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# Noncommutative Hardy algebras, multipliers, and quotients

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# NONCOMMUTATIVE HARDY ALGEBRAS, MULTIPLIERS, AND QUOTIENTS

by

Jonas R. Meyer

An Abstract

Of a thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Mathematics in the Graduate College of The University of Iowa

July 2010

Thesis Supervisor: Professor Paul Muhly

## ABSTRACT

The principal objects of study in this thesis are the noncommutative Hardy algebras introduced by Muhly and Solel in 2004, also called simply "Hardy algebras," and their quotients by ultraweakly closed ideals. The Hardy algebras form a class of nonselfadjoint dual operator algebras that generalize the classical Hardy algebra, the noncommutative analytic Toeplitz algebras introduced by Popescu in 1991, and other classes of operator algebras studied in the literature.

It is known that a quotient of a noncommutative analytic Toeplitz algebra by a weakly closed ideal can be represented completely isometrically as the compression of the algebra to the complement of the range of the ideal, but there is no known general extension of this result to Hardy algebras. An analogous problem on representing quotients of Hardy algebras as compressions of images of induced representations is considered in Chapter 2. Using Muhly and Solel's generalization of Beurling's theorem together with factorizations of weakly continuous linear functionals on infinite multiplicity operator spaces, it is shown that compressing onto the complement of the range of an ultraweakly closed ideal in the space of an infinite multiplicity induced representation yields a completely isometric isomorphism of the quotient.

A generalization of Pick's interpolation theorem for elements of Hardy algebras evaluated on their spaces of representations was proved by Muhly and Solel. In Chapter 3, a general theory of reproducing kernel $W^*$-correspondences and their multipliers is developed, generalizing much of the classical theory of reproducing kernel Hilbert space. As an application, it is shown using the generalization of Pick's theorem that the function space representation of a Hardy algebra is isometrically isomorphic (with its quotient norm) to the multiplier algebra of a reproducing kernel

$W^*$-correspondence constructed from a generalization of the Szegő kernel on the unit disk. In Chapter 4, properties of polynomial approximation and analyticity of these functions are studied, with special attention given to the noncommutative analytic Toeplitz algebras.

In the final chapter, the canonical curvatures for a class of Hermitian holomorphic vector bundles associated with a $C^*$-correspondence are computed. The Hermitian metrics are closely related to the generalized Szegő kernels, and when specialized to the disk, the bundle is the Cowen-Douglas bundle associated with the backward shift operator.

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the July 2010 graduation.

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## TABLE OF CONTENTS


CHAPTER

1 INTRODUCTION, BACKGROUND, AND PRELIMINARIES . . . 1

2 DISTANCE TO IDEALS: QUOTIENTS AS COMPRESSIONS ON INDUCED REPRESENTATION SPACES . . . . . . . . . . . . . . . 15

3 MULTIPLIERS OF REPRODUCING KERNEL $W^*$-MODULES . . 23

- 3.1 $W^*$-algebra-valued kernels . . . . . . . . . . . . . . . . . . . . . . . 23
- 3.2 Completely positive kernels and multipliers . . . . . . . . . . . 30

4 FUNCTION SPACE REPRESENTATIONS OF HARDY ALGEBRAS 47

- 4.1 Approximation by polynomials . . . . . . . . . . . . . . . . . . . . 47
- 4.2 Generic matrices and polynomial identities in noncommutative analytic Toeplitz algebras . . . . . . . . . . . 52

5 CURVATURE OF A VECTOR BUNDLE OVER THE UNIT BALL OF A DUAL CORRESPONDENCE . . . . . . . . . . . . . . . . . . . 57

REFERENCES . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 62

# CHAPTER 1
# INTRODUCTION, BACKGROUND, AND PRELIMINARIES

The noncommutative Hardy algebras introduced by Muhly and Solel [31] form a class of nonselfadjoint dual operator algebras generalizing many other classes in the literature, including the classical Hardy algebra, the noncommutative analytic Toeplitz algebras introduced by Popescu [37], analytic crossed products, and operator path algebras. This thesis is largely concerned with studying the quotients of these algebras by ultraweakly closed ideals.

In Chapter 2, an analogue of a result obtained independently by Davidson and Pitts [12] and by Arias and Popescu [2] on realizing quotients of noncommutative analytic Toeplitz algebras as compressions is derived in the setting of quotients of Hardy algebras by arbitrary ultraweakly closed ideals. The result relies on representing the algebra on what is called an induced representation space, such that the resulting algebra has infinite multiplicity.

In Chapter 3, a theory of reproducing kernel $W^*$-modules and $W^*$-correspondences is developed, together with an extension of the classical theory of multipliers of reproducing kernel Hilbert spaces. The results are applied to the function space representations of the Hardy algebras introduced in [31] to realize such spaces isometrically isomorphically as multiplier algebras of reproducing kernel correspondences associated with generalizations of the Szegő kernel on the unit disk. The results of Paschke and Rieffel on self-dual modules are key to the development.

Further study of these algebras of functions and the corresponding ideals is taken up in Chapter 4. A notion of "polynomial approximation" for these ideals is studied, followed by some discussion of the holomorphic mapping properties of the functions. It is shown how the theory of polynomial identities in ring theory relates to the function space representations of the noncommutative analytic Toeplitz algebras.

Chapter 5 describes a Hermitian holomorphic vector bundle that is associated with a dual correspondence, and whose Hermitian metric is closely related to the Szegő kernel. The canonical connection and curvature are given, along with a couple of simple special cases.

Further introduction and motivation for the problems considered are given in the individual chapters. In the remainder of this chapter some of the basic objects are defined and some preliminary notation and results are laid out. Most of these results can be found in the following works, and often only a citation of a standard reference is provided. Some basic references for the theory of selfadjoint operator algebras are [15], [14], [9], and [44]. Some of the nonselfadjoint theory included in [9] will also be useful. For Hilbert $C^*$-modules, the basic references are [24] and [26]. While much of the theory from Paschke's article [35] can now be found in [26], on several occasions it will be useful to refer to the original. For the bimodules considered here, referred to below as "correspondences," good references for the basics are [40], [4], and [30]. For background on the Fock modules and the noncommutative tensor and Hardy algebras of a correspondence, the main references are [27], [28], and [31].

Much of the theory below is developed for both $C^*$-algebras and $W^*$-algebras, but it is the $W^*$ setting that will be most prevalent in subsequent chapters. A $C^*$-algebra is, briefly, a Banach $*$-algebra satisfying the identity $\|x^*x\| = \|x\|^2$. It is assumed that the reader is familiar with some of the basic properties and definitions concerning their theory, including the fundamentals of $*$-representations and positive elements. The terminology used here and throughout is that a $W^*$-algebra is an abstract $C^*$-algebra with a Banach predual, using Sakai's abstract characterization of von Neumann algebras. The term von Neumann algebra is thus reserved for "concrete" $W^*$-algebras. That is, a von Neumann algebra is an ultraweakly closed, selfadjoint, nondegenerate subalgebra of the algebra of bounded operators on a Hilbert space. Sakai's theorem says that if $M$ is a $W^*$-algebra, then $M$ has a faithful normal representation $\pi : M \to B(H)$ on some Hilbert space $H$ [44, Theorem 1.16.7]. The adjective "normal" can be taken to mean weak-$*$ continuous, and it can also be characterized in terms of preserving least upper bounds of bounded increasing nets of selfadjoint elements. The image $\pi(M)$ is a von Neumann algebra, and $\pi$ is a homeomorphism from the weak-$*$ topology on $M$ to the ultraweak topology on $\pi(M)$. The weak-$*$ topology on $M$ is also called its ultraweak topology, and the predual $M_*$ is isometrically identified as the space of normal linear functionals on $M$ (where here "normal" again means ultraweakly continuous). When $N$ and $M$ are $W^*$-algebras, $B_*(N, M)$ will denote the space of bounded linear maps that are continuous with respect to the ultraweak topologies on $N$ and $M$. Note that if $N$ is finite dimensional, then $B_*(N, M) = B(N, M)$, because the weak-$*$ and norm topologies on $N$ coincide.

Given a $C^*$-algebra $A$, an inner product module over $A$ is first of all a right $A$-module $E$. This means, as usual for modules over algebras, that $E$ is a complex vector space and the module action $E \times A \to E$ is bilinear in addition to satisfying the identity $(xa)b = x(ab)$ for all $x \in E$, $a, b \in A$. Secondly, there is an $A$-valued inner product $\langle \cdot, \cdot \rangle$ on $E \times E$, such that the following conditions are satisfied for all $x, y, z \in E$, $\lambda \in \mathbb{C}$, and $a \in A$.

1. Linearity in the second variable:

$$\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle,$$

$$\langle x, \lambda y \rangle = \lambda \langle x, y \rangle, \text{ and}$$

$$\langle x, ya \rangle = \langle x, y \rangle a;$$

2. Hermitian symmetry: $\langle x, y \rangle = \langle y, x \rangle^*$;

3. Positivity: $\langle x, x \rangle \geq 0$; and

4. Definiteness: $\langle x, x\rangle = 0$ implies $x = 0$.

If $E$ is an inner product module over $A$, then a norm on $E$ is defined by $\|x\| = \|\langle x, x\rangle\|^{1/2}$. If $E$ is complete with respect to this norm, then $E$ is called a *Hilbert $C^*$-module* over $A$. If $E$ satisfies all of the conditions of an inner product module except possibly definiteness, then $E$ is called a pre-inner product module over $A$, and $E$ is a semi-normed space. Note that the map $(x, a) \mapsto xa$ from $E \times A$ to $E$ is continuous, due to the inequality $\|xa\| \leq \|x\|\|a\|$. When several Hilbert $C^*$-modules are being considered, the inner product of $E$ will sometimes be denoted $\langle \cdot, \cdot\rangle_E$ to remove possible ambiguity.

**Lemma 1** (Cauchy-Schwarz inequality)**.** *Let $E$ be a pre-inner product module over $A$. For all $x$ and $y$ in $E$, $\langle y, x\rangle\langle x, y\rangle \leq \|x\|^2\langle y, y\rangle$, and consequently $\|\langle x, y\rangle\| \leq \|x\|\|y\|$.*

Proofs can be found in all of the references on Hilbert $C^*$-modules mentioned above, for example [24, Proposition 1.1]. One of the important uses of this inequality is in constructing a Hilbert $C^*$-module from a pre-inner product module. It implies that $\mathcal{N} = \{x \in E : \langle x, x\rangle = 0\}$ is a submodule of $E$ and that the pre-inner product of $E$ induces an inner product on the quotient $E/\mathcal{N}$, which can in turn be completed to a Hilbert $C^*$-module [24, pages 3-4].

**Example 2.** The most basic (nontrivial) example of a Hilbert $C^*$-module over a $C^*$-algebra $A$ is $A$ itself, with module action given by ordinary multiplication in $A$, and with $\langle a, b\rangle = a^*b$. This module plays a fundamental role in much of theory, because if $E$ is another inner product module over $A$, then it is often necessary to consider $A$-module maps from $E$ to $A$. The Hilbert $C^*$-submodules of $A$ are the closed right ideals.

The *dual module* of an inner product module $E$ over $A$ is denoted $E'$, and defined to be the set of all bounded $A$-module maps from $E$ to $A$. It is, as the name indicates, also given a module structure over $A$. The actions of $\mathbb{C}$ and $A$ on

$E'$ are defined by $(\lambda f)(x) = \overline{\lambda} f(x)$ and $(fa)(x) = a^* f(x)$ for $\lambda \in \mathbb{C}$, $f \in E'$, $a \in A$, and $x \in E$. The reason for these choices is that they allow a canonical imbedding of $E$ into $E'$ to be linear rather than conjugate linear, just as in the Hilbert space setting. Given $x \in E$, define $\tilde{x} \in E'$ by $\tilde{x}(y) = \langle x, y \rangle$. If $A$ is a $W^*$-algebra, then $E'$ can be given an inner product that extends the inner product of $E$, and with which it is a Hilbert $C^*$-module over $A$. The operator norm and the norm coming from the inner product coincide, and with its Banach space structure $E'$ is a Banach dual space. Its weak-$*$ topology will be discussed below, but for now it should be noted that the inner product on $E'$ is uniquely determined by its weak-$*$ continuity in each variable together with the condition that it extends the inner product of $E$, i.e., $\langle \tilde{x}, \tilde{y} \rangle_{E'} = \langle x, y \rangle_E$ for all $x$ and $y$ in $E$. A Hilbert $C^*$-module is called *self-dual* if the map $x \mapsto \tilde{x}$ is onto. For more details and proofs of everything mentioned in this paragraph, see [35] or [26].

Unlike in the Hilbert space case, not all bounded linear maps on a Hilbert $C^*$-module are adjointable. The condition that a bounded linear map also be a module map is still not enough in general. Let $E$ and $F$ be Hilbert $C^*$-modules over $A$, and let $T : E \to F$ be a map. Then $T$ is called *adjointable* if there is a map $T^* : F \to E$ such that for all $x \in E$ and $y \in F$, $\langle y, Tx \rangle_F = \langle T^* y, x \rangle_E$. An adjointable map is automatically a bounded module map [24, page 8]. The set of all adjointable maps from $E$ to $F$ is denoted $\mathcal{L}(E, F)$, and the set of all bounded $A$-module maps from $E$ to $F$ is denoted $B_A(E, F)$. With the operator norm, $B_A(E, F)$ is a Banach space, and $\mathcal{L}(E, F)$ is a closed subspace and thus also a Banach space. If $E$ and $F$ are self-dual modules, then $\mathcal{L}(E, F) = B_A(E, F)$ (and in fact it is enough for $E$ to be self-dual [35, Proposition 3.4]).

The set of adjointable maps $\mathcal{L}(E) = \mathcal{L}(E, E)$ on a Hilbert $C^*$-module $E$ forms a $C^*$-algebra [24, page 8]. If $E$ is a self-dual Hilbert $C^*$-module over a $W^*$-algebra $M$, $\mathcal{L}(E)$ is also a $W^*$-algebra. In this case $E$ is called a self-dual $W^*$-module.

The dual space structure on $\mathcal{L}(E)$ is described using the "conjugate Banach space" $\tilde{E}$, which is equal to $E$ as a set and has the same additive structure and norm, but with scalar multiplication given by $\lambda \cdot x = \overline{\lambda}x$. Each $T$ in $\mathcal{L}(E)$ determines a functional $\tilde{T}$ on the projective Banach space tensor product $M_* \otimes \tilde{E} \otimes E$ satisfying $\tilde{T}(f \otimes x \otimes y) = f(\langle x, Ty \rangle)$ on simple tensors. The map $T \mapsto \tilde{T}$ is an isometric imbedding of $\mathcal{L}(E)$ onto a weak-$*$ closed subspace of $(M_* \otimes \tilde{E} \otimes E)^*$ [35, Proposition 3.10], thus showing that $\mathcal{L}(E)$ is the dual space of a quotient of $M_* \otimes \tilde{E} \otimes E$ by standard duality results for Banach spaces [42, Theorems 4.7 and 4.8]. The following simple consequence will be used in a subsequent chapter.

**Lemma 3.** *Let $E$ be a self-dual $W^*$-module over $M$, and let $x$ and $y$ be elements of $E$. Then the map $\theta : T \mapsto \langle x, Ty \rangle$ from $\mathcal{L}(E)$ to $M$ is continuous with respect to the ultraweak topologies on $\mathcal{L}(E)$ and $M$. That is, $\theta$ is in $B_*(\mathcal{L}(E), M)$.*

*Proof.* From the discussion above of the dual space structure on $\mathcal{L}(E)$, it is immediate that for each $f \in M_*$, $f \circ \theta$ is in $\mathcal{L}(E)_*$. Thus if $\{T_i\}$ is a net in $\mathcal{L}(E)$ that converges ultraweakly to $T$, then $\{f(\theta(T_i))\}$ converges to $f(\theta(T))$. Since the ultraweak topology on $M$ is the coarsest topology that makes every element of $M_*$ continuous, it follows that $\{\theta(T_i)\}$ converges ultraweakly to $\theta(T)$ in $M$. $\square$

While the following result appears to be standard, a precise reference is elusive, and thus a proof is sketched for convenience.

**Lemma 4.** *Let $N$ and $M$ be $W^*$-algebras. Then $B_*(N, M)$ is norm closed in $B(N, M)$.*

*Proof.* Let $\{T_k\}$ be a sequence in $B_*(N, M)$ that converges in norm to $T \in B(N, M)$. Then for all $f \in M_*$, $\{f \circ T_k\}$ converges in norm to $f \circ T$. Each $f \circ T_k$ is in $N_*$, and it follows that $f \circ T$ is in $N_*$ because $N_*$ is norm closed in $N^*$. The rest follows as in the proof of the preceding lemma. $\square$

Let $E$ be a self-dual $W^*$-module over $M$. As mentioned above, $E$ is a dual

Banach space. This was proved by Paschke by showing that $E$ can be isometrically imbedded as a weak-$*$ closed subspace of the dual of the projective Banach space tensor product $M_* \otimes \tilde{E}$ [35, Proposition 3.8]. Following Baillet et al. [4], the weak-$*$ topology on $E$ that comes from this duality is called the $\sigma$ topology. A net $\{x_i\}$ in $E$ converges to $x \in E$ in the $\sigma$ topology if and only if for each pair of sequences $\{f_k\} \subset M_*$ and $\{y_k\} \subset E$ such that $\sum_k \|f_k\|\|y_k\| < \infty$, the net $\{\sum_k f_k(\langle y_k, x_i\rangle)\}$ converges to $\sum_k f_k(\langle y_k, x\rangle)$. In particular, it follows by an argument similar to the proof of Lemma 3 that for each $x \in E$, $\langle x, \cdot\rangle$ is continuous from the $\sigma$ topology on $E$ to the ultraweak topology on $M$.

If $E$ is an inner product module over a $W^*$-algebra $M$, then $E'$ is a self-dual $W^*$-module over $M$ [35, Theorem 3.2]. The canonical imbedding of $E$ into $E'$ maps onto a $\sigma$ dense subset [4, Lemma 1.5], and $E'$ is called the *self-dual completion* of $E$. It follows that a $\sigma$ closed submodule of a self-dual $W^*$-module is self-dual. This all makes the following lemma straightforward, and it will be useful in subsequent chapters.

**Lemma 5.** *Let $E$ be a self-dual $W^*$-module over $M$, and let $S$ be a subset of $E$. The following are equivalent:*

1. *The self-dual $M$-submodule of $E$ generated by $S$ is $E$.*

2. *The $M$-submodule of $E$ generated by $S$ is $\sigma$ dense in $E$.*

3. *If $x \in E$ is orthogonal to every element of $S$, then $x = 0$.*

*Proof.* Assume 1 and let $E_0$ denote the $M$-submodule of $E$ generated by $S$. Let $F$ denote the $\sigma$ closure of $E_0$ in $E$. Then $F$ is a self-dual $M$-submodule of $E$ containing $S$, and by 1 it follows that $F = E$, proving 2.

Now assume 2. By $\sigma$ continuity of the maps $\langle x, \cdot\rangle$ on $E$, if $\langle x, y\rangle = 0$ for all $y \in S$, then $\langle x, y\rangle = 0$ for all $y$ in the $\sigma$ closed submodule of $E$ generated by $S$. By hypothesis, this means $\langle x, y\rangle = 0$ for all $y \in E$. In particular $\langle x, x\rangle = 0$, and thus

$x = 0$, proving 3.

Finally, it is shown that 3 implies 1 by contraposition. Suppose that the self-dual $M$-submodule $F$ of $E$ generated by $S$ is not equal to $E$. By self-duality of $F$, $E$ is the orthogonal direct sum $E = F \oplus F^\perp$ [26, Proposition 2.5.4]. It follows that $0 \neq F^\perp \subseteq S^\perp$. $\square$

**Definition 6.** A subset $S$ of a self-dual $W^*$-module $E$ satisfying the equivalent conditions of Lemma 5 is called a *total* subset of $E$.

It has already been mentioned that bounded module maps on self-dual modules are automatically adjointable. Fortunately, bounded module maps between arbitrary inner product modules over a $W^*$-algebra behave well with respect to self-dual completions, as seen in the following result of Paschke [35, Proposition 3.6 and Corollary 3.7].

**Proposition 7.** *Let $E$ and $F$ be inner-product modules over a $W^*$-algebra $M$, and let $T : E \to F$ be a bounded module map. Then $T$ has a unique extension to a bounded module map $\tilde{T} : E' \to F'$, and $\|\tilde{T}\| = \|T\|$. If $E = F$, then the map $T \mapsto \tilde{T}$, restricted to the algebra of adjointable operators on $E$, is a (faithful) $*$-homomorphism from $\mathcal{L}(E)$ to $\mathcal{L}(E')$.*

If $E$ is a Hilbert $C^*$-module over $B$, and $A$ is another $C^*$-algebra, then $E$ is called a *$C^*$-correspondence from $A$ to $B$* if $E$ is also a left $A$-module such that the left action is determined by a $*$-homomorphism $\varphi : A \to \mathcal{L}(E)$. In the case when $A = B$, $E$ is called a $C^*$-correspondence over $A$. If $N$ and $M$ are $W^*$-algebras and $E$ is a Hilbert $C^*$-correspondence from $N$ to $M$, then $E$ is called a $W^*$-correspondence from $N$ to $M$ if in addition $E$ is a self-dual module over $M$ and the left action of $N$, $\varphi : N \to \mathcal{L}(E)$, is normal.

A key feature of $C^*$ and $W^*$-correspondences is the internal tensor product. If $E$ is a $C^*$-correspondence from $A$ to $B$ with left action $\varphi$, and $F$ is a $C^*$-correspondence from $B$ to $C$ with left action $\psi$, then on the algebraic tensor product

$E \otimes_{alg} F$ one forms a $C$-valued pre-inner product satisfying $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle y_1, \psi(\langle x_1, x_2 \rangle) y_2 \rangle$ on simple tensors. The usual completion process yields the internal Hilbert $C^*$-module tensor product of $E$ and $F$, denoted $E \overline{\otimes} F$ or $E \overline{\otimes}_\psi F$, which is a $C^*$-correspondence from $A$ to $C$ with left action $\varphi(\cdot) \otimes I_F$. In the case where $A$ and $B$ are $W^*$-algebras, taking the self-dual completion yields the self-dual tensor product, denoted by $E \otimes F$ or $E \otimes_\psi F$, which is a $W^*$-correspondence from $A$ to $C$. The subscript $\psi$ is used when it is important to emphasize what action is being "tensored over," and the ordinary Hilbert $C^*$-module tensor product is given the extra adornment because self-dual tensor products will appear more frequently below. In the special case when $C = \mathbb{C}$, that is, when $F$ is a Hilbert space together with a representation $\psi$ of $B$ on $F$, then $E \otimes_\psi F$ is sometimes called an *induced representation space*, and $E$ is said to "induce" the representation $\varphi(\cdot) \otimes I_F$ of $A$ on the Hilbert space $E \otimes_\psi F$ from the representation $\psi$ of $B$.[1] The induced representation of $\mathcal{L}(E)$ will be denoted $\psi^E$, so that $\psi^E \circ \varphi = \varphi(\cdot) \otimes I_F$.

The following result, due to Rieffel, is an extension of Paschke's result on self-dual completions to accommodate normal left actions [40, Proposition 6.10].

**Proposition 8.** *Let $N$ and $M$ be $W^*$-algebras, and let $E$ be a $C^*$-correspondence from $N$ to $M$ satisfying the additional condition that for all $x$ and $y$ in $E$, the map $a \mapsto \langle x, a \cdot y \rangle$ is in $B_*(N, M)$. Then the left action of $N$ extends uniquely to a normal action on the self-dual completion $E'$ of $E$, making $E'$ a $W^*$-correspondence from $N$ to $M$.*

Paschke developed a notion of self-dual direct sums of Hilbert $C^*$-modules over a $W^*$-algebra. Manuilov and Troitsky point out that these sums can be obtained as self-dual completions of ordinary Hilbert $C^*$-module direct sums [26, page 63], but the more explicit description of Paschke will be useful below, and thus it is included here.

[1] The terminology comes from Rieffel [39], who used this construction to generalize Mackey's theory of induced representations of groups.

**Definition 9.** Let $\{E_i\}_{i\in I}$ be a family of self-dual $W^*$-modules over $M$. Then the ultraweak direct sum of $\{E_i\}_{i\in I}$ is the subset $X$ of the Cartesian product of $\{E_i\}_{i\in I}$ such that $\{x_i\}$ is in $X$ if and only if there is a uniform bound on the norms of the finite sums $\sum_{i\in F}\langle x_i, x_i\rangle$ as $F$ ranges over all finite subsets of $I$. This is equivalent to the ultraweak convergence of the sum $\sum_{i\in I}\langle x_i, x_i\rangle$ in the sense of convergence of the net of finite partial sums. The inner product on $X$ is defined by $\langle\{x_i\},\{y_i\}\rangle = \sum_i\langle x_i, y_i\rangle$, where the sum converges ultraweakly in $M$. With coordinatewise operations, $X$ is a self-dual $W^*$-module over $M$ [35, pages 457-458]. This direct sum is denoted $\bigoplus_i^{uw} E_i$, or simply $\bigoplus_i E_i$ when it is clear from context that only self-dual modules are being considered.

The Fock module of a correspondence is fundamental to the definition of the noncommutative Hardy algebra. Let $E$ be a $W^*$-correspondence over $M$ with (normal) left action $\varphi : M \to \mathcal{L}(E)$. For $k \in \mathbb{N}$ let $E^{\otimes k}$ denote the self-dual $k^{\text{th}}$ internal tensor power of $E$ over $M$, which is a $W^*$-correspondence over $M$ with left action denoted $\varphi_n$. For $k$ greater than 2, $E^{\otimes k}$ is defined recursively by $E^{\otimes k} = E \otimes E^{\otimes(k-1)}$. It is sometimes convenient to write $E^{\otimes 0} = M$, considered as a $W^*$-correspondence over $M$ with left action given by left multiplication. Thus, $\varphi_0(a) = L_a$ is the operator of left multiplication by $a$ on $M$. The *full Fock module of* $E$, denoted $\mathcal{F}(E)$, is defined to be the ultraweak direct sum of $W^*$-modules $M \oplus E \oplus E^{\otimes 2} \oplus \cdots$. The module $\mathcal{F}(E)$ is also a $W^*$-correspondence over $M$, whose left action of $M$ is denoted $\varphi_\infty : M \to \mathcal{L}(\mathcal{F}(E))$. When thinking of elements of $\mathcal{L}(\mathcal{F}(E))$ as having matrix decompositions relative to the summands in the definition of $\mathcal{F}(E)$, $\varphi_\infty(a)$ can be written as $\operatorname{diag}(L_a, \varphi(a), \varphi_2(a), \varphi_3(a), \ldots)$.

The noncommutative tensor and Hardy algebras are constructed as algebras of operators on $\mathcal{F}(E)$. For each $\zeta \in E$, first let the operators $T_\zeta^{(1)}$, $T_\zeta^{(2)}$, $T_\zeta^{(3)}$, $\ldots$ be defined as follows. The map $T_\zeta^{(1)} : M \to E$ sends $a \in M$ to $\zeta a$. For $k > 1$, the map $T_\zeta^{(k)} : E^{\otimes(k-1)} \to E^{\otimes k}$ sends $\eta \in E^{\otimes(k-1)}$ to $\zeta \otimes \eta$. Straightforward computations show that each $T_\zeta^{(k)}$ is adjointable and has norm at most $\|\zeta\|$. Let $T_\zeta \in \mathcal{L}(\mathcal{F}(E))$ denote the operator that sends $(a, \eta_1, \eta_2, \ldots)$ to $(0, T_\zeta^{(1)}a, T_\zeta^{(2)}\eta_1, T_\zeta^{(3)}\eta_2) = (0, \zeta a, \zeta \otimes \eta_1, \zeta \otimes \eta_2, \ldots)$. Each $T_\zeta$ is adjointable with $||T_\zeta|| = ||\zeta||$ (because $\|T_\zeta^{(1)}\| = \|\zeta\|$), and $T_\zeta$ is called a *left creation operator* on $\mathcal{F}(E)$. The *tensor algebra of* $E$, denoted by $\mathcal{T}_+(E)$, is the norm closed subalgebra of $\mathcal{L}(\mathcal{F}(E))$ generated by $\varphi_\infty(M)$ and $\{T_\zeta : \zeta \in E\}$. The *Hardy algebra of* $E$, denoted by $H^\infty(E)$, is defined to be the ultraweak closure of $\mathcal{T}_+(E)$ [31, pages 358-359].

The reason for the term "Hardy algebra" is that in the case when $M = \mathbb{C}$ and $E$ is a one dimensional Hilbert space, $H^\infty(E)$ is completely isometrically isomorphic to the Hardy algebra $H^\infty$ of bounded holomorphic functions on the unit disk in the complex plane. In fact, in this case $H^\infty(E)$ is an ultraweakly closed, unital, concrete operator algebra generated by a unilateral shift of multiplicity one, which up to unitary equivalence may be thought of as "multiplication by $z$" on the Hardy space $H^2$ of holomorphic functions on the disk whose sequences of Taylor series coefficients at 0 lie in $\ell^2$. When $H^\infty$ is viewed as a concrete operator algebra in this way, it is often called the *analytic Toeplitz algebra*. Allowing $E$ to have dimension greater than one then yields noncommutative "multivariable" analogues of the Hardy algebra, which were introduced by Popescu [37], and dubbed *noncommutative analytic Toeplitz algebras* by Davidson and Pitts [11]. Muhly and Solel thus initiated a wide-ranging generalization by allowing $E$ to be an arbitrary $W^*$-correspondence over an arbitrary $W^*$-algebra $M$. While this generalization allows for interesting new phenomena and applies to many classes of operator algebras studied in the literature, it is remarkable that it is not so wide-ranging as to lose many of the interesting properties of the classical case. Some examples of these properties will be seen below, and the reader is referred to the work of Muhly and Solel for more background and examples [31, 32, 33].

The *dual correspondences* of a $W^*$-correspondence $E$ over $M$, quite separate from the notion of dual modules discussed above, are of fundamental importance in describing the representation theory of $H^\infty(E)$, among other things. Let $E$ be a $W^*$-correspondence over a $W^*$-algebra $M$ with left action given by a normal $*$-homomorphism $\varphi : M \to \mathcal{L}(E)$, and let $\sigma : M \to B(H)$ be a nondegenerate normal $*$-representation, making $H$ a $W^*$-correspondence from $M$ to $\mathbb{C}$. The $\sigma$ dual of $E$ is defined to be the set $E^\sigma$ of bounded left module maps from $H$ to $E \otimes_\sigma H$. That is, $E^\sigma = \{T \in B(H, E \otimes_\sigma H) : T\sigma(a) = (\varphi(a) \otimes I_H)T \text{ for all } a \in M\}$. The commutant of $\sigma(M)$, $\sigma(M)'$, is a von Neumann algebra, and $E^\sigma$ is a $W^*$-correspondence over $\sigma(M)'$ (see [31, Proposition 3.2] and [40, Theorem 6.5]). The inner product and module actions are defined as follows for $S, T \in E^\sigma$ and $X \in \sigma(M)'$: $\langle S, T\rangle = S^*T$, $X \cdot S = (I_E \otimes X)S$, and $S \cdot X = SX$. The $*$-homomorphism from $\sigma(M)'$ to $\mathcal{L}(E^\sigma)$ giving the left action will be denoted by $\varphi^\sigma$. If $\sigma$ is faithful and $\iota$ denotes the identity representation of $\sigma(M)'$, then Muhly and Solel showed that $E$ is isomorphic to the $W^*$-correspondence $(E^\sigma)^\iota$ over $\sigma(M)'' = \sigma(M) \cong M$ [31, Theorem 3.6], which among other things gives a concrete realization of $W^*$-correspondences as spaces of bounded intertwining operators of two normal representations of a $W^*$-algebra (see [40, Proposition 6.12] for another approach).

The set of adjoints of elements of $E^\sigma$ is denoted $E^{\sigma*}$. Each element $z$ of the unit ball[2] of $E^{\sigma*}$, denoted $\mathbb{D}(E^{\sigma*})$, determines an ultraweakly continuous completely contractive represention of $H^\infty(E)$ on $H$, often denoted $\sigma \times z$. Here it will be denoted $\psi_z$, and it is determined by the following properties. For each $\zeta \in E$ and $h \in H$, $\psi_z(T_\zeta)h = z(\zeta \otimes h)$, and $\psi_z \circ \varphi_\infty = \sigma$ [31, Theorem 2.9 and Corollary 2.14]. By viewing each $\psi_z$ as a point evaluation at $z \in \mathbb{D}(E^{\sigma*})$, a representation of $H^\infty(E)$ as a space of functions on $\mathbb{D}(E^{\sigma*})$ is obtained. For each $F \in H^\infty(E)$, let $\hat{F} : \mathbb{D}(E^{\sigma*}) \to B(H)$ be defined by $\hat{F}(z) = \psi_z(F)$. Let $H^\infty(E, \sigma)$ be the range of the map $F \mapsto \hat{F}$, and let $\Gamma_\sigma : H^\infty(E) \to H^\infty(E, \sigma)$ be defined by $\Gamma_\sigma(F) = \hat{F}$.

[2]The notation $\mathbb{D}(E^{\sigma*})$ is from Muhly and Solel [31] and is intended to be reminiscent of the notation $\mathbb{D}$ for the unit disk in $\mathbb{C}$.

Thus $H^\infty(E,\sigma)$ is an algebra of $B(H)$-valued functions on $\mathbb{D}(E^{\sigma*})$ with pointwise operations, and $\Gamma_\sigma$ is a homomorphism. Properties and some examples of these algebras of functions will be explored in Chapters 3 and 4.

For each $t$ in $\mathbb{R}$ there is an ultraweakly continuous completely isometric automorphism $\gamma_t$ of $H^\infty(E)$ uniquely determined by the conditions that $\gamma_t \circ \phi_\infty = \phi_\infty$ and for each $\zeta \in E$, $\gamma_t(T_\zeta) = e^{it}T_\zeta$. Thus $\{\gamma_t\}_{t\in\mathbb{R}}$ is a ($2\pi$-periodic) one parameter group of automorphisms of $H^\infty(E)$, called the *gauge group.* A subset of $H^\infty(E)$ that is invariant under each $\gamma_t$ is called *gauge invariant.* For each nonnegative integer $k$, let $\Phi_k \in B(H^\infty(E))$ be defined by the weak-$*$ integral formula $\Phi_k(F) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \gamma_t(F)e^{-ikt}dt$. Then $\Phi_k$ is the projection onto the ultraweakly closed span of the set $\{T_{\zeta_1}T_{\zeta_2}\dots T_{\zeta_k} | \zeta_j \in E, 1 \le j \le k\}$, i.e., $\Phi_k$ is an idempotent map of norm 1. Every gauge invariant ultraweakly closed subspace of $H^\infty(E)$ is also invariant under each $\Phi_k$. Let $Q_k \in \mathcal{L}(\mathcal{F}(E))$ denote the orthogonal projection onto $E^{\otimes k}$. An alternative description of $\Phi_k$ is given by the formula $\Phi_k(F) = \displaystyle\sum_{j=0}^{\infty} Q_{j+k}FQ_j$, $F \in H^\infty(E)$. The element $\Phi_k(F)$ of $H^\infty(E)$ can be thought of as the degree $k$ term in the "Fourier series" of $F$, and $F$ can be recovered from the ultraweak Cesàro sum[3] of its Fourier series:

$$F = uw\text{-}\lim_{k\to\infty} \sum_{j=0}^{k-1} \left(1 - \frac{j}{k}\right) \Phi_k(F).$$

It will also be convenient to write $F_k = \Phi_k(F)$. The results mentioned in this paragraph are from [31, pages 365-366], and will be used in Chapter 4.

The following criteria for positivity of matrices over $C^*$-algebras are well known. A proof of the first part can be found in [35, Proposition 6.1], and a proof of the third part is in [24, Lemma 4.2].

**Lemma 10.** *Let $A$ be a $C^*$-algebra.*

[3] One could also use Abel summation. To see why ordinary summation will not work in general, consider the case of the classical Hardy algebra, $H^\infty(\mathbb{D})$. The sequence of partial sums of the Taylor series of a bounded analytic function $f$ on the unit disk converges to $f$ in the weak-$*$ topology of $H^\infty(\mathbb{D})$ if and only if the sequence is bounded in the sup norm. Around a century ago Fejér gave an example where this doesn't happen [16],[25, Section 3].

1. *Let* $(b_{ij})$ *be an element of the* $C^*$*-algebra* $M_n(A)$*. Then* $(b_{ij})$ *is positive if and only if for all* $a_1, \ldots, a_n \in A$*, the sum* $\sum_{i,j=1}^{n} a_i^* b_{ij} a_j$ *is positive in* $A$*.*

2. *A matrix* $P \in M_n(A)$ *is positive if and only if* $P$ *is a finite sum of matrices of the form* $(a_i^* a_j)$*.*

3. *If* $E$ *is an inner product module over* $A$ *and* $x_1, x_2, \ldots, x_n$ *are elements of* $E$*, then the matrix* $(\langle x_i, x_j \rangle)$ *is positive in* $M_n(A)$*.*

4. *If* $P$ *is positive in* $M_n(A)$*, and if* $X$ *is an* $n$*-by-*$m$ *matrix over* $A$*, then* $X^*PX$ *is positive in* $M_m(A)$

*Proof sketches for 2 and 4.* For 2, assume that $P$ is positive, write $P = S^*S$, and expand the right-hand side in terms of the rows of $S$. Conversely, sums of positive elements are positive, and if $R \in M_n(A)$ denotes the matrix that is zero below the first row and whose first row has $j^{\text{th}}$ entry $a_j$, then $(a_i^* a_j) = R^*R$

For 4, again write $P = S^*S$, so that $X^*PX = (SX)^*SX$. A way to see that this is positive without working outside of $C^*$-algebras is by adding enough zero rows and columns so that all matrices are $\max(m, n)$ by $\max(m, n)$, and then compressing to the appropriate $m$-by-$m$ corner if needed. $\square$

# CHAPTER 2
# DISTANCE TO IDEALS: QUOTIENTS AS COMPRESSIONS ON INDUCED REPRESENTATION SPACES

Arias and Popescu [2, Corollary 2.10] and (independently) Davidson and Pitts [12, Corollary 2.2] showed that a quotient of a noncommutative analytic Toeplitz algebra (in a finite or countably infinite number of variables) by an ultraweakly closed ideal is completely isometrically isomorphic to the compression to the orthogonal complement of the range of the ideal in the Fock space. Muhly and Solel [29, Theorem 2.9] proved an analogous result for quotients of tensor algebras by ideals satisfying certain hypotheses. Recently, Viselter extended the result of Muhly and Solel to a larger class of ideals [46, Example 5.2]. Inspired by these results and a desire to better understand quotients of Hardy algebras, this chapter provides a representation theorem for quotients of noncommutative Hardy algebras as compressions on induced representation spaces.

Somewhat more precisely, it is shown that a quotient of a noncommutative Hardy algebra by an ultraweakly closed ideal is realized completely isometrically as a compression of the representation induced by an infinite multiplicity faithful normal representation of the coefficient $W^*$-algebra. The assumption of infinite multiplicity is made because it allows factorization of functionals as vector functionals with good estimates, which assists in proving the needed distance formula. The proof of the distance formula is inspired by Davidson and Pitts's proof cited above. While writing the proof, I was informed that Maxim Gurevich at Technion had obtained the same result, using similar methods, at about the same time.

A result of Muhly and Solel [28, Theorem 4.7], generalizing Beurling's Theorem to the setting of tensor algebras of correspondences, is an essential part of the proof. The following adaptation of this result to noncommutative Hardy algebras is the precise version needed below.

**Lemma 11.** *Let $\pi : M \to B(H)$ be a faithful normal $*$-representation of a $W^*$-algebra $M$, let $E$ be a $W^*$-correspondence over $M$, and let $\rho : H^\infty(E) \to B(\mathcal{F}(E) \otimes_\pi H)$ be the representation given by $\rho(T) = T \otimes I_H$. If $K \subset \mathcal{F}(E) \otimes_\pi H$ is a closed invariant subspace for $\rho(H^\infty(E))$, then there are partial isometries $\{V_i\}_{i \in I}$ in the commutant of $\rho(H^\infty(E))$ with pairwise orthogonal ranges whose sum is $K$.*

*Proof.* Let $\sigma : M \to B(\mathcal{F}(E) \otimes_\pi H)$ be the (faithful, normal) representation given by $\sigma(a) = \varphi_\infty(a) \otimes I_H$. The map $\Phi : \sigma(M) \to \pi(M)$ given by $\Phi(\sigma(a)) = \pi(a)$ is an isomorphism of von Neumann algebras. It follows from the decomposition theorem for such isomorphisms [14, Corollary 1, I.4.3; Theorem 3, I.4.4] that $\sigma$ and $\pi$ are quasi-equivalent representations of $M$ [15, Proposition 5.3.1]. In other words, $\pi$ is quasi-invariant with respect to $E$ in the terminology of [28, Definition 4.4], and the result follows from Muhly and Solel's analogue of Beurling's theorem [28, Theorem 4.7] upon noting that $\rho(\mathcal{T}_+(E))$ and $\rho(H^\infty(E))$ have the same commutant. This holds because $\rho$ is an ultraweak homeomorphism onto its range [31, Lemma 2.1] and thus $\rho(\mathcal{T}_+(E))$ is ultraweakly dense in $\rho(H^\infty(E))$. $\square$

*Remark* 12. If $V$ is a partial isometry in the commutant of $\rho(H^\infty(E))$ as in the preceding lemma, then the initial projection $V^*V$ is also in the commutant [28, Lemma 4.2].

For the remainder of this chapter, $\pi : M \to B(H)$ will denote a fixed faithful, normal, infinite multiplicity representation of $M$. Let $\pi^{\mathcal{F}(E)} : \mathcal{L}(\mathcal{F}(E)) \to B(\mathcal{F}(E) \otimes_\pi H)$ denote the $*$-representation satisfying $\pi^{\mathcal{F}(E)}(T)(x \otimes h) = (Tx) \otimes h$ for all $T$ in $\mathcal{L}(\mathcal{F}(E))$, $x$ in $\mathcal{F}(E)$, and $h$ in $H$. So $\pi^{\mathcal{F}(E)}(T) = T \otimes I_H$. This is an induced representation of $\mathcal{L}(\mathcal{F}(E))$ in the sense of Rieffel [39], as mentioned in Chapter 1. For convenience let $\rho$ denote the restriction of $\pi^{\mathcal{F}(E)}$ to $H^\infty(E)$. Muhly and Solel showed[1] [31, Lemma 2.1] that $\pi^{\mathcal{F}(E)}$ is an ultraweak homeomorphism

[1] Muhly and Solel remark that the result is contained in Rieffel's work [40], but their version is

onto its range, so in particular $\rho(H^\infty(E))$ is ultraweakly closed and ultraweakly homeomorphic to $H^\infty(E)$.

Because $\pi$ has infinite multiplicity, so does $\pi^{\mathcal{F}(E)}$. In a little more detail, there is a faithful, normal representation $\pi_0$ of $M$ on a Hilbert space $H_0$ and a separable infinite dimensional Hilbert space $K$ such that $H = H_0 \otimes K$ and $\pi(\cdot) = \pi_0(\cdot) \otimes I_K$. This yields a canonical identification of the Hilbert modules $\mathcal{F}(E) \otimes_\pi H$ and $(\mathcal{F}(E) \otimes_{\pi_0} H_0) \otimes K$ over $\mathcal{L}(\mathcal{F}(E))$, so that $\pi^{\mathcal{F}(E)}$ is an infinite multiple of the induced representation $\pi_0^{\mathcal{F}(E)}$.

Let $J$ be an ultraweakly closed two-sided ideal in $H^\infty(E)$. Let $\mathcal{N}_J$ denote the orthogonal complement of the range of $\rho(J)$, i.e.,

$$\mathcal{N}_J = (\rho(J)(\mathcal{F}(E) \otimes_\pi H))^\perp.$$

This notation will be fixed throughout the remainder of the chapter. The main result of this chapter is as follows.

**Theorem 13.** *Compression by $P_{\mathcal{N}_J}$ yields a completely isometric isomorphism of $H^\infty(E)/J$ onto $P_{\mathcal{N}_J}\rho(H^\infty(E))P_{\mathcal{N}_J}$. That is, the map $T \mapsto P_{\mathcal{N}_J}\rho(T)P_{\mathcal{N}_J}$ is an algebra homomorphism of $H^\infty(E)$ onto $P_{\mathcal{N}_J}\rho(H^\infty(E))P_{\mathcal{N}_J}$ with kernel $J$, and the resulting isomorphism $\dot{T} \mapsto P_{\mathcal{N}_J}\rho(T)P_{\mathcal{N}_J}$ of $H^\infty(E)/J$ onto $P_{\mathcal{N}_J}\rho(H^\infty(E))P_{\mathcal{N}_J}$ is a complete isometry.*

The remainder of the chapter will build toward proving this theorem, beginning with a series of remarks on some of its more straightforward aspects.

*Remark* 14.

1. The matrix norm structure on the quotient is the usual one induced by the identification of $M_k(H^\infty(E)/J)$ with $M_k(H^\infty(E))/M_k(J)$, while $M_k(H^\infty(E))$ inherits the norm from the $C^*$-algebra $M_k(\mathcal{L}(E))$

2. Because $J$ is a left ideal, $\mathcal{N}_J^\perp$ is an invariant subspace for $\rho(H^\infty(E))$. From this it follows that the compression in the theorem is a homomorphism of $H^\infty(E)$ onto $P_{\mathcal{N}_J}\rho(H^\infty(E))P_{\mathcal{N}_J}$ [9, Lemma 35.6], and that in fact $P_{\mathcal{N}_J}\rho(T)P_{\mathcal{N}_J} =$

cited because it is precisely the version used here.

$P_{\mathcal{N}_J}\rho(T)$ for all $T \in H^\infty(E)$.

3. It is immediate that $P_{\mathcal{N}_J}\rho(T) = 0$ for all $T \in J$. This follows from the fact that $\rho(T)(x)$ is in $\mathcal{N}_J^\perp$ for all $x \in \mathcal{F}(E) \otimes_\pi H$, by the definition of $\mathcal{N}_J$. Thus $J$ is in the kernel as claimed.

4. The map is completely contractive, because $\rho$ is completely isometric and compressions are completely contractive.

5. Because $\rho : H^\infty(E) \to \rho(H^\infty(E))$ is a completely isometric isomorphism, it suffices to work exclusively with $\rho(H^\infty(E))$ and its ideal $\rho(J)$.

6. All that remains to be shown is that for each $m \in \mathbb{N}$ and each matrix $A \in M_m(H^\infty(E))$, the norm of $\rho_m(A) + M_m(\rho(J))$ in $M_m(\rho(H^\infty(E)))/M_m(\rho(J))$ is no greater than $\|(P_{\mathcal{N}_J} \otimes I_m)\rho_m(A)\|$. In particular, this implies that the kernel of the map is precisely $J$.

One of the tools to be used in proving the inequality is "factorization" of ultraweakly continuous linear functionals on $\rho(H^\infty(E))$. An ultraweakly closed space $\mathcal{S}$ of operators on a Hilbert space $\mathcal{H}$ is said to have property $\mathbb{A}_n(1)$ if for each $\epsilon > 0$ and each ultraweakly continuous linear functional $\phi$ on $M_n(\mathcal{S})$, there are elements $x$ and $y$ of $\mathcal{H}^{(n)}$, thought of as column vectors, such that $\|x\|\|y\| < (1+\epsilon)\|\phi\|$ and $\phi(A) = \langle x, Ay\rangle$ for all $A \in M_n(\mathcal{S})$. Thus $\mathcal{S}$ has property $\mathbb{A}_n(1)$ if and only if $M_n(\mathcal{S})$ has property $\mathbb{A}_1(1)$. The same definition applies when $n = \aleph_0$, if $M_n(\mathcal{S})$ is taken to mean $\mathcal{S} \otimes B(\ell^2)$. Of course, property $\mathbb{A}_{\aleph_0}(1)$ implies property $\mathbb{A}_n(1)$ for all finite $n$. Further discussion of this topic can be found in [7, Chapter 2] and [9, Chapter 8]. The needed factorization result comes from the following lemma.

**Lemma 15.** *Let $\mathcal{H}$ be a Hilbert space. An infinite multiple of an ultraweakly closed subspace of $B(\mathcal{H})$ has property $\mathbb{A}_{\aleph_0}(1)$.*

*Proof.* Let $S$ be an ultraweakly closed subspace of $B(\mathcal{H})$. The fact that $S^{(\infty)}$ has property $\mathbb{A}_1(1)$ follows from the fact that $B(\mathcal{H})^{(\infty)}$ has property $\mathbb{A}_1(1)$ [9, Proposition 59.6] and the fact that the predual of $S^{(\infty)}$ is isometrically isomorphic to the quotient of the predual of $B(\mathcal{H})^{(\infty)}$ by the preannihilator of $S^{(\infty)}$ [14, page 43]. If $\mathcal{L}$ is a separable Hilbert space, then $S^{(\infty)} \otimes B(\mathcal{L}) \cong (S \otimes B(\mathcal{L}))^{(\infty)}$ via a unitarily implemented isomorphism.[2] This shows that $S^{(\infty)} \otimes B(\mathcal{L})$ is unitarily equivalent to an infinite multiple of an ultraweakly closed subspace of $B(\mathcal{H} \otimes \mathcal{L})$ and therefore has property $\mathbb{A}_1(1)$. $\square$

**Corollary 16.** *With the setup as above, $\rho(H^\infty(E))$ has property $\mathbb{A}_{\aleph_0}(1)$.*

The following lemma yields the final piece of the puzzle. It should be noted that its proof does not require that $J$ is a left ideal, a fact which may be useful in further study of the right ideals of $H^\infty(E)$.

**Lemma 17.** *For each $m \in \mathbb{N}$ and each matrix $A \in M_m(\rho(H^\infty(E)))$,*

$$\|\operatorname{dist}(A, M_m(\rho(J)))\| \leq \|(P_{\mathcal{N}_J} \otimes I_m)A\|.$$

*Proof.* Let $\epsilon > 0$ and suppose that $\|\operatorname{dist}(A, M_m(\rho(J)))\| = 1$. Then there is [19, Lemma 2.4] an ultraweakly continuous linear functional $\phi$ on $M_m(\rho(H^\infty(E)))$ that annihilates $M_m(\rho(J))$ such that

$$1 - \epsilon < |\phi(A)| \leq \|\phi\| = 1.$$

By Corollary 16, there are vectors $\eta = (\eta_i)_{i=1}^m$ and $\zeta = (\zeta_i)_{i=1}^m$ in $(\mathcal{F}(E) \otimes_\pi H)^{(m)}$ (thought of as column space) such that $\phi(\rho_m(T)) = \langle \zeta, \rho_m(T)\eta \rangle$ for all $T \in M_m(H^\infty(E))$ and $\|\eta\|\|\zeta\| < 1 + \epsilon$.

Let $\mathcal{N}$ and $\mathcal{N}_0$ be the subspaces of $\mathcal{F}(E) \otimes_\pi H$ defined by

$$\mathcal{N} = \overline{\sum_{i=1}^m \rho(H^\infty(E))\eta_i}$$

and

$$\mathcal{N}_0 = \overline{\sum_{i=1}^m \rho(J)\eta_i}.$$

[2]The tensor products here are spacial tensor products.

Because $\mathcal{N}$ is an invariant subspace for $\rho(H^\infty(E))$, by Lemma 11 there are partial isometries $\{V_i\}_{i\in I}$ in the commutant of $\rho(H^\infty(E))$ with pairwise orthogonal ranges such that $\mathcal{N} = \sum_{i\in I}^{\oplus} V_i(\mathcal{F}(E)\otimes_\pi H)$. Using the facts that $J$ is a right ideal and $\rho(J)$ commutes with each $V_i$,

$$\begin{aligned}\mathcal{N}_0 = \overline{\sum_{i=1}^{m}\rho(J)\eta_i} &= \overline{\sum_{i=1}^{m}\rho(J)\rho(H^\infty(E))\eta_i}\\ &= \overline{\rho(J)\mathcal{N}}\\ &= \overline{\rho(J)\sum_{i\in I}\oplus V_i(\mathcal{F}(E)\otimes_\pi H)}\\ &= \sum_{i\in I}\oplus\overline{V_i\rho(J)(\mathcal{F}(E)\otimes_\pi H)}\\ &= \sum_{i\in I}\oplus\overline{V_i\mathcal{N}_J^\perp}.\end{aligned}$$

It follows that $\mathcal{N}\ominus\mathcal{N}_0 = \sum_i \oplus(V_i(\mathcal{F}(E)\otimes_\pi H)\ominus\overline{V_i\mathcal{N}_J^\perp}) \subseteq \cap_i(V_i\mathcal{N}_J^\perp)^\perp$, so that[3] for each $i \in I$,

$$V_i^*(\mathcal{N}\ominus\mathcal{N}_0) \subseteq \mathcal{N}_J^{\perp\perp} = \mathcal{N}_J. \tag{2.1}$$

Let $\zeta' \in \mathcal{N}^{(m)}$ be the orthogonal projection of $\zeta$ onto $\mathcal{N}^{(m)}$. Because $\mathcal{N}$ is invariant for $\rho(H^\infty(E))$ and $\eta$ is in $\mathcal{N}^{(m)}$, for each $T \in M_m(\rho(H^\infty(E)))$ it follows that

$$\phi(T) = \langle\zeta, T\eta\rangle = \langle\zeta, P_{\mathcal{N}^{(m)}}T\eta\rangle = \langle P_{\mathcal{N}^{(m)}}\zeta, T\eta\rangle = \langle\zeta', T\eta\rangle. \tag{2.2}$$

In particular, if $T$ is in $M_m(\rho(J))$, then $\langle\zeta', T\eta\rangle = 0$, so $\zeta'$ is in $(\mathcal{N}\ominus\mathcal{N}_0)^{(m)}$. For each $i$, define $\xi_i = V_i^{*(m)}\zeta'$ and $\nu_i = V_i^{*(m)}\eta$. Because $\{V_i\}_{i\in I}$ is a set of partial isometries with pairwise orthogonal ranges that add up to $\mathcal{N}$, and because $\zeta'$ and

[3] Observe that $V_i^*(\mathcal{N}\ominus\mathcal{N}_0) \subseteq V_i^*(\cap_j(V_j\mathcal{N}_J^\perp)^\perp) \subseteq \cap_j V_i^*((V_j\mathcal{N}_J^\perp)^\perp) \subseteq V_i^*((V_i\mathcal{N}_J^\perp)^\perp)$, and generally if $X$ is an operator on a Hilbert space of which $M$ is a subset, then $X^*((XM)^\perp) \subseteq M^\perp$.

$\eta$ are in $\mathcal{N}^{(m)}$, it follows that

$$\zeta' = \sum_i \oplus V_i^{(m)} \xi_i,$$
$$\eta = \sum_i \oplus V_i^{(m)} \nu_i,$$
$$\|\zeta'\| = (\sum_i \|\xi_i\|^2)^{1/2}, \text{ and}$$
$$\|\eta\| = (\sum_i \|\nu_i\|^2)^{1/2}.$$

Define $\psi$ to be the normal linear functional on $M_m(B(\mathcal{F}(E) \otimes_\pi H))$ given by $\psi(T) = \sum_i \langle \xi_i, T\nu_i \rangle$. Then

$$\|\psi\| \leq \sum_i \|\xi_i\| \|\nu_i\| \leq (\sum_i \|\xi_i\|^2)^{1/2} (\sum_i \|\nu_i\|^2)^{1/2} = \|\zeta'\| \|\eta\| \leq \|\zeta\| \|\eta\| < 1 + \epsilon,$$

and for $T \in M_m(\rho(H^\infty(E)))$,

$$\begin{aligned}
\phi(T) &= \langle \zeta', T\eta \rangle \\
&= \left\langle \sum_i \oplus V_i^{(m)} \xi_i, T \sum_j \oplus V_j^{(m)} V_j^{*(m)} \eta \right\rangle \\
&= \sum_{i,j} \langle \xi_i, V_i^{*(m)} T V_j^{(m)} V_j^{*(m)} \eta \rangle \\
&= \sum_{i,j} \langle \xi_i, V_i^{*(m)} V_j^{(m)} T V_j^{*(m)} \eta \rangle \\
&= \sum_i \langle \xi_i, V_i^{*(m)} V_i^{(m)} T V_i^{*(m)} \eta \rangle \\
&= \sum_i \langle \xi_i, T V_i^{*(m)} V_i^{(m)} V_i^{*(m)} \eta \rangle \\
&= \sum_i \langle \xi_i, T V_i^{*(m)} \eta \rangle \\
&= \sum_i \langle \xi_i, T\nu_i \rangle = \psi(T),
\end{aligned}$$

which shows that $\phi$ is the restriction of $\psi$ to $M_m(\rho(H^\infty(E)))$. The calculation uses the fact that $\{V_i^{(m)}\}_{i \in I}$ is a set of partial isometries in the commutant of $M_m(\rho(H^\infty(E)))$ with pairwise orthogonal ranges along with Remark 12.

By (2.1), each $\xi_i$ is in $\mathcal{N}_J^{(m)}$, and therefore for all $T$, $\psi(T) = \psi(P_{\mathcal{N}_J^{(m)}} T)$. Putting this all together with $A$ from above,

$$1 - \epsilon < |\phi(A)| = |\psi(A)| = |\psi(P_{\mathcal{N}_J^{(m)}} A)| \leq \|\psi\| \|P_{\mathcal{N}_J^{(m)}} A\| \leq (1 + \epsilon) \|P_{\mathcal{N}_J^{(m)}} A\|.$$

Therefore $\|P_{\mathcal{N}_J^{(m)}} A\| \geq 1$, which is what was claimed. $\square$

# CHAPTER 3
# MULTIPLIERS OF REPRODUCING KERNEL $W^*$-MODULES

This chapter concerns the theory of self-dual reproducing kernel $W^*$-modules and their multipliers. A major source of motivation is the Pick-like interpolation theorem for $H^\infty(E)$. The complex-valued theory of reproducing kernel Hilbert spaces provides an operator theoretic framework for stating, proving, and generalizing Pick's theorem as a statement about multipliers of a reproducing kernel Hilbert space. However, a general theory of multipliers for reproducing kernel $W^*$-correspondences appropriate for the Muhly and Solel setting is still in its infancy.

In the first section, some of the basic theory is generalized to the case where the scalars are replaced with a von Neumann algebra $M$ and the Hilbert spaces are replaced with self-dual Hilbert $C^*$-modules over $M$. In the second and final section of this chapter, the theory is extended to $W^*$-correspondences, and connections to Muhly and Solel's theory are shown.

## 3.1 $W^*$-algebra-valued kernels

Positive $C^*$-algebra-valued kernel functions and the corresponding reproducing kernel Hilbert $C^*$-modules have been been considered by several authors for various applications [22, 21, 34, 20, 6, 5]. The kernels considered in [6] and [5] provide the most direct motivation for the development in this and the next section, generalizing much of the classical theory of reproducing kernel Hilbert spaces [1]. This section provides what may be seen as the most straightforward generalization to the $C^*$-algebra-valued setting, namely self-dual $W^*$-modules with $W^*$-algebra-valued kernels, because these correspondences share more in common with Hilbert spaces than do general Hilbert $C^*$-modules. More importantly, these are the tools that will relate more directly to the representation theory of noncommutative Hardy algebras in the next section.

There is some overlap with the cited works. Namely, the definition of positive kernel function and the correspondence between kernels and reproducing kernel Hilbert $C^*$-modules goes back at least to Kakihara [22], and can be found in some of the other references cited above. Among the new results presented in this section are a generalization of the expression of a kernel in terms of an orthonormal basis, and of the basic theory of multipliers. Most of the techniques in this section are similar to those used in the classical setting, and results of Paschke on self-dual modules are used to effect the generalizations.

**Definition 18.** Let $Z$ be a set and let $M$ be a $W^*$-algebra. A function $K : Z \times Z \to M$ is called a positive $M$-valued kernel if for all $n \in \mathbb{N}$ and points $z_1, \ldots, z_n \in Z$, the matrix $(K(z_i, z_j))_{i,j=1}^n$ is positive in $M_n(M)$.

In this section such kernels will be constructed from certain $M$-modules of functions on $Z$, and conversely it will be shown that each positive kernel determines such an $M$-module.

**Definition 19.** Let $E$ be a self-dual $W^*$-module over $M$ whose elements are functions from $Z$ to $M$. Then $E$ is called a *reproducing kernel $W^*$-module* over $M$ (or *reproducing kernel $M$-module*) if each point evaluation on $Z$ is a bounded module map to $M$ with its standard right module structure.

*Remark* 20. It should be emphasized that the assumption that elements of $E$ are functions means that equality of elements of $E$ is the usual equality of functions. In particular, if all point evaluations of an element $f$ of $E$ yield $0 \in M$, then $f$ is the zero element of $E$. Because $E$ is self-dual in the definition, the condition that point evaluations are bounded module maps is equivalent to the condition that for each $z \in Z$ there is a $k_z \in E$ such that for all $f \in E$, $f(z) = \langle k_z, f \rangle$. Regardless, the condition that point evaluations are module maps is another way of saying that the module operations are pointwise. That is, $(f + g)(z) = f(z) + g(z)$ and $(f \cdot a)(z) = f(z)a$ for all $f, g \in E$, $a \in M$, and $z \in Z$.

The name "reproducing kernel" is given because to each reproducing kernel $W^*$-module $E_M$ on a set $Z$ corresponds a positive $M$-valued kernel that "reproduces" values of functions in $E$, as will be made precise shortly.

**Definition 21.** Let $E$ be a reproducing kernel $M$-module on a set $Z$, and for each $z \in Z$ let $k_z$ be the element of $E$ such that for all $f \in E$, $f(z) = \langle k_z, f \rangle$. Define $K_E : Z \times Z \to M$ by $K_E(w, z) = \langle k_w, k_z \rangle$. Then $K_E$ is called the *kernel associated with* $E$.

*Remark* 22.

1. The kernel associated with a reproducing kernel $M$-module is positive by part 3 of Lemma 10. Furthermore, it is immediate from the definition that $K_E(\cdot, z) \in E$ for each $z \in Z$, and for each $f \in E$ and $z \in Z$, $f(z) = \langle K_E(\cdot, z), f \rangle$. Thus the kernel function can be used to "reproduce" the evaluations of the functions in $E$.

2. It will be useful in the following proposition to note that $\{k_z : z \in Z\}$ is a total subset of $E$ (see Lemma 5 and the subsequent definition). This follows from the fact that taking the inner product with $k_z$ yields point evaluation at $z$, and all point evaluations of an element $f$ of $E$ are 0 only if $f$ is 0 (see Remark 20).

**Proposition 23.** *Let $M$ be a $W^*$-algebra, let $Z$ be a set, and let $K : Z \times Z \to M$ be a positive kernel. Then there is a unique reproducing kernel $M$-module $E_K$ on $Z$ such that $K$ is the kernel associated with $E_K$; i.e., $K_{E_K} = K$.*

*Proof.* Let $E_{00}$ be the free right $M$-module generated by the set $\{k_z : z \in Z\}$. (It will be seen below that $k_z$ represents the function $K(\cdot, z)$, but for now it is a formal generator.) That is, $E_{00}$ is the set of formal finite sums of terms of the form $k_z a$ with $z \in Z$ and $a \in M$. An $M$-valued pre-inner product can be defined on $E_{00}$ by $\left\langle \sum_i k_{w_i} a_i, \sum_j k_{z_j} b_j \right\rangle = \sum_{i,j} a_i^* K(w_i, z_j) b_j$. By positivity of $K$, this definition satisfies all of the properties of an $M$-valued inner product except perhaps that $\langle x, x \rangle = 0$ may not imply that $x = 0$. In the usual way [24, pages 3-4], using the Cauchy-Schwarz inequality allows us to mod out by the zero length vectors to obtain an inner product on the quotient and complete to a Hilbert $C^*$-module $E_0$ over $M$.

Let $E$ denote the self-dual completion of $E_0$. For each $z \in Z$, the element of $E$ corresponding to $k_z$ after these completions will be denoted the same. Each $x \in E$ determines a function $\hat{x} : Z \to M$ by the formula $\hat{x}(z) = \langle k_z, x \rangle$, and $\hat{k}_w(z) = \langle k_z, k_w \rangle = K(z, w)$ by construction. If $\hat{x}$ vanishes on $Z$, then $\langle k_z, x \rangle = 0$ for all $z \in Z$, which by $\sigma$ density of the $M$-module span of $\{k_z : z \in Z\}$ in $E$ forces $x = 0$ by Lemma 5. Thus $E$ can be identified with the set of functions $\{\hat{x} : x \in E\}$. Making this identification, and writing $E_K = E$, it follows that $E_K$ is a reproducing kernel $M$-module on $Z$ with kernel $K_{E_K} = K$.

Suppose that $F$ is a reproducing kernel $M$-module on $Z$ that also has kernel $K$. Denoting by $F_{00}$ the $M$-submodule of $F$ spanned by $\{k_z : z \in Z\}$, the map from $F_{00}$ to $E_0$ defined by $\sum_i k_{z_i} a_i \mapsto \sum_i k_{z_i} a_i$ is an isometric module map, and thus extends to an isometric module map $U$ from the norm closure $F_0$ of $F_{00}$ to $E_0$. Because the restriction of $U$ to $F_{00}$ already has dense range, $U$ is onto, and therefore adjointable [24, Theorem 3.5]. Given $f \in F_0$, the computation $(Uf)(z) = \langle k_z, Uf \rangle_{E_0} = \langle U^* k_z, f \rangle_{F_0} = \langle k_z, f \rangle_{F_0} = f(z)$ shows that $U$ is the identity map on $F_0 = E_0$. The self-dual module generated by $F_0$ is $F$, and the self-dual module generated by $E_0$ is $E_K$. Thus the identification $F_0 = E_0$ carries over to $F = E_K$, including the same point evaluations because $F$ and $E_K$ have the same kernel, which proves the uniqueness.

$\square$

*Remark* 24. It follows from the uniqueness in Proposition 23 that if $E$ is a reproducing kernel $W^*$-module, then $E_{K_E} = E$. To be explicit, the map $K \mapsto E_K$ is a bijection from the set of positive $M$-valued kernels on $Z \times Z$ onto the set of reproducing kernel $M$-modules on $Z$, and $E \mapsto K_E$ is its inverse.

Paschke used analogues of orthonormal bases to give a convenient direct sum decomposition of each self-dual Hilbert $W^*$-module, and this will be used in the next proposition. A subset $\{e_i : i \in I\}$ of a $W^*$-module $E$ will be called an *orthonormal basis* for $E$ if $\langle e_i, e_i \rangle$ is a nonzero projection for all $i$, $\langle e_i, e_j \rangle = 0$ for all $i \neq j$, and $\{e_i : i \in I\}$ is a maximal set with these properties. Using Zorn's lemma and an analogue of polar decomposition [35, Proposition 3.11] Paschke showed that that every self-dual $W^*$ module has an orthonormal basis [35, proof of Theorem 3.12]. The following result shows that the usual representation of a complex-valued positive kernel function in terms of an orthonormal basis for the corresponding reproducing kernel Hilbert space [1, Proposition 2.18] has a direct analogue in the $W^*$-algebra-valued setting.

**Proposition 25.** *Let $M$ be a $W^*$-algebra, let $Z$ be a set, let $K : Z \times Z \to M$ be a positive kernel, and let $\{e_i : i \in I\}$ be an orthonormal basis for $E_K$. Then for each $w$ and $z$ in $Z$, $K(w, z) = \sum_i e_i(w)e_i(z)^*$, ultraweakly. That is, the net of finite partial sums converges to $K(w, z)$ in the ultraweak topology of $M$.*

*Proof.* Set $p_i = \langle e_i, e_i \rangle$, so that each $p_i$ is a projection in $M$, and computing $\langle e_i p_i - e_i, e_i p_i - e_i \rangle$ quickly shows that $e_i p_i = e_i$. By Paschke's theorem on ultraweak direct sum decompositions [35, Theorem 3.12] and its proof, $E_K$ is isomorphic to the ultraweak direct sum $X = \bigoplus_i^{uw} p_i M$, with an isometric isomorphism $T : E_K \to X$ given by $Tf = \{\langle e_i, f \rangle\}$, giving an analogue of Parseval's identity. Note that $p_i \langle e_i, f \rangle = \langle e_i p_i, f \rangle = \langle e_i, f \rangle$, so that $\langle e_i, f \rangle$ is in fact in $p_i M$. For each $w$ and $z$ in $Z$, this isomorphism yields

$$\begin{aligned} K(w, z) &= \langle k_w, k_z \rangle_{E_K} = \langle Tk_w, Tk_z \rangle_X \\ &= \langle \{\langle e_i, k_w \rangle\}, \{\langle e_i, k_z \rangle\} \rangle_X \\ &= \langle \{e_i(w)^*\}, \{e_i(z)^*\} \rangle_X \\ &= \sum_i e_i(w)e_i(z)^*, \end{aligned}$$

where the last equality and the ultraweak convergence of the sum follows from the definition of the ultraweak direct sum (Definition 9). □

In the classical setting, this formula is useful in finding the kernel function when the space and an orthonormal basis are already in sight. For instance, the Hardy space on the unit disk, $H^2(\mathbb{D})$, is a Hilbert space of functions with continuous point evaluations, and therefore is a reproducing kernel Hilbert space, whose kernel is called the Szegő kernel. Using the fact that the nonnegative powers of the coordinate function provide an orthonormal basis, one sums the geometric series to find that the Szegő kernel is $K_{H^2}(w, z) = \frac{1}{1-w\bar{z}}$. A similar but slightly more complicated analysis provides the kernel function for the Bergman space, and for many other "concrete" Hilbert function spaces. One might expect a similar analysis to be useful in the $W^*$-setting, but for the present work the case of spaces constructed from given kernel functions is of more relevance.

A primary motivation for the development in this chapter is to study certain quotients of the noncommutative Hardy algebras as multiplier algebras of reproducing kernel $W^*$-modules (see Section 3.2 below). In the previous works relating to reproducing kernel $C^*$-modules cited above, no general theory of multipliers has been developed, although there is a notable special case in [5] that will be discussed in the next section. The remainder of this section lays down some fundamentals on multipliers appropriate for the present setting.

**Definition 26.** A function $\phi : Z \to M$ is called a multiplier of the reproducing kernel $M$-module $E$ on $Z$ if for all $f \in E$ the function $\phi(\cdot)f(\cdot)$ is in $E$.

**Lemma 27.** *Let $\phi$ be a multiplier of $E$. Then the map $M_\phi : E \to E$ defined by $M_\phi f = \phi(\cdot)f(\cdot)$ is in $\mathcal{L}(E)$, and for all $z \in Z$, $M_\phi^* k_z = k_z \phi(z)^*$.*

*Proof.* The closed graph theorem will be used to show that $M_\phi$ is bounded. Suppose that $\{f_k\}_k$ is a sequence in $E$ such that the sequence $\{(f_k, M_\phi f_k)\}$ converges in norm to a point $(f, g) \in E \times E$. Fix $z \in Z$. Because evaluation at $z$ is bounded and

$M_\phi f_k \to g$, it follows that $\phi(z)f_k(z) \to g(z)$ in $M$. Similarly, evaluation at $z$ and left multiplication by $\phi(z)$ are bounded, so the convergence of $\{f_k\}$ to $f$ implies that $\phi(z)f_k(z) \to \phi(z)f(z)$. Since $z$ was arbitrary, it follows that $g = M_\phi f$, showing that $M_\phi$ is bounded. It is immediate from the definition that $M_\phi$ is a module map, and it follows from the self-duality of $E$ that $M_\phi$ is in $\mathcal{L}(E)$.

Given $w$ and $z$ in $Z$, the straightforward computation

$$\begin{aligned}(M_\phi^* k_z)(w) &= \langle k_w, M_\phi^* k_z\rangle = \langle M_\phi k_w, k_z\rangle \\ &= \langle k_z, M_\phi k_w\rangle^* = (\phi(z)k_w(z))^* \\ &= k_w(z)^*\phi(z)^* = \langle k_w, k_z\rangle \phi(z)^* \\ &= k_z(w)\phi(z)^*\end{aligned}$$

proves the last part of the lemma. □

*Remark* 28. This lemma says that the functions $k_z$ are "eigenvectors" for the adjoints of multiplication operators on $E$, with $M$-valued "eigenvalues". Taking advantage of this fact is an important ingredient in the proof of the next proposition, which gives a necessary and sufficient positivity criterion for a function to be multiplier. It would be interesting to see how this notion of eigenvalue relates to other operator-valued spectra in the literature.

**Proposition 29.** *A function $\phi : Z \to M$ is a multiplier of the reproducing kernel $M$-module $E = E_K$ with $\|M_\phi\| \le 1$ if and only if the map $K_\phi : Z \times Z \to M$ defined by $K_\phi(w,z) = K(w,z) - \phi(w)K(w,z)\phi(z)^*$ is a positive kernel.*

*Proof.* Suppose that $\phi$ is a multiplier of $E$ with $\|M_\phi\| \le 1$. Positivity of $K_\phi$ follows by computation from the fact that $I_E - M_\phi M_\phi^*$ is positive. For all $w_1, \dots, w_n \in Z$ and $a_1, \dots, a_n \in M$, compute

$$\begin{aligned}0 &\le \left\langle \sum_{i=1}^n k_{w_i} a_i, (I_E - M_\phi M_\phi^*)\left(\sum_{j=1}^n k_{w_j} a_j\right)\right\rangle \\ &= \sum_{i,j=1}^n a_i^* \left(\langle k_{w_i}, k_{w_j}\rangle - \langle k_{w_i}, M_\phi M_\phi^* k_{w_j}\rangle\right) a_j\end{aligned}$$

$$= \sum_{i,j=1}^{n} a_i^* \left( \langle k_{w_i}, k_{w_j} \rangle - \langle M_\phi^* k_{w_i}, M_\phi^* k_{w_j} \rangle \right) a_j$$

$$= \sum_{i,j=1}^{n} a_i^* \left( \langle k_{w_i}, k_{w_j} \rangle - \langle k_{w_i} \phi(w_i)^*, k_{w_j} \phi(w_j)^* \rangle \right) a_j$$

$$= \sum_{i,j=1}^{n} a_i^* \left( \langle k_{w_i}, k_{w_j} \rangle - \phi(w_i) \langle k_{w_i}, k_{w_j} \rangle \phi(w_j)^* \right) a_j$$

$$= \sum_{i,j=1}^{n} a_i^* \left( K(w_i, w_j) - \phi(w_i) K(w_i, w_j) \phi(w_j)^* \right) a_j.$$

By part 1 of Lemma 10, $K_\phi$ is positive.

Conversely, suppose that $K_\phi$ is positive. The idea is first to show that there is a contractive map $T \in \mathcal{L}(E)$ satisfying $T(k_z) = k_z \phi(z)^*$ for all $z \in Z$, and then to show that $T^* = M_\phi$. If such a $T$ exists, then it must send a sum $\sum_{i=1}^n k_{w_i} a_i$ to $\sum_{i=1}^n k_{w_i} \phi(w_i)^* a_i$. Using positivity of $K_\phi$ yields

$$\left\langle \sum_{i=1}^{n} k_{w_i} \phi(w_i)^* a_i, \sum_{j=1}^{n} k_{w_j} \phi(w_j)^* a_j \right\rangle = \sum_{i,j=1}^{n} a_i^* \phi(w_i) K(w_i, w_j) \phi(w_j)^* a_j$$

$$\leq \sum_{i,j=1}^{n} a_i^* K(w_i, w_j) a_j = \left\langle \sum_{i=1}^{n} k_{w_i} a_i, \sum_{j=1}^{n} k_{w_j} a_j \right\rangle.$$

It follows that

$$\left\| \sum_{i=1}^{n} k_{w_i} \phi(w_i)^* a_i \right\| \leq \left\| \sum_{i=1}^{n} k_{w_i} a_i \right\|,$$

which shows that $\sum_{i=1}^n k_{w_i} a_i \mapsto \sum_{i=1}^n k_{w_i} \phi(w_i)^* a_i$ defines a contractive module map on the (uncompleted) submodule $E_0$ spanned over $M$ by $\{k_z : z \in Z\}$. Because $E = E_0'$, the self-dual completion of $E_0$, the map has an extension $T \in \mathcal{L}(E)$ with $\|T\| \leq 1$ by Proposition 7. For each $f \in E$ and $z \in Z$, $(T^* f)(z) = \langle k_z, T^* f \rangle = \langle T k_z, f \rangle = \langle k_z \phi(z)^*, f \rangle = \phi(z) \langle k_z, f \rangle = \phi(z) f(z)$. Thus $\phi$ is a multiplier and $\|M_\phi\| = \|T^*\| = \|T\| \leq 1$.

□

### 3.2 Completely positive kernels and multipliers

The multiplier criterion in the previous section generalizes the well known Hilbert space case [1, Corollary 2.37], and it is closely related to Pick's interpolation theorem. An analogue of Proposition 29 for function space representations of noncommutative Hardy algebras is already known (cf. [32, Theorems 3.3 and 3.6]). The proof of Muhly and Solel has an intermediate step involving transfer functions, and a basic result in this section (see the proof of Theorem 45) gives a more direct argument from the interpolation theorem [31, Theorem 5.3]. A major distinction between elements of the Schur class of Muhly and Solel and the multipliers of the previous section is that there is a priori no reproducing kernel $W^*$-module of which the former are multipliers. One of the main goals of this section is to bridge this gap by developing the appropriate theory of reproducing kernel $W^*$-correspondences and their multipliers.

It should be noted that a description of the Schur class (Definition 44 below) of Muhly and Solel as an algebra of multipliers of a certain *Hilbert space* of functions was given by Ball et al. [5, Theorems 4.9 and 5.1], and there is some similarity between their work and some of the work here. However, there are several points of departure here that should be made clear. Their results on multipliers depend on first tensoring with a Hilbert space to work in a more classical framework. This process of tensoring can be seen as inducing representations of the multiplier algebra. Here, the multiplier criterion is developed for arbitrary reproducing kernel $W^*$-correspondences, as defined below. Furthermore, while some of the basics of the theory of reproducing kernel $C^*$-correspondences are given by Ball et al. [5, Section 3], their paper does not address the multipliers of such objects apart from the theorems mentioned above relating to the noncommutative Hardy algebras. In other directions their work goes far beyond what is presented here, and it was influential on the following.

The next order of business is to extend the theory from the previous section to the case of completely positive kernels and the associated reproducing kernel $W^*$-correspondences, starting with the following definition.

**Definition 30.** Let $Z$ be a set, and let $N$ and $M$ be $W^*$-algebras. A map $K$ :

$Z \times Z \to B_*(N, M)$ is called a *normal completely positive kernel* from $N$ to $M$ (for short, cp kernel), if for all $n \in \mathbb{N}$ and points $z_1, \ldots, z_n \in Z$, the matrix $(K(z_i, z_j))_{i,j=1}^n$ represents a (normal) completely positive map from $M_n(N)$ to $M_n(M)$.

*Remark* 31. This definition, without the assumption of normality (i.e., without the assumption of ultraweak continuity of the values of $K$) was introduced by Barreto et al. [6, Definition 3.2.2], and the following lemma providing alternative descriptions of cp kernels is subordinate to a lemma of theirs [6, Lemma 3.2.1].

**Lemma 32.** *Let $Z$ be a set, let $N$ and $M$ be $W^*$-algebras, and let $K : Z \times Z \to B_*(N, M)$ be a function. The following are equivalent.*

1. *$K$ is a cp kernel.*

2. *For all $n \in \mathbb{N}$ and points $z_1, \ldots, z_n \in Z$, the matrix $(K(z_i, z_j))_{i,j=1}^n$ represents a positive map from $M_n(N)$ to $M_n(M)$.*

3. *The map $K' : (N \times Z) \times (N \times Z) \to M$ defined by $K'((a_1, z_1), (a_2, z_2)) = K(z_1, z_2)(a_1 a_2^*)$ is a positive kernel.*

Part of the usefulness of cp kernels comes from the fact that they are closed under pointwise composition, also called Schur products [6]. If $K_1$ is a cp kernel from $N$ to $M$ on $Z$, and $K_2$ is a cp kernel from $L$ to $N$ on $Z$, then then map $K : Z \times Z \to B_*(L, M)$ defined by $K(w, z) = K_1(w, z) \circ K_2(w, z)$ is a cp kernel from $L$ to $M$ [6, Theorem 3.4.2], called the *Schur product* of $K_1$ and $K_2$, and denoted by $K = K_1 * K_2$.

Although Barreto et al. also discuss positive $C^*$-algebra-valued kernels in the sense of the previous section, they do not appear to have made explicit how they are a special case of cp kernels between two $C^*$-algebras. Namely, under the identification of a $W^*$-algebra $M$ with $B_*(\mathbb{C}, M)$, a positive $M$-valued kernel becomes a cp kernel from $\mathbb{C}$ to $M$, as shown in the following simple proposition.

**Proposition 33.** *Let $Z$ be a set, let $M$ be a $W^*$-algebra, and let $K : Z \times Z \to M$ be a function. Then $K$ is a positive kernel if and only if the function $\overline{K} : Z \times Z \to B_*(\mathbb{C}, M)$ defined by $\overline{K}(z_1, z_2)(\lambda) = \lambda K(z_1, z_2)$ is a cp kernel.*

*Proof.* Suppose $\overline{K}$ is a cp kernel, and for each positive integer $n$ let $E_n$ denote the matrix in $M_n(\mathbb{C})$ each of whose entries is 1. Given points $z_1, \ldots, z_n \in Z$, the definition of $\overline{K}$ immediately yields $(K(z_i, z_j))_{i,j=1}^n = (\overline{K}(z_i, z_j))_{i,j=1}^n(E_n)$. Since $E_n$ is positive, it follows that $K$ is a positive kernel.

To prove the converse, a little juggling will be used to reduce the problem to the complete positivity of a Schur product map on complex matrices. By Lemma 32, to show that complete positivity of $\overline{K}$ follows from positivity of $K$, it will suffice to observe that if $n$ is a positive integer and $A = (a_{ij})_{i,j=1}^n$ is a positive element of $M_n(M)$, then the "Schur product" map $S_A : M_n(\mathbb{C}) \to M_n(M)$ defined by $S_A((\lambda_{ij})_{i,j=1}^n) = (\lambda_{i,j} a_{i,j})_{i,j=1}^n$ is a positive map. This in turn is equivalent to showing the positivity of the Schur product map $S_\Lambda : (a_{ij})_{i,j=1}^n \mapsto (\lambda_{i,j} a_{i,j})_{i,j=1}^n$ on $M_n(M)$ corresponding to a fixed positive element $\Lambda = (\lambda_{ij})_{i,j=1}^n$ of $M_n(\mathbb{C})$. For this, denote by $\tilde{S}_\Lambda : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ the usual Schur product map corresponding to $\Lambda$. It is a standard result that $\tilde{S}_\Lambda$ is completely positive [36, Theorem 3.7], and identifying $M_n(M)$ with $M \otimes M_n(\mathbb{C})$ yields $S_\Lambda = I_M \otimes \tilde{S}_\Lambda$. It follows that $S_\Lambda$ is completely positive,[1] and in particular positive. □

One can always obtain an example of an $M$-valued positive kernel by evaluating each value of a cp kernel from $N$ to $M$ at the identity element of $N$. An example of such a kernel specialized to the diagonal arises in Chapter 5. This observation is also the point of the easier direction in the preceding proposition. The following is the main motivating class of examples of cp kernels in this chapter.

**Example 34.** Let $E$ be a $W^*$-correspondence over $M$, let $\sigma : M \to B(H)$ be

[1] For instance, a Stinespring representation of $\tilde{S}_\Lambda$ lifts to one for $S_\Lambda$.

a normal representation of $M$, and let $E^\sigma$ be the $\sigma$ dual of $E$, which is a $W^*$-correspondence over $\sigma(M)'$. For each $w$ and $z$ in $\mathbb{D}(E^{\sigma *})$, let $\theta_{w^*,z^*} : \sigma(M)' \to \sigma(M)'$ be defined by $\theta_{w^*,z^*}(a) = \langle w^*, a \cdot z^* \rangle_{E^\sigma}$. By ultraweak continuity of the left action $\varphi^\sigma$ of $\sigma(M)'$ and Lemma 3, $\theta_{w^*,z^*}$ is in $B_*(\sigma(M)')$. Because $\|\theta_{w^*,z^*}\| \leq \|w\|\|z\| < 1$, it is possible to define an analogue of the Szegő kernel on $\mathbb{D}(E^{\sigma *})$, $K_S : \mathbb{D}(E^{\sigma *}) \times \mathbb{D}(E^{\sigma *}) \to B_*(\sigma(M)')$, by $K_S(w, z) = \mathrm{id} + \theta_{w^*,z^*} + \theta^2_{w^*,z^*} + \theta^3_{w^*,z^*} + \cdots = (\mathrm{id} - \theta_{w^*,z^*})^{-1}$, where id denotes the identity operator on $\sigma(M)'$. The values of $K_S$ are ultraweakly continuous by Lemma 4, and complete positivity of $K_S$ will be shown below.

**Lemma 35.** *Let $E$ be an inner product module over a $C^*$-algebra $A$, let $n$ be a positive integer, and let $x_1, x_2, \ldots, x_n$ be elements of $E$. Then the map $\Theta : M_n(\mathcal{L}(E)) \to M_n(A)$ defined by $\Theta((T_{ij})) = (\langle x_i, T_{ij} x_j \rangle)$ is positive.*

*Proof.* Using part 2 of Lemma 10, it is enough to see that if $T_1, T_2, \ldots, T_n$ are in $\mathcal{L}(E)$, then $\Theta((T_i^* T_j))$ is positive. Observe that $\Theta((T_i^* T_j)) = (\langle T_i x_i, T_j x_j \rangle)$, so the result follows from part 3 of Lemma 10. □

**Proposition 36.** *The kernel $K_S$ of example 34 is a cp kernel.*

*Proof.* It was already observed that the values of $K_S$ are normal, so it only remains to show that $K_S$ satisfies the positivity condition. Define $K_1 : Z \times Z \to B_*(\sigma(M)')$ by $K_1(w, z) = \theta_{w^*,z^*}$. Then given $z_1, z_2, \ldots, z_n \in \mathbb{D}(E^{\sigma *})$, positivity of $(K_1(z_i, z_j))$ follows from Lemma 35 and complete positivity of $\varphi^\sigma$. By Lemma 32, $K_1$ is a cp kernel. For each integer $m > 1$, let $K_m$ be defined recursively by $K_m = K_1 * K_{m-1}$, and define $K_0$ to be the constant cp kernel from $\sigma(M)'$ to itself with value id. Each $K_m$ is a cp kernel because it is a Schur product of cp kernels. Since positivity of maps between $C^*$-algebras is closed under addition and norm limits, it follows from Lemma 32 that $K_S = K_0 + K_1 + K_2 + \cdots$ is a cp kernel. □

*Remark* 37. Proposition 36 is not new, but this proof is. A proof using the criterion of "Kolmogorov factorizations" from [6] was provided by Muhly and Solel [33, page 229].

The extension of the material from the last section to the setting of cp kernels requires the notion of a reproducing kernel $W^*$-correspondence from $N$ to $M$. The third part of Lemma 32 indicates that the appropriate notion might involve reproducing kernel $M$-modules of functions defined on $N \times Z$. The following definition is an adaptation and consolidation of the one found in [5, Definition 3.3].

**Definition 38.** Let $Z$ be a set, and let $N$ and $M$ be $W^*$-algebras. A reproducing kernel $M$-module $E$ of $M$-valued functions on $N \times Z$ is called a *reproducing kernel $W^*$-correspondence* from $N$ to $M$ if $E$ has a normal left action of $N$ that satisfies $(a \cdot f)(b, z) = f(ba, z)$ for all $a, b \in N$, $f \in E$, and $z \in Z$.

*Remark* 39. Similarly to the convention of the previous section, $k_{a,z}$ will denote the element of $E$ that induces point evaluation at $(a, z) \in N \times Z$. That is, $f(a, z) = \langle k_{a,z}, f \rangle$ for all $f \in E$, $a \in N$, and $z \in Z$. The kernel function associated with $E$ is denoted by $K_E : (N \times Z) \times (N \times Z) \to M$, and defined by $K_E((a, w), (b, z)) = \langle k_{a,w}, k_{b,z} \rangle$.

**Lemma 40.** *Let $E$ be a reproducing kernel correspondence from $N$ to $M$, and let $\{k_{a,z} : a \in N, z \in Z\}$ be the elements of $E$ that induce the point evaluations.*

1. *The action of $b \in N$ on $k_{a,z}$ is given by $b \cdot k_{a,z} = k_{ab^*,z}$.*

2. *Each element $f$ of $E$ is linear in its first variable. Furthermore, if $f \in E$ and $z \in Z$ are fixed, then the map $a \mapsto f(a, z)$ is in $B_*(N, M)$.*

3. *Let $\tilde{K}_E$ be the function on $Z \times Z$ with values mapping from $N$ to $M$ defined by $\tilde{K}_E(w, z)(a) = K_E((a, w), (1, z))$. Then $\tilde{K}_E$ takes values in $B_*(N, M)$, and $\tilde{K}_E$ is a cp kernel from $N$ to $M$.*

*Proof.* Part 1 follows by a computation. For each $c \in N$ and $w \in Z$, note that

$$\begin{aligned}(b \cdot k_{a,z})(c,w) &= \langle k_{c,w}, b \cdot k_{a,z}\rangle = \langle b^* \cdot k_{c,w}, k_{a,z}\rangle \\ &= \langle k_{a,z}, b^* \cdot k_{c,w}\rangle^* = ((b^* \cdot k_{c,w})(a,z))^* \\ &= (k_{c,w}(ab^*, z))^* = \langle k_{ab^*,z}, k_{c,w}\rangle^* \\ &= \langle k_{c,w}, k_{ab^*,z}\rangle = k_{ab^*,z}(c,w).\end{aligned}$$

To prove 2, let $f$ and $z$ be fixed as in the statement, and define $\psi : N \to M$ by $\psi(a) = f(a,z)$. Then by 1, $\psi(a) = \langle k_{1,z}, a \cdot f\rangle$, so $\psi$ is in $B_*(N,M)$ by Lemma 3 and ultraweak continuity of the left action of $N$.

This leaves only 3. Given $w, z \in Z$ and $a \in N$, the evaluation $\tilde{K}_E(w,z)(a) = K_E((a,w),(1,z)) = \langle k_{a,w}, k_{1,z}\rangle = k_{1,z}(a,w)$ shows that $\tilde{K}_E$ takes values in $B_*(N,M)$ by 2. For all $a, b \in N$ and $w, z \in Z$, it follows from 1 that

$$\begin{aligned}K_E((a,w),(b,z)) &= \langle k_{a,w}, k_{b,z}\rangle = \langle k_{a,w}, b^* \cdot k_{1,z}\rangle \\ &= \langle b \cdot k_{a,w}, k_{1,z}\rangle = \langle k_{ab^*,w}, k_{1,z}\rangle \\ &= \tilde{K}_E(w,z)(ab^*),\end{aligned}$$

so that $K_E = \tilde{K}'_E$, and the complete positivity now follows from Lemma 32.

□

A reproducing kernel $W^*$-correspondence can be constructed from a cp kernel in a unique way. This is of particular interest in the motivating class of examples in Example 34, where a cp kernel $K_S$ is given, and the resulting space and its multipliers are to be studied.

**Proposition 41.** *Let $Z$ be a set, let $N$ and $M$ be $W^*$-algebras, and let $K : Z \times Z \to B_*(N,M)$ be a cp kernel from $N$ to $M$. Then there is a unique reproducing kernel $W^*$-correspondence $E_K$ from $N$ to $M$ on $N \times Z$ whose $M$-valued kernel function is $K' : (N \times Z) \times (N \times Z) \to M$, defined by $K'((a,w),(b,z)) = K(w,z)(ab^*)$.*

*Proof.* Let $K'$ be as in the statement. By Lemma 32, $K'$ is a positive $M$-valued kernel. Thus by Proposition 23, there is a unique reproducing kernel $M$-module $E_K$

of functions on $N \times Z$ such that $K_{E_K} = K'$. From the definition of reproducing kernel correspondence, there is only one way to define the action of $N$. Namely, for $a \in N$ and $f \in E_K$, $a \cdot f$ should be defined by $(a \cdot f)(b, z) = f(ba, z)$. Thus the uniqueness is already proven, but for existence there remains some work to show that the candidate action of $N$ exists.

Let $E_0$ denote the $M$-submodule of $E_K$ generated by $\{k_{a,z} : (a, z) \in N \times Z\}$, and fix $\sum_{j=1}^{n} k_{a_j,z_j} x_j \in E_0$. Looking at Part 1 of Lemma 40, if the left action of $N$ exists, $a \in N$ must send $\sum_{j=1}^{n} k_{a_j,z_j} x_j$ to $\sum_{j=1}^{n} k_{a_j a^*,z_j} x_j$. Let $(a^*a)_{i,j=1}^{n} \in M_n(N)$ denote the matrix each of whose elements is $a^*a$, let $E_n \in M_n(N)$ denote the matrix each of whose elements is $1 \in N$, let $A \in M_n(N)$ denote the matrix that is zero outside the first column and whose first column has $j^{\text{th}}$ entry $a_j$, and let $X$ denote the column matrix over $M$ whose $j^{\text{th}}$ entry is $x_j$. A computation ensues:

$$\begin{aligned}
\left\langle \sum_{i=1}^{n} k_{a_i a^*,z_i} x_i, \sum_{j=1}^{n} k_{a_j a^*,z_j} x_j \right\rangle &= \sum_{i,j=1}^{n} x_i^* K'((a_i a^*, z_i), (a_j a^*, z_j)) x_j \\
&= X^*(K'((a_i a^*, z_i), (a_j a^*, z_j)))_{i,j=1}^{n} X \\
&= X^*(K(z_i, z_j)(a_i a^* a a_j^*))_{i,j=1}^{n} X \\
&= X^*(K(z_i, z_j))_{i,j=1}^{n}((a_i a^* a a_j^*)_{i,j=1}^{n}) X \\
&= X^*(K(z_i, z_j))_{i,j=1}^{n}(A(a^*a)_{i,j=1}^{n} A^*) X \\
&\leq X^*(K(z_i, z_j))_{i,j=1}^{n}(A\|a\|^2 E_n A^*) X \\
&= \|a\|^2 X^*(K(z_i, z_j))_{i,j=1}^{n}(A E_n A^*) X \\
&= \|a\|^2 \left\langle \sum_{i=1}^{n} k_{a_i,z_i} x_i, \sum_{j=1}^{n} k_{a_j,z_j} x_j \right\rangle.
\end{aligned}$$

The inequality follows from two applications of part 4 of Lemma 10, positivity of the map $(K(z_i, z_j))_{i,j=1}^{n}$, and the fact that $(a^*a)_{i,j=1}^{n} \leq \|a\|^2 E_n$. The last equality is just working backwards through the computation with 1 in place of $a$. Taking norms shows that

$$\left\| \sum_{i=1}^{n} k_{a_i a^*,z_i} x_i \right\| \leq \|a\| \left\| \sum_{i=1}^{n} k_{a_i,z_i} x_i \right\|.$$

Thus each $a$ determines a bounded module map on $E_0$ that sends $k_{b,z}$ to $k_{ba^*,z}$, which by Proposition 7 extends to a map $\varphi(a) \in \mathcal{L}(E)$. To check that $\varphi$ is a $*$-homomorphism, it is enough to restrict the image of $\varphi$ to the generating set $\{k_{a,z} : (a,z) \in N \times Z\}$. Multiplicativity is immediate. To see that $\varphi$ is $*$-preserving, note that

$$\langle k_{c,z}, \varphi(a^*)k_{b,w}\rangle = \langle k_{c,z}, k_{ba,w}\rangle = K(z,w)(ca^*b) = \langle k_{ca^*,z}, k_{b,w}\rangle = \langle \phi(a)k_{c,z}, k_{b,w}\rangle.$$

The verification of linearity is similar, and thus $\varphi$ is a $*$-homomorphism.

The action of $a \in N$ on $f \in E_K$ is given by

$$(a \cdot f)(b,z) = \langle k_{b,z}, a \cdot f\rangle = \langle a^* \cdot k_{b,z}, f\rangle = \langle k_{ba,z}, f\rangle = f(ba,z),$$

as required, and all that remains is to show that $\varphi$ is normal. Because $E_K$ is the self-dual completion of $E_0$, it suffices by Lemma 4 and Proposition 8 to show that for each $x$ and $y$ in $E_0$, the map $a \mapsto \langle x, a \cdot y\rangle$ is in $B_*(N,M)$. So let $\sum_i k_{b_i,w_i}x_i$ and $\sum_j k_{c_j,z_j}y_j$ be fixed elements of $E_0$, and note that for each $a \in N$,

$$a \mapsto \left\langle \sum_i k_{b_i,w_i}x_i, a \cdot \sum_j k_{c_j,z_j}y_j \right\rangle = \sum_{i,j} x_i^* K(w_i,z_j)(b_i a c_j^*)y_j$$

is an ultraweakly continuous function because multiplication maps by fixed elements of $W^*$-algebras are ultraweakly continuous, and each $K(w_i,z_j)$ is ultraweakly continuous by hypothesis. This completes the proof. □

*Remark* 42. One of the main things the preceding results add to the theory is the extension of the results on $C^*$-correspondences to $W^*$-correspondences. For the considerations of Ball et al. [5], which involved tensoring with a Hilbert space over a representation, self-duality was not relevant, and although they sometimes considered modules over $W^*$ algebras, they did not consider ultraweak continuity of the left action.

Motivation for the introduction of multipliers of reproducing kernel $W^*$-correspondences will come from the following review of the interpolation theorem and positivity criterion for the algebras of functions obtained from the noncommutative Hardy algebras by evaluating on adjoints of elements of dual correspondences.

Let $E$ be a $W^*$-correspondence over $M$, and let $\sigma : M \to B(H)$ be a normal representation of $M$ on a Hilbert space $H$, to be fixed throughout the remainder of this section. Then, as discussed in Chapter 1, $H^\infty(E,\sigma)$ is the algebra of $B(H)$-valued functions on $\mathbb{D}(E^{\sigma*})$ obtained from $H^\infty(E)$ by evaluating $F \in H^\infty(E)$ at $z \in \mathbb{D}(E^{\sigma*})$ as $\hat{F}(z) = \psi_z(F) = (\sigma \times z)(F)$. The kernel $J_\sigma$ of the corresponding homomorphism $\Gamma_\sigma : H^\infty(E) \to H^\infty(E,\sigma)$ is ultraweakly closed, because it is the intersection of the kernels of the ultraweakly continuous maps $\{\psi_z : z \in \mathbb{D}(E^\sigma)\}$. It follows that $H^\infty(E,\sigma)$ can be given the norm and ultraweak topologies induced by the algebraic isomorphism $H^\infty(E,\sigma) \cong H^\infty(E)/J_\sigma$, thus making $H^\infty(E,\sigma)$ a dual operator algebra. Recall also the cp kernel $K_S : \mathbb{D}(E^{\sigma*}) \times \mathbb{D}(E^{\sigma*}) \to B_*(\sigma(M)')$ from example 34, defined by $K_S(w,z) = (\mathrm{id} - \theta_{w^*,z^*})^{-1}$. It will be convenient to extend the codomain of the values of $K_S$ to $B(H)$ without changing the notation, and this will be highlighted in cases where confusion may arise. For fixed operators $X$ and $Y$ in $B(H)$, let $\mathrm{Ad}(X,Y) : B(H) \to B(H)$ denote the bounded operator defined by $\mathrm{Ad}(X,Y)(T) = XTY^*$.

Part of Muhly and Solel's interpolation theorem [31, Theorem 5.3] can be stated as follows.

**Theorem 43.** *Given $z_1, \ldots, z_n \in \mathbb{D}(E^{\sigma*})$ and operators $C_1, \ldots, C_n \in B(H)$, there is an element $F$ of $H^\infty(E)$ with $\|F\| \le 1$ such that $\hat{F}(z_k) = C_k$ for each $k$ if and only if the matrix $((\mathrm{id} - \mathrm{Ad}(C_i, C_j)) \circ K_S(z_i, z_j))$ defines a completely positive map from $M_n(\sigma(M)')$ to $M_n(B(H))$.*

This positivity condition provides motivation for what Muhly and Solel call Schur class operator functions on $\mathbb{D}(E^{\sigma*})$, defined as follows.

**Definition 44.** Let $f$ be a $B(H)$-valued function on $\mathbb{D}(E^{\sigma*})$ Then $f$ is called a *Schur class operator function* on $\mathbb{D}(E^{\sigma*})$ if for each positive integer $k$ and each $k$-tuple of elements $z_1, \ldots, z_n \in \mathbb{D}(E^{\sigma*})$, the matrix $((\mathrm{id} - \mathrm{Ad}(f(z_i), f(z_j)))\circ K_S(z_i, z_j))$ defines a completely positive map from $M_n(\sigma(M)')$ to $M_n(B(H))$. The set of all Schur class functions on $\mathbb{D}(E^{\sigma *})$ will be denoted $S_{E,\sigma}$.

An immediate consequence of Theorem 43 is that if $F$ is an element of the closed unit ball of $H^\infty(E)$, then $\hat{F}$ is in $S_{E,\sigma}$. In fact, using analogues of systems matrices and transfer functions, Muhly and Solel proved the converse, namely that each element of $S_{E,\sigma}$ is $\hat{F}$ for some $F$ in the closed unit ball of $H^\infty(E)$ [32, Theorems 3.3 and 3.6]. Here is another proof, which takes advantage of the dual space structure of $H^\infty(E)$ to argue directly from the interpolation theorem.

**Theorem 45.** *The image of the closed unit ball of $H^\infty(E)$ under the map $F \mapsto \hat{F}$ is precisely $S_{E,\sigma}$.*

*Proof.* As already mentioned, and as pointed out in [32, Theorem 3.2], one direction is immediate from the interpolation theorem. If $F$ is in the closed unit ball of $H^\infty(E)$ and $z_1, \dots, z_n$ are in $\mathbb{D}(E^{\sigma *})$, then by taking $C_k = \hat{F}(z_k)$ in Theorem 43, the existence of the interpolating function $\hat{F}$ coming from $F$ in the closed unit ball of $H^\infty(E)$ implies the complete positivity of the matrix function $((\mathrm{id} - \mathrm{Ad}(\hat{F}(z_i), \hat{F}(z_j))) \circ K_S(z_i, z_j))$. Thus $\hat{F}$ is in $S_{E,\sigma}$.

Conversely, suppose that $f$ is in $S_{E,\sigma}$. Let $\Lambda$ denote the set of finite subsets of $\mathbb{D}(E^{\sigma *})$, directed by inclusion. For each $\lambda = \{z_1, \dots, z_n\} \in \Lambda$, by hypothesis the map $((\mathrm{id} - \mathrm{Ad}(f(z_i), f(z_j)) \circ K_S(z_i, z_j))$ is completely positive. Thus by Theorem 43, there is an $F_\lambda$ in the closed unit ball of $H^\infty(E)$ such that $\hat{F}_\lambda|_\lambda = f|_\lambda$. Consider the net $\{F_\lambda\}_{\lambda \in \Lambda}$ in the closed unit ball of $H^\infty(E)$. Since $H^\infty(E)$ is an ultraweakly closed subspace of $\mathcal{L}(\mathcal{F}(E))$, the ultraweak topology restricted to $H^\infty(E)$ is a weak-$*$ topology coming from a quotient of the predual $\mathcal{L}(\mathcal{F}(E))_*$ of $\mathcal{L}(\mathcal{F}(E))$ [42, Theorems 4.7 and 4.8]. By Alaoglu's theorem, the closed unit ball of $H^\infty(E)$ is compact in the relative ultraweak topology. Since a topological space is compact if and only if each net in the space has a convergent subnet, there is an $F$ in the closed unit ball of $H^\infty(E)$ and a subnet $\{G_i\}_{i \in I}$ of $\{F_\lambda\}_{\lambda \in \Lambda}$ that converges ultraweakly to $F$. The claim is that $\hat{F} = f$.

Saying that $\{G_i\}_{i\in I}$ is a subnet of $\{F_\lambda\}_{\lambda\in\Lambda}$ means that there is a function $T : I \to \Lambda$ such that $G_i = F_{T(i)}$ for all $i \in I$, and for each $\lambda_0 \in \Lambda$ there is an $i_0 \in I$ such that $i \geq i_0$ in $I$ implies $T(i) \geq \lambda_0$ in $\Lambda$. Let $z$ be an element of $\mathbb{D}(E^{\sigma *})$, and let $\lambda_0 = \{z\} \in \Lambda$. Then there exists an $i_0$ such that for all $i \geq i_0$, $T(i) \geq \lambda_0$. This means that for all $i \geq i_0$, $z$ is in $T(i)$, and thus $\hat{G}_i(z) = \hat{F}_{T(i)}(z) = f(z)$. Hence, the net $\{\hat{G}_i(z)\}_{i\in I}$ is eventually constant with value $f(z)$, so that $\{\hat{G}_i(z)\}_{i\in I}$ converges to $f(z)$. On the other hand, by ultraweak continuity of the "point evaluation map" at $z$, $\psi_z : H^\infty(E) \to B(H)$, the fact that $\{G_i\}_{i\in I}$ converges to $F$ implies that $\{\hat{G}_i(z)\}_{i\in I}$ converges to $\hat{F}(z)$. Since the ultraweak topology is Hausdorff, it follows that $f(z) = \hat{F}(z)$, as claimed. □

The relationship between a Schur class operator function and the cp kernel $K_S$ on $\mathbb{D}(E^{\sigma *}) \times \mathbb{D}(E^{\sigma *})$ is reminiscent of the relationship shown in Proposition 29 between a contractive multiplier of a reproducing kernel self-dual $W^*$-module $E_K$ and its corresponding positive kernel $K$. This prompts the question of whether elements of $S_{E,\sigma}$, and therefore elements of $H^\infty(E,\sigma)$, are the multipliers (in an appropriate sense) of a reproducing kernel $W^*$-correspondence, and leads to the following definition.

**Definition 46.** Let $N$ and $L$ be $W^*$-correspondences, and let $E_K$ be a reproducing kernel $W^*$-correspondence from $N$ to $L$ with associated cp kernel $K : Z \times Z \to B_*(N, L)$. Then a function $\phi : Z \to L$ is called a *multiplier* of $E_K$ if for each $f \in E_K$, the function $(a, z) \mapsto \phi(z)f(a, z)$ is in $E_K$.

The next two results are extensions of Lemma 27 and Proposition 29 to reproducing kernel $W^*$-correspondences.

**Lemma 47.** *Let $\phi$ be a multiplier of $E_K$. Then the map $M_\phi : E_K \to E_K$ defined by $(M_\phi f)(a, w) = \phi(w)f(a, w)$ is in $\mathcal{L}(E)$ and commutes with the left action of $N$. For all $a \in N$ and $w \in Z$, $M_\phi^* k_{a,z} = k_{a,z}\phi(z)^*$.*

*Proof.* The proof of Lemma 27 goes through line by line, except for the condition of commuting with the left action of $N$. The latter follows from the computation $(M_\phi(a\cdot f))(b,z) = \phi(z)(a\cdot f)(b,z) = \phi(z)f(ba,z) = (M_\phi f)(ba,z) = (a\cdot(M_\phi f))(b,z)$, which concludes the proof. $\square$

**Theorem 48.** *Let $E_K$ be a reproducing kernel $W^*$-correspondence from $N$ to $L$ with cp kernel $K : Z \times Z \to B_*(N,L)$. Then a function $\phi : Z \to L$ is a multiplier of $E_K$ with $\|M_\phi\| \leq 1$ if and only if the map $K_\phi : Z \times Z \to B_*(N,L)$ defined by $K_\phi(w,z) = (\mathrm{id} - \mathrm{Ad}(\phi(w),\phi(z))) \circ K(w,z)$ is a cp kernel.*

*Proof.* The values of $K_\phi$ are normal because the values of $K(w,z)$ are normal and multiplication operators by fixed elements of $L$ are normal on $L$. The remainder of the proof is similar to that of Proposition 29. The details are provided for completeness, and because it should be ensured that everything goes through with the additional structure. Recall that $K'$ is the $L$-valued kernel function associated with $E_K$ on $(N \times Z) \times (N \times Z)$ that satisfies $K'((a,w),(b,z)) = K(w,z)(ab^*) = \langle k_{a,w}, k_{b,z}\rangle$. Similarly, $K'_\phi$ denotes the $L$-valued function on $(N \times Z) \times (N \times Z)$ defined by $K'_\phi((a,w),(b,z)) = K_\phi(w,z)(ab^*)$. First suppose that $\phi$ is a multiplier of $E_K$ with $\|M_\phi\| \leq 1$. Then $I_E - M_\phi M_\phi^*$ is positive in $\mathcal{L}(E_K)$. For all $a_1,\ldots,a_n \in N$, $z_1,\ldots,z_n \in Z$, and $b_1,\ldots,b_n \in L$, compute

$$
\begin{aligned}
0 &\leq \left\langle \sum_{i=1}^{n} k_{a_i,z_i} b_i, (I_E - M_\phi M_\phi^*) \left( \sum_{j=1}^{n} k_{a_j,z_j} b_j \right) \right\rangle \\
&= \sum_{i,j=1}^{n} b_i^* \left( \langle k_{a_i,z_i}, k_{a_j,z_j} \rangle - \langle k_{a_i,z_i}, M_\phi M_\phi^* k_{a_j,z_j} \rangle \right) b_j \\
&= \sum_{i,j=1}^{n} b_i^* \left( \langle k_{a_i,z_i}, k_{a_j,z_j} \rangle - \langle M_\phi^* k_{a_i,z_i}, M_\phi^* k_{a_j,z_j} \rangle \right) b_j \\
&= \sum_{i,j=1}^{n} b_i^* \left( \langle k_{a_i,z_i}, k_{a_j,z_j} \rangle - \langle k_{a_i,z_i} \phi(z_i)^*, k_{a_j,z_j} \phi(z_j)^* \rangle \right) b_j \\
&= \sum_{i,j=1}^{n} b_i^* \left( \langle k_{a_i,z_i}, k_{a_j,z_j} \rangle - \phi(z_i) \langle k_{a_i,z_i}, k_{a_j,z_j} \rangle \phi(z_j)^* \right) b_j \\
&= \sum_{i,j=1}^{n} b_i^* \left( K'((a_i,z_i),(a_j,z_j)) - \phi(z_i) K'((a_i,z_i),(a_j,z_j)) \phi(z_j)^* \right) b_j.
\end{aligned}
$$

This shows that $K'_\phi$ is a positive $L$-valued kernel by part 1 of Lemma 10, and hence $K_\phi$ is a cp kernel by Lemma 32.

Conversely, suppose that $K_\phi$ is a cp kernel. Similarly to the proof of Proposition 29, inspiration comes from the "eigenvalue" property of Lemma 47, and the first goal is to show that there is a map $T \in \mathcal{L}(E)$ with $\|T\| \leq 1$ satisfying $T(k_{a,z}) = k_{a,z}\phi(z)^*$ for all $a \in N$ and $z \in Z$. Thus, if it exists, $T$ sends a sum $\sum_{i=1}^n k_{a_i,z_i} b_i$ to $\sum_{i=1}^n k_{a_i,z_i}\phi(z_i)^* b_i$, with each $a_i \in N$, $z_i \in Z$, and $b_i \in L$. By Lemma 32, $K'_\phi$ is a positive $L$-valued kernel, whence

$$
\begin{aligned}
&\left\langle \sum_{i=1}^{n} k_{a_i,z_i} \phi(z_i)^* b_i, \sum_{j=1}^{n} k_{a_j,z_j} \phi(z_j)^* b_j \right\rangle \\
&= \sum_{i,j=1}^{n} b_i^* \phi(z_i) K'((a_i,z_i),(a_{,j},z_j)) \phi(z_j)^* b_j \\
&\leq \sum_{i,j=1}^{n} b_i^* K'((a_i,z_i),(a_j,z_j)) b_j \\
&= \left\langle \sum_{i=1}^{n} k_{a_i,z_i} b_i, \sum_{j=1}^{n} k_{a_j,z_j} b_j \right\rangle.
\end{aligned}
$$

Taking norms shows that $\sum_{i=1}^n k_{a_i,z_i} b_i \mapsto \sum_{i=1}^n k_{a_i,z_i}\phi(z_i)^* b_i$ defines a contractive module map on the $L$-submodule of $E_K$ generated by $\{k_{a,z} : a \in N, z \in Z\}$. By

Proposition 7, this map has an extension $T \in \mathcal{L}(E)$ with $\|T\| \leq 1$. Let $f$ be an element of $E_K$. Then for each $a \in N$ and $z \in Z$, $(T^*f)(a,z) = \langle k_{a,z}, T^*f\rangle = \langle Tk_{a,z}, f\rangle = \langle k_{a,z}\phi(z)^*, f\rangle = \phi(z)\langle k_{a,z}, f\rangle = \phi(z)f(a,z)$. Thus $\phi$ is a multiplier with $M_\phi^* = T$, and $\|M_\phi\| = \|T\| \leq 1$. $\square$

The theorem can now be applied to the algebras $H^\infty(E,\sigma)$ to yield a direct connection between the noncommutative Hardy algebras and reproducing kernel correspondences. Let $K_S : \mathbb{D}(E^{\sigma *}) \times \mathbb{D}(E^{\sigma *}) \to B_*(\sigma(M)', B(H))$ be the cp kernel defined in Example 34, $K_S(w,z) = (\mathrm{id} - \theta_{w^*,z^*})^{-1}$, with the codomain of the values extended to $B(H)$. Let $\mathrm{Mult}(E_{K_S})$ denote the algebra of multipliers of the reproducing kernel $W^*$-correspondence $E_{K_S}$ from $M$ to $B(H)$ associated with $K_S$.

**Corollary 49.** *Let $E$ be a $W^*$-correspondence over $M$, and let $\sigma : M \to B(H)$ be a normal representation of $M$ on a Hilbert space $H$. Then $H^\infty(E,\sigma) = \mathrm{Mult}(E_{K_S})$. Furthermore, for each $f \in H^\infty(E,\sigma)$, there is an $F \in H^\infty(E)$ such that $f = \hat{F}$ and $\|M_f\| = \|F\|$.*

*Proof.* Let $f$ be an element of $H^\infty(E,\sigma)$. Then for some $F \in H^\infty(E)$, $f = \Gamma_\sigma(F)$. Consider the function $g = \frac{1}{\|F\|} f = \Gamma_\sigma(\frac{1}{\|F\|}F) \in H^\infty(E,\sigma)$. By Theorem 45, $g$ is a Schur class operator function on $\mathbb{D}(E^{\sigma *})$. By Theorem 48, $g$ is a multiplier of $E_{K_S}$. Thus, so is $f = \|F\| g$, and $H^\infty(E,\sigma)$ is contained in the algebra of multipliers of $E_{K_S}$.

Now suppose that $\phi : \mathbb{D}(E^{\sigma *}) \to B(H)$ is a multiplier of $E_{K_S}$. Then applying the other directions in Theorems 48 and 45 in that order shows that $\frac{1}{\|M_\phi\|}\phi = \Gamma_\sigma(F)$ for some $F$ in the closed unit ball of $H^\infty(E)$, so in particular $\phi$ is in $H^\infty(E,\sigma)$, which concludes the proof of the first statement.

For the final part of the corollary, let $f$ be an element of $H^\infty(E,\sigma)$, and let $g = \frac{1}{\|M_f\|} f$. Then $g$ is a multiplier of $E_K$ with $\|M_g\| = 1$, so by Theorem 48 $g$ is a Schur class operator function, and by Theorem 45 there is an element $F$ of $H^\infty(E)$ with $\|F\| \leq 1$ such that $g = \Gamma_\sigma(F)$. Suppose, to reach a contradiction, that

$\|F\| < 1$. Then $\frac{1}{\|F\|}g$ is a multiplier of norm strictly greater than one, so by Theorem 48, $\frac{1}{\|F\|}g$ is not a Schur class function. On the other hand, $\frac{1}{\|F\|}g = \Gamma_\sigma(\frac{1}{\|F\|}F)$, so by Theorem 45, $\frac{1}{\|F\|}g$ is a Schur class function. This contradiction shows that in fact $\|F\| = 1$. Thus, $\|M_f\|F$ has norm $\|M_f\|$, and $f = \Gamma_\sigma(\|M_f\|F)$. □

**Corollary 50.** *If $H^\infty(E,\sigma)$ is given the quotient norm induced by the algebraic isomorphism $H^\infty(E,\sigma) \cong H^\infty(E)/\ker(\Gamma_\sigma)$, then the identification of $H^\infty(E,\sigma)$ with $\mathrm{Mult}(E_{K_S})$ is isometric. In particular, the closed unit ball of $H^\infty(E,\sigma)$ is precisely the set $S_{E,\sigma}$ of Schur class operator functions. Furthermore, the sup norm that $H^\infty(E,\sigma)$ has as an algebra of $B(H)$-valued functions is dominated by the quotient (or multiplier) norm.*

*Proof.* By definition of the quotient norm on $H^\infty(E,\sigma)$, the norm of an element $f \in H^\infty(E,\sigma)$ is $\|f\| = \inf\{\|F\| : \Gamma_\sigma(F) = f\}$. By the last part of the preceding corollary and its proof, this is the same as $\|M_f\|$ (and in fact, the infimum is always attained). The second statement of the corollary follows by Theorems 45 and 48.

For the last statement, let $f$ be an element of $H^\infty(E,\sigma)$, let $z$ be an element of $\mathbb{D}(E^{\sigma*})$, and let $F_0$ be an element of $H^\infty(E)$ such that $f(z) = \hat{F}_0(z)$. Then the fact that $\psi_z$ is contractive yields the inequality $\|f(z)\| = \|\hat{F}_0(z)\| \le \|F_0\|$. Consequently,

$$\|f(z)\|_{B(H)} \le \inf_{\hat{F}(z)=f(z)} \|F\|_{H^\infty(E)} \le \inf_{\hat{F}=f} \|F\|_{H^\infty(E)} = \|f\|_{H^\infty(E,\sigma)},$$

and taking the sup over $z \in \mathbb{D}(E^{\sigma*})$ concludes the proof. □

*Remark* 51.

1. Tensoring $E_{K_S}$ with $H$ over the identity representation id of $B(H)$ yields an induced representation of $\mathcal{L}(E_{K_S})$ on $E_{K_S} \otimes_{\mathrm{id}} H$, which in particular restricts to a representation of $\mathrm{Mult}(E_{K_S})$. This is closer to the perspective studied by Ball et al. [5].

2. One area of future interest is to explore how $E_{K_S}$ may be described more concretely as a space of functions. This correspondence may be obtained by "extending the scalars" from the reproducing kernel correspondence $\tilde{E}_{K_S}$ over

$\sigma(M)'$ that is obtained when one restricts the codomain of the values of $K_S$ to $\sigma(M)'$. It follows from work of Muhly and Solel [33, page 229] that $\tilde{E}_{K_S}$ may be naturally realized as a sub-correspondence of the Fock space of the $\sigma$-dual of $E$, $\mathcal{F}(E^\sigma)$. However, the function theoretic aspects of $\mathcal{F}(E^\sigma)$ and this submodule remain largely unexplored.

# CHAPTER 4
# FUNCTION SPACE REPRESENTATIONS OF HARDY ALGEBRAS

Recall from Chapters 1 and 3 that if $E$ is a $W^*$-correspondence over $M$ and $\sigma : M \to B(H)$ is a normal representation, then $\Gamma_\sigma : H^\infty(E) \to H^\infty(E,\sigma)$ is the homomorphism defined by $\Gamma_\sigma(F)(z) = \hat{F}(z) = \psi_z(F)$, and $J_\sigma = \ker(\Gamma_\sigma)$. As already mentioned, $J_\sigma$ is ultraweakly closed because each $\psi_z$ is ultraweakly continuous, and

$$J_\sigma = \bigcap_{z\in\mathbb{D}(E^{\sigma *})} \ker(\psi_z).$$

In this chapter, further properties of $J_\sigma$ and the quotient $H^\infty(E,\sigma)$ are studied. In section 4.1, the observation that $J_\sigma$ is gauge invariant is used to deduce results on "polynomial approximation," and elements of $H^\infty(E,\sigma)$ are described from the perspective of holomorphic function theory on Banach spaces. The noncommutative analytic Toeplitz algebras provide a motivating class of examples in section 4.2, in which it is shown that the considerations of the previous section lead to connections with algebras of polynomial identities and generic matrices.

## 4.1 Approximation by polynomials

As seen in Chapter 1, the gauge automorphisms $\{\gamma_t\}_{t\in\mathbb{R}}$ are the ultraweakly continuous and completely isometric automorphisms of $H^\infty(E)$ satisfying $\gamma_t \circ \varphi_\infty = \varphi_\infty$ and $\gamma_t(T_\zeta) = e^{it}T_\zeta$ for all $t \in \mathbb{R}$ and $\zeta \in E$. The first observation of this section is that the ideal $J_\sigma$ is gauge invariant. This invariance will in turn imply that the partial sums of the Fourier series of elements of $J_\sigma$ also lie in $J_\sigma$, thus allowing ultraweak "polynomial approximation" in $J_\sigma$. The details are as follows.

**Proposition 52.** *The ideal $J_\sigma$ is gauge invariant.*

*Proof.* Recall from Chapter 1 that if $z$ is in $\mathbb{D}(E^{\sigma *})$, then $\psi_z : H^\infty(E) \to B(H)$ is the ultraweakly continuous completely contractive representation determined by the conditions that $\psi_z \circ \varphi_\infty = \sigma$ on $M$ and $\psi_z(T_\zeta)h = z(\zeta \otimes h)$ for all $\zeta \in E$ and

$h \in H$. For each $t \in \mathbb{R}$, $\psi_z \circ \gamma_t$ is an ultraweakly continuous completely contractive representation of $H^\infty(E)$ satisfying

$$(\psi_z \circ \gamma_t) \circ \varphi_\infty = \psi_z \circ (\gamma_t \circ \varphi_\infty) = \psi_z \circ \varphi_\infty = \sigma,$$

and

$$(\psi_z \circ \gamma_t)(T_\zeta)h = \psi_z(e^{it}T_\zeta)h = e^{it}\psi_z(T_\zeta)h = e^{it}z(\zeta \otimes h)$$

for all $\zeta \in E$ and $h \in H$. On the other hand, so is $\psi_{(e^{it}z)}$, and thus $\psi_{(e^{it}z)} = \psi_z \circ \gamma_t$. Therefore, since $e^{it}\mathbb{D}(E^{\sigma *}) = \mathbb{D}(E^{\sigma *})$,

$$J_\sigma = \bigcap_{z \in \mathbb{D}(E^{\sigma *})} \ker \psi_z = \bigcap_{z \in \mathbb{D}(E^{\sigma *})} \ker \psi_{(e^{it}z)} = \bigcap_{z \in \mathbb{D}(E^{\sigma *})} \ker(\psi_z \circ \gamma_t),$$

and it follows that $\gamma_t(J_\sigma) \subseteq J_\sigma$. $\square$

Proposition 52 is a simple result, but has useful consequences. It can be interpreted as saying that $J_\sigma$ is a "homogeneous" ideal, thus allowing the "grading" of $H^\infty(E)$ to be passed onto the quotient, as will be made precise below. As discussed in Chapter 1, each element $F$ of $H^\infty(E)$ has a Fourier expansion, $F \sim \sum_{k=0}^\infty F_k$, the series being ultraweakly Cesàro summable to $F$. If $Q_j \in \mathcal{F}(\mathcal{L}(E))$ denotes the orthogonal projection onto $E^{\otimes j}$, then

$$F_k = \Phi_k(F) = \frac{1}{2\pi}\int_0^{2\pi} \gamma_t(F)e^{-ikt}dt = \sum_{j=0}^\infty Q_{j+k}FQ_j.$$

Intuitively, this means that elements of $H^\infty(E)$ have well behaved "polynomial" approximations. It is convenient to give this intuition a formal status in the following definition.

**Definition 53.** Let $F$ be an element of $H^\infty(E)$.

1. $F$ is called a *polynomial* if there exists a nonnegative integer $k$ such that $F_n = 0$ for all $n > k$. If $F$ is a nonzero polynomial, then the smallest such $k$ is called the *degree* of $F$.

2. If $F = F_k$, then $F$ is called a *$k$-homogeneous* polynomial.

3. In general, $F_k$ is called the *$k$-homogeneous part* of $F$, whether or not $F$ is a polynomial.

4. The set of all polynomials in $H^\infty(E)$ will be denoted by $\mathcal{P}(E)$.

5. The set of all $k$-homogeneous polynomials, $\Phi_k(H^\infty(E)) = \Phi_k(\mathcal{P}(E))$, will be denoted by $\mathcal{P}_k(E)$.

The algebra $H^\infty(E)$ is in a sense a dual operator algebraic version of a graded algebra. The set of polynomials, $\mathcal{P}(E)$, is a graded complex algebra in the usual algebraic sense; it is the vector space internal direct sum of its homogeneous parts, $\mathcal{P}(E) = \sum_{k=0}^{\infty} \mathcal{P}_k(E)$, with $\mathcal{P}_j(E)\mathcal{P}_k(E) \subseteq \mathcal{P}_{j+k}(E)$ for all nonnegative integers $j$ and $k$. The existence of ultraweakly Cesàro summing Fourier expansions of elements of $H^\infty(E)$ implies in particular that $\mathcal{P}(E)$ is ultraweakly dense in $H^\infty(E)$, but that is also apparent from the definition of $H^\infty(E)$. More importantly, these polynomial approximations carry over to ultraweakly closed gauge invariant ideals.

**Proposition 54.** *Let $J$ be an ultraweakly closed gauge invariant ideal in $H^\infty(E)$. Then $J \cap \mathcal{P}(E)$ is a homogeneous ideal in $\mathcal{P}(E)$ that is ultraweakly dense in $J$. The graded algebra $\mathcal{P}(E)/(J \cap \mathcal{P}(E))$ imbeds canonically into $H^\infty(E)/J$ as an ultraweakly dense subalgebra.*

*Proof.* Let $F$ be an element of $J$, and let $k$ be a nonnegative integer. Then $F_k = \frac{1}{2\pi}\int_0^{2\pi} \gamma_t(F)e^{-ikt}dt$ is in $J$, because each value of the integrand is in $J$, and the integral is an ultraweak limit of finite linear combinations of values of the integrand. In particular, the partial Cesàro sums of the Fourier expansion of $F$ are in $J \cap \mathcal{P}(E)$ and converge ultraweakly to $F$, so $J \cap \mathcal{P}(E)$ is ultraweakly dense in $J$. Furthermore, the fact that $F_k$ is in $J \cap \mathcal{P}(E)$ whenever $F$ is in $J \cap \mathcal{P}(E)$ shows that $J \cap \mathcal{P}(E)$ is a homogeneous ideal in $\mathcal{P}(E)$, and thus the grading passes to the quotient algebra $\mathcal{P}(E)/(J \cap \mathcal{P}(E))$. The canonical isomorphism $\mathcal{P}(E)/(J \cap \mathcal{P}(E)) \to (\mathcal{P}(E) + J)/J$ maps onto an ultraweakly dense subalgebra of $H^\infty(E)/J$ because $\mathcal{P}(E)$ is ultraweakly dense in $H^\infty(E)$. $\square$

*Remark* 55. Various notions of Banach algebras graded by the nonnegative integers have appeared in the literature [8, 13]. The graded algebras in [8] are assumed to have $\ell^1$-summable homogeneous parts, which makes them far removed from the Hardy or tensor algebra setting. Dixon's definition [13] is closer to capturing our situation, but there it is assumed that the "polynomials" (in the terminology used here) are norm dense in the algebra. In fact, the tensor algebras $\mathcal{T}_+(E)$ provide examples of graded Banach algebras in Dixon's sense, but these will not be explored here.

The remainder of this section will be concerned with the holomorphic nature of the functions in $H^\infty(E,\sigma)$. For the general theory of analytic mappings between Banach spaces, a good reference is Upmeier's book [45, Section 1]. First, note that if $F = F_k$ is a homogeneous polynomial of degree $k$, then $\hat{F}_k$ extends to a continuous $k$-homogeneous polynomial from $E^{\sigma *}$ to $B(H)$. For the special case where $F_k = T_{\zeta_1} \cdots T_{\zeta_k}$, the function $\hat{F}_k : E^{\sigma *} \to B(H)$ is defined by

$$\begin{aligned}\hat{F}_k(z)h &= z(I_E \otimes z)\cdots(I_{E^{\otimes(k-1)}} \otimes z)(\zeta_1 \otimes \cdots \otimes \zeta_k \otimes h)\\ &= z(\zeta_1 \otimes z(\cdots z(\zeta_{k-1} \otimes z(\zeta_k \otimes h))\cdots))\\ &= z^{(k)}(\zeta_1 \otimes \cdots \otimes \zeta_k \otimes h),\end{aligned}$$

where $z^{(k)} : E^{\otimes k} \otimes H \to H$ is the $k^{\text{th}}$ generalized power of $z$ in the terminology of [31, page 363],[1] defined by $z^{(k)} = z(I_E \otimes z)\cdots(I_{E^{\otimes(k-1)}} \otimes z)$. For a general $F_k \in \mathcal{P}_k(E)$, the extension of $\hat{F}_k$ to $E^{\sigma *}$ is given by $\hat{F}_k(z)h = z^{(k)}(F_k(1) \otimes h)$, 1 denoting the identity of $M = E^{\otimes 0}$ as an element of $\mathcal{F}(E)$. Thus if $G : (E^{\sigma *})^k \to B(H)$ is the continuous multilinear map defined by

$$G(z_1, z_2, \ldots, z_k)h = z_1(I_E \otimes z_2)\cdots(I_{E^{\otimes(k-1)}} \otimes z_k)(F_k(1) \otimes h),$$

then $\hat{F}_k(z) = G(z, z, \ldots, z)$, and $\|\hat{F}_k(z)\| \le \|F_k\|\|z\|^k$, justifying the remark above that $\hat{F}_k$ is a continuous $k$-homogeneous polynomial from $E^{\sigma *}$ to $B(H)$ [45, Definition 1.1]. The norm of $\hat{F}_k$ as a homogeneous polynomial is defined by $\|\hat{F}_k\|_\infty =$

[1] Generalized powers of elements of $E^\sigma$ will also appear in Chapter 5. It was for elements of $E^\sigma$ that the generalized powers were originally defined in [31], but the same terminology is used here for the adjoints, and context will make it clear which is intended.

$\sup\{\|\hat{F}_k(z)\| : \|z\| \leq 1\}$, and therefore the estimate above shows that $\|\hat{F}_k\|_\infty \leq \|F_k\|$. In fact, by Corollary 50 and continuity of $\hat{F}_k$, the potentially sharper inequality $\|\hat{F}_k\|_\infty \leq \|\hat{F}_k\|_{H^\infty(E,\sigma)}$ holds.

Now let $F$ be an element of $H^\infty(E)$ with Fourier series $\sum_{k=0}^\infty F_k$. Then the series $\sum_{k=0}^\infty \hat{F}_k$ is a power series from $E^{\sigma*}$ to $B(H)$ [45, Definition 1.3]. Its radius of convergence $R$ satisfies

$$\frac{1}{R} = \limsup_{k\to\infty}(\|\hat{F}_k\|_\infty)^{\frac{1}{k}}.$$

For each $k$, $\|\hat{F}_k\|_\infty \leq \|F_k\| = \|\Phi_k(F)\| \leq \|F\|$, so that $R \geq 1$. Thus $z \mapsto \hat{F}(z) = \sum_{k=0}^\infty \hat{F}_k(z)$ is a holomorphic mapping from $\mathbb{D}(E^{\sigma*})$ to $B(H)$, with the series converging uniformly on each ball centered at $0 \in E^{\sigma*}$ of radius strictly less than 1 [45, Proposition 1.5, Definition 1.6, and Corollary 1.15].

*Remark* 56.

1. The algebras $H^\infty(E,\sigma)$ form a class of operator-valued holomorphic functions defined on the bounded symmetric homogeneous domains [18] $\mathbb{D}(E^{\sigma*})$ that deserve extensive further study, both for their own interest and for their connections to the noncommutative Hardy algebras. Such a study was begun by Muhly and Solel [32]. They gave transfer function realizations of elements of the unit ball of $H^\infty(E,\sigma)$, analogues of the notion from mathematical system theory, to which a gentle introduction can be found in [43]. In the case when $\sigma$ is faithful, they used these to describe holomorphic automorphisms of $\mathbb{D}(E^{\sigma*})$ that induce automorphisms of $H^\infty(E)$, and much more.

2. In another direction, Popescu has undertaken an extensive study of what he calls "free holomorphic functions" [38]. These are power series in noncommuting variables that define operator-valued holomorphic functions on spaces of row operators, generalizing the functions obtained from the function space representations of the noncommutative analytic Toeplitz algebras discussed in the next section. One area of future research interest is to extend his ideas

to the setting of holomorphic functions on domains in $E^{\sigma *}$ that come from power series of the form $\sum_{k=0}^{\infty} \hat{F}_k$, with each $\hat{F}_k \in \mathcal{P}_k(E)$ being a continuous $k$-homogeneous polynomial extended to all of $E^{\sigma *}$ as above, but not necessarily coming from an element $\hat{F}$ of $H^\infty(E, \sigma)$. This would involve extending $\mathcal{P}(E)$ to an algebra of formal power series, and looking at all of the possible representations of these power series as functions on domains in $E^{\sigma *}$ as $\sigma$ ranges over the normal representations of $M$.

## 4.2 Generic matrices and polynomial identities in noncommutative analytic Toeplitz algebras

In this section attention is focused on the noncommutative analytic Toeplitz algebras. By applying the observations of the previous section, it is seen that $J_\sigma$ in this case can be viewed as a completion of an algebra of polynomial identities for the $n$-by-$n$ matrices, and that the resulting algebra of functions can be viewed as a completion of an algebra of generic $n$-by-$n$ matrices. First, some preliminaries are given on the algebras and some of their finite dimensional representations.

Let $d$ and $n$ be positive integers. Set $M = \mathbb{C}$, $E = \mathbb{C}^d$ with the standard inner product and standard basis denoted $\{e_1, \ldots, e_d\}$, and set $\sigma : \mathbb{C} \to B(\mathbb{C}^n)$ to be the only nondegenerate representation of $\mathbb{C}$ on $\mathbb{C}^n$. Then $H^\infty(\mathbb{C}^d)$ is the ultraweakly closed unital subalgebra of $B(\mathcal{F}(\mathbb{C}^d)))$ generated by the creation operators $S_1, \ldots, S_d$ defined by $S_j(x) = e_j \otimes x$; i.e., $S_j = T_{e_j}$ in the general notation above.[2] The algebra $H^\infty(\mathbb{C}^d)$ is called the *noncommutative analytic Toeplitz algebra*, introduced by Popescu [37] and since studied by several authors, including Arias, Davidson, Pitts, and Popescu [2, 11, 12, 38].

Using the fact that a basis has been fixed for $\mathbb{C}^d$, the set $E^{\sigma *}$ of adjoints of

[2]This algebra appears with various names and notation in the literature. For example, it is sometimes called the noncommutative Hardy algebra or the noncommutative analytic Toeplitz algebra, and it has been denoted by $\mathcal{F}_d^\infty$ or $\mathcal{L}_d$. To avoid further overloading the term "noncommutative Hardy algebra," the latter terminology is used, and for consistency with the general notation for the noncommutative Hardy algebra of a correspondence, the algebra will be denoted by $H^\infty(\mathbb{C}^d)$.

elements of the $\sigma$ dual of $E = \mathbb{C}^d$ is identified as follows:

$$E^{\sigma*} = (\mathbb{C}^d)^{\sigma*} = B(\mathbb{C}^d \otimes \mathbb{C}^n, \mathbb{C}^n) \cong B((\mathbb{C}^n)^{(d)}, \mathbb{C}^n) \cong \mathrm{Row}_d(B(\mathbb{C}^n)).$$

Elements of $E^{\sigma*}$ will be treated as lying in the latter space, so if $T$ is in $E^{\sigma*}$, then $T = (T_1, T_2, \ldots, T_d)$ with each $T_j$ in $B(\mathbb{C}^n)$ and $\|T\| = \|\sum_{j=1}^{d} T_j T_j^*\|$. Note that $T$ is still thought of as an element of $B((\mathbb{C}^n)^{(d)}, \mathbb{C}^n)$, but with its "coordinates" chosen. Following Davidson and Pitts [11], the shorthand notation $\mathbb{B}_{d,n} = \mathbb{D}(\mathrm{Row}_d(B(\mathbb{C}^n)))$ will be used, the case $n = 1$ being the standard unit ball $\mathbb{B}_d = \mathbb{B}_{d,1}$ in $\mathbb{C}^d$, thought of as a row Hilbert space, or as the dual of $\mathbb{C}^d$.

The Fock space $\mathcal{F}(\mathbb{C}^d)$ and the series expansions of elements of $H^\infty(\mathbb{C}^d)$ can be described using the free monoid on $d$ generators. Let $\mathbb{F}_d^+$ denote the free monoid generated by the symbols $g_1, \ldots, g_d$ with identity $\emptyset$. Elements of $\mathbb{F}_d^+$ are "words" of the form $w = g_{i_1} g_{i_2} \ldots g_{i_k}$, and the length of $w$ is $|w| = k$. Thus $\emptyset$ is the empty word and has length zero. The space $\mathcal{F}(\mathbb{C}^d)$ has a canonical orthonormal basis indexed by $\mathbb{F}_d^+$. The "vacuum" in $\mathcal{F}(\mathbb{C}^d)$ will be denoted by $e_\emptyset = (1, 0, 0, \ldots)$, and if $w = g_{i_1} g_{i_2} \ldots g_{i_k}$, then $e_w$ denotes $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \in (\mathbb{C}^d)^{\otimes k} \subset \mathcal{F}(\mathbb{C}^d)$ (considering $(\mathbb{C}^d)^{\otimes k}$ as a subspace of $\mathcal{F}(\mathbb{C}^d)$ in the obvious way). If $(T_1, \ldots, T_d)$ is a $d$-tuple of bounded operators, then $T_w$ denotes $T_{i_1} T_{i_2} \ldots T_{i_k}$. If $F$ is in $H^\infty(\mathbb{C}^d)$ and $F e_\emptyset = \sum_{w \in \mathbb{F}_d^+} a_w e_w$, then the "Fourier series"[3] associated with $F$ is $\sum_{w \in \mathbb{F}_d^+} a_w S_w$. Thus, with $F_k = \Phi_k(F)$ denoting the $k$-homogeneous part of $F$ as in the previous section, $F_k = \sum_{|w|=k} a_w S_w$, and

$$F = uw\text{-}\lim_{k\to\infty} \sum_{j=0}^{k-1} \left(1 - \frac{j}{k}\right) F_j = uw\text{-}\lim_{k\to\infty} \sum_{|w|<k} \left(1 - \frac{|w|}{k}\right) a_w S_w.$$

It is now easy to describe the function $\hat{F} : \mathbb{B}_{d,n} \to B(\mathbb{C}^n)$. Given $z = (z_1, \ldots, z_d) \in \mathbb{B}_{d,n}$, for each $k \in \{1, \ldots, d\}$ and $h \in \mathbb{C}^d$, reflecting on how the coordinates for $\mathbb{B}_{d,n}$ were chosen leads to the evaluation $\psi_z(S_k)h = z(e_k \otimes h) = z_k(h)$. Thus,

[3] In case the reader is curious, the ordering on $\mathbb{F}_d^+$ for summation can be determined first by length and second by lexicographic ordering on words of fixed length. For example, $g_2 < g_1^2 < g_1 g_2 < g_2 g_1 < g_2^2 < g_1^3$. This is not particularly relevant, as there will be no considerations of conditional convergence.

$\psi_z(S_k) = z_k$, and more generally it follows that $\psi_z(S_w) = z_w$ for $w \in \mathbb{F}_d^+$. Therefore, the holomorphic function $\hat{F} : \mathbb{B}_{d,n} \to B(\mathbb{C}^n)$ is given by $\hat{F}(z) = \sum_{w \in \mathbb{F}_d^+} a_w z_w$, where the series converges uniformly on compact subsets of $\mathbb{B}_{d,n}$, as follows from the more general discussion in the previous section.

To describe how the material from the previous section relates to the present setting, some algebraic preliminaries are needed from the theory of polynomial identity rings [41, 17]. Let $\mathbb{C}\langle X_1, \ldots, X_d \rangle$ denote the free unital associative complex algebra in $d$ variables $X_1, \ldots, X_d$. Let $\mathrm{PI}_{d,n}$ denote the subset of $\mathbb{C}\langle X_1, \ldots, X_d \rangle$ consisting of all polynomial identities for $M_n(\mathbb{C})$ in $d$ variables. That is, $p(X_1, \ldots, X_d)$ is in $\mathrm{PI}_{d,n}$ if and only if specializing the variables to a $d$-tuple of $n$-by-$n$ matrices $A_1, \ldots, A_d \in M_n(\mathbb{C})$ always results in $p(A_1, \ldots, A_d) = 0$. Let $\mathbb{C}_n\langle Y_1, \ldots, Y_d \rangle$ denote the algebra of $d$ generic $n$-by-$n$ complex matrices. This means that $\mathbb{C}_n\langle Y_1, \ldots, Y_d \rangle$ is the subalgebra of the matrix algebra $M_n(\mathbb{C}[x_{ij}(k)])$ over the polynomial ring in $dn^2$ commuting variables $\{x_{ij}(k) : i, j \in \{1, \ldots, n\}, k \in \{1, \ldots, d\}\}$ generated by the *generic matrices* $Y_k = (x_{ij}(k))_{i,j}$. Then the homomorphism $\mathbb{C}\langle X_1, \ldots, X_d \rangle \to \mathbb{C}_n\langle Y_1, \ldots, Y_d \rangle$ sending $X_k$ to $Y_k$ is surjective, and its kernel is $\mathrm{PI}_{d,n}$. The algebra $\mathbb{C}_n\langle Y_1, \ldots, Y_d \rangle$ is free among algebras satisfying the polynomial identities of the $n$-by-$n$ matrices, and it has no zero divisors [17, pages 18-19].

For the present setting where $E = \mathbb{C}^d$ and $\sigma : \mathbb{C} \to B(\mathbb{C}^n)$, the notation $H^\infty_{d,n}$, $J_{d,n}$, and $\Gamma_{d,n}$ will be used in place of $H^\infty(E, \sigma)$, $J_\sigma$, and $\Gamma_\sigma$, respectively.

**Proposition 57.**

1. *The map $\alpha : p(X_1, \ldots, X_d) \mapsto p(S_1, \ldots, S_d)$ is an isomorphism from the free algebra $\mathbb{C}\langle X_1, \ldots, X_d \rangle$ onto the algebra of polynomials $\mathcal{P}(\mathbb{C}^d)$ in $H^\infty(\mathbb{C}^d)$. The ideal $\mathrm{PI}_{d,n}$ is mapped under $\alpha$ onto $J_{d,n} \cap \mathcal{P}(\mathbb{C}^d)$, which is ultraweakly dense in $J_{d,n}$.*

2. *The unique homomorphism $\beta : \mathbb{C}_n\langle Y_1, \ldots, Y_d \rangle \to H^\infty_{d,n}$ sending $Y_k$ to $\hat{S}_k$ maps $\mathbb{C}_n\langle Y_1, \ldots, Y_d \rangle$ isomorphically onto $(\mathcal{P}(\mathbb{C}^d) + J_{d,n})/J_{d,n}$, which is ultraweakly dense in $H^\infty_{d,n}$.*

*Proof.* It is clear that $\alpha$ is onto $\mathcal{P}(\mathbb{C}^d)$. Since elements of $H^\infty(\mathbb{C}^d)$ are uniquely determined by their Fourier series, $\alpha$ is also injective. Suppose that $p$ is in $\mathrm{PI}_{d,n}$. Then for all $z = (z_1, \ldots, z_d) \in \mathbb{B}_{d,n}$, $\Gamma_{d,n}(\alpha(p))(z) = p(z_1, \ldots, z_d) = 0$, by definition of $\mathrm{PI}_{d,n}$, and therefore $\alpha(p)$ is in $J_{d,n}$. Conversely, if $F$ is in $J_{d,n} \cap \mathcal{P}(\mathbb{C}^d)$, then $\alpha^{-1}(F)$ is a polynomial that vanishes when evaluated on all $d$-tuples of $n$-by-$n$ matrices in an open subset of $M_n(\mathbb{C})^d$. By the identity theorem for holomorphic functions, $\alpha^{-1}(F)$ induces the 0 map on $M_n(\mathbb{C})^d$, and hence $\alpha^{-1}(F)$ is in $\mathrm{PI}_{d,n}$. Thus $\alpha(\mathrm{PI}_{d,n}) = J_{d,n} \cap \mathcal{P}(\mathbb{C}^d)$, which is ultraweakly dense in $J_{d,n}$ by Propositions 52 and 54. This concludes the proof of part 1.

For part 2, the ultraweak density of $(\mathcal{P}(\mathbb{C}^d) + J_{d,n})/J_{d,n}$ in $H^\infty_{d,n}$ follows from the fact that $\mathcal{P}(\mathbb{C}^d)$ is dense in $H^\infty(\mathbb{C}^d)$, as in Proposition 54. The uniqueness and existence of $\beta$ follows from the fact that $H^\infty_{d,n}(E)$, being an algebra of $B(\mathbb{C}^n)$-valued functions, satisfies all of the polynomial identities of the $n$-by-$n$ matrices. However, it will be helpful to be more explicit. By part 1, $\alpha$ induces an isomorphism $\mathbb{C}\langle X_1, \ldots, X_d\rangle / \mathrm{PI}_{d,n} \to \mathcal{P}(\mathbb{C}^d)/(J_{d,n} \cap \mathcal{P}(\mathbb{C}^d)) \cong (\mathcal{P}(\mathbb{C}^d) + J_{d,n})/J_{d,n}$ that sends $[X_k]$ to $\hat{S}_k$, and precomposing with the isomorphism from $\mathbb{C}_n\langle Y_1, \ldots, Y_d\rangle$ to $\mathbb{C}\langle X_1, \ldots, X_d\rangle / \mathrm{PI}_{d,n}$ sending $Y_k$ to $[X_k]$ simultaneously yields $\beta$ and shows that it is an isomorphism. □

*Remark* 58.

1. The result of part 1 has some overlap with a result of Davidson and Pitts [11, Proposition 2.4] when $n = 1$, in which they proved that $J_{d,1}$ is the weak closure of the commutator ideal of $H^\infty(\mathbb{C}^d)$. However, they also allow $d$ to be infinite, and the above arguments would need some adjustment to accommodate this case. Namely, it would no longer be true that $\alpha$ is onto if $d$ were infinite.

2. While Popescu has studied the function theory of the algebras $H^\infty(\mathbb{C}^d)$, most of the attention so far has been spent on the case when $n$ is infinite. In this case, $J_{d,n} = (0)$, and it turns out that if $H^\infty_{d,n}$ is given the sup norm, then it is

completely isometrically isomorphic to $H^\infty(\mathbb{C}^d)$ [38, Theorem 3.1].

3. The algebras $H^\infty_{d,1} = H^\infty_d$ have been studied from several perspectives. Applying the results of Chapter 3 to this setting leads to a realization of $H^\infty_d$ as the multiplier algebra of the symmetric Fock space $\mathcal{F}^s(\mathbb{C}^d)$, viewed as a reproducing kernel Hilbert space of holomorphic functions on the unit ball. This is part of the original perspective of Arveson [3], who was not working with noncommutative analytic Toeplitz algebras. Davidson and Pitts observed the connection to representations of the noncommutative analytic Toeplitz algebras [12]. Arveson, among other things, identified the $C^*$-envelope of $H^\infty_d$ [3, Theorem 8.15].

4. The above results show that the algebra $H^\infty_{d,n}$ of holomorphic functions in $d$ matrix variables can be thought of as a completion of a generic matrix algebra, and by the results of Chapter 3 it may be studied as the multiplier algebra of a reproducing kernel correspondence over $B(\mathbb{C}^n)$ consisting of $B(\mathbb{C}^n)$-valued functions on $\mathbb{B}_{d,n}$. There remains much to be studied concerning the algebraic and function theoretic properties of $H^\infty_{d,n}$. In particular, the identification of its $C^*$-envelope is still an open problem.

# CHAPTER 5
# CURVATURE OF A VECTOR BUNDLE OVER THE UNIT BALL OF A DUAL CORRESPONDENCE

In this section, a Hermitian holomorphic vector bundle is associated with each dual correspondence in such a way that the metric is related to an operator-valued positive kernel function that generalizes the Szegő kernel on the open unit disk of $\mathbb{C}$. The canonical connection and curvature of these bundles are given. Specialized to the disk, the Cowen-Douglas bundle of the backward shift is recovered, also yielding the Poincaré metric.

Let $E$ be a $C^*$-correspondence over a $C^*$-algebra $A$, and let $\sigma : A \to B(H)$ be a nondegenerate representation. The Cauchy kernel [33] is defined by assigning to each $z$ in the unit disk $\mathbb{D}(E^\sigma)$ of the dual correspondence $E^\sigma$ over $\sigma(A)'$ an element $C(z)$ in the dual correspondence of the Fock module of $E$, $\mathcal{F}(E)^\sigma$, as follows. First, for each positive integer $k$, define the $k^{\text{th}}$ generalized power of $z$, $z^{(k)} : H \to E^{\otimes k} \otimes_\sigma H$, by $z^{(k)} = (I_{E^{\otimes(k-1)}} \otimes z) \cdots (I_E \otimes z)z$. The map $C(z) : H \to \mathcal{F}(E) \otimes_\sigma H$ is defined by its block column form $C(z) := [z^{(0)}, z^{(1)}, z^{(2)}, \ldots]^\top$. Each map $C(z)$ is injective and has closed range by [33, Proposition 10], so $C(z)H$ is an element of the Grassmannian space $\mathcal{G}(\dim H, \mathcal{F}(E) \otimes_\sigma H)$ of closed subspaces of $\mathcal{F}(E) \otimes_\sigma H$ whose dimensions are equal to that of $H$. While many of the computations below are valid in the general setting, for the remainder of this section it will be assumed that $E^\sigma$ and $H$ are finite-dimensional.[1] Finite-dimensionality of $E^\sigma$ allows the theory of finite dimensional complex geometry to be applied to vector bundles over $\mathbb{D}(E^\sigma)$, and finite-dimensionality of $H$ removes possible topological complications involving the meaning of "frame" for a vector bundle whose fibres are isomorphic to $H$.

[1] Thus, the remark that $C(z)H$ is closed becomes superfluous.

Let $n = \dim H < \infty$. The map $z \mapsto C(z)H$ determines a Hermitian holomorphic vector bundle $Y$ over $\mathbb{D}(E^\sigma)$ as a pullback of the universal bundle over the Grassmannian space $\mathcal{G}(n, \mathcal{F}(E) \otimes_\sigma H)$ [47, pages 17 and 24]. The bundle $Y$ is trivial as a holomorphic vector bundle, and a global frame can be given by choosing a basis for $H$ as follows. Let $\{e_j\}_{j=1}^n$ be an orthonormal basis for $H$, and for each $j$ let $\gamma_j : \mathbb{D}(E^\sigma) \to Y$ be defined[2] by $\gamma_j(z) = C(z)e_j$. Then $\{\gamma_j\}_{j=1}^n$ is a global frame of holomorphic sections for $Y$.

Once this is done it is possible to determine the coordinate form of the canonical connection and curvature of the bundle with respect to this frame and a choice of coordinates on $\mathbb{D}(E^\sigma)$, following Section III.2 of [47]. The matrix of the Hermitian metric at $z \in \mathbb{D}(E^\sigma)$ with respect to the frame (i.e., the Grammian of the frame at $z$) is the matrix of $h(z) = C(z)^*C(z) = (\mathrm{id} - \theta_{z,z})^{-1}(I_H)$, where id denotes the identity operator on $\sigma(A)'$ and $\theta_{z,z} \in B(\sigma(A)')$ is the operator defined by $\theta_{z,z}(a) = z^*(I_E \otimes a)z = \langle z, a \cdot z\rangle_{E^\sigma}$.

*Remark* 59. Although it will not play a direct role in what follows, it is worth noting that $h$ is obtained from $K_S$ of Chapter 3 by evaluating $K_S$ on the diagonal (on adjoints) and evaluating the resulting values at the identity.

The first step in computing the curvature and connection will be to compute the Fréchet derivative of the function $h : \mathbb{D}(E^\sigma) \to \sigma(A)'$, which as above is given by $h(z) = (\mathrm{id} - \theta_{z,z})^{-1}(I_H)$. This derivative, $Dh$, assigns to each point $z \in \mathbb{D}(E^\sigma)$ a bounded, real linear operator from $E^\sigma$ to $\sigma(A)'$, which will be denoted by $Dh(z, \cdot)$.[3] It will be useful to first extend the notation $\theta_{z,z}$ used above. For each $w_1$ and $w_2$ in $E^\sigma$ let $\theta_{w_1,w_2}$ denote the operator in $B(\sigma(A)')$ defined by $\theta_{w_1,w_2}(a) = {w_1}^*(I_E \otimes a)w_2 = \langle w_1, a \cdot w_2\rangle_{E^\sigma}$. Notice that $h$ can be written as a composition of elementary mappings.

[2]Technically $\gamma_j(z)$ should be the pair $(z, C(z)e_j)$, but this abuse of notation can be used where the context makes it clear.

[3]Under the standing assumption that $E^\sigma$ is finite-dimensional it is superfluous to mention boundedness of $Dh(z, \cdot)$, but everything in this paragraph works in the general case.

For example, evaluation at the identity is linear, so its derivative is the constant map with value equal to evaluation at the identity. The map $f : a \mapsto (1-a)^{-1}$ is differentiable on the unit ball of any unital Banach algebra, with $Df(a,b) = (1-a)^{-1}b(1-a)^{-1}$ (a straightforward computation). As for $g : z \mapsto \theta_{z,z}$, it is the specialization of a bilinear map to the diagonal, and $Dg(z,\eta) = \theta_{\eta,z} + \theta_{z,\eta}$. Combining these facts, the result of the computation is that $Dh(z,\eta) = (\mathrm{id} - \theta_{z,z})^{-1} \circ (\theta_{\eta,z} + \theta_{z,\eta})(h(z))$.

Fix an orthonormal basis $\{e_j\}_{j=1}^n$ for $H$ along with the associated frame $\{\gamma_j\}_{j=1}^n$ as above. Assume that $\mathbb{D}(E^\sigma)$ is given (complex) coordinates $z_1, z_2, \ldots, z_N$ corresponding to a (complex) basis $v_1, v_2, \ldots, v_N$ for $E^\sigma$. By [47, page 79], the formulas for the connection and curvature in terms of $h$ are $\Theta(z) = h(z)^{-1}\partial h(z)$ and $K(z) = \bar{\partial}\Theta(z)$, respectively, where $\partial$ and $\bar{\partial}$ act on differential forms of type $(p,q)$ by exterior differentiation followed by projection onto the forms of type $(p+1,q)$ and $(p,q+1)$, respectively. In coordinate form, $\partial = \sum_{j=1}^N \frac{\partial}{\partial z_j} dz_j$ and $\bar{\partial} = \sum_{j=1}^N \frac{\partial}{\partial \bar{z}_j} d\bar{z}_j$. So $\partial h(z) = \sum_{j=1}^N \frac{\partial h}{\partial z_j}(z) dz_j$, and each partial derivative $\frac{\partial h}{\partial z_j}$ is obtained from $Dh$ as $\frac{\partial h}{\partial z_j}(z) = \frac{1}{2}(\frac{\partial h}{\partial x_j}(z) - i\frac{\partial h}{\partial y_j}(z)) = \frac{1}{2}(Dh(z,v_j) - iDh(z,iv_j)) = (\mathrm{id} - \theta_{z,z})^{-1} \circ \theta_{z,v_j}(h(z))$. Thus the connection is

$$\Theta(z) = h(z)^{-1} \sum_{j=1}^N (\mathrm{id} - \theta_{z,z})^{-1} \circ \theta_{z,v_j}(h(z)) dz_j. \tag{5.1}$$

To compute the curvature $K(z) = \bar{\partial}\Theta$, the partial derivatives of the coefficients of $\Theta$ are needed, and again they can be obtained by first computing the Fréchet derivatives. Fix $j \in \{1, 2, \ldots, N\}$ and let $F_j : \mathbb{D}(E^\sigma) \to \sigma(A)'$ be the function defined by $F_j(z) = h(z)^{-1}(\mathrm{id} - \theta_{z,z})^{-1} \circ \theta_{z,v_j}(h(z))$. At first sight $F_j$ is significantly more complicated than $h$, but again the calculation of the derivative can be broken up into routine parts. For instance, the derivative of $h$ was already computed. The map $z \mapsto \theta_{z,v_j}$ is linear and thus presents no problem. The inverse map $\mathrm{Inv} : a \mapsto a^{-1}$ on the invertible elements of any Banach algebra has derivative $D\,\mathrm{Inv}(a,b) = -a^{-1}ba^{-1}$ (a straightforward computation). Finally, to put it all together, note that the component parts are related by multiplication or composition, and thus $F_j$ is a multilinear combination of functions whose derivatives are straightforward to compute. The result of the somewhat tedious computation shows that

$$\begin{aligned} DF_j(z,\eta) = h(z)^{-1}\big[&-(\mathrm{id}-\theta_{z,z})^{-1}\circ(\theta_{\eta,z}+\theta_{z,\eta})(h(z))h(z)^{-1}(\mathrm{id}-\theta_{z,z})^{-1}\circ\theta_{z,v_j}(h(z)) \\ &+(\mathrm{id}-\theta_{z,z})^{-1}\circ[(\theta_{\eta,z}+\theta_{z,\eta})\circ(\mathrm{id}-\theta_{z,z})^{-1}\circ\theta_{z,v_j} \\ &+\theta_{z,v_j}\circ(\mathrm{id}-\theta_{z,z})^{-1}\circ(\theta_{\eta,z}+\theta_{z,\eta})+\theta_{\eta,v_j}](h(z))\big]. \end{aligned}$$

For each $k\in\{1,2,\ldots,N\}$, $\frac{\partial F_j}{\partial \bar z_k}(z)=\frac{1}{2}(DF_j(z,v_k)+iDF_j(z,iv_k))$. Thus the formula for $DF_j$ yields

$$\begin{aligned} \frac{\partial F_j}{\partial \bar z_k}(z) = h(z)^{-1}\big[&-(\mathrm{id}-\theta_{z,z})^{-1}\circ\theta_{v_k,z}(h(z))h(z)^{-1}(\mathrm{id}-\theta_{z,z})^{-1}\circ\theta_{z,v_j}(h(z)) \\ &+(\mathrm{id}-\theta_{z,z})^{-1}\circ[\theta_{v_k,z}\circ(\mathrm{id}-\theta_{z,z})^{-1}\circ\theta_{z,v_j} \\ &+\theta_{z,v_j}\circ(\mathrm{id}-\theta_{z,z})^{-1}\circ\theta_{v_k,z}+\theta_{v_k,v_j}](h(z))\big]. \end{aligned}$$

Finally, using the fact that $K=\bar\partial\Theta=\sum_{k=1}^N\sum_{j=1}^N\frac{\partial F_j}{\partial\bar z_k}d\bar z_k\wedge dz_j$, the formula for the curvature at $z\in\mathbb{D}(E^\sigma)$ is

$$\begin{aligned} K(z) = h(z)^{-1}\sum_{1\le j,k\le N}\big[&-(\mathrm{id}-\theta_{z,z})^{-1}\circ\theta_{v_k,z}(h(z))h(z)^{-1}(\mathrm{id}-\theta_{z,z})^{-1}\circ\theta_{z,v_j}(h(z)) \\ &+(\mathrm{id}-\theta_{z,z})^{-1}\circ[\theta_{v_k,z}\circ(\mathrm{id}-\theta_{z,z})^{-1}\circ\theta_{z,v_j} \\ &+\theta_{z,v_j}\circ(\mathrm{id}-\theta_{z,z})^{-1}\circ\theta_{v_k,z}+\theta_{v_k,v_j}](h(z))\big]\,d\bar z_k\wedge dz_j. \end{aligned}$$

Here are a couple of examples illustrating special cases of the formulas above.

**Example 60.** Consider the case when $A=E=H=\mathbb{C}$. The map $\sigma$ is then just the identification of the algebra $\mathbb{C}$ with $B(\mathbb{C})$. Identifying $E\otimes_\sigma H=\mathbb{C}\otimes_{\mathbb{C}}\mathbb{C}$ with $\mathbb{C}$ in the usual way, $E^\sigma$ is identified with $B(\mathbb{C})$, which in turn is canonically identified with $\mathbb{C}$. Similarly, all of the tensor powers of $\mathbb{C}$ will be identified with $\mathbb{C}$ in the usual way; both $\mathcal{F}(E)$ and $\mathcal{F}(E)\otimes_\sigma H$ are therefore identified with $\ell^2_{\mathbb{Z}_+}$. For $z$ in the unit disk $\mathbb{D}$ of the complex plane, $C(z):\mathbb{C}\to\ell^2$ is given by $C(z)\lambda=\lambda(1,z,z^2,z^3,\ldots)$. So the fibre of the vector bundle over $z$ is the span of $(1,z,z^2,\ldots)$. This is precisely the vector bundle associated with the backward shift $U_+^*$ in [10] because the span of $(1,z,z^2,\ldots)$ is $\ker(U_+^*-z)$. In this case it is easier to compute the curvature directly than to specialize the formulas above. Either way, the result is

$$\begin{aligned} h(z) &= \frac{1}{1-|z|^2}, \\ \Theta(z) &= \frac{\bar{z}}{1-|z|^2}dz, \text{ and} \\ K(z) &= \frac{1}{(1-|z|^2)^2}d\bar{z}\wedge dz. \end{aligned}$$

Note that $h$ is the Poincaré metric on the disk [23, page 40].

**Example 61.** Let $A = H = \mathbb{C}$ again, but this time let $E = \mathbb{C}^d$ for some $d \in \mathbb{N}$. There are identifications of $E^\sigma$ with $\mathbb{C}^d$ and of $\mathcal{F}(E) \otimes_\sigma H$ with $\mathcal{F}(\mathbb{C}^d)$, and the standard basis of $\mathbb{C}^d$ will be used. Here the fibre over a point $z$ in the unit ball of $\mathbb{C}^d$ is the span of $(1, z, z^{\otimes 2}, z^{\otimes 3}, \ldots)$. Direct computation or specialization of the above formulas yield

$$\begin{aligned} h(z) &= \frac{1}{1-\langle z, z\rangle}, \\ \Theta(z) &= \sum_{j=1}^{d} \frac{\bar{z}_j}{1-\langle z, z\rangle}dz_j, \text{ and} \\ K(z) &= \sum_{1\le j,k\le d} \left(\frac{\bar{z}_j z_k}{(1-\langle z, z\rangle)^2} + \frac{\delta_{jk}}{1-\langle z, z\rangle}\right) d\bar{z}_k \wedge dz_j. \end{aligned}$$

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