Summer 2010

Freeness of hyperplane arrangement bundles and local homology of arrangement complements

Amanda C. Hager
University of Iowa

FREENESS OF HYPERPLANE ARRANGEMENT BUNDLES AND LOCAL HOMOLOGY OF ARRANGEMENT COMPLEMENTS

by

Amanda C. Hager

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy degree
in Mathematics in
the Graduate College of
The University of Iowa

July 2010

Thesis Supervisor: Professor Richard Randell

ABSTRACT

A recent result of Salvetti and Settepanella gives, for a complexified real arrangement, an explicit description of a minimal CW decomposition as well as an explicit algebraic complex which computes local system homology. We apply their techniques to discriminantal arrangements in two dimensional complex space and calculate the boundary maps which will give local system homology groups given any choice of local system. This calculation generalizes several known results; examples are given related to Milnor fibrations, solutions of KZ equations, and the LKB representation of the braid group.

Another algebraic object associated to a hyperplane arrangement is the module of derivations. We analyze the behavior of the derivation module for an affine arrangement over an infinite field and relate its derivation module to that of its cone. In the case of an arrangement fibration, we analyze the relationship between the derivation module of the total space arrangement and those of the base and fiber arrangements. In particular, subject to certain restrictions, we establish freeness of the total space arrangement given freeness of the base and fiber arrangements.

Abstract Approved: ______________________________
Thesis Supervisor

____
Title and Department

____
Date

FREENESS OF HYPERPLANE ARRANGEMENT BUNDLES AND LOCAL HOMOLOGY OF ARRANGEMENT COMPLEMENTS

by

Amanda C. Hager

A thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy degree in Mathematics
in the Graduate College of
The University of Iowa

July 2010

Thesis Supervisor: Professor Richard Randell

Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Amanda C. Hager

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the July 2010 graduation.

Thesis Committee: ____________________
Richard Randell, Thesis Supervisor

____
Fred Goodman

____
John Robinson

____
Jonathan Simon

____
Julianna Tymoczko

ABSTRACT

A recent result of Salvetti and Settepanella gives, for a complexified real arrangement, an explicit description of a minimal CW decomposition as well as an explicit algebraic complex which computes local system homology. We apply their techniques to discriminantal arrangements in two dimensional complex space and calculate the boundary maps which will give local system homology groups given any choice of local system. This calculation generalizes several known results; examples are given related to Milnor fibrations, solutions of KZ equations, and the LKB representation of the braid group.

Another algebraic object associated to a hyperplane arrangement is the module of derivations. We analyze the behavior of the derivation module for an affine arrangement over an infinite field and relate its derivation module to that of its cone. In the case of an arrangement fibration, we analyze the relationship between the derivation module of the total space arrangement and those of the base and fiber arrangements. In particular, subject to certain restrictions, we establish freeness of the total space arrangement given freeness of the base and fiber arrangements.

TABLE OF CONTENTS

LIST OF FIGURES

Figure

CHAPTER 1
INTRODUCTION

1.1 Background and Motivation

This thesis focuses on the topology of complex hyperplane arrangement complements. A hyperplane arrangement is a finite collection of hyperplanes ($(\ell-1)$-dimensional subspaces) in any ℓ-dimensional vector space. An arrangement can be viewed as the zero locus of a defining polynomial $Q(z_1,\ldots,z_\ell)$ which is a product of n linear forms. When the vector space is $\mathbb{C}^\ell$, the complement of the union of an arrangement is topologically interesting. For instance, the complement of a non-empty arrangement has non-trivial fundamental group; the pure braid groups are an example of groups which are realizable as the fundamental group of an arrangement [8]. Arrangement groups are not yet fully understood, even though there are several interesting presentations of the fundamental group of an arrangement available [5, 9, 17]. For example, it is not known whether an arrangement group must be torsion free.

It is also unknown if the integral homology groups of certain covers of the arrangement complement must be torsion free. The Milnor fiber [15] is one such cover; it is an n-fold cyclic cover of the complement of the corresponding projective arrangement, where n is the number of hyperplanes in the arrangement. There is a canonical subgroup of index n of the fundamental group corresponding to this Milnor fiber. The fibration that produces the Milnor fiber F is actually the defining polynomial Q, which at this point is assumed to be homogeneous. This fiber turns out to be a cyclic n-fold cover of the projectivized complement of the arrangement, which is the base space of a restriction of the Hopf fibration $h:\mathbb{C}^{\ell+1}\to\mathbb{P}^\ell$. These maps are shown in Figure 1.1.

Torsion in the homology of the Milnor fiber would provide an invariant of the

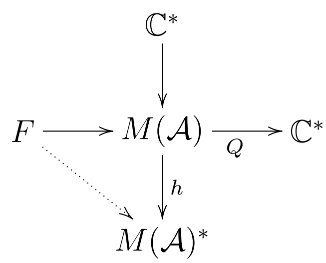


Figure 1.1: Diagram of the Hopf and Milnor fibrations for $\mathcal{A}$ central

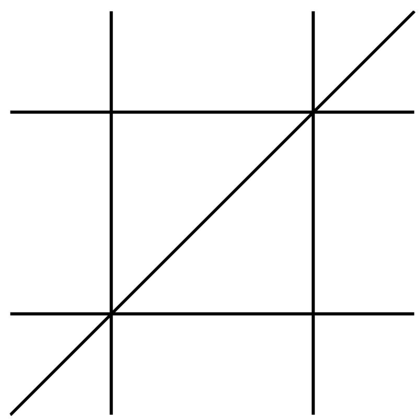


Figure 1.2: A diagram of a 2-arrangement with $Q(\mathcal{A}) = (x^2 - 1)(y^2 - 1)(x - y)$

fundamental group of the original arrangement, and homology groups of covers correspond to homology groups of the original space with coefficients in a local system. To this end, we may use a combinatorial method of computing homology groups with local coefficients developed by Salvetti and Settepanella [22]; this method uses a finite CW complex which has the homotopy type of the complement [21] and collapses it to a minimal CW decomposition using Morse theory.

In this thesis we look at a special family of arrangements, *discriminantal arrangements*, which are of use in computing hypergeometric integrals [6]. The simplest example of a discriminantal arrangement in dimension two is depicted in Figure 1.2. In two dimensional space, we are able to compute all local systems for which the free ranks of the homology groups are interesting; usually, the boundary maps have full rank which forces the first local homology group to be trivial, but for certain local systems we have nontrivial first homology.

In dimensions higher than two, the problem becomes more difficult. The special CW complex is straightforward to write down, but the process of collapsing the complex to a minimal one and writing down the local homology boundary maps is much more difficult.

Another opportunity for geometric analysis lies in the module of derivations associated to the arrangement. This module may be viewed as the set of linear maps on ℓ-variable polynomials which satisfy the Leibniz rule and which preserve the ideal in $\mathbb{C}[z_1, \ldots, z_\ell]$ generated by the defining polynomial Q. Alternatively, we could view derivations as vector fields on $\mathbb{C}^\ell$ which are tangent to each of the hyperplanes in the arrangement [16]. These derivations form a module over $S = \mathbb{C}[z_1, \ldots, z_\ell]$.

Saito [20] discovered that this module was dual to the S-module of logarithmic 1-forms with poles along the variety defined by Q which was already known to generate the cohomology algebra of the complement [1, 3]. Terao [23] found that the modules of derivations for most of the examples of classical interest, such as the braid arrangements and reflection arrangements, are *free* S-modules. However, no one knows precisely how the topology of the arrangement complement or the combinatorics of the arrangement and this module are related. Specifically, the well-known Terao conjecture [25] states that the combinatorics of the arrangement, or the way the hyperplanes intersect each other, determine the module of derivations; this remains unsolved. We do not even know if the topology determines the structure of the module. However, several results are available which establish ties between the module and the topology: freeness of the module of derivations implies factorization of the Poincaré polynomial of the complement [24] (which is computable from the intersection lattice), and freeness of the module behaves well under certain operations on hyperplane arrangements which are used to dissect and understand them [23].

In many of the most interesting examples of arrangements, the complement admits a fibration in which the base space and the generic fiber are both arrangement complements [10]. Terao [25] showed that the existence of these fibrations comes from the existence of modular elements in the intersection lattice of the arrangement.

1.2 Overview

In Chapter 2, we provide definitions and background information about hyperplane arrangement combinatorics, the Salvetti Complex, discrete Morse theory, rank-one local systems, and the module of derivations.

In Chapter 3, we compute the Morse complex for a particular family of arrangements and use this complex to compute local homology groups for any choice of twisted coefficients. We also present several interesting examples of choices of local systems; the resulting computations reproduce known results.

In Chapter 4, we extend the definition of freeness to affine arrangements in vector spaces over infinite fields, and prove that an affine arrangement is free in this sense if and only if its cone is free. Also, subject to some restrictions, we show that in the case of an arrangement fibration, the total space arrangement is free only if the base and fiber arrangements are free. This is a partial answer to a conjecture first made by Falk and Proudfoot [11].

CHAPTER 2
BACKGROUND AND DEFINITIONS

For a more complete introduction to background material, the reader is referred to Orlik and Terao's book [16].

2.1 The Intersection Lattice

Let $\mathbb{K}$ be a field and let V be an ℓ-dimensional vector space over $\mathbb{K}$.

Definition 2.1.1. A *hyperplane* is an affine $(\ell-1)$-dimensional subspace of V. A *hyperplane arrangement* $\mathcal{A}$ in V (or *arrangement* or *ℓ-arrangement* for short) is a finite collection of hyperplanes.

A hyperplane may be thought of as the zero locus of a linear polynomial $\alpha \in \mathbb{K}[x_1, x_2, \ldots, x_\ell]$. Unless otherwise specified, we will assume that all of the hyperplanes contain the origin, which makes the arrangement *central*. Otherwise, the arrangement is *affine*.

Definition 2.1.2. Given a hyperplane H in V, a *defining form* $\alpha \in \mathbb{K}[x_1, x_2, \ldots, x_\ell]$ for H is a linear polynomial whose zero locus is H. If $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ is an arrangement, and if for $1 \leq i \leq n$ a defining form for H_i is α_i, then we call $Q(\mathcal{A}) = \prod \alpha_i$ the *defining polynomial* of $\mathcal{A}$.

We may use *coning* to compare affine and central arrangements.

Definition 2.1.3. Let $\mathcal{A}$ be an affine ℓ-arrangement which is defined by $Q = Q(\mathcal{A}) \in \mathbb{K}[x_1, \ldots, x_\ell]$. Let Q' be the polynomial Q homogenized using a new variable $x_{\ell+1}$. The *cone* of $\mathcal{A}$, denoted $\mathbf{c}\mathcal{A}$, is the central $(\ell+1)$-arrangement defined by the polynomial $x_{\ell+1}Q'$. Note that $\mathbf{c}\mathcal{A}$ has one extra hyperplane, which we call the *hyperplane at infinity* or equivalently the *additional hyperplane*. Similarly, given any central $\ell+1$-arrangement $\mathcal{A}$ defined by a homogeneous polynomial Q, we may define the *decone* $\mathbf{d}\mathcal{A}$ of $\mathcal{A}$ by choosing a distinguished hyperplane H_0, linearly changing coordinates in $\mathbb{K}^{\ell+1}$ so that the form defining H_0 is $x_{\ell+1}$, and substituting 1 for $x_{\ell+1}$

in the changed polynomial Q.

Example 2.1.4. Let $\mathcal{A}$ be an affine arrangement in $\mathbb{C}^1$ given by $Q(\mathcal{A}) = x(x-1)(x+1)$. Then $Q(\mathbf{c}\mathcal{A}) = x(x-y)(x+y)y$. A handy way to visualize the cone/decone is by embedding the affine ℓ-arrangement in the hyperplane $x_{\ell+1} = 1$ in $\ell+1$-space, drawing all of the hyperplanes spanned by the origin and each of the $\ell - 1$ dimensional spaces, and adding the $x_{\ell+1} = 0$ hyperplane. This is illustrated for the above example in Figure 2.1.

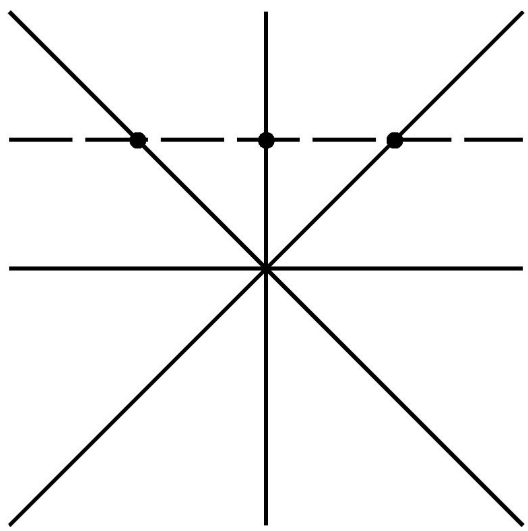


Figure 2.1: An affine $\mathbb{C}^1$ arrangement and its cone in $\mathbb{C}^2$

Example 2.1.5. We note that the polynomial obtained by deconing depends on the choice of distinguished hyperplane. Let $Q(\mathcal{A}) = x(x+2y)(x-2y)y$. If we choose the hyperplane defined by $y = 0$ to be the distinguished hyperplane, then $Q(\mathbf{d}\mathcal{A}) = x(x+2)(x-2)$. However, if we decide that $x = 0$ is the distinguished hyperplane, then $Q(\mathbf{d}\mathcal{A}) = (1+2y)(1-2y)y$, and these polynomials are not equal up to scalar.

Let $M(\mathcal{A}) = V - \cup H_i$ denote the complement of the arrangement $\mathcal{A}$. In the case that $\mathbb{K} = \mathbb{C}$ and $V = \mathbb{C}^\ell$, then $M(\mathcal{A})$ is an open subset of $\mathbb{C}^\ell$ and is therefore a smooth ℓ-dimensional complex manifold.

Definition 2.1.6. A *projective arrangement* is a finite collection of codimension-1 subspaces in $\mathbb{P}^\ell$.

Note 2.1.7. Another way of viewing coning and deconing is through projectivization. If Q is a homogeneous polynomial in $\ell+1$ variables, then it is a well-defined function on projective ℓ-space $\mathbb{P}^\ell$. We use M^* to denote the complement of the zero set of Q in $\mathbb{P}^\ell$. Since the complement of a projective hyperplane in $\mathbb{P}^\ell$ is affine ℓ-space $\mathbb{K}^\ell$, the complement of $n+1$ hyperplanes in $\mathbb{P}^\ell$ is the complement of n hyperplanes in $\mathbb{K}^\ell$.

In the case $\mathbb{K} = \mathbb{C}$, we have that the restriction of the Hopf bundle map $p : \mathbb{C}^{\ell+1} \to \mathbb{P}^\ell$ to the complement of a central arrangement $\mathcal{A}$ is a trivial fiber bundle [16], so we have

$$M(\mathcal{A}) \cong M^* \times \mathbb{C}^* \cong M(\mathbf{d}\mathcal{A}) \times \mathbb{C}^*$$

This means that we can study the topology of affine, central, or projective arrangement complements and frequently obtain information about the other two.

We are particularly interested in arrangements in $\mathbb{C}^\ell$ whose defining forms have all real coefficients. If this is the case, we say that the arrangement is a *complexified real arrangement*. Frequently, we will draw the corresponding real arrangement as a diagram to illustrate the combinatorics of the complexified real arrangement; the two intersection lattices are isomorphic. See Figure 2.2 for an example of a complexified real arrangement in $\mathbb{C}^2$ which is drawn in $\mathbb{R}^2$. Note that for complex arrangements, codimension means complex codimension; the hyperplanes in $\mathcal{A}$ are all isomorphic to $\mathbb{C}^1$.

Example 2.1.8. Let $\mathcal{A}_{\ell,n}$ be the ℓ-arrangement defined by the polynomial:

$$Q_{\ell,n} = \prod_{i=1}^{\ell}\prod_{j=1}^{n}(z_i - j) \prod_{1\le p<q\le \ell} (z_p - z_q)$$

We call these arrangements *discriminantal arrangements*; an example may be found in Figure 2.1.

We wish to understand the topology of the complement $M(\mathcal{A})$, and to this

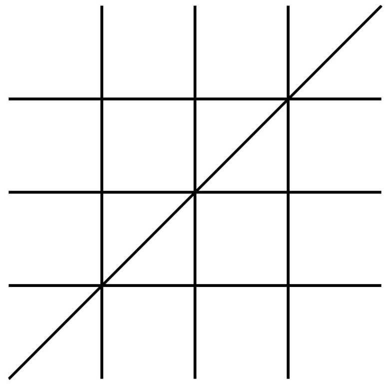


Figure 2.2: Discriminantal arrangement with $\ell = 2$, $n = 3$

end, we frequently employ a geometric semi-lattice associated to the arrangement which is a fairly powerful combinatorial invariant.

Definition 2.1.9. Let $L(\mathcal{A})$ be the set of all nonempty intersections of hyperplanes in $\mathcal{A}$. Define a partial order on $L(\mathcal{A})$ by reverse inclusion: $X \leq Y \Leftrightarrow Y \subset X$. $L(\mathcal{A})$ is called the *intersection lattice* of $\mathcal{A}$. Define a *rank* function on $L(\mathcal{A})$ by $r(X) = codim_V(X)$. Let $r(\mathcal{A}) := \max\{r(X) \,|\, X \in L(\mathcal{A})\}$. There are two operations on $L(\mathcal{A})$: the *meet* of X and Y is defined to be $X \wedge Y := \cap\{Z \in L(\mathcal{A}) \,|\, Z \leq X, Z \leq Y\}$, and the *join* of X and Y is defined to be $X \vee Y := X \cap Y$.

Note that $r(V) = 0$ and that V is the unique element of rank 0. If $\mathcal{A}$ is central, then the join of two elements always exists as a lattice element, the intersection of all H_i is the unique element of maximal rank, and $L(\mathcal{A})$ is a geometric lattice. If $r(\mathcal{A}) = \ell$, we say the arrangement is *essential*.

Definition 2.1.10. Let $\mathcal{A}$ be central, so the intersection lattice is a geometric lattice. An intersection lattice element $X \in L(\mathcal{A})$ is called *modular* if for all $Y, Z \in L(\mathcal{A})$ with $Z \leq Y$, we have $Z \vee (X \wedge Y) = (Z \vee X) \wedge Y$.

Modularity can be defined for an element in any geometric lattice, but in the case of an intersection lattice for an arrangement, there is a topological interpretation as well. A proof of the following lemma may be found in Orlik and Terao's book [16]:

Lemma 2.1.11. *Let $X \in L(\mathcal{A})$. The following are equivalent:*

1. *X is modular.*

2. *For all $Y \in L(\mathcal{A})$, $r(X) + r(Y) = r(X \vee Y) + r(X \wedge Y)$*

3. *For all $Y \in L(\mathcal{A})$, $X \wedge Y = X + Y$.*

4. *For all $Y \in L(\mathcal{A})$, $X + Y \in L$.*

Example 2.1.12. Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^3$ defined by $Q(\mathcal{A}) = (x - y)(x - z)(y - z)xy$. Figure 2.3 shows the intersection lattice for $\mathcal{A}$, where each number i denotes the hyperplane defined by the ith form in the above expression of Q, and concatenated numbers indicated intersections of hyperplanes. For example, 24 means $H_2 \cap H_4$.

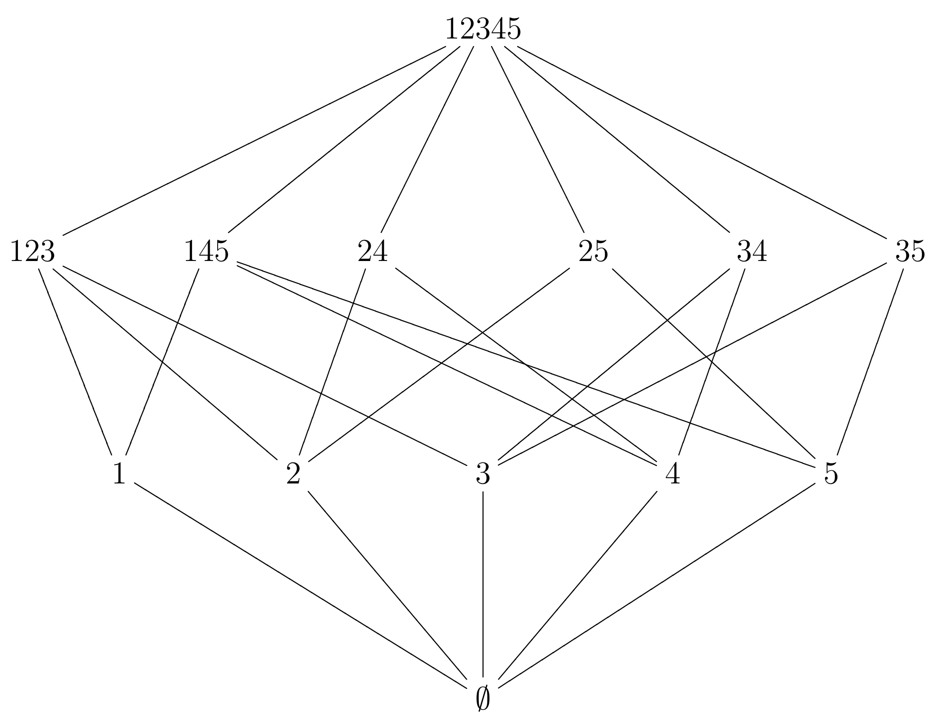


Figure 2.3: $L(\mathcal{A})$ when $Q(\mathcal{A}) = (x - y)(x - z)(y - z)xy$

Note that the element marked "24" is $H_2 \cap H_4$ which is the y-axis. This element is not modular since $24 + 35 = H_2 \cap H_4 + H_3 \cap H_5$ is the xy-plane which is the zero set of the form z, and this form is not a factor of Q. However, it can be checked that the bottom and top elements are always modular, as are the hyperplanes on the next to bottom level. Also, note that the elements 123 and 145 are modular, although this is slightly tedious to check.

Sometimes there is a maximal chain of modular elements in the lattice:

Definition 2.1.13. Let $\mathcal{A}$ be an arrangement with $r(\mathcal{A}) = \ell$ (so $\mathcal{A}$ is essential). Then $\mathcal{A}$ is called *supersolvable* if there exists a maximal chain of modular elements

$$V = X_0 < X_1 < \cdots < X_\ell = \{0\}$$

If X is modular, then Terao [25] found that the orthogonal projection map which collapses X down to the origin turns out to be a fiber bundle projection map, where the base space is the complement of a central $r(X)$-arrangement and the generic fiber is the complement of an affine $(\ell - r(X))$-arrangement. In particular, if X has rank $\ell - 1$ or is a one-dimensional subspace, then each fiber of the projection mapping is the complex line $\mathbb{C}$ with a fixed number of points removed.

Supersolvability also has an interesting topological interpretation. The following definition is due to Falk and Randell [10]:

Definition 2.1.14.

1. The arrangement $\{0\}$ in $\mathbb{C}^1$ is a *fiber-type* arrangement.
2. Suppose that, after suitable linear coordinate change, projection to the first $(\ell - 1)$ coordinates is a fiber bundle projection $M \to M'$, where M' is the complement of a fiber-type arrangement in $\mathbb{C}^{\ell-1}$. Then $\mathcal{A}$ is a *fiber-type* arrangement.

Theorem 2.1.15. *[25] A hyperplane arrangement is fiber-type if and only if it is supersolvable.*

This theorem establishes a strong tie between combinatorics and topology of arrangements. Note that the arrangement in Example 2.1.12 is supersolvable since

$\emptyset \leq 1 \leq 123 \leq 12345$ and these elements are all modular.

2.2 Discrete Morse Theory on the Salvetti Complex

If $\mathcal{A}$ is a complex arrangement, then the complement of $\mathcal{A}$ admits a minimal CW decomposition; this was found independently by Dimca and Papadima [7] and Randell [19]. Salvetti [21] discovered a CW decomposition for the complement of a complexified real arrangement which is easy to define and use, but it is far from minimal. Yoshinaga [26] used the Lefschetz hyperplane section theorem to describe attaching maps for a minimal decomposition of the complement which did not rely on the Salvetti complex or the combinatorics of the arrangement. Then, in their 2007 paper, Salvetti and Settepanella [22] described a way to collapse Salvetti's complex to a minimal one in order to more easily compute local homology. Their method makes use of combinatorial Morse theory; for a user-friendly introduction to the topic, see papers by Forman [12, 13].

We remark that if $\mathcal{A}$ is a complexified real arrangement, then the corresponding real arrangement can help us determine certain topological invariants. In particular, we may be able to compute invariants which are combinatorially determined, such as betti numbers, since the real and complex arrangements have identical combinatorics. Salvetti and Settepanella's technique (and even the definition of the Salvetti complex itself) relies heavily on the corresponding real arrangement.

The technique described below may be summarized in the following steps:

1. Define the face poset $\mathcal{S}$ and the Salvetti complex $\mathbf{S}$.

2. Give a total ordering on $\mathcal{S}$ using polar coordinates.

3. Use the ordering to define a combinatorial Morse function on $\mathbf{S}$.

4. Use the combinatorial gradiant field from the Morse function to collapse $\mathbf{S}$ to a minimal decomposition.

Assume, then, that $\mathcal{A}$ is a complexified real arrangement with corresponding real arrangement $\mathcal{A}_{\mathbb{R}}$.

Definition 2.2.1. Let $X \in L(\mathcal{A}_{\mathbb{R}})$ be any intersection lattice element. A *face* of $\mathcal{A}_{\mathbb{R}}$ is a connected component of $M(\mathcal{A}_{\mathbb{R}}^X)$, where

$$\mathcal{A}_{\mathbb{R}}^X = \{X \cap H \mid H \in \mathcal{A}_{\mathbb{R}}, X \nsubseteq H, \text{and } X \cap H \neq \emptyset\}$$

is the restriction of $\mathcal{A}_{\mathbb{R}}$ to X. Let

$$\mathcal{S} := \{F^k\}$$

be the stratification of $\mathbb{R}^\ell$ into codimension-k facets F^k (connected components of intersection lattice elements). Then $\mathcal{S}$ has a partial ordering

$$F^i \prec F^j \Leftrightarrow \overline{F^i} \supset F^j$$

We call $\mathcal{S}$ the *face poset* of $\mathcal{A}_{\mathbb{R}}$.

Theorem 2.2.2. *[21] There is a CW complex (called the* Salvetti complex*) with the homotopy type of $M(\mathcal{A})$ and whose k-cells correspond bijectively with pairs $[C \prec F^k]$ where C is a chamber (or 0-face) in $\mathcal{S}$ and $F^k \in \mathcal{S}$.*

A cell $[C \prec F^k]$ is in the boundary of $[D \prec G^j]$ if and only if $F^k \prec G^j$ and the chambers C and D are contained in the same chamber of the subarrangement $\{H \in \mathcal{A}_{\mathbb{R}} \mid F \subset H\}$.

Notation 2.2.3. Given a chamber C and a facet F, denote by $C.F$ the unique chamber containing F in its closure and lying in the same chamber as C in the subarrangement $\{H \in \mathcal{A}_{\mathbb{R}} \mid F \subset H\}$. Given two facets F and G, the notation $C.F.G$ shall be read as $(C.F).G$.

In using the real arrangement $\mathcal{A}_{\mathbb{R}}$ to do Morse theory, we will need to describe points in $\mathbb{R}^\ell$ using polar coordinates. The coordinate changes are as follows:

$$\begin{aligned} \rho &= \sqrt{x_1^2 + \cdots + x_\ell^2} \\ \cos^2(\theta_1) &= \tfrac{x_1^2}{x_1^2+\cdots+x_\ell^2} \\ \vdots &\vdots \vdots \\ \cos^2(\theta_i) &= \tfrac{x_i^2}{x_i^2+\cdots+x_\ell^2} \\ \vdots &\vdots \vdots \\ \cos^2(\theta_{\ell-1}) &= \tfrac{x_{\ell-1}^2}{x_{\ell-1}^2+x_\ell^2} \end{aligned}$$

By convention, for any point $x = (x_1, x_2, \ldots, x_\ell)$ and any i with $x_i = \cdots = x_\ell = 0$, we set $\theta_i = 0$.

Notation 2.2.4. Let

$$V_i(\bar{\theta}_i, \ldots, \bar{\theta}_{\ell-1}) := \{P = (\rho, \theta_1, \ldots, \theta_{\ell-1}) \,|\, \theta_i = \bar{\theta}_i, \ldots, \theta_{\ell-1} = \bar{\theta}_{\ell-1}\}$$

By convention, we let $\rho = \theta_0$ if necessary. This space is an i-dimensional open half-subspace of $\mathbb{R}^\ell$. We will denote by $|V_i(\bar{\theta})|$ the linear span of $V_i(\bar{\theta})$.

Definition 2.2.5. A system of polar coordinates in $\mathbb{R}^\ell$, defined by an origin O and a basis $\mathbf{e}_1, \ldots, \mathbf{e}_\ell$ is *generic* with respect to the arrangement $\mathcal{A}_\mathbb{R}$ if it satisfies the following conditions:

1. the origin O is contained in a chamber C_0 of $\mathcal{A}_\mathbb{R}$;

2. there exists $\delta \in (0, \pi/2)$ such that the union of the bounded facets is contained in the open cone

$$\tilde{B}(\delta) := \{P = (\rho, \theta_1, \ldots, \theta_{\ell-1}) \in \mathbb{R}^\ell \,|\, \theta_i \in (0, \delta) \,\forall\, 1 \leq i \leq n-1, \rho > 0\}$$

3. subspaces $V_i(\bar{\theta}) = V_i(\bar{\theta}_i, \ldots, \bar{\theta}_{\ell-1})$ which intersect the closure of $\tilde{B}$ are generic with respect to $\mathcal{A}_\mathbb{R}$, in the sense that, for each k-codimensional subspace $L \in L(\mathcal{A}_\mathbb{R})$,

$$i \geq k \Rightarrow V_i(\bar{\theta}) \cap L \cap clos(\tilde{B}) \neq \emptyset \text{ and } dim(|V_i(\bar{\theta})| \cap L) = i - k$$

Example 2.2.6. Let $\mathcal{A}$ be a real 2-arrangement given by $Q = (3x - y - 5)(x - 3)(x - y - 1)$. Suppose we choose standard polar coordinates, so the origin is at

$(0,0)$ and our orthonormal basis is $\mathbf{v_1} = (1,0), \mathbf{v_2} = (1,0)$. Then the origin is in an unbounded chamber of $\mathcal{A}$, and the union of the bounded facets of $\mathcal{A}$ is contained in an open cone $\tilde{B}(\pi/3)$. However, $V_1(\pi/4)$ does not intersect the hyperplane H defined by the form $x - y - 1$, so these coordinates fail to be generic. See Figure 2.4.

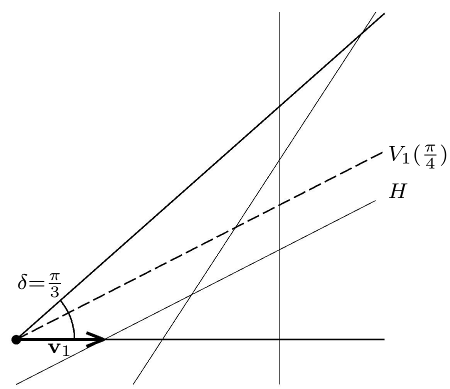


Figure 2.4: Non-generic polar coordinates in $\mathbb{R}^2$

These generic polar coordinates always exist; in fact, genericity is an open condition in $\mathbb{R}^\ell$, in the sense that the set of points of $\mathbb{R}^\ell$ with respect to which there exists a choice of polar coordinates which is generic is open in $\mathbb{R}^\ell$.

Now, a total order on $\mathcal{S}$ will be calculated which is induced by these polar coordinates. Given a codimensional-k facet $F \in \mathcal{S}$, and given $\theta = (\theta_i, \ldots, \theta_{\ell-1})$ with $\theta_j \in [0, \delta]$ for all $j = i, \ldots, \ell - 1$, denote by

$$F(\theta_i, \ldots, \theta_{\ell-1}) := F \cap V_i(\theta)$$

We allow $(\theta_i, \ldots, \theta_{\ell-1})$ to be empty; in this case $F(\emptyset) = F \cap V_\ell$. Note that by genericity conditions, if $i \geq k$ then $F(\theta)$ is either empty or it is a facet of codimension $\ell - (i - k)$ (with respect to $\mathbb{R}^\ell$) contained in $V_i(\theta)$.

Also, for any facet $F(\theta)$, let

$$i_{F(\theta)} = min\{j \geq 0 \,|\, V_j \cap \overline{F(\theta)} \neq \emptyset\}$$

where V_j is the linear subspace generated by $\mathbf{e}_1, \ldots, \mathbf{e}_j$.

Definition 2.2.7. Given any facet $F(\theta)$ (assume that $F(\theta) \neq \emptyset$), let the *minimum vertex* $P_{F(\theta)}$ be the unique 0-dimensional facet in its boundary which satisfies the following:

1. If $i_{F(\theta)} \geq i$ (meaning $\overline{F(\theta)} \cap V_{i-1} = \emptyset$), then $P_{F(\theta)}$ is the unique vertex such that
$$\theta_{i-1}(P) = min\{\theta_{i-1}(Q) \,|\, Q \in \overline{F(\theta)}\}$$

2. If $i_{F(\theta)} < i$, then the point $P_{F(\theta)}$ is either the origin O or it is the unique one such that
$$\theta_{i_{F(\theta)}-1}(P) = min\{\theta_{i_{F(\theta)}-1}(Q) \,|\, Q \in \overline{F(\theta)} \cap V_{i_{F(\theta)}}\}$$

The uniqueness of this vertex is implied by the genericity conditions. We will associate to the facet $F(\theta)$ the n-vector of polar coordinates of $P_{F(\theta)}$
$$\Theta(F(\theta)) := (\theta_0(F(\theta)), \ldots, \theta_{i_{F(\theta)}-1}(F(\theta)), 0, \ldots, 0)$$

Definition 2.2.8. Given $F, G \in \mathcal{S}$, and given $\bar{\theta} = (\bar{\theta}_i, \ldots, \bar{\theta}_{n-1})$, $0 \leq i \leq n$, $\bar{\theta}_j \in [0, \delta]$ for $j \in i, \ldots, n-1$, ($\bar{\theta} = \emptyset$ for $i = n$) such that $F(\bar{\theta}), G(\bar{\theta}) \neq \emptyset$, we set
$$F(\bar{\theta}) \lhd G(\bar{\theta})$$
if and only if one of the following cases occurs:

1. $P_{F(\bar{\theta})} \neq P_{G(\bar{\theta})}$. Then $\Theta(F(\bar{\theta})) < \Theta(G(\bar{\theta}))$ according to the anti-lexicographic ordering of the coordinates (i.e., the lexicographic ordering starting from the last coordinate).

2. $P_{F(\bar{\theta})} = P_{G(\bar{\theta})}$. Then either

   (a) $F(\bar{\theta})$ is a vertex, but $G(\bar{\theta})$ is not.

   (b) neither F nor G is a vertex. Let $i_0 := i_{F(\bar{\theta})} = i_{G(\bar{\theta})}$.

      i. If $i_0 \geq i$, then one can write
$$\Theta(F(\bar{\theta})) = \Theta(G(\bar{\theta})) = (\tilde{\theta}_0, \ldots, \tilde{\theta}_{i-1}, \bar{\theta}_i, \ldots, \bar{\theta}_{i_0-1}, 0, \ldots, 0)$$

Then, for all ϵ, $0 < \epsilon \ll \delta$, it must happen that

$$F(\tilde{\theta}_{i-1} + \epsilon, \bar{\theta}_i, \ldots, \bar{\theta}_{i_0-1}, 0, \ldots, 0) \lhd G(\tilde{\theta}_{i-1} + \epsilon, \bar{\theta}_i, \ldots, \bar{\theta}_{i_0-1}, 0, \ldots, 0)$$

ii. If $i_0 < i$, then one can write

$$\Theta(F(\bar{\theta})) = \Theta(G(\bar{\theta})) = (\tilde{\theta}_0, \ldots, \tilde{\theta}_{i_0-1}, 0, \ldots, 0)$$

Then, for all ϵ, $0 < \epsilon \ll \delta$, it must happen that

$$F(\tilde{\theta}_{i_0-1} + \epsilon, 0, \ldots, 0) \lhd G(\tilde{\theta}_{i_0-1} + \epsilon, 0, \ldots, 0)$$

This relation is irreflexive and transitive, so we have

Theorem 2.2.9. *The relation $\lhd$ is a total ordering on the facets of $V_i(\bar{\theta})$, for any given $\bar{\theta}$. In particular, it gives a total ordering on $\mathcal{S}$.*

Example 2.2.10. Let $\mathcal{A}$ be the arrangement of Example 2.2.6, so $\mathcal{A}$ is defined by $Q = (3x - y - 5)(x - 3)(x - y - 1)$. If we choose polar coordinates for $\mathbb{R}^2$ with the origin at $(-13/9, 1/2)$ and $\mathbf{v}_1 = (1, 0)$, then we get generic polar coordinates with $\delta = \arctan .9 < \pi/4$; see Figure 2.5(a). The unbounded chamber containing the origin intersects V_0 nontrivially, and so that chamber will be marked with a 1 (note that the minimum vertex of that chamber is the origin). There are three 1-facets and three chambers which intersect V_1 nontrivially, and so those are ordered next. Note that chamber 3 comes before edge 4 according to condition 1 of Definition 2.2.8, and also that edge 2 comes before chamber 3 according to condition 2a. Finally, the vertex P_1 is the minimum vertex for four different facets; those are ordered according to condition 2b. Figure 2.5(b) shows the ordering of all facets in $\mathcal{S}$.

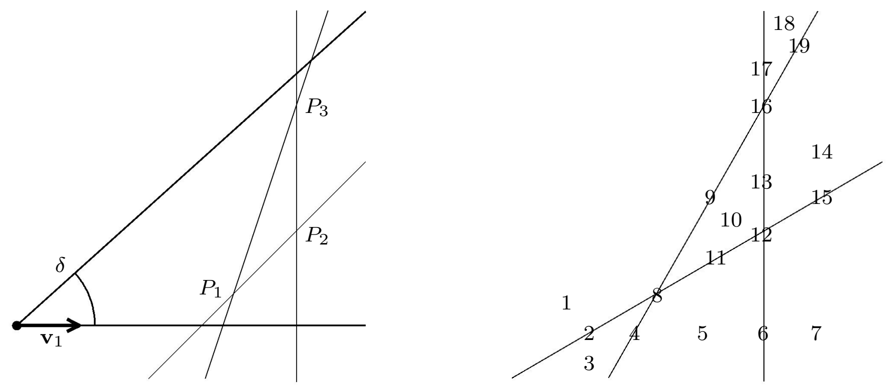
We will use this polar ordering of facets to define a combinatorial gradient vector field Φ over $\mathbf{S}$ which is the gradient of a combinatorial Morse function.

Definition 2.2.11. For $j = 0, \ldots, n-1$, let Φ_{j+1} be the collection of pairs of cells

$$\Phi_{j+1} = \{([C \prec F^j], [C \prec F^{j+1}]) \in \mathbf{S}_j \times \mathbf{S}_{j+1}\}$$

(where $\mathbf{S}_j$ is the j-skeleton of $\mathbf{S}$) so that

1. $F^j \prec F^{j+1}$



(a) Generic Polar Coordinates for $\mathcal{A}$ (b) Polar Ordering for $\mathcal{A}$

Figure 2.5: Polar ordering for affine 2-arrangements

2. $F^{j+1} \lhd F^j$

3. $([C \prec F^{j-1}], [C \prec F^j]) \notin \Phi_j \ \forall \ F^{j-1} \prec F^j$

Let $\Phi = \displaystyle\bigsqcup_{j=0}^{n-1} \Phi_{j+1}$.

Theorem 2.2.12. *One has:*

1. *Φ is a combinatorial vector field on $\mathbf{S}$ which is the gradient field of a combinatorial Morse function.*

2. *The pair $([C \prec F^j], [C \prec F^{j+1}])$ belongs to Φ if and only if the following conditions hold:*

   (a) $F^j \prec F^{j+1}$

   (b) $F^{j+1} \lhd F^j$

   (c) $\forall F^{j-1}$ such that $C \prec F^{j-1} \prec F^j$, one has $F^{j-1} \lhd F^j$.

3. *Given $F^j \in \mathcal{S}$, there exists a chamber C such that the cell $[C \prec F^j]$ is the "head" of arrow if and only if there exists $F^{j-1} \prec F^j$ with $F^j \lhd F^{j-1}$. More precisely, the pair which is in Φ is $([C \prec \bar{F}^{j-1}], [C \prec F^j])$, where $\bar{F}^{j-1}$ is the maximum $(j-1)$-facet (with respect to polar ordering) satisfying*

$$C \prec \bar{F}^{j-1} \prec F^j, \ F^j \lhd \bar{F}^{j-1}$$

4. *The set of k-dimensional critical cells is given by*

$$Sing_k(\mathbf{S}) = \{[C \prec F^k] \,|\, F^k \cap V_k = \emptyset, F^j \lhd F^k \forall C \prec F^j \precneqq F^k\}$$

Equivalently, $F^k \cap V_k$ *is the maximum (in polar ordering) among all facets of* $C \cap V_k$*.*

Corollary 2.2.13. *Once a polar ordering is assigned, the set of singular cells is described only in terms of it by:*
$Sing_k(\mathbf{S}) := \{[C \prec F^k] :$

1. $F^k \lhd F^{k+1} \; \forall \; F^{k+1}$ *s.t.* $F^k \prec F^{k+1}$

2. $F^{k-1} \lhd F^k \; \forall \; F^{k-1}$ *s.t.* $C \prec F^{k-1} \prec F^k$ $\}$

We note that the integral boundary of the Morse complex generated by these singular cells is zero, so we have minimality of this CW decomposition of the complement. We can use this minimal CW complex to combinatorially compute homology groups with local coefficients.

2.3 Rank-One Local Systems

If X is a path-connected space having a universal cover $\tilde{X}$ and fundamental group π, then given any left $\mathbb{Z}[\pi]$-module M, we may define homology groups with local coefficients in M. The module structure of M can come from any representation of π on M. The chain groups are $C_n(X; M) := C_n(\tilde{X}) \otimes_\pi M$, where $C_n(\tilde{X})$ is viewed as a right $\mathbb{Z}[\pi]$-module. The boundary maps are $\partial \otimes Id$.

In the case that we are taking a representation on $\mathbb{C}^n$, we say that we have a *rank-n local system*. In particular, we can choose a representation $\rho : \pi_1(M(\mathcal{A})) \to GL_1(\mathbb{C}) = \mathbb{C}^*$, in which case we have a rank-one local system. It is known that the fundamental group of an arrangement complement is generated by n transverse loops around each hyperplane in the arrangement, but the fundamental group is in general not abelian. For examples of presentations of the fundamental group see papers by Arvola [2], Cohen-Suciu [5], Falk [9], and Randell [17] [18]. However,

$GL_1(\mathbb{C})$ is abelian, so any representation chosen factors through the first integral homology group of the complement. In effect, then, a choice of representation is equivalent to the assignment of a non-zero complex number (which represents an automorphism of $\mathbb{C}$) to each hyperplane in $\mathcal{A}$. Once a choice of complex weights is made, we may use a minimal CW decomposition to combinatorially compute the local homology groups.

We are interested in computing homology with local coefficients because it tells us about ordinary homology of the *Milnor fiber* $F = Q^{-1}(1)$ of the fibration given by $Q : M(\mathcal{A}) \to \mathbb{C}^*$. A more thorough discussion is given in a paper by Cohen and Suciu [4]. F is a cyclic n-fold cover of the projectivized complement M^* of the arrangement $\mathcal{A}$, where n is the number of hyperplanes in the arrangement, or the degree of Q. Using a Leray-Serre argument on the fibration $p : F \to M^*$, we find that $H_p(F;\mathbb{C}) = H_p(M^*; H_0(p^{-1}(x_0);\mathbb{C}))$, or that $H_*(F;\mathbb{C}) = H_*(M^*;\mathcal{L})$, where $\mathcal{L}$ is a rank n local system given by a representation $\rho : \pi_1(M^*) \to GL_n(\mathbb{C})$. This particular representation decomposes into a direct sum of rank-one representations $\rho = \bigoplus_{k=0}^{n-1} \rho_k$, where ρ_k maps each generator to ξ^k or the kth power of a primitive nth root of unity. We use $\mathcal{L}_k$ to denote the kth rank-one local system. In short, we are interested in rank-one local systems because

$$H_*(F;\mathbb{C}) = \bigoplus_{k=0}^{n-1} H_*(M^*;\mathcal{L}_k)$$

2.4 The Module of Derivations

The final background notion we will need (in Chapter 4) is the module of derivations, which we now define.

If V is an ℓ-dimensional vector space over a field $\mathbb{K}$ (usually we will be working with $\mathbb{C}^\ell$ over $\mathbb{C}$), then let $x_1, x_2, \ldots, x_\ell$ be a basis for V^*, and let $S = S(V^*) \cong \mathbb{K}[x_1, x_2, \ldots, x_\ell]$ be the symmetric algebra of V^* over $\mathbb{K}$. If we wish to emphasize the dimension of V we will write $S(V_\ell^*)$.

Definition 2.4.1. We say that $\theta : S \to S$ is a *derivation* of S if θ is multilinear and satisfies the Leibniz rule: $\theta(fg) = \theta(f)g - f\theta(g)$ for all $f, g \in S$. Let $\mathrm{Der}_{\mathbb{K}}(S)$ be the free S-module of derivations of S.

A basis of $\mathrm{Der}_{\mathbb{K}}(S)$ may be given by the maps $\frac{\partial}{\partial x_i}$ for $1 \leq i \leq \ell$, where $\frac{\partial}{\partial x_i}(x_j) = \delta^i_j$. Therefore any derivation may be written in the form $f_1\frac{\partial}{\partial x_1} + f_2\frac{\partial}{\partial x_2} + \cdots + f_\ell\frac{\partial}{\partial x_\ell}$, where $f_1, f_2, \ldots, f_\ell \in \mathbb{K}[x_1, x_2, \ldots, x_\ell]$.

Definition 2.4.2. Let $\mathcal{A}$ be a hyperplane arrangement in V with defining polynomial Q. Then the *module of $\mathcal{A}$-derivations* is defined to be

$$D(\mathcal{A}) := \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \,|\, \theta(Q) \in QS\}$$

Although $D(\mathcal{A})$ is a submodule of a free S-module, it is not necessarily a free S-module itself. We are interested in situations in which $D(\mathcal{A})$ is a free S-module.

Definition 2.4.3. We say that $\mathcal{A}$ is *free* if $D(\mathcal{A})$ is a free S-module.

Example 2.4.4. The module of derivations for the empty arrangement Φ_ℓ in V^ℓ is actually all of $\mathrm{Der}_{\mathbb{K}}(S)$, which is of course free.

The following lemma is often handy in determining whether a given derivation is contained in $D(\mathcal{A})$:

Lemma 2.4.5. *[16] Let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ be a central arrangement and let α_i denote the linear form which determines H_i for $1 \leq i \leq n$. If $\theta \in \mathrm{Der}_{\mathbb{K}}(S)$, then $\theta \in D(\mathcal{A})$ if and only if $\theta \in D(\alpha_i)$ for all i. That is, θ is an $\mathcal{A}$-derivation if and only if $\theta(\alpha_i)$ is divisible by α_i for all i.*

Given a collection of $\mathcal{A}$-derivations $\theta_1, \theta_2, \ldots, \theta_\ell$, we will want to know if these derivations comprise an S-module basis for $D(\mathcal{A})$.

Definition 2.4.6. Let $\theta_i = f_{i1}\frac{\partial}{\partial x_1} + f_{i2}\frac{\partial}{\partial x_2} + \cdots + f_{i\ell}\frac{\partial}{\partial x_\ell} \in D(\mathcal{A})$ for $1 \leq i \leq \ell$. We may arrange the coefficient polynomials into a matrix, which we will call the *coefficient matrix*:

$$\mathsf{M} = \mathsf{M}(\theta_1, \ldots, \theta_\ell) := \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1\ell} \\ f_{21} & f_{22} & \cdots & f_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ f_{\ell 1} & f_{\ell 2} & \cdots & f_{\ell\ell} \end{bmatrix}$$

The following result is due to Saito [20] and is often referred to as *Saito's criterion*:

Theorem 2.4.7. *Given a central ℓ-arrangement $\mathcal{A}$, let $\theta_1, \theta_2, \ldots, \theta_n \in D(\mathcal{A})$ with $\theta_i = f_{i1}\frac{\partial}{\partial x_1} + f_{i2}\frac{\partial}{\partial x_2} + \cdots + f_{i\ell}\frac{\partial}{\partial x_\ell}$. Let M denote the ℓ by ℓ matrix of coefficient polynomials. Then $\{\theta_1, \theta_2, \ldots, \theta_\ell\}$ is a basis for $D(\mathcal{A})$ if and only if* $\det(\mathsf{M}) \doteq Q(\mathcal{A})$.

Example 2.4.8. Let $\mathcal{A}$ be a 2-arrangement defined by $Q(\mathcal{A}) = xy(x-y)$. Then

$$f_1\frac{\partial}{\partial x} + f_2\frac{\partial}{\partial y} \in D(\mathcal{A}) \Leftrightarrow f_1 \in xS,\ f_2 \in yS,\ \text{and } f_1 - f_2 \in (x-y)S$$

Two derivations which satisfy these conditions are $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ (this is known as the *Euler derivation* and is always in the module of derivations for any central arrangement) and $y(x-y)\frac{\partial}{\partial y}$. Using Saito's criterion, it is clear that these derivations form a basis, since

$$\det \begin{bmatrix} x & 0 \\ y & y(x-y) \end{bmatrix} = xy(x-y) = Q(\mathcal{A})$$

We note that this theorem also works for affine arrangements.

CHAPTER 3
LOCAL SYSTEM HOMOLOGY GROUPS FOR DISCRIMINANTAL ARRANGEMENTS

3.1 A Minimal CW-Decomposition

Let $A_{\ell,n}$ be the arrangement defined by

$$Q_{\ell,n} = \prod_{i=1}^{\ell}\prod_{j=1}^{n}(z_i - j) \prod_{1\le p<q\le \ell}(z_p - z_q)$$

as in section 2.1.

In this section a system, of polar coordinates will be determined which is suitable for application of Salvetti's and Settepanella's technique.

Choose

$$O = (n^2 - 2n + 3, 1 - 1/n)$$
$$\mathbf{v}_1 = (-n^2 + 3n - 3 + 1/n, n - 1 + 1/n),\ \mathbf{e}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$$
$$\delta = \arccos\frac{\mathbf{v}_1\cdot\mathbf{w}}{\|\mathbf{v}_1\|\|\mathbf{w}\|},\ \text{where } \mathbf{w} = (-n^2 + 2n - 2, 1/2n)$$

as a generic polar coordinate system for $\mathcal{A}$. The second basis vector $\mathbf{e}_2$ is simply $\mathbf{e}_1$ rotated $\pi/2$ radians in the positive direction.

Example 3.1.1. When $n = 2$, $\mathcal{A}$ contains 31 facets, which are ordered as in Figure 3.1.

Example 3.1.2. When $n = 2$, and given the above polar ordering, the Salvetti complex $\mathbf{S}$ for $\mathcal{A}$ has 12 critical cells (see Figure 3.1):

- 0-cells: $[1 \prec 1]$
- 1-cells: $[1 \prec 2], [3 \prec 4], [5 \prec 6], [7 \prec 8], [9 \prec 10]$
- 2-cells: $[3 \prec 22], [5 \prec 12], [7 \prec 12], [8 \prec 18], [14 \prec 26], [16 \prec 26]$

3.2 The Morse Complex for Local Homology

Using the critical cells obtained above, it is possible to compute the *Morse complex* and compute homology with local coefficients.

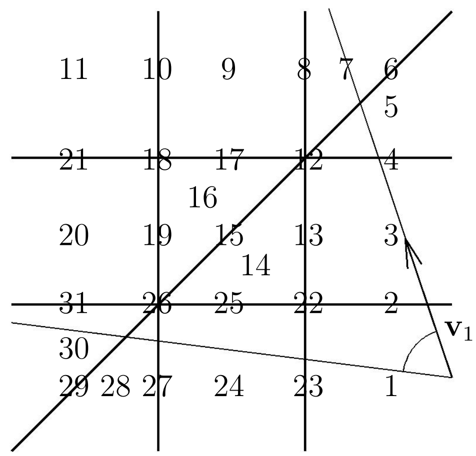


Figure 3.1: Polar Ordering for $\mathcal{A}$ when $n = 2$

Let L be the local system over $M(\mathcal{A})$ where $L = \mathbb{C}$, with base ring $\mathbb{Z}[\pi/\pi']$. Let $O \in C_0$ be the basepoint for the fundamental group. Take as generators for the fundamental group $2n+1$ small loops, one for each hyperplane and transverse to it, which are composed with minimal paths from the loop to O in order to produce representatives for $2n+1$ fundamental group elements. These generators act on L as automorphisms, so choosing a homomorphism from π_1 to $Aut(\mathbb{C})$ amounts to assigning a non-zero complex number to each hyperplane (and thus each π_1 generator) in $\mathcal{A}$. Denote these complex numbers as $z_1, z_2, \ldots, z_{2n+1}$, where for $1 \leq i \leq n$, z_i corresponds to the hyperplane $y = i$, z_{2n+2-i} corresponds to the hyperplane $x = i$, and z_{n+1} corresponds to the hyperplane $y = x$.

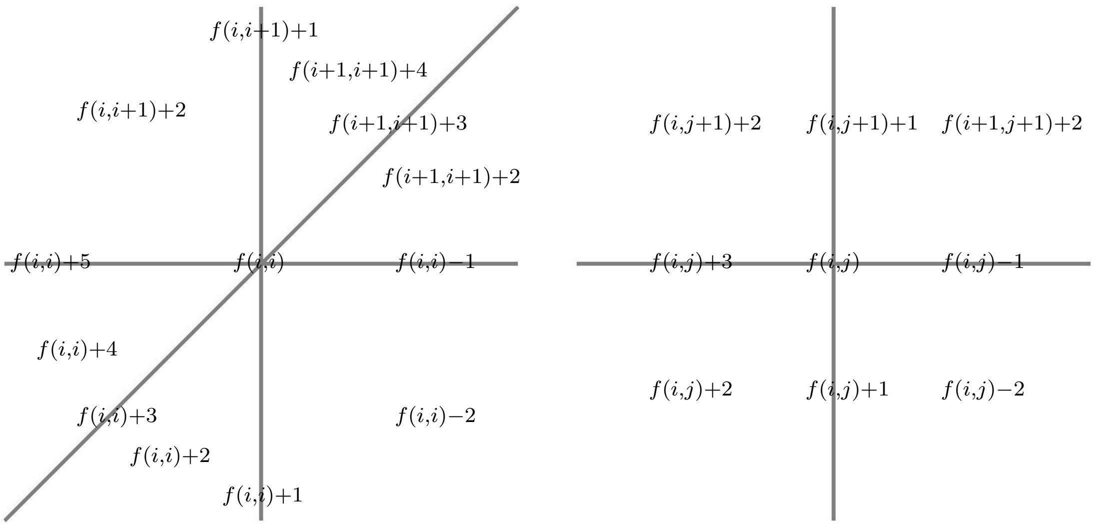
To simplify much of the notation and calculations that follow, define a function f as follows:

$$f : \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \to \{1, 2, \ldots, 4n^2 + 6n + 3\}$$

$$f(i,j) = 4n^2 + 6n + 2 - 4(n(y-1) + x) - 2\left(\left[\frac{n(y-1)+x}{n+1}\right]\right)$$

where $[x]$ is the greatest integer that is less than or equal to x. f is a function which assigns to (i,j) the polar ordering of the 2-facet located at $x = i$, $y = j$.

Locally, then, the facets around a double- and triple-point in the arrangement can be described as shown in Figure 3.2. Note that $f(n,n) < f(n-1,n) < \cdots < f(1,n) < f(n,n-1) < \cdots < f(1,n-1) < \cdots < f(1,1)$.



(a) Triple point (b) Double point

Figure 3.2: Polar ordering with respect to nearby 2-facet

Definition 3.2.1. A cell $[C \prec F] \in \mathbf{S}$ is called *locally critical* if F is the maximum, with respect to $\lhd$, of all facets in the interval $\{F' : C \prec F' \prec F\}$.

In particular, a 0-cell in $\mathbf{S}$ is always locally critical, and a 1-cell $[C \prec F] \in \mathbf{S}$ is locally critical if and only if $C \lhd F$, since $\{F' : C \prec F' \prec F\} = \{C, F\}$.

Notation 3.2.2. Recall that galleries of adjacent chambers in $\mathcal{A}$ uniquely correspond to positive paths in the 1-skeleton of $\mathbf{S}$. Also, for each $C \in \mathbf{S}$ there exists a unique positive path $\Gamma(C)$ which connects the origin O to the vertex of $\mathbf{S}$ corresponding to C [22].

Given an ordered sequence of (possibly not adjacent) chambers $C_1, \ldots, C_t$ let $u(C_i, C_{i+1})$ be a minimal positive path induced by a minimal gallery starting in C_i and ending in C_{i+1}, and let $u(C_1, \ldots, C_t)$ be the rel-homotopy class of $u(C_1, C_2)u(C_2, C_3) \cdots u(C_{t-1}, C_t)$.

Let

$$\overline{u}(C_1, \ldots, C_t) \in \pi_1(M(\mathcal{A}), O)$$

be the homotopy class of the path

$$\overline{u}(C_1, \ldots, C_t) := (\Gamma(C_1))^{-1} u(C_1, \ldots, C_t)\Gamma(C_t)$$

Finally, let

$$\overline{u}(C_1, \ldots, C_t)_* \in Aut(L)$$

be the automorphism induced by $\overline{u}(C_1, \ldots, C_t)$.

In order to state Salvetti's and Settepanella's theorem on the homology of the Morse complex, two more definitions will be needed.

Definition 3.2.3. Given a codimensional-k facet F^k such that $F^k \cap V_k \neq \emptyset$, an *ordered admissible k-sequence* is a sequence of pairwise different codimensional-$(k-1)$ facets

$$\mathcal{F}(F^k) := (F_{i_1}^{(k-1)}, \cdots, F_{i_m}^{(k-1)}),\ m \geq 1$$

such that

1. $F_{i_j}^{(k-1)} \prec F^k, \forall j$
2. $F^k \lhd F_{i_j}^{(k-1)}$ for $j < m$ while for the last element $F_{i_m}^{(k-1)} \lhd F^k$
3. $F_{i_1}^{(k-1)} \lhd \cdots \lhd F_{i_{m-1}}^{(k-1)}$

In order to compose two admissible k-sequences

$$\mathcal{F}(F^k) := (F_{i_1}^{k-1}, \cdots, F_{i_m}^{k-1})$$
$$\mathcal{F}(F'^k) := (F'^{k-1}_{j_1}, \cdots, F'^{k-1}_{j_l})$$

it must hold that

$$F_{i_m}^{k-1} \prec F'^k$$

If it happens that $F_{i_m}^{k-1} = F'^{k-1}_{j_1}$, write the facet only once, so there are no repetitions in the composition.

Definition 3.2.4. Given a critical k-cell $[C \prec F^k] \in \mathbf{S}$ and a critical $(k-1)$-cell

$[D \prec G^{k-1}] \in \mathbf{S}$, an *admissible sequence*

$$\mathcal{F} = \mathcal{F}_{([C\prec F^k],[D\prec G^{k-1}])}$$

for the given pair of critical cells is a sequence of pairwise different codimensional-$(k-1)$ facets

$$\mathcal{F} := (F_{i_1}^{(k-1)}, \cdots, F_{i_h}^{(k-1)})$$

obtained as a composition of ordered admissible k-sequences

$$\mathcal{F}(F_{j_1}^k) \cdots \mathcal{F}(F_{j_s}^k)$$

such that

1. $F_{j_1}^k = F^k$
2. $F_{i_h}^{(k-1)} = G^{(k-1)}$
3. $C.F_{i_1}^{(k-1)}. \cdots .F_{i_h}^{(k-1)} = D$
4. for all $j = 1, \ldots, h$ the $(k-1)$-cell $[C.F_{i_1}^{(k-1)}. \cdots .F_{i_j}^{(k-1)} \prec F_{i_j}^{(k-1)}]$ is locally critical.

Notation 3.2.5. Let $Seq := Seq([C \prec F^k], [D \prec G^{(k-1)}])$ denote the set of all admissible sequences for the given pair of critical cells. Let $s = (F_{i_1}^{(k-1)}, \cdots, F_{i_h}^{(k-1)}) \in Seq$ denote an element of Seq. Let

$$u(s) = u(C, C.F_{i_1}^{(k-1)}, \cdots, C.F_{i_1}^{(k-1)}. \cdots .F_{i_h}^{(k-1)})$$
$$\overline{u}(s) = \overline{u}(C, C.F_{i_1}^{(k-1)}, \cdots, C.F_{i_1}^{(k-1)}. \cdots .F_{i_h}^{(k-1)})$$

Finally, let $l(s) := h$ be the length of s, and let $b(s)$ be the number of k-sequences comprising s.

We are now ready to state Salvetti and Settepanella's main theorem [22].

Theorem 3.2.6. *The homology groups with local coefficients*

$$H_k(M(\mathcal{A}); L)$$

are computed by the algebraic complex (C_*, ∂_*) *such that, in dimension* k*,*

$$C_k := \oplus L.e_{[C\prec F^k]}$$

where there is one generator for each critical cell $[C \prec F^k]$ *in* $\mathbf{S}$ *of dimension* k.

The boundary operator is given by

$$\partial_k(l.e_{[C\prec F^k]}) = \sum A^{[C\prec F^k]}_{[D\prec G^{k-1}]}(l).e_{[D\prec G^{k-1}]}$$

where the incidence coefficient is given by

$$A^{[C\prec F^k]}_{[D\prec G^{k-1}]} := \sum_{s\in Seq} (-1)^{l(s)-b(s)}\overline{u}(s)_*$$

For the arrangement $\mathcal{A}$ given in Section 2, then:

1. C_0 is generated by one 0-cell, $\{[1 \prec 1]\}$.

2. C_1 is generated by $2n+1$ 1-cells: $\{[2m+1 \prec 2m+2] : 0 \le m \le 2n\}$.

3. C_2 is generated by n^2+n 2-cells:

$$\{[f(i+1,j+1)+2 \prec f(i,j)] : 1 \le i,j \le n-1\}$$
$$\cup\{[f(i+1,i+1)+4 \prec f(i,i)] : 1 \le i \le n-1\}$$
$$\cup\{[2j+1 \prec f(n,j)] : 1 \le j \le n\}$$
$$\cup\{[4n+3-2i \prec f(i,n)] : 1 \le i \le n\}$$

Lemma 3.2.7. *For* $n \ge 2$, $\partial_1(l.e_{[2m+1\prec 2m+2]}) = z_1z_2\cdots z_m(1-z_{m+1})l.e_{[1\prec 1]}$, *for* $1 \le m \le 2n$.

Proof. An admissible sequence for the pair $([2m+1 \prec 2m+2], [1 \prec 1])$ is obtained as a composition of admissible 1-sequences

$$\mathcal{F}(F^1_{j_1})\cdots\mathcal{F}(F^1_{j_s}) = (F^0_{i_1}, \cdots, F^0_{i_h})$$

such that

1. $F^1_{j_1} = 2m$

2. $F^0_{i_h} = 1$ and $2m+1.F^0_{i_1}.\cdots.F^0_{i_h} = 1$

3. $\forall 1 \le l \le h$ the 0-cell $[2m+1.F^0_{i_1}.\cdots.F^0_{i_l} \prec F^0_{i_l}]$ is locally critical

Note that admissible 1-sequences only exist for F^1 such that $F^1 \cap V_1 \neq \emptyset$, so admissible 1-sequences only exist for facets $2, 4, 6, \ldots, 2n$. Also, recall that 0-cells are always locally critical and that $C_1.C_2 = C_2$ for any two chambers C_1, C_2.

For the 1-facet $2m+2$, there are two choices for $\mathcal{F}(F^1_{j_1})$: $(2m+1)$ and $(2m+3, 2m+1)$. If $m=0$, then no further compositions need to be made. Otherwise, in order to compose $\mathcal{F}(F^1_{j_1})$ with some $\mathcal{F}(F^1_{j_2})$, the facet $F^1_{j_2}$ must satisfy:

1. $2k-1 \prec F^1_{j_2}$
2. $F^1_{j_2} \in \{2, 4, 6, \ldots, 2n\}$ (so that $\mathcal{F}(F^1_{j_2})$ will exist)

No repeats are allowed in the $F^1_{j_l}$. This implies that $F^1_{j_2} = 2m$, and $\mathcal{F}(F^1_{j_2}) = (2m+1, 2m-1)$ or $(2m-1)$. Therefore, $\mathcal{F}(F^1_{j_1})\mathcal{F}(F^1_{j_2}) = (2m+3, 2m+1, 2m-1)$ or $(2m+1, 2m-1)$.

Repeat this process m times. There are ultimately two admissible sequences:

$$s_1 = (2m+3, 2m+1, \ldots, 1) \quad \overline{u}(s_1) = \overline{u}(2m+1, 2m+3, 2m+1, \ldots, 1)$$
$$s_2 = (2m+1, 2m-1, \ldots, 1) \quad \overline{u}(s_2) = \overline{u}(2m+1, 2m+1, 2m-1, \ldots, 1)$$
$$l(s_1) = m+2 \quad b(s_1) = m+1$$
$$l(s_2) = m+1 \quad b(s_2) = m+1$$

$\overline{u}(s_1)$ is the homotopy class of the path which is the composition of minimal positive paths from 1 to $2m+3$ to $2m+1$ to $\ldots$ to 1, and this is the same as the homotopy class of the composition of the first $m+1$ generators of π_1. This implies that $\overline{u}(s_1)_* = z_1 \cdots z_{m+1}$. Similarly, $\overline{u}(s_2)_* = z_1 \cdots z_m$.

Therefore, $A^{[2m+1 \prec 2m+2]}_{[1 \prec 1]} = z_1 \cdots z_m(1 - z_{m+1})$. □

Note that if $z_1 = z_2 = \cdots = z_{2n+1} = 1$, then $H_0(M(\mathcal{A}), \mathbb{C})$ is isomorphic to $\mathbb{C}$. Otherwise, if $z_i \neq 1$ for any i, then

$$\partial_1 \left(\frac{1}{z_1 \cdots z_{i-1}(1 - z_i)} . e_{[2i-1 \prec 2i]} \right) = 1.e_{[1 \prec 1]}$$

and $H_0(M(\mathcal{A}), \mathbb{C})$ is trivial.

Now ∂_2 must be calculated. Let $[C \prec f(i,j)]$ denote a critical 2-cell in $\mathbf{S}$, where $1 \leq i, j \leq n$. Depending on the values of i and j, C can take one of three forms: $f(i+1,j+1)+2$, $f(i+1,j+1)+4$, or $2m+1$. Unless $i=j$, the polar ordering of C is uniquely determined, and its numeric value will not be important.

Claim 3.2.8. *Given $[C \prec f(i,j)]$, there are up to two types of admissible sequences pertaining to the pair $([C \prec f(i,j)], [2k+1 \prec 2k+2])$, where $1 \leq k \leq 2n$:*

1. *There are admissible sequences of the form*

$$\mathcal{F}(f(i,j))\mathcal{F}(f(i,j+1))\cdots\mathcal{F}(f(i,i))$$
$$\mathcal{F}(f(i+1,i+1))\cdots\mathcal{F}(f(N,N))\mathcal{F}(f(N+1,N))\cdots\mathcal{F}(f(n,N))$$

   where $N = min\{k+1, n\}$, if and only if $j \leq i \leq n-1$ and $i \leq k \leq n$

2. *There are admissible sequences of the form*

$$\mathcal{F}(f(i,j))\mathcal{F}(f(i,j+1))\cdots\mathcal{F}(f(i,N))\mathcal{F}(f(i+1,N))\cdots\mathcal{F}(f(n,N))$$

   if and only if one of the following holds:

   (a) $j-1 \leq k \leq n-1$

   (b) $k = 2n+1-i$

   (c) $k = i = n$

Proof. Let i,j,k be given. Since $F^2_{j_1} = f(i,j)$, the last 1-facet $F^1_{i_m}$ in $\mathcal{F}(f(i,j))$ must have lower polar order than $f(i,j)$, and $F^1_{i_m} \prec F^2_{j_2}$, it must hold that $F^2_{j_2} = f(i+1,j), f(i,j+1)$, or $f(i+1,i+1)$ in the case $i=j$. The same holds true for each $F^2_{j_l}$, provided those 2-facets exist.

Suppose a portion of a sequence of 1-facets is equal to

$$\ldots\mathcal{F}(f(i,j))\mathcal{F}(f(i+1,j))\mathcal{F}(f(i+1,j+1))\ldots$$

Then the last 1-facet of $\mathcal{F}(f(i,j))$ must be $f(i,j)-1$, so $C.F^1_{i_1}.\cdots.(f(i,j)-1) = f(i,j)-2$ or $f(i+1,j+1)+2$. Barring repeats in the list of 1-facets, there are two admissible 1-sequences to choose from (4 in the case $i=j$) and all of them result in non-locally critical 1-cells, making the sequence inadmissible. For example, in Figure 3.1, if $\mathcal{F}(f(i,j))\mathcal{F}(f(i+1,j))\mathcal{F}(f(i+1,j+1)) = \mathcal{F}(26)\mathcal{F}(22)\mathcal{F}(12)$, the last 1-facet of $\mathcal{F}(26)$ is 25, so $C.F^1_{i_1}.\cdots.25 = 14$ or 24, but $[24.23 \prec 23], [24.13 \prec 13], [14.23 \prec 23], [14.13 \prec 13]$ are all non-locally critical. The same argument holds for sequences containing strings of the form $\mathcal{F}(f(j-1,j))\mathcal{F}(f(j,j))\mathcal{F}(f(j+1,j+1))$ or $\mathcal{F}(f(i,i))\mathcal{F}(f(i+1,i+1))\mathcal{F}(f(i+1,i+2))$.

Now, assume that there exists an admissible sequence of the form

$$\mathcal{F}(f(i,j))\mathcal{F}(f(i,j+1))\cdots\mathcal{F}(f(i,i))$$
$$\mathcal{F}(f(i+1,i+1))\cdots\mathcal{F}(f(N,N))\mathcal{F}(f(N+1,N))\cdots\mathcal{F}(f(n,N))$$

Assume $i < N$ (otherwise the sequence is identical to the admissible sequence of the second type). Then $j \le i \le n-1$ and $i \le k \le n+1$. Suppose $k = n+1$, however. Then $N = n$, so the last two admissible 2-sequences in the composition are $\mathcal{F}(f(n-1,n-1))\mathcal{F}(f(n,n))$, and the last 1-facet in $\mathcal{F}(f(n-1,n-1))$ must be $f(n,n)+3$. There are then four possibilities for $\mathcal{F}(f(n,n))$ (again, eliminating repeats):

$$[(f(n,n)+2).(f(n,n)+1) \prec f(n,n)+1]$$
$$[(f(n,n)+2).(f(n,n)+5).(2n+4) \prec 2n+4]$$
$$[(f(n,n)+4).(f(n,n)+1) \prec f(n,n)+1]$$
$$[(f(n,n)+4).(2n+4) \prec 2n+4]$$

But these are all non-locally critical, contradicting the admissibility of the sequence. Therefore, $k \le n$.

Assume that there exists an admissible sequence of the form

$$\mathcal{F}(f(i,j))\mathcal{F}(f(i,j+1))\cdots\mathcal{F}(f(i,N))\mathcal{F}(f(i+1,N))\cdots\mathcal{F}(f(n,N))$$

Then $i \le N = min\{k+1,n\}$. If $N = n$, then $k+1 \ge n$ and one of three cases happens: either $k = n-1$, $k = n$, or $k = 2n+1-i$ for some i. If $k = n-1$, then it must hold that $j-1 \le k$. If $k = n$, then $i = n$, otherwise an argument similar to those above shows that the sequence violates the fourth condition of Definition 3.2.4 whenever $i < n = k$. Finally, if $N = k+1$, then $k \le n-1$, and since $j-1 \le k$, the conditions are all satisfied.

Now let i, j, k be given such that $j \leq i \leq n-1$ and $i \leq k \leq n$. Then

$$(f(i,j+1)+1, f(i,j+2)+1, \ldots, f(i,i)+1,$$
$$f(i+1,i+1)+3, f(i+2,i+2)+3, \ldots, f(N,N)+3,$$
$$f(N,N)-1, f(N+1,N)-1, \ldots, f(n-1,N)-1, 2k+2)$$

is an admissible sequence for the given pair.

Let i, j, k be given such that $j-1 \leq k \leq n-1$. Then

$$(f(i,j+1)+1, f(i,j+2)+1, \ldots, f(i,N)+1,$$
$$f(i,N)-1, f(i+1,N)-1, \ldots, f(n-1,N)-1, 2k+2)$$

is an admissible sequence for the pair.

If i, j, k are such that $k = 2n+1-i$ or $k = i = n$, then

$$(f(i,j+1)+1, f(i,j+2)+1, \ldots, f(i,n)+1, 2k+2)$$

is an admissible sequence for the pair. □

It remains to calculate $Seq([C \prec F^k], [D \prec G^{(k-1)}])$ along with the induced coefficient $A^{[C \prec F^k]}_{[D \prec G^{k-1}]}$ for each pair of critical cells which satisfies at least one of the above conditions.

First, assume that $([C \prec F^k], [D \prec G^{(k-1)}])$ satisfies the first condition of Claim 3.2.8 but not the second, so $j \leq i \leq n-1$ and $k = n$. If $j < i$, then there are four admissible sequences:

1. $(f(i,j+1)+1, f(i,j+2)+1, \ldots, f(i,i)+1,$
   $f(i+1,i+1)+3, f(i+2,i+2)+3, \ldots, f(n,n)+3, 2n+2)$

2. $(f(i,j+1)+1, f(i,j+2)+1, \ldots, f(i,i)+1,$
   $f(i,i)+3, f(i+1,i+1)+3, \ldots, f(n,n)+3, 2n+2)$

3. $(f(i,j)+1, f(i,j+1)+1, \ldots, f(i,i)+1,$
   $f(i+1,i+1)+3, f(i+2,i+2)+3, \ldots, f(n,n)+3, 2n+2)$

4. $(f(i,j)+1, f(i,j+1)+1, \ldots, f(i,i)+1,$

$f(i,i)+3, f(i+1,i+1)+3, \ldots, f(n,n)+3, 2n+2)$

These admissible sequences induce the following four galleries:

1. $(C, f(i,j+1)-2, f(i,j+2)-2, \ldots, f(i,i)-2,$
   $f(i+1,i+1)+2, f(i+2,i+2)+2, \ldots, f(n,n)+2, 2n+1)$

2. $(C, f(i,j+1)-2, f(i,j+2)-2, \ldots, f(i,i)-2,$
   $f(i,i)+2, f(i+1,i+1)+2, \ldots, f(n,n)+2, 2n+1)$

3. $(C, f(i,j)-2, f(i,j+1)-2, \ldots, f(i,i)-2,$
   $f(i+1,i+1)+2, f(i+2,i+2)+2, \ldots, f(n,n)+2, 2n+1)$

4. $(C, f(i,j)-2, f(i,j+1)-2, \ldots, f(i,i)-2,$
   $f(i,i)+2, f(i+1,i+1)+2, \ldots, f(n,n)+2, 2n+1)$

These galleries induce the following four fundamental group elements whose representatives are minimal paths from the basepoint in chamber 1 to each chamber in the gallery and finally back to the basepoint:

1. $z_1 \cdots z_n z_{n+2} \cdots z_{2n+1-i}$

2. $z_j z_1 \cdots z_n z_{n+2} \cdots z_{2n+1-i}$

3. $z_{2n+2-i} z_1 \cdots z_n z_{n+2} \cdots z_{2n+1-i}$

4. $z_j z_{2n+2-i} z_1 \cdots z_n z_{n+2} \cdots z_{2n+1-i}$

Note that $(-1)^{l(s)-b(s)}$ equals 1 for the first and fourth sequences and -1 for the second and third. Therefore,

$$A^{[C \prec F^k]}_{[D \prec G^{k-1}]} = (z_j - 1)(z_{2n+2-i} - 1) z_1 \cdots z_n z_{n+2} \cdots z_{2n+1-i}$$

If $j = i$, then either $C = f(i+1,i+1)+2$ or $C = f(i+1,i+1)+4$. If $C = f(i+1,i+1)+2$, then there are four admissible sequences:

1. $(f(i+1,i+1)+3, f(i+2,i+2)+3, \ldots, f(n,n)+3, 2n+2)$

2. $(f(i,i)+1, f(i+1,i+1)+3, f(i+2,i+2)+3, \ldots, f(n,n)+3, 2n+2)$

3. $(f(i,i)+3, f(i+1,i+1)+3, f(i+2,i+2)+3, \ldots, f(n,n)+3, 2n+2)$

4. $(f(i,i)+1, f(i,i)+3, f(i+1,i+1)+3, f(i+2,i+2)+3, \ldots, f(n,n)+3, 2n+2)$

The third and fourth sequences cancel because they induce identical π_1-elements, but their lengths $l(s)$ differ by 1, so their signs are opposite. In other words the fourth sequence is the m-extension of the third sequence by the 1-facet $f(i,i)+1$ [22, Theorem 8]. Therefore,

$$A^{[C \prec F^k]}_{[D \prec G^{k-1}]} = (1 - z_i) z_1 \cdots z_n z_{n+2} \cdots z_{2n+1-i}$$

If $C = f(i+1, i+1) + 4$, then there are two admissible sequences:

1. $(f(i,i)+1, f(i+1,i+1)+3, f(i+2,i+2)+3, \ldots, f(n,n)+3, 2n+2)$

2. $(f(i,i)+1, f(i,i)+3, f(i+1,i+1)+3, f(i+2,i+2)+3, \ldots, f(n,n)+3, 2n+2)$

Therefore

$$A^{[C \prec F^k]}_{[D \prec G^{k-1}]} = (z_{2n+2-i} - 1) z_1 \cdots z_{i-1} {z_i}^2 z_{i+1} \cdots z_{2n+1-i}$$

Now, assume $([C \prec F^k], [D \prec G^{(k-1)}])$ satisfies both of the conditions in Claim 3.2.8, so $j \leq i \leq n-1$, $i \leq k \leq n$ and $j - 1 \leq k \leq n-1$. Then:

$$A^{[C \prec F^k]}_{[D \prec G^{k-1}]} = \begin{cases} (z_j - 1)(z_{2n+2-i} - 1) z_1 \cdots z_k z_{n+2} \cdots z_{2n+1-i} & \text{if } j < i \\ (z_i - 1)(z_{n+1} z_{2n+2-i} - 1) z_1 \cdots z_k z_{n+2} \cdots z_{2n+1-i} & \text{if } i = j \text{ and } C = \\ & f(i+1, i+1) + 2 \\ (z_i - 1)(z_{2n+2-i} - 1) z_1 \cdots z_k z_{n+1} \cdots z_{2n+1-i} & \text{if } i = j \text{ and } C = \\ & f(i+1, i+1) + 4 \end{cases}$$

Finally, assume that $([C \prec F^k], [D \prec G^{(k-1)}])$ satisfies the second condition in Claim 3.2.8 but not the first, so either $i < j$, $i = n$, $i > k$, or $k > n$. If $k = i = n$, then:

$$A^{[C \prec F^k]}_{[D \prec G^{k-1}]} = \begin{cases} (z_j - 1)(z_{n+2} - 1) z_1 \cdots z_n & \text{if } j \leq n-1 \\ (1 - z_n) z_1 \cdots z_n & \text{if } j = n \text{ and } C = 2n+1 \\ (z_{n+2} - 1) z_1 \cdots z_{n-1} {z_n}^2 z_{n+1} & \text{if } j = n \text{ and } C = 2n+3 \end{cases}$$

$$\begin{bmatrix} a(d-1) & a(a-1)(d-1) & ab(a-1)(d-1) & abc(1-a) & 0 \\ 0 & ab(cd-1) & ab(1-b) & abc(1-b) & 0 \\ 0 & abc(d-1) & ab^2c(d-1) & abc(1-bc) & 0 \\ 0 & abcd(e-1) & 0 & 0 & abcd(1-b) \\ ad(ce-1) & ad(a-1)(ce-1) & abd(1-a) & 0 & abcd(1-a) \\ acd(e-1) & acd(a-1)(e-1) & a^2bcd(e-1) & 0 & abcd(1-ac) \end{bmatrix}$$

Figure 3.3: Transpose of matrix for ∂_2 when $n = 2$, with variable substitutions

If $k = 2n + 1 - i$, then:

$$A^{[C \prec F^k]}_{[D \prec G^{k-1}]} = \begin{cases} (1 - z_j)z_1 \cdots z_{2n+1-i} & \text{if } C = f(i+1, j+1) + 2 \\ (1 - z_j z_{n+1})z_1 \cdots z_{2n+1-i} & \text{if } i = j \text{ and } C = f(i+1, i+1) + 4 \end{cases}$$

If $j - 1 < k \leq n - 1$, then either $i < j$ or $i > k$, and:

$$A^{[C \prec F^k]}_{[D \prec G^{k-1}]} = \begin{cases} (z_j - 1)(z_{2n+2-i} - 1)z_1 \cdots z_k z_{n+2} \cdots z_{2n+1-i} & \text{if } i > k \\ (z_j - 1)(z_{2n+2-i} - 1)z_1 \cdots z_k z_{n+1} \cdots z_{2n+1-i} & \text{if } i < j \end{cases}$$

Finally, if $j - 1 = k \leq n - 1$, then either $i < j$ or $i > k$, and:

$$A^{[C \prec F^k]}_{[D \prec G^{k-1}]} = \begin{cases} (z_{2n+2-i} - 1)z_1 \cdots z_k z_{n+1} \cdots z_{2n+1-i} & \text{if } i < j \\ (z_{n+1} z_{2n+2-i} - 1)z_1 \cdots z_i z_{n+2} \cdots z_{2n+1-i} & \text{if } i > k,\ i = j, \text{ and} \\ & \quad C = f(i+1, i+1) + 2 \\ (z_{2n+2-i} - 1)z_1 \cdots z_i z_{n+1} \cdots z_{2n+1-i} & \text{if } i > k,\ i = j, \text{ and} \\ & \quad C = f(i+1, i+1) + 4 \\ (z_{n+2} - 1)z_1 \cdots z_j & \text{if } i > k \text{ and } i = n \\ (z_{2n+2-i} - 1)z_1 \cdots z_j z_{n+2} \cdots z_{2n+1-i} & \text{if } i > k \text{ and } j < i < n \end{cases}$$

Example 3.2.9. When $n = 2$, there are five critical 1-cells and six critical 2-cells. The matrix representing the second boundary operator is showing in Figure 3.3, where the 1-cells and 2-cells are ordered as in Example 3.1.2. For display purposes, the following substitutions have been made: $z_1 = a$, $z_2 = b$, $z_3 = c$, $z_4 = d$, and $z_5 = e$.

3.3 Computations and Examples

Theorem 3.3.1. *Let n be given, along with automorphisms $\{z_1, z_2, \ldots, z_{2n+1}\}$ as in Section 3.2. Then the rank of the matrix representing the second boundary map is as follows:*

1. *If $z_1 = z_2 = \cdots = z_{2n+1} = 1$, then $\mathrm{rk}[\partial_2] = 0$, and the Betti numbers for the complement are $b_0(M(\mathcal{A})) = 1$, $b_1(M(\mathcal{A})) = 2n+1$, and $b_2(M(\mathcal{A})) = n^2 + n$.*

2. *If $z_1 = z_2 = \cdots = z_{n+1} = 1$, and if there exists some i, $n+2 \le i \le 2n+1$, such that $z_i \ne 1$, then $\mathrm{rk}[\partial_2] = n+1$. Similarly, if $z_{n+1} = \cdots = z_{2n+1} = 1$, and if there exists some i, $1 \le i \le n$, such that $z_i \ne 1$, then $\mathrm{rk}[\partial_2] = n+1$. In this case, the Betti numbers are $b_0(M(\mathcal{A})) = 0$, $b_1(M(\mathcal{A})) = n-1$, and $b_2(M(\mathcal{A})) = n^2 - 1$.*

3. *If $z_1, z_2, \ldots, z_{2n+1}$ do not satisfy the above two cases, and if:*

   (a) there are two indices i, j in $\{1, 2, \ldots, n\}$ satisfying: if $k \ne i, j$, then $z_k = z_{2n+2-k} = 1$, and at least one of z_i, z_j, z_{2n+2-i}, or z_{2n+2-i} is not equal to 1.

   (b) at each double point in the arrangement, either $z_i = z_j$ or $z_i = 1$ or $z_j = 1$,

   (c) at each triple point in the arrangement, either $z_i z_{n+1} z_{2n+2-i} = 1$ or $z_i = z_{2n+2-i} = 1$,

   (d) either $\prod_{i=1}^{2n+1} z_i = 1$ or $\prod_{i=1}^{n+1} z_i = \prod_{i=n+1}^{2n+1} z_i = 1$,

   then $\mathrm{rk}[\partial_2] = 2n-1$. In this case, the Betti numbers are $b_0(M(\mathcal{A})) = 0$, $b_1(M(\mathcal{A})) = 1$, and $b_2(M(\mathcal{A})) = n^2 - n + 1$.

4. If $z_1, z_2, \ldots, z_{2n+1}$ do not satisfy the above three cases, then $\mathrm{rk}[\partial_2] = 2n$*, and the Betti numbers are* $b_0(M(\mathcal{A})) = 0$*,* $b_1(M(\mathcal{A})) = 0$*, and* $b_2(M(\mathcal{A})) = n^2 - n$*.*

Proof. Fix the matrix representing the second boundary map by ordering the critical 1 cells and 2 cells in ascending order with respect to the polar ordering of C in $[C \prec F]$. See Example 3.1.2 and Figure 3.3.

If $z_1 = \cdots = z_{2n+1} = 1$, then $[\partial_2]$ is the zero matrix.

If $z_1 = \cdots = z_{n+1} = 1$, and if there exists some i, $n+2 \leq i \leq 2n+1$, such that $z_i \neq 1$, then the bottom n rows of the matrix are identically zero. If $z_{n+2} \neq 0$, then the minor formed by the first $n+1$ rows and the first $n+1$ columns is an upper triangular matrix with determinant $(z_{n+2} - 1)^{n+1} \neq 0$. Otherwise, let $i \geq 2$ be the *least* integer such that $z_{n+i} \neq 0$. Then the minor formed by rows $\{1, 2, \ldots, n+1\}$ and columns $\{n + (i-1), 2n + (i-1), \ldots, (i-1)n + (i-1)\} \cup \{(i-1)(n+1) - 1, (i-1)(n+1) - 1 + n, \ldots, (i-1)(n+1) - 1 + (n-i+1)n\}$ has determinant equal to $\pm(z_{n+i} - 1)^{n+1} \neq 0$. Therefore the rank of $[\partial_2]$ must be equal to $n+1$.

If $z_{n+1} = \cdots = z_{2n+1} = 1$, and if there exists some i, $1 \leq i \leq n$, such that $z_i \neq 1$, then the top n rows of the matrix are identically zero. A similar argument finding $(n+1) \times (n+1)$ minors with nonzero determinants shows that the rank of $[\partial_2]$ must be $n+1$.

Now assume that $z_1, \ldots, z_{2n+1}$ satisfy condition (3). Let i, j be the special indices.

If $z_{2n+2-i} = z_{2n+2-j} = 1$, then $z_{n+1} \neq 1$ (otherwise the automorphisms satisfy case (2)) and without loss of generality $z_i \neq 1$. By condition (3c), then, $z_{n+1} = \frac{1}{z_i}$, and by condition (3d), $z_j = 1$. With these automorphisms, rows i, $n{+}1$, and $2n{+}2{-}i$ in $[\partial_2]$ are linearly dependent, so $\mathrm{rk}[\partial_2] \leq 2n - 1$. Further, the $(2n-1) \times (2n-1)$ minor given by taking columns $\{n + k(n+1) : 0 \leq k \leq n-1\} \cup \{(n+1)(n+1-k) : 1 \leq k \leq n, k \neq i\}$ and striking out rows $n+1$ and $2n+2-i$ has determinant equal to $\frac{\pm(z_i - 1)^{2n-1}}{{z_i}^{n-2+i}} \neq 0$. A similar argument holds if $z_i = z_j = 1$.

$$\begin{bmatrix} (z-1)z & 0 & 0 & 0 & (1-z)z & \frac{z(y-1)}{y} \\ (z-1)^2 z & z(1-y) & z-1 & z(y-1) & -(z-1)^2 z & \frac{(y-1)(z-1)z}{y} \\ yz(z-1)^2 & (1-y)yz & y(z-1) & 0 & (1-z)yz^2 & (y-1)z^2 \\ 1-z & 1-y & \frac{z-1}{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-y)z & (1-z)z & \frac{(y-1)z}{y} \end{bmatrix}$$

Figure 3.4: ∂_2 for $n = 2$, automorphisms $(z, y, \frac{1}{yz}, z, y)$

If $z_i \neq 1$ and $z_j \neq 1$, then $z_{2n+2-i} = \frac{1}{z_i z_{n+1}}$ and $z_{2n+2-j} = \frac{1}{z_j z_{n+1}}$ by condition (3c). Then, by conditions (3d), either $z_i z_j z_{n+1} = 1$ or $z_{n+1} = 1$. If $z_{n+1} = 1$, then by condition (3b), $z_i z_j = 1$, which implies that $z_i z_j z_{n+1} = 1$. In either case, substitute $z_{n+1} = \frac{1}{z_i z_j}$, $z_{2n+2-i} = z_j$, and $z_{2n+2-j} = z_i$. With this choice of automorphisms, many of the columns of $[\partial_2]$ are linearly dependent, but there are 2^{2n-1} choices of $2n-1$ columns which have non-zero determinant (after the appropriate rows have been stricken out). For example, see Figure 3.4, which shows $[\partial_2]$ for automorphisms $(z, y, 1/(zy), z, y)$. Note the linear dependence of columns 5,6 and columns 2,3. There will always be two sets of two linearly dependent columns (their indices will vary with i and j), and for $n > 2$, there will be $2n-4$ sets of 3 pairwise-linearly dependent columns and $n^2 - 5n + 6$ zero columns. Now, since $n^2 + n - (n^2 + 5n - 6) - 2(2n-4) - 2 = 2n$, there are $2n$ columns that are possibly linearly independent. At this point, one could check all $2n+1$ minors of the submatrix to see that they are all zero. Therefore, $\mathrm{rk}[\partial_2] = 2n - 1$.

Finally, if $z_i \neq 1$ and $z_j = 1$, then $z_{2n+2-i} = \frac{1}{z_i z_{n+1}}$, which implies that either $z_{2n+2-j} = 1$ or $\frac{1}{z_{n+1}}$. An argument similar to the previous one will work in the latter case, since $z_{2n+2-j} = \frac{1}{z_{n+1}}$ implies that $z_{n+1} = \frac{1}{z_i}$ and therefore $z_{2n+2-i} = 1$. On the other hand, if $z_{2n+2-j} = 1$, there will be $2n-1$ sets of two linearly dependent columns, and all other columns will be identically zero. This implies that the rank of the matrix is at most $2n-1$, and if one of each of these sets of columns is chosen (and the i-th and $n+1$-th rows struck out), there is a minor of size $2n-1 \times 2n-1$ with determinant equal to a polynomial whose only roots are $z_i = 1$ and $z_i z_{n+1} = 1$.

Now, assume that the automorphisms fall into the fourth category. Then there exists $i \leq n+1$ and $j \geq n+1$ such that $z_i \neq 1$ and $z_j \neq 1$, and the automorphisms must fail one of the condition in part (3) of the theorem. In each case, a $2n \times 2n$ minor will be found in the matrix representing the boundary which has non-zero determinant. Note that the rank cannot be $2n+1$ since the matrix represents a homology boundary map.

Assume the automorphisms fail condition (3b), so there exist $1 \leq i < j \leq n$ such that $z_i \neq 1$, $z_{2n+2-j} \neq 1$ and $z_i \neq z_{2n+2-j}$. If $j = n$, then the minor formed by columns $\{1, 2, \ldots, n-1, n+1, n+2, \ldots, 2n, n^2+n-1-(i-1)(n+1)\}$ with the $2n+2-j$-th row stricken out has a determinant which is non-zero. If $j < n$, then the minor formed by columns $\{i, n^2+n-i, n^2+n-i-n, \ldots, n^2+n-i-n(i-1), 2n+1-i, 3n+1-i, \ldots, (n-i+1)n+1-i, n^2+n-(n+1)(j-1), n^2+n-j-n(i-1)\}$ with the $2n+2-j$-th row stricken out has a non-zero determinant.

□

Example 3.3.2. Let each z_i be equal to the same mth root of unity, $\eta = e^{2\pi i/m}$. Then the rank of $[\partial_2]$ is $2n$ unless $n = 2$ and $m = 3$. This is the example shown in Figure 3.1. This partially recovered a known result of Cohen and Suciu [4], in which they compute ranks of homology groups with local coefficients using the same representations (which correspond to cyclic covers) but for an arbitrary arrangement.

Example 3.3.3. Let $z_1 = \cdots = z_n = z_{n+2} = \cdots = z_{2n+1} = z \neq 1$, and let $z_{n+1} = y$. Then the rank of $[\partial_2]$ is $2n$ unless $n = 2$ and $y = \frac{1}{z^2}$, in which case the rank is $2n-1$. This was a result obtained in [6], and it is used to compute a faithful representation of the braid group called the LKB representation.

Example 3.3.4. In this example, let $z_{2n+2-i} = z_i$, for $1 \leq i \leq n$, and let $z_{n+1} = y$. Then, if one or two indices are chosen such that $z_i = z \neq 1$, if all other $z_k = 1$ and if $y = \frac{1}{z^2}$, then the rank of $[\partial_2]$ is equal to $2n-1$, and the first local homology group is nontrivial. Otherwise, the rank is $2n$. These computations are useful in computing solutions to KZ-equations.

CHAPTER 4
FREENESS OF ARRANGEMENT BUNDLES

4.1 Freeness of Affine Hyperplane Arrangements

Throughout this section, let $\mathbb{K}$ be an infinite field. We wish to extend known results about central arrangements over $\mathbb{K}$ to affine arrangements over $\mathbb{K}$.

Recall that V_ℓ is an ℓ-dimensional vector space over $\mathbb{K}$, and that $S(V_\ell^*)$ is the symmetric algebra over V_ℓ^*, so that $S(V_\ell^*) \cong \mathbb{K}[x_1, \ldots, x_\ell]$. When we do not need to emphasize the dimension of the vector space, we will simply write S. Recall also that $\mathrm{Der}_{\mathbb{K}}(S)$ is the free S-module of $\mathbb{K}$-linear maps which satisfy the Leibniz rule. Any of these maps, or derivations, may be written in the form $\theta = \sum_{i=1}^{\ell} f_i \frac{\partial}{\partial x_i}$, where $f_i \in S$. Corresponding to an arrangement, there is a submodule of $\mathrm{Der}_{\mathbb{K}}(S)$ called the *module of $\mathcal{A}$-derivations*, defined as follows:

$$D(\mathcal{A}) := \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(Q) \in QS\}$$

In the case of a central arrangement, if this module is free as an S-module, then we say that $\mathcal{A}$ is a *free arrangement.*

The following definition may be found in Orlik and Terao's book [16]:

Definition 4.1.1. A nonzero element $\theta \in \mathrm{Der}_{\mathbb{K}}(S)$ is *homogeneous of polynomial degree* p if $\theta = \sum_{i=1}^{\ell} f_i \frac{\partial}{\partial x_i}$ and f_i is homogeneous of degree p for $1 \le i \le \ell$. In this case we write $\mathrm{pdeg}\theta = p$.

We will extend the definition of polynomial degree to all derivations as follows:

Definition 4.1.2. A nonzero element $\theta \in \mathrm{Der}_{\mathbb{K}}(S)$ is *of polynomial degree* p if $\theta = \sum_{i=1}^{\ell} f_i \frac{\partial}{\partial x_i}$ and $\max\{\deg f_i \mid 1 \le i \le \ell\} = p$.

Note 4.1.3. Definition 4.1.1 can be used to define a grading on $\mathrm{Der}_{\mathbb{K}}(S)$ by defining $\mathrm{Der}_{\mathbb{K}}(S)_p$ to be the set of all derivations which are homogeneous of degree p when

$p \geq 0$ and defining $\mathrm{Der}_{\mathbb{K}}(S)_p = 0$ if $p < 0$, so

$$\mathrm{Der}_{\mathbb{K}}(S) = \bigoplus_{p \in \mathbb{Z}} \mathrm{Der}_{\mathbb{K}}(S)_p$$

The extended definition does not give us any such structure, and we will not need it. Therefore, in what follows, we will use pdeg to mean polynomial degree, with the understanding that any derivation which is of polynomial degree p and whose coefficients are all homogeneous of degree p is "homogeneous of polynomial degree p" in the sense of Orlik and Terao.

We note also that Saito's criterion (see Theorem 2.4.7) remains valid for affine arrangements; we will use it to prove the following theorem.

Theorem 4.1.4. *Let $\mathcal{A}$ be an affine ℓ-arrangement over an infinite field $\mathbb{K}$ defined by a polynomial Q. Then the cone $\mathbf{c}\mathcal{A}$ is free as a central $(\ell + 1)$-arrangement if and only if $D(\mathcal{A})$ is a free $S(V_\ell^*)$-module and if there is a basis $\{\theta_1, \theta_2, \ldots, \theta_\ell\}$ for $D(\mathcal{A})$ such that $\sum_{i=1}^{\ell}$ pdeg $\theta_i = \deg Q$.*

Proof. ($\Leftarrow$) Assume $\mathcal{A}$ is a free central $(\ell + 1)$-arrangement. We will show that any decone of $\mathcal{A}$, $\mathbf{d}\mathcal{A}$, is a free affine ℓ-arrangement.

Without loss of generality, assume that $Q(\mathcal{A}) = x_{\ell+1} \prod \alpha_H$, where the form $x_{\ell+1}$ corresponds to the distinguished hyperplane H_0. For any $H \in \mathcal{A}$, we may write $\alpha_H = c_1 x_1 + \cdots + c_\ell x_\ell + c_{\ell+1} x_{\ell+1}$, and for $H \neq H_0$ we have $c_i \neq 0$ for at least one i between 1 and ℓ.

Since $\mathcal{A}$ is free, there exists a homogeneous basis $\{\theta_1, \ldots, \theta_\ell, \theta_E\}$ for $D(\mathcal{A})$, where $\theta_E = \sum_{i=1}^{\ell} x_i \frac{\partial}{\partial x_i}$ is the Euler derivation and $\theta_i = \sum_{j=1}^{\ell+1} F_{j,i} \frac{\partial}{\partial x_j}$. Since $\theta_i(\alpha_H) \in \alpha_H S(V_{\ell+1}^*)$ for all $H \in \mathcal{A}$, we have $\theta_i(\alpha_{H_0}) = \theta_i(x_{\ell+1}) = F_{\ell+1,i} \in x_{\ell+1} S(V_{\ell+1}^*)$ for all $1 \leq i \leq \ell + 1$. We write $F_{\ell+1,i} = x_{\ell+1} f_{\ell+1,i}$ for each i. The matrix of coefficients for this basis, denoted $\mathsf{M} = \mathsf{M}(\theta_1, \ldots, \theta_\ell, \theta_E)$, is given by

$$\mathsf{M} = \begin{bmatrix} F_{1,1} & F_{1,2} & \dots & F_{1,\ell} & x_1 \\ F_{2,1} & F_{2,2} & \dots & F_{2,\ell} & x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{\ell,1} & F_{\ell,2} & \dots & F_{\ell,\ell} & x_\ell \\ x_{\ell+1}f_{\ell+1,1} & x_{\ell+1}f_{\ell+1,2} & \dots & x_{\ell+1}f_{\ell+1,\ell} & x_{\ell+1} \end{bmatrix}$$

Another basis for $D(\mathcal{A})$ is therefore given by

$$\mathsf{M}' = \begin{bmatrix} F_{1,1} - x_1 f_{\ell+1,1} & F_{1,2} - x_1 f_{\ell+1,2} & \dots & F_{1,\ell} - x_1 f_{\ell+1,\ell} & x_1 \\ F_{2,1} - x_2 f_{\ell+1,1} & F_{2,2} - x_2 f_{\ell+1,2} & \dots & F_{2,\ell} - x_2 f_{\ell+1,\ell} & x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{\ell,1} - x_\ell f_{\ell+1,1} & F_{\ell,2} - x_\ell f_{\ell+1,2} & \dots & F_{\ell,\ell} - x_\ell f_{\ell+1,\ell} & x_\ell \\ 0 & 0 & \dots & 0 & x_{\ell+1} \end{bmatrix}$$

Fixing i and suppressing some subscripts, let $\Psi := (F_1 - x_1 f_{\ell+1})\frac{\partial}{\partial x_1} + \cdots + (F_\ell - x_\ell f_{\ell+1})\frac{\partial}{\partial x_\ell}$. Ψ is a derivation in $\mathrm{Der}_{\mathbb{K}}(S(V_{\ell+1}^*))$ which we will use to create a derivation $\psi \in \mathrm{Der}_{\mathbb{K}}(S(V_\ell^*))$. To do this, we will simply substitute $x_{\ell+1} = 1$ in all of the coefficient polynomials. Note that the mapping $\nu : \mathbb{K}[x_1, \dots, x_{\ell+1}] \to \mathbb{K}[x_1, \dots, x_\ell]$ that does this is a ring homomorphism, and $Q(\mathbf{d}\mathcal{A}) = \prod \nu(\alpha_H) = \nu(Q(\mathcal{A}))$. If $g \in \mathbb{K}[x_1, \dots, x_{\ell+1}]$ satisfies $\Psi(\alpha_H) = \sum c_j(F_j - x_j f_{\ell+1}) = \alpha_H g$ for any $H \neq H_0$ in $\mathcal{A}$, then

$$\begin{aligned} \psi(\nu(\alpha_H)) &= \textstyle\sum_{j=1}^{\ell} c_j \nu(F_j - x_j f_{\ell+1}) \\ &= \nu\left(\textstyle\sum_{j=1}^{\ell} c_j (F_j - x_j f_{\ell+1})\right) \\ &= \nu(\alpha_H g) \\ &= \nu(\alpha_H)\nu(g) \in \nu(\alpha_H) S(V_\ell^*) \end{aligned}$$

and this implies that $\psi \in D(\mathbf{d}\mathcal{A})$. If we take the determinant of the matrix of coefficients for $\{\psi_1, \dots, \psi_\ell\}$, we get $\nu(\det \mathsf{M}') \doteq \nu(Q(\mathcal{A})) = Q(\mathbf{d}\mathcal{A})$. By Saito's criterion for affine arrangements, then, $\{\psi_1, \dots, \psi_\ell\}$ form a basis for $D(\mathbf{d}\mathcal{A})$.

We also need to check that $\sum_{i=1}^{\ell} \text{pdeg}\psi_i = \deg Q(\mathbf{d}\mathcal{A})$. We have that $\text{pdeg}\psi_i \leq \text{pdeg}\Psi_i$ for $1 \leq i \leq \ell$. Ψ_i is homogeneous of some fixed polynomial degree, so we have $\text{pdeg}\psi_i < \text{pdeg}\Psi_i$ if and only if every coefficient polynomial is divisible by $x_{\ell+1}$. Therefore, if there is some i such that $\text{pdeg}\psi_i < \text{pdeg}\Psi_i$, then $\det \mathsf{M}'$ is divisible by $x_{\ell+1}^2$, which is impossible. Thus we have $\text{pdeg}\psi_i = \text{pdeg}\Psi_i$ for $1 \leq i \leq \ell$, so $\sum_{i=1}^{\ell} \text{pdeg}\psi_i = \deg Q(\mathcal{A}) - 1 = \deg Q(\mathbf{d}\mathcal{A})$.

($\Rightarrow$) Let $\mathcal{A}$ be an affine ℓ-arrangement. Let $\theta \in D(\mathcal{A})$, $\theta = \sum_{i=1}^{\ell} f_i \frac{\partial}{\partial x_i}$, where $f_i \in S(V_\ell^*) \cong \mathbb{K}[x_1, \ldots, x_\ell]$. Let $d = \max\{\deg f_i \,:\, 1 \leq i \leq \ell\}$.

Let $F_1, \ldots, F_\ell \in \mathbb{K}[x_1, \ldots, x_{\ell+1}]$ be homogeneous polynomials, all of the same degree, so that $\nu(F_i) = f_i$ for all $1 \leq i \leq \ell$. Further, let $F_1, \ldots, F_\ell$ be the polynomials of least degree with this property. For example, the polynomial $f_1(x, y) = x^3 y + 2xy^2 - 5$ is of degree 4, $f_2(x, y) = x^2$ is of degree 2, and $F_1(x, y, z) = x^3 y + 2xy^2 z - 5z^4$, $F_2(x, y, z) = x^2 z^2$. This defines a set map $\Gamma : \text{Der}_{\mathbb{K}}(S(V_\ell^*)) \to \text{Der}_{\mathbb{K}}(S(V_{\ell+1}^*))$, where $\Gamma\left(\sum_{i=1}^{\ell} f_i \frac{\partial}{\partial x_i}\right) = \sum_{i=1}^{\ell} F_i \frac{\partial}{\partial x_i}$. We will show that $\Gamma(\theta) \in D(\mathbf{c}\mathcal{A})$.

Recall that $S(V^*)$ is a graded ring, with $S(V^*)_i$ the additive subgroup consisting of the zero polynomial and all homogeneous polynomials of degree i. We will write $S(V^*)_{\leq i}$ to denote the additive subgroup consisting of the zero polynomial and all polynomials of degree i or less. Consider $\mu_d : S(V_\ell^*)_{\leq d} \to S(V_{\ell+1}^*)_d$, where $\mu_d\left(\sum_\alpha c_\alpha x^\alpha\right) = \sum_\alpha c_\alpha x^\alpha x_{\ell+1}^{d - \deg x^\alpha}$. Then μ_d is a group homomorphism and $\mu_d(f_i) = F_i$ for all $1 \leq i \leq \ell$ since we chose $d = \max\{\deg f_i \,:\, 1 \leq i \leq \ell\}$.

Let $H \in \mathbf{c}\mathcal{A}$ with defining form α_H. Then $\theta(\nu(\alpha_H)) \in \nu(\alpha_H) S(V_\ell^*)$. This implies that there exists some $g \in S(V_\ell^*)$ so that

$$\left(\sum_{i=1}^{\ell} f_i \frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{\ell} c_j x_j + c_{\ell+1}\right) = \sum_{i=1}^{\ell} c_i f_i = \nu(\alpha_H) g$$

We need $\Gamma(\theta)(\alpha_H) = \left(\sum_{i=1}^{\ell} F_i \frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{\ell+1} c_j x_j\right) = \sum_{i=1}^{\ell} c_i F_i \in \alpha_H S(V_{\ell+1}^*)$.

But if we note that $\deg g \leq i-1$ which implies that $\mu_{d-1}(g)$ exists and is homogeneous, we have

$$\begin{aligned}
\textstyle\sum_{i=1}^{\ell} c_i F_i &= \mu_d\left(\textstyle\sum_{i=1}^{\ell} c_i f_i\right) \\
&= \mu_d(\nu(\alpha_H)g) \\
&= \mu_d\left(\left(\textstyle\sum_{i=1}^{\ell} c_i x_i\right) g + c_{\ell+1} g\right) \\
&= \textstyle\sum_{i=1}^{\ell} c_i \mu_d(x_i g) + c_{\ell+1}\mu_d(g) \\
&= \textstyle\sum_{i=1}^{\ell} c_i x_i \mu_{d-1}(g) + c_{\ell+1} x_{\ell+1} \mu_{d-1}(g) \\
&= \alpha_H \mu_{d-1}(g)
\end{aligned}$$

Since this is true for all $H \in \mathbf{c}\mathcal{A}$, we have that $\Gamma(\theta) \in D(\mathbf{c}\mathcal{A})$.

Now assume that we have ℓ derivations $\theta_1, \ldots, \theta_\ell \in D(\mathcal{A})$ which form a basis for $D(\mathcal{A})$ and such that $\sum_{i=1}^{\ell} \mathrm{pdeg}\theta_i = \deg Q(\mathcal{A})$. Then $\Gamma(\theta_1), \ldots, \Gamma(\theta_\ell) \in D(\mathbf{c}\mathcal{A})$ along with the Euler derivation θ_E, and therefore $\det \mathsf{M}(\Gamma(\theta_1), \ldots, \Gamma(\theta_\ell), \theta_E) \in Q(\mathbf{c}\mathcal{A})S(V_{\ell+1}^*)$.

Note that the last $\ell+1$st coefficient is zero in each of $\Gamma(\theta_1), \ldots, \Gamma(\theta_\ell)$, so the matrix of coefficients is block upper diagonal:

$$\mathsf{M}(\Gamma(\theta_1), \ldots, \Gamma(\theta_\ell), \theta_E) = \left[\begin{array}{cccc|c}
F_{1,1} & F_{1,2} & \ldots & F_{1,\ell} & x_1 \\
F_{2,1} & F_{2,2} & \ldots & F_{2,\ell} & x_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
F_{\ell,1} & F_{\ell,2} & \ldots & F_{\ell,\ell} & x_\ell \\
\hline
0 & 0 & \ldots & 0 & x_{\ell+1}
\end{array}\right]$$

We have $\det \mathsf{M} = Q(\mathbf{c}\mathcal{A}) \cdot g$ for some $g \in S(V_{\ell+1}^*)$. Let M' denote the upper left block in the matrix, so we have

$$\begin{aligned} \det \mathsf{M}' \cdot x_{\ell+1} &= \det \mathsf{M} \\ &= Q(\mathbf{c}\mathcal{A}) \cdot g \\ &= \tfrac{Q(\mathbf{c}\mathcal{A})}{x_{\ell+1}} x_{\ell+1} \cdot g \end{aligned}$$

This implies that $\det \mathsf{M}' = \frac{Q(\mathbf{c}\mathcal{A})}{x_{\ell+1}} \cdot g$. Apply ν to both sides to get $\det \mathsf{M}(\theta_1, \ldots, \theta_\ell) = Q(\mathcal{A}) \cdot \nu(g)$. By Saito's criterion, this implies that $\nu(g)$ is a nonzero constant, and therefore g is a nonzero polynomial in $S(V_{\ell+1}^*)$. Thus $\{\Gamma(\theta_1), \ldots, \Gamma(\theta_\ell), \theta_E\}$ are linearly independent, and since $\sum_{i=1}^{\ell} \text{pdeg}\Gamma(\theta_i) + \text{pdeg}\theta_E = \sum_{i=1}^{\ell} \text{pdeg}\theta_i + 1 = \deg Q(\mathcal{A}) + 1 = \deg Q(\mathbf{c}\mathcal{A})$, we have a basis for $D(\mathbf{c}\mathcal{A})$.

□

We are led to define freeness for affine arrangements in the following way, a similar version of which may also be found in a paper by Jambu [14].

Definition 4.1.5. Let $\mathcal{A}$ be an arbitrary ℓ-arrangement over some field $\mathbb{K}$ with defining polynomial Q. Then $\mathcal{A}$ is *free* if $D(\mathcal{A})$ is a free module over $S(V_\ell^*)$ and if there is a basis $\{\theta_1, \theta_2, \ldots, \theta_\ell\}$ for $D(\mathcal{A})$ such that $\sum_{i=1}^{\ell} \text{pdeg } \theta_i = \deg Q$.

Note that this definition coincides with the original definition of freeness for a central arrangement. The following example shows that the condition on the polynomial degrees of the derivations cannot be omitted in the theorem.

Example 4.1.6. Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^2$ with variables x and y defined by $Q(\mathcal{A}) = xy(x + y - 1)$. Then $\theta_1 = xy\frac{\partial}{\partial x} + y(y-1)\frac{\partial}{\partial y}$ and $\theta_2 = x(x-1)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial x}$ are both in $D(\mathcal{A})$, and

$$\det \begin{bmatrix} xy & x(x-1) \\ y(y-1) & xy \end{bmatrix} = Q(\mathcal{A})$$

By Saito's criterion, these derivations form a basis for $D(\mathcal{A})$. Note that $\text{pdeg}\theta_1 + \text{pdeg}\theta_2 = 4 > 3 = \deg Q(\mathcal{A})$.

However, when we try to create a basis for the cone by "homogenizing" the derivations by applying Γ to θ_1 and θ_2 and appending the Euler derivation, we get

$$\det \begin{bmatrix} xy & x(x-z) & x \\ y(y-z) & xy & y \\ 0 & 0 & z \end{bmatrix} = xyz^2(x+y-z) = Q(\mathbf{c}\mathcal{A})z$$

In fact, the arrangement $\mathbf{c}\mathcal{A}$ is not free at all. Because it is a 3-arrangement, it is free if and only if it is supersolvable, and $L(\mathbf{c}\mathcal{A})$ does not have any rank 2 elements which are modular.

Example 4.1.7. Even when an affine arrangement is free, we do not know as much about an arbitrary basis for $D(\mathcal{A})$ as we do in the central case. Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^2$ with variables x and y defined by $Q(\mathcal{A}) = x$. Then it is clear that $D(\mathcal{A}) = \{\theta = f_1\frac{\partial}{\partial x} + f_2\frac{\partial}{\partial y} \in Der_{\mathbb{C}}(S(V_2^*)) \mid f_1 \in x \cdot S(V_2^*)\}$. Then the following two derivations are in $D(\mathcal{A})$:

$$\begin{aligned} \theta_1 &= (x^2+x)\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \\ \theta_2 &= x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \end{aligned}$$

Saito's criterion shows that these derivations form a basis for $D(\mathcal{A})$, since

$$\det \begin{bmatrix} x^2+x & x \\ x & 1 \end{bmatrix} = (x^2+x)(1) - (x)(x) = x$$

The defining polynomial of $\mathbf{c}\mathcal{A}$ in $\mathbb{C}^3$ is $Q(\mathbf{c}\mathcal{A}) = xz$, and

$$\begin{aligned} \Gamma(\theta_1) &= (x^2+xz)\frac{\partial}{\partial x} + xz\frac{\partial}{\partial y} \\ \Gamma(\theta_2) &= x\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} \end{aligned}$$

When we take the determinant of the coefficient matrix for $\{\Gamma(\theta_1), \Gamma(\theta_2), \theta_E\}$, we have

$$\det \begin{bmatrix} x^2+xz & x & x \\ xz & z & y \\ 0 & 0 & z \end{bmatrix} = xz^3$$

which implies that the derivations are linearly independent, but they do not span $D(\mathbf{c}\mathcal{A})$. Note that these derivations are not even a basis for the multiarrangement defined by xz^3 since the Euler derivation fails to satisfy the condition $\theta_E(z) \in (z^3) \cdot S(V_3^*)$ and is not even in the module of derivations for the multiarrangement.

4.2 Arrangement Bundles

Let $\mathcal{A}$ be a complexified real arrangement in $\mathbb{C}^\ell$ with modular element $X \in L(\mathcal{A})$ of rank k. Orthogonal projection $\pi : \mathbb{C}^\ell \to \mathbb{C}^\ell/X$ restricts to a fiber bundle projection map with total space $M(\mathcal{A})$. The base space of this projection mapping in an arrangement complement in $\mathbb{C}^k$; this arrangement consists of all hyperplanes in $\mathcal{A}$ containing X. The generic fiber is the complement of an affine arrangement in $\mathbb{C}^{\ell-k}$. We will use $\mathcal{B}$ to denote the base space arrangement and $\mathcal{F}$ to denote the arrangement corresponding to a generic fiber.

Let $D(\mathcal{A})$, $D(\mathcal{B})$, and $D(\mathcal{F})$ be the modules of derivations for each of the arrangements. Note that since $\mathcal{F}$ is an affine arrangement, the coefficients of derivations in $D(\mathcal{F})$ will not generally be homogeneous multivariable polynomials. However, we may assume that after linear change of coordinates the modular element $X = \{(0, \ldots, 0, z_{k+1}, \ldots, z_\ell) \mid z_{k+1}, \ldots, z_\ell \in \mathbb{C}\}$ and the projection mapping is the standard projection map forgetting $\ell - k$ coordinates, so $\pi(z_1, \ldots, z_\ell) = (z_1, \ldots, z_k)$.

We wish to establish exactly which hyperplanes in $\mathcal{A}$ correspond to hyperplanes in $\mathcal{B}$ and which correspond to hyperplanes in $\mathcal{F}$.

Definition 4.2.1. Let $\mathcal{A}$ be a hyperplane arrangement in a vector space V, and let X and Y be elements in the intersection lattice $L(\mathcal{A})$. Then Y is *horizontal* with respect to X if $X+Y=V$.

If $\alpha = c_1z_1 + c_2z_2 + \cdots + c_\ell z_\ell$ is a linear form for a hyperplane H in $\mathcal{A}$, then H corresponds to a hyperplane in $\mathcal{B}$ if and only if H is not horizontal with respect to X. This is equivalent to H containing X and, because of our assumption about X, is also equivalent to the condition $c_{k+1} = \cdots = c_\ell = 0$. The horizontal hyperplanes are exactly those for which there exists a $j > k$ such that $c_j \neq 0$; these are the hyperplanes which correspond to hyperplanes in the fiber arrangement.

We will therefore partition the linear forms comprising the defining polynomial Q into two sets. We will denote by α_i those forms for which the corresponding hyperplane is horizontal with respect to X and β_j will denote those for which the corresponding hyperplane is non-horizontal with respect to X. Therefore $Q = \prod_i \alpha_i \prod_j \beta_j$.

Note that the α_i naturally correspond to linear forms on $\mathbb{C}^k$. As a set, then, the complement of $\mathcal{B}$ is defined by $M(\mathcal{B}) := \{(z_1,\ldots,z_k) \,|\, \alpha_i(z_1,\ldots,z_k) \neq 0 \,\forall\, i\}$. If we fix one of these points $(\bar{z}_1,\ldots,\bar{z}_k) \in M(\mathcal{B})$, then the complement of $\mathcal{F}$ is defined by $M(\mathcal{F}) := \{z = (\bar{z}_1,\ldots,\bar{z}_k, z_{k+1},\ldots,z_\ell) \,|\, \beta_j(z) \neq 0 \,\forall\, j\}$.

Example 4.2.2. Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^4$ with

$$Q(\mathcal{A}) = xy(x^2-y^2)(x^2-z^2)(x^2-w^2)(y^2-z^2)(y^2-w^2)(z^2-w^2)$$

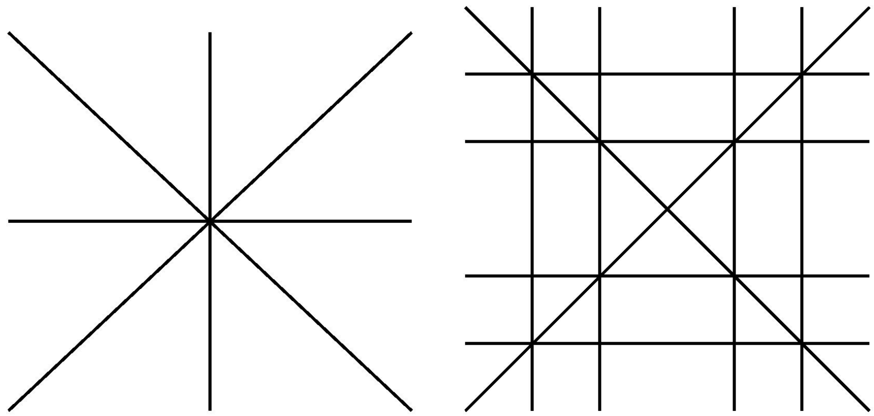
This arrangement is not supersolvable since there are no modular rank 3 elements in its intersection lattice. There is a single modular rank 2 element: the intersection of the first four hyperplanes, or the subspace of $\mathbb{C}^4$ such that $x=0$ and $y=0$. The projection mapping $\pi(x,y,z,w) = (x,y)$ is therefore a fiber bundle projection mapping, and we have

$$Q(\mathcal{B}) = xy(x^2-y^2)$$

and

$$Q(\mathcal{F}) = (x_0^2-z^2)(x_0^2-w^2)(y_0^2-z^2)(y_0^2-w^2)(z^2-w^2)$$

for a generic fiber $\pi^{-1}(x_0,y_0)$. See Figure 4.2.2.



(a) Base Arrangement (b) Fiber Arrangement

Figure 4.1: Diagrams of $\mathcal{B}$ and $\mathcal{F}$ in two-dimensional space

For any fiber $\pi^{-1}(x_0, y_0)$, $\mathcal{F}$ is free as an affine arrangement, with basis given by

$$\theta_1 = z(x_0^2 - z^2)(y_0^2 - z^2)\frac{\partial}{\partial z} + w(x_0^2 - w^2)(y_0^2 - w^2)\frac{\partial}{\partial w}$$

and

$$\theta_2 = w(x_0^2 - z^2)(y_0^2 - z^2)\frac{\partial}{\partial z} + z(x_0^2 - w^2)(y_0^2 - w^2)\frac{\partial}{\partial w}$$

The matrix of coefficients for this basis is

$$\begin{bmatrix} z(x_0^2 - z^2)(y_0^2 - z^2) & w(x_0^2 - z^2)(y_0^2 - z^2) \\ w(x_0^2 - w^2)(y_0^2 - w^2) & z(x_0^2 - w^2)(y_0^2 - w^2) \end{bmatrix}$$

The base space is free (it is even supersolvable) with matrix of coefficients for a basis given by

$$\begin{bmatrix} x & 0 \\ y & y(x^2 - y^2) \end{bmatrix}$$

Interestingly, these bases may be "extended" to a single basis for $D(\mathcal{A})$ whose coefficient matrix is block lower-diagonal:

$$\left[\begin{array}{cc|cc} x & 0 & 0 & 0 \\ y & y(x^2-y^2) & 0 & 0 \\ \hline z & z(x^2-z^2) & z(x^2-z^2)(y^2-z^2) & w(x^2-z^2)(y^2-z^2) \\ w & w(x^2-w^2) & w(x^2-w^2)(y^2-w^2) & z(x^2-w^2)(y^2-w^2) \end{array}\right]$$

This suggests that it may be possible to use Saito's criterion and the freeness of the base and fiber arrangements to construct a basis for the total arrangement $\mathcal{A}$.

Lemma 4.2.3. *Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^\ell$ with modular element $X \in L(\mathcal{A})$ of rank or codimension k. Let $\pi : \mathbb{C}^\ell \to \mathbb{C}^\ell/X \cong \mathbb{C}^{\ell-k}$ be a fiber bundle projection corresponding to X. Suppose that there is a k-parameter family of derivations in $Der_{\mathbb{C}}S((\mathbb{C}^{\ell-k})^*)$ such that*

$$\theta = \sum_{j=k+1}^{\ell} f_j(a_1, \ldots, a_k, x_{k+1}, \ldots, x_\ell)\frac{\partial}{\partial x_j}$$

with f_j a polynomial in k variables $a_1, \ldots, a_k, x_{k+1}, \ldots, x_\ell$ and such that for any fixed point $(a_1, \ldots, a_k)$ in the complement of $\mathcal{B}$ we have $\theta \in D(\mathcal{F})$. Then $\bar{\theta} \in D(\mathcal{A})$ where

$$\bar{\theta} = \sum_{j=k+1}^{\ell} f_j(x_1, \ldots, x_k, x_{k+1}, \ldots, x_\ell)\frac{\partial}{\partial x_j}$$

Proof. Recall that we may write $Q(\mathcal{A}) = \prod_i \alpha_i \prod_j \beta_j$, where the α_i are forms corresponding to "base hyperplanes" and the β_j correspond to "fiber hyperplanes." Also, recall that for every i, $\alpha_i = c_1x_1 + \cdots + c_kx_k$. Therefore, $\bar{\theta}(\alpha_i) = 0$, which is in the ideal generated by α_i in $\mathrm{Der}_{\mathbb{C}}S(\mathbb{C}^{\ell*})$.

Now consider $\beta_j = c_1x_1+\cdots+c_kx_k+c_{k+1}x_{k+1}+\cdots+c_\ell x_\ell$. For any $(a_1, \ldots, a_k) \in M(\mathcal{B})$, we have $\beta_j' = c_1a_1 + \cdots + c_ka_k + c_{k+1}x_{k+1} + \cdots + c_\ell x_\ell$ as a form which defines a hyperplane in the fiber arrangement $\mathcal{F}$. Then because $\theta(\beta_j') \in \beta_j' S(V^*_{(\ell-k)})$, there exists some $g \in \mathbb{C}[x_{k+1}, \ldots, x_\ell]$ such that

$$\begin{aligned} \theta(\beta_j') &= c_{k+1}f_{k+1}(a_1, \ldots, a_k, x_{k+1}, \ldots, x_\ell) + \cdots + c_\ell f_\ell(a_1, \ldots, a_k, x_{k+1}, \ldots, x_\ell) \\ &= (c_1a_1 + \cdots + c_ka_k + c_{k+1}x_{k+1} + \cdots + c_\ell x_\ell)g(x_{k+1}, \ldots, x_\ell) \end{aligned}$$

Simply substituting x_s for a_s for all s, we get

$$\begin{aligned}\bar{\theta}(\beta_j) &= c_{k+1}f_{k+1}(x_1,\ldots,x_k,x_{k+1},\ldots,x_\ell)+\cdots+c_\ell f_\ell(x_1,\ldots,x_k,x_{k+1},\ldots,x_\ell)\\ &= (c_1x_1+\cdots+c_kx_k+c_{k+1}x_{k+1}+\cdots+c_\ell x_\ell)g(x_{k+1},\ldots,x_\ell)\end{aligned}$$

Therefore, $\bar{\theta}(\beta_j)$ is in the ideal generated by β_j, so $\bar{\theta}\in D(\mathcal{A})$. □

Suppose that there is a k-parameter family of sets of $\ell-k$ derivations in $\mathrm{Der}_{\mathbb{C}}S(\mathbb{C}^{\ell-k^*})$, say $\theta_1,\theta_2,\ldots,\theta_{\ell-k}$, such that the derivations satisfy the conditions of Lemma 4.2.3 and $\mathcal{F}$ is a free affine arrangement with basis $\theta_1,\ldots,\theta_{\ell-k}$. Then by Saito's criterion the determinant of the matrix of the coefficients for $\theta_1,\theta_2,\ldots,\theta_{\ell-k}$ is up to scalar equal to $Q(\mathcal{F})$, and the set of derivations $\bar{\theta}_1,\ldots,\bar{\theta}_{\ell-k}$ is linearly independent. Unfortunately, in an arbitrary module, we cannot necessarily extend an linearly independent set of elements to a basis.

Theorem 4.2.4. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$ with modular element $X\in L(\mathcal{A})$ of rank k. Assume that the base arrangement $\mathcal{B}$ and fiber arrangement $\mathcal{F}$ corresponding to X are free arrangements. Suppose that the conditions of Lemma 4.2.3 hold and that there exists a basis $\{\psi_1,\ldots,\psi_k\}$ for $D(\mathcal{B})$ and polynomial functions $F_{ij}(x_1,\ldots,x_\ell)$ for $k+1\le i\le\ell$ and $1\le j\le k$ such that for each j $\psi_j+\sum_{i=k+1}^{\ell}F_{ij}\frac{\partial}{\partial x_i}\in D(\mathcal{A})$. Then $\mathcal{A}$ is free with basis $\{\psi_1+\sum_{i=k+1}^{\ell}F_{i1}\frac{\partial}{\partial x_i},\ldots,\psi_k+\sum_{i=k+1}^{\ell}F_{ik}\frac{\partial}{\partial x_i},\bar{\theta}_1,\ldots,\bar{\theta}_{\ell-k}\}$.*

Proof. Saito's criterion. □

Example 4.2.5. Let $Q(\mathcal{A})=(x-y)(x-z)(x-w)(y-z)(y-w)(z-w)$ be the braid arrangement in $\mathbb{C}^4$. This arrangement is fiber-type, and we can choose forgetful mappings for all of the fibrations. That is, the first projection map is $p_4:\mathbb{C}^4\to\mathbb{C}^3$ defined by $p_4(x,y,z,w)=(x,y,z)$, $p_3(x,y,z)=(x,y)$, and $p_2(x,y)=x$. In all three cases, the fibrations satisfy the conditions of Theorem 4.2.4.

The base space for $p_2:\mathbb{C}^2\to\mathbb{C}^1$ is the empty arrangment, and $p_2^{-1}(x_0)=\{(x_0,y)\,|\,y\neq x_0\}$. A basis for $D(\mathcal{B})$ is given by $\frac{\partial}{\partial x}$ which is extendable to $\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$ and a one-parameter family of bases for the modules of the fibers is given by $(x-y)\frac{\partial}{\partial y}$.

These make up the following basis for the arrangement defined by $x - y$ in $\mathbb{C}^2$:

$$\begin{bmatrix} 1 & 0 \\ 1 & x-y \end{bmatrix}$$

These two derivations may be extended to derivations in the module for the arrangement defined by $(x-y)(x-z)(y-z)$, and $(x_0 - z)(y_0 - z)\frac{\partial}{\partial z} \in D(\mathcal{F}_{(x_0,y_0)})$ for any $(x_0, y_0) \in M(\mathcal{B})$. We then get the following matrix for a basis for the arrangement in $\mathbb{C}^3$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & x-y & 0 \\ 1 & x-z & (x-z)(y-z) \end{bmatrix}$$

Repeating the process one more time, we find the following basis for the 4-dimensional braid arrangement:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & x-y & 0 & 0 \\ 1 & x-z & (x-z)(y-z) & 0 \\ 1 & x-w & (x-w)(y-w) & (x-w)(y-w)(z-w) \end{bmatrix}$$

We note that, in general, we will only see a block lower diagonal matrix like this if the fiber bundle projections are all forgetful mappings, which only happens if the modular elements are all of the form $(0, \ldots, 0, x_{k+1}, \ldots, x_\ell)$. As this cannot be guaranteed via a linear coordinate change, we do not necessarily get a lower diagonal coefficient matrix for fiber-type arrangements.

REFERENCES

[1] V. I. Arnol'd. The cohomology ring of the colored braid group. *Mat. Zametki*, 5(2):227–231, 1969.

[2] William A. Arvola. The fundamental group of the complement of an arrangement of complex hyperplanes. *Topology*, 31(4):757–765, 1992.

[3] Egbert Brieskorn. Sur les groupes de tresses [d'après V. I. Arnol'd]. In *Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401*, pages 21–44. Lecture Notes in Math., Vol. 317. Springer, Berlin, 1973.

[4] Daniel C. Cohen and Alexander I. Suciu. On Milnor fibrations of arrangements. *J. London Math. Soc. (2)*, 51(1):105–119, 1995.

[5] Daniel C. Cohen and Alexander I. Suciu. The braid monodromy of plane algebraic curves and hyperplane arrangements. *Comment. Math. Helv.*, 72(2):285–315, 1997.

[6] Daniel C. Cohen and Alexander N. Varchenko. Resonant local systems on complements of discriminantal arrangements and $\mathfrak{sl}_2$ representations. *Geom. Dedicata*, 101:217–233, 2003.

[7] Alexandru Dimca and Stefan Papadima. Hypersurface complements, Milnor fibers and higher homotopy groups of arrangments. *Ann. of Math. (2)*, 158(2):473–507, 2003.

[8] Edward Fadell and Lee Neuwirth. Configuration spaces. *Math. Scand.*, 10:111–118, 1962.

[9] Michael Falk. Homotopy types of line arrangements. *Invent. Math.*, 111(1):139–150, 1993.

[10] Michael Falk and Richard Randell. The lower central series of a fiber-type arrangement. *Invent. Math.*, 82(1):77–88, 1985.

[11] Michael J. Falk and Nicholas J. Proudfoot. Parallel connections and bundles of arrangements. *Topology and its Applications*, 118:65–83, 2002.

[12] Robin Forman. Morse theory for cell complexes. *Adv. Math.*, 134(1):90–145, 1998.

[13] Robin Forman. A user's guide to discrete Morse theory. *Sém. Lothar. Combin.*, 48:Art. B48c, 35 pp. (electronic), 2002.

[14] Michel Jambu. Fiber-type arrangements and factorization properties. *Adv. Math.*, 80(1):1–21, 1990.

[15] John W. Milnor. *Singular points of complex hypersurfaces.* Number 61 in Ann. of Math. Studies. Princeton University Press, 1968.

[16] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[17] Richard Randell. The fundamental group of the complement of a union of complex hyperplanes. *Invent. Math.*, 69(1):103–108, 1982.

[18] Richard Randell. Correction: "The fundamental group of the complement of a union of complex hyperplanes" [Invent. Math. **69** (1982), no. 1, 103–108; MR0671654 (84a:32016)]. *Invent. Math.*, 80(3):467–468, 1985.

[19] Richard Randell. Morse theory, Milnor fibers and minimality of hyperplane arrangements. *Proc. Amer. Math. Soc.*, 130(9):2737–2743 (electronic), 2002.

[20] Kyoji Saito. Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(2):265–291, 1980.

[21] M. Salvetti. Topology of the complement of real hyperplanes in $\mathbf{C}^N$. *Invent. Math.*, 88(3):603–618, 1987.

[22] Mario Salvetti and Simona Settepanella. Combinatorial Morse theory and minimality of hyperplane arrangements. *Geom. Topol.*, 11:1733–1766, 2007.

[23] Hiroaki Terao. Arrangements of hyperplanes and their freeness. I. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(2):293–312, 1980.

[24] Hiroaki Terao. Generalized exponents of a free arrangement of hyperplanes and Shepherd-Todd-Brieskorn formula. *Invent. Math.*, 63(1):159–179, 1981.

[25] Hiroaki Terao. Modular elements of lattices and topological fibration. *Adv. Math.*, 62:135–154, 1986.

[26] Masahiko Yoshinaga. Hyperplane arrangements and Lefschetz's hyperplane section theorem. *Kodai Math. J.*, 30(2):157–194, 2007.