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Bifurcation theory for a class of second order differential equations

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BIFURCATION THEORY FOR A CLASS OF SECOND ORDER DIFFERENTIAL EQUATIONS

by

Alvaro Correa

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

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Thesis Supervisor: Professor Yi Li

ABSTRACT

We consider the existence of positive solutions of the nonlinear two point boundary value problem $u'' + \lambda f(u) = 0$, $u(-1) = u(1) = 0$, where $f(u) = u(u-a)(u-b)(u-c)(1-u)$, $0 < a < b < c < 1$, as the parameter λ varies through positive values. Every solution $u(x)$ is an even function, and when it exists, it is uniquely identified by $\alpha = u(0)$. We study how the number of solutions changes when the parameter varies, i.e. we will be focusing on the locations of *bifurcation points.*

The authors P. Korman, Y. Li and T. Ouyang ("Computing the location and the direction of bifurcation", *Mathematical Research Letters, accepted*), prove that a necessary and sufficient condition for α to be a bifurcation point is

$$G(\alpha) \equiv F(\alpha)^{1/2} \int_0^\alpha \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\, d\tau - 2 = 0,$$

where $F(\alpha) = \int_0^\alpha f(u)\, du$. We will prove that $G(\alpha)$ has vertical asymptotes at $\alpha = b$, $\alpha = 1$ and at any point $\alpha \in (0, 1)$ for which $\int_0^\alpha f(u)\, du = 0$. We will use the asymptotic behavior of G to estimate intervals where $G(\alpha) \neq 0$, that is, intervals where there is no bifurcation point.

Abstract Approved: ______________________________
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Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the May 2011 graduation.

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where $F(\alpha) = \int_0^\alpha f(u)\, du$. We will prove that $G(\alpha)$ has vertical asymptotes at $\alpha = b$, $\alpha = 1$ and at any point $\alpha \in (0,1)$ for which $\int_0^\alpha f(u)\, du = 0$. We will use the asymptotic behavior of G to estimate intervals where $G(\alpha) \neq 0$, that is, intervals where there is no bifurcation point.

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CHAPTER 1
INTRODUCTION

In this thesis, we study exact bifurcation diagrams and the existence of multiple positive solutions to the Dirichlet problem

$$u''(x) + \lambda f(u(x)) = 0 \text{ on } (-1, 1),\ u(-1) = u(1) = 0, \tag{1.1}$$

depending on a positive parameter λ. We recall that solutions of (1.1) are even functions, with $u'(x) < 0$ for $x > 0$, and hence any solution is uniquely identified by $\alpha = u(0)$, see [12]. A bifurcation occurs where the number of solutions change as the parameter λ changes. Since the value of $u(0) = \alpha$ uniquely identifies $u(x)$, then $u(0) = \alpha$ stands for u. In order to draw solution curves, one puts the λ on the horizontal axis and the parameter α on the vertical axis. It is customary to refer to these curves as *bifurcation diagrams.*

In applications, information about the shape of solution branches is needed. Branches of (1.1) are smooth. We are then interested in whether or not these curves have turns with respect to the λ-direction, and if so, how many bifurcations there are and where they are located.

The problem (1.1) arises in many different physical situations, for example, in the theory of thermal ignition of gases, in quantum field theory and in population dynamics (see e.g. [18]). The nonlinearity of f in combustion problems, describes intermediate steady states of the temperature distribution u, λ then measures the amount of unburnt substance. Here the bifurcation points have significant physical implications, and that is important whether or not these exist (see e.g. [22]).

We study the open problem where $f(u) = u(u-a)(u-b)(u-c)(d-u)$, and f does not depend explicitly on x and whose roots satisfy $0 < a < b < c < d < +\infty$. A necessary condition for a positive solution of (1.1) to exist is that $\int_0^b f(t)\,dt > 0$, and $\int_b^c f(t)\,dt > 0$. Roughly speaking, these hypotheses imply that the graph of f has two positive humps and two negative bumps, where each positive bump has a bigger area than the previous negative hump. We want to show how bumps of the function f may affect the "bifurcation diagrams" or the multiplicity results. The convexity properties of $f(u)$ will be important to determine the location of the bifurcation points.

This problem, when $f(u) = (u-a)(u-b)(c-u)$, i.e. a cubic polynomial whose roots are three distinct constants $a \leq b < c$, was studied by J. Smoller and A. Wasserman [19], who attempted to solve the problem in general, and succeeded in solving it for $a = 0$. Later S.-H. Wang [20] solved the problem under some restrictions on a. Both papers used phase-plane analysis. Then P. Korman, Y. Li and T. Ouyang [12], [13] used bifurcation theory to attack the problem. Their technique was a careful use of the Crandall and Rabinowitz theorem but again some restrictions were necessary (all of the above mentioned papers covered more general types of $f(u)$, behaving like cubic functions). Finally in [14] the problem is completely understood for all such cubic polynomials.

In order to study the number of solutions of (1.1) and bifurcations, we must study the qualitative shape of the bifurcation diagram. The shape of such bifurcation diagram is determined by its turning points.

Our main tool comes from [14] where P. Korman, Y. Li and T. Ouyang present a formula, where the relevant point is that it allows us to compute all $\alpha's$ where a turn may occur, and another formula, which allows to compute the direction of the turn.

1.1 Overview

In Chapter 2, we provide definitions and background information. Necessary lemmas will be announced and some will be proved. All of the results cited in this chapter are based on the papers [7], [8], [9], [12], [13], [14], [15].

In Chapter 3, we derive a region that satisfies necessary and sufficient conditions for the existence of positive solution. We end the chapter with an application of Contraction Mapping Theorem.

Chapter 4 is the heart of this thesis. Using delicate integral estimates we will study the asymptotic behavior of G, where $G(\alpha) \equiv F(\alpha)^{1/2} \int_0^\alpha \frac{f(\alpha)-f(\tau)}{[F(\alpha)-F(\tau)]^{3/2}}\, d\tau - 2$ and $F(\alpha) = \int_0^\alpha f(u)\, du$. Such asymptotic analysis enable us to narrow the range of possible bifurcation points.

In Chapter 5, we explore and prove a series of properties which restrict the location of a bifurcation point by studying the change of concavity of the graph of f and the points where the rays from 0 and b touche the graph of f.

CHAPTER 2
DEFINITIONS AND BACKGROUND

We present exact multiplicity results for the boundary value problem of the type

$$u''(x) + \lambda f(u) = 0 \text{ on } (-1,1), \quad u(-1) = u(1) = 0, \tag{2.1}$$

with the nonlinearity $f(u) = -u(u-a)(u-b)(u-c)(u-d)$, $0 < a < b < c < d < +\infty$ and λ a positive parameter. It is convenient to consider the problem on the interval $(-1, 1)$, because positive solutions are even in this interval. By shifting and scaling, we can replace the interval $(-1, 1)$ for any interval (a, b). Most results in the literature are obtained when $f = f(u)$, i.e. when the problem is autonomous. Changes in the sign of $f(u)$ lead to multiple positive solutions of (2.1). All of the results cited in this chapter are based on the papers [7], [8], [9], [12], [13], [14], [15].

For the quintic nonlinearity in problem (2.1), by letting $u = d\nu$, we may assume that $d = 1$, so that our nonlinearity is $f(u) = u(u-a)(u-b)(u-c)(1-u)$, with new a, b, c, *i.e.* we consider

$$u''(x) + \lambda u(u-a)(u-b)(u-c)(1-u) = 0 \text{ on } (-1,1), \; u(-1) = u(1) = 0. \tag{2.2}$$

This substitution allows us to "compactify" the parameter set, since now $0 < a < b < c < 1$. Note that $u \equiv 0$ is a trivial solution of (2.2).

2.1 Bifurcation approach

The implicit function theorem for Banach spaces is the basic tool for continuation of solutions.The following version for Banach spaces is given by M.G. Crandall and P.H. Rabinowitz [2].

Theorem 2.1.1. *Let* X*,* Λ *and* Z *be Banach spaces,and* $f(x,\lambda)$ *a continuous mapping of an open set* $\mathrm{U} \subset \mathrm{X} \times \Lambda \to \mathrm{Z}$*. Assume that* f *has a Frechet derivative with respect to* x*,* $f_x(x,\lambda)$ *which is continuous on* U*. Assume that* $f(x_0,\lambda_0) = 0$ *for some* $(x_0,\lambda_0) \in \mathrm{U}$*. If* $f_x(x_0,\lambda_0)$ *is an isomorphism (i.e. 1:1 and onto) of* X *onto* Z*, then there is a ball* $B_r = \{\lambda : \| \lambda - \lambda_0 \|\} < r$ *and unique continuous map* $x(\lambda) : B_r(\lambda_0) \to \mathrm{X}$*, such that* $f(x(\lambda),\lambda) \equiv 0$*,* $x(\lambda_0) = x_0$*. If* f *is the class* C^p*, so is* $x(\lambda)$*,* $p \geq 1$*.*

We call (x_0,λ_0) a regular solution if it satisfies the conditions of Theorem 2.1.1, otherwise we call it a singular solution. The main tool for Philip Korman, Yi Li, Tiancheng Ouyang's papers has been the M.G. Crandall and P.H. Rabinowitz Theorem. It gives us conditions to have continuation of solutions through a critical point.

Next we state a bifurcation theorem of Crandall-Rabinowitz [3].

Theorem 2.1.2. *Let* X *and* Y *be Banach spaces. Let* $(\overline{\lambda},\overline{x}) \in \mathbf{R} \times X$ *and let* F *be a continuously differentiable mapping of an open neighborhood of* $(\overline{\lambda},\overline{x})$ *into* Y*. Let the null-space* $N(F_x(\overline{\lambda},\overline{x})) =$ *span* x_0 *be one-dimensional and codim* $R(F_x(\overline{\lambda},\overline{x})) = 1$*. Let* $F_\lambda(\overline{\lambda},\overline{x}) \notin R(F_x(\overline{\lambda},\overline{x}))$*. If* Z *is a complement of span* x_0 *in* X*, then the solutions of* $F(\lambda,x) = F(\overline{\lambda},\overline{x})$ *near* $(\overline{\lambda},\overline{x})$ *form a curve* $(\lambda(s),x(s)) = (\overline{\lambda}+\tau(s),\overline{x}+sx_0+z(s))$*,* *where* $s \to (\tau(s), z(s)) \in \mathbf{R} \times Z$ *is a continuously differentiable function near* $s = 0$ *and* $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.

I will denote derivatives of $u(x)$ by either $u'(x)$ or u_x and mix both notations to make our proofs more transparent (u'_x will denote the second derivative of $u(x)$, when convenient.)

2.2 General properties

It is well known that the method of supersolution and subsolution yields not only existence of a solution but it also locates the solution between given bounds. To use this method, one must be able to construct a supersolution γ and an sub-solution ψ of (2.1) (with the appropriate boundary conditions) so that $0 \leq \psi \leq \gamma$. Remember that a function $\gamma(x) \in C^2(-1,1) \cap C^0[-1,1]$ is called a supersolution of (2.1) if

$$\gamma'' + \lambda f(\gamma) \leq 0 \quad on\ (-1,1), \quad \gamma(-1) \geq 0, \quad \gamma(1) \geq 0. \tag{2.3}$$

A subsolution $\psi(x)$ is defined by reversing the inequalities in (2.3). The following result is standard.

Lemma 2.2.1. *Let* $\gamma(x)$ *and* $\psi(x)$ *be respectively super- and subsolutions of (2.1), and* $\gamma(x) \geq \psi(x)$ *on* $(-1,1)$ *with* $\gamma(x) \not\equiv \psi(x)$, *then* $\gamma(x) > \psi(x)$ *on* $(-1,1)$.

We shall often use this lemma with either $\gamma(x)$ or $\psi(x)$ or both being solution of (2.1). The following lemma is a consequence of the first.

Lemma 2.2.2. *Let* $u(x)$ *be a nontrivial solution of (2.1) with* $f(u) \equiv 0$. *If* $u(x) \geq 0$ *on* $(-1,1)$ *then* $u > 0$ *on* $(-1,1)$.

Lemma 2.2.3. *Let $\xi \in (-1,1)$ be any critical point of $u(x)$,* i.e. $u'(\xi) \equiv 0$. *Then $u(x)$ is symmetric with respect to ξ.*

Proof. Let $\nu(x) = u(2\xi - x)$. Then $\nu''(x) + \lambda f(\nu) \equiv 0$ on the interval $(-1,1) \cap (2\xi - 1, 2\xi + 1)$. Then both $u(x)$ and $\nu(x)$ satisfy the equation (2.1) and $\nu(\xi) = u(\xi)$ and $\nu'(\xi) = u'(\xi) = 0$. By uniqueness of initial value problems, $u(x) \equiv \nu(x)$, and the proof follows. □

From the lemmas above, it follows that any positive solution of (2.1) is an even function, moreover $u'(x) > 0$ on $(-1,0)$, and $u'(x) < 0$ on $(0,1)$. Thus, $\alpha \equiv u(0)$ is the maximal value of solution.

The next lemma shows that it is impossible that two solutions of (2.1) share the same α.

Lemma 2.2.4. *The value of $u(0) = \alpha$ uniquely identifies the solutions pair $(\lambda, u(x))$ (i.e. there is at most one λ, with at most one solution $u(x)$, so that $u(0) = \alpha$).*

Proof. Assume on the contrary that we have two solution pairs $(\lambda, u(x))$ and $(\mu, \nu(x))$, with $u(0) = \nu(0) = \alpha$. Clearly, $\lambda \neq \mu$, since otherwise we have a contradiction with uniqueness of initial value problems. (Recall that $u'(0) = \nu'(0) = 0$). Then $u(\frac{1}{\sqrt{\lambda}}x)$ and $\nu(\frac{1}{\sqrt{\lambda}}x)$ are both solutions of the same initial value problem

$$u'' + f(u) = 0, \qquad u(0) = \alpha, \quad u'(0) = 0,$$

and hence $u(\frac{1}{\sqrt{\lambda}}x) \equiv \nu(\frac{1}{\sqrt{\lambda}}x)$, but this is impossible, since the first function vanishes at $x = \sqrt{\lambda}$, while the second at $x = \sqrt{\mu}$. □

To use the bifurcation theory approach, we need to linearize (2.1)

$$\omega''(x) + \lambda f'(u(x))\omega = 0 \ \text{ on } (-1,1), \quad \omega(-1) = \omega(1) = 0, \tag{2.4}$$

where $u(x)$ is a solution of (2.1). If (2.4) has a nontrivial solution, then we call $u(x)$ a singular solution of (2.1). If $\omega(x) \equiv 0$ is the only solution of (2.4), we say that the solution $u(x)$ is nonsingular.

Lemma 2.2.5. *Let $u(x)$ be a positive solution of (2.1), with*

$$u'(1) < 0. \tag{2.5}$$

If the problem (2.4) admits a nontrivial solution, then it does not change sign, i.e. we may assume that $\omega(x) > 0$ on $(-1, 1)$.

Proof. The function $u'(x)$ also satisfies the linear equation (2.4). By the condition (2.5), $u'(x)$ is not a multiple of $\omega(x)$. Hence its roots are interlaced with those of $\omega(x)$. If $\omega(x)$ had a root ξ inside say $(-1, 0)$, then $u'(x)$ would have to vanish on $(-1, \xi)$, which is impossible. □

Condition (2.5) will hold for any positive solution, provided that

$$f(0) \geq 0, \tag{2.6}$$

see, e.g., p.107 in M. Renardi and R.C. Rogers [17]. If $f(u) < 0$, then we can have $u'(1) = 0$.

Lemma 2.2.6. *If the problem (2.4) admits nontrivial solutions, then the solution set is one dimensional. If moreover $u(x)$ is a positive solution, satisfying (2.5), then ω is an even function.*

Proof. By uniqueness of the initial value problem the value of $w'(1)$ uniquely determines $\omega(x)$, and hence the null space is one dimensional. Turning to the second claim, $u(x)$ is positive, then it is even. Hence $\omega(-x)$ also solve (2.4). Since the null space is one dimensional, $\omega(-x) = c\omega(x)$ for some constant c. Evaluating this relation at $x = 0$, we conclude that $c = 1$ (since $\omega(0) > 0$ by the previous lemma), which is the desired symmetry. □

Lemma 2.2.7. *Any two positive solutions of (2.1) do not intersect inside* $(-1, 1)$ (i.e., *they are strictly ordered on* $(-1, 1)$).

Proof. Let $u(x)$ and $\nu(x)$ be two intersecting solutions. Since both of them are even functions, they intersect on the half-interval $(0, 1)$ as well. Let $0 < \xi < \eta < 1$ be two consecutive intersection points. If $\nu(x) > u(x)$ on (ξ, η), then $\mid u'(\xi) \mid > \mid \nu'(\xi) \mid$, while $\mid u'(\eta) \mid < \mid \nu'(\eta) \mid$. The energy $E(x) + \frac{1}{2}u'(x)^2 + \lambda F(u(x))$ is constant for any solution $u(x)$. But at ξ, $u(x)$ has higher energy than $\nu(x)$, and at η the order is reversed, a contradiction. □

The next theorem was proved in [12] where f satisfies

$$f(u) = u(u - b)(c - u) \quad \text{for} \quad -1 < x < 1, \quad u(-1) = u(1) = 0, \tag{2.7}$$

with $0 < b < c < +\infty$ and

$$\int_0^c f(u)du > 0. \tag{2.8}$$

Theorem 2.2.8. *Under the conditions (2.7) , (2.8) there is a critical* $\lambda_0 > 0$ *such that for* $\lambda < \lambda_0$ *the problem (2.1) has no nontrivial solutions, it has exactly one nontrivial* *solution for* $\lambda = \lambda_0$*, and exactly two nontrivial solutions for* $\lambda > \lambda_0$*. Moreover, all solutions lie on a single solution curve, which for* $\lambda > \lambda_0$ *has two branches denoted by* $u^-(x, \lambda) < u^+(x, \lambda)$*, with* $u^+(x, \lambda)$ *strictly monotone increasing in* λ*,* $u^-(0, \lambda)$ *strictly monotone decreasing in* λ*, and* $\lim_{\lambda \to \infty} u^+(x, \lambda) = c$*,* $\lim_{\lambda \to \infty} u^-(x, \lambda) = 0$ *for* $x \in (-1, 1) \backslash \{0\}$*, while* $u^-(0, \lambda) > b$ *for all* $\lambda > \lambda_0$*.*

CHAPTER 3
GENERAL STUDY OF THE PARAMETRIC SPACE

This chapter studies necessary and sufficient conditions for the existence of positive solutions of the quintic nonlinearity problem

$$u''(x) + \lambda u(u-a)(u-b)(u-c)(1-u) = 0 \text{ on } (-1,1),\ u(-1) = u(1) = 0, \quad (3.1)$$

as well as some basic behaviours of positive solutions.

P. Hess showed that $\int_0^b f(s)\,ds > 0$ and $\int_b^1 f(s)\,ds > 0$ was a sufficient condition for the existence of positive solution of the problem (3.1), i.e. he showed how bumps of the function f affect the existence of positive solutions. We are able to prove this using the argument due by A. Ambrosetti and P.H.Rabiniwitz [8, p.12] that was outlined in the Theorem 2.2.8.

3.1 Region that satisfies the sufficient condition

In the next theorem, we will derive a region that satisfies sufficient condition for the existence of positive solution. This region will be very important for numerical computation.

Theorem 3.1.1. *Let* $f(s) = s(s-a)(s-b)(s-c)(1-s)$ *with* $0 < a < b < c < 1$, *then* $\int_0^b f(s)\,ds > 0$ *and* $\int_b^1 f(s)\,ds > 0$ *is equivalent to*

$$\Big(0 < a < \frac{1}{3}, \quad \frac{-2a+3a^2}{-1+2a} < b \leq 1 + a\sqrt{1-3a+a^2}\Big)$$

and

$$\Big(\frac{-5ab+3b^2+3ab^2-2b^3}{-10a+5b+5ab-3b^2} < c < \frac{-2+3a-3b+4ab-3b^2+ab^2-2b^3}{-3+5a-4b+5ab-3b^2}\Big),$$

or

$$\Big(0 < a < \frac{1}{3} \quad , \quad (1 + a - \sqrt{1 - 3a + a^2} < b < 1\Big)$$

and

$$\Big(b < c < \frac{-2 + 3a - 3b + 4ab - 3b^2 + 3ab^2 - 2b^3}{-3 + 5a - 4b + 5ab - 3b^2}\Big).$$

Proof. Since $\int_0^b f(s)\,ds = \frac{1}{60}b^3(b(2b^2+5c-3b(1+c))+a(-3b^2-10c+5b(1+c))$. Then $\int_0^b f(s)\,ds > 0$ is equivalent to $\frac{1}{60}b^3(b(2b^2+5c-3b(1+c))+a(-3b^2-10c+5b(1+c) > 0$. Multiplying the last inequality by $\frac{60}{b^3}$, and grouping in a and b we obtain

$$b(2b^2 + 5c - 3b(1 + c)) + a(-3b^2 - 10c + 5b(1 + c)) > 0,$$

so $a(-3b^2 - 10c + 5b(1 + c)) > -b(2b^2 + 5c - 3b(1 + c))$, and also notice that $(-3b^2 - 10c + 5b(1 + c)) < 0$, hence

$$a < \frac{-b(2b^2 + 5c - 3b(1 + c)}{-3b^2 - 10c + 5b(1 + c)}. \tag{3.2}$$

Denoting $Ł(b, c) = \frac{-3b^2-3b^2c+2b^3+5bc}{3b^2+10c-5b(1+c)}$, we have

$$a < Ł(b, c). \tag{3.3}$$

We will prove that Ł is increasing in (b, c) and $Ł(c, c) < \frac{1}{3}$. Taking derivative, we obtain

$$\frac{\partial Ł}{\partial b} = \frac{6b^4 + 50c^2 - 20b^3(1 + c) - 60bc(1 + c) + 15b^2(1 + 5c + c^2)}{(3b^2 + 10c - 5b - 5bc)^2}.$$

Define $N(b) = 6b^4 + 50c^2 - 20b^3(1 + c) - 60bc(1 + c) + 15b^2(1 + 5c + c^2)$, and $b = \epsilon c$,

$0 < \epsilon < 1$, then

$$\begin{aligned} N(\epsilon c) &= 50c^2 - 60c^2(1+c)\epsilon + 15c^2(1+5c+c^2)\epsilon^2 - 20c^3(1+c)\epsilon^3 + 60c^4\epsilon^4 \\ &= 50c^2 - 60c^2\epsilon - 60c^3\epsilon + 15c^2\epsilon^2 + 75c^3\epsilon^2 + 15c^4\epsilon^2 - 20c^3\epsilon^3 - 20c^4\epsilon^3 + 6c^4\epsilon^4 \\ &= (15\epsilon^2 - 20\epsilon^3 + 6\epsilon^4)c^4 + (-60\epsilon + 75\epsilon^2 - 20\epsilon^3)c^3 + (50 - 60\epsilon + 15\epsilon^2)c^2 \\ &= c^2[(15\epsilon^2 - 20\epsilon^3 + 6\epsilon^4)c^2 + (-60\epsilon + 75\epsilon^2 - 20\epsilon^3)c + (50 - 60\epsilon + 15\epsilon^2)]. \end{aligned}$$

Consider

$$M(c) = (15\epsilon^2 - 20\epsilon^3 + 6\epsilon^4)c^2 + (-60\epsilon + 75\epsilon^2 - 20\epsilon^3)c + (50 - 60\epsilon + 15\epsilon^2). \quad (3.4)$$

Let $h(\epsilon) = 15 - 20\epsilon + 6\epsilon^2$, (3.4)then $h'(\epsilon_0) = 0 \Leftrightarrow \epsilon_0 = \frac{5}{3} > 1$.

Since $\epsilon_0 \notin (0,1)$ and h is a quadratic function in ϵ, to know the sign of h in $(0,1)$ is necessary to check the values of h in the extreme of the interval $(0,1)$. But $h(0) = 15$ and $h(1) = 1$, then

$$h(\epsilon) = 15 - 20\epsilon + 6\epsilon^2 > 0 \ \ in \ \ (0,1). \quad (3.5)$$

From (3.4), $M'(c) = 2(15\epsilon^2 - 20\epsilon^3 + 6\epsilon^4)c + (-60\epsilon + 75\epsilon^2 - 20\epsilon^3)$, so $M'(c_0) = 0$ if and only if $c_0 = \frac{60-75\epsilon+20\epsilon^2}{2\epsilon h(\epsilon)}$.

Claim 1. $60 - 75\epsilon + 20\epsilon^2 > 2\epsilon(15 - 20\epsilon + 6\epsilon^2)$.

Proof. Define $k(\epsilon) = -12\epsilon^3 + 60\epsilon^2 - 105\epsilon + 60$, $\epsilon \in (0,1)$, then $k'(\epsilon) = -36\epsilon^2 + 120\epsilon - 105 < 0$, so $k(\epsilon)$ is decreasing and $3 = k(1) \le k(\epsilon)$ for all $\epsilon \in (0,1)$. Hence $k(\epsilon) > 0$. Then we may conclude the *claim*.

Using the claim and (3.5), we obtain that $c_0 > 1$.

Claim 2. $t(\epsilon) = 15\epsilon^2 - 60\epsilon + 50 > 0$.

Proof. $t'(\epsilon_0) = 0 \Leftrightarrow \epsilon_0 = 2 \notin (0,1)$. But $t(0) = 50$ and $t(1) = 5$, so $t(\epsilon) > 0$ in $(0,1)$. Using (3.4) and (3.5), M is a quadratic function in c, where the principal coefficient $15\epsilon^2 - 20\epsilon^3 + 6\epsilon^4 > 0$, and its critical point $c_0 \notin (0,1)$. To determine the sign of M in $(0,1)$ is enough to evaluate M at $c = 0$ and $c = 1$. Note that $M(0) = 15\epsilon^2 - 60\epsilon + 50 = l(\epsilon)$, so $l'(\epsilon_0) = 0$ if and only if $\epsilon_0 = 2$, but $l(0) = 50$ and $l(1) = 5$, therefore $M(0) > 0$. Also, it is easy to verify that $M(1) = 6\epsilon^4 - 40\epsilon^3 + 105\epsilon^2 - 120\epsilon + 50 = H(\epsilon) > 0$. This follows because $H'(\epsilon) = 24\epsilon^3 - 120\epsilon^2 + 210\epsilon - 120$, $H''(\epsilon) = 72\epsilon^2 - 240\epsilon + 210$ and $H'''(\epsilon_0) = 0$ if and only if $\epsilon_0 = \frac{240}{144} > 1$, so $H''(0) = 210 > 0$ and $H''(1) = 42 > 0$ therefore $H''(\epsilon)$ is increasing, then $H'(0) < H'(\epsilon) < H'(1)$ for all $\epsilon \in (0,1)$, but $H'(1) = -6$, consequently $H'(\epsilon) < 0$, so H is decreasing, then $H(1) < H(\epsilon) < H(0)$. Hence $M(1) = H(\epsilon) > 0$. Consequently $M(c) > 0$.

But $N(\epsilon c) = c^2 M(c) > 0$, so $N(b) > 0$. Then

$$\frac{\partial Ł}{\partial b} = \frac{N(b)}{(3b^2 + 10c - 5b - 5bc)^2} > 0. \tag{3.6}$$

We conclude that Ł is increasing in b, so

$$Ł(b,c) < Ł(c,c) = \frac{2c^2 - c^3}{5c - 2c^2}. \tag{3.7}$$

Claim 3. $Ł(c,c) \leq \frac{1}{3}$

Proof. Define $w(c) = 3c^2 - 8c + 5$ with $c \in (0,1)$. Then $w'(c_0) = 0 \Leftrightarrow c_0 = \frac{8}{6} > 1$. Since $w(1) = 0$, we conclude that $w(c) \geq 0$. Therefore $6c^2 - 3c^3 \leq 5c - 2c^2$. Consequently $3(2c^2 - c^3) \leq (5c - 2c^2)$. So $Ł(c,c) < \frac{1}{3}$.

From (3.3), (3.7) and *Claim* 3, we have

$$0 < a < \frac{1}{3}.$$

The rest of the proof of this theorem is similar. □

3.2 Basic Behaviour of Solutions

The structure of the set of solution of the problem (3.1) is richer in properties. For a first simple approach, we notice that solutions of (3.1) are fixed points of the operator $T : L_2[-1,1] \to L_2[-1,1]$ by $Tu = h$, where $h(x) = \int_{-1}^{1} Z(x,y) f(u(y))\, dy$ and Z is the Green's function corresponding to the problem (3.1).

We want to find out properties of the Dirichlet-branches. The next properties are very useful to understand the behavior of these branches.

Lemma 3.2.1. *Any nontrivial positive solution of (3.1) satisfies* $u'(1) \neq 0$.

Proof. It follows from the fact that $f(0) = 0$. □

Lemma 3.2.2. *If u is a nontrivial positive solution of (3.1) with maximum value $\alpha = u(0)$, then $\alpha \in (0,1)$.*

Proof. Suppose that $\alpha = 0$, then $u(0) = 0$ and $u'(0) = 0$. By uniqueness of initial value problems, $u \equiv 0$, a contradiction.

Suppose that $\alpha = 1$, then $u(0) = 1$ and $u'(0) = 0$. By uniqueness of initial value problems, $u \equiv 1$, a contradiction.

Consider $\alpha > 1$, therefore $u''(0) + \lambda f(u(0)) = 0$, so $u''(0) = -\lambda f(u(0)) = -\lambda f(\alpha) > 0$, a contradiction. □

Lemma 3.2.3. *Let u a nontrivial positive solution of (3.1), then* $\int_0^\alpha f(s)\, ds > 0$.

Proof. Suppose that u is a nontrivial positive solution of (3.1), then

$$\frac{1}{2}(u'(x))^2 = \lambda[F(\alpha) - F(u(x))]. \tag{3.8}$$

Let $x = 1$, substituting this in(3.8), we obtain

$$\begin{aligned} \frac{1}{2}(u'(1))^2 &= \lambda[F(\alpha) - F(u(1))] \\ &= \lambda F(\alpha). \end{aligned} \tag{3.9}$$

From Lemma (3.2.1) $F(\alpha) > 0$. □

Lemma 3.2.4. *Let u a nontrivial positive solution of (3.1) then $\alpha \in (a, b) \cup (c, 1)$.*

Proof. From Lemma (3.2.2) $\alpha \in (0, a] \cup (a, b) \cup [b, c] \cup (c, 1)$. If $\alpha = a$ then $u \equiv a$, a contradiction.

If $\alpha \in (0, a)$ then $u''(0) > 0$, a contradiction.

The same proof applies if $\alpha \in [b, c]$. □

Lemma 3.2.5. *Let $\alpha = u(0)$, then $f(\alpha) > 0$.*

Proof. It is clear that the proof of the lemma follows from the Lemma (3.2.4). □

Lemma 3.2.6. *If $\alpha \in (a, b)$ then $\int_0^b f(s)\,ds > 0$.*

Proof. $\int_0^b f(s)\,ds = \int_0^\alpha f(s)\,ds + \int_\alpha^b f(s)\,ds$. □

Lemma 3.2.7. *If $\alpha \in (c, 1)$ then $\int_0^1 f(s)\,ds > 0$ and $\int_b^1 f(s)\,ds > 0$.*

Proof. Note that $b \in [u(1), u(0)]$, so there is $x_0 \in (0, 1)$ such that $u(x_0) = b$. Therefore

$$u'^2(x_0) = \lambda[F(\alpha) - F(b)] > 0,$$

so $F(\alpha) > F(b)$. Suppose that $\int_b^1 f(s)\,ds \leq 0$, then $F(1) \leq F(b)$, so $F(\alpha) > F(1)$, a contradiction. □

Lemma 3.2.8. *If $\alpha \in (a, b)$ then $\alpha \in (\gamma_0, b)$ where γ_0 is such that $\int_0^{\gamma_0} f(s)\,ds = 0$.*

Proof. $\int_0^{\alpha} f(s)\,ds = \int_0^{\gamma_0} f(s)\,ds + \int_{\gamma_0}^{b} f(s)\,ds > 0$. □

Lemma 3.2.9. *If $\alpha \in (c, 1)$ then $\alpha \in (\gamma_1, 1)$ where γ_1 is such that $\int_b^{\gamma_1} f(s)\,ds = 0$.*

Proof. Because $F(\alpha) > F(b)$, $\alpha \in (\gamma_1, 1)$. □

We can use Green's function to turn (3.1) into an equivalent integral equation. Applying the Green's function to the equation

$$-u''(x) = \lambda f_1(u) \tag{3.10}$$

gives

$$u(x) = \lambda \int_{-1}^{1} Z(x, y) f_1(u(y))\,dy, \tag{3.11}$$

where Z is the Green's function for (3.10), i.e.,

$$Z(x, y) = \begin{cases} \frac{1}{2}(x+1)(1-y), & -1 \leq x \leq y \leq 1, \\ \frac{1}{2}(y+1)(1-x), & -1 \leq y \leq x \leq 1, \end{cases} \tag{3.12}$$

and

$$f_1(u) = \begin{cases} f'(0)u, & u \leq 0, \\ u(u-a)(u-b)(u-c)(1-u), & 0 \leq u \leq 1, \\ f'(1)(u-1), & u > 0. \end{cases} \tag{3.13}$$

Define $\sigma_f = max\{|f'(u(x))| : -1 \leq x \leq 1\}$, $P^2 = \int_{-1}^{1}\int_{-1}^{1} Z^2(x,y)\,dy\,dx$ and $\bar{\lambda_0} = \frac{1}{P\sigma_f}$. Elementary computations show that $P^2 = \frac{4}{45}$. The next two theorems say that there is a unique solution if $\lambda < \bar{\lambda_0}$, but the equation (3.1) always has the trivial solution.

Theorem 3.2.10. *Assume $\alpha = u(0) \in (\gamma_0, b)$, then the integral equation (3.11) possesses a unique solution $u(x) \in L_2[-1,1]$ for every small value of the parameter λ that satisfies $\lambda < \bar{\lambda_0}$, i.e. $u \equiv 0$.*

Proof. Define the mapping $T : L_2[-1,1] \to L_2[-1,1]$ by $Tu = h$, where $h(x) = \lambda \int_{-1}^{1} Z(x,y) f_1(u(y))\,dy$. This map is well defined for each $u \in L_2[-1,1]$, $h \in L_2[-1,1]$, because Z is bounded. Let $t(x) = \int_{-1}^{1} Z(x,y) f_1(u(y))\,dy$. By Cauchy-Schwarz inequality

$$\begin{aligned}\left|\int_{-1}^{1} Z(x,y) f_1(u(y))\,dy\right| &\leq \int_{-1}^{1} |Z(x,y) f_1(u(y))|\;dy \\ &\leq \left(\int_{-1}^{1} |Z(x,y)|^2\,dy\right)^{1/2} \left(\int_{-1}^{1} |f_1(u(y))|^2\,dy\right)^{1/2}\end{aligned}$$

then

$$|t(x)|^2 \leq \left(\int_{-1}^{1} |Z(x,y)|^2\,dy\right)\left(\int_{-1}^{1} |f_1(u(y))|^2\,dy\right)$$

therefore

$$\begin{aligned}\int_{-1}^{1} |t(x)|^2\,dx &\leq \int_{-1}^{1} \left(\left(\int_{-1}^{1} |Z(x,y)|^2\,dy\right)\left(\int_{-1}^{1} |f_1(u(y))|^2\,dy\right)\right)\,dx \\ &= \left(\int_{-1}^{1}\int_{-1}^{1} |Z(x,y)|^2\,dy\,dx\right)\left(\int_{-1}^{1} |f_1(u(y))|^2\,dy\right) \\ &< \infty,\end{aligned}$$

so

$$t(x) \in L_2[-1,1]. \tag{3.14}$$

We know that $L_2[-1,1]$ is a complete metric space with metric

$$d(u_1,u_2) = \left(\int_{-1}^{1} |u_1(x) - u_2(x)|^2\, dx\right)^{1/2}.$$

Note that $d(Tu_1,Tu_2) = d(h_1,h_2)$, where $Tu_i = h_i$ and $h_i(x) = \lambda\int_{-1}^{1} Z(x,y) f_1(u_i(y))\, dy$ for $i = 1,2$. We will show that T is a contraction mapping for small λ.

This can be seen as follows,

$$\begin{aligned}
d(Tu_1,Tu_2) &= \lambda\left(\int_{-1}^{1}\left|\int_{-1}^{1} Z(x,y)[f_1(u_1(y)) - f_1(u_2(y))]\, dy\right|^2 dx\right)^{1/2} \\
&\le \lambda\left(\int_{-1}^{1}\left[\int_{-1}^{1} Z(x,y)\left|f_1(u_1(y)) - f_1(u_2(y))\right|\, dy\right]^2 dx\right)^{1/2} \\
&\le \lambda\left(\int_{-1}^{1}\left(\int_{-1}^{1} Z(x,y)^2\, dy\right)\left(\int_{-1}^{1}\left|f_1(u_1(y)) - f_1(u_2(y))\right|^2 dy\right) dx\right)^{1/2} \\
&\le \lambda\left(\int_{-1}^{1}\int_{-1}^{1} Z(x,y)^2\, dy\, dx\right)^{1/2}\left(\int_{-1}^{1}\left|f_1(u_1(y)) - f_1(u_2(y))\right|^2 dy\right)^{1/2} \\
&\le \lambda P\left(\int_{-1}^{1}\left|f_1(u_1(y)) - f_1(u_2(y))\right|^2 dy\right)^{1/2} \\
&\le \lambda P\left(\int_{-1}^{1}\left|\sigma_f\left(u_1(y) - u_2(y)\right)\right|^2 dy\right)^{1/2},
\end{aligned}$$

so

$$\begin{aligned}
d(Tu_1,Tu_2) &\le \lambda P\sigma_f\left(\int_{-1}^{1}\left|u_1(y) - u_2(y)\right|^2 dy\right)^{1/2} \\
&\le \lambda P\sigma_f\left|d(u_1,u_2)\right|.
\end{aligned}$$

If $\lambda < \bar{\lambda}_0$, then $d(Tu_1,Tu_2) < d(u_1,u_2)$, therefore T is a contraction mapping, so there is a unique $u \in L_2[-1,1]$ such that $Tu = u$. $\square$

Theorem 3.2.11. *Assume* $\alpha = u(0) \in (\gamma_1, 1)$, *then the integral equation (3.11) possesses a unique solution* $u(x) \in L_2[-1, 1]$ *for every small value of the parameter* λ *that satisfies* $\lambda < \bar{\lambda_0}$.

Proof. The same proof works. $\square$

CHAPTER 4
ASYMPTOTIC BEHAVIOR OF G

The shape of any bifurcation diagram is determined by its turning points. In this chapter, I will use a theorem derived in [14] that provides a necessary and sufficient condition on α for be a bifurcation point. The theorem will be used to find places where there are no bifurcation points. Also, the formula given in [14] can be used to compute numerically all turning points.

We will assume throughout this chapter that $\int_0^b f(t)\,dt > 0$, and $\int_b^1 f(t)\,dt > 0$. It is clear that there exists $\gamma_0 \in (a,b)$ and $\gamma_1 \in (c,1)$ such that $\int_0^{\gamma_0} f(t)\,dt = 0$, and $\int_b^{\gamma_1} f(t)\,dt = 0$.

We will adopt the following notation

$$J_k = max\{|f^{(k)}(u)| : u \in [0,1]\} > 0, k = 0,1,2,3.$$

$$j_0 = min\{f(u) : u \in [0,1]\} < 0.$$

In addition, let $x_0 \in (a,b)$ such that $f'(x_0) = 0$.

4.1 On the Asymptotic Behavior of $G(\alpha)$

The following result was proved by P. Korman, Y. Li, T. Ouyang in [14].

Theorem 4.1.1. *A solution of problem (2.1) with the maximal value $\alpha = u(0)$ is a bifurcation point if and only if*

$$G(\alpha) \equiv F(\alpha)^{1/2} \int_0^\alpha \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\,d\tau - 2 = 0, \tag{4.1}$$

with $F(u) = \int_0^u f(t)\,dt$.

An intuitive study of (4.1) tells us, for example, that $G(\alpha)$ approaches to $+\infty$ when $\alpha \to \gamma_0^+$. We want to explore and provide a rigorous proof of the asymptotic behavior of G when α is close to γ_0, b, γ_1 and 1. The next eight theorems are the main results of this thesis.

Theorem 4.1.2. $\lim_{\alpha \to b^-} G(\alpha) = -\infty$, *i.e., to the left of* b *but near* b *there is no bifurcation point.*

Proof. For τ close to α, f and F can be expressed

$$f(\tau) \;=\; f(\alpha) + (\tau - \alpha) f'(\alpha) + \frac{(\tau-\alpha)^2}{2} f''(\alpha) + O((\tau-\alpha)^3), \tag{4.2}$$

$$F(\tau) \;=\; F(\alpha) + (\tau - \alpha) F'(\alpha) + \frac{(\tau-\alpha)^2}{2} F''(\alpha) + O((\tau-\alpha)^3). \tag{4.3}$$

From (4.2), (4.3) and using the fact that $F' = f$ and $F'' = f'$, we obtain

$$f(\alpha) - f(\tau) \;=\; (\alpha - \tau) f'(\alpha) - \frac{(\alpha-\tau)^2}{2} f''(\alpha) + O((\alpha-\tau)^3), \tag{4.4}$$

and

$$F(\alpha) - F(\tau) \;=\; (\alpha - \tau) f(\alpha) - \frac{(\alpha-\tau)^2}{2} f'(\alpha) + O((\alpha-\tau)^3). \tag{4.5}$$

Then

$$\begin{aligned}
\frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}} &= \frac{(\alpha-\tau)[f'(\alpha) - \frac{(\alpha-\tau)}{2} f''(\alpha) + O((\alpha-\tau)^2)]}{(\alpha-\tau)^{3/2}[f(\alpha) - \frac{(\alpha-\tau)}{2} f'(\alpha) + O((\alpha-\tau)^2)]^{3/2}} \\
&= \frac{[f'(\alpha) - \frac{(\alpha-\tau)}{2} f''(\alpha) + O((\alpha-\tau)^2)]}{(\alpha-\tau)^{1/2}[f(\alpha) - \frac{(\alpha-\tau)}{2} f'(\alpha) + O((\alpha-\tau)^2)]^{3/2}} \\
&= \frac{[f'(\alpha) - \frac{(\alpha-\tau)}{2} f''(\alpha) + f_1]}{(\alpha-\tau)^{1/2}[f(\alpha) - \frac{(\alpha-\tau)}{2} f'(\alpha) + f_2]^{3/2}},
\end{aligned} \tag{4.6}$$

where $f_1, f_2 = O((\alpha-\tau)^2)$. Therefore, there are constants $C_1, C_2, \delta_1, \delta_2 > 0$ such that

$$|f_1| \leq C_1|(\alpha-\tau)^2| \ \text{ for } \ |\alpha-\tau| < \delta_1$$

and

$$|f_2| \leq C_2|(\alpha-\tau)^2| \ \text{ for } \ |\alpha-\tau| < \delta_2.$$

Fix $\epsilon > 0$ such that

$$\epsilon < min\{\frac{-f'(b)}{2J_2}, \frac{-f'(b)}{2C_2}, \sqrt{\frac{-f'(b)}{4C_1}}, \frac{b-x_0}{2}, \frac{b-\gamma_0}{2}, \delta_1, \delta_2\}. \tag{4.7}$$

Let $\alpha \in (b-\epsilon, b)$. Then

$$\begin{aligned} f'(\alpha) &= f'(b) + (\alpha-b)f''(z), \quad z \in (\alpha, b) \\ &= f'(b) + (b-\alpha)(-f''(z)) \\ &\leq f'(b) + \epsilon J_2, \end{aligned} \tag{4.8}$$

from (4.7), $f'(\alpha) + \frac{1}{2}(-f'(b)) < 0$ for all $\alpha \in (b-\epsilon, b)$. Also

$$\left|\frac{\alpha-\tau}{2}f''(\alpha)\right| < \frac{1}{4}\left(-f'(b)\right), \tag{4.9}$$

$$\left|(\alpha-\tau)^2\right| < \frac{1}{4C_1}\left(-f'(b)\right), \tag{4.10}$$

$$|\alpha-\tau| < \frac{1}{2C_2}(-f'(b)), \tag{4.11}$$

for any $\tau \in (\alpha-\epsilon, \alpha)$.

Define

$$I_1(\alpha) = \int_0^{\alpha-\epsilon} \frac{f(\alpha)-f(\tau)}{[F(\alpha)-F(\tau)]^{3/2}}\, d\tau, \tag{4.12}$$

$$I_2(\alpha) = \int_{\alpha-\epsilon}^{\alpha} \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\, d\tau, \tag{4.13}$$

$$K(\alpha) = \frac{2f'(\alpha) - f'(b)}{2} \int_{\alpha-\epsilon}^{\alpha} \frac{d\tau}{(\alpha-\tau)^{1/2}[f(\alpha) + \frac{(\alpha-\tau)}{2}(-f'(\alpha) - f'(b))]^{3/2}}, \tag{4.14}$$

$$S(\alpha) = \frac{J_0 - j_0}{[F(\alpha) - F(\alpha - \epsilon)]^{3/2}}(\alpha - \epsilon), \tag{4.15}$$

and

$$T(\alpha) = F(\alpha)^{1/2}[S(\alpha) + K(\alpha)] - 2. \tag{4.16}$$

Therefore from (4.1), (4.12) and (4.13) we obtain

$$G(\alpha) = F(\alpha)^{1/2}\big[I_1(\alpha) + I_2(\alpha)\big] - 2. \tag{4.17}$$

From (4.10) and (4.11)

$$|f_1| \leq \frac{1}{4}(-f'(b)), \tag{4.18}$$

and

$$|f_2| \leq \frac{\alpha - \tau}{2}(-f'(b)). \tag{4.19}$$

We will prove that $I_1(\alpha) \leq S(\alpha)$. For every $\tau \in (0, \alpha - \epsilon)$ we have that

$$F(\tau) \leq F(\alpha - \epsilon),$$

and

$$f(\alpha) - f(\tau) \leq J_0 - j_0.$$

Therefore

$$\frac{f(\alpha)-f(\tau)}{[F(\alpha)-F(\tau)]^{3/2}} \leq \frac{J_0-j_0}{[F(\alpha)-F(\alpha-\epsilon)]^{3/2}}.$$

Hence

$$\begin{aligned} I_1(\alpha) &= \int_0^{\alpha-\epsilon} \frac{f(\alpha)-f(\tau)}{[F(\alpha)-F(\tau)]^{3/2}}\, d\tau \\ &\leq \int_0^{\alpha-\epsilon} \frac{J_0-j_0}{[F(\alpha)-F(\alpha-\epsilon)]^{3/2}}\, d\tau \\ &= \frac{J_0-j_0}{[F(\alpha)-F(\alpha-\epsilon)]^{3/2}}(\alpha-\epsilon). \end{aligned}$$

It follows from (4.15) that $I_1(\alpha) \leq S(\alpha)$. Note that

$$\lim_{\alpha\to b^-} S(\alpha) = \frac{J_0-j_0}{[F(b)-F(b-\epsilon)]^{3/2}}(b-\epsilon) \quad \text{is finite.} \tag{4.20}$$

Next, we will be working with $I_2(\alpha)$. Observe that from (4.6), (4.13), (4.18) and (4.19) follows that

$$\begin{aligned} I_2(\alpha) &= \int_{\alpha-\epsilon}^{\alpha} \frac{[f'(\alpha)-\frac{(\alpha-\tau)}{2}f''(\alpha)+f_1]}{(\alpha-\tau)^{1/2}[f(\alpha)-\frac{(\alpha-\tau)}{2}f'(\alpha)+f_2]^{3/2}}\, d\tau \\ &\leq \int_{\alpha-\epsilon}^{\alpha} \frac{f'(\alpha)-\frac{1}{2}f'(b)}{(\alpha-\tau)^{1/2}[f(\alpha)-\frac{\alpha-\tau}{2}f'(\alpha)-\frac{\alpha-\tau}{2}f'(b)]^{\frac{3}{2}}}\, d\tau \\ &= \left(f'(\alpha)-\frac{1}{2}f'(b)\right)\int_{\alpha-\epsilon}^{\alpha} \frac{d\tau}{(\alpha-\tau)^{1/2}[f(\alpha)+\frac{(\alpha-\tau)}{2}(-f'(\alpha)-f'(b))]^{3/2}}. \end{aligned}$$

From (4.14) we obtain $I_2(\alpha) \leq K(\alpha)$. Consequently $G(\alpha) \leq T(\alpha)$. Let $A = f(\alpha)$ and $L = -f'(\alpha) - f'(b)$, then

$$K(\alpha) = \left(f'(\alpha)-\frac{1}{2}f'(b)\right)\int_{\alpha-\epsilon}^{\alpha} \frac{d\tau}{(\alpha-\tau)^{1/2}[A+\frac{(\alpha-\tau)}{2}L]^{3/2}},$$

Take $u = 2\sqrt{\alpha - \tau}$, then $du = -\frac{d\tau}{\sqrt{\alpha-\tau}}$. therefore

$$\begin{aligned} K(\alpha) &= \Big(f'(\alpha) - \frac{1}{2}f'(b)\Big)\int_{2\sqrt{\epsilon}}^{0} \frac{-du}{[A + \frac{L}{8}u^2]^{3/2}} \\ &= \Big(f'(\alpha) - \frac{1}{2}f'(b)\Big)\int_{0}^{2\sqrt{\epsilon}} \frac{du}{[A + \frac{L}{8}u^2]^{3/2}}. \\ &= \frac{\Big(f'(\alpha) - \frac{1}{2}f'(b)\Big)\sqrt{8\epsilon}}{\sqrt{2A^3 + \epsilon L A^2}} \\ &= \frac{\Big(f'(\alpha) - \frac{1}{2}f'(b)\Big)\sqrt{8\epsilon}}{\sqrt{2f(\alpha)^3 + \epsilon f(\alpha)^2\Big(-f'(\alpha) - f'(b)\Big)}}. \end{aligned}$$

Hence $\lim_{\alpha\to b^-} T(\alpha) = -\infty$. Consequently $\lim_{\alpha\to b^-} G(\alpha) = -\infty$. □

Theorem 4.1.3. $\lim_{\alpha\to 1^-} G(\alpha) = -\infty$, *i.e., near* 1, *there is no bifurcation point.*

Proof. The proof is similar to the previous theorem. □

Theorem 4.1.4. $\lim_{\alpha\to\gamma_0^+} G(\alpha) = +\infty$, *i.e., to the right of* γ_0 *but near* γ_0, *there is no bifurcation point.*

Proof. Fix $\epsilon > 0$ such that

$$\epsilon < min\{\frac{a}{2}, \frac{f''(0)}{J_3}, \frac{b-\gamma_0}{4}, \gamma_0 - a\}. \tag{4.21}$$

From (4.21) it follows that $f''(x) > 0$ for any $x \in (0, \epsilon)$. Let $\alpha \in (\gamma_0, \gamma_0 + \epsilon)$. Define the following integrals

$$I_1(\alpha) = \int_0^{\epsilon} \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\, d\tau, \tag{4.22}$$

$$I_2(\alpha) = \int_{\epsilon}^{a} \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\, d\tau, \tag{4.23}$$

$$I_3(\alpha) = \int_a^{\gamma_0-\epsilon} \frac{f(\alpha)-f(\tau)}{[F(\alpha)-F(\tau)]^{3/2}}\,d\tau, \tag{4.24}$$

and

$$I_4(\alpha) = \int_{\gamma_0-\epsilon}^{\alpha} \frac{f(\alpha)-f(\tau)}{[F(\alpha)-F(\tau)]^{3/2}}\,d\tau, \tag{4.25}$$

then

$$G(\alpha) = F(\alpha)^{1/2}\big[I_1(\alpha)+I_2(\alpha)+I_3(\alpha)+I_4(\alpha)\big]-2. \tag{4.26}$$

Next, we will study every of these integrals separately. For every $\tau \in [\epsilon, a]$, we have that $f(\tau) \leq 0$ and $F(a) \leq F(\tau)$. Therefore

$$\begin{aligned} I_2(\alpha) &= \int_\epsilon^a \frac{f(\alpha)-f(\tau)}{[F(\alpha)-F(\tau)]^{3/2}}\,d\tau \\ &\geq \int_\epsilon^a \frac{f(\alpha)}{[F(\alpha)-F(\tau)]^{3/2}}\,d\tau \\ &\geq \int_\epsilon^a \frac{f(\alpha)}{[F(\alpha)-F(a)]^{3/2}}\,d\tau \\ &\geq \frac{f(\alpha)}{[F(\alpha)-F(a)]^{3/2}}(a-\epsilon). \end{aligned} \tag{4.27}$$

So

$$I_2(\alpha) \geq \frac{f(\alpha)}{[F(\alpha)-F(a)]^{3/2}}(a-\epsilon) \tag{4.28}$$

Note that

$$F(\alpha) > 0, \tag{4.29}$$

for every $\gamma_0 < \alpha$.

Also, for all $\tau \in [a, \gamma_0 - \epsilon]$, $f(\tau) \leq J_0$ and $F(\tau) \leq F(\gamma_0 - \epsilon)$. Consequently

$$
\begin{aligned}
I_3(\alpha) &= \int_a^{\gamma_0-\epsilon} \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\, d\tau \\
&\geq \int_a^{\gamma_0-\epsilon} \frac{-J_0}{[F(\alpha) - F(\tau)]^{3/2}}\, d\tau \\
&\geq \int_a^{\gamma_0-\epsilon} \frac{-J_0}{[F(\alpha) - F(\gamma_0 - \epsilon)]^{3/2}}\, d\tau \\
&\geq \frac{-J_0}{[F(\alpha) - F(\gamma_0 - \epsilon)]^{3/2}}(\gamma_0 - a - \epsilon). \qquad (4.30)
\end{aligned}
$$

Using Mean Value Theorem, there are $c_0, c_1 \in (\tau, \alpha)$ such that

$$
f(\alpha) - f(\tau) = (\alpha - \tau) f'(c_0) \qquad (4.31)
$$

and

$$
F(\alpha) - F(\tau) = (\alpha - \tau) f(c_1). \qquad (4.32)
$$

Hence

$$
\begin{aligned}
I_4(\alpha) &= \int_{\gamma_0-\epsilon}^{\alpha} \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\, d\tau \\
&= \int_{\gamma_0-\epsilon}^{\alpha} \frac{(\alpha - \tau) f'(c_0)}{[(\alpha - \tau) f(c_1)]^{3/2}}\, d\tau \\
&= \int_{\gamma_0-\epsilon}^{\alpha} \frac{f'(c_0)}{(\alpha - \tau)^{1/2} [f(c_1)]^{3/2}}\, d\tau \\
&= \frac{f'(c_0)}{f(c_1)^{3/2}} \int_{\gamma_0-\epsilon}^{\alpha} \frac{1}{(\alpha - \tau)^{1/2}}\, d\tau.
\end{aligned}
$$

Let $u = (\alpha - \tau)^{1/2}$, then

$$
\begin{aligned}
I_4(\alpha) &= \frac{f'(c_0)}{f(c_1)^{3/2}} \int_{\sqrt{\alpha-\gamma_0+\epsilon}}^{0} -2\, du \\
&= \frac{2f'(c_0)}{f(c_1)^{3/2}} \int_0^{\sqrt{\alpha-\gamma_0+\epsilon}} du \\
&= \frac{2f'(c_0)}{f(c_1)^{3/2}} \sqrt{\alpha - \gamma_0 + \epsilon}. \qquad (4.33)
\end{aligned}
$$

The Taylor expansion of F around 0 is

$$F(\tau) = F(0) + \tau F'(0) + \frac{\tau^2}{2}F''(0) + \frac{\tau^3}{6}F'''(c_3), \tag{4.34}$$

where $c_3 \in (0, \tau)$. Note that $F(0) = F'(0) = 0$. Since $\tau \in [0, \epsilon]$, then $f''(c_3) > 0$. We have

$$F(\tau) = \frac{\tau^2}{2}f'(0) + \frac{\tau^3}{6}f''(c_3), \tag{4.35}$$

then

$$\begin{aligned} F(\alpha) - F(\tau) &= F(\alpha) - \frac{\tau^2}{2}f'(0) - \frac{\tau^3}{6}f''(c_3) \\ &< F(\alpha) + \tau^2(-f'(0)). \end{aligned} \tag{4.36}$$

Therefore

$$I_1(\alpha) = \int_0^\epsilon \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\, d\tau \quad \geq \quad \int_0^\epsilon \frac{f(\alpha)}{[F(\alpha) + \tau^2(-f'(0))]^{3/2}}\, d\tau. \tag{4.37}$$

Define

$$\begin{aligned} H(\alpha) &\equiv F(\alpha)^{1/2}[\frac{f(\alpha)}{[F(\alpha) - F(a)]^{3/2}}(a - \epsilon) \\ &+ \frac{f(\alpha) - f_{max}}{[F(\alpha) - F(\gamma_0 - \epsilon)]^{3/2}}(\gamma_0 - a - \epsilon) \\ &+ \frac{2f'(c_0)}{f(c_1)^{3/2}}(\alpha - \gamma_0 + \epsilon) \\ &+ \int_0^\epsilon \frac{f(\alpha)}{[F(\alpha) + \tau^2(-f'(0))]^{3/2}}\, d\tau] - 2. \end{aligned}$$

From (4.26), (4.27), (4.30), (4.33) and (4.37) we have

$$G(\alpha) \geq H(\alpha). \tag{4.38}$$

On the other hand

$$\begin{aligned}
\lim_{\alpha\to\gamma_0^+} H(\alpha) &= \frac{F(\gamma_0)^{1/2} f(\gamma_0)}{[F(\gamma_0)-F(a)]^{3/2}}(a-\epsilon) \\
&+ \frac{F(\gamma_0)^{1/2}(f(\gamma_0)-f_{max})}{[F(\gamma_0)-F(\gamma_0-\epsilon)]^{3/2}}\sqrt{\gamma_0-a-\epsilon} \\
&+ \frac{2F(\gamma_0)^{1/2} f'(c_0)}{f(c_1)^{3/2}}\epsilon \\
&+ \lim_{\alpha\to\gamma_0^+} F(\alpha)^{1/2} f(\alpha)\int_0^{\epsilon}\frac{d\tau}{[F(\alpha)+\tau^2(-f'(0))]^{3/2}} - 2.
\end{aligned}$$

Define $A_0 = (-f'(0))$. Hence

$$\begin{aligned}
& \lim_{\alpha\to\gamma_0^+} F(\alpha)^{1/2} f(\alpha)\int_0^{\epsilon}\frac{1}{[F(\alpha)+\tau^2 A_0]^{3/2}}\,d\tau \\
= & \lim_{\alpha\to\gamma_0^+} F(\alpha)^{1/2} f(\alpha)\int_0^{\epsilon}\frac{1}{F(\alpha)^{3/2}[1+\frac{\tau^2 A_0}{F(\alpha)}]^{3/2}}\,d\tau \\
= & \lim_{\alpha\to\gamma_0^+} \frac{f(\alpha)}{F(\alpha)}\int_0^{\epsilon}\frac{1}{[1+\frac{\tau^2 A_0}{F(\alpha)}]^{3/2}}\,d\tau.
\end{aligned}$$

Let $\frac{A_0}{F(\alpha)}\tau = t$, then $d\tau = \sqrt{\frac{F(\alpha)}{A_0}}dt$. Therefore

$$\begin{aligned}
\lim_{\alpha\to\gamma_0^+} \frac{f(\alpha)}{F(\alpha)}\int_0^{\epsilon}\frac{1}{[1+\frac{\tau^2 A_0}{F(\alpha)}]^{3/2}}\,d\tau &= \lim_{\alpha\to\gamma_0^+}\frac{f(\alpha)}{A_0^{1/2}F(\alpha)^{1/2}}\int_0^{\epsilon\sqrt{\frac{A_0}{F(\alpha)}}}\frac{dt}{(1+t^2)^{3/2}} \\
&= +\infty.
\end{aligned}$$

Hence $\lim_{\alpha\to\gamma_0^+} H(\alpha) = +\infty$. Consequently from (4.38) $\lim_{\alpha\to\gamma_0^+} G(\alpha) = +\infty$ □

Theorem 4.1.5. $\lim_{\alpha\to\gamma_1^+} G(\alpha) = +\infty$, *i.e., near* γ_1^+, *there is no bifurcation point.*

Proof. The proof is similar to previous theorem. □

4.2 Further Analysis of Asymptotic Behavior, Part I.

From the last section, we knew that close to γ_0, b, γ_1 and 1, bifurcations can not occur. In this section, we want to describe four intervals that do not contain bifurcation points. For numerical studies we want to avoid them. The following five lemmas are preparation for the main result of this section. We will assume through this section ϵ is fixed and it satisfies

$$0 < \epsilon < min\{-\frac{9J_2}{2J_3} + \frac{3}{J_3}\sqrt{\frac{9J_2^2}{4} + \frac{-J_3 f'(b)}{3}}, \frac{-3f'(b)}{4J_2}, \frac{b - x_0}{2}\}. \tag{4.39}$$

In addition, we define

$$\delta = min\{\frac{2}{11}\epsilon, \frac{-f'(b)}{J_2}, \frac{\left[F(b) - F(b-\epsilon)\right]^{3/2}}{4\sqrt{2}(1+a)(1+b)(1+c)(b-\epsilon)(-f'(b) + \frac{J_2}{6}\epsilon)^{1/2}}\}. \tag{4.40}$$

We assume throughout this section that $\alpha \in (b - \delta, b)$.

Lemma 4.2.1. $J_0 - j_0 \leq 2(1+a)(1+b)(1+c)$.

Proof. Since $f(u) = -u^5 + (a+b+c+1)u^4 + (-a-b-c-ab-ac-bc)u^3 + (ab + ac + bc + abc)u^2 - abcu$, a direct calculation reveals

$$\begin{aligned} f(u) &\leq (a+b+c+1) + (ab+ac+bc+abc) \\ &= (1+a) + (b+c) + a(b+c) + bc(1+a) \\ &= (1+a)(1+b)(1+c). \end{aligned}$$

Also

$$\begin{aligned} f(u) &\geq -1 - (a+b+c+ab+ac+bc) - abc \\ &= -(1+a)(1+b)(1+c), \end{aligned}$$

so $J_0 - j_0 \leq 2(1+a)(1+b)(1+c)$. □

Lemma 4.2.2. $F(\alpha) - F(\alpha - \epsilon) \geq F(b) - F(b - \epsilon)$.

Proof. Define $g(x) = F(x) - F(x-\epsilon)$. Then $g'(x) = f(x) - f(x-\epsilon) < 0$ for all $x \in [b-\epsilon, b]$, so $g(\alpha) \geq g(b)$, therefore $F(\alpha) - F(\alpha-\epsilon) \geq F(b) - F(b-\epsilon)$. $\square$

Lemma 4.2.3. *$f^{'}(\alpha) - \frac{(\alpha-\tau)}{2} f^{''}(\alpha) + \frac{(\alpha-\tau)^2}{6} f^{'''}(\bar{\alpha}) < \frac{f^{'}(b)}{2}$, for every $\tau, \bar{\alpha} \in (\alpha-\epsilon, \alpha)$.*

Proof. There exists a number $\bar{\bar{\alpha}}$ between a and b such that

$$\begin{aligned} f^{'}(\alpha) = f^{'}(b) + (\alpha - b) f^{''}(\bar{\bar{\alpha}}) &= f^{'}(b) - (b-\alpha) f^{''}(\bar{\bar{\alpha}}) \\ &\leq f^{'}(b) + (b-\alpha) J_2 \\ &\leq f^{'}(b) + \epsilon J_2. \qquad (4.41) \end{aligned}$$

On the other hand, for any $\tau, \bar{\alpha} \in (\alpha-\epsilon, \alpha)$

$$\begin{aligned} f^{'}(\alpha) - \frac{(\alpha-\tau)}{2} f^{''}(\alpha) + \frac{(\alpha-\tau)^2}{6} f^{'''}(\bar{\alpha}) &\leq f^{'}(\alpha) + \frac{(\alpha-\tau)}{2} J_2 + \frac{(\alpha-\tau)^2}{6} J_3 \\ &\leq f^{'}(\alpha) + \frac{\epsilon}{2} J_2 + \frac{\epsilon^2}{6} J_3. \qquad (4.42) \end{aligned}$$

From (4.41) and (4.42), we obtain

$$f^{'}(\alpha) - \frac{(\alpha-\tau)}{2} f^{''}(\alpha) + \frac{(\alpha-\tau)^2}{6} f^{'''}(\bar{\alpha}) \leq f^{'}(b) + \frac{3J_2}{2}\epsilon + \frac{J_3}{6}\epsilon^2.$$

Since $0 < \epsilon < -\frac{9J_2}{2J_3} + \frac{3}{J_3}\sqrt{\frac{9J_2^2}{4} + \frac{-J_3 f^{'}(b)}{3}}$, then $J_3\epsilon^2 + 9J_2\epsilon + 3f^{'}(b) < 0$, therefore

$$f^{'}(b) + \frac{3J_2}{2}\epsilon + \frac{J_3}{6}\epsilon^2 < \frac{f^{'}(b)}{2}. \qquad (4.43)$$

So $f^{'}(\alpha) - \frac{(\alpha-\tau)}{2} f^{''}(\alpha) + \frac{(\alpha-\tau)^2}{6} f^{'''}(\bar{\alpha}) < \frac{f^{'}(b)}{2}$. $\square$

Lemma 4.2.4. *Let $A = f(\alpha)$ and $B = \frac{-f^{'}(\alpha)}{2} + \frac{J_2}{6}\epsilon$, then $\frac{\epsilon B}{A} > 1$.*

Proof. We can write $f^{'}$ in the following form

$$f^{'}(\alpha) \quad = \quad f^{'}(b) + (\alpha - b)f^{''}(\bar{b}),$$

where $\bar{b} \in (\alpha, b)$. Hence

$$f^{'}(\alpha) \quad \leq \quad f^{'}(b) + (b - \alpha)J_2.$$

So

$$f^{'}(\alpha) \quad \leq \quad f'(b) + \epsilon J_2. \tag{4.44}$$

From our hypothesis, $\epsilon < \frac{-3f'(b)}{4J_2}$, therefore $f'(\alpha) < 0$. Using (4.44) we obtain

$$(\frac{-f^{'}(\alpha)}{2} + \frac{J_2}{6}\epsilon)\epsilon \geq (\frac{-f^{'}(b)}{2} - \frac{J_2}{3}\epsilon)\epsilon. \tag{4.45}$$

Similarly

$$f(\alpha) \quad = \quad f(b) + (\alpha - b)f^{'}(b) + \frac{(\alpha - b)^2}{2}f^{''}(\bar{\bar{b}}),$$

where $\bar{\bar{b}} \in (\alpha, b)$. So

$$\begin{aligned} f(\alpha) \quad &= \quad -f^{'}(b)(b - \alpha) + \frac{(b - \alpha)^2}{2}f^{''}(\bar{\bar{b}}) \\ &= \quad [-f^{'}(b) + \frac{(b - \alpha)}{2}f^{''}(\bar{\bar{b}})](b - \alpha), \end{aligned}$$

consequently

$$f(\alpha) \quad < \quad [-f^{'}(b) + \frac{J_2}{2}\epsilon](b - \alpha). \tag{4.46}$$

On other hand, since $\epsilon < \frac{-3f^{'}(b)}{4J_2}$, it implies that

$$\frac{f^{'}(b)}{4} < \frac{-J_2}{3}\epsilon, \tag{4.47}$$

$$\frac{f^{'}(b)}{2} < \frac{-J_2}{3}\epsilon \tag{4.48}$$

and

$$\frac{-3f^{'}(b)}{8} > \frac{J_2}{2}\epsilon. \tag{4.49}$$

Using (4.49), we obtain

$$-f^{'}(b) + \frac{J_2}{2}\epsilon < -f^{'}(b) - \frac{3f^{'}(b)}{8}. \tag{4.50}$$

From (4.47), (4.48) and (4.50) we get

$$\begin{aligned} \frac{(\frac{-f^{'}(b)}{2} - \frac{J_2}{3}\epsilon)\epsilon}{(-f^{'}(b) + \frac{J_2}{2}\epsilon)} &> \frac{(\frac{-f^{'}(b)}{2} - \frac{J_2}{3}\epsilon)\epsilon}{(-f^{'}(b) - \frac{3}{8}f^{'}(b))} \\ &> \frac{2}{11}\epsilon. \end{aligned}$$

Therefore

$$(\frac{-f^{'}(b)}{2} - \frac{J_2}{3}\epsilon)\epsilon > (-f^{'}(b) + \frac{J_2}{2}\epsilon)(b-\alpha). \tag{4.51}$$

From (4.46) and (4.51)

$$(\frac{-f^{'}(b)}{2} - \frac{J_2}{3}\epsilon)\epsilon > f(\alpha). \tag{4.52}$$

Using (4.45) and (4.52), we obtain

$$(\frac{-f^{'}(\alpha)}{2} + \frac{J_2}{6}\epsilon)\epsilon > f(\alpha).$$

So $\frac{\epsilon B}{A} > 1$. $\square$

Lemma 4.2.5. $f(\alpha) \leq -2(b-\alpha)f^{'}(b)$.

Proof.

$$\begin{aligned} f(\alpha) &= f(b) + (\alpha - b)f'(b) + \frac{(\alpha - b)^2}{2} f''(\bar{b_0}) \\ &= -(b-\alpha)f'(b) + \frac{(\alpha - b)^2}{2} f''(\bar{b_0}) \end{aligned} \tag{4.53}$$

where $\bar{b_0} \in (\alpha, b)$. Since

$$b - \alpha \leq -\frac{2f'(b)}{J_2},$$

we get

$$\frac{(\alpha - b)^2}{2} f''(\bar{b_0}) \leq -(b-\alpha)f'(b).$$

It follows from (4.53) $f(\alpha) \leq -2(b-\alpha)f'(b)$. □

Theorem 4.2.6. $G(\alpha) < 0$ *for every* $\alpha \in (b - \delta, b)$.

Proof. Let $\alpha \in (b-\delta, b)$. Consider, using ϵ as in (4.39), the following integrals

$$I_1(\alpha) = \int_0^{\alpha-\epsilon} \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\, d\tau, \tag{4.54}$$

and

$$I_2(\alpha) = \int_{\alpha-\epsilon}^{\alpha} \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\, d\tau. \tag{4.55}$$

It follows from (4.1) that $G(\alpha) = F(\alpha)^{1/2}\big[I_1(\alpha) + I_2(\alpha)\big] - 2$. For any $\tau \in (0, \alpha - \epsilon)$, $F(\tau) \leq F(\alpha - \epsilon)$, so from (4.54) we get

$$\begin{aligned} I_1(\alpha) &= \int_0^{\alpha-\epsilon} \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\, d\tau \\ &\leq \int_0^{\alpha-\epsilon} \frac{J_0 - j_0}{[F(\alpha) - F(\tau)]^{3/2}}\, d\tau \\ &\leq \int_0^{\alpha-\epsilon} \frac{J_0 - j_0}{[F(\alpha) - F(\alpha-\epsilon)]^{3/2}}\, d\tau \end{aligned}$$

So

$$I_1(\alpha) \quad \leq \quad \frac{J_0 - j_0}{\big[F(\alpha) - F(\alpha - \epsilon)\big]^{3/2}}(\alpha - \epsilon). \tag{4.56}$$

From Lemma 4.2.1, Lemma 4.2.2 and (4.56) we obtain

$$I_1(\alpha) \quad = \quad \int_0^{\alpha-\epsilon} \frac{f(\alpha) - f(\tau)}{\big[F(\alpha) - F(\tau)\big]^{3/2}}\, d\tau \leq \frac{J_0 - j_0}{\big[F(b) - F(b - \epsilon)\big]^{3/2}}(b - \epsilon). \tag{4.57}$$

From Taylor's formula we get

$$f(\tau) \quad = \quad f(\alpha) + (\tau - \alpha)f'(\alpha) + \frac{(\tau - \alpha)^2}{2}f''(\alpha) + \frac{f'''(\bar{\alpha})}{6}(\tau - \alpha)^3, \tag{4.58}$$

and

$$F(\tau) \quad = \quad F(\alpha) + (\tau - \alpha)F'(\alpha) + \frac{(\tau - \alpha)^2}{2}F''(\alpha) + \frac{F'''(\bar{\alpha_0})}{6}(\tau - \alpha)^3, \tag{4.59}$$

where $\bar{\alpha}, \bar{\alpha_0} \in (\tau, \alpha)$. From (4.58), (4.59) and using the fact that $F' = f$ and $F'' = f'$, we obtain

$$f(\alpha) - f(\tau) \quad = \quad (\alpha - \tau)\Big(f'(\alpha) - \frac{(\alpha - \tau)}{2}f''(\alpha) + \frac{f'''(\bar{\alpha})}{6}(\tau - \alpha)^2\Big) \tag{4.60}$$

and

$$F(\alpha) - F(\tau) \quad = \quad (\alpha - \tau)\Big(f(\alpha) - \frac{(\alpha - \tau)}{2}f'(\alpha) + \frac{f''(\bar{\alpha_0})}{6}(\tau - \alpha)^2\Big). \tag{4.61}$$

It follows from Lemma 4.2.3 and (4.60) that

$$f(\alpha) - f(\tau) \leq \frac{(\alpha - \tau)f'(b)}{2}. \tag{4.62}$$

Hence, by (4.60), (4.61) and (4.62)

$$\frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}} \quad \leq \quad \frac{f'(b)}{2(\alpha - \tau)^{1/2}\big[f(\alpha) + (\alpha - \tau)(\frac{-f'(\alpha)}{2} + \frac{f''(\bar{\alpha_0})}{6}(\alpha - \tau))\big]^{3/2}}.$$

Since

$$f(\alpha) + (\alpha - \tau)(\frac{-f'(\alpha)}{2} + \frac{f''(\bar{\alpha_0})}{6}(\alpha - \tau)) \leq f(\alpha) + (\alpha - \tau)(\frac{-f'(\alpha)}{2} + \frac{J_2}{6}\epsilon).$$

then

$$\frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}} \leq \frac{f'(b)}{2(\alpha - \tau)^{1/2}\left[f(\alpha) + (\alpha - \tau)(\frac{-f'(\alpha)}{2} + \frac{J_2}{6}\epsilon)\right]^{3/2}} \tag{4.63}$$

From (4.55) and (4.63) and using the notation given in Lemma (4.2.4), it follows

$$I_2(\alpha) \leq \frac{1}{2} f'(b) \int_{\alpha-\epsilon}^{\alpha} \frac{d\tau}{(\alpha - \tau)^{1/2}[A + (\alpha - \tau)B]^{3/2}}.$$

Let

$$u = \sqrt{\alpha - \tau}, \quad \text{then} \quad du = -\frac{d\tau}{2\sqrt{\alpha - \tau}}, \tag{4.64}$$

so

$$\begin{aligned} I_2(\alpha) &\leq f'(b) \int_{\sqrt{\epsilon}}^{0} \frac{-du}{[A + Bu^2]^{3/2}} \\ &= f'(b) \int_{0}^{\sqrt{\epsilon}} \frac{du}{[A + Bu^2]^{3/2}} \end{aligned}$$

Let $u = \sqrt{\frac{A}{B}}s$ then $A + Bu^2 = A(1 + s^2)$ and $du = \sqrt{\frac{A}{B}}ds$. Therefore

$$I_2(\alpha) \leq \frac{f'(b)}{AB^{1/2}} \int_{0}^{\sqrt{\frac{B\epsilon}{A}}} \frac{ds}{[1 + s^2]^{3/2}} \tag{4.65}$$

From Lemma 4.2.4 it follows that

$$\begin{aligned} I_2(\alpha) &\leq \frac{f'(b)}{AB^{1/2}} \int_{0}^{1} \frac{ds}{[1 + s^2]^{3/2}} \\ &= \frac{f'(b)}{AB^{1/2}} \frac{s}{1 + s^2}\Big|_0^1. \end{aligned}$$

Hence

$$I_2(\alpha) \leq \frac{f^{'}(b)}{\sqrt{2}AB^{1/2}}. \tag{4.66}$$

Using Mean Value Theorem, we obtain that

$$\begin{aligned} -f^{'}(\alpha) &= -f^{'}(b) - (\alpha - b)f^{''}(\bar{b}), \text{ where } \bar{b} \in (\alpha, b). \\ &\leq -f^{'}(b) + \epsilon J_2, \end{aligned}$$

and consequently

$$\frac{-f^{'}(\alpha)}{2} \leq \frac{-f^{'}(b)}{2} + \frac{J_2}{2}\epsilon. \tag{4.67}$$

Since $\epsilon < \frac{-3f^{'}(b)}{4J_2}$, we get

$$\frac{-f^{'}(b)}{2} + \frac{J_2}{2}\epsilon < -f^{'}(b). \tag{4.68}$$

Hence, by (4.67) and (4.68)

$$B^{1/2} < \Big(-f^{'}(b) + \frac{J_2}{6}\epsilon\Big)^{1/2}. \tag{4.69}$$

By Lemma 4.2.5, (4.66) and (4.69) it follows that

$$I_2(\alpha) < -\frac{1}{2\sqrt{2}(b-\alpha)(-f^{'}(b) + \frac{J_2}{6}\epsilon)^{1/2}}. \tag{4.70}$$

From Lemma 4.2.1, (4.57) and (4.70) we obtain

$$I_1(\alpha) + I_2(\alpha) < \frac{2(1+a)(1+b)(1+c)}{[F(b) - F(b-\epsilon)]^{3/2}}(b-\epsilon) - \frac{1}{2\sqrt{2}(b-\alpha)(-f^{'}(b) + \frac{J_2}{6}\epsilon)^{1/2}}.$$

Then

$$G(\alpha) < 0 \text{ holds for all } \alpha \in (b-\delta, b).$$

$\square$

Theorem 4.2.7. *There exists $\delta > 0$ such that $G(\alpha) < 0$ for every $\alpha \in (1 - \delta, 1)$.*

Proof. The proof is similar to previous theorem. □

4.3 Further Analysis of Asymptotic Behavior, Part II.

The next Theorem will study the asymptotic behavior of $G(\alpha)$ as α gets close to γ_0 from the right. Fix ϵ such that

$$0 < \epsilon < min\{\frac{f''(0)}{J_3}, \frac{f(\gamma_0)}{4J_1}, \frac{\gamma_0 - a}{2}, \frac{a}{4}, \frac{b-a}{2}\}. \tag{4.71}$$

We will assume through this section, as before, that $A_0 = -f'(0)$. In addition, we define

$$\delta = min\Big\{\epsilon, \ \frac{A_0\epsilon^2}{J_0}, \ \frac{f(\gamma_0)^2}{16A_0J_0\Big(\frac{\sqrt{2F(b)}J_0}{(f(\gamma_0)\epsilon)^{3/2}}(b-a-\epsilon) + \frac{8\sqrt{\epsilon F(b)}J_1}{f(\gamma_0)^{3/2}} + 1\Big)^2}\Big\}. \tag{4.72}$$

We start with the following lemma that is needed for the proof of the Theorem of this section.

Lemma 4.3.1. *For all $x \in (0, \epsilon)$, $f''(x) > 0$.*

Proof. It follows from (4.71). □

Theorem 4.3.2. *$G(\alpha) > 0$ for all $\alpha \in (\gamma_0, \gamma_0 + \delta)$.*

Proof. Let $\alpha \in (\gamma_0, \gamma_0 + \delta)$, therefore $\alpha - \gamma_0 < \frac{A_0\epsilon^2}{J_0}$ and $\alpha - \gamma_0 < \epsilon$. Define

$$I_1(\alpha) = \int_0^{\epsilon} \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\, d\tau, \tag{4.73}$$

$$I_2(\alpha) = \int_{\epsilon}^{a} \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\, d\tau, \tag{4.74}$$

$$I_3(\alpha) = \int_a^{\alpha-\epsilon} \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\, d\tau, \tag{4.75}$$

$$I_4(\alpha) = \int_{\alpha-\epsilon}^{\alpha} \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\, d\tau. \tag{4.76}$$

There exist $y_1, y_2 \in (\gamma_0, \alpha)$, and $y_3 \in (\alpha - \epsilon, \alpha)$ such that

$$f(\alpha) = f(\gamma_0) + f'(y_2)(\alpha - \gamma_0), \tag{4.77}$$

$$F(\alpha - \epsilon) = F(\alpha) - \epsilon F'(y_3), \tag{4.78}$$

$$F(\alpha) = F(\gamma_0) + (\alpha - \gamma_0)F'(y_1), \tag{4.79}$$

and in addition for $\tau \in (0, \epsilon)$, there is $y \in (0, \tau) \subset (0, \epsilon)$ such that

$$F(\tau) = F(0) + \tau F'(0) + \frac{\tau^2}{2} F''(0) + \frac{\tau^3}{6} F'''(y). \tag{4.80}$$

We also note that $F(\gamma_0) = 0$, $F'(y_1) = f(y_1)$ and $(\alpha - \gamma_0)J_0 < A_0\epsilon^2$.

From (4.79) follows

$$\begin{aligned} F(\alpha) &= (\alpha - \gamma_0) f(y_1) \\ &\leq (\alpha - \gamma_0) J_0 \\ &\leq A_0 \epsilon^2 \end{aligned} \tag{4.81}$$

Since $F(0) = F'(0) = 0$, $F''(0) = f'(0)$ and $F'''(y) = f''(y)$, it follows from (4.80)

$$F(\alpha) - F(\tau) = F(\alpha) - \frac{\tau^2}{2} f'(0) - \frac{\tau^3}{6} f''(y). \tag{4.82}$$

From Lemma 4.3.1 and (4.82) we get that for any $\tau \in (0, \epsilon)$

$$F(\alpha) - F(\tau) \leq F(\alpha) - \tau^2 f'(0), \tag{4.83}$$

so

$$\frac{1}{F(\alpha)-F(\tau)} \geq \frac{1}{F(\alpha)-\tau^2 f'(0)}. \tag{4.84}$$

From (4.84) we obtain

$$\begin{aligned} I_1(\alpha) &= \int_0^{\epsilon} \frac{f(\alpha)-f(\tau)}{[F(\alpha)-F(\tau)]^{3/2}}\, d\tau \\ &\geq \int_0^{\epsilon} \frac{f(\alpha)}{[F(\alpha)-\tau^2 f'(0)]^{3/2}}\, d\tau \\ &= \frac{f(\alpha)}{F(\alpha)^{3/2}} \int_0^{\epsilon} \frac{d\tau}{\left[1+\frac{A_0}{F(\alpha)}\tau^2\right]^{3/2}}. \end{aligned} \tag{4.85}$$

Let $\sqrt{\frac{A_0}{F(\alpha)}}\tau = t$, then $d\tau = \sqrt{\frac{F(\alpha)}{A_0}}dt$. Therefore

$$I_1(\alpha) \geq \frac{f(\alpha)}{A_0^{1/2} F(\alpha)} \int_0^{\sqrt{\frac{A_0}{F(\alpha)}}\epsilon} \frac{dt}{(1+t^2)^{3/2}}. \tag{4.86}$$

From (4.81) and (4.86) we get

$$I_1(\alpha) \geq \frac{f(\alpha)}{A_0^{1/2} F(\alpha)} \int_0^{1} \frac{dt}{(1+t^2)^{3/2}}, \tag{4.87}$$

from which follows

$$I_1(\alpha) \geq \frac{f(\alpha)}{\sqrt{2A_0}F(\alpha)}. \tag{4.88}$$

Also, from (4.77), $f(\alpha) \geq f(\gamma_0) - J_1\epsilon$. Since $\epsilon < \frac{f(\gamma_0)}{2J_1}$, this implies

$$f(\alpha) > \frac{f(\gamma_0)}{2}. \tag{4.89}$$

Using (4.81), we have

$$F(\alpha) \leq (\alpha-\gamma_0)J_0. \tag{4.90}$$

Consequently

$$F(\alpha)^{1/2} I_1(\alpha) > \frac{f(\gamma_0)}{2\sqrt{2A_0 J_0}(\alpha-\gamma_0)^{1/2}}. \tag{4.91}$$

Note that on $[\epsilon, a]$, $f(\tau) \leq 0$ and $F(a) \leq F(\tau)$. Therefore we have

$$\begin{aligned} I_2(\alpha) &= \int_\epsilon^a \frac{f(\alpha)-f(\tau)}{[F(\alpha)-F(\tau)]^{3/2}}\, d\tau \\ &\geq \int_\epsilon^a \frac{f(\alpha)}{[F(\alpha)-F(\tau)]^{3/2}}\, d\tau \\ &\geq \int_\epsilon^a \frac{f(\alpha)}{[F(\alpha)-F(a)]^{3/2}}\, d\tau \\ &\geq \frac{f(\alpha)}{[F(\alpha)-F(a)]^{3/2}}(a-\epsilon). \end{aligned} \tag{4.92}$$

So

$$F(\alpha)^{1/2} I_2(\alpha) \geq 0. \tag{4.93}$$

By (4.78)

$$F(\alpha-\epsilon) = F(\alpha) - \epsilon f(y_3). \tag{4.94}$$

Similarly, $f(y_3) = f(\gamma_0) + f'(y_4)(y_3 - \gamma_0)$, where $y_4 \in (min\{y_3, \gamma_0\}, max\{y_3, \gamma_0\})$. So

$$\begin{aligned} f(y_3) &\geq f(\gamma_0) - J_1\epsilon \\ &> \frac{f(\gamma_0)}{2}, \end{aligned} \tag{4.95}$$

therefore

$$F(\alpha) - F(\alpha-\epsilon) > \frac{f(\gamma_0)}{2}\epsilon. \tag{4.96}$$

But

$$I_3(\alpha) = \int_a^{\alpha-\epsilon} \frac{f(\alpha)-f(\tau)}{[F(\alpha)-F(\tau)]^{3/2}}\, d\tau,$$

then

$$
\begin{aligned}
I_3(\alpha) &\geq \frac{-J_0}{[F(\alpha) - F(\alpha - \epsilon)]^{3/2}}(\alpha - \epsilon - a) \\
&\geq \frac{-2\sqrt{2}J_0}{f(\gamma_0)^{3/2}\epsilon^{3/2}}(b - a - \epsilon).
\end{aligned} \tag{4.97}
$$

Since $F(\alpha)^{1/2} \leq F(b)^{1/2}$, (4.97) implies that

$$
F(\alpha)^{1/2} I_3(\alpha) \geq \frac{-2\sqrt{2}J_0 F(b)^{1/2}}{f(\gamma_0)^{3/2}\epsilon^{3/2}}(b - a - \epsilon). \tag{4.98}
$$

On the other hand, by definition of $I_4(\alpha)$

$$
I_4(\alpha) = \int_{\alpha-\epsilon}^{\alpha} \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\, d\tau.
$$

Now

$$
f(\alpha) - f(\tau) = (\alpha - \tau) f'(y_5) \tag{4.99}
$$

and

$$
F(\alpha) - F(\tau) = (\alpha - \tau) f(y_6) \tag{4.100}
$$

where $y_5, y_6 \in (\tau, \alpha) \subset (\alpha - \epsilon, \alpha)$. Note that for any $x \in (\alpha - \epsilon, \alpha)$

$$
f(\alpha) \leq f(x) + \epsilon J_1. \tag{4.101}
$$

From (4.71), (4.89) and (4.101) we get

$$
f(x) > \frac{f(\gamma_0)}{4}. \tag{4.102}
$$

Hence

$$\begin{aligned}
I_4(\alpha) &= \int_{\alpha-\epsilon}^{\alpha} \frac{f'(y_5)}{(\alpha-\tau)^{1/2} f(y_6)^{3/2}}\, d\tau \\
&= \frac{2\sqrt{\epsilon} f'(y_5)}{f(y_6)^{3/2}} \\
&\geq \frac{-2\sqrt{\epsilon} J_1}{f(y_6)^{3/2}} \\
&> \frac{-16\sqrt{\epsilon} J_1}{f(\gamma_0)^{3/2}}.
\end{aligned}$$

Then we have

$$F(\alpha)^{1/2} I_4(\alpha) \quad > \quad \frac{-16\sqrt{\epsilon} F(b)^{1/2} J_1}{f(\gamma_0)^{3/2}}. \tag{4.103}$$

We conclude from (4.91), (4.98) and (4.103) that

$$G(\alpha) > \frac{f(\gamma_0)}{2\sqrt{2A_0 J_0}(\alpha-\gamma_0)^{1/2}} - \frac{2\sqrt{2F(b)} J_0}{(f(\gamma_0)\epsilon)^{3/2}}(b-a-\epsilon) - \frac{16\sqrt{\epsilon F(b)} J_1}{f(\gamma_0)^{3/2}} - 2$$

From (4.72) $G(\alpha) > 0$. □

Theorem 4.3.3. *There exists $\delta > 0$ such that $G(\alpha) > 0$ for every $\alpha \in (\gamma_1, \gamma_1 + \delta)$.*

Proof. The proof is similar to previous theorem.

□

CHAPTER 5
LOCATION OF BIFURCATIONS

5.1 Applications of Theorem 2.2.8

Inspired by Theorem 2.2.8, we conjecture that there are two values λ_0, λ_1, of the parameter λ for which one of the following holds:

Case 1: If $\lambda_0 \neq \lambda_1$, then problem (3.1) has exactly one positive solution for $0 < \lambda = min\{\lambda_0, \lambda_1\}$, exactly two positive solutions for $min\{\lambda_0, \lambda_1\} < \lambda < max\{\lambda_0, \lambda_1\}$, exactly three positive solutions for $\lambda = max\{\lambda_0, \lambda_1\}$, and exactly four nontrivial positive solutions for $\lambda > max\{\lambda_0, \lambda_1\}$.

Case 2: If $\lambda_0 = \lambda_1$, the problem (3.1) has exactly one positive solution for $0 < \lambda = \lambda_0 = \lambda_1$, and exactly four nontrivial positive solutions for $\lambda > \lambda_0 = \lambda_1$.

Moreover, we conjecture as well that all solutions lie on two smooth curves which are parabola-like, having exactly one turn to the right on such curve. The first curve Γ_0, the lower curve, has a turning point at λ_0 with $\lim_{\lambda\to\infty,\Gamma_0^+} u(0,\lambda) = b$ while $\lim_{\lambda\to\infty,\Gamma_0^-} u(0,\lambda) = \gamma_0$. The second curve Γ_1, the upper curve, has a turning point at λ_1 with $\lim_{\lambda\to\infty,\Gamma_1^+} u(0,\lambda) = 1$ while $\lim_{\lambda\to\infty,\Gamma_1^-} u(0,\lambda) = \gamma_1$, where γ_0 is the unique root of $\int_0^{\gamma_0} f(s)\,ds = 0$ in (a,b) and γ_1 is the unique root of $\int_b^{\gamma_1} f(s)\,ds = 0$ in $(c,1)$.

5.2 General Behaviour of Solutions

We begin by deriving some lemmas. By our assumptions on the function f, it is clear that there are exactly three points where a ray starting at the origin touches the graph of $f(u)$. We denote the first point by β_1, i.e. $\beta_1 \in (a,b)$ is the first solution of equation

$$f'(\beta) = \frac{f(\beta)}{\beta}. \tag{5.1}$$

We recall, from the analysis in [10], that turning (or singular) points of (2.1) can occur only when (2.4) has a nontrivial solution $w(x)$. Furthermore in such a case we can choose $w(x)$ to be strictly positive on $(-1, 1)$.

The next lemma was proved in [12] for the cubic case, $f(u) = u(u-a)(u-b)$. Fortunately, the same proof applies for (3.1), which we present for completeness.

Lemma 5.2.1. *Let $u(x)$ be any critical point of (3.1),* $0 < u(x) < b$. *Then*

$$u(0) > \beta_1. \tag{5.2}$$

Proof. We will show that if $u(0) \leq \beta_1$, then the only solution of (2.4) is $w \equiv 0$. First, we claim that

$$f'(u) > \frac{f(u)}{u} \text{ for } 0 < u < \beta_1. \tag{5.3}$$

Indeed, denote $p(u) = uf'(u) - f(u)$. Then $p(0) = p(\beta_1) = 0$, and $p'(u) = uf''(u)$. It follows that $p'(u) > 0$ near $u = 0$, and $p'(u) < 0$ near $u = \beta_1$. Since $p(u)$ has no roots in $(0, \beta_1)$ (since solution of (5.1) is unique) it follows that $p(u) > 0$ on $(0, \beta_1)$, establishing (5.3). We now rewrite (2.1) in the form

$$u'' + \lambda \frac{f(u)}{u} u = 0.$$

Using the Sturm comparison theorem and (5.3), we conclude that (2.4) cannot have a positive solution $w(x)$. (By (5.3) any solution of (2.4) would have to vanish on

$(-1,1)$). Since any nontrivial solution of (2.4) has to be positive and even, the proof is complete. □

Suppose that the function $f(u)$ is concave up at b, that is $f''(b) \geq 0$. It is clear that there is exactly one point where a ray out of the point b touches to the right the graph of $f(u)$. We denote this point by β_2, i.e. β_2 is the only solution of equation

$$f'(\beta) = \frac{f(\beta)}{\beta - b}, \quad \text{for } \beta > b.$$

Theorem 5.2.2. *Suppose that $f''(b) \geq 0$, and let $u(x)$ be any singular solution of (3.1), and that $\gamma_1 < u(0) < 1$. Then $u(0) > \beta_2$.*

Proof. Suppose that $u(0) \leq \beta_2$. We shall reach a contradiction. Note that $f'(u) > \frac{f(u)}{u-b}$ for all $b < u \leq \beta_2$. Also,

$$\begin{aligned} 0 &= u'' + \lambda f(u) \\ &= u'' + \frac{\lambda f(u)}{u-b}(u-b) \\ &= u'' + \frac{\lambda f(u)}{u-b}u - \frac{\lambda f(u)}{u-b}b. \end{aligned}$$

So,

$$u'' + \frac{\lambda f(u)}{u-b}u = \frac{\lambda f(u)}{u-b}b. \tag{5.4}$$

We define,

$$r_1(x) = \lambda f'(u), \tag{5.5}$$

$$r_2(x) = \tfrac{\lambda f(u)}{u-b}, \tag{5.6}$$

and

$$g(x) = \lambda \frac{f(u)}{u-b} b. \tag{5.7}$$

By (5.4), (5.6) and (5.7) we have

$$u'' + r_2(x)u = g(x) \tag{5.8}$$

and, since the linearized must have positive solution w, we have that

$$w'' + r_1(x)w = 0. \tag{5.9}$$

We define the new dependent variable

$$\nu = \frac{u-b}{w}. \tag{5.10}$$

So, $u - b = \nu w$. A simple computation shows that

$$u'' = \nu'' w + 2\nu' w' + \nu w''. \tag{5.11}$$

Remember that by Lemma 2.8, if the problem (2.4) admits a nontrivial solution w, then it does not change sign, i.e. we may assume that $w(x) > 0$ on $(-1, 1)$. We get

$$\nu'' + \frac{2w'}{w}\nu' + \frac{w''}{w}\nu = \frac{u''}{w}. \tag{5.12}$$

on $(-1, 1)$.

From (5.8)

$$u'' = g(x) - r_2(x)u. \tag{5.13}$$

Combining (5.12) and (5.13), we get

$$\nu'' + \frac{2w'}{w}\nu' + \frac{w''}{w}\nu = \frac{g(x) - r_2(x)u}{w}. \tag{5.14}$$

We find then that

$$\begin{aligned}
\frac{g(x) - r_2(x)u}{w} &= \frac{\frac{\lambda f(u)}{u-b}b - \frac{\lambda f(u)}{u-b}u}{w} \\
&= -\frac{\lambda f(u)}{u-b}\frac{(u-b)}{w}
\end{aligned}$$

so we can write

$$\frac{g(x) - r_2(x)u}{w} = -r_2(x)\nu. \tag{5.15}$$

As before, dividing (5.9) by the positive quantity w, we get

$$\frac{w''}{w} = -r_1(x). \tag{5.16}$$

Replacing (5.15) and (5.16) into (5.14), we obtain

$$\nu'' + \frac{2w'}{w}\nu' - r_1(x)\nu = -r_2(x)\nu \tag{5.17}$$

therefore,

$$\nu'' + \frac{2w'}{w}\nu' + (r_2(x) - r_1(x))\nu = 0. \tag{5.18}$$

Note that $r_2(x) < r_1(x)$, since $f'(u) > \frac{f(u)}{u-b}$.

Choose x_0 such that $u(x_0) = b$. If $x_0 = 1$, then $u(x_0) = u(1) = 0 = b$, a contradiction.

Similarly $x_0 \neq -1$, so $x_0 \in (-1, 1)$.

Consider the equation

$$\nu'' + \frac{2w'}{w}\nu' + (r_2(x) - r_1(x))\nu = 0 \;\; on \;\; [-x_0, x_0]. \tag{5.19}$$

ν is a even function, since $\nu(-x_0) = \frac{u(-x_0)-b}{w(-x_0)} = \nu(x_0)$.

Since ν is continuous on $[-x_0, x_0]$, ν attains a maximum value at some number d in

$[-x_0, x_0]$. Let $\nu(d) = max\{\nu(x):\ x \in [-x_0, x_0]\}$. Then there are two possibilities for d:

1. if $d = x_0$, in such case $\nu(d) = 0$, a contradiction, since $\nu(0) = \frac{u(0)-b}{w(0)} > 0$.

2. if $d \in (-x_0, x_0)$, then $\nu''(d) \leq 0$ and $\nu'(d) = 0$, so

$$\begin{aligned} 0 &= \nu''(d) + \frac{2w'(d)}{w(d)}\nu'(d) + (r_2(d) - r_1(d))\nu \\ &= \nu''(d) + (r_2(d) - r_1(d))\nu \\ &< 0, \end{aligned}$$

a contradiction. This finishes the proof of the theorem. □

Also, note that the function $f(u)$ could be, for example, concave up at b' with $b' < b$. Let $b^* = max\{b, b'\}$. It is clear that there is exactly one point where a ray out of the point b^* touches to the right the graph of $f(u)$. We denote this point by β_3, i.e., $\beta_3 \in (c, 1)$ is the only solution of equation

$$f'(\beta) = \frac{f(\beta) - f(b^*)}{\beta - b^*} \quad \text{for} \quad \beta > b^*.$$

Theorem 5.2.3. *Let $u(x)$ be any singular solution of (3.1) with $0 < u(0) < 1$. Suppose that $f''(b^*) \geq 0$, then*

$$u(0) > \beta_3. \tag{5.20}$$

Proof. Suppose that $u(0) \leq \beta_3$. We shall reach a contradiction. Note that $f'(u) > \frac{f(u)-f(b^*)}{u-b^*}$ for all $b^* < u \leq \beta_3$. Each step that follows has a straightforward analog in the setting of Theorem **??**. □

Theorem 5.2.4. *Suppose that* $\alpha = u(0) \in (\gamma_0, b)$. *Let* $\lambda_0 = \frac{\gamma_0^2}{\int_a^b f(s)\, ds}$. *If* $0 \leq \lambda \leq \lambda_0$, *then the problem (3.1) does not possess any positive solutions.*

Proof. Suppose that (3.1) has a positive solution for some λ, $0 < \lambda \leq \lambda_0$, then

$$\frac{1}{2}[u'(s)]^2 + \lambda \int_{u(0)}^{u(s)} f(x)\, dx = 0.$$

Thus,

$$\frac{1}{2\lambda}[u'(s)]^2 = -\int_{u(0)}^{u(s)} f(x)\, dx = \int_{u(s)}^{u(0)} f(x)\, dx \leq \int_a^b f(x)\, dx.$$

Define $\epsilon = \int_a^b f(s)\, ds$. Using the last equation, we obtain $[u'(s)] \leq (2\epsilon\lambda)^{\frac{1}{2}}$. Solving this last inequality and using the fact that $u'(s) < 0$ on $(0, 1)$, we obtain

$$u'(s) \geq -(2\epsilon\lambda)^{\frac{1}{2}}. \tag{5.21}$$

Integrating (5.21),

$$\int_0^1 u'(s)\, ds \geq -(2\epsilon\lambda)^{\frac{1}{2}}. \tag{5.22}$$

Consequently,

$$u(1) - u(0) \geq -(2\epsilon\lambda)^{\frac{1}{2}}. \tag{5.23}$$

So,

$$\lambda \geq \frac{u(0)^2}{2\epsilon} > \frac{\gamma_0^2}{2\epsilon} = \lambda_0, \tag{5.24}$$

and therefore $\lambda > \lambda_0$. This contradicts the assumption $0 \leq \lambda \leq \lambda_0$. $\square$

Theorem 5.2.5. *Assume that* $\alpha = u(0) \in (\gamma_1, 1)$. *Let* $\lambda_1 = \frac{\gamma_1^2}{\int_a^b f(s)\, ds + \int_c^1 f(s)\, ds}$. *If* $0 \leq \lambda \leq \lambda_1$, *then problem (3.1) does not possess any positive solutions.*

Proof. Suppose that (3.1) has a positive solution for some λ, $0 < \lambda \leq \lambda_1$, then

$$\frac{1}{2}[u'(s)]^2 + \lambda \int_{u(0)}^{u(s)} f(x)\,dx = 0.$$

Thus,

$$\frac{1}{2\lambda}[u'(s)]^2 = -\int_{u(0)}^{u(s)} f(x)\,dx = \int_{u(s)}^{u(0)} f(x)\,dx \leq \int_a^b f(x)\,dx + \int_c^1 f(s)\,ds.$$

Define $\epsilon = \int_a^b f(s)\,ds + \int_c^1 f(s)\,ds$. Using the last equation, we obtain $[u'(s)] \leq (2\epsilon\lambda)^{\frac{1}{2}}$. Solving this last inequality and using the fact that $u'(s) < 0$ on $(0, 1)$, we obtain

$$u'(s) \geq -(2\epsilon\lambda)^{\frac{1}{2}}. \tag{5.25}$$

Integrating (5.25),

$$\int_0^1 u'(s)\,ds \geq -(2\epsilon\lambda)^{\frac{1}{2}}. \tag{5.26}$$

Consequently,

$$u(1) - u(0) \geq -(2\epsilon\lambda)^{\frac{1}{2}}. \tag{5.27}$$

So,

$$\lambda \geq \frac{u(0)^2}{2\epsilon} > \frac{\gamma_1^2}{2\epsilon} = \lambda_1, \tag{5.28}$$

therefore $\lambda > \lambda_1$. This contradicts the assumption $0 \leq \lambda \leq \lambda_1$. $\square$

Lemma 5.2.6. *Consider $\alpha = u(0) \in (\gamma_0, b)$, and let $\lambda_0 = \frac{\gamma_0}{M}$ and $M = max\{|f(u)| : 0 < u < b\}$. If $0 \leq \lambda \leq \lambda_0$, then problem (3.1) does not possess any positive solutions.*

Proof. u satisfies (3.1) if and only if u satisfies the integral equation

$$u(x) = \lambda \int_{-1}^{1} G(x,y) f(u(y))\, dy \tag{5.29}$$

where G is the Green's function for (3.1), i.e.,

$$G(x,y) = \begin{cases} \frac{1}{2}(x+1)(1-y), & -1 \leq x \leq y \leq 1, \\ \\ \frac{1}{2}(y+1)(1-x), & -1 \leq y \leq x \leq 1, \end{cases} \tag{5.30}$$

from which $G(x,y) \geq 0$ on $[-1,1] \times [-1,1]$ and $\int_{-1}^{1} G(x,y)\, dy \leq 1$.

Hence

$$\begin{aligned} |u(x)| &\leq \lambda \int_{-1}^{1} G(x,y) |f(u(y))|\, dy \\ &\leq \lambda\, max\{|f(u)| :\ 0 \leq u \leq b\} \int_{-1}^{1} G(x,y)\, dy \\ &\leq \lambda M. \end{aligned}$$

Hence, if $\gamma_0 < u(0) < b$, then

$$\begin{aligned} \gamma_0 &\leq |u(x)| \\ &\leq \lambda M. \end{aligned}$$

so,

$$\lambda > \frac{\gamma_0}{M}.$$

$\square$

Lemma 5.2.7. *Suppose that $\alpha = u(0) \in (\gamma_1, 1)$, and let $\lambda_1 = \frac{\gamma_1}{k}$, and $M = max\{|f(u)| :\ 0 < u < 1\}$. If $0 \leq \lambda \leq \lambda_1$, then problem (3.1) does not possess any positive solutions.*

Under our assumptions on f, there is a constant $\mu > 0$, such that $f(u) \leq \mu u$ for all $u > 0$.

Lemma 5.2.8. *If u is a non-trivial solution of (3.1), then* $\lambda \geq \frac{\pi^2}{4\mu}$

Proof.

$$\lambda\mu \int_{-1}^{1} u^2 \, dx \geq \lambda \int_{-1}^{1} f(u)udu = \int_{-1}^{1} u'^2 \, dx \geq \frac{\pi^2}{4} \int_{-1}^{1} u^2 \, dx,$$

so $\lambda \geq \frac{\pi^2}{4\mu}$. □

Lemma 5.2.9. *If $\alpha = u(0) \in (\gamma_0, b)$, then problem (3.1) has no non-trivial solutions for $0 \leq \lambda < max\{\frac{\pi^2}{4\mu}, \frac{\gamma_0}{M}, \frac{\gamma_0^2}{\int_a^b f(s)\,ds}\}$.*

Lemma 5.2.10. *If $\alpha = u(0) \in (\gamma_1, 1)$, then problem (3.1) has no non-trivial solutions for $0 \leq \lambda < max\{\frac{\pi^2}{4\mu}, \frac{\gamma_1}{M}, \frac{\gamma_1^2}{\int_a^b f(s)\,ds + \int_c^1 f(s)\,ds}\}$.*

5.3 Some generalizations

We shall be interested in the following Dirichlet boundary problem,

$$u'' + u(u - b_1)(u - b_2)...(u - b_{2n-1})(1 - u) = 0, \quad x \in (-1, 1),$$

$$u(-1) = u(1) = 0, \tag{5.31}$$

with constants $0 = b_0 < b_1 < b_2 < ...... < b_{2n-1} < b_{2n} = 1$, a positive parameter λ, and $n \geq 1$. The polynomial $f(u) = u(u - b_1)(u - b_2).....(u - b_{2n-1})(1 - u)$ has a negative hump over $(0, b_1)$ followed by a positive hump (b_1, b_2). We shall refer to them as a pair of humps. This polynomial has n pairs of humps. It is well known that each solution branch has its maximum value inside a single positive hump, and that it is necessary to have $\int_0^{b_2} f(u)\,du > 0$ in order for solutions with maximum value in (b_1, b_2) to exist.We shall assume this condition to hold, and similarly for the other humps, otherwise we can combine pairs of humps.

To use the bifurcation theory approach, we need to linearize (5.31). Consider the linearized problem for (5.31),

$$w''(x) + \lambda f'(u(x))w = 0 \text{ on } (-1,1), \quad w(-1) = w(1) = 0, \tag{5.32}$$

where $u(x)$ is a solution of (5.31). If $w(x)$ is a nontrivial solution of (5.32), then we call $u(x)$ a singular solution of (5.31). If $w(x) \equiv 0$ is the only solution of (5.32), we say that the solution $u(x)$ is nonsingular. In this section, we generalize some results from the previous section for the problem (5.31). We may repeat each proof with some modifications to conclude each of the followings results.

Lemma 5.3.1. *Any nontrivial positive solution of (3.1) satisfies* $u(0) \in \bigsqcup_{k=1}^{n} (\gamma_{k-1}, b_{2k})$, $n \geq 1$.

Lemma 5.3.2. *if* $u(0) \in (b_{2k-1}, b_{2k})$, *then* $\gamma_{k-1} < u(0) < b_{2k}$, $k = 1, 2, ..., n,\ \ n \geq 1$.

Suppose that the function $f(u)$ is concave up at b_{2k-2}, that is $f''(b_{2k-2}) = 0$, for some $k = 1, 2, ..., n$, $n \geq 1$ It is clear that there is exactly one point where a ray out of the point b_{2k-2} touches to the right the graph of $f(u)$. We denote this point by β_k, i.e. β_k is the only solution of equation

$$f'(\beta) = \frac{f(\beta)}{\beta - b_{2k-2}}.$$

Theorem 5.3.3. *Let $u(x)$ be any critical point of (5.32) with $u(x) \in (0, b_{2k})$, and suppose that $f''(b_{2k-2}) \leq 0$. Then*

$$u(0) > \beta_k.$$

From our assumptions there is $\gamma_{k-1} \in (b_{2k-1}, b_{2k})$ that satisfies $\int_{b_{2k-2}}^{\gamma_{k-1}} f(s)\,ds = 0$, $n \geq 1$, $k = 1, 2, ..., n$.

Theorem 5.3.4. $\lim_{\alpha \to b_{2k}^-} G(\alpha) = -\infty$, *i.e., near b_{2k}, there is no bifurcation point, for all $n \geq 1$, $k = 1, 2, ..., n$.*

Theorem 5.3.5. $\lim_{\alpha \to \gamma_{k-1}^+} G(\alpha) = +\infty$, *i.e., near γ_{k-1}^+, there is no bifurcation point, for all $n \geq 1$, $k = 1, 2, ..., n$.*

Lemma 5.3.6. *Suppose that $\alpha = u(0) \in (b_{2k-1}, b_{2k})$ and $\gamma_{k-1} \in (b_{2k-1}, b_{2k})$. Define $\lambda_{k-1} = \frac{\gamma_{k-1}^2}{\sum_{j=1}^{k} \int_{b_{2j-1}}^{b_{2j}} f(s)\,ds}$. If $0 \leq \lambda \leq \lambda_{k-1}$, then problem (3.1) does not possess any positive solutions.*

Lemma 5.3.7. *Assume $\alpha = u(0) \in (\gamma_k, b_{2k})$ and define $\lambda_k = \frac{\gamma_k}{M}$, where $M = max\{|f(u)| : \; 0 < u < b_{2k}\}$, $n \geq 1$, $k = 1, 2, ..., n$. If $0 \leq \lambda \leq \lambda_k$, then problem (5.31) does not possess any positive solutions.*

Lemma 5.3.8. *Suppose $\alpha = u(0) \in (\gamma_k, b_{2k})$, then the problem (5.31) possesses no non-trivial solutions for $0 \leq \lambda < max\{\frac{\pi^2}{4\mu}, \frac{\gamma_k}{M}, \frac{\gamma_k^2}{\sum_{j=1}^{k} \int_{b_{2j-1}}^{b_{2j}} f(s)\,ds}\}$, where $M = max\{|f(u)| : \; 0 < u < b_{2k}\}$, $n \geq 1$, $k = 1, 2, ..., n$.*

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