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Weak solutions to a Monge-Ampère type equation on Kähler surfaces

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WEAK SOLUTIONS TO A MONGE-AMPÈRE TYPE EQUATION ON KÄHLER SURFACES

by

Arvind Satya Rao

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

May 2010

Thesis Supervisor: Assistant Professor Hao Fang

ABSTRACT

In the context of moment maps and diffeomorphisms of Kähler manifolds, Donaldson introduced a fully nonlinear Monge-Ampère type equation. Among the conjectures he made about this equation is that the existence of solutions is equivalent to a positivity condition on the initial data. Weinkove later affirmed Donaldson's conjecture using a gradient flow for the equation in the space of Kähler potentials of the initial data. The topic of this thesis is the case when the initial data is merely semipositive and the domain is a closed Kähler surface. Regularity techniques for degenerate Monge-Ampère equations, specifically those coming from pluripotential theory, are used to prove the existence of a bounded, unique, weak solution. With the aid of a Nakai criterion, due to Lamari and Buchdahl, it is shown that this solution is smooth away from some curves of negative self-intersection.

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the May 2010 graduation.

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CHAPTER 1
INTRODUCTION

1.1 The Problem

Let (M, ω) be a closed Kähler manifold. Suppose that $\chi \in H^2(M; \mathbb{R})$ is another positive $(1,1)$-form. In the context of moment maps and diffeomorphisms Donaldson [12] suggested the study of

$$\omega \wedge \left(\chi + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^{n-1} = c\left(\chi + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^{n}. \tag{1.1}$$

Equation (1.1) is a fully nonlinear elliptic PDE of Monge-Ampère type. In local coordinates,

$$\omega = \frac{\sqrt{-1}}{2\pi} g_{i\bar{j}} dz^i \wedge dz^{\bar{j}} \quad \text{and} \quad \chi = \frac{\sqrt{-1}}{2\pi} h_{i\bar{j}} dz^i \wedge dz^{\bar{j}}.$$

Suppose $\tilde{\chi} = \chi + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ solves (1.1), then in normal coordinates at a point equation (1.1) has the form

$$nc = \sum_{j=1}^{n} \frac{1}{\lambda_j}; \tag{1.2}$$

where $\{\lambda_j\}$ are eigenvalues of $[h_{i\bar{j}} + \varphi_{i\bar{j}}]$. Dropping all but one of the terms in (1.2) yields

$$nc\lambda_j - 1 > 0 \quad \text{for each} \quad j = 1, \ldots, n. \tag{1.3}$$

Inequality (1.3) is equivalent to the class condition $[nc\tilde{\chi} - \omega] > 0$. In the interior of the cone determined by this class condition, Weinkove [32] proved the so-called

$\mathcal{J}$-flow of (1.1) converges to a smooth solution. Not long after, Song and Weinkove [30] proved the same result when there is a $\chi' \sim \chi$ for which

$$nc\chi'^{n-1} - (n-1)\chi'^{n-2} \wedge \omega > 0; \tag{1.4}$$

essentially they showed that the $\mathcal{J}$-flow converges to a smooth metric in a larger cone of admissible metrics. When $n = 2$ class conditions (1.3) and (1.4) coincide. It is worth noting that Fang, Lai, and Ma [16] have used a flow in a cone of positive $(1,1)$-forms to obtain smooth metrics $\tilde{\chi}$ that satisfy

$$\omega^k \wedge \tilde{\chi}^{n-k} = c_k \tilde{\chi}^n \quad \text{where } k = 1, \ldots, n$$

if and only if the initial data, χ and ω, satisfies a certain class condition. Furthermore, this class of fully nonlinear equations includes equation (1.1) and, in fact, both the $\mathcal{J}$-flow and condition (1.4) are identical to the $k = 1$ flow and class condition which appear in [16].

Song and Weinkove [30] also proved that on Kähler surfaces the $\mathcal{J}$-flow blows up over some curves of negative self-intersection, as conjectured by Donaldson [12]. The purpose of this thesis is to prove equation (1.1), with degenerate initial data, has weak solutions on Kähler surfaces which admit curves of negative self-intersection, and to investigate the regularity properties of these solutions. Specifically, the following is proved.

Theorem 1.1.1. *Let M^2 be a closed Kähler surface with two Kähler classes $[\omega]$ and $[\chi]$.*

$$c = \frac{[\chi] \cdot [\omega]}{[\chi]^2}$$

is a constant of integration. If $[2c\chi - \omega]$ is semipositive and not Kähler, then

$$\omega \wedge \left(\chi + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\right) = c\left(\chi + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\right)^2 \tag{1.5}$$

has a unique bounded solution, u, in the sense of currents that is smooth away from $E \subsetneq M$, a finite union of irreducible curves of negative self-intersection.

$$E = \bigcup_{j=0}^{m} E_j \qquad \textit{and} \qquad E_j \cdot E_j = -1 \quad \textit{for} \quad j = 1 \ldots m.$$

The conditions imposed on $[2c\chi - \omega]$ by theorem (1.1.1) guarantee that it is numerical effective and not Kähler–meaning that it lies in the boundary of the Kähler cone. On Kähler manifolds Nef(M), the numerically effective cone (nef cone), is the closure of $\mathcal{K}(M) \subseteq H^{1,1}(M;\mathbb{R})$, the Kähler cone: $\mathrm{Nef}(M) = \overline{\mathcal{K}(M)}$, see [27] and [11]. Requiring $[2c\chi - \omega] \notin \mathcal{K}(M)$ is necessary because, as stated earlier, Wienkove [32] has already proved that (1.5) has a smooth solution when $[2c\chi - \omega]$ is Kähler. Due to Lamari [26] and Buchdahl [6] there is a more geometric characterization for semipositive classes in $\partial\mathcal{K}(M) \subseteq H^{1,1}(M;\mathbb{R})$ on compact complex surfaces. Namely, $[\chi - \omega]$ must vanish on some effective divisor. A slightly more detailed discussion of this appears in lemma (.0.4) of the appendix.

1.2 Overview

Nondegenerate Monge-Ampère equations serve as a prototype for the kind of degenerate Monge-Ampère equations studied in this thesis. The C^2 estimate for equation (1.5) is essentially the C^2 estimate that appears in the nondegenerate setting. Existence and regularity of solutions for nondegenerate complex Monge-Ampère equations is the content of chapter 2. The Calabi conjecture, a now classical application for nondegenerate Monge-Ampère equations, asserts that Kähler-Einstein metrics exist for manifolds with definite first Chern class; and it is also covered in chapter 2.

Theorem (1.1.1) is proved in chapter 3. By converting (1.5) into a degenerate Monge-Ampère equation we have access to the pluripotential techniques which are used to obtain L^∞ estimates for such equations. On Kähler manifolds, Kołodziej [24] used pluripotential theory to prove L^∞ estimates and solve the Dirichlet problem on domains in $\mathbb{C}^n$ for Monge-Ampère equations with non-negative and L^p bounded right-hand-side. Later Demailly, Eyssidieux, Guedj, Pali, Zeriahi, and others adapted Kołodziej's ideas for Monge-Ampère equations with semipositive initial data. In chapter 3 relevant concepts from pluripotential theory are reviewed and used to obtain L^∞ estimates for equations of the later type.

CHAPTER 2
NONDEGENERATE MONGE-AMPÈRE EQUATIONS

Consider the class of Monge-Ampère equations

$$\left(\alpha + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\right)^n = F(u, \nabla u)\Omega, \tag{2.1}$$

on (M, ω), a closed Kähler manifold. Ω is a smooth volume form and $[\alpha] \in H^{1,1}(M; \mathbb{R})$. The initial data is the class $[\alpha]$ and function F. An equation in the class of equations (2.1) for which $[\alpha]$ is Kähler, meaning it can be represented by a Kähler form, and F is positive and at least C^3, is called a nondegenerate Monge-Ampère equation.

A necessary condition for there to be solutions to (2.1) is

$$\int_M (\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u)^n = \int_M F(u, \nabla u)\Omega. \tag{2.2}$$

So we assume that the initial data, F and ω together satisfy (2.2). Suppose that u solves (2.1). It is not necessary for $\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u$ to be positive definite; if n is even, a negative definite metric would also solve. Nevertheless we seek a positive solution metric because the linearization of the Monge-Ampère equation at $\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u > 0$ is elliptic. In fact this is true everywhere in the open set of positive metrics that are cohomolgous to ω. Together these positive metrics form the cone of Kähler metrics in $[\omega]$,

$$\mathbf{Ka}([\omega]) := \left\{\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u \;\middle|\; \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u > 0 \quad \text{and} \quad \int_M u\;\Omega = 0\right\},$$

which is an open and convex set within $[\omega]$. The normalization condition $\int_M u\,\Omega = 0$ is included to ensure uniqueness–more on this in section (2.2). Modulo constants the space of Kähler potentials,

$$\mathcal{H} := \left\{ \varphi \in C^4(M) \;\middle|\; \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi > 0 \right\},$$

is in one-to-one correspondence with the Kähler cone. A quick calculation shows that the linearization of (2.1) is elliptic in $\mathbf{Ka}([\omega])$. Let $\varphi(s) : (-\epsilon, \epsilon) \longrightarrow \mathbf{Ka}([\omega])$ be a C^1 curve so that $\varphi(0) = u$ and $v = \dot{\varphi}(0)$, then

$$\begin{aligned} \frac{1}{\omega_u^n}\frac{d}{ds}\left\{ \left(\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi(s) \right)^n \right\}\Big|_{s=0} &= \frac{1}{\omega_u^n} \cdot n\omega_u^{n-1} \wedge \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}v \\ &= \Delta_u v. \end{aligned}$$

In this chapter the following theorem is proved.

Theorem 2.0.1. (Yau) *Suppose $F \in C^k(M)$ $(k \geq 3)$ is positive and Ω is a smooth volume form on M. Then*

$$(\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u)^n = \exp\{F + cu\}\Omega, \quad c \geq 0 \textit{ is a constant,} \tag{2.3}$$

has a unique solution $u \in C^{k+1,\alpha}(M)$ for any $\alpha \in [0, 1)$.

Equation (2.3) is perturbed in order to prove existence of solutions, this is the method of continuity, and it is described in section (2.5). To make the argument work *a priori* estimates up to the second order are derived, section (2.3). The bootstrapping technique, section (2.5.2), provides the higher regularity needed to show that the perturbed solutions are compact in $C^{k+1}(M)$. Various applications of theorem (2.0.1) are presented in section (2.6); in particular, existence of Kähler-Einstein metrics for Kähler manifolds with nonpositive Ricci curvature is proved. The following presentation is based primarily on material from Yau [33] and Siu [29].

2.1 Preliminaries

(M^n, ω) is a compact Kähler manifold of complex dimension n. ω is a positive (1,1)-form and given locally as

$$\omega = \frac{\sqrt{-1}}{2\pi} g_{i\bar{j}}\, dz^i \wedge dz^{\bar{j}}.$$

Here and throughout repetition of indices denotes summation. Both $\tilde{\omega}$ and ω_u are defined to be

$$\tilde{\omega} = \omega_u := \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u.$$

Locally, they have the following form

$$\tilde{\omega} = \frac{\sqrt{-1}}{2\pi}\tilde{g}_{i\bar{j}}\, dz^i \wedge dz^{\bar{j}} \quad \text{and} \quad \omega = \frac{\sqrt{-1}}{2\pi} g_{i\bar{j}}\, dz^i \wedge dz^{\bar{j}}.$$

Metrics that solve the Monge-Ampère equation will be in the Kähler cone of $[\omega]$. Its definition is repeated below.

$$\mathbf{Ka}([\omega]) := \Big\{\omega_u \;\Big|\; \omega_u > 0 \quad \text{and} \quad \int_M u\ \Omega = 0\Big\}.$$

The Riemann curvature tensor and its traces (Ricci curvature tensor and scalar curvature) when written in local coordinates with respect to ω are

$R_{i\bar{j}k\bar{l}}$ (resp. $R_{i\bar{j}}$ and R). Anything different is dependent on $\tilde{\omega}$. Again suppose $v \in \mathcal{H}$. Metric Laplacians associated to ω and ω_v act on $C^2(M)$ functions and are denoted

$$\Delta := g^{i\bar{j}}\nabla_i\nabla_{\bar{j}} \quad \text{and} \quad \Delta_v := g_v^{i\bar{j}}\nabla_i\nabla_{\bar{j}},$$

respectively. The Laplacian associated to ω_u, where u is a solution to equation (2.3), is denoted

$$\tilde{\Delta} := \tilde{g}^{i\bar{j}}\nabla_i\nabla_{\bar{j}}.$$

2.2 Uniqueness

Solutions to equation (2.3) are unique up to constants. To see this suppose $u, v \in C^2(M)$ and $\omega_u^n = \omega_v^n$.

$$\begin{aligned} 0 = \omega_u^n - \omega_v^n &= \int_0^1 \frac{d}{dt}\Big(t\omega_u + (1-t)\omega_v\Big)^n \, dt \\ &= \left\{ n \int_0^1 \Big(t\omega_u + (1-t)\omega_v\Big)^{n-1} \, dt \right\} \wedge \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(u-v). \end{aligned} \tag{2.4}$$

Since ω_u and ω_v are known and positive definite, $\Big(t\omega_u + (1-t)\omega_v\Big)^{n-1}$ is positive as a $(n-1, n-1)$-form $\forall\ t \in [0,1]$. Furthermore, (2.4) is a linear elliptic equation. By the maximum principle for linear elliptic equations, $\sup_M u - v = \sup_{\partial M} u - v$, otherwise it achieves its maximum in the interior which implies $u - v$ is constant. $\partial M = \emptyset$ so $u - v$ is constant. If u and v are mean zero, then the integral of $u - v$ over M is zero, so necessarily $u = v$. Therefore, a mean zero potential which solves (2.3) is unique.

2.3 C^2 Estimates

To prove the existence of a solution, u, to (2.3), *a priori* estimates up to $C^{2,\alpha}$ are needed. The uniform $C^{2,\alpha}$ estimates of Evans and Krylov require uniform estimates of the real second partials of u. Instead of directly estimating all n^2 mixed second partials, $\{u_{i\bar{j}}\}$, it will suffice to bound the eigenvalues of $\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u$, the solution Kähler form. Essentially, this is a consequence of the Cauchy-Schwarz inequality.

Lemma 2.3.1. *Suppose $A = [a_{i\bar{j}}] \in GL(n, \mathbb{C})$ is Hermitian and positive. Also, there is a positive constant $\Lambda > 1$, so that $\Lambda^{-1} < tr\{A\} < \Lambda$. Then $|a_{i\bar{j}}| < \Lambda^2$ for $i, j = 1, \ldots, n$.*

Proof. Suppose that $\{e_1, \cdots, e_n\}$ is an orthonormal frame of $T_p'(M)$. Let $g(e_i, e_{\bar{j}}) = a_{i\bar{j}}$. Because A is Hermitian, positive, and bounded, $g(v, w) := [w]^*A[v]$ is a nondegenerate inner product on $T_p'(M)$. If $v, w \in T_p'(M)$ are unit vectors, the Cauchy-Schwarz inequality and bounds on A imply that

$$0 \leq |g(v, w)|^2 \leq g(v, v) \cdot g(w, w) \leq \Lambda^2.$$

In particular,

$$0 \leq |a_{i\bar{j}}|^2 \leq a_{i\bar{i}} \cdot a_{j\bar{j}} \leq \Lambda^2.$$

$\square$

2.3.1 C^2 Lower Bound

The right-hand-side of (2.3) is smooth and M is compact so by the extreme value theorem $\exists \varepsilon > 0$ so that $\tilde{\omega}^n = n! \cdot \det(\tilde{g}_{i\bar{j}}) > \epsilon \quad \forall p \in M$. Letting $\{\lambda_i\}$ denote the eigenvalues of $\tilde{g}$ and the statement becomes:

$$n! \prod_{i=1}^{n} \lambda_i > \epsilon.$$

Suppose there is a global constant $C > 0$ so that $C > \lambda_i$ for $i = 1, \ldots, n$. Then

$$\lambda_i > \frac{\varepsilon}{n! C^{n-1}} \quad \text{for} \quad i = 1 \ldots n.$$

So, a uniform C^2 lower bound would follow naturally from a uniform C^2 upper bound.

2.3.2 C^2 Upper Bound

So far the task of estimating the mixed second partials of the solution to (2.3), u, has been reduced to estimating the supremum norm of the largest eigenvalue of $\partial\bar{\partial}u$. Equivalently, we can uniformly bound $\mathrm{tr}\{g^{-1}\tilde{g}\} = n + \Delta u$. Before proceeding the method is briefly outlined.

$\tilde{\Delta} f(n + \Delta u)$, where $f \in C^\infty(M)$ is

$$\tilde{\Delta} f(n+\Delta u) = \boxed{\text{1st Order in } u} + \boxed{\text{2nd Order in } u} + \boxed{\text{3rd Order in } u} + \boxed{\text{4th Order in } u}. \tag{2.5}$$

Application of a suitable Laplacian to equation (2.3) yields an expression with a fourth order term identical to the one which appears in (2.5). Then subtracting this second expression from $\tilde{\Delta} f(n+\Delta u)$ eliminates the fourth order terms. A Schwarz inequality is used to handle the third order term of (2.5) and the maximum principle for Laplacians finishes the estimate.

To simplify the ensuing calculations normal coordinates at an arbitrary point $p \in M$ are taken so that $\partial g = \bar{\partial} g = 0$ at $p \in M$. This property of Kähler of metrics is proven in lemma (.0.3) of the appendix. Two derivatives of the logarithm of the Monge-Ampère equation (2.3) are taken.

$$\begin{aligned}
\partial\bar{\partial}\Big\{ \log\det(\tilde{g}) - F - cu - \log\det(g)\Big\} &= \partial \mathrm{tr}\left\{g_u^{-1}\bar{\partial} g_u\right\} - \partial\bar{\partial}F - c\partial\bar{\partial}u - \partial\mathrm{tr}\left\{g^{-1}\bar{\partial}g\right\} \\
&= \mathrm{tr}\left\{\partial g_u^{-1} \wedge \bar{\partial} g_u\right\} + \underbrace{\mathrm{tr}\left\{g_u^{-1}\partial\bar{\partial} g_u\right\}}_{A} - \partial\bar{\partial}F \\
&\quad - c\partial\bar{\partial}u - \mathrm{tr}\left\{\underbrace{\partial g^{-1} \wedge \partial g}_{=0}\right\} - \mathrm{tr}\left\{g^{-1}\partial\bar{\partial}g\right\}.
\end{aligned}$$

$$\begin{aligned}
\partial\bar{\partial}\mathrm{tr}\left\{g^{-1}g_u\right\} &= \mathrm{tr}\left\{g_u\partial\bar{\partial}g^{-1} - \bar{\partial}g^{-1}\wedge\partial g_u + \partial g^{-1}\wedge\bar{\partial}g_u + g^{-1}\partial\bar{\partial}g_u\right\} \\
&= \mathrm{tr}\left\{\partial\bar{\partial}g^{-1}g_u\right\} + \underbrace{\mathrm{tr}\left\{g^{-1}\partial\bar{\partial}g_u\right\}}_{B}.
\end{aligned}$$

A and B are fourth order terms. Note

$$\tilde{g}^{k\bar{l}}B_{k\bar{l}} = g^{k\bar{l}}B_{k\bar{l}}\mathrm{tr}_{k\bar{l}}A_{k\bar{l}}.$$

Then,

$$\begin{aligned}
\mathrm{tr}_{k\bar{l}}\left\{ g_u^{-1}\partial\bar{\partial}\mathrm{tr}\left\{g^{-1}g_u\right\} \right\} &= \mathrm{tr}_{k\bar{l}}\left\{g_u^{-1}\mathrm{tr}\left\{g_u\partial\bar{\partial}g^{-1}\right\}\right\} + \mathrm{tr}_{k\bar{l}}\left\{g_u^{-1}B\right\} \\
&= \mathrm{tr}_{k\bar{l}}\left\{ g_u^{-1}\mathrm{tr}\left\{g_u\partial\bar{\partial}g^{-1}\right\} \right\} + \Delta F + c\Delta u \\
&+ \mathrm{tr}_{k\bar{l}}\left\{ g^{-1}\mathrm{tr}\left\{g^{-1}\partial\bar{\partial}g\right\} \right\} - \mathrm{tr}_{k\bar{l}}\left\{ g^{-1}\left\{\partial g_u^{-1}\wedge\bar{\partial}g_u\right\} \right\}.
\end{aligned}$$

In local coordinates the above formula becomes

$$\begin{aligned}
\tilde{\Delta}\left(n+\Delta u\right) &= \tilde{g}^{k\bar{l}}\tilde{g}_{i\bar{j}}R^{i\bar{j}}{}_{k\bar{l}} + \Delta F + c\Delta u - g^{k\bar{l}}\partial_k\tilde{g}^{i\bar{j}}\partial_{\bar{l}}\tilde{g}_{i\bar{j}} + g^{i\bar{j}}g^{k\bar{l}}R_{i\bar{j}k\bar{l}} \\
&= \tilde{g}^{k\bar{l}}\tilde{g}_{i\bar{j}}R^{i\bar{j}}{}_{k\bar{l}} + \Delta F + c\Delta u + g^{k\bar{l}}\tilde{g}^{i\bar{a}}\tilde{g}^{b\bar{j}}\partial_k\tilde{g}_{a\bar{b}}\partial_{\bar{l}}\tilde{g}_{i\bar{j}} + R. \qquad (2.6)
\end{aligned}$$

Since there are no fourth order terms in (2.6), the only thing left to do is handle the third order term; this is facilitated by the following lemma.

Lemma 2.3.2.

$$g^{k\bar{l}}\tilde{g}^{i\bar{a}}\tilde{g}^{b\bar{j}}\partial_k\tilde{g}_{a\bar{b}}\partial_{\bar{l}}\tilde{g}_{i\bar{j}} \geq \frac{|\tilde{\nabla}\Delta u|^2}{n+\Delta u}. \qquad (2.7)$$

Proof. The following proof is contained in chapter 2 of Siu [29]. In normal coordinates for g so that $\tilde{g}$ is diagonal at $p \in M$ (see lemma (.0.3)), the right-hand-side of (2.7) can be rewritten so the Cauchy-Schwarz inequality can be applied.

$$
\begin{aligned}
|\tilde{\nabla}\Delta u|^2 &= \tilde{g}^{i\bar{j}}\partial_i \mathrm{tr}\big(g^{-1}\tilde{g}\big)\partial_{\bar{j}}\mathrm{tr}\big(g^{-1}\tilde{g}\big) \\
&= \tilde{g}^{i\bar{j}}\mathrm{tr}\big(g^{-1}\partial_i\tilde{g}\big)\mathrm{tr}\big(g^{-1}\partial_{\bar{j}}\tilde{g}\big), \quad \text{because } \partial_c g_{a\bar{b}} = \partial_{\bar{c}} g_{a\bar{b}} = 0, \\
&= g^{a\bar{b}} g^{c\bar{d}} \tilde{g}^{i\bar{j}} \partial_i \tilde{g}_{c\bar{d}} \partial_{\bar{j}} \tilde{g}_{a\bar{b}} \\
&\leq g^{c\bar{d}}\Big(\tilde{g}^{i\bar{j}}\partial_i\tilde{g}_{c\bar{d}}\partial_{\bar{j}}\tilde{g}_{c\bar{d}}\Big)^{\frac{1}{2}} \cdot g^{a\bar{b}}\Big(\tilde{g}^{k\bar{l}}\partial_k\tilde{g}_{a\bar{b}}\partial_{\bar{l}}\tilde{g}_{a\bar{b}}\Big)^{\frac{1}{2}}, \quad \text{by Cauchy-Schwarz inequality,} \\
&= \bigg(g^{a\bar{b}}\Big(\tilde{g}^{i\bar{j}}\partial_i\tilde{g}_{a\bar{b}}\partial_{\bar{j}}\tilde{g}_{a\bar{b}}\Big)^{\frac{1}{2}}\bigg)^2 \\
&= \bigg(\sum_a \sqrt{\tilde{g}_{a\bar{a}}}\Big(\sum_i \tilde{g}^{a\bar{a}}\tilde{g}^{i\bar{i}}\partial_i\tilde{g}_{a\bar{a}}\partial_{\bar{i}}\tilde{g}_{a\bar{a}}\Big)^{\frac{1}{2}}\bigg)^2, \quad \text{because } \tilde{g} \text{ is diagonal,} \\
&= \bigg(\sum_a \sqrt{\tilde{g}_{a\bar{a}}}\Big(\tilde{g}^{a\bar{a}}|\partial\tilde{g}_{a\bar{a}}|^2_{\tilde{\omega}}\Big)^{\frac{1}{2}}\bigg)^2 \\
&\leq \Big(\sum_a \tilde{g}_{a\bar{a}}\Big) \cdot \Big(\sum_a \tilde{g}^{a\bar{a}}|\partial\tilde{g}_{a\bar{a}}|^2_{\tilde{\omega}}\Big), \quad \text{by Cauchy-Schwarz inequality ,} \\
&\leq \Big(n + \Delta u\Big) g^{a\bar{b}}\tilde{g}^{k\bar{l}}\tilde{g}^{i\bar{j}}\partial_a\tilde{g}_{i\bar{l}}\partial_{\bar{b}}\tilde{g}_{k\bar{j}}.
\end{aligned}
$$

□

In the last step a property of the Kähler metric $\tilde{\omega}$ is used:

$$
d\tilde{\omega} = 0 \quad \Longrightarrow \quad \partial_a\tilde{g}_{b\bar{c}} = \partial_b\tilde{g}_{a\bar{c}} \quad \text{and} \quad \partial_{\bar{a}}\tilde{g}_{b\bar{c}} = \partial_{\bar{c}}\tilde{g}_{b\bar{a}}.
$$

Now, Lemma (2.3.2) is used to compare third order terms which appear in the right hand side of (2.6).

$$\begin{aligned}
\tilde{\Delta}\log(n+\Delta u) &= \frac{\tilde{\Delta}(n+\Delta u)}{n+\Delta u} - \frac{|\tilde{\nabla}(n+\Delta u)|^2}{(n+\Delta u)^2} \\
&= \frac{1}{n+\Delta u}\Bigg(\tilde{g}^{k\bar{l}}\tilde{g}_{i\bar{j}}R^{i\bar{j}}{}_{k\bar{l}} + \Delta F + c\Delta u \\
&\qquad + \underbrace{g^{k\bar{l}}\tilde{g}^{i\bar{a}}\tilde{g}^{b\bar{j}}\partial_k\tilde{g}_{a\bar{b}}\partial_{\bar{l}}\tilde{g}_{i\bar{j}} - \frac{|\tilde{\nabla}(n+\Delta u)|^2}{n+\Delta u}}_{\geq 0} + R\Bigg) \\
&\geq \frac{1}{n+\Delta u}\Bigg(\tilde{g}^{k\bar{l}}\tilde{g}_{i\bar{j}}R^{i\bar{j}}{}_{k\bar{l}} + \Delta F - nc + R\Bigg).
\end{aligned}$$

The curvature term $R^{i\bar{j}}{}_{k\bar{l}}$ is bounded so $\exists C > 0$ so that $R^{i\bar{j}}{}_{k\bar{l}} \geq -Cg^{i\bar{j}}g_{k\bar{l}}$. Also, the extreme value theorem and smoothness of f and R imply that they are both bounded. All constants used in this section are dependent on ω and M.

$$\begin{aligned}
\tilde{\Delta}\log(n+\Delta u) &\geq \frac{1}{n+\Delta u}\Bigg(-C\cdot(n+\Delta u)\cdot\tilde{g}^{k\bar{l}}g_{k\bar{l}} + \inf\Delta F - nc + R\Bigg) \\
&= \underbrace{-C\tilde{g}^{k\bar{l}}g_{k\bar{l}}}_{D} + \frac{\inf\{\Delta F - nc + R\}}{n+\Delta u} \qquad (2.8) \\
&= -(C+1)n + (C+1)\tilde{\Delta}u + \tilde{g}^{k\bar{l}}g_{k\bar{l}} + \frac{\inf\{\Delta F - nc + R\}}{n+\Delta u}. \qquad (2.9)
\end{aligned}$$

In the last step the formula

$$\tilde{g}^{k\bar{l}}g_{k\bar{l}} + \tilde{\Delta}u = n$$

is used to rewrite term D of equation (2.8). At $p \in M$, $g_{i\bar{j}} = \delta_{i\bar{j}}$ and $\tilde{g}_{i\bar{j}} = \big(1+u_{i\bar{i}}\big)\delta_{i\bar{j}}$.

So,

$$\tilde{\Delta}\log(n+\Delta u) = -(C+1)n + (C+1)\tilde{\Delta}u + \sum_{i=1}^{n}\frac{1}{1+u_{i\bar{i}}} + \frac{\inf\{\Delta F - nc + R\}}{n+\Delta u}.$$

After using

$$\left(\sum_{i=1}^{n}\frac{1}{1+u_{i\bar{i}}}\right)^{n-1} \geq \frac{\sum_{i=1}^{n} 1+u_{i\bar{i}}}{\Pi_{i=1}^{n} 1+u_{i\bar{i}}},$$

$$\begin{aligned}\tilde{\Delta}\log(n+\Delta u) &\geq -(C+1)n + (C+1)\tilde{\Delta}u + \left(\frac{\sum_{i=1}^{n} 1+u_{i\bar{i}}}{\Pi_{i=1}^{n} 1+u_{i\bar{i}}}\right)^{\frac{1}{n-1}} \\ &\quad + \frac{\inf\{\Delta F - nc + R\}}{n+\Delta u} \\ &\geq -(C+1)n + (C+1)\tilde{\Delta}u + \Big(n+\Delta u\Big)^{\frac{1}{n-1}} \cdot \exp\Big(-\frac{F+cu}{n-1}\Big) \\ &\quad + \frac{\inf\{\Delta F - nc + R\}}{n+\Delta u}.\end{aligned}$$

The Monge-Ampère equation (2.3) is used in the last line. At the maximum point of $\log(n+\Delta u) - (C+1)u$ we have

$$\begin{aligned}(C+1)n - \frac{\inf\{\Delta F - nc + R\}}{n+\Delta u} &\geq \Big(n+\Delta u\Big)^{\frac{1}{n-1}} \cdot \exp\Big(-\frac{F+cu}{n-1}\Big) \qquad \Longrightarrow \\ 2\cdot\max\{(C+1)n, \frac{|\inf\{\Delta F - nc + R\}|}{n+\Delta u}\} &\geq \Big(n+\Delta u\Big)^{\frac{1}{n-1}} \cdot \exp\Big(-\frac{F+cu}{n-1}\Big).\end{aligned} \tag{2.10}$$

This implies that either

$$2(C+1)n \cdot \exp\Big(\frac{F+cu}{n-1}\Big) \geq \Big(n+\Delta u\Big)^{\frac{1}{n-1}} \quad \text{or} \quad 2\frac{|\inf\{\Delta F - nc + R\}|}{n+\Delta u} \geq \Big(n+\Delta u\Big)^{\frac{1}{n-1}}.$$

The left-hand-side of the first inequality is *a priori* bounded because potential functions of metrics in $\mathbf{Ka}([\omega])$ have *a priori* supremum bounds, this fact is proved in section (2.4). Both inequalities are upper bounds for $(n + \Delta u)(p)$. Suppose the maximum of $\log(n + \Delta u) - (C + 1)u$ occurs at $p \in M$. Then,

$$(n + \Delta u)\exp\Big\{ - (C+1)u\Big\}(q) \leq C_1 \cdot \exp\Big\{ - (C+1)\inf_{x \in M} u\Big\} \quad \forall q \in M. \tag{2.11}$$

And

$$0 < n + \Delta u \leq C_1 \exp\Big\{(C+1)\Big(u - \inf_{x\in M} u\Big)\Big\}.$$

Therefore, the eigenvalues of $\tilde{g}$ are uniformly bounded above in terms of known constants and $\inf_M u$ and $\sup_M u$. To complete the C^2 estimate for (2.3), $\sup_M |u|$ is estimated.

2.4 L^∞ Estimate

The C^2 estimates derived in the previous section are incomplete without uniform estimates of the supremum norm of solutions to (2.3). In this section L^∞ estimates for solutions are derived. The strategy employed is a Moser-iteration argument due to Aubin [2], Bourguignon [1], and Kazdan [20]. An *a priori* L^1 estimate of u is needed for the iteration to work, so we'll first calculate this estimate. Let $G(p, q)$ denote the Green's function of Δ. Since M is compact there is a constant $K > 0$ so that $G(p, q) \geq -K$. For more information about Green's functions on Riemannian manifolds see Aubin [3]. u is mean zero so we have

$$\begin{aligned}
u(p) = -\int_M G(p,q)\Delta u(q)\ \omega^n(q) &= -\int_M (G(p,q)+K)\Delta u(q)\ \omega^n(q) \\
&\le \int_M (G(p,q)+K)\cdot n\ \omega^n(q)\ , \qquad \text{because } n+\Delta u > 0.
\end{aligned}$$

Therefore,

$$\sup_{p\in M} u(p) \le n\cdot \sup_{p\in M}\int_M (G(p,q)+K)\ \omega^n(q). \tag{2.12}$$

By using (2.12), $\|u\|_1$ can be estimated. Note that u mean zero implies

$$0 = \int_M u = \int_M u^+ - u^- \quad \Longrightarrow \quad \int_M u^+ = \int_M u^-.$$

Then,

$$\begin{aligned}
\int_M |u|\ \omega^n = \int_M \left(u^+ + u^-\right)\ \omega^n &= 2\int_M u^+\ \omega^n \\
&\le 2[\omega]^n \sup_{x\in M} u \\
&\le 2n[\omega]^n\cdot \sup_{p\in M}\int_M (G(p,q)+K)\ \omega^n(q).
\end{aligned}$$

So, there is a constant $C = C(M,\omega)$ for which $\|u\|_1 \le C$.

Lemma 2.4.1. *Suppose that $u \in C^4$ is a solution of*

$$\begin{cases} \left(\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\right)^n = f(x,u)\Omega \\ \frac{df}{dt} \geq 0,\ \ f > 0,\ \ and\ f \in C^k(M \times \mathbb{R}) \\ \exists C > 0\ so\ that\ \|f(x,u)\|_\infty \leq C. \end{cases} \tag{2.13}$$

Then there is a constant $C = C(\mathit{Vol}(M), \omega, \sup_{y \in M \times \mathbb{R}} f)$ *so that*

$$\|u\|_\infty \leq C.$$

Proof. Equation (2.3) fits into the class of equations defined by (2.13). Suppose $h : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing function, to be defined later. Then by Stokes' theorem we derive,

$$0 = \int_M \partial\left(h\omega_{su}^{n-1} \wedge \bar{\partial}u\right) = \int_M \partial h \wedge \omega_{su}^{n-1} \wedge \bar{\partial}u + \int_M h\ \omega_{su}^{n-1} \wedge \partial\bar{\partial}u \quad \Longrightarrow$$

$$\int_M h\ \omega_{su}^{n-1} \wedge \partial\bar{\partial}u = -\int_M \partial h \wedge \omega_{su}^{n-1} \wedge \bar{\partial}u. \tag{2.14}$$

Furthermore,

$$\begin{aligned} \int_M h(u)\left\{\omega_u^n - \omega^n\right\} &= \int_0^1 \int_M h(u)\frac{d}{ds}\left\{(\omega + s\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u)^n\right\}\ ds \\ &= n\int_0^1 \int_M h(u)(\omega + s\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u)^{n-1} \wedge \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\ ds \\ &= -n\int_0^1 \int_M h'(u)\Big(s\tilde{\omega} + (1-s)\omega\Big)^{n-1} \wedge \partial u \wedge \frac{\sqrt{-1}}{2\pi}\bar{\partial}u\ ds, \\ &\qquad \text{by (2.14)}, \\ &= n\int_0^1 \int_M h'(u)\Big(s\omega + (1-s)\tilde{\omega}\Big)^{n-1} \wedge \partial u \wedge \frac{\sqrt{-1}}{2\pi}\bar{\partial}u\ ds. \end{aligned}$$

Because the Kähler cone is convex, $\Big(s\omega + (1-s)\tilde{\omega}\Big)^{n-1}$ is positive as a $(n-1, n-1)$-form $\forall s \in [0,1]$.

$$0 < \Big(s\omega + (1-s)\tilde{\omega}\Big)^{n-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} (1-s)^j s^{n-j-1} \tilde{\omega}^j \wedge \omega^{n-j-1}. \tag{2.15}$$

Each term in the left-hand-side of (2.15) is positive because $s \in [0,1]$, and ω and $\tilde{\omega}$ are positive. Moreover,

$$\Big(s\omega + (1-s)\tilde{\omega}\Big)^{n-1} \wedge \eta \wedge \bar{\eta} \geq s^{n-1}\omega^{n-1} \wedge \eta \wedge \bar{\eta}.$$

Then,

$$\begin{aligned} \int_M h(u)\Big\{\omega_u^n - \omega^n\Big\} &\geq n \int_0^1 \int_M h'(u) s^{n-1}\omega^{n-1} \wedge \partial u \wedge \frac{\sqrt{-1}}{2\pi}\bar{\partial} u \; ds \\ &= \int_M h'(u)\omega^{n-1} \wedge \partial u \wedge \frac{\sqrt{-1}}{2\pi}\bar{\partial} u, \quad \text{after integrating in } s, \\ &= \frac{1}{n}\int_M h'(u)|\nabla u|^2 \omega^n. \end{aligned} \tag{2.16}$$

Define $h := u|u|^\alpha$. By the hypothesis of lemma (2.4.1), the left-hand-side of (2.16) is bounded above and

$$\int_M h(u)\Big\{\omega_u^n - \omega^n\Big\} \leq \Big| \int_M h(u)(f-1)\;\omega^n \Big| \leq \sup_M\{|f-1|\} \cdot \int_M |u|^{\alpha+1}\;\omega^n. \tag{2.17}$$

Differentiating h yields

$$\int_M h'(u)|\nabla u|^2\omega^n = (\alpha+1)\int_M |u|^\alpha |\nabla u|^2\omega^n = 4\frac{\alpha+1}{(\alpha+2)^2}\int_M \left|\nabla u|u|^{\frac{\alpha}{2}}\right|^2 \ \omega^n. \quad (2.18)$$

By combining equation (2.17) and equation (2.18) with inequality (2.16),

$$4\frac{\alpha+1}{(\alpha+2)^2}\|\nabla u|u|^{\frac{\alpha}{2}}\|_2^2 \leq n \cdot \sup_M\{|f-1|\} \cdot \|u\|_{\alpha+1}^{\alpha+1}. \quad (2.19)$$

Now, we proceed with the iteration part of the argument. The Sobolev embedding theorem for $k = p = 2$ states

$$W^{2,2}(M) \hookrightarrow L^{\frac{2n}{n-1}}(M).$$

When written as an inequality it is

$$\|v\|_{\frac{2n}{n-1}} \leq C\Big(\|\nabla v\|_2 + \|v\|_2\Big) \ \text{ where } \ C = C(n) > 0.$$

Set $v = u|u|^{\frac{\alpha}{2}}$ in the Sobolev inequality above. When the Cauchy inequality is applied, see [14],

$$\begin{aligned}
\|u|u|^{\frac{\alpha}{2}}\|_{\frac{2n}{n-1}}^2 &\leq 2C^2\left(\|\nabla u|u|^{\frac{\alpha}{2}}\|_2^2 + \|u\|_{\alpha+2}^{2(\alpha+2)}\right)\\
&\leq 2C^2\left(n \cdot \sup_M\{|f-1|\} \cdot \frac{(\alpha+2)^2}{4(\alpha+1)}\|u\|_{\alpha+1}^2 + \int_M |u|^{\alpha+2}\Omega\right), \ \text{ by (2.19)},\\
&\leq 2C^2\Bigg(n \cdot \sup_M\{|f-1|\} \cdot (\alpha+2)\mathrm{Vol}(M)^{\frac{1}{\alpha+2}} \cdot \Big(\int_M |u|^{\alpha+2}\Omega\Big)^{\frac{\alpha+1}{\alpha+2}} \qquad (2.20)\\
&\qquad + \int_M |u|^{\alpha+2}\Omega\Bigg).
\end{aligned}$$

Hölder's inequality is used in the last line. To simplify expression (2.20) define $\tilde{C}(f, \mathrm{Vol}(M)) = n \cdot \sup_M\{|f-1|\}\mathrm{Vol}(M)^{\frac{1}{\alpha+2}}$. It is assumed that $\tilde{C} \geq \max\{4C^2, 1\}$. Also, set $\beta = \frac{n}{n-1}$ and $p := \alpha + 2 \geq 2$. After noticing that $x^\varepsilon \leq x + 1$ $\forall x \geq 0$ and for each $\varepsilon \leq 1$, we get

$$\begin{aligned} \| u|u|^{\frac{\alpha}{2}} \|^2_{\frac{2n}{n-1}} = \|u\|^p_{p\beta} &\leq \tilde{C}p\left(1 + \int_M |u|^p\Omega\right) \\ &\leq 2\tilde{C}p \max\{1, \int_M |u|^p\Omega\}. \end{aligned} \tag{2.21}$$

When u is replaced by $u^{\beta^{k-1}}$ in inequality (2.21), for each $k \in \mathbb{Z}^+$

$$\|u^{p\beta^{k-1}}\|_\beta \leq 2\tilde{C}p \max\{1, \int_M |u|^{p\beta^{k-1}}\Omega\}. \tag{2.22}$$

After the logarithm of the $p\beta^{k-1}$th root of each side of (2.22) is taken, for each $k \in \mathbb{Z}^+$

$$\log \max\{1, \|u\|_{p\beta^k}\} \leq \frac{1}{p\beta^{k-1}} \log 2\tilde{C} + \frac{1}{p\beta^{k-1}} \log p + \log \max\{1, \|u\|_{p\beta^{k-1}}\}. \tag{2.23}$$

By inductively applying inequality (2.23) and summing,

$$\log \max\{1, \|u\|_{p\beta^k}\} \leq \frac{1}{p}\Big(\sum_{j=0}^{k-1} \beta^{-j}\Big) \log 2\tilde{C} + \frac{1}{p}\Big(\sum_{j=0}^{k-1} \beta^{-j}\Big) \log p + \log \max\{1, \|u\|_p\};$$

the limit as $k \longrightarrow \infty$ and $p = 2$ is

$$\log \max\{1, \|u\|_\infty\} \leq \frac{n}{p} \log \tilde{C}p + \log \max\{1, \|u\|_2\}.$$

So, it suffices to find a uniform upper bound for $\|u\|_2$. Because u is mean zero,

u is not in the kernel of $-\Delta$. Let $\lambda_1 > 0$ be the smallest non-zero eigenvalue of $-\Delta$. Then integration-by-parts yields

$$(\nabla u, \nabla u) = -(\Delta u, u) \geq \lambda_1(u,u) \quad \Longrightarrow \quad \int_M |\nabla u|^2 \ \geq \lambda_1 \int_M |u|^2. \tag{2.24}$$

For $\alpha = 0$, equation (2.19) is

$$\int_M |\nabla u|^2 \leq n \cdot \sup_M\{|f-1|\} \cdot \|u\|_1. \tag{2.25}$$

After combining (2.24) with (2.25) and using Hölder inequality (with $p = q = \frac{1}{2}$),

$$\int_M |u|^2 \ \leq n \cdot \frac{\sup_M\{|f-1|\}}{\lambda_1} \cdot \int_M |u| \ \Omega \leq n \cdot \mathrm{Vol}(M)^{\frac{1}{2}} \frac{\sup_M\{|f-1|\}}{\lambda_1} \Big(\int_M |u|^2 \Big)^{\frac{1}{2}}.$$

Finally,

$$\|u\|_2 \leq n \cdot \mathrm{Vol}(M)^{\frac{1}{2}} \frac{\sup_M\{|f-1|\}}{\lambda_1},$$

and $\|u\|_\infty$ is uniformly bounded. □

2.5 Existence of a Solution to Equation (2.3)

With the C^2 estimate for solutions to (2.3) in hand we can now describe the continuity approach used to prove the existence of a solution. The method of continuity associates a family of Monge-Ampère equations,

$$\Big(\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_t\Big)^n = A(t)e^{tF+cu}\omega^n \ \text{ for } c \geq 0 \text{ and } t \in [0,1] \ , \tag{$\star_t$}$$

to equation (2.3). Where

$$A(t) = \mathrm{Vol}(M) \cdot \Big(\int_M e^{tF+cu} \ \omega^n \Big)^{-1}$$

is a compatibility constant for each t. It will eventually be shown that $(\star_t)$ is solvable $\forall t$, so it is necessary that integrals over M of both sides of $(\star_t)$ be equal for each $t \in [0,1]$. Note the equation at $t = 1$ is (2.3), the equation we want to solve. By showing $S = \Big\{ t \in [0,1] \ \mid \ (\star_t) \text{ has a solution} \Big\}$ is nonempty, open, and closed the existence of a solution is established. $S \neq \emptyset$ because $u_0 \in \mathbb{R}$ a constant solves the equation at $t = 0$.

2.5.1 S is Open

Suppose $t_0 \in S$; then u_{t_0} solves equation $\star_{t_0}$. Let $B_k = \Big\{ v \in C^{k+1,\alpha}(M) \mid \int_M v \ \omega^n = 0 \text{ and } \omega_v > 0 \Big\}$ and $D_k = \Big\{ f \in C^{k-1,\alpha}(M) \mid \int_M f \ \omega^n = \mathrm{Vol}(M) \Big\}$. Define $\mathbf{MA} : B_k \longrightarrow D_k$ to be

$$\mathbf{MA}(u) := \log\det(g_{i\bar{j}} + u_{i\bar{j}}) - \log\det(g_{i\bar{j}}) - cu.$$

MA is a map of Banach spaces and it has a differential. We will show that its differential at u_{t_0}, the solution to $\star_{t_0}$, is an isomorphism of $T_{u_{t_0}}(B_k)$ with $T_{\mathbf{MA}(u_{t_0})}(D_k)$. To compute the differential of **MA** at u_{t_0} consider a C^1 curve in B_k,

$$\phi(s) : (-\epsilon, \epsilon) \longrightarrow B_k \quad \text{so that} \quad \phi(0) = u_{t_0} \text{ and } \dot{\phi}(0) = v \in T_{u_{t_0}}(B_k).$$

It is not important to completely characterize the tangent space of B_k, but we will need one fact; namely, $\dot{\phi}(s)$ is in B_k for each s. By the definition of B_k

$$0 = \int_M \phi(s)\ \omega^n.$$

Then taking the derivative of both sides shows that

$$0 = \int_M \dot{\phi}(s)\Big|_{s=0}\ \omega^n = \int_M v\ \omega^n,$$

and $v \in T_{u_{t_0}}(B_k)$ is mean zero. By a similar argument $w \in T_{u_{t_0}}(D_k)$ is mean zero: Let $\beta(s) : (-\epsilon, \epsilon) \longrightarrow D_k$ be a C^1 curve so that $\dot{\beta}(0) = w$. Then by differentiating

$$\mathrm{Vol}(M) = \int_M \beta(s)\ \omega^n,$$

$$0 = \int_M \dot{\beta}(s)\Big|_{t=0}\ \omega^n = \int_M w\ \omega^n.$$

With our minimal characterization of the tangent spaces of B_k and D_k we can proceed. The linearization of **MA** is:

$$\begin{aligned}
D_{u_{t_0}}\mathbf{MA}(v) &= \frac{d}{ds}\Big\{ \log \bigwedge_{j=1}^{n} \left(g_{i\bar{j}} + \phi(s)_{i\bar{j}}\right) - \log\det(g_{i\bar{j}}) - c\phi(s) \Big\}\Big|_{s=0} \\
&= \frac{n}{\det(g_{i\bar{j}} + u^{t_0}_{i\bar{j}})} \cdot \Big(\bigwedge_{j=1}^{n-1} \left(g_{i\bar{j}} + \phi(s)_{i\bar{j}}\right) \Big) \wedge \dot{\phi}(s)_{i\bar{j}}\Big|_{s=0} - c\dot{\phi}(s)|_{s=0} \\
&= \frac{n}{\det(g_{i\bar{j}} + u^{t_0}_{i\bar{j}})} \cdot \Big(\bigwedge_{j=1}^{n-1} g_{i\bar{j}} + u^{t_0}_{i\bar{j}} \Big) \wedge v_{i\bar{j}} - cv \\
&= \Delta^{u_{t_0}} v - cv
\end{aligned}$$

The $\omega_{u_{t_0}}$-Laplacian has negative spectrum and c is positive, so the spectrum of $\Delta^{u_{t_0}} - c$ is negative; moreover it is a bijection on mean zero functions. Then the linearization of $\mathbf{MA}$, as claimed, is an isomorphism and we can use the implicit function theorem for Fréchet differentiable maps, see [17], to show that S is open. When this implicit function theorem is applied to $\mathbf{MA}$ at u_{t_0} we get an open set $U \subset B_k$ with $u_{t_0} \in U$, and another open set $V \subset D_k$ with $\log A(t_0) + t_0 F \in V$, for which $\mathbf{MA} : U \longrightarrow V$ is a bijection. Furthermore, there exists a $\delta > 0$ so that $\log A(t) + tF \in V \quad \forall t \in (t_0 - \delta, t_0 + \delta)$. Using the just established fact that $\mathbf{MA}$ is invertible on U, we see that

$$\mathbf{MA}^{-1}(\log A(t) + tF) = u_t$$

solves $(\star_t)$ for $t \in (t_0 - \delta, t_0 + \delta)$. Therefore, S is open.

2.5.2 S is Closed

Uniform ellipticity and convexity of (2.3), and uniform C^2 estimates for solutions imply uniform $C^{2,\alpha}$ estimates for solutions $\{u_t\}$, see Evans [13]. Therefore, $S \subset C^{2,\alpha}(M)$ is a bounded set. Repeated applications of Schauder estimates show that S is uniformly bounded in $C^{k+1,\alpha}(M)$. Take a sequence of solutions $\{u_t\} \subseteq S$. Around a point in M there is an open set $U \subset M$ for which $\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}v$ and $v \in C^\infty(U)$. On U the logarithm of equation $(\star_t)$ is

$$0 = \log\det\left(v_{i\bar{j}} + \frac{\partial^2 u_t}{\partial z^i \partial z^{\bar{j}}}\right) - \log\det(v_{i\bar{j}}) - cu_t - tF - \log A(t). \tag{2.26}$$

Differentiating (2.26) in the z_l (or $\bar{z}_l$) direction (this is okay because we have assumed that $u_t \in C^4(M)$) yields

$$0 = \tilde{\Delta}\frac{\partial(v+u_t)}{\partial z^l} - c\frac{\partial u_t}{\partial z^l} - \Delta\frac{\partial v}{\partial z^l} - tF_l \quad \Longrightarrow \quad \tilde{\Delta}\frac{\partial(v+u_t)}{\partial z^l} - c\frac{\partial(v+u_t)}{\partial z^l} = \Delta v_l - cv_l + tF_l.$$

This calculation is very similar to the linearization of the Monge-Ampère operator in section (2.5.1). The regularity assumption made in theorem (2.0.1) for the right-hand-side of equation (2.3) implies that $F_l \in C^{k-1}(M)$. Also, the linear operator $\tilde{\Delta} - c$ has uniformly bounded coefficients in $C^{0,\alpha}(M)$, because u_t is uniformly bounded in $C^{2,\alpha}(M)$. Then by Schauder interior estimates, theorem 6.2 of [17], we have the following $C^{2,\alpha}$-norm estimate of $\partial_l u_t$:

$$\|\partial_l u_t\|_{C^{2,\alpha}(U)} \leq C\Big(\|\partial_l u_t\|_{L^\infty(M)} + \|tF_l + \Delta v_l - cv_l\|_{C^\alpha(M)}\Big), \tag{2.27}$$

where $C = C(\omega, U, M, \alpha)$. Furthermore, (2.27) is a $C^{3,\alpha}(U)$ estimate of u_t, and the coefficients of $\tilde{\Delta} - c$ lie in $C^{1,\alpha}(U)$, which is one derivative higher than its previously established regularity. $k-2$ more applications of Schauder interior estimates to (2.27) show that $\|u_t\|_{C^{k+1,\alpha}(U)} \leq C(\|F\|_{C^k(M)}, \omega, U)$. M is compact, so it can be covered by a finite number of open sets on which Schauder estimates are valid. Therefore, the local estimates derived above can be made global; u_t is uniformly bounded in $C^{k+1,\alpha}(M)$. Suppose $C > 0$ is the $C^{k+1,\alpha}(M)$ bound for S, then for any $f \in S$

$$|D^\beta f(x) - D^\beta f(y)| \leq C \cdot \mathrm{d}(x,y)^\alpha \quad \text{where } \beta \subset \{1, \ldots, n\} \quad \text{and} \quad |\beta| = k+1.$$

Consequently, for each $\epsilon > 0$ there is a $\delta = \left(\frac{\epsilon}{C}\right)^{\alpha}$ for which $|D^{\beta}f(x) - D^{\beta}f(y)| \leq \epsilon$ for every $f \in S$ and $d(x,y) \leq \delta$. This is precisely the definition of equicontinuity. Then by *Arzelá-Ascoli* $S \subset C^{k+1}(M)$ is sequentially compact, and thus compact and closed. Suppose $u = \lim_{t\to 1} u_t \in C^{k+1}(M) \cap S$ solves equation (2.3). Another application of Schauder estimates shows that u is uniformly bounded in $C^{k+1,\alpha}(M)$.

2.6 Kähler-Einstein Metrics

In this section theorem (2.0.1) is applied to prove the Calabi conjecture when $c_1(M)$ is negative definite and $c_1(M)$ is cohomologous to zero.

Theorem 2.6.1. *Let (M,ω) be a compact Kähler manifold. If $\Gamma \in H^{1,1}(M,\mathbb{C})$ represents $c_1(M)$, then there exists a unique Kähler metric $\tilde{\omega} \in [\omega]$ so that Ric$(\tilde{\omega}) = \Gamma$.*

Proof. $[\mathrm{Ric}(\omega)] = c_1(M)$, see Griffiths and Harris [18]. Suppose there is a metric $\tilde{\omega}$ for which $\mathrm{Ric}(\tilde{\omega}) = \Gamma$. Then,

$$\Gamma \in c_1(M) \quad \Longleftrightarrow \quad \mathrm{Ric}(\omega) - \mathrm{Ric}(\tilde{\omega}) \sim 0.$$

By the $\partial\bar{\partial}$-lemma, Griffiths and Harris [18], there is a function $F \in C^{\infty}(M)$ so that

$$\partial\bar{\partial}F = \mathrm{Ric}(\omega) - \mathrm{Ric}(\tilde{\omega}) = -\partial\bar{\partial}\log\det(g_{i\bar{j}}) + \partial\bar{\partial}\log\det(\tilde{g}_{i\bar{j}}).$$

In the last line, the local representation of Ricci curvature is used, see Kobayashi and Nomizu [23].

$$
\begin{aligned}
&\partial\bar{\partial}\left\{ F + \log\det(g_{i\bar{j}}) - \log\det(\tilde{g}_{i\bar{j}}) \right\} = 0 \\
&\qquad \iff \omega^{n-1} \wedge \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\left\{ F + \log\det(g_{i\bar{j}}) - \log\det(\tilde{g}_{i\bar{j}}) \right\} = 0 \\
&\qquad\qquad \iff \quad \Delta\left\{ F + \log\omega^n - \log\tilde{\omega}^n \right\} = 0.
\end{aligned}
$$

M is compact so by the maximum principle $F + \log\omega^n - \log\tilde{\omega}^n$, a harmonic function, must be a constant, say $b \in \mathbb{R}$. Therefore, finding a Kähler metric, $\tilde{\omega} \in [\omega]$ so that $\Gamma = \mathrm{Ric}(\tilde{\omega})$ is equivalent to solving a Monge-Ampère equation:

$$
\begin{cases}
\tilde{\omega}^n = e^{F-b}\omega^n \\
\tilde{\omega} = \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u > 0 \qquad \text{and} \qquad \int_M e^F\omega^n = \mathrm{Vol}(M).
\end{cases} \tag{2.28}
$$

The normalization $\int_M e^F\omega^n = \mathrm{Vol}(M)$ and Stokes' theorem implies that $b = 0$:

$$
\mathrm{Vol}(M) = \int_M e^{F-b}\omega^n = e^b \int_M e^F\omega^n = e^b\mathrm{Vol}(M) \quad \Longrightarrow \quad b = 0.
$$

By Theorem (2.0.1) with $c = 0$, equation (2.28) has a unique mean zero solution $u \in C^\infty(M)$. □

If $c_1(M)$ is positive definite, then it has positive representative $\Gamma \in c_1(M)$. By theorem (2.6.1) there is a metric $\tilde{\omega} \in [\omega]$ for which $\mathrm{Ric}(\tilde{\omega}) = \Gamma > 0$; so ω can be deformed in its class to $\tilde{\omega}$, a metric of positive Ricci curvature. A theorem due to Kobayashi [22] states that Ricci positive compact Kähler manifolds are simply connected. Composing theorem (2.6.1) with Kobayashi's theorem we get:

Theorem 2.6.2. *If (M, ω) is a closed Kähler manifold with positive first Chern class, then M is simply connected.*

Another application of theorem (2.6.1) is the following result.

Theorem 2.6.3. (Calabi's Conjecture) *(M, ω) is a closed Kähler manifold. If $c_1(M)$ is cohomologous to zero or negative definite, then M is a Kähler-Einstein manifold.*

Proof. Suppose $c_1(M) < 0$ (resp. ~ 0) then there is a $(1,1)$-form $\Gamma > 0$ so that $c_1(M) = c \cdot [\Gamma]$ where $c = -1$ (resp. $c = 0$). Γ is a positive $(1,1)$-form. Moreover, (M, Γ) is a Kähler manifold with $c_1(M) = c \cdot [\Gamma]$. So it can be assumed that $c_1(M) = c \cdot [\omega]$. By theorem (2.6.1) there is a positive form, $\tilde{\omega} = \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u > 0$, so that $\mathrm{Ric}(\tilde{\omega}) = c\omega$. In the case $c = 0$, $\mathrm{Ric}(\tilde{\omega}) = 0$ and $\tilde{\omega}$ is a flat Kähler-Einstein metric.

To understand the $c = -1$ case start by deriving the associated Monge-Ampère equation from the curvature relation $\mathrm{Ric}(\tilde{\omega}) = c\tilde{\omega}$.

$$
\begin{aligned}
\mathrm{Ric}(\omega) - \mathrm{Ric}(\tilde{\omega}) &= \mathrm{Ric}(\omega) - c\tilde{\omega} \\
&= \mathrm{Ric}(\omega) - c\omega - c\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u \\
&= \partial\bar{\partial}\Big(F - cu\Big).
\end{aligned}
\tag{2.29}
$$

The assumption that $c\omega \in c_1(M)$ is used in the second line of (2.29). From the local formulas of $\mathrm{Ric}(\tilde{\omega})$ and $\mathrm{Ric}(\omega)$ we get the following Monge-Ampère equation:

$$\begin{cases} \tilde{\omega}^n = e^{F-cu}\omega^n \\ \tilde{\omega} = \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u > 0 \quad \text{and} \quad \int_M e^{F-cu}\omega^n = \mathrm{Vol}(M). \end{cases} \tag{2.30}$$

By Theorem (2.0.1), equation (2.30) has a unique mean zero solution $u \in C^\infty(M)$ when $c = -1$.

□

Remark. *In addition to formulating the conjecture, Calabi also proved that solutions to the associated Monge-Ampère equation are unique. Little progress was made until Aubin [2] and Yau [33]* $c_1(M)$ *independently proved the negative definite case. Yau proved the* $c_1(M) \sim 0$ *case [33]. In both instances there is an alternative proof due to Cao [7], that uses Ricci flow to deform the initial metric,* ω*, to a solution metric* ω_∞*.*

When $c_1(M)$ is positive the Calabi conjecture is false, because there are known examples of Kähler surfaces with positive first Chern class that do not admit Kähler-Einstein metrics (*e.g.* $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$). In fact, the continuity method and *a priori* estimates derived in the previous sections fail in this case. When $c_1(M) > 0$ the Monge-Ampère equation is

$$\tilde{\omega}^n = e^{f-u}\omega^n \quad \text{and it is assumed that} \quad \int_M \tilde{\omega}^n = \int_M e^{f-u}\omega^n. \tag{2.31}$$

A uniform supremum bound for

$$\int_M \tilde{\omega}^n - \omega^n = \int_M \left(e^{f-u} - 1\right)\omega^n \tag{2.32}$$

is a required element of the Moser iteration technique used in section (2.4). Since there is no *a priori* infimum estimate for u, e^{f-u} does not have an *a priori* supremum bound–without which there is no upper bound for the right-hand-side of (2.31). Also there is no bound on the right-hand-side of (2.32), so the Moser iteration technique fails. Since the C^2 estimates obtained in section (2.3) require *a priori* estimates for $\|u\|_\infty$, the regularity and existence arguments described in this chapter do not work. Though, the failure of the continuity method when $c_1(M) > 0$ does not necessarily preclude the existence of Kähler-Einstein metrics for certain subclasses of Kähler manifolds with positive first Chern class.

CHAPTER 3
DEGENERATE MONGE-AMPÈRE EQUATIONS

Let (M, ω) be a Kähler manifold. Again consider the class of Monge-Ampère equations:

$$\left(\alpha + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\right)^n = F(u, \nabla u)\Omega. \tag{3.1}$$

Equations of the type (3.1) for which $[\alpha]$ is merely non-negative and $F \in L^p(M)$, for $p > 1$, will be called degenerate. Lately there has been much interest in such equations, especially in connection with finding canonical metrics; for instance, constructing singular Kähler-Einstein metrics via Kähler-Ricci flow. The main focus of this chapter will be to apply known regularity and existence techniques for degenerate Monge-Ampère equations to find a singular canonical metric determined by

$$\omega \wedge \left(\chi + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\right) = c\left(\chi + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\right)^2 \quad \text{and} \quad [2c\chi - \omega] \geq 0. \tag{3.2}$$

Our analysis begins by converting (3.2) into a Monge-Ampère equation. Set $\tilde{\chi} := \chi + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u$. In terms of $\tilde{\chi}$, (3.2) is

$$0 = c\tilde{\chi}^2 - \omega \wedge \tilde{\chi} = c\left(\tilde{\chi} - \frac{\omega}{2c}\right)^2 - \frac{\omega^2}{4c}$$

after completing the square. Scale ω by $\frac{1}{2c}$ so the new ω is $\frac{\omega}{2c}$ and we arrive at

$$\left(\chi - \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\right)^2 = \omega^2 \quad \text{where } [\chi - \omega] \geq 0. \tag{3.3}$$

Equation (3.3) is a degenerate Monge-Ampère equation, because the initial data, $\chi - \omega$, is semipositive. Put another way, the eigenvalues of $\chi - \omega$ lie in the boundary of the cone of ellipticity of equation (3.3). In this chapter the following result is proved.

Theorem 3.0.4. *Let M^2 be a closed Kähler surface with two Kähler classes $[\omega]$ and $[\chi]$.*

$$c = \frac{[\chi] \cdot [\omega]}{[\chi]^2}$$

is a constant of integration. If $[2c\chi - \omega] \geq 0$ is semipositive and not Kähler, then

$$\omega \wedge \Big(\chi + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\Big) = c\Big(\chi + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\Big)^2 \tag{3.4}$$

has a unique bounded solution, u, in the sense of currents that is smooth away from $E \subsetneq M$, a finite union of irreducible curves of negative self-intersection.

$$E = \bigcup_{j=0}^{m} E_j \quad \textit{and } E_j \cdot E_j = -1 \quad \textit{for} \quad j = 1 \ldots m.$$

As stated in the introduction much of the proof rests on techniques from pluripotential theory, which were originally used to study degenerate Monge-Ampère equations. These newer ideas are used to prove existence and uniqueness of weak solutions. In chapter (2) a compactness result, *Arzelá-Ascoli*, was used to prove existence. By contrast, something like the Perron method (see [21] or [17]) is used to prove existence for degenerate equations of the type studied in this chapter. Essentially the unique solution will be constructed by taking the upper envelope of a set of plurisubharmonic functions. Also, in this chapter a very different L^∞ estimate is proved; it facilitates the proof of existence for weak solutions and is used later to obtain higher regularity. All of these ideas are the content of sections (3.1) and (3.2).

The same strategy employed in chapter (2) to obtain higher regularity for solutions, C^2 estimates plus Evans-Krylov and Schauder estimates (the bootstrap technique) is used here; and this is covered in section (3.3). As was the case for non-degenerate Monge-Ampére equations the C^2 estimates require *a priori* L^∞ estimates. Therefore, sections (3.1) and (3.2), and section (3.3) are highly dependent.

3.1 L^∞ Estimates

In this section we review the known methods for estimating the L^∞ norm of solutions to Monge-Ampère equations with semipositive initial data. The following material has been drawn from Demailly and Pali [10], Eyssidieux *et al.* [15], and Kołodziej [24]. Suppose γ is a smooth, semipositive, and *big* $(1,1)$-form and Ω is a smooth volume form on M. Instead of working directly with (3.3), we will consider degenerate Monge-Ampère equations of the type:

$$\left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\right)^n = \Omega. \tag{3.5}$$

Equation (3.3) falls into this class of equations. Here the initial data, γ, lies in the boundary of the cone of positive $(1,1)$-forms. Instead of assuming that $u \in C^4(M)$, as was the case in chapter 2 we consider weak solutions. Equation (3.5) is perturbed with ω, a positive $(1,1)$-form. The resulting family of equations is

$$\begin{cases} \left(\gamma + t\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_t\right)^n = F_t\Omega \\ F_t := \int_M (\gamma + t\omega)^n \Big/ \int_M \Omega. \end{cases} \tag{$\star_t$}$$

By Yau [33], these equations admit smooth solutions, which are denoted u_t. As pluripotential theory plays a necessary role in the derivation of L^∞ bounds for solutions to $(\star_t)$, the relevant concepts and theorems from this theory are reviewed below.

Definition 3.1.1 (*Currents*)**.** A continuous linear functional on $\mathcal{A}_c^{(n-p,n-p)}(M)$, the space of smooth and compactly supported $(n-p,n-p)$-forms, is a **current** of *type* (p,p) or *bidegree* $(n-p,n-p)$. T, a current, is *real* if $\overline{T(\varphi)} = T(\overline{\varphi})$ for all $\varphi \in \mathcal{A}_c^{(n-p,n-p)}(M)$. A real current is *positive* (resp. *non-negative*) if

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^p T(\eta \wedge \bar{\eta}) > 0 \text{ (resp. } \geq 0) \quad \forall \eta \in \mathcal{A}_c^{(p,0)}.$$

Let $I, J \subset \{1, 2, \dots, n\}$. I^c and J^c are complements of I and J, respectively. T, a current, has the following local representation.

$$T = \sum_{I,J} T_{IJ} dz^I \wedge d\bar{z}^J.$$

Each coefficient

$$T_{IJ}(\varphi) := T(\varphi dz^{I^c} \wedge d\bar{z}^{J^c}), \quad \text{where} \quad \varphi \in C_c^\infty(M),$$

is a distribution. The space of (p,p)-currents is equipped with the weak* star topology. So, $\lim_j T_j = T$ *iff* $\lim_j T_j(\varphi) = T(\varphi)$ for each $\varphi \in \mathcal{A}_c^{(n-p,n-p)}(M)$.

Examples.

1. Any smooth form on M is a current. In particular, ω^{n-p} is a smooth current of *bidegree* (p,p).

2. On $\mathbb{C}^n$, $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|z|^2$ is a $(1,1)$-current.

3. $V \subseteq M$ is subvariety of dimension $k \leq n$.

$$\mathcal{A}_c^{(k,k)}(M) \ni \varphi \longmapsto \int_V \varphi$$

   is a current of integration, and it is sometimes denoted by $[V]$.

4. $$\mathrm{PSH}(\gamma) := \left\{ u : M \to [-\infty, \infty) \text{ u.s.c. } \mid \gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u \geq 0 \text{ as a current} \right\}$$

   is the set of γ-plurisubharmonic functions.

Remark. *Currents can be approximated by smooth forms. This is done by mollifying the coefficients of T. Suppose that $\rho \in C_c^\infty(B_r(0))$ is a radially symmetric function and $\int_{B_r} \rho = 1$. Define $\rho_\epsilon(z) := \epsilon^{2n}\rho(\frac{z}{\epsilon})$ and $T_\epsilon(z) := T(\rho_\epsilon(z-w))$. The smoothed coefficients of T, $\{T_{IJ}^\epsilon\}$, are decreasing sequences and $T_\epsilon \to T$ in the sense of currents. The value of smoothing T is that many times statements about currents can be proved by working with sequences of smooth forms and taking limits.*

The next lemma establishes two important *a priori* estimates for plurisubharmonic functions which will be used later.

Lemma 3.1.1.

1. *There are constants* $\alpha > 0$ *and* $C = C(M, \omega)$ *so that*

$$\int_M \exp\{-\alpha(\varphi - \sup_{x \in M} \varphi)\}\omega^n \leq C \qquad \forall \varphi \in PSH(\omega). \tag{3.6}$$

2. *Suppose that* $\gamma, \omega \in H^{1,1}_{DR}(M)$ *and* γ *is semipositive and* ω *is strictly positive. Then all* γ*-plurisubharmonic functions are uniformly bounded above.*

Proof. The first estimate is due to Tian, lemma 2.1 of [31], which is an extension of lemma 4.4 in Hörmander [19]. The statement in [31] is for C^2, ω-plurisubharmonic functions. To show that (3.6) holds for all functions in PSH(ω) take a sequence of smooth functions $\{\varphi_\epsilon\} \subset$ PSH(ω) so that $\varphi_\epsilon \searrow \varphi$ as $\epsilon \to 0$, see [5]. Then by Tian's L^1 estimate for C^2, ω-plurisubharmonic functions,

$$\int_M \exp\{-\alpha(\varphi_\epsilon - \sup_{x \in M} \varphi_\epsilon)\}\omega^n \leq C \qquad \forall \epsilon > 0.$$

The Lebesgue monotone convergence theorem (see [28]) implies

$$\lim_{\epsilon \to 0} \int_M \exp\{-\alpha(\varphi_\epsilon - \sup_{x \in M} \varphi_\epsilon)\}\omega^n = \int_M \exp\{-\alpha(\varphi - \sup_{x \in M} \varphi)\}\omega^n \leq C.$$

Proof of #2: Again consider the smooth sequence $\varphi_\epsilon \searrow \varphi$ with the added property that each φ_ϵ is mean zero. By the hypothesis $\gamma, \omega \in H^{1,1}_{DR}(M)$ and γ is semipositive and $\omega > 0$ is strictly positive. Then $\varphi_\epsilon \in \mathrm{PSH}(\gamma)$ implies $\Delta^\omega \varphi_\epsilon \geq -\sup_{x\in M} \mathrm{tr}\{\omega^{-1}\gamma\} = -C'$. $G(p,q)$ is the Green's function of Δ^ω. Since M is compact $G(p,q)$ has a minimum: there is a constant $K > 0$ for which $G(p,q) \geq -K$ for all $(p,q) \in M \times M$.

$$\begin{aligned}\varphi_\epsilon(p) &= -\int_M G(p,q)\Delta^\omega \varphi_\epsilon(q)\, dV(q) = -\int_M (G(p,q)+K)\Delta^\omega \varphi_\epsilon(q)\, dV(q)\\ &\leq C' \int_M (G(p,q)+K)\, dV(q) \qquad \text{because } -\Delta^\omega \varphi_\epsilon \leq C'.\end{aligned}$$

Since $\varphi_\epsilon \searrow \varphi$ pointwise,

$$\varphi(p) = \lim_{\epsilon\to 0} \varphi_\epsilon(p) \leq C' \cdot \sup_{p\in M} \int_M (G(p,q)+K)\, dV(q).$$

The right-hand-side of the inequality above is a uniform supremum estimate for functions in $\mathrm{PSH}(\gamma)$. □

Comparison Principle. *If $\gamma \in H^{1,1}_{DR}(M)$ is big and semipositive, and φ, $\psi \in$ PSH(γ) on M, then*

$$\int_{\{\varphi<\psi\}} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\psi\right)^n \leq \int_{\{\varphi<\psi\}} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi\right)^n.$$

Proof. For a proof of the comparison principle see Demailly and Pali [10]. □

Currents have an intrinsic (local) norm. Suppose E is a Borel set in a chart (U, z), where $z = (z_1, \ldots, z_n)$ are local coordinates on U. Then

$$\|T\|_E := \sum_{I,J} |T_{IJ}|,$$

where $|T_{IJ}|$ is the total variation of T_{IJ} as a measure.

Lemma 3.1.2. *Let T be a positive (p,p)-current and $K \subset\subset (U,z)$ be a compact set. Then*

$$\|T\|_K \leq C \int_K T \wedge \beta^{n-p} \quad \textit{where } \beta = \partial\bar{\partial}|z|^2,$$

and $C = C(n,p)$.

Proof. The Riesz representation theorem associates a measure, μ_{IJ}, to T_{IJ}. The measure of compact sets, such as K, by μ_{IJ} is

$$\mu_{IJ}(K) := \inf\left\{ T(f dz^{I^c} \wedge d\bar{z}^{J^c}) \;\mid\; f \in C^\infty(U) \text{ and } \operatorname{supp}(f) \subseteq K \text{ and } 0 \leq f \leq 1 \right\},$$

see the proof of the Riesz representation theorem in Rudin [28]. χ_K is the indicator function for K.

$$\begin{aligned}
\|T_{IJ}\| &= \sum_{I,J} \mu_{IJ}(K) \leq \sum_{I,J} T(\chi_K \; dz^{I^c} \wedge d\bar{z}^{J^c}) \\
&= \left(\binom{n}{n-p} \right)^{-1} T(\chi_K \beta^{n-p}) \\
&\leq \int_K T \wedge \beta^{n-p}.
\end{aligned}$$

Here, $T \wedge \beta^{n-p}$ is interpreted to be its associated measure given by the Riesz representation theorem. □

Chern-Levine-Nirenberg. *Suppose $K \subset\subset (U, z)$ is compact and U is a bounded domain with compact closure. Also, $u_0, \dots, u_p \in PSH(\gamma) \bigcap L^\infty(M)$ for $p \le n$, and T is a closed and positive current and $\gamma \ge 0$. Then the following inequalities are valid for some constant $C = C(K, U)$.*

(a)

$$\|(\frac{\sqrt{-1}}{2\pi})^p \partial\bar{\partial} u_1 \wedge \partial\bar{\partial} u_2 \wedge \dots \wedge \partial\bar{\partial} u_p \wedge T\|_K$$
$$\le C\|u_1\|_{L^\infty(U)}\|u_2\|_{L^\infty(U)} \cdots \|u_p\|_{L^\infty(U)}\|T\|_U$$

(b)

$$\|u_0(\frac{\sqrt{-1}}{2\pi})^p \partial\bar{\partial} u_1 \wedge \partial\bar{\partial} u_2 \wedge \dots \wedge \partial\bar{\partial} u_p \wedge T\|_K$$
$$\le C\|u_0\|_{L^1(U)}\|u_2\|_{L^\infty(U)} \cdots \|u_p\|_{L^\infty(U)}\|T\|_U$$

Proof. Choose $\rho \in C_0^\infty(U)$ so that $0 \le \rho \le 1$ and $K \subseteq \text{supp}(\rho) \subseteq U$. Also choose an increasing sequence of compact sets $K = K_1 \subset K_2 \subset \dots \subset K_n \subset U$. Again, $\beta = \partial\bar{\partial}|z|^2$.

$$\begin{aligned} \|\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} u_1 \wedge T\|_K &\le \int_K \rho \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} u_1 \wedge T \wedge \beta^{n-p-1}, \text{ by lemma (3.1.2)},\\ &\le \int_U u_1 \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\rho \wedge T \wedge \beta^{n-p-1}, \text{ by Stokes' Theorem for currents},\\ &\le C\|u_1\|_{L^\infty(U)}\|T\|_U. \end{aligned}$$

Here C is dependent on the second derivatives of ρ. Repeating the above argument p times yields inequality (a). Cover K with a finite number of balls $\{B_{R_j}(a_j)\}_{j \le N}$

contained in coordinate patch (U, z). For notational convenience set $B_j = B_{R_j}(a_j)$ for $j \leq N$. An exhaustion function of B_j is a plurisubharmonic function h with the property $\lim_{x \to \partial B_J} h = +\infty$. For example, $h = -\log\{R_j^2 - |z - a_j|^2\}$ is of an exhaustion function for B_j. In general, any domain in $\mathbb{C}^n$ that admits an exhaustion function is called pseudoconvex. Let $h \in C^\infty(B_j)$ be an exhaustion function for B_j. The functions $u_1, \ldots, u_p$ are bounded, so after subtracting a suitably large constant from each function it is assumed that each u_i is non-positive. Since h blows up near the boundary, h restricted to $B_{\epsilon R_j}(a_j) \subsetneq B_j$ for some $\epsilon < 1$, will take its maximum on $\partial B_{\epsilon R_j}(a_j)$. h is known, so a suitably large constant subtracted from h ensures that $h \leq u_i \ \ \forall i \leq p$ on $B_{\epsilon R_j}(a_j) \subsetneq B_j$. After setting $u_i = \max\{h, u_i\}$, $h = u_i$ near the boundary of B_j, because h blows-up at the boundary.

$$
\begin{aligned}
\|u_0 \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} u_1 \wedge T\|_{B_j \bigcap K} &\leq -\int_{B_j} u_0 \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} u_1 \wedge T \wedge \beta^{n-p-1}, \text{ by lemma (3.1.2)}, \\
&\leq -\int_{B_j} u_0 \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}(u_1 - h) \wedge T \wedge \beta^{n-p-1} \\
&\quad - \int_{B_j} u_0 \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} h \wedge T \wedge \beta^{n-p-1} \\
&\leq -\int_{B_j} (u_1 - h) \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} u_0 \wedge T \wedge \beta^{n-p-1} \\
&\quad - \int_{B_j} u_0 \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} h \wedge T \wedge \beta^{n-p-1} \\
&\leq -\int_{B_j} u_0 \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} h \wedge T \wedge \beta^{n-q-1}.
\end{aligned}
$$

In the last line we used the fact that $0 < \gamma \leq -\sqrt{-1}\partial\bar{\partial} u$ and $u_i - h \geq 0$ on

B_i. Successive applications of the above inequality give

$$
\begin{aligned}
&-\int_{B_j} u_0(\frac{\sqrt{-1}}{2\pi})^p\partial\bar{\partial}u_1\wedge\cdots\wedge\partial\bar{\partial}u_p\wedge T\wedge\beta^{n-p-q}\\
&\qquad\leq -\int_U u_0(\frac{\sqrt{-1}}{2\pi})^p(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}h)^p\wedge T\wedge\beta^{n-q-1}\leq C\cdot\|T\|_U\int_U|u_0|\beta^n.
\end{aligned}
$$

Here C is dependent on the second derivatives of h and the supremum norms of u_i for each $i\leq p$. After summing over j, inequality (b) is obtained. □

Definition 3.1.2. The Monge-Ampére capacity of a Borel set $K\subset M$ with respect to γ, a semipositive and big $(1,1)$-form, is

$$
\mathrm{Cap}_\gamma(K) := \sup\left\{\int_K(\gamma+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u)^n \;\mid\; u\in\mathrm{PSH}(\gamma)\text{ and }0\leq u\leq 1\right\}.
$$

The next proposition collects some facts about this capacity.

Proposition 3.1.1.

Suppose A and B are Borel sets in M.

(a) If $A\subseteq B$, then $Cap_\gamma(A)\leq Cap_\gamma(B)$.

(b) $Cap_\gamma(A\cup B)\leq Cap_\gamma(A)+Cap_\gamma(B)$.

(c) Suppose $\{E_j\}$ is an increasing sequence of Borel sets and $E=\bigcup_{i=1}^\infty E_j$, then

$$
Cap_\gamma(E)=\lim_{j\to\infty}Cap_\gamma(E_j).
$$

Proof. Let $v \in \mathrm{PSH}(\gamma)$ and $0 \leq v \leq 1$. By the definition of capacity and $A \subseteq B$,

$$\int_A \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}v\right)^n \leq \int_B \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}v\right)^n \quad \Longrightarrow \quad \mathrm{Cap}_\gamma(A) \leq \mathrm{Cap}_\gamma(B).$$

Similar reasoning proves (b).

$$\int_{A\cup B} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}v\right)^n \leq \int_A \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}v\right)^n + \int_B \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}v\right)^n$$

implies,

$$\mathrm{Cap}_\gamma(A \cup B) = \sup_v \int_{A\bigcup B} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}v\right)^n \leq \mathrm{Cap}_\gamma(A) + \mathrm{Cap}_\gamma(B).$$

The following proves (c):

$$\mathrm{Cap}_\gamma(E) = \sup_v \int_E (\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}v)^n = \sup_{j\ ,\ v} \int_{E_j} (\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}v)^n = \lim_{j\to\infty} \mathrm{Cap}_\gamma(E_j).$$

$\square$

Proposition 3.1.2. *If $\varphi \in PSH(\gamma)$ is normalized so that $sup_M\varphi \leq 0$, then there is a constant $C = C(n, \gamma, M)$ for which*

$$Cap_\gamma\left(\{\varphi < -s\}\right) \leq \frac{C}{s} \qquad \text{where } s \in \mathbb{R}^+.$$

Proof. M has a finite pseudoconvex cover: M is covered by coordinate charts $\{(U_p, z_p)\}_{p\in M}$ for which $z_p(p) = 0$. Assume that $B_1(0) \subset \mathrm{Img}(z_p)$ for every $p \in M$. The preimage of $B_1(0)$ under z_p, hereafter denoted V_p, is pseudoconvex. To see this compose a plurisubharmonic exhaustion function for $B_1(0)$ (*i.e.* $h = -\log\{1 - |z|^2\}$)

with z_p; then $h_p := h \circ z_p$ is an exhaustion function for V_p. Under biholomorphic changes of coordinates h_p remains plurisubharmonic, because positivity of currents is invariant under bi-holomorphic coordinate changes. By compactness of M there is a finite subcover $\{V_\alpha\}$ of $\{V_p\}$. We may further stipulate that $\gamma = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_\alpha$ and $u_\alpha = 0$ on ∂V_α, for some $u_\alpha \in C^\infty(V_\alpha)$. Let $v \in \mathrm{PSH}(\gamma)$ and $0 \le v \le 1$, and normalize φ so that $\sup_M \varphi = 0$. Further suppose $K \subset \{\varphi < -s\}$ is compact and $\{V'_\alpha\}$ is another cover of M, subordinate to $\{V_\alpha\}$ for which $\overline{V'_\alpha} \subset V_\alpha$, then

$$\int_{K\bigcap\{\varphi<-s\}} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}v\right)^n \le \sum_\alpha \left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_{K\bigcap\overline{V'_\alpha}} \left(\partial\bar{\partial}\left(u_\alpha + v\right)\right)^n.$$

$K\bigcap\overline{V'_\alpha} \subseteq \{\varphi < -s\}$ implies

$$\begin{aligned}\int_{K\bigcap\overline{V'_\alpha}} \left(\frac{\sqrt{-1}}{2\pi}\right)^n \left(\partial\bar{\partial}\left(u_\alpha + v\right)\right)^n &\le \frac{1}{s}\left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_{K\bigcap\overline{V'_\alpha}} -\varphi\left(\partial\bar{\partial}\left(u_\alpha + v\right)\right)^n \\ &\le C_1 \cdot \frac{\|\varphi\|_{L^1(M)}}{s}\ \|u_\alpha + v\|^n_{L^\infty(V_\alpha)}.\end{aligned}$$

The proof of part (b) of the Chern-Levine-Nirenberg inequalities justifies the last inequality. Summing over α gives

$$\int_{K\bigcap\{\varphi<-s\}} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}v\right)^n \le C_1\frac{\|\varphi\|_1}{s}.$$

Here C is dependent on γ and the cover $\{V_\alpha\}$. Monge-Ampére capacity is inner-regular [8], so

$$\mathrm{Cap}_\gamma\left(\{\varphi < -s\}\right) = \sup_{K\subset\{\varphi<-s\}} \mathrm{Cap}_\gamma\left(K \cap \{\varphi < -s\}\right).$$

Because the constant C_1 is independent of K, we immediately get

$$\mathrm{Cap}_\gamma\left(\{\varphi < -s\}\right) \leq C_1 \frac{\|\varphi\|_1}{s}.$$

The proof is completed by an application of Jensen's inequality to (3.6),

$$\log C \geq \log\left(\int_M \exp\left\{-\alpha\left(\varphi - \sup_{x\in M}\varphi\right)\right\}\gamma^n\right) \geq \int_M -\alpha\left(\varphi - \sup_{x\in M}\varphi\right).$$

When combined with the hypothesis $\sup_{x\in M}\varphi \leq 0$ we get an *a priori* L^1 bound,

$$\frac{1}{\alpha}\log C_2 \geq \int_M -\varphi\,\omega^n = \|\varphi\|_1.$$

□

Proposition 3.1.3. *Let $\varphi \in PSH(\gamma)\bigcap L^\infty(M)$, then for all $s > 0$ and $0 \leq r \leq 1$*

$$r^n\, Cap_\gamma(\{\varphi < -s - r\}) \leq \int_{\{\varphi<-s\}} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^n.$$

Proof. Suppose $v \in \mathrm{PSH}(\gamma)$ and $0 \leq v \leq 1$.

$$\begin{aligned} r^n \int_{\{\varphi<-s-r\}} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}v\right)^n &= \int_{\{\varphi<-s-r\}} \left(r\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}rv\right)^n \\ &\leq \int_{\{\varphi<-s-r\}} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}rv\right)^n, \end{aligned}$$

because $r\gamma \leq \gamma$. Since $\{\varphi < -s-r\} \subseteq \{\varphi < -s-r+rv\}$,

$$\int_{\{\varphi<-s-r\}} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} rv\right)^n \leq \int_{\{\varphi<-s-r+rv\}} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(rv-s-r)\right)^n.$$

By the comparison principle,

$$\int_{\{\varphi<-s-r+rv\}} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(rv-s-r)\right)^n \leq \int_{\{\varphi<-s-r+rv\}} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^n.$$

Finally, $\{\varphi < -s-r+rv\} \subseteq \{\varphi < -s\}$ implies

$$\int_{\{\varphi<-s-r+rv\}} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^n \leq \int_{\{\varphi<-s\}} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^n.$$

$\square$

Proposition 3.1.4. *Suppose $U \subseteq M$ is an open set and Ω is a smooth volume form. Then*

$$\int_U \Omega \leq C \cdot \exp\left\{-\alpha \left[\frac{[\gamma]^n}{Cap_\gamma(U)}\right]^{\frac{1}{n}}\right\}.$$

Here $\alpha = \alpha(M)$ is the α invariant of M and $C = C(\alpha, PSH(\gamma), M)$.

Proof. Consider the following extremal function:

$$\Psi_U(x) := \sup\left\{\varphi(x) \;\mid\; \varphi \in \mathrm{PSH}(\gamma) \text{ and } \varphi|_U \leq 0\right\}.$$

$\Psi_U \geq 0$ everywhere on M and $\Psi_U \equiv 0$ on U, because $0 \in \mathrm{PSH}(\gamma)$. The supremum of a collection of plurisubharmonic functions is plurisubharmonic, so $\Psi_U \in \mathrm{PSH}(\gamma)$. By lemma (3.1.1) Ψ_U is bounded, so $\Psi_U \in L^\infty(M)$. However, Ψ_U may not be upper semicontinuous. Instead consider the upper semicontinuous regularization of Ψ_U, which is defined $\Psi_U^* := \limsup_{y\to x} \Psi_U(y)$. Ψ_U^* is non-negative and $\Psi_U^* \in L^\infty(M)$.

Also, Ψ_U^* is a γ-plurisubharmonic function, because it satisfies a maximum principle. Let v be a smooth potential function for γ in some open set $V \subset M$ and $h + v$ is harmonic with respect to the metric Laplacian. Ψ_U is a γ-plurisubharmonic function *iff* for every compact set $K \subset\subset V$ and every harmonic function $v + h$, the following is true:

$$v + \Psi_U \leq h + v \quad \text{on } \partial K \qquad \Longrightarrow \qquad v + \Psi_U \leq h + v \quad \text{on } K. \tag{3.7}$$

The usually maximum principle for $C^2(M) \bigcap \mathrm{PSH}(\gamma)$ is equivalent to condition (3.7), see Klimek [21]. By the definition of Ψ_U^*,

$$\begin{aligned} v(a) + \Psi_U^*(a) = \limsup_{x \to a} \ (v(x) + \Psi_U(x)) \leq v(a) + \limsup_{x \to a} \ h(x) = v(a) + h(a) \\ \Longrightarrow \quad v + \Psi_U^* \leq h + v. \end{aligned}$$

So, the inequalities stated in (3.7) are preserved by lim sup, and $\Psi_U^* \in \mathrm{PSH}(\gamma)$. For a more detailed discussion of Ψ_U and its upper semicontinuous regularization see Klimek [21]. A well known corollary to a theorem of Bedford and Taylor [4] shows that Ψ_U^* is the homogeneous solution to: $\left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\right)^n = 0$ on $M - \bar{U}$. We may now proceed with the proof of the proposition. By equation(3.6) of lemma (3.1.1) there are two positive constants C and α so that

$$\int_U \Omega = \int_U \exp\{-\alpha\Psi_U^*\}\ \Omega, \qquad \text{because } \Psi_U^* \equiv 0 \text{ on } U,$$
$$\leq \exp\left\{-\alpha \sup_{x\in M} \Psi_U^*\right\} \cdot \int_M \exp\left\{-\alpha\left(\Psi_U^* - \sup_{x\in M}\Psi_U^*\right)\right\}\ \Omega$$
$$\leq C \cdot \exp\left\{-\alpha \sup_{x\in M} \Psi_U^*\right\}. \tag{3.8}$$

Set $A := \sup_{x\in M} \Psi_U^*$. If $A > 1$, the definition of Monge-Ampère capacity and the identity $\left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\Psi_U^*\right)^n = 0$ on $M - \bar{U}$ lead to

$$\begin{aligned}\mathrm{Cap}_\gamma\left(\bar{U}\right) &\geq \int_{\bar{U}} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\frac{\Psi_U^*}{A}\right)^n = \left(\frac{1}{A}\right)^n \cdot \int_{\bar{U}} \left(A\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\Psi_U^*\right)^n \\ &\geq \left(\frac{1}{A}\right)^n \cdot \int_{\bar{U}} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\Psi_U^*\right)^n + \left(\frac{1}{A}\right)^n \cdot \int_{M-\bar{U}} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\Psi_U^*\right)^n \\ &= \left(\frac{1}{A}\right)^n \cdot \int_M \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\Psi_U^*\right)^n \\ &\geq \frac{[\gamma]^n}{A^n}.\end{aligned}$$

Then

$$A = \sup_{x\in M} \Psi_U^* \geq \left(\frac{[\gamma]^n}{\mathrm{Cap}_\gamma\left(\bar{U}\right)}\right)^{\frac{1}{n}},$$

which implies

$$\exp\left\{-\alpha \sup_{x\in M} \Psi_U^*\right\} \leq e^\alpha \cdot \exp\left\{-\alpha\left[\frac{[\gamma]^n}{\mathrm{Cap}_\gamma(\bar{U})}\right]^{\frac{1}{n}}\right\};$$

which, when combined with (3.8), proves the lemma. Consider the other case, $A \leq 1$. By Stokes' theorem $[\gamma]^{-n}\mathrm{Cap}_\gamma(\bar{U}) \leq 1$. Again, using the definition of capacity we have

$$[\gamma]^{-n} \cdot \mathrm{Cap}_\gamma\left(\bar{U}\right) \geq [\gamma]^{-n} \int_{\bar{U}} \left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\Psi_U^*\right)^n = 1 \qquad \Longrightarrow \qquad \frac{[\gamma]^n}{\mathrm{Cap}_\gamma\left(\bar{U}\right)} = 1.$$

Furthermore, $e^{-\alpha A} \leq 1$ implies

$$\exp\left\{-\alpha \sup_{x\in M} \Psi_U^*\right\} \leq e^{\alpha} \cdot \exp\left\{-\alpha \left[\frac{[\gamma]^n}{\mathrm{Cap}_\gamma(\bar{U})}\right]^{\frac{1}{n}}\right\},$$

which, when combined with (3.8), gives the proposition. □

We now have the tools to uniformly bound $\{u_t\}$, the solutions to $(\star_t)$. The following lemma is crucial.

Lemma 3.1.3. *$f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a decreasing right-continuous function with the following properties:*

1. $\lim_{s\to\infty} f(s) = 0$.
2. *$\exists \alpha, A > 0$ so that* $\quad rf(s+r) \leq Af(s)^{1+\alpha} \quad \forall s > 0$ *and* $0 \leq r < 1$.

Then there exists a constant $S_\infty = S_\infty(\alpha, A, r)$ so that $f(s) = 0$ for all $s \geq S_\infty$.

Proof. After taking the logarithm of the inequality in #2 of the hypothesis,

$$\log f(s+r) \leq \log A - \log r + (1+\alpha)\log f(s). \tag{3.9}$$

Fix $r < 1$ and iterate inequality (3.9) to get

$$\begin{aligned}
\log f(s + \sum_{j=1}^{k} r^j) &\le \log A - k \log r + (1+\alpha) \log f(s + \sum_{j=1}^{k-1} r^j) \\
&\le [1 + (1+\alpha)] \log A - [k + (k-1)(1+\alpha)] \log r \\
&\qquad\qquad + (1+\alpha) \log f(s + \sum_{j=1}^{k-2} r^j) \\
&\vdots \\
&\le \sum_{j=0}^{k-1} (1+\alpha)^j \log A - [(1+\alpha) \sum_{j=0}^{k-1} (k-j)(1+\alpha)^{j-1}] \log r \\
&\qquad\qquad + (1+\alpha)^k \log f(s).
\end{aligned}$$

The final inequality is the one of interest, so it is repeated below.

$$\begin{aligned}
\log f(s + \sum_{j=1}^{k} r^j) \le \sum_{j=0}^{k-1} (1+\alpha)^j \log A \\
- [(1+\alpha) \sum_{j=0}^{k} (k-j)(1+\alpha)^{j-1}] \log r + (1+\alpha)^k \log f(s).
\end{aligned} \tag{3.10}$$

The summations in the first two terms of equation (3.10) have closed form expressions. Recall the following formula.

$$\sum_{j=0}^{l} x^j = \frac{x^{l+1} - 1}{x - 1}. \tag{3.11}$$

Also,

$$\frac{d}{dx}\Big(\sum_{j=0}^{l} x^j\Big) = \sum_{j=0}^{l} jx^{j-1}$$

has the same form as the coefficient of $\log r$ in inequality (3.10). By differentiating the left-hand-side of (3.11),

$$\begin{aligned}\sum_{j=1}^{l} jx^{j-1} &= \frac{d}{dx}\Big(\frac{x^{l+1}-1}{x-1}\Big) = \frac{(l+1)x^l(x-1)-(x^{l+1}-1)}{(x-1)^2}\\ &= \frac{lx^{l+1}-(l+1)x^l+1}{(x-1)^2}.\end{aligned}$$

Letting $x = 1+\alpha$ we get

$$\sum_{j=0}^{k-1}(1+\alpha)^j = \frac{(1+\alpha)^k-1}{\alpha} \text{ and } \sum_{j=0}^{k-1} j(1+\alpha)^{j-1} = \frac{(k-1)(1+\alpha)^k - k(1+\alpha)^{k-1}+1}{\alpha^2},$$

from which we calculate

$$\begin{aligned}\sum_{j=0}^{k-1}(k-j)(1+\alpha)^j &= k\sum_{j=0}^{k-1}(1+\alpha)^j - (1+\alpha)\sum_{j=1}^{k-1} j(1+\alpha)^{j-1}\\ &= k\frac{(1+\alpha)^k-1}{\alpha} - \frac{(k-1)(1+\alpha)^{k+1}-k(1+\alpha)^k+\alpha+1}{\alpha^2}.\end{aligned}$$

Then (3.10) becomes

$$\begin{aligned}\log f(s+\sum_{j=1}^{k} r^j) \leq{}& \frac{(1+\alpha)^k-1}{\alpha}\log A\\ &+\Big\{\frac{(k-1)(1+\alpha)^{k+1}-k(1+\alpha)^k+\alpha+1}{\alpha^2} - k\frac{(1+\alpha)^k-1}{\alpha}\Big\}\log r\\ &+(1+\alpha)^k \log f(s).\end{aligned}$$

Finally,

$$\log f(s+\sum_{j=1}^{k} r^j) \leq \frac{(1+\alpha)^k}{\alpha}\left\{\log A + \alpha \log f(s) - \frac{1+\alpha}{\alpha}\log r\right\} + \frac{k+1+\alpha}{\alpha}\log r - \frac{1}{\alpha}\log A. \tag{3.12}$$

α, A, and r are constant, so by #1 of the hypothesis there is a $s_0 \in \mathbb{R}^+$ so that

$$\left\{\log A + \alpha \log f(s_0) - \frac{1+\alpha}{\alpha}\log r\right\} < 0.$$

By (3.12),

$$\lim_{k\to\infty} f(s_0+\sum_{j=1}^{k} r^j) = f(s_0+\frac{r}{1-r}) < 0.$$

Therefore, $f(s) = 0$ for all $s > S_\infty = s_0 + \frac{r}{1-r}$. $\square$

Lemma 3.1.4. *$\{u_t\}$ are solutions to $(\star_t)$. There is a constant $C = C(\omega, \gamma, M, n)$ so that $\|u_t - sup_{x\in M} u_t\|_\infty \leq C$ independent of t.*

Proof. Set $\gamma_t := \gamma + t\omega$. If

$$f_t(s) := \left[\mathrm{Cap}_{\gamma_t}(\{u_t < -s\})\right]^{\frac{1}{n}}$$

satisfies the hypothesis of lemma (3.1.3) uniformly in t, then the lemma is proved. Right continuity of f_t follows from (c) of proposition (3.1.1). Note $\mathrm{PSH}(\gamma_t) \subseteq \mathrm{PSH}(\gamma+\omega)$ implies $\mathrm{Cap}_{\gamma_t}(K) \leq \mathrm{Cap}_{\gamma+\omega}(K)$ on any Borel set, which in turn shows that $f_t(s) \leq$

$f_1(s)$ for each $t \in [0,1]$. Since $u_t \in \mathrm{PSH}(\omega+\gamma)$ for all $t \in [0,1]$ and $\sup_{x\in M} u_t$ is uniformly bounded independent of t, by lemma (3.1.1), proposition (3.1.2) implies

$$f_t(s) \le f_1(s) \le \frac{C}{s},$$

where $C = C(\gamma, \omega, M)$. Therefore, $\lim_{s\to\infty} f_t(s) = 0$ uniformly in t, and f_t satisfies #1 of lemma (3.1.3) uniformly.

Now, property #2 of lemma (3.1.3) for $f_t(s)$ is verified. Let $0 \le r \le 1$ and $s \ge 0$, then

$$\begin{aligned}
[rf_t(s+r)]^n &= r^n \mathrm{Cap}_{\gamma_t}\left(\{u_t < -s-r\}\right) \\
&\le \int_{\{u_t<-s\}} \left(\gamma_t + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_t\right)^n, \text{ by proposition (3.1.3),} \\
&= \int_{\{u_t<-s\}} F_t\Omega, \text{ because } u_t \text{ solves } (\star_t), \\
&\le \int_{\{u_t<-s\}} \Omega,
\end{aligned}$$

because $F_t \le 1$ for each $t \in [0,1]$. Proposition (3.1.4) implies

$$[rf_t(s+r)]^n \le C \cdot \exp\left\{-\alpha\left[\frac{[\gamma_t]^n}{\mathrm{Cap}_{\gamma_t}\left(\{u_t<-s\}\right)}\right]^{\frac{1}{n}}\right\}. \tag{3.13}$$

The constant which appears in (3.13) is independent of t. Also, the right-hand-side of (3.13) looks like $\exp\{-\frac{1}{x}\}$ where $x \sim \mathrm{Cap}_{\gamma_t}(\{u_t < -s\})$. There is a constant B for which $Bx^m \ge \exp\{-\frac{1}{x}\}$ and $m > 0$, for a proof see lemma (.0.2). Note that B has no relation to t. With this fact, inequality (3.13) becomes

$$rf_t(s+r) \le \tilde{C} \cdot f_t(s)^{1+\alpha} \quad \forall s > 0\ .$$

$\tilde{C}$ is independent of t. The hypothesis of proposition (3.1.2) is verified, therefore the set of solutions to $(\star_t)$ is uniformly bounded in $L^\infty(M)$.

□

With uniform L^∞ estimates it is now possible to prove existence of solutions, in the sense of current for equation (3.5). We close this section with an existence argument which is based on Kołodziej [25], and a proof of uniqueness.

Lemma 3.1.5. (Existence) *Set $\gamma_j := \gamma + \frac{1}{j}\omega$. By Yau [33] there are functions $u_j \in$ PSH$(\gamma_j)\bigcap C^\infty(M)$, which solve the equations*

$$\begin{cases} \left(\gamma_j + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_j\right)^n = G_j\Omega \\ G_j = [\gamma_j]^n \Big/ \int_M \Omega \qquad \text{and} \qquad [\gamma]^n = \int_M \Omega. \end{cases} \tag{3.14}$$

The sequence $\{G_j\}$ is bounded below by 1 and $\lim_{j\to\infty} G_j = 1$.

Then $u = \left(\limsup_{j\to\infty} u_j\right)^*$ is a bounded solution of

$$\left(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\right)^n = \Omega.$$

Proof. Define a new set of functions, $\psi_k := \left(\sup_{k\le j} u_j\right)^*$. If $l \ge k$ then $\frac{1}{k} \ge \frac{1}{l}$. So, $u_l \in \mathrm{PSH}(\gamma_k)$ for each $l \ge k$. It follows that $\psi_k \in \mathrm{PSH}(\gamma_k)$ and $\psi_k \to u$ uniformly as $k \to \infty$. Moreover,

$$\left(\gamma_k + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_l\right)^n \ge \left(\gamma_l + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_l\right)^n = G_l\Omega \qquad \forall l \ge k\ . \tag{3.15}$$

Given γ-plurisubharmonic functions f and g, their associated Monge-Ampère measures satisfy

$$\Big(\gamma_k + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\max\{u_k, u_{k+1}\}\Big)^n$$
$$\geq \chi_{\{u_k \geq u_{k+1}\}}\Big(\gamma_k + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_k\Big)^n + \chi_{\{u_{k+1}\geq u_k\}}\big(\gamma_k + \partial\bar{\partial}u_{k+1}\big)^n \geq G_{k+1}\Omega,$$

by theorem 1.8 of [25]. The later inequality follows from (3.15). Using induction it can be shown that

$$\Big(\gamma_k + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\psi_k\Big)^n \geq \Omega. \tag{3.16}$$

By lemma (3.1.4) the set $\{\psi_k\}_{k\in\mathbb{Z}^+}$ is uniformly bounded. It suffices to show that u is a local solution. Locally a decreasing sequence of potential functions $\{v_j\}$ can be produced from the potential functions of γ so that $\gamma_j = \partial\bar{\partial}v_j$. Furthermore, these potential functions are uniformly bounded and the sequence $\{v_k + \psi_k\}$ is uniformly bounded and decreasing. Then the Bedford-Taylor [4] monotone convergence theorem combined with the measure inequality (3.16) yields

$$\Omega \leq \lim_{k\to\infty}\big(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\left(v_k + \psi_k\right)\big)^n = \big(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\left(v_\infty + u\right)\big)^n.$$

And the notion of convergence used here is weak convergence. The integral compatibility condition and Stokes' theorem imply $\Omega = \Big(\gamma + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\Big)^n$.

$\square$

3.2 Uniqueness

Lemma 3.2.1. *If $u, v \in PSH(\gamma)\bigcap L^\infty(M)$ both solve* (3.5) *for $n = 2$, then u and v differ by a constant.*

Proof. Set $\theta := u - v$. Recall

$$\alpha^k - \beta^k = (\alpha - \beta) \wedge \sum_{j=1}^{k-1} \alpha^j \wedge \beta^{k-j-1}. \tag{3.17}$$

In the case $k = 2$, Stokes' theorem for currents implies

$$\begin{aligned} 0 &= \int_M \theta \left(\gamma_u^2 - \gamma_v^2\right) = \int_M \theta \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\theta \wedge (\gamma_u + \gamma_v)\,, \text{ by (3.17)}, \\ &= \int_M \frac{\sqrt{-1}}{2\pi} \partial\theta \wedge \bar{\partial}\theta \wedge \gamma_u + \int_M \frac{\sqrt{-1}}{2\pi} \partial\theta \wedge \bar{\partial}\theta \wedge \gamma_v. \end{aligned}$$

Both γ_u and γ_v are semipositive, so each term in the second equality is zero,

$$0 = \int_M \frac{\sqrt{-1}}{2\pi} \partial\theta \wedge \bar{\partial}\theta \wedge \gamma_u \quad \text{and} \quad 0 = \int_M \frac{\sqrt{-1}}{2\pi} \partial\theta \wedge \bar{\partial}\theta \wedge \gamma_v. \tag{3.18}$$

$$\begin{aligned} \int_M \frac{\sqrt{-1}}{2\pi} \partial\theta \wedge \bar{\partial}\theta \wedge \gamma &= \int_M \frac{\sqrt{-1}}{2\pi} \partial\theta \wedge \bar{\partial}\theta \wedge \gamma_u - \int_M \frac{\sqrt{-1}}{2\pi} \partial\theta \wedge \bar{\partial}\theta \wedge \partial\bar{\partial}u \\ &= -\int_M \frac{\sqrt{-1}}{2\pi} \partial\theta \wedge \bar{\partial}u \wedge \partial\bar{\partial}\theta \\ &= \int_M \frac{\sqrt{-1}}{2\pi} \partial\theta \wedge \bar{\partial}u \wedge \partial\bar{\partial}v - \int_M \frac{\sqrt{-1}}{2\pi} \partial\theta \wedge \bar{\partial}u \wedge \partial\bar{\partial}u \\ &= \int_M \frac{\sqrt{-1}}{2\pi} \partial\theta \wedge \bar{\partial}u \wedge \gamma_v - \int_M \frac{\sqrt{-1}}{2\pi} \partial\theta \wedge \bar{\partial}u \wedge \gamma_u. \end{aligned}$$

In the last line $\int_M \frac{\sqrt{-1}}{2\pi} \partial\theta \wedge \bar{\partial}u \wedge \gamma$ is added and subtracted. Let f be u or v. Then the Schwarz inequality [25] implies

$$\int_M \frac{\sqrt{-1}}{2\pi} \partial\theta \wedge \bar{\partial}u \wedge \gamma_f \leq \left(\int_M \frac{\sqrt{-1}}{2\pi} \partial\theta \wedge \bar{\partial}\theta \wedge \gamma_f\right) \cdot \left(\int_M \frac{\sqrt{-1}}{2\pi} \partial u \wedge \bar{\partial}u \wedge \gamma_f\right) = 0.$$

Identity (3.18) is used to get the later equality. Therefore,

$$\int_M \frac{\sqrt{-1}}{2\pi}\partial\theta \wedge \bar{\partial}\theta \wedge \gamma_f = 0.$$

γ_f is semipositive so $\frac{\sqrt{-1}}{2\pi}\partial\theta\wedge\bar{\partial}\theta\wedge\gamma_f = 0$. However, γ_f may be zero on a set of nonzero measure. By lemma 4.1 of Demailly [10], we can assume that γ_f is a Kähler metric away from Z, a complex analytic set (codimension ≥ 1). Z is necessarily pluripolar and measure zero. Plurisubharmonic functions have unique extensions over pluripolar sets [9], so θ is constant everywhere on M. $\square$

3.3 C^2 Estimates & Higher Regularity

To begin, perturb (3.3) by adding $t\omega$ to $\chi - \omega$,

$$\Big(\chi + (t-1)\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_t\Big)^2 = C_t\ \omega^2. \tag{3.19}$$

Determine C_t by integrating both sides of (3.19),

$$\begin{aligned}
\int_M \Big(\chi + (t-1)\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\Big)^2 &= \int_M \Big(\chi - \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\Big)^2 \\
&\quad + \int_M 2\Big(\chi - \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\Big) \wedge t\omega + t^2\omega^2 \\
&= \int_M (1+t^2)\omega^2 + \int_M 2\Big(\chi - \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\Big) \wedge t\omega\ , \\
&\quad \text{by (3.3),} \\
&= \int_M (1+t^2)\omega^2 + 2t\int_M \Big(\chi - \omega\Big) \wedge \omega\ , \\
&\quad \text{by Stokes' Theorem.}
\end{aligned}$$

Then,

$$C_t = (1+t^2) + 2t\frac{[\chi-\omega]\cdot[\omega]}{[\omega]^2}$$
$$= (t-1)^2 + t\frac{[\chi]^2}{[\omega]^2}, \quad \text{because } c = \frac{1}{2}.$$

And the perturbed equations are

$$\begin{cases} \left(\chi + (t-1)\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_t\right)^2 = C_t\ \omega^2 \\ C_t \text{ are constants which are uniformly bounded in } t \in [0,1]. \end{cases} \qquad (\star_t)$$

On Kähler surfaces Donaldson [12] remarked that $[2c\chi - \omega] \geq 0$ (since $c = \frac{1}{2}$ was chosen earlier, $[\chi - \omega] \geq 0$) is equivalent to the existence of curves of negative self-intersection. It is due to Lamari [26] and Buchdahl [6] that there must exist curves of negative self-intersection when the initial data is semipositive and not Kähler. Furthermore, they also prove that there exists a finite number of irreducible curves of negative self-intersection, E, for which the difference of the first Chern class of E and $[2c\chi - \omega]$ is a Kähler class, see also Song and Weinkove [30]. Moreover, the proposition which appears in [30] facilitates the estimation of the second derivatives of weak solutions to (3.4), away from E.

Proposition 3.3.1. (Buchdahl-Larmari) *Let M^2 be a Kähler surface with a Kähler class $\beta \in H^{1,1}(M,\mathbb{R})$. If $\alpha \in H^{1,1}(M,\mathbb{R})$ satisfies*

$$\alpha^2 > 0 \quad \textit{and} \quad \alpha\cdot\beta > 0,$$

then either α is a Kähler class or there exists a positive integer m and curves of negative self-intersection $E_1, \ldots, E_m$ and positive real numbers $a_1, \ldots, a_m$ so that

$$\alpha - \sum_{j=1}^{m} a_j \cdot PD[E_j] > 0$$

is a Kähler class. $PD[E_j]$ represents the Poincaré dual of E_j.

Equations $(\star_t)$ for $t \neq 0$ are nondegenerate Monge-Ampère equations, so by Yau [33] each has a unique solution $u_t \in C^\infty(M)$ $\forall t > 0$. The calculation below shows that $\alpha = [\chi - \omega]$ satisfies the hypothesis of proposition (3.3.1) with $\beta = [\chi]$. As stated earlier $[\chi - \omega]$ is *big*, so $\alpha^2 > 0$.

$$\begin{aligned}
\alpha \cdot \beta &= [\chi]^2 - [\omega] \cdot [\chi] \\
&= 2\frac{[\chi] \cdot [\omega]}{[\chi]^2}[\chi]^2 - [\omega] \cdot [\chi], \text{ by the definition of } c = \frac{1}{2}, \\
&= [\omega] \cdot [\chi] > 0.
\end{aligned}$$

Also,

$$\alpha \cdot \beta = [\chi]^2 - [\omega] \cdot [\chi] \quad \text{ and } \alpha^2 = [\chi]^2 - 2[\omega] \cdot [\chi] + [\omega]^2.$$

Then by proposition (3.3.1),

Fact 1. *There is a $m \in \mathbb{Z}^+$ and irreducible curves of negative self-intersection $E_1, \ldots, E_m$ and positive real numbers $a_1, \ldots, a_m$ so that*

$$[\chi - \omega] - \sum_{j=1}^{m} a_j PD[E_j] > 0.$$

is Kähler. For notational convenience let D and E be

$$D = \sum_{j=1}^{m} a_j E_j \quad \text{and} \quad E = \bigcup_{j=1}^{m} E_j.$$

$c_1(E_j) = PD[E_j]$, for a proof see proposition 1 of chapter 1 in Griffiths and Harris [18]. In particular, there are sections $\sigma_j \in [E_j]$, each necessarily vanishing on E_j with Hermitian metrics h_j, so that

$$a_j c_1(E_j) = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log|\sigma_j|^2_{h_j}$$

as smooth $(1,1)$-forms on $M - E$ and currents everywhere on M. By fact (1) there is a Kähler form $\kappa > 0$ so that

$$\kappa \in [\chi - \omega] - \sum_{j=1}^{m} a_j c_1(E_j) > 0.$$

Moreover, the $\partial\bar{\partial}$-lemma implies that there is a unique $f \in C^\infty(M)$, which is only dependant on κ, χ, ω, and E so that

$$\begin{aligned}
\kappa &= \chi - \omega - \sum_{j=1}^{m} \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log|\sigma_j|^2_{h_j} + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} f \\
&= \chi - \omega - \sum_{j=2}^{m} \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log|\sigma_j|^2_{h_j} + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log|\sigma_1|^2_{h_1} + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log e^f \\
&= \chi - \omega - \sum_{j=2}^{m} \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log|\sigma_j|^2_{h_j} + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log e^f |\sigma_1|^2_{h_1}.
\end{aligned}$$

Set $h_1 := e^f h_1$ to be the Hermitian metric on E_1, therefore

$$\chi - \omega = \kappa + \sum_{j=1}^{m} \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log|\sigma_j|^2_{h_j}$$

as smooth $(1,1)$-forms on $M - E$ and currents everywhere on M. Equations $(\star_t)$ can be rewritten with κ:

$$\begin{cases} \Big(\kappa + t\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\{u_t + \sum_{j=1}^{m} \log|\sigma_j|^2_{h_j}\}\Big)^2 = F_t\Big(\kappa + t\omega\Big)^2 \\ F_t = C_t\omega^2 \Big/ \Big(\kappa + t\omega\Big)^2 \quad \text{uniformly bounded in } t \in [0,1]. \end{cases} \tag{3.20}$$

Here equality is as smooth forms on $M - E$ and currents everywhere on M. Before the required estimates are stated and proved, the notation to be used is codified. Let

$$\kappa_t = \kappa + t\omega \quad \text{ and } \quad \tilde{\kappa}_t = \kappa + t\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\{u_t + \log\prod_{j=1}^{m}|\sigma_j|^2_{h_j}\},$$

then there are real constants $\Lambda, \lambda > 0$ so that

$$\lambda \le \kappa_t \le \Lambda \qquad \forall t \in [0,1].$$

Locally, κ_t and $\tilde{\kappa}_t$ are

$$\kappa_t = \frac{\sqrt{-1}}{2\pi} g^t_{i\bar{j}} dz^i \wedge dz^{\bar{j}} \quad \text{and} \quad \tilde{\kappa}_t = \frac{\sqrt{-1}}{2\pi} \tilde{g}^t_{i\bar{j}} dz^i \wedge dz^{\bar{j}}.$$

The Laplacians that we will encounter are

$$\Delta_t f := g_t^{i\bar{j}} \nabla^t_i \nabla^t_{\bar{j}} f \quad \text{and} \quad \tilde{\Delta}_t f := \tilde{g}_t^{i\bar{j}} \nabla^t_i \nabla^t_{\bar{j}} f.$$

In normal coordinates at $q \in M$, $\nabla^t_i u = \partial_i u := u_i$ and $u_{i\bar{j}} = \partial_i \partial_{\bar{j}} u = \nabla^t_i \nabla^t_{\bar{j}} u$. All covariant derivatives are taken with respect to ∇^t, the unique connection on TM, which is compatible with the Hermitian metric $g^t_{i\bar{j}} dz^i \bigotimes dz^{\bar{j}}$.

R^t, $R^t_{i\bar{j}}$, and $R^t_{i\bar{j}k\bar{l}}$ denote the scalar curvature, Ricci curvature, and Riemannian curvature tensors determined by κ_t, respectively. The κ_t-curvatures are completely dependent on κ_t up to its second partials and continuously dependent on $t \in [0,1]$. Furthermore, the $\{\kappa_t\}$ are known and have positive eigenvalues which are uniformly bounded for all $t \in [0,1]$. It follows that R^t, $R^t_{i\bar{j}}$, and $R^t_{i\bar{j}k\bar{l}}$ are uniformly bounded, which is collected as a fact to be used later.

Fact 2. (Curvature Bounds) *There is a constant* $C_0 = C(\lambda, \Lambda)$, *for which*

$$\sup_M |R^t_{i\bar{j}k\bar{l}}|, \quad \sup_M |R^t_{i\bar{j}}|, \quad \sup_M |R^t| \le C_0 \quad \forall t \in [0,1].$$

Lemma 3.3.1. *Suppose that* u_t *is a weak solution to*

$$\begin{cases} \left(\kappa_t + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\{u_t + \log \prod_{j=1}^m |\sigma_j|^2_{h_j}\}\right)^2 = F_t \kappa_t^2 \\ F_t = f_t \omega^2 \Big/ \kappa_t^2 \quad \text{is uniformly bounded in } t \in [0,1]. \end{cases}$$

Then on a compact set $K \subset M - E$, *there is a constant*

$C = C(K, \lambda, \Lambda, \sup_{M\times[0,1]} F_t, \sup_{M\times[0,1]} \nabla^2 F_t)$, *independent of* t, *such that*

$$2 + \Delta_t\Big(u_t + \log\prod_{j=1}^m |\sigma_j|^2_{h_j}\Big) \le C \exp\{\sup_M u_t - \inf_M u_t\} \sup_K \Big|\prod_{j=1}^m |\sigma_j|^2_{h_j}\Big|^C \Big/ \inf_K \Big(\prod_{j=1}^m |\sigma_j|^2_{h_j}\Big)^C.$$

Proof. For notational convenience u_t will simultaneously denote itself and $u_t + \log\prod_{j=1}^m |\sigma_j|^2_{h_j}$. Also, t subscripts and superscripts are suppressed with the understanding that t is fixed. By lemma (.0.3) there are normal coordinates for g at $q \in K$ for which $\tilde{g}_{i\bar{j}} = (1 + u_{i\bar{j}})\delta_{i\bar{j}}$ and $0 = \partial_k g_{i\bar{j}}$ and $0 = \partial_{\bar{k}} g_{i\bar{j}}$. We begin by differentiating the term to be estimated,

$$\partial_i\partial_{\bar{j}}tr\{g^{-1}\tilde{g}\} = tr\{\partial_i\partial_{\bar{j}}(g^{-1})\tilde{g} + g^{-1}\partial_i\partial_{\bar{j}}\tilde{g}\} = R^{k\bar{l}}{}_{i\bar{j}}\tilde{g}_{k\bar{l}} - g^{k\bar{l}}\tilde{R}_{k\bar{l}i\bar{j}}.$$

Differentiating the Monge-Ampère equation twice gives

$$\begin{aligned}
\partial_i\partial_{\bar{j}}\log(F_t) &= \partial_i\partial_{\bar{j}}\log\det(\tilde{g}) - \partial_i\partial_{\bar{j}}\log\det(g) \\
&= \partial_i tr\{\tilde{g}^{-1}\partial_{\bar{j}}\tilde{g}\} - tr\{g^{-1}\partial_i\partial_{\bar{j}}(g)\} \\
&= tr\{-\tilde{g}^{-1}\partial_i(\tilde{g})\tilde{g}^{-1}\partial_{\bar{j}}(\tilde{g}) + \tilde{g}^{-1}\partial_i\partial_{\bar{j}}(\tilde{g})\} - tr\{g^{-1}\partial_i\partial_{\bar{j}}(g)\} \\
&= -\tilde{g}^{a\bar{q}}\tilde{g}^{p\bar{s}}\partial_i(\tilde{g}_{p\bar{q}})\partial_{\bar{j}}(\tilde{g}_{a\bar{s}}) + \tilde{g}^{p\bar{q}}\partial_i\partial_{\bar{j}}(\tilde{g}_{p\bar{q}}) + R_{i\bar{j}} \\
&= -\tilde{g}^{a\bar{q}}\tilde{g}^{p\bar{s}}\partial_i(\tilde{g}_{p\bar{q}})\partial_{\bar{j}}(\tilde{g}_{a\bar{s}}) - \tilde{g}^{p\bar{q}}\tilde{R}_{p\bar{q}i\bar{j}} + R_{i\bar{j}}.
\end{aligned}$$

After setting $i = k,\ j = l,\ p = i,\ q = j$, and taking the trace of $\nabla^2\log(F_t)$ and $\nabla^2\mathrm{tr}\{g^{-1}\tilde{g}\}$, we have

$$\Delta\log(F_t) = -g^{k\bar{l}}\tilde{g}^{a\bar{j}}\tilde{g}^{k\bar{s}}\partial_k(\tilde{g}_{i\bar{j}})\partial_{\bar{l}}(\tilde{g}_{a\bar{s}}) - g^{k\bar{l}}\tilde{g}^{i\bar{j}}\tilde{R}_{i\bar{j}k\bar{l}} + g^{k\bar{l}}R_{k\bar{l}}$$

and

$$\tilde{\Delta}tr\{g^{-1}\tilde{g}\} = \tilde{g}^{i\bar{j}}R^{k\bar{l}}{}_{i\bar{j}}\tilde{g}_{k\bar{l}} - \tilde{g}^{i\bar{j}}g^{k\bar{l}}\tilde{R}_{k\bar{l}i\bar{j}}.$$

Summing the previous two formulas eliminates the fourth order terms and we get

$$\tilde{\Delta}tr\{g^{-1}\tilde{g}\} = \tilde{g}^{i\bar{j}}R^{k\bar{l}}{}_{i\bar{j}}\tilde{g}_{k\bar{l}} + \Delta\log(F_t) + g^{k\bar{l}}\tilde{g}^{a\bar{j}}\tilde{g}^{k\bar{s}}\partial_k(\tilde{g}_{i\bar{j}})\partial_{\bar{l}}(\tilde{g}_{a\bar{s}}) - R.$$

The curvature term $R^{k\bar{l}}{}_{i\bar{j}}$ is bounded, so $R^{k\bar{l}}{}_{i\bar{j}} \geq -C_0 g^{k\bar{l}}g_{i\bar{j}}$ for some $C_0 > 0$. Similarly, the extreme value theorem and smoothness of F_t and R imply that they too are bounded. Applying lemma (2.3.2) to the 3rd order term of $\tilde{\Delta}\log tr\{g^{-1}\tilde{g}\}$ gives

$$
\begin{aligned}
\tilde{\Delta}\log tr\{g^{-1}\tilde{g}\} &\geq \frac{1}{tr\{g^{-1}\tilde{g}\}}\Big\{\tilde{g}^{i\bar{j}}R^{k\bar{l}}{}_{i\bar{j}}\tilde{g}_{k\bar{l}} + \Delta\log(F_t) - R\Big\} \\
&\geq \frac{1}{tr\{g^{-1}\tilde{g}\}}\Big\{-C_0\tilde{g}^{i\bar{j}}g^{k\bar{l}}g_{i\bar{j}}\tilde{g}_{k\bar{l}} + \Delta\log(F_t) - R\Big\}, \text{ by fact(2),} \\
&\geq -C_0\tilde{g}^{i\bar{j}}g_{i\bar{j}} + \frac{1}{tr\{g^{-1}\tilde{g}\}}\Big\{\Delta\log(F_t) - R\Big\} \\
&= -(C_0+1)n + (C_0+1)\tilde{\Delta}\{u_t + \log\prod_{j=1}^{m}|\sigma_j|^2_{h_j}\} + \tilde{g}^{k\bar{l}}g_{k\bar{l}} \\
&\qquad + \frac{\inf_M\Big\{\Delta\log(F_t) - R\Big\}}{tr\{g^{-1}\tilde{g}\}}.
\end{aligned}
$$

In the last step the formula below is used.

$$
\tilde{g}^{k\bar{l}}g_{k\bar{l}} + \tilde{\Delta}\{u_t + \log\prod_{j=1}^{m}|\sigma_j|^2_{h_j}\} = n.
$$

$\tilde{g} = \mu_j\delta_{i\bar{j}}$ and $g = \delta_{i\bar{j}}$ at $q \in K$. Also,

$$
\begin{aligned}
\tilde{g}^{k\bar{l}}g_{k\bar{l}} &= \frac{\mu_1+\mu_2}{\mu_1\mu_2} = \frac{tr\{g^{-1}\tilde{g}\}}{\det(\tilde{g})} \\
&= \frac{tr\{g^{-1}\tilde{g}\}}{F_t\det(g)}, \quad \text{by (3.20),} \\
&= \frac{tr\{g^{-1}\tilde{g}\}}{F_t}, \quad \text{because } \det(g) = 1 \text{ at } q \in K.
\end{aligned}
$$

Then we arrive at

$$
\begin{aligned}
\tilde{\Delta}\Big\{\log tr\{g^{-1}\tilde{g}\} - (C_0+1)\{u_t + \log\prod_{j=1}^{m}|\sigma_j|^2_{h_j}\}\Big\} &\geq -(C_0+1)n \\
&\quad + \frac{tr\{g^{-1}\tilde{g}\}}{F_t} + \frac{\inf_{x\in M}\Big\{\Delta\log(F_t) - R\Big\}}{tr\{g^{-1}\tilde{g}\}}.
\end{aligned}
$$

Suppose that the maximum point of the term on which $\tilde{\Delta}$ acts occurs at $q \in K$, then we have

$$n(C_0+1) - \frac{\inf_{x\in M}\Big\{\Delta \log(F_t) - R\Big\}}{tr\{g^{-1}\tilde{g}\}} \geq \frac{tr\{g^{-1}\tilde{g}\}}{F_t} \qquad \Longrightarrow$$

$$2F_t \cdot \max\left\{n(C_0+1), \frac{\sup_{x\in M}\Big|\Delta\log(F_t) - R\Big|}{tr\{g^{-1}\tilde{g}\}}\right\} \geq tr\{g^{-1}\tilde{g}\}. \quad (3.21)$$

Because F_t is uniformly bounded in t and fact (2), there is a constant

$$C_1 = \max\left\{1, \sup_{y\in M\times[0,1]} F_t, \ \sup_{x\in M}\Big|\Delta\log(F_t) - R\Big|\right\},$$

independent of t, so that either

$$2nC_1(C_0+1) \geq tr\{g^{-1}\tilde{g}\},$$

or

$$2C_1^2\frac{1}{tr\{g^{-1}\tilde{g}\}} \geq 2F_t \cdot \frac{\sup_M\Big|\Delta\log(F_t) - R\Big|}{tr\{g^{-1}\tilde{g}\}} \geq tr\{g^{-1}\tilde{g}\}.$$

In either case $tr\{g^{-1}\tilde{g}\}(q)$ has an upper bound $C > 0$, where $C = C(n, \lambda, \Lambda, K, \sup F_t, \nabla^2 F_t)$ that is independent of $t \in [0,1]$. Also for convenience assume that $C > C_0 + 1$. At $q \in K$ we have

$$tr\{g^{-1}\tilde{g}\}e^{Cu_t}\Big(\prod_{j=1}^m |\sigma_j|^2_{h_j}\Big)^C \leq Ce^{Cu_t}\Big(\prod_{j=1}^m |\sigma_j|^2_{h_j}\Big)^C.$$

Hence, everywhere on K

$$tr\{g^{-1}\tilde{g}\}e^{C\inf_M u_t}\inf_{x\in K}\Big(\prod_{j=1}^m|\sigma_j|^2_{h_j}\Big)^C \le Ce^{Cu_t}\Big(\prod_{j=1}^m|\sigma_j|^2_{h_j}\Big)^C$$
$$\implies \quad tr\{g^{-1}\tilde{g}\} \le Ce^{C\{\sup_M u_t-\inf_M u_t\}}\sup_{x\in K}\Big(\prod_{j=1}^m|\sigma_j|^2_{h_j}\Big)^C \Big/ \inf_{x\in K}\Big(\prod_{j=1}^m|\sigma_j|^2_{h_j}\Big)^C.$$

Then,

$$tr\{g^{-1}\tilde{g}\} \le C\exp\Big\{\sup_{x\in M}u_t - \inf_{x\in M}u_t\Big\}\cdot\sup_{x\in K}\Big(\prod_{j=1}^m|\sigma_j|^2_{h_j}\Big)^C \Big/ \inf_{x\in K}\Big(\prod_{j=1}^m|\sigma_j|^2_{h_j}\Big)^C. \quad (3.22)$$

□

The L^∞ estimate of lemma (3.1.4) uniformly bounds the term $\sup_{x\in M}u_t - \inf_{x\in M}u_t$; therefore, the eigenvalues of $\tilde{\kappa_t}$ are uniformly bounded uniformly in t by lemma (3.3.1). These uniform $C^2(K)$ estimates for solutions imply uniform $C^{2,\alpha}(K)$ estimates via Evans-Krylov [13]. The bootstrap technique described in chapter (2) provides uniform $C^{k,\alpha}(K)$ estimates for all $k > 0$. By Arzelá-Ascoli there is a subsequence of $\{u_t\}$ which converges to u in $C^k(K)$ for every $k \in \mathbb{Z}^+$. Therefore, u is a smooth function on K.

APPENDIX
KÄHLER COORDINATES

In chapter 2 the following inequality was used in the proof of lemma (3.1.4).

Lemma .0.2. *There is a constant $B > 0$ so that*

$$Bx^m \geq e^{-\frac{1}{x}} \quad \textit{for} \quad m > 0. \tag{23}$$

Proof. We will show that $f = \frac{e^{-\frac{1}{x}}}{x^m}$ has a maximum on $\mathbb{R}^+$.

$$\begin{aligned} f' &= \frac{-m}{x^{m+1}} e^{-\frac{1}{x}} + \frac{1}{x^{m+2}} e^{-\frac{1}{x}} \\ &= e^{-\frac{1}{x}} \left(\frac{1 - mx}{x^{m+2}} \right). \end{aligned}$$

f is increasing on $(0, \frac{1}{m})$ and decreasing on $(\frac{1}{m}, +\infty)$. Therefore, $f(\frac{1}{m}) = m^m e^{-m}$ is a maximum for f on $\mathbb{R}^+$. Any $B \geq m^m e^{-m}$ would make inequality (23) true. □

In this thesis Kähler coordinates are used to simplify calculations for C^2 estimates of nondegenerate and degenerate Monge-Ampère equations. Below is a proof of their existence.

Lemma .0.3. *Let (M, ω) be a Kähler manifold. In local coordinates $\omega = \frac{\sqrt{-1}}{2\pi} g_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$. Then at a point $p \in M$ coordinates can be found so that*

(a) $g_{i\bar{j}}(p) = \delta_{i\bar{j}}$.

(b) $\partial_k g_{i\bar{j}}(p) = \partial_{\bar{k}} g_{i\bar{j}}(p) = 0$ *for* $k = 1 \dots n$.

(c) $\tilde{g}_{i\bar{j}}(p) = (1 + u_{i\bar{i}})\delta_{i\bar{j}}$.

Proof. Let $z = (z^1, \dots, z^n)$ be holomorphic coordinates around some point $p \in M$ for which $z(p) = (0, \dots, 0)$. We can always change coordinates linearly at a point so that $g_{i\bar{j}}(0) = \delta_{i\bar{j}}$, these are called normal coordinates. Hermitian form $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + u_{i\bar{j}}$ can be diagonalized, because Hermitian matrices are unitarily diagonalizable. Suppose $\mathcal{O} \in U(n)$ diagonalizes $\partial\bar{\partial}u$ at a point, then the new coordinates are $\tilde{z}_j = \mathcal{O}^i_j z_i$ and

$$\tilde{g}\left(\mathcal{O}X, \overline{\mathcal{O}Y}\right)(0) = (\mathcal{O}Y)^*(I + [u_{i\bar{j}}])\mathcal{O}X = Y^*(\mathcal{O}^*\mathcal{O} + \mathcal{O}^*[u_{i\bar{j}}]\mathcal{O})X = \delta_{i\bar{j}}(1 + u_{i\bar{i}})X^i\overline{Y^j}.$$

To prove (b) consider a linear change of coordinates $z = F(w^1, \dots, w^n)$ defined by

$$F^l(w) = w^l + \frac{1}{2}A^l_{ij}w^i w^j.$$

Here the coefficients A^l_{ij} are constant and $A^l_{ij} = A^l_{ji}$. The derivatives of this change of coordinates are

$$F^l_k = \frac{\partial F^l}{\partial w^k} = \delta^l_k + A^l_{kj}w^j \quad \text{and} \quad F^l_{km} = A^l_{km}.$$

Define vector fields

$$W_i := \frac{\partial}{\partial w^i} \quad \text{and} \quad Z_i := \frac{\partial}{\partial z^i}.$$

These vector fields are related by $W_i = F_i^l Z_l$.

$$g\left(W_i, \overline{W}_j\right) = g\left(F_i^l Z_l, \overline{F_j^s Z_s}\right) = \left(\delta_i^l + A_{ib}^l w^b\right) \overline{\left(\delta_j^s + A_{jc}^s w^c\right)} g\left(Z_l, \overline{Z}_s\right)$$
$$= g\left(Z_i, \overline{Z}_j\right) + \overline{A_{jc}^s w^c} g\left(Z_i, \overline{Z}_s\right) + A_{ib}^l w^b g\left(Z_l, \overline{Z}_j\right) + A_{ib}^l \overline{A_{jc}^s} w^b \overline{w}^c g\left(Z_l, \overline{Z}_s\right).$$

In particular, $g\left(W_i, \overline{W}_j\right)(0) = g\left(Z_i, \overline{Z}_j\right)(0)$.

$$\frac{\partial}{\partial w^k} g\left(W_i, \overline{W}_j\right) = F_k^l \frac{\partial}{\partial z^l} g\left(Z_i, \overline{Z}_j\right) + \overline{A_{jc}^s w^c} F_k^l \frac{\partial}{\partial z^l} g\left(Z_i, \overline{Z}_s\right) + A_{ik}^l g\left(Z_l, \overline{Z}_j\right)$$
$$+ A_{ib}^l w^b F_k^l \frac{\partial}{\partial z^l} g\left(Z_l, \overline{Z}_j\right) + A_{ik}^l \overline{A_{jc}^s} \overline{w}^c g\left(Z_l, \overline{Z}_s\right)$$
$$+ A_{ib}^l \overline{A_{jc}^s} w^b \overline{w}^c F_k^l \frac{\partial}{\partial z^l} g\left(Z_l, \overline{Z}_s\right).$$

At $z = 0$ the first derivatives of the metric in the $w-$coordinates are

$$\frac{\partial}{\partial w^k} g\left(W_i, \overline{W}_j\right)(0) = \delta_k^l \frac{\partial}{\partial z^l} g\left(Z_i, \overline{Z}_j\right)(0) + A_{ik}^l \delta_l^j$$
$$= \frac{\partial}{\partial z^k} g\left(Z_i, \overline{Z}_j\right)(0) + A_{ik}^j.$$

If $A_{ik}^j = -\frac{\partial}{\partial z^k} g\left(Z_i, \overline{Z}_j\right)(0)$, then the first derivatives of g in the $w-$ coordinates vanish at $z = 0$. A similar conclusion is reached for the anti-holomorphic derivatives of g. Note the condition $A_{ij}^l = A_{ji}^l$ is natural because $d\omega = 0$ implies that $\partial_k g_{i\bar{j}} = \partial_i g_{k\bar{j}}$ and $\partial_{\bar{k}} g_{i\bar{j}} = \partial_{\bar{j}} g_{i\bar{k}}$.

□

Lemma .0.4. *Suppose (M, ω) is a closed Kähler surface and χ is a positive and $(1,1)$-form. Further suppose $[\chi - \omega]$ is semipositive and $[\chi - \omega]^2 = [\omega]^2$. Then $[\chi - \omega]$ is not Kähler if and only if it there is an effective divisor, D, for which $[\chi - \omega] \cdot D = 0$.*

Proof. The product on cohomology classes which appears in the hypothesis is called the intersection product. It is defined $\alpha \cdot \beta = \int_M \alpha \wedge \beta$ if $\alpha,\ \beta \in H^2(M;\mathbb{R})$–so it is well-defined. This product is symmetric, bilinear, and nondegenerate ($\alpha \cdot \beta = 0\ \forall \beta$ implies α is exact). By Poincaré duality it is possible to define the intersection of divisors and line bundles. For instance, if $L \in \mathrm{Pic}(M)$ is a line bundle and D is a divisor then $\alpha \cdot L = \int_M \alpha \wedge c_1(L)$, $\alpha \cdot C = \int_C \alpha$, and $D \cdot L := \int_D c_1(L)$.

The lemma follows directly from a Nakai criterion proved independently by Buchdahl [6] and Lamari [26].

Theorem .0.1. *Let (M^2, ω) be a closed Kähler surface and $\rho \in H^{1,1}(M;\mathbb{R})$ satisfies the following conditions:*

(a) $\rho \cdot \rho > 0$.

(b) $\rho \cdot [\eta] > 0$*, for η a closed positive* $(1,1)$*-form.*

(c) $\rho \cdot [D] > 0$ *for every effective divisor $D \subset M$.*

Then ρ can be represented by a closed positive $(1,1)-$*form.*

Integrating both sides of (3.3) shows that $[\chi - \omega]^2 > 0$. By the hypothesis of theorem (1.1.1) $[\chi - \omega] \cdot [\chi] = [\omega] \cdot [\chi] > 0$ when $c = \frac{1}{2}$. Since $[\chi - \omega]$ is at most semipositive and it satisfies (a) and (b) of theorem (.0.1), there is a curve D for which $[\chi - \omega] \cdot D = 0$. Implicit in this statement is that every representative of $[\chi - \omega]$ vanishes on D. To see this suppose that $\chi' \sim (\chi - \omega)$ and $\chi - \omega$ is semipositive. By the $\partial\bar{\partial}$-lemma there is a smooth function f so that $\chi' - (\chi - \omega) = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f$. Stokes' theorem and compact support of f imply

$$\int_D \chi' = \int_D \chi - \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f = \int_D \chi - \omega = 0.$$

So, χ' is a most semipositive, and every representative of $[\chi - \omega]$ must vanish on some divisor. The class conditions $[\chi - \omega] \geq 0$ and not Kähler, imposed by theorem (1.1.1), are equivalent to the more geometric restriction: $[\chi - \omega]$ is semipositive and it vanishes on some effective divisor D.

$\square$

REFERENCES

[1] *Première classe de Chern et courbure de Ricci: preuve de la conjecture de Calabi*, volume 58 of *Astérisque*. Société Mathématique de France, Paris, 1978. Lectures presented at a Séminaire held at the Centre de Mathématiques de l'École Polytechnique, Palaiseau, March–June 1978.

[2] Thierry Aubin. Équations du type Monge-Ampère sur les variétés kählériennes compactes. *Bull. Sci. Math. (2)*, 102(1):63–95, 1978.

[3] Thierry Aubin. *Some nonlinear problems in Riemannian geometry*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[4] Eric Bedford and B. A. Taylor. The Dirichlet problem for a complex Monge-Ampère equation. *Invent. Math.*, 37(1):1–44, 1976.

[5] Zbigniew Błocki and Sławomir Kołodziej. On regularization of plurisubharmonic functions on manifolds. *Proc. Amer. Math. Soc.*, 135(7):2089–2093 (electronic), 2007.

[6] Nicholas Buchdahl. On compact Kähler surfaces. *Ann. Inst. Fourier (Grenoble)*, 49(1):vii, xi, 287–302, 1999.

[7] Huai Dong Cao. Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds. *Invent. Math.*, 81(2):359–372, 1985.

[8] Urban Cegrell. *Capacities in complex analysis*. Aspects of Mathematics, E14. Friedr. Vieweg & Sohn, Braunschweig, 1988.

[9] Jean-Pierre Demailly. *Complex Analytic and Differential Geometry*. http://www-fourier.ujf-grenoble.fr/demailly/books.html, 1997.

[10] Jean-Pierre Demailly and Nefton Pali. Degenerate complex Monge-Ampère equations over compact Kähler manifolds. *preprint*, (arXiv:0710.5109v3 [math.DG]), March 24, 2009.

[11] Jean-Pierre Demailly and Mihai Paun. Numerical characterization of the Kähler cone of a compact Kähler manifold. *Ann. of Math. (2)*, 159(3):1247–1274, 2004.

[12] S. K. Donaldson. Moment maps and diffeomorphisms. *Asian J. Math.*, 3(1):1–15, 1999. Sir Michael Atiyah: a great mathematician of the twentieth century.

[13] Lawrence C. Evans. Classical solutions of fully nonlinear, convex, second-order elliptic equations. *Comm. Pure Appl. Math.*, 35(3):333–363, 1982.

[14] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[15] Philippe Eyssidieux, Vincent Guedj, and Ahmed Zeriahi. Singular Kähler-Einstein metrics. *J. Amer. Math. Soc.*, (PII: S 0894-0347(09)00629-8 (To appear in print)), Posted February 6, 2009.

[16] Hao Fang, Mijia Lai, and Xinan Ma. On a class of fully nonlinear flow in kähler geometry. *preprint*, 2009.

[17] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[18] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

[19] Lars Hörmander. *An introduction to complex analysis in several variables*, volume 7 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, third edition, 1990.

[20] Jerry L. Kazdan. A remark on the preceding paper of S. T. Yau: "On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I". *Comm. Pure Appl. Math.*, 31(3):413–414, 1978.

[21] Maciej Klimek. *Pluripotential theory*, volume 6 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York, 1991. Oxford Science Publications.

[22] Shoshichi Kobayashi. On compact Kähler manifolds with positive definite Ricci tensor. *Ann. of Math. (2)*, 74:570–574, 1961.

[23] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. II*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1996. Reprint of the 1969 original, A Wiley-Interscience Publication.

[24] Sławomir Kołodziej. The complex Monge-Ampère equation. *Acta Math.*, 180(1):69–117, 1998.

[25] Sławomir Kołodziej. The complex Monge-Ampère equation and pluripotential theory. *Mem. Amer. Math. Soc.*, 178(840):x+64, 2005.

[26] A. Lamari. Le cône kählérien d'une surface. *J. Math. Pures Appl. (9)*, 78(3):249–263, 1999.

[27] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[28] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[29] Yum Tong Siu. *Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics*, volume 8 of *DMV Seminar*. Birkhäuser Verlag, Basel, 1987.

[30] Jian Song and Ben Weinkove. On the convergence and singularities of the J-flow with applications to the Mabuchi energy. *Comm. Pure Appl. Math.*, 61(2):210–229, 2008.

[31] Gang Tian. On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$. *Invent. Math.*, 89(2):225–246, 1987.

[32] Ben Weinkove. On the J-flow in higher dimensions and the lower boundedness of the Mabuchi energy. *J. Differential Geom.*, 73(2):351–358, 2006.

[33] Shing Tung Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. *Comm. Pure Appl. Math.*, 31(3):339–411, 1978.