^2\sigma^2)$$

The probability density function of $l$ is given by

$$p(l, y) = \frac{1}{\sqrt{(2\pi\alpha^2\sigma^2)}}\ exp\left(\frac{-(l - \ln p)^2}{2\alpha^2\sigma^2}\right) \tag{2.5}$$

and the log-likelihood function is given by

$$\ln\left[p(l, y)\right] = -ln\left[2\pi\alpha^2\sigma^2\right] - \frac{(l - \ln p)^2}{2\alpha^2\sigma^2} \tag{2.6}$$

Taking the derivative of log-likelihood function, we obtain

$$\frac{\partial \ln [p(l,y)]}{\partial p} = \frac{(l - \ln p)}{p\alpha^2\sigma^2} \tag{2.7}$$

The Fisher information is given by

$$\begin{aligned} E\left[\left(\frac{\partial \ln [p(l,y)]}{\partial p}\right)^2\right] &= E\left[\frac{(l-\ln p)^2}{p^2\alpha^4\sigma^4}\right] \\ &= E\left[\frac{(\alpha w)^2}{p^2\alpha^4\sigma^4}\right] \\ &= \frac{1}{p^2\alpha^2\sigma^2} \end{aligned}$$

The CRLB is given by the inverse of Fisher information. Thus CRLB for this estimation problem is

$$CRLB = p^2\alpha^2\sigma^2 \tag{2.8}$$

Observe that the CRLB for this estimation problem grows linearly with the noise power. It provides the lower bound on the achievable variances by unbiased estimators. The efficiency of an unbiased estimator provides the closeness of estimator's variance to the CRLB. An efficient estimator is an unbiased estimator that meets the CRLB. The next step is to investigate whether there exists an efficient estimator for this estimation problem. From [20], if the observations are perturbed by additive Gaussian noise then an efficient estimator exists if and only if the signal is affine in the parameter to be estimated. From (2.4), it can be observed that the signal has a *nonaffine* dependence on $p$ and an affine dependence on Gaussian noise $w$. Hence, no efficient estimator exists for this problem.

This leads us to investigate the statistical properties of the Maximum Likelihood Estimator (MLE). The MLE of $p$ is obtained by finding the value of $p$ that maximizes the likelihood function given in (2.5). Since logarithm is a continuously increasing function, the value which maximizes the likelihood function will also maximize its log-likelihood function. Hence, the MLE is given by

$$\hat{p}_{ML} = \underset{p}{\operatorname{argmax}}\ \ln\left[p(l, y)\right] \tag{2.9}$$

From (2.4) and (2.7), the MLE of $p$:

$$\hat{p}_{ML} = z \tag{2.10}$$

From (2.3) and using the fact that for any $a$

$$\begin{aligned} E\left[e^{aw}\right] &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{aw} e^{\frac{-w^2}{2\sigma^2}} dw \\ &= e^{\frac{a^2\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(w+a\sigma^2)^2}{2\sigma^2}} dw \\ &= e^{a^2\sigma^2/2} \end{aligned}$$

we obtain the bias of MLE as:

$$\begin{aligned} E\left[\hat{p}_{ML}\right] - p &= E[z] - p \\ &= p\left(E[e^{-\alpha w}] - 1\right) \\ &= p\left(e^{\alpha^2\sigma^2/2} - 1\right) \end{aligned}$$

and the Mean Square Error(MSE) is given by:

$$\begin{aligned} E[(\hat{p}_{ML} - p)^2] &= p^2 E\left[(e^{-\alpha w} - 1)^2\right] \\ &= p^2 E\left[e^{-2\alpha w} - 2e^{-\alpha w} + 1\right] \\ &= p^2\left(e^{2\alpha^2\sigma^2} - 2e^{\alpha^2\sigma^2/2} + 1\right). \end{aligned}$$

Observe that both bias and MSE of the MLE increase exponentially with $\sigma^2$.

## 2.3 The Best Unbiased Estimate

In previous chapter, we showed that there is no efficient estimator for this problem. This motivates us to ask the question: What are the achievable error variances by the class of unbiased estimators? To this end, we completely characterize the class of unbiased estimators for this problem and derive the error variances for this class. We then present the result that there is a unique unbiased estimator for $p$ and show that the MSE of this only unbiased estimate grows exponentially with noise power. The result presented is negative in the sense that this unique unbiased estimator yields a poor MSE.

Our goal is to obtain an estimator of the form $f(z, \alpha, \sigma^2)$ whose mean is $p$ for all $p > 0$, $\alpha$ and $\sigma^2$, i.e.

$$E\left[f(z, \alpha, \sigma^2)\right] = p. \tag{2.11}$$

From now onwards we will drop the arguments $\alpha$ and $\sigma^2$ from the list of arguments of $f$. Using (2.3), observe that (2.11) requires that for all $p > 0$ there hold:

$$p = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} f\left(pe^{-\alpha w}\right) e^{\frac{-w^2}{2\sigma^2}} dw \tag{2.12}$$

Then because of (2.4) we have that for all $p > 0$ there holds,

$$p = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \frac{f(z)}{z} exp\left(-\frac{(\ln z - \ln p)^2}{2\alpha^2\sigma^2}\right) dz. \tag{2.13}$$

Now define

$$t = \ln z \quad \text{and} \quad v = \frac{\ln p}{\sigma^2\alpha^2} \tag{2.14}$$

Then for all $v$, (2.13) becomes:

$$e^{\alpha^2\sigma^2 v} = \frac{1}{\sqrt{2\pi\alpha^2\sigma^2}} \int_{-\infty}^{\infty} f\left(e^t\right)\ e^{\frac{-t^2}{2\sigma^2\alpha^2}}\ e^{vt} dt\ e^{\frac{-v^2\sigma^2\alpha^2}{2}}. \tag{2.15}$$

i.e. for all $v$, there holds:

$$\exp\left(\alpha^2\sigma^2 v + \frac{v^2\alpha^2\sigma^2}{2}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} f\left(e^t\right)\ e^{\frac{-t^2}{2\sigma^2\alpha^2}}\ e^{vt} dt. \tag{2.16}$$

Thus with $v$ and $t$ as domain variables

$$exp\left(\alpha^2\sigma^2 v + \frac{v^2\sigma^2\alpha^2}{2}\right) \tag{2.17}$$

and

$$\frac{1}{\sqrt{2\pi\alpha^2\sigma^2}} f\left(e^t\right) e^{\frac{-t^2}{2\sigma^2\alpha^2}} \tag{2.18}$$

are Laplace pairs. This clearly proves that $f(z)$ is unique. Observe that (2.17) is the moment generating function of the normal distribution obeying:

$$N\left(\alpha^2\sigma^2, \alpha^2\sigma^2\right)$$

Thus using the definition of moment generating function and the uniqueness of Laplace transforms, the following relationship holds:

$$\begin{aligned} \frac{1}{\sqrt{2\pi\alpha^2\sigma^2}} f\left(e^t\right) e^{\frac{-t^2}{2\sigma^2\alpha^2}} &= \frac{1}{\sqrt{2\pi\alpha^2\sigma^2}} \exp\left(-\frac{(t-\alpha^2\sigma^2)^2}{2\alpha^2\sigma^2}\right) \\ &= \frac{1}{\sqrt{2\pi\alpha^2\sigma^2}} e^{\frac{-t^2}{2\sigma^2\alpha^2}} e^t e^{-\frac{\alpha^2\sigma^2}{2}} \end{aligned}$$

The above relationship directly establishes that

$$f\left(e^t\right) = e^{-\frac{\alpha^2\sigma^2}{2}} e^t \tag{2.19}$$

Thus one obtains that there is a unique unbiased estimate of $p$ and is given by

$$\hat{p}_u = e^{-\frac{\alpha^2\sigma^2}{2}} z \tag{2.20}$$

The next point of interest is to examine the MSE of this only unbiased estimate. The MSE of $\hat{p}_u$:

$$\begin{aligned} E\left[(\hat{p}_u - p)^2\right] &= E\left[\left(e^{-\frac{\alpha^2\sigma^2}{2}} z - p\right)^2\right] \\ &= p^2 E\left[\left(e^{-\frac{\alpha^2\sigma^2}{2}} e^{-\alpha w} - 1\right)^2\right] \\ &= p^2 E\left[e^{-\alpha^2\sigma^2} e^{-2\alpha w} - 2e^{-\frac{\alpha^2\sigma^2}{2}} e^{-\alpha w} + 1\right] \\ &= p^2\left(e^{-\alpha^2\sigma^2} e^{2\alpha^2\sigma^2} - 2e^{-\frac{\alpha^2\sigma^2}{2}} e^{\frac{\alpha^2\sigma^2}{2}} + 1\right) \\ &= p^2\left(e^{\alpha^2\sigma^2} - 1\right) \end{aligned}$$

This shows that the MSE of the only unbiased estimate grows exponentially with noise power. In this section we showed that there is a unique unbiased estimate of $p$ whose MSE rises exponentially with noise power. We also presented the MLE for this problem and showed that both bias and MSE grow exponentially with $\sigma^2$. On the other hand, CRLB for this problem grows linearly with $\sigma^2$.

## 2.4 The Linear MMSE Estimate

Observe that the only unbiased estimate of $p$ is linear in $z$. This behooves us to derive the linear MMSE estimate for this problem. In general, linear estimators have well understood properties and easy to obtain because of the linearity property.

In this section, we derive the linear MMSE estimate of $p$, with linearity being in the observation $z$. Suppose $\hat{p}_v = bz$ is the linear MMSE estimate, then the goal is to find a $b$ that minimizes

$$E\left[(bz-p)^2\right] \tag{2.21}$$

From (2.3) and using the fact that $E\left[aw\right] = e^{a^2\sigma^2/2}$, we obtain:

$$\begin{aligned} E\left[(bz-p)^2\right] &= p^2 E\left[(be^{-\alpha w}-1)^2\right] \\ &= p^2 E\left[b^2 e^{-2\alpha w} - 2be^{-\alpha w} + 1\right] \end{aligned}$$

As $p > 0$, we further obtain:

$$\frac{\partial E\left[(bz-p)^2\right]}{\partial p} = 0 \Rightarrow bE[e^{-2\alpha w}] - E[e^{-\alpha w}] = 0$$

Therefore, the minimizing $b$ obeys:

$$b = \frac{E[e^{-2\alpha w}]}{E[e^{-\alpha w}]} = e^{-\frac{3\alpha^2\sigma^2}{2}} \tag{2.22}$$

Thus the linear MMSE estimate that we seek is

$$\hat{p}_v = e^{-\frac{3\alpha^2\sigma^2}{2}} z \tag{2.23}$$

Its bias is given by

$$\begin{aligned}
E[\hat{p}_v] - p &= e^{-\frac{3\alpha^2\sigma^2}{2}} E[z] - p \\
&= p\left(e^{-\frac{3\alpha^2\sigma^2}{2}} E[e^{-\alpha w}] - 1\right) \\
&= p\left(e^{-\alpha^2\sigma^2} - 1\right)
\end{aligned}$$

Further, the MSE of the estimate is:

$$\begin{aligned}
E\left[(\hat{p}_v - p)^2\right] &= E\left[\left(e^{-\frac{3\alpha^2\sigma^2}{2}} z - p\right)^2\right] \\
&= p^2 E\left[\left(e^{-\frac{3\alpha^2\sigma^2}{2}} e^{-\alpha w} - 1\right)^2\right] \\
&= p^2\left(e^{-3\alpha^2\sigma^2} E[e^{-2\alpha w}] - 2e^{-\frac{3\alpha^2\sigma^2}{2}} E[e^{-\alpha w}] + 1\right) \\
&= p^2\left(e^{-3\alpha^2\sigma^2} e^{2\alpha^2\sigma^2} - 2e^{-\frac{3\alpha^2\sigma^2}{2}} e^{\frac{\alpha^2\sigma^2}{2}} + 1\right) \\
&= p^2\left(1 - e^{-\alpha^2\sigma^2}\right)
\end{aligned}$$

Observe that the MSE of $\hat{p}_v$ is bounded by $p^2$ which is better than the MSE of the MLE $\hat{p}_{ML}$ or the unbiased estimate $\hat{p}_u$ which grow exponentially with $\sigma^2$. The CRLB for this estimation problem grows linearly with $\sigma^2$. Hence for large values of $\sigma^2$, the MSE of Linear MMSE estimate is better than the CRLB. However, this is not surprising as the underlying estimate $\hat{p}_v$ is biased. Observe that the bias of $\hat{p}_v$ is bounded by $p$ in magnitude which is better than the bias of $\hat{p}_{ML}$ which increases exponentially with $\sigma^2$. Another interesting observation is that the bias of $\hat{p}_v$ is always negative while the bias of $\hat{p}_{ML}$ is always positive.

Figure 2.1 shows the MSEs of the unbiased estimate, the linear MMSE estimator and the CRLB with the assumption of unit distance estimation. We assumed that the path loss coefficient $\beta$ to be three. In a clean environment, $\beta$ is two. The MSE's and CRLB are calculated for $\sigma$ ranging from zero to two. We can observe that Linear MMSE estimator performs better than the unbiased estimate. In fact as expected, for larger noise levels the linear MMSE estimator significantly out performs the CRLB.

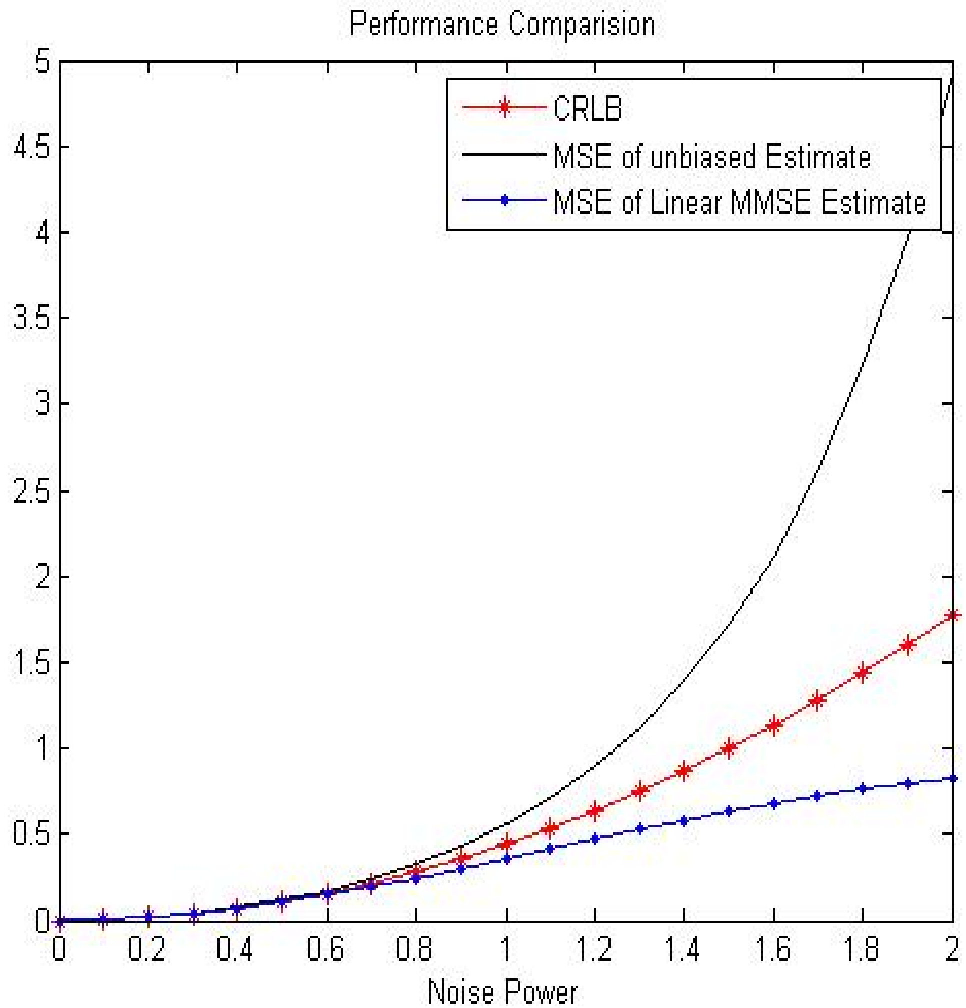


Figure 2.1: Comparison of MSEs of the unbiased estimator and Linear MMSE estimator with CRLB

## 2.5 Conclusion

In this chapter, we have considered the problem of estimating the powers of distance from RSS corrupted by log-normal shadowing. We have studied the statistical properties of the estimation problem with a particular focus on bias and MSE. We have shown that the CRLB for this problem grows linearly with $\sigma^2$ and that there is no unbiased estimate which meets CRLB. We have also demonstrated that both bias and MSE of the MLE grow exponentially with $\sigma^2$. We have also proved that there exists an unique unbiased estimator whose MSE grows exponentially with the noise power. The linear MMSE estimator whose bias and the MSE are bounded in noise $\sigma^2$ has been provided. The use of distance estimation as a tool for source localization and the impact of the distance estimation results on source localization will be discussed in the next chapter.

# CHAPTER 3
# SOURCE LOCALIZATION

## 3.1 Introduction

The previous chapter presented the statistical analysis of distance estimation from RSS under log-normal shadowing. We have shown that there is a unique unbiased distance estimator from RSS values and its variance increases exponentially with noise variance. Further, we have derived the linear MMSE estimate of powers of distances whose both bias and mean square error are bounded in the noise power.This chapter explores the structural relationship between estimation of source location $y$ and the powers of distance. We will show the direct implication of results on distance estimation to source localization. Later, we use the results obtained in the previous chapter to directly propose an estimate of $y$ whose bias and MSE are bounded in $\sigma^2$.

In section 3.2 we derive the CRLB for estimation of $y$ from $s_i$ under (1.3). It is argued that the estimation problem is inefficient, as there is no unbiased estimate of $y$ which meets CRLB. This motivates us to investigate the statistical properties of the class of unbiased estimators of $y$. In section 3.3, we demonstrate that the problem of source localization is structurally related to distance estimation, in particular to the estimation of quantities like $d_i^2 - d_j^2$. The interesting fact that there is an isomorphism between the unbiased estimation of $y$ and the unbiased estimation of these quantities is demonstrated in this section. In section 3.4 we characterize the class of unbiased estimators of $y$. We show that the variance of the whole class of unbiased estimators increases exponentially with noise power. In section 3.5 we exploit the structural dependence of the estimation of $y$ on the squares of distances and propose another class of estimators of $y$ based on the Linear MMSE estimator of distances presented in the previous chapter. We show that the proposed class of estimators have both bias and error variance bounded in noise power.

### 3.2 Preliminaries

Our goal is to investigate the statistical properties of the estimators, in particular the potential biases and error variances of the estimators. The first step in this regard is to consider the class of unbiased estimators. As the CRLB provides the lower bound on the error variance achievable by any unbiased estimator, the first logical step is to obtain the CRLB.

We first make some definitions. A set of $N > 2$ vectors in $\mathbb{R}^2$ belongs to the class $\mathcal{X}_N$, if the members of the set are non-collinear. For notational convenience, define

$$\alpha = \frac{2}{\beta}, z_l = \left(\frac{A}{s_l}\right)^{\alpha} \tag{3.1}$$

and

$$p_l = d_l^2 \tag{3.2}$$

Then (1.3) can be written as

$$z_l = e^{-\alpha w_l} d_l^2 \tag{3.3}$$

The underlying estimation problem is to estimate $y$ from the observation of $\{s_1, \ldots, s_N\}$, given $\{x_1, \ldots, x_N\}$, $A$,$\beta$ and $\sigma^2$. Given $z_i$, for $i = \{1, \ldots, N\}$ $(N > 2)$ under (3.1) and (3.3), our goal is to obtain an estimator of the form

$$\hat{y} = f(x_1, \ldots, x_N, z_1, \ldots, z_N, A, \beta, \sigma^2) \tag{3.4}$$

The estimator that we obtain must work for all $\{x_1, \ldots, x_N\} \in \mathcal{X}_N$, all $y \in \mathbb{R}^2$, $A$,

$\beta$ and $\sigma^2$. Now we consider the derivation of CRLB for this estimation problem. Taking the logarithm of (3.3), we obtain:

$$\ln z_l = \ln p_l - \alpha w_l. \tag{3.5}$$

Now define

$$L = [\ln z_1, \dots, \ln z_N].$$

and observe that $\{\ln z_1, \dots, \ln z_N\}$ are jointly normal and independent. Therefore, it follows that for every $l \in \{1, \dots, N\}$:

$$ln z_l \sim N(ln p_l, \alpha^2\sigma^2)$$

The probability density function is given by

$$p(L, y) = \frac{1}{(2\pi\alpha^2\sigma^2)^{N/2}} \; exp\left(\sum_{i=1}^{N} \frac{-(\ln z_i - \ln p_i)^2}{2\alpha^2\sigma^2}\right)$$

and the likelihood function is given by

$$l(L, y) = -\frac{N}{2}\ln 2\pi\alpha^2\sigma^2 - \sum_{i=1}^{N} \frac{(\ln z_i - \ln p_i)^2}{2\alpha^2\sigma^2}$$

Using

$$\frac{\partial p_i}{\partial y} = \frac{\partial \left\|x_i - y\right\|^2}{\partial y} = \frac{-2(x_i - y)}{\left\|x_i - y\right\|^2}$$

we obtain that

$$\frac{\partial l(L, y)}{\partial y} \quad = \quad \sum_{i=1}^{N} \frac{(\ln z_i - \ln p_i)}{\alpha^2\sigma^2} \frac{2(x_i - y)}{\left\|x_i - y\right\|^2}$$

From (3.5) and (3.1), we have that

$$\frac{\partial l(L, y)}{\partial y} = \sum_{i=1}^{N} \frac{-\beta w_i}{\sigma^2} \frac{(y - x_i)}{\left\|x_i - y\right\|^2}$$

For notational convenience define

$$U = \left[\frac{(x_1 - y)}{\left\|x_1 - y\right\|^2}, \dots, \frac{(x_N - y)}{\left\|x_N - y\right\|^2}\right]$$

Then it follows that

$$\frac{\partial l(L, y)}{\partial y} = -\frac{\beta}{\sigma^2} U w$$

where $w = [w_1, \cdots, w_N]^T$, and

$$E\left[\frac{\partial l(L,y)}{\partial y}\frac{\partial l(L,y)}{\partial y}^T\right] = \frac{\beta^2}{\sigma^4}UE[ww^T]U^T$$

From the assumption that $w_i$ are iid Gaussian random variables with zero mean and variance $\sigma^2$, it follows that $E[ww^T] = \sigma^2 I$, where $I$ is an identity matrix. Therefore, it follows that

$$E\left[\frac{\partial l(L,y)}{\partial y}\frac{\partial l(L,y)}{\partial y}^T\right] = \frac{\beta^2}{\sigma^2}UU^T$$

The above calculations provide the Fisher Information Matrix for this estimation problem as follows:

$$F = \frac{\beta^2}{\sigma^2}\sum_{i=1}^{N}\frac{(x_i - y)(x_i - y)^T}{\|x_i - y\|^4} \tag{3.6}$$

The CRLB comprises the diagonal elements of the inverse of the Fisher Information. Thus observe that the CRLB for this problem grows linearly with noise power. As the CRLB provides the lower bound on the achievable error variances by unbiased estimators, the next step is to investigate whether there exists an efficient estimator, i.e. an unbiased estimator that meets CRLB. From [20], if the observations are perturbed by additive Gaussian noise then an efficient estimator exists if and only if the signal is affine in the parameter to be estimated. From (3.5), it can be observed that the signal has a *nonaffine* dependence on $y$ and an affine dependence on Gaussian noise $w_i$. Hence, no efficient estimator exists for this problem. This motivates us to investigate the statistical properties of the class of unbiased estimators of $y$, in particular the achievable error variances by the whole class of unbiased estimators of $y$.

## 3.3 Characterization of Unbiased Estimates

The main aim of this section is to characterize the whole class of unbiased estimators of $y$ and analyze their statistical properties. In section 3.3.1 we first relate the problem of source localization to distance estimation. We show that source localization is structurally connected to the estimation of linear functions of quantities such as $d_i^2 - d_j^2$. We prove that the class of unbiased estimators of $y$ is limited by the class of unbiased estimators of quantities of the form $d_i^2 - d_j^2$. In section 3.3.2, we prove that for a given $i$ and $j$, there is a unique unbiased estimator for $d_i^2 - d_j^2$. In section 3.3.3, we use the results obtained so far to characterize the whole class of unbiased estimators of $y$. We also provide and analyze the statistical quantities of bias and error variance of this class of unbiased estimators.

### 3.3.1 Source Localization and Distance Estimation

We first make the following definitions useful in establishing the relationship between localization and distance estimation. Recall that a set of $N > 2$ vectors in $\mathbb{R}^2$ belongs to the class $\mathcal{X}_N$, if the members of the set are non-collinear. For a given $N > 2$ and the set of vectors $\{x_1, x_2, \cdots, x_N\} \in \mathcal{X}_N$, consider the set of matrices $\mathcal{R}$ with members of the following form

$$R = [\ x_{i1} - x_{j1},\ x_{i2} - x_{j2},\ \cdots,\ x_{ik} - x_{jk}\ ]^T \tag{3.7}$$

where the members of $R$ satisfy the following properties

a) All the indices $i_l$, $j_m$ belong to $\{1, 2, \cdots, N\}$.

b) For each $l$, $i_l \neq j_l$.

c) For each $l \neq m$, $\{i_l, j_l\} \neq \{i_m, j_m\}$

d) It is a rank 2 matrix.

Observe that (d) is satisfied as long as the sensors are noncollinear. We define the following vectors corresponding to a given $R \in \mathcal{R}$

$$D = [\ d_{i1}^2 - d_{j1}^2,\ d_{i2}^2 - d_{j2}^2,\ \cdots,\ d_{ik}^2 - d_{jk}^2\ ]^T \tag{3.8}$$

and

$$K = [\ \|x_{i1}\|^2 - \|x_{j1}\|^2,\ \|x_{i2}\|^2 - \|x_{j2}\|^2,\ \cdots,\ \|x_{ik}\|^2 - \|x_{jk}\|^2\ ]^T \tag{3.9}$$

and

$$Z = [\ z_{i1} - z_{j1},\ z_{i2} - z_{j2},\ \cdots,\ z_{ik} - z_{jk}\ ]^T \tag{3.10}$$

For a given $R \in \mathcal{R}$, the corresponding $D$ defined in (3.8) is called an admissible and the corresponding $Z$ defined in (3.10) is called an observation vector. We now establish a relation between $y$ and $R \in \mathcal{R}$, and its corresponding admissible $D$ and $K$. Consider a set of vectors $\{x_1, x_2, \cdots, x_N\} \in \mathcal{X}_N$ and equations

$$\|x_{i1} - y\|^2 = d_{i1}^2 \quad and \quad \|x_{j1} - y\|^2 = d_{j1}^2$$

where $i1, j1 \in \{1, \ldots, N\}$. Subtracting one equation from the another gives the following equation,

$$2\,(x_{i_1} - x_{j_1})^T\, y = \|x_{i_1}\|^2 - \|x_{j_1}\|^2 + d_{j_1}^2 - d_{i_1}^2$$

Performing the above analysis repeatedly and writing the equations in the form of a matrix gives the following relationship

$$2Ry = K + D \tag{3.11}$$

Since $R$ (having rank 2) has full column rank, its pseudo-inverse exists and is given by

$$R^+ = (R^T R)^{-1} R^T \tag{3.12}$$

Then, the position of the source can be given by the following relationship

$$y = \frac{1}{2}R^{+}K + \frac{1}{2}R^{+}D \tag{3.13}$$

The above relations (3.11),(3.13) directly prove the following theorem.

**Theorem 3.3.1** *Assume that $N > 2$ and (3.5) holds for $i = \{1, \ldots, N\}$. Suppose that $f(x_1, \ldots, x_N, z_1, \ldots, z_N, A, \beta, \sigma^2)$ is an unbiased estimator of $y$ for all $\{x_1, \ldots, x_N\} \in \mathcal{X}_N$, all $y \in \mathbb{R}^2$, $A$, $\beta$ and $\sigma^2$. Then for every $R \in \mathcal{R}$ and corresponding $K$ and $D$,*

$$2Rf(x_1, \ldots, x_N, z_1, \ldots, z_N, A, \beta, \sigma^2) - K \tag{3.14}$$

*is an estimate of corresponding $D$. Conversely, suppose $g(z_1, \ldots, z_N, A, \beta, \sigma^2)$ is an estimator of $D$ for all $A$, $\beta$ and $\sigma^2$. Then for corresponding $R \in \mathcal{R}$ and corresponding $K$,*

$$\frac{1}{2}R^{+}K + \frac{1}{2}R^{+}g(z_1, \ldots, z_N, A, \beta, \sigma^2) \tag{3.15}$$

*is an unbiased estimate of $y$ for all $\{x_1, \ldots, x_N\} \in \mathcal{X}_N$, all $y \in \mathbb{R}^2$, $A$, $\beta$ and $\sigma^2$*

The above theorem establishes that the class of estimators of $y$ is limited by the class of estimators of the set of admissibles corresponding to the set of matrices $\mathcal{R}$. Every element of an admissible $D$ corresponding to any $R \in \mathcal{R}$ is of the form $d_i^2 - d_j^2$, where $i, j \in \{1, \ldots, N\}$. This motivates us to investigate the class of unbiased estimators of the quantity $d_i^2 - d_j^2$, for a given $i, j \in \{1, \ldots, N\}$.

### 3.3.2 Unbiased Estimation of $d_i^2 - d_j^2$

Observe that under (1.1), the resulting sets of distances $\{d_1, \ldots, d_N\}$ cover *almost all* $\mathbb{R}_+^N$ as we vary the elements of the set of anchors $\{x_1, \ldots, x_N\}$ over all $\mathcal{X}_N$ and the source $y$ over all $\mathbb{R}^2$. For convenience, define the following vectors,

$$w = [w_1, \ldots, w_N]', z = [z_1, \ldots, z_N]' \tag{3.16}$$

$$p = [p_1, \ldots, p_N]' = [d_1^2, \ldots, d_N^2]' \tag{3.17}$$

Consider (3.1),(3.2) and (3.3) for $i = \{1, \ldots, N\}$ where $N > 2$, then for a given $i$,$j$ our goal is to obtain an estimator of the form $h(z, \alpha, \sigma^2)$ such that

$$E\left[h(z, A, \beta, \sigma^2)\right] = p_i - p_j \tag{3.18}$$

The estimator that we obtain must work for almost all $p_i \geq 0$, $p_j \geq 0$ and all $A$, $\beta$ and $\sigma^2$. From now onwards, we will drop the arguments $A$, $\beta$ and $\sigma^2$ from the list of arguments of $h$.

**Lemma 3.3.1** *Consider (3.3) for $i = \{1, \ldots, N\}$ where $N > 2$ and $w_l \sim N(0, \sigma^2)$ are mutually uncorrelated Gaussian random variables. Suppose $h(z)$ is an unbiased estimate of $p_i - p_j$ for almost all $p_i \geq 0$,$p_j \geq 0$. Then $h(z)$ is unique and is given by*

$$h(z) = e^{-\alpha^2\sigma^2/2}(z_i - z_j) \tag{3.19}$$

**Proof:** Suppose $h(z)$ is an unbiased estimate of $p_i - p_j$ for almost all $p_i \geq 0$, $p_j \geq 0$. This requires that for almost all possible values of $p_i > 0$ and $p_j > 0$ there holds:

$$p_i - p_j = \frac{1}{(2\pi\sigma^2)^{N/2}} \int_{\mathbb{R}^N} h(z) \exp\left(-\sum_{i=1}^{N} \frac{w_i^2}{2\sigma^2}\right) dw \tag{3.20}$$

Then because of (3.3), we have that for almost all non-negative $p_i$ and $p_j$, there holds:

$$p_i - p_j = \frac{(-1)^N}{(2\pi\sigma^2\alpha^2)^{N/2}} \int_{\mathbb{R}_+^N} \frac{h(z)}{\prod_{i=1}^{N} z_i} \exp\left(-\sum_{i=1}^{N} \frac{(\ln p_i - \ln z_i)^2}{2\sigma^2\alpha^2}\right) dz \tag{3.21}$$

Now define

$$t_i = \ln z_i \quad , \quad v_i = \frac{\ln p_i}{\sigma^2\alpha^2} \tag{3.22}$$

and also

$$t = [t_1, \ldots, t_N]' \quad , \quad v = [v_1, \ldots, v_N]'.$$

Then for all possible $v_i$, (3.21) becomes:

$$\begin{aligned} e^{\sigma^2\alpha^2 v_i} - e^{\sigma^2\alpha^2 v_j} &= \frac{(-1)^N}{(2\pi\sigma^2\alpha^2)^{N/2}} \int_{\mathbb{R}^N} h(e^{t_1}, \cdots, e^{t_N}) \exp\left(-\frac{\left\|v\sigma^2\alpha^2 - t\right\|^2}{2\sigma^2\alpha^2}\right) dt \\ &= \frac{(-1)^N}{(2\pi\sigma^2\alpha^2)^{N/2}} \int_{\mathbb{R}^N} h(e^{t_1}, \cdots, e^{t_N})\ e^{\frac{-t't}{2\sigma^2\alpha^2}}\ e^{v't} dt\ e^{\frac{-v'v\sigma^2\alpha^2}{2}} \end{aligned}$$

i.e. for all possible $v_i$, there holds:

$$\frac{(-1)^N}{(2\pi\sigma^2\alpha^2)^{N/2}} \int_{\mathbb{R}^N} h(e^{t_1}, \cdots, e^{t_N})\ e^{\frac{-t't}{2\sigma^2\alpha^2}}\ e^{v't} dt\ = \left(e^{\sigma^2\alpha^2 v_i} - e^{\sigma^2\alpha^2 v_j}\right)\left(e^{\frac{v'v\sigma^2\alpha^2}{2}}\right)$$

Thus with $v$ and $t$ as the two domain variables

$$\left(e^{\sigma^2\alpha^2 v_i} - e^{\sigma^2\alpha^2 v_j}\right)\left(e^{\frac{v'v\sigma^2\alpha^2}{2}}\right)$$

and

$$\frac{(-1)^N}{(2\pi\sigma^2\alpha^2)^{N/2}} h(e^{t_1}, \cdots, e^{t_N})\ e^{\frac{-t't}{2\sigma^2\alpha^2}}$$

may be viewed as the Laplace pairs. This clearly indicates that $h(z)$ is unique. Now it suffices to show that (3.19) is an unbiased estimate. From (3.3) and using the fact that

$$E\left[e^{-\alpha w_i}\right] = e^{\alpha^2\sigma^2/2} \tag{3.23}$$

we obtain that

$$E\left[h(z)\right] = e^{-\alpha^2\sigma^2/2}\left(p_i E[e^{-\alpha w_i}] - p_j E[e^{-\alpha w_j}]\right) = p_i - p_j \tag{3.24}$$

Hence the lemma holds.

### 3.3.3 Class of Unbiased Estimators of $y$

In section 3.3.1, we showed that the class of unbiased estimates is limited by an isomorphism with the class of unbiased estimators of quantities like $d_i^2 - d_j^2$. In Section 3.3.2, we proved that there is a unique unbiased estimator for $d_i^2 - d_j^2$ for a given $i$ and $j$. Now combining Theorem 3.3.1 and Lemma 3.3.1 would directly prove the following theorem which characterizes all unbiased estimators of $y$.

**Theorem 3.3.2** *Assume that $N > 2$ and (1.3) holds for $i = \{1, \ldots, N\}$. Suppose that $\hat{y}$ is an unbiased estimator of $y$ for all $\{x_1, \ldots, x_N\} \in \mathcal{X}_N$, all $y \in \mathbb{R}^2$, $A$, $\beta$ and $\sigma^2$. Then for every $R \in \mathcal{R}$ and corresponding $K$ and $D$,*

$$\hat{y} = \frac{1}{2} R^+ \left( K + e^{-\alpha^2\sigma^2/2} Z \right) \tag{3.25}$$

Proof: From theorem 3.3.1,

$$\frac{1}{2} R^+ K + \frac{1}{2} R^+ g(z, A, \beta, \sigma^2) \tag{3.26}$$

characterizes all the unbiased estimates of $y$, where $R \in \mathcal{R}$ and its corresponding $K$ and $g(z_1, \ldots, z_N, A, \beta, \sigma^2)$ is unbiased estimate of corresponding $D$. From lemma 3.3.1, $g(z, A, \beta, \sigma^2)$ is unique and is given by

$$g(z, A, \beta, \sigma^2) = e^{-\alpha^2\sigma^2} Z$$

Hence the theorem holds and (3.26) characterizes the whole class of unbiased estimators of $y$.

The next point of interest is the mean square error or the error variances that are achievable by the class of unbiased estimators. Let us first make the following definitions. Define the matrix $X$ of dimension $N \times 2$ with the positions of the sensors as follows.

$$X = [\ x_1,\ x_2,\ \cdots,\ x_N\ ]^T \tag{3.27}$$

For some $R \in \mathcal{R}$ defined in (3.7), let us define the following corresponding matrix $V$ based on the indices $i_l, j_l$ of distances in matrix $R$. The elements of matrix $V$ are given by

$$V(l, m) = \begin{cases} 1 & i_l = m \\ -1 & j_l = m \\ 0 & \text{elsewhere} \end{cases} \tag{3.28}$$

Lemma  in Appendix-A gives the relationship between the matrices $X$, $R$ and its corresponding matrix $V$ as follows

$$R = VX \tag{3.29}$$

With the similarity between the structure of $R$ defined in (3.7) and the structure of corresponding $D$ defined in (3.8) and corresponding $Z$ defined in (3.10), the analogy of Lemma  holds for matrices $D$ and $Z$. Hence the following relationships follow:

$$D = Vp \tag{3.30}$$

$$Z = Vz \tag{3.31}$$

where vectors $p$ and $z$ are defined in (3.16) and (3.17). From (3.26), it follows that

$$\hat{y} - y = \frac{1}{2} R^{+} \left( e^{-\alpha^2\sigma^2/2} Z \right) \tag{3.32}$$

Thus, the mean square error is given by

$$E\left[(\hat{y} - y)(\hat{y} - y)^T\right] = \frac{1}{4} R^{+} C_1 R^{+T} \tag{3.33}$$

where

$$C_1 = E\left[(e^{-\alpha^2\sigma^2/2} Z - D)(e^{-\alpha^2\sigma^2/2} Z - D)^T\right] \tag{3.34}$$

Because of 3.30 and 3.31, $C_1$ can be re-written as

$$\begin{aligned} C_1 &= E\left[V(e^{-\alpha^2\sigma^2/2} z - p)(e^{-\alpha^2\sigma^2/2} z - p)^T V^T\right] \\ &= VUV^T \end{aligned} \tag{3.35}$$

where

$$U = E\left[(e^{-\alpha^2\sigma^2/2} z - p)(e^{-\alpha^2\sigma^2/2} z - p)^T\right] \tag{3.36}$$

From the definition of $p$ and $z$ defined in (3.17) and (3.16), the elements of matrix $U$ can be written as

$$U(i,j) = E\left[(e^{-\alpha^2\sigma^2/2} z_i - d_i^2)(e^{-\alpha^2\sigma^2/2} z_j - d_j^2)\right] \tag{3.37}$$

From equation (3.3) and using the fact that $E\left[e^{aw_i}\right] = e^{a^2\sigma^2/2}$, the $i^{th}$ diagnol element of $C_1$ is

$$\begin{aligned} U(i,i) &= E\left[(e^{-\alpha^2\sigma^2/2}z_i - d_i^2)^2\right] \\ &= d_i^4\ E\left[(e^{-\alpha^2\sigma^2/2}e^{-\alpha w_i} - 1)^2\right] \\ &= d_i^4\left(e^{-\alpha^2\sigma^2}E\left[e^{-2\alpha w_i}\right] - 2e^{-\alpha^2\sigma^2/2}E\left[e^{-\alpha w_i}\right] + 1\right) \\ &= d_i^4(e^{\alpha^2\sigma^2} - 1) \end{aligned}$$

Using similar analysis as above, the off-diagnol element $U(i,j)$ of $C_1$ is given by

$$\begin{aligned} U(i,j) &= E\left[(e^{-\alpha^2\sigma^2/2}z_i - d_i^2)(e^{-\alpha^2\sigma^2/2}z_j - d_j^2)\right] \\ &= d_i^2 d_j^2\ E\left[(e^{-\alpha^2\sigma^2/2}e^{-\alpha w_i} - 1)(e^{-\alpha^2\sigma^2/2}e^{-\alpha w_j} - 1)\right] \end{aligned}$$

Because $w_i$ are mutually uncorrelated and using the fact that $E[aw_i] = e^{a^2\sigma^2/2}$, it follows that:

$$\begin{aligned} U(i,j) &= d_i^2 d_j^2 E\left[(e^{-\alpha^2\sigma^2/2}e^{-\alpha w_i} - 1)\right] E\left[(e^{-\alpha^2\sigma^2/2}e^{-\alpha w_j} - 1)\right] \\ &= 0 \end{aligned}$$

From the above analysis, the elements of $U$ are given by

$$U(i,j) = \begin{cases} 0 & \text{for } i \neq j \\ d_i^4(e^{\alpha^2\sigma^2} - 1) & \text{for } i = j \end{cases} \tag{3.38}$$

Thus $U = (e^{\alpha^2\sigma^2} - 1)\ \mathrm{diag}\,(d_1^2, \ldots, d_N^2)$

From (3.33), (3.35) and (3.38), the mean square error is given by

$$E\left[(\hat{y} - y)(\hat{y} - y)^T\right] = \frac{(e^{\alpha^2\sigma^2} - 1)}{4} R^+ V\ \mathrm{diag}\left(d_1^2, \ldots, d_N^2\right) V^T R^{+T} \tag{3.39}$$

In view of this, the mean square error of the class of unbiased estimators of $y$ defined in (3.25) grows exponentially with $\sigma^2$.

### 3.4 The Proposed Biased Estimate

In this section, we exploit the structural relationship between the estimation of $y$ and the distance estimation, and we propose the class of estimators of $y$ based on the linear MMSE estimator of $d_i^2$ presented in the previous chapter. In section 2.4, We showed that the linear MMSE estimate of $d_i^2$ is $e^{-3\alpha^2\sigma^2/2}z_i$. The proposed class of estimators of $y$ based on this estimate of $d_i^2$ is given by

$$\hat{y}_v = \frac{1}{2}R^+\left(K + e^{-3\alpha^2\sigma^2/2}Z\right) \tag{3.40}$$

Now let us investigate the bias and MSE for the proposed class of estimators. Using the fact that $E\left[e^{-\alpha w_i}\right] = e^{\alpha^2\sigma^2/2}$, the bias of the proposed estimator is:

$$\begin{aligned}
\hat{y}_v - y &= \frac{1}{2}R^+\left(e^{-3\alpha^2\sigma^2/2}E[Z] - D\right) \\
&= \frac{1}{2}R^+\left(e^{-3\alpha^2\sigma^2/2}e^{\alpha^2\sigma^2/2}D - D\right) \\
&= \frac{1}{2}R^+D\left(e^{-\alpha^2\sigma^2} - 1\right)
\end{aligned}$$

Observe that proposed estimate has bias bounded in $\sigma^2$. The mean square error is given by

$$E\left[(\hat{y}_v - y)(\hat{y}_v - y)^T\right] = \frac{1}{4}R^+C_2R^{+T} \tag{3.41}$$

where

$$C_2 = E\left[(e^{-3\alpha^2\sigma^2/2}Z - D)(e^{-3\alpha^2\sigma^2/2}Z - D)^T\right] \tag{3.42}$$

Because of (3.30) and (3.31), $C_2$ can be re-written as

$$\begin{aligned}
C_2 &= E\left[V(e^{-3\alpha^2\sigma^2/2}z - p)(e^{-3\alpha^2\sigma^2/2}z - p)^TV^T\right] \\
&= VWV^T
\end{aligned} \tag{3.43}$$

where

$$W = E\left[(e^{-3\alpha^2\sigma^2/2}z - p)(e^{-3\alpha^2\sigma^2/2}z - p)^T\right] \tag{3.44}$$

From the definition of $p$ and $z$ defined in (3.17) and (3.16), the elements of matrix W can be written as follows

$$W(i,j) = E\left[(e^{-3\alpha^2\sigma^2/2}z_i - d_i^2)(e^{-3\alpha^2\sigma^2/2}z_j - d_j^2)\right] \tag{3.45}$$

From equation (3.3) and using the fact that $E\left[e^{aw_i}\right] = e^{a^2\sigma^2/2}$, the $i^{th}$ diagnol element of $C_2$ is

$$\begin{aligned} W(i,i) &= E\left[(e^{-3\alpha^2\sigma^2/2}z_i - d_i^2)^2\right] \\ &= d_i^4\ E\left[(e^{-3\alpha^2\sigma^2/2}e^{-\alpha w_i} - 1)^2\right] \\ &= d_i^4\left(e^{-3\alpha^2\sigma^2}E\left[e^{-2\alpha w_i}\right] - 2e^{-3\alpha^2\sigma^2/2}E\left[e^{-\alpha w_i}\right] + 1\right) \\ &= d_i^4(1 - e^{-\alpha^2\sigma^2}) \end{aligned}$$

Using similar analysis as above, the off-diagnol element $W(i,j)$ of $C_2$ is given by

$$\begin{aligned} W(i,j) &= E\left[(e^{-3\alpha^2\sigma^2/2}z_i - d_i^2)(e^{-3\alpha^2\sigma^2/2}z_j - d_j^2)\right] \\ &= d_i^2 d_j^2\ E\left[(e^{-3\alpha^2\sigma^2/2}e^{-\alpha w_i} - 1)(e^{-3\alpha^2\sigma^2/2}e^{-\alpha w_j} - 1)\right] \end{aligned}$$

Because of $w_i$ are mutually correlated and using the fact that $E[aw_i] = e^{a^2\sigma^2/2}$, it follows:

$$\begin{aligned} W(i,j) &= d_i^2 d_j^2 E\left[(e^{-3\alpha^2\sigma^2/2}e^{-\alpha w_i} - 1)\right] E\left[(e^{-3\alpha^2\sigma^2/2}e^{-\alpha w_j} - 1)\right] \\ &= d_i^2 d_j^2(1 - e^{\alpha^2\sigma^2})^2 \end{aligned}$$

Hence the mean square error of the proposed estimate is given by

$$E\left[(\hat{y}_v - y)(\hat{y}_v - y)^T\right] = \frac{1}{4}R^+VWV^TR^{+T} \tag{3.46}$$

where

$$W(i,j) = \begin{cases} d_i^2 d_j^2(1 - e^{-\alpha^2\sigma^2})^2 & \text{for } i \neq j \\ d_i^4(1 - e^{-\alpha^2\sigma^2}) & \text{for } i = j \end{cases} \quad (3.47)$$

Observe that both the diagonal and off-diagonal elements of $W$ are bounded in $\sigma^2$, unlike the diagonal elements of $U$ in case of unbiased estimators. This is even better than the linear dependence of CRLB on noise power. However, this is not surprising as the underlying estimate is biased. Under large noise variances, the proposed class of estimators performs better than CRLB.

## 3.5 Conclusion

In this chapter, we have considered statistical issues involved in estimating the source location from RSS measurements affected by log-normal shadowing. We have derived the CRLB for this problem and have shown that there is no unbiased estimator which meets the CRLB. We have also completely characterized the class of unbiased estimators of $y$ and showed that the MSE of each of its members grows exponentially with $\sigma^2$. We have proposed another class of biased estimators based on the linear MMSE estimator of the squares of distances. Finally, we have carried out the statistical analysis of this proposed class and showed that each of its member has both bias and variance bounded in $\sigma^2$.

# CHAPTER 4
# CONCLUSION

As sensor networks emerge as a key technology with localization as a fundamental component of wide range of applications, the study of area of localization has become important. In this thesis we have considered source localization from RSS under log-normal shadowing. In summary, the literature survey work showed that study of statistical issues in localization using RSS measurements under the practical log-normal fading is rarely presented and many algorithms presented earlier assume that distance measurements are available even though in general, distance information is not directly available and needs to be obtained through various readily available information like TOA, TDOA information or RSS measurements. All these factors motivated the development of this thesis.

We have studied the statistical properties of distance estimation and its implication to source localization from RSS measurements under log-normal shadowing. The first major contribution of this thesis is the study of the problem of distance estimation under the assumed noise model. We have shown that the underlying estimation problem is inefficient. Continuing further, we have presented results showing that the only unbiased estimator and ML estimator of distances has an exponentially growing MSE indicating unacceptable performance. As a remedy, we have derived the linear MMSE estimate and have demonstrated that it enjoys a superior MSE. In the main problem of this thesis, we have studied source localization directly. We characterized the class of all unbiased estimators for this problem and showed that their error variances grow exponentially with noise power. Then we considered a class of biased estimators of $y$ based on the linear MMSE estimator of squares of distances and have demonstrated that both its bias and variance are bounded in the noise variance.

Concerning future research, an interesting topic would be to consider the class of estimators with minimum error variances subject to the constrain that the norm of the bias gradient is upper bounded by a constant. The norm of the bias gradient is particularly interesting since it directly provides the maximum variation of bias over the neighborhood of parameter to be estimated and is unaffected by constant bias terms. Even if the constant bias terms are large, they can be easily removed. Other future topics include investigation of class of linear MMSE estimators of $y$ and the study of their statistical properties.

# APPENDIX
# SELECTED LEMMA'S

**Lemma .0.1** *Given the matrix $X$ with the positions of the sensors as defined in* (3.27)*. Consider some $R \in \mathcal{R}$ as defined in (3.7) and the corresponding $V$ matrix as defined in (3.28), then following relationship holds*

$$R = VX \tag{.1}$$

**Proof**: The result is clearly obvious, but yet formally proved here.

Consider any arbitrary $l \in \{1, 2, \cdots, k\}$. From the properties the members of $\mathcal{R}$ satisfy (as defined in (3.7)), we have that for $i_l, j_l \in \{1, 2, \cdots, n\}$ and $i_l \neq j_l$. Hence, by the structure of matrix $V$, as defined in (3.28), we have that

1. There are exactly two non-zero elements in the $l^{th}$ row of matrix $V$.

2. Those two non-zero elements are $V(l, i_m)$ and $V(l, j_m)$.

3. $V(l, i_m) = 1$ and $V(l, j_m) = -1$

From the definition of matrix multiplication, the $l^{th}$ row of product of matrices $V$ and $X$ is given by $\sum_m V(l,m)x_m^T$. From the above described properties of elements of $V$, we have that the $l^{th}$ row of $VX$ is $(x_{il}^T - x_{jl}^T)$ , which is equal to the $l^{th}$ row of the matrix $R$.

We have essentially shown that the $l^{th}$ row of $VX$ is equal to the $l^{th}$ row of $R$. Since, $l$ was arbitrary, it is true for all $l \in \{1, 2, \cdots, k\}$. Hence, we have that

$$R = VX$$

Thus, the lemma holds.

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