Spring 2011

TQFT diffeomorphism invariants and skein modules

Paul Harlan Drube
University of Iowa

TQFT DIFFEOMORPHISM INVARIANTS AND SKEIN MODULES

by

Paul Harlan Drube

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

May 2011

Thesis Supervisor: Professor Charles Frohman

ABSTRACT

There is a well-known correspondence between two-dimensional topological quantum field theories (2-D TQFTs) and commutative Frobenius algebras. Every 2-D TQFT also gives rise to a diffeomorphism invariant of closed, orientable two-manifolds, which may be investigated via the associated commutative Frobenius algebras. We investigate which such diffeomorphism invariants may arise from TQFTs, and in the process uncover a distinction between two fundamentally different types of commutative Frobenius algebras ("weak" Frobenius algebras and "strong" Frobenius algebras). These diffeomorphism invariants form the starting point for our investigation into marked cobordism categories, which generalize the local cobordism relations developed by Dror Bar-Natan during his investigation of Khovanov's link homology.

We subsequently examine the particular class of 2-D TQFTs known as "universal $sl(n)$ TQFTs". These TQFTs are at the algebraic core of the link invariants known as $sl(n)$ link homology theories, as they provide the algebraic structure underlying the boundary maps in those homology theories. We also examine the 3-manifold diffeomorphism invariants known as skein modules, which were first introduced by Marta Asaeda and Charles Frohman. These 3-manifold invariants adapt Bar-Natan's marked cobordism category (as induced by a specific 2-D TQFT) to embedded surfaces, and measure which such surfaces may be embedded within in 3-manifold (modulo Bar-Natan's local cobordism relations). Our final results help to characterize the structure of such skein modules induced by universal $sl(n)$ TQFTs.

Abstract Approved: ______________________________
Thesis Supervisor

____
Title and Department

____
Date

TQFT DIFFEOMORPHISM INVARIANTS AND SKEIN MODULES

by

Paul Harlan Drube

A thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

May 2011

Thesis Supervisor: Professor Charles Frohman

Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Paul Harlan Drube

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the May 2011 graduation.

Thesis Committee:

Charles Frohman, Thesis Supervisor

Dan Anderson

Keiko Kawamuro

Richard Randell

Maggy Tomova

ACKNOWLEDGEMENTS

I'd like to thank my family for all of the support that they have given me throughout my mathematical education. I'd also like to thank my advisor, Charles Frohman, for all of the helpful suggestions and feedback. Without the encouragement of the people around me, this thesis would not have been possible.

ABSTRACT

There is a well-known correspondence between two-dimensional topological quantum field theories (2-D TQFTs) and commutative Frobenius algebras. Every 2-D TQFT also gives rise to a diffeomorphism invariant of closed, orientable two-manifolds, which may be investigated via the associated commutative Frobenius algebras. We investigate which such diffeomorphism invariants may arise from TQFTs, and in the process uncover a distinction between two fundamentally different types of commutative Frobenius algebras ("weak" Frobenius algebras and "strong" Frobenius algebras). These diffeomorphism invariants form the starting point for our investigation into marked cobordism categories, which generalize the local cobordism relations developed by Dror Bar-Natan during his investigation of Khovanov's link homology.

We subsequently examine the particular class of 2-D TQFTs known as "universal $sl(n)$ TQFTs". These TQFTs are at the algebraic core of the link invariants known as $sl(n)$ link homology theories, as they provide the algebraic structure underlying the boundary maps in those homology theories. We also examine the 3-manifold diffeomorphism invariants known as skein modules, which were first introduced by Marta Asaeda and Charles Frohman. These 3-manifold invariants adapt Bar-Natan's marked cobordism category (as induced by a specific 2-D TQFT) to embedded surfaces, and measure which such surfaces may be embedded within in 3-manifold (modulo Bar-Natan's local cobordism relations). Our final results help to characterize the structure of such skein modules induced by universal $sl(n)$ TQFTs.

TABLE OF CONTENTS

LIST OF FIGURES

Figure

CHAPTER 1
INTRODUCTION

1.1 Background and Motivation

Modern knot theory owes much of its vitality to the introduction of the Jones Polynomial, as first presented by V. F. R. Jones in [8]. That link invariant assigns a Laurent polynomial to every equivalence class of oriented links, and was unique at the time in that it was completely determined by a so-called local skein relation. That skein relation related the Jones polynomials of link diagrams that differed only at a single crossing, allowing the polynomial to be calculated (with relative ease) in a purely combinatorial matter.

In [11], Mikhail Khovanov introduced his groundbreaking homology theory for links. This theory, which assigned an entire sequence of homology groups to an equivlance class of links, was influential largely in that it represented a "categorification" of the Jones polynomial. Here the term "categorification" means that the Euler characteristic of the link homology groups recovers the link's Jones polynomial. In particular, Khovanov's homology carries strictly more information than the Jones polynomial, distinguishing all knots that are distinguishable by the Jones polynomial as well as many knots that aren't distinguishable by the Jones polynomial.

Khovanov's link homology directly influences the topics of this thesis via the boundary maps in its chain complex. As originally noticed by Dror Bar-Natan in [3], those boundary maps could be interpreted as two-dimensional surfaces (cobordisms) connecting a pair of closed one-dimensional manifolds (two state-spaces of the link that result from "smoothing" all crossings). The mathematical structure underlying these boundary maps were then interpreted via two-dimensional topological quantum field theory, which functorially assigns algebraic data to all such 2-D cobordisms.

Variations on Khovanov's link homology soon appeared [12],[18],[14], with many of the new homology theories being "reverse engineered" from the particular two-dimensional topological quantum field theories that were chosen for their boundary maps. All of these related theories shared many basic features, and they collectively became known as $sl(n)$ link homology theories because of their relationship to the $sl(n)$ link polyonomials. Similarly, the topological quantum field theories underlying their boundary maps were all of a particular form, here conveniently referred to as "universal $sl(n)$ topological quantum field theories".

The underlying topological quantum field theories soon developed into an interesting topic in their own right, and this is the primary stream of research that we follow in this thesis. In particular, Bar-Natan's cobordism interpretation of Khovanov's boundary operator spawned the study of local cobordism relations and skein module diffeomorphism invariants of three-manifolds. This field of study was pioneered by Marta Asaeda and Charles Frohman in [1], and this thesis culminates in a generalization of several fundamental results from their work.

1.2 Basic Definitions and Theorems

In this section we introduce the basic theoretical framework that informs my own work from Chapters 2 and 3. We begin in Subsection 1.2.1 with the formal definition of an n-dimensional topological quantum field theory (TQFT), which requires a modest amount of category theory. Wishing to focus upon TQFTs of dimension $n = 2$, Subsection 1.2.2 introduces commutative Frobenius algebras and recaps the foundational results that relate such algebras to 2-D TQFTs. Subsection 1.2.3 introduces skein modules as well as Bar-Natan's marked cobordism category, which will play a central role in Chapter 3.

1.2.1 Topological Quantum Field Theories

The notion of a Topological Quantum Field Theory, or TQFT, was formally introduced by Atiyah in [2]. Here we introduce the technical definition of such a theory as a symmetric monoidal functor from the n-dimensional oriented cobordism category to the category of vector spaces over some fixed field k. For a more detailed development of these topics, see [2] or [17]; for a rigorous treatment of basic concepts from category theory, see [19].

We begin by recalling the definition of a **symmetric monoidal category**, which is a category $\mathbf{C}$ in possession of a commutative and associative product functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ with unit object $1_{\mathbf{C}} \in \mathbf{C}$. In particular, there exist isomorphisms:

1. $C \otimes C' \simeq C' \otimes C$ for all $C, C' \in \mathbf{C}$.

2. $(C \otimes C') \otimes C'' \simeq C \otimes (C' \otimes C'')$ for all $C, C', C'' \in \mathbf{C}$.

3. $C \otimes 1_{\mathbf{C}} \simeq C$ for all $C \in \mathbf{C}$.

Note that, as the map $\otimes$ is a functor, it can also be used to take the product of two morphisms in $\mathbf{C}$. As the conditions above feature isomorphisms instead of equals signs, also note that we are actually working with a non-strict symmetric monoidal category. As such, the isomorphisms themselves are required to satisfy a set of coherence conditions that take the form of commutative diagrams (see [19]). Luckily, these additional considerations do not impact this thesis, and in fact one may show that every non-strict symmetric monoidal category is naturally equivalent to a strict monoidal category [19].

Given symmetric monoidal categories $\mathbf{C}$ and $\mathbf{D}$, equipped (respectively) with products $\otimes_{\mathbf{C}}$ and $\otimes_{\mathbf{D}}$, a **symmetric monoidal functor** is a functor $F : \mathbf{C} \to \mathbf{D}$ that preserves symmetric monoidal structure. Namely, there exist isomorphisms:

1. $F(C \otimes_{\mathbf{C}} C') \simeq F(C) \otimes_{\mathbf{D}} F(C')$ for all $C, C' \in \mathbf{C}$.

2. $F(1_{\mathbf{C}}) \simeq 1_{\mathbf{D}}$.

Here we distinguish the two tensor product operations for emphasis. From now on, we will suppress the subscript when it is clear from context.

The two symmetric monoidal categories that play a central role in topological quantum field theory are $\mathbf{Vect(k)}$ and $\mathbf{nCob}$. For k a field, $\mathbf{Vect(k)}$ is the usual category of vector spaces over k. $\mathbf{Vect(k)}$ has a symmetric monoidal structure with respect to normal tensor product of vector spaces (over k). The standard isomorphism $k \otimes_k V \simeq V$ shows that $1_{\mathbf{Vect(k)}} = k$ serves as the unit object with respect to this product. **nCob**, the n-dimensional cobordism category, is defined as follows:

Definition 1.2.1. *For any positive integer n, define a symmetric monoidal category* ***nCob*** *with:*

- *Objects are closed, oriented (n-1)-manifolds.*

- *Given objects $M, N \in$* ***nCob****, morphisms from M to N are n-dimensional cobordisms from M to N. If $\bar{M}$ denotes the manifold M with orientation reversed, such a cobordism is an oriented n-manifold B with an (orientation-preserving) diffeomorphism $\partial B \simeq \bar{M} \sqcup N$. All bordisms are taken modulo orientation-preserving diffeomorphisms $B \simeq B'$ that extend to the obvious diffeomorphism $\partial B \simeq \partial B'$.*

- *Given morphisms $B : M \to M'$ and $B' : M' \to M''$, composition is given by gluing along the common boundary: $B' \circ B : M \to M''$, $B' \circ B = B \coprod_{M'} B'$.*

- *Symmetric monoidal structure given by disjoint union of manifolds, with the empty (n-1)-manifold acting as the unit object.*

One may visualize a cobordism $B : M \to N$ as an n-dimensional surface embedded in $\mathbb{R}^n \times I$, "going up" from $\partial_0 B = B \cap (\mathbb{R}^n \times 0) = M$ to $\partial_1 B = B \cap (\mathbb{R}^n \times 1) = N$. Composition of morphisms in **nCob** is then simply vertical "stacking" of compatible corbordisms, while the identity morphism $id_M : M \to M$ is diffeomorphic to the cylinder $M \times I$. To ensure that gluing is well behaved, in the embedded case we further require that B is tubular near both of its ends: $B \cap (\mathbb{R}^n \times [0, \delta]) = M \times [0, \delta]$

and $B \cap (\mathbb{R}^n \times [1-\delta, 1]) = N \times [1-\delta, 1]$ for some small $\delta > 0$. For the majority of our discussion any sort of embedding will be forgotten so that such technical gluing conditions may be ignored (although "top" and "bottom" will still be identifiable via the convention that the orientation on B disagrees with the orientation on the "in" boundary M and agrees with the orientation on the "out" boundary N).

We are now ready for the definition of a TQFT:

Definition 1.2.2. *A k-valued* **n-dimensional topological quantum field theory** *(n-D TQFT) is a symmetric monoidal functor from* ***nCob*** *to* ***Vect(k)***. *In particular, it is a functor Z such that:*

- *For an (n-1)-dimensional closed oriented manifold M, $Z(M)$ is a k-vector space.*
- *For an n-dimensional cobordism $B : M \to N$, $Z(B)$ is a k-linear map from $Z(M)$ to $Z(N)$.*
- *For a composition of n-dimensional cobordisms $M' \circ M$, there is a composition of k-linear maps $Z(M') \circ Z(M)$.*
- *For any disjoint union $M \sqcup N$, $Z(M \sqcup N) = Z(M) \otimes_k Z(N)$.*
- *$Z(\emptyset) = k$.*

Note that if a pair of oriented n-manifolds M and N are diffeomorphic via a orientation-preserving diffeomorphism that extends to their boundaries, then $Z(M) = Z(N)$ by the definition of **nCob**. Functoriality of Z also ensures that $Z(A) \simeq Z(B)$

for a pair of diffeomorphic closed, oriented (n-1)-manifolds, although here we don't necessarily have equality.

Now take any closed, oriented (n-1)-manifold M and consider the n-manifold $B = M \times I$, a manifold with boundary $\partial B = M \sqcup \bar{M}$. A TQFT Z offers several basic identifications for $Z(B)$, as there is a distinct cobordism for each partition of the two boundary components of B. The simplest identification for $Z(B)$ follows from the "cylinder" bordism from M to M (or equivalently from $\bar{M}$ to $\bar{M}$), which corresponds to the identity map 1_M (respectively $1_{\bar{M}}$). We may also identify $Z(B)$ as an "evaluation" map $ev_{Z(M)}$ from $Z(M \sqcup \bar{M}) \simeq Z(M) \otimes Z(\bar{M})$ to $Z(\emptyset) \simeq k$, resulting in a bilinear pairing of $Z(M)$ with $Z(\bar{M})$, or as a "coevaluation" map $coev_{Z(M)}$ from k to $Z(M) \otimes Z(\bar{(}M))$. The bilinear pairing of $ev_{Z(M)}$ prompts the following well-known proposition, whose proof I have based largely upon the one from [17]:

Proposition 1.2.3. *Let Z be an n-dimensional TQFT, and let M be a closed, oriented (n-1)-manifold. If $\bar{M}$ is M with orientation reversed, then $Z(\bar{M}) \simeq Z(M)^*$, where $Z(M)^*$ is the dual space of $Z(M)$.*

Proof. Fix M and let $B = M \times I$. If we define $ev_{Z(M),b} : Z(M) \to k$ by $ev_{Z(M),b}(a) = ev_{Z(M)}(a, b)$ for any $b \in Z(\bar{M})$, the pairing $ev_{Z(M)}$ induces a k-linear map $\alpha : Z(\bar{M}) \to Z(M)^*$ via $\alpha(b)(a) = ev_{Z(M),b}(a)$. Utilizing the natural bilinear pairing $\gamma : Z(M)^* \otimes Z(M) \to k$, we then define $\beta : Z(M)^* \to Z(\bar{M})$ as the composition: $(\gamma \otimes 1_{Z(\bar{M})}) \circ (1_{Z(M)^*} \otimes coev_{Z(M)}) : Z(M)^* \otimes k \to Z(M)^* \otimes Z(M) \otimes Z(\bar{M}) \to k \otimes Z(\bar{M})$. It can be shown that β is the inverse of α, making α an isomorphism. $\square$

Another well-known result that will immediately prove useful is the following. Once again, my proof follows that from [17]

Proposition 1.2.4. *Let Z be an n-dimensional TQFT, and let M be a closed, oriented (n-1)-manifold. Then $Z(M)$ is a finite-dimensional vector space over k.*

Proof. Using the same notation as from Proposition 1.2.3, we focus upon the middle space $Z(M)^* \otimes Z(M) \otimes Z(\bar{M})$ in the definition of β. As we may rewrite any element $v \in Z(M) \otimes Z(\bar{M})$ so that $v \in Z(M) \otimes W$ for some finite-dimensional subspace $W \subseteq Z(\bar{M})$, the image of the last part of β in $Z(\bar{M})$ is finite-dimensional. As β is already known to be an isomorphism by 1.2.3, it follows that $Z(\bar{M}) \simeq Z(M)^*$ and hence that $Z(M)$ is finite-dimensional. $\square$

More generally, now consider any oriented n-manifold B along with a partition of its boundary into two disjoint components: $\partial B = M_1 \sqcup M_2$. A TQFT Z may then be used to identify B with a k-linear map $Z(M_1) \to Z(M_2)$. Of particular interest is the case where $M_1 = \partial B$ and $M_2 = \emptyset$, which allows us to identify $Z(B)$ with a k-linear map $Z(\partial B) \to k$. This is simply an element of the dual vector space $Z(\partial B)^*$, which is isomorphic to $Z(\partial B)$ by Proposition 1.2.4. Thus we have the identification $Z(B) \in Z(\partial B)$ for any oriented n-manifold B. If B is closed, this becomes an identification of $Z(B)$ with an element of the base field k. In other words, Z gives a k-valued diffeomorphism invariant of closed n-manifolds. What types of k-valued diffeomorphism invariants may arise from a TQFT in this manner will be the primary topic of Chapter 2.

1.2.2 2-D TQFTs and Commutative Frobenius Algebras

In this thesis we will focus upon TQFTs of dimension $n = 2$, which we know to be symmetric monoidal functors $Z : \mathbf{2Cob} \to \mathbf{Vect(k)}$. An object of **2Cob** is an closed, oriented 1-manifold, which is necessarily a disjoint union of copies of S^1. It follows that any 2-D TQFT Z is completely determined, on the object level, by the assignment $Z(S^1) = A \in \mathbf{Vect(k)}$. In particular, $Z(M) \simeq A^{\otimes J}$ for the 1-manifold consisting of J disjoint circles. Morphisms in **2Cob** are cobordisms between J copies of S^1 and I copies of S^1 for some non-negative integers I, J. Following Khovanov [11], we let S^I_J denote the (diffeomorphism class of the) connected cobordism of minimal genus from J circles to I circles. Some such elementary cobordisms are shown below in Figure 1.1. Note that we distinguish S^1_2 and S^2_1, despite the fact that both are diffeomorphic to a copy S^2 with three boundary components, as we wish to identify each with a different k-linear map between its "in" and "out" boundary components:

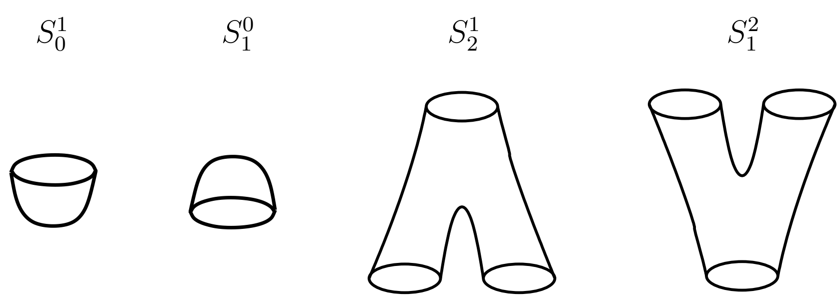


Figure 1.1: Elementary Cobordisms

Our TQFT sends these surfaces to the k-linear maps:

$$Z(S^1_0) = \iota : k \to A \qquad Z(S^0_1) = \varepsilon : A \to k$$

$$Z(S^1_2) = m : A \otimes A \to A \qquad Z(S^2_1) = \Delta : A \to A \otimes A$$

The maps above grant A both a multipication m and a comultiplication δ, with unit map ι and counit map (trace) ε. It can be shown that these operations are commutative and associative, ensuring that A has additional structure as both a k-algebra and a k-coalgebra (see [15] for a thorough discussion). As we will see in Theorem 1.2.7, A has the full structure of a commutative Frobenius algebra. This prompts the following definition:

Definition 1.2.5. *Let k be a field. A* **commutative Frobenius algebra** *is a finite-dimensional, commutative k-algebra that satisfies any of the following, equivalent, properties:*

1. *There exists a k-linear map $\varepsilon : A \to k$ such that $Null(\varepsilon)$ contains no nontrivial left ideals.*

2. *There exists a nondegenerate bilinear pairing $\beta : A \otimes A \to k$.*

3. *There exists a left (equivalently right) A-linear isomorphism between A and A^*.*

The map ε is known as the **Frobenius form**, or trace map, and is said to be nondegenerate if it satisfies the condition above. The map β is known the Frobenius pairing and is defined in terms of the Frobenius form as $\beta(a, a') = \varepsilon(aa')$. The equivalence of the conditions above is widely available in references such as [15]. One important consequence of these conditions, as proven in [15], is that every commutative Frobenius algebra (A, ε) admits a cocommutative comultiplication $\Delta : A \to A \otimes A$ whose counit $\varepsilon : A \to k$ is the Frobenius form ε.

If A is an n-dimensional Frobenius algebra ($n < \infty$) with basis $\{x_1, ..., x_n\}$, nondegeneracy of ε is equivalent to the invertibility of the matrix $\lambda = [[\varepsilon(x_i x_j)]]$. We henceforth refer to λ as the **Frobenius matrix** of ε. Given a nondegenerate Frobenius form ε and a basis $\{x_1, ..., x_n\}$, λ can be used to determine a dual-basis relative to ε: a second k-linear basis such that $\varepsilon(x_i y_i) = 1$ and $\varepsilon(x_i y_j) = 0$ whenever $i \neq j$. A commutative Frobenius algebra A, along with a nondegenerate trace ε and a choice of dual-bases (x_i, y_i), is collectively referred to as the **Frobenius system**. We write the Frobenius system in compact form as $(A, \varepsilon, (x_i, y_i))$, or simply as (A, ε) when our results are independent of a choice of basis. For any choice of dual-bases (x_i, y_i), we always have $\Delta(1) = \sum_i x_i \otimes y_i$.

An equivalence of Frobenius systems (A, ε) and $(\tilde{A}, \tilde{\varepsilon})$ is a k-algebra isomorphism $\phi : A \to \tilde{A}$ such that $\varepsilon \equiv \tilde{\varepsilon} \circ \phi$. It is important to note that, since a fixed algebra A may admit numerous non-equivalent Frobenius structures, it is actually the entire Frobenius systems that form the object set in the category of Frobenius algebras. It may be shown (see [15]) that every Frobenius equivalence extends to a coalgebra isomorphism: that $\varepsilon \equiv \tilde{\varepsilon} \circ \phi$ ensures $\Delta \equiv \tilde{\Delta} \circ \phi$ for the respective comultiplication maps.

Here we pause to point out several additional properties of commutative Frobenius algebras. The first of these properties follows from condition 3 in Definition 1.2.5, which prompts a k-linear isomorphism $A \otimes A \cong A \otimes A^* \cong End(A)$ given by

$a \otimes b \mapsto a\varepsilon(b\_)$. This linear isomorphism actually extends to a k-algebra isomorphism if one defines a multiplication operation (called the ε-multiplication) on $A \otimes A$ by $(a \otimes b)(a' \otimes b') = a\varepsilon(ba') \otimes b' = a \otimes \varepsilon(ba')b'$. One final property satisfied by any commutative Frobenius algebra that follows from a consideration of the ε-multiplication is the following, which is carefully established in standard resources such as [9]:

Proposition 1.2.6. *Given a Frobenius system (A, ε) as well as any choice of dual-bases (x_i, y_i), then $a = \sum_i x_i \varepsilon(y_i a) = \sum_i \varepsilon(a x_i) y_i$ for all $a \in A$*

This property will prove to be especially important in Subsection 1.2.3 and then again in Chapter 3, where it will directly relate to the "neck-cutting" relations of the associated marked cobordism category. A related consequence from our definition of ε-multiplication is that $\varepsilon = \tilde{\varepsilon}$ iff $\sum(x_i \otimes y_i) = \sum(\tilde{x}_i \otimes \tilde{y}_i)$, as both sums serve as the unit element for the ε-multiplication.

Our interest in commutative Frobenius algebras is the following theorem, the idea of which dates back to Dijkgraaf [6]. For a full proof of the result (a completely constructive and rather tedious exercise), see standard texts such as [15]

Theorem 1.2.7. *For any 2-D TQFT Z, $Z(S^1) = A$ is a commutative Frobenius algebra. Furthermore, the category of 2-D TQFTs over k is equivalent to the category of commutative Frobenius k-algebras.*

Proof. Equivalence follows by taking Z to $Z(S^1) = A$ and the "cap" cobordism S^0_1 to the Frobenius form ε. $\square$

In Chapters 2 and 3, we will use Theorem 1.2.7 to recast many topological results concerning TQFTs in purely algebraic terms. Yet in order to motivate this approach, it is helpful to introduce Bar Natan's marked cobordism category and local skein relations.

1.2.3 Skein Modules

As first presented by Bar-Natan [3], the correspondence between 2-D TQFTs and Frobenius systems may be used to pictorially represent algebraic properties of a Frobenius system $(A, \varepsilon, (x_i, y_i))$ in terms of local relations on "marked" 2-D surfaces.

Bar-Natan's approach utilizes a modification of the category **2Cob** that consists of "decorated" cobordisms $\mathbf{2Cob}_A$. The objects of $\mathbf{2Cob}_A$ are the same as those in **2Cob**, but the morphisms in this new category may be "marked" by elements of A (a surface is said to be "unmarked" if marked by $1 \in A$). These markings may be moved around upon a fixed component of a surface and may be multiplied together (or factored) on that component, but may not "jump" between distinct components of the same cobordism. As A is a k-algebra, we also take $\mathbf{2Cob}_A$ to be a k-linear category. By convention, we choose to write elements of $k \subseteq A$ in front of the surface. This convention is followed even if the given cobordism has multiple components: recall that a disjoint union of cobordisms corresponds to tensor product, and as tensor product is k bilinear it doesn't matter which component of the cobordism a marking in k originated from.

In order to accommodate the various algebraic properties of $(A, \varepsilon, (x_i, y_i))$, a quotient category $\mathbf{2Cob}_A/l$ is formed via three sets of local relations l. The effect of these relations is to identify (marked) surfaces that are evaluated similarly by the associated 2-D TQFT. All of this was originally done with respect to the rank-2 Frobenius algebra underlying Khovanov's link homology, a system with underlying algebra $A = \mathbb{C}[x]/(x^2)$ and trace ε defined on the standard basis $\{1, x\}$ by $\varepsilon(1) = 0$, $\varepsilon(x) = 1$. This structure gives a particularly simple dual basis of $\{(1, x), (x, 1)\}$. Using Bar-Natan's original Frobenius system as a clarifying example, we describe these three sets of local relations below.

1) Sphere Relations: In Subsection 1.2.2 we noted how a 2-D TQFT identifies the "cap" cobordism S_1^0 with the Frobenius map $\varepsilon : A \to k$. Similarly, the "cup" cobordism S_0^1 is identified with the unit map $u : k \to A$. As $a = a * u(1)$ for any $a \in A$, a cup marked by a is interpreted as precomposition by a (here $*$ simply denotes multiplication in A). These observations combine to yield the so-called "sphere relations", which associate a sphere marked by a with the element $\varepsilon(a) \in k$. More formally, they say that we may remove an unmarked sphere component from any cobordism at the cost of multiplying the rest of the cobordism by $\varepsilon(a) \in k$. In Bar-Natan's original Frobenius system we have $\varepsilon(1) = 0$ and $\varepsilon(x) = 1$ for the k-linear basis $\{1, x\}$, so that all "sphere relations" are generated by the two relations below. Note that, following Bar-Natan's original notation, we use a dot to denote a surface marked by $x \in A$.

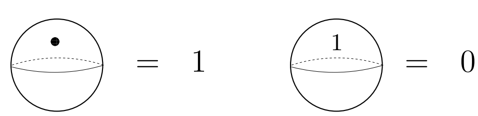


Figure 1.2: Bar-Natan's Sphere Relations

2) "Dot Reduction" Relation: This relation allows us to re-label any component of a cobordism by an equivalent element of A. In Bar-Natan's case, there is a single generating relation in $x^2 = 0$, and hence all such re-labelings are generated by the double "dot reduction" below:

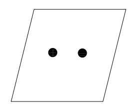
 $= \quad 0$

Figure 1.3: Bar-Natan's "Dot Reduction" Relation

3) Neck-Cutting Relation: The so-called "neck-cutting" relation may be applied to any cobordism component N that admits a compression disk- a two-dimensional disk D^2 such that $\delta D^2 \subset N$ and the interior of D^2 is disjoint from N. Formally, the effected component is "cut" along the compression disk D^2, with a regular neighborhood of D^2 (an annulus with core D^2) being replaced by two-copies of D^2 that are glued in along the two new boundary components [1].

Bar-Natan's original neck-cutting relation is shown below. Algebraically, it is a direct consequence of the Frobenius algebra property from Proposition 1.2.6, which states that $a = \sum_i x_i \varepsilon(y_i a)$ for all $a \in A$. We interpret this as an equality between two endomorphisms of A. The left-hand side of the equation is simply the identity

[1] If the cobordisms on the left- and right-hand sides of the resulting equality are interpreted as the two boundary components of a 3-manifold with corners (a 3-cobordism between the left and right sides), then this neck-cutting operation represents nothing more than the attachment of a 2-handle.

map $1_A : A \to A$, which is mapped to a cylinder via the associated TQFT. The right-hand side of the equation is a sum of terms that apply $\varepsilon(x_i\_)$ to the input and then output $y_i = y_i u(1)$. Via the associated TQFT, these respectively correspond to a cap decorated by x_i and then a cup decorated by y_i.

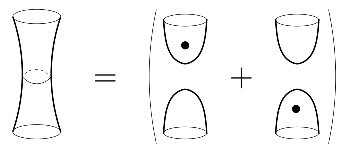


Figure 1.4: Bar-Natan's Neck-Cutting Relation

In Chapter 3 we deal with generalizations of these local relations to the entire family of "universal $sl(n)$ Frobenius extensions", while even broader generalizations of the relations to an arbitrary Frobenius system will be implicit in much of what we do in Chapter 2. In both chapters, the cobordisms of interest will be closed. No matter the underlying Frobenius algebra, if we consider these cobordisms abstractly (i.e.- not embedded within some higher-dimensional space) such local relations allow us to evaluate any closed cobordism (marked or unmarked) to an element of k. Neck-cutting is used to compress all surfaces down to a disjoint union of spheres, which may then be evaluated to k via the dot-reduction relation and spheres relations. By construction, for unmarked closed surfaces M this evaluation always coincides with the value $Z(M)(1) \in k$ from the diffeomorphism invariant of the associated 2-D TQFT Z.

One application of Bar-Natan's marked cobordism category are the 3-manifold diffeomorphism invariants known as skein modules, which were originally introduced by Asaeda and Frohman in [1]. Given a 3-manifold M and a Frobenius system (A, ε), the skein module $K_A(M)$ of M induced by (A, ε) is a k-module generated by the isotopy classes of marked surfaces embedded in M, and taken mudolo the same relations as from the marked cobordism category $\mathbf{2Cob}_A/l$. As such, these invariants identify marked, closed cobordisms that "evaluate similarly" via the 2-D TQFT associated to (A, ε), although now our cobordisms are no longer in ambient space and thus the choice of embedding matters.

The shift to embedded surfaces presents two significant differences when dealing with Bar-Natan's original local relations. First off, the sphere relations may only be applied to spheres that bound balls. Secondly, many higher genus surfaces may not possess a compression disk. These differences are actually what make the skein modules interesting: if we were able to compress all closed surfaces down to spheres and then apply the sphere relations, all skein modules would be trivial!

In [1], Asaeda and Frohman explore skein modules induced by the specific Frobenius system underlying Khovanov homology. Thus the local relations on their surfaces are exactly the same as Bar-Natan's from earlier in this subsection. The existence of skein modules $K_A(M)$ for any underlying Frobenius system (A, ε) were thoroughly established by Kaiser in [10] [2]. At the end of Chapter 3, we will ap-

[2]Kaiser treated the more general case of Frobenius extensions over an arbitrary commutative ring R (as opposed to a field $R = k$)- an approach we will develop in Section 3.1.

ply many of our new results about "universal $sl(n)$ Frobenius extensions" towards the skein modules induced by those Frobenius systems. The result will be several theorems that greatly generalize facts about the "Bar-Natan skein module" in [1].

1.3 Thesis Outline

This thesis presents two related investigations into the realm of 2-D TQFTs and their associated commutative Frobenius algebras. In Chapter 2, we focus upon the observation from the end of Subsection 1.2.1 that every 2-D TQFT comes equipped with diffeomorphism invariant of closed, oriented 2-manifolds. Which diffeomorphism invariants may arise in such a way are comprehensively characterized. This topic, which (shockingly) hasn't been touched upon anywhere in the literature, is especially interesting in that it reveals a deep indebtedness of topological quantum field theory to combinatorics and symmetric polynomials.

In Chapter 3 we alternatively expand upon the marked cobordism category and skein module discussion of Subsection 1.2.3. The results above are generalized to an entire class of "universal $sl(n)$ TQFTs". This work will require us to generalize the notion of a commutative Frobenius algebra to ring extensions known as Frobenius extensions, which allow us to replace the base field k by an arbitrary commutative ring (with 1) R. Our exploration of "universal $sl(n)$ skein modules" will exhibit a direct application of our work from Chapter 2, as the diffeomorphism invariant underlying any 2-D TQFTs forms a fundamental part of any marked cobordism category. We close with several results categorizing "universal $sl(n)$ skein modules".

CHAPTER 2
DIFFEOMORPHISM INVARIANTS FROM TQFT

2.1 Introduction

At the end of Subsection 1.2.1 we noted how, for any oriented n-manifold B, an n-dimensional TQFT Z allows us to identify $Z(B)$ with an element of the k-vector space $Z(\delta B)$. In the case of a closed n-manifold we have $Z(\delta B) = Z(\emptyset) = k$ and hence an identification $Z(B) \in k$. As morphisms in the cobordism category **nCob** are taken modulo orientation-preserving diffeomorphisms, it follows that every n-dimensional TQFT comes equipped with a k-valued diffeomorphism invariant for closed orientable n-manifolds. Our primary goal in this chapter is characterize which such invariants may arise from TQFTs in the case of dimension $n = 2$, as well as to investigate what these invariants can tell us about the underlying TQFT Z. Some of our results here will require that k be algebraically closed, so for the remainder of this chapter we will take $k = \mathbb{C}$.

We begin by noting that, as every closed orientable 2-manifold is diffeomorphic to the genus-i surface Σ_i (with positive orientation) for some integer $i \geq 0$, our invariant takes the form of a function $f : \mathbb{N} \to k$. If $f = f_Z$ is the diffeomorphism invariant associated to the 2-D TQFT Z, we henceforth refer to f_Z as the **Frobenius function** of Z (or equivelently as the Frobenius function of the associated Frobenius system). Restated, our primary goal of this chapter is then:

Question 2.1.1. *Which functions $f : \mathbb{N} \to k$ may be realized as the Frobenius func-*

tion of some 2-D TQFT Z?

In Section 2.3 we will quickly see that any potential Frobenius function f must possess a recurrence relation of degree equal to the value $f(1) = n$. Beyond this, our investigation will reveal two fundamentally different classes of Frobenius systems that yield easily distinguishable types of Frobenius functions: "strong" systems whose functions possess a minimal-order recurrence of degree precisely n, and "weak" systems whose functions admit a lower-order recurrence. The answer to Question 2.1.1 will be drastically different depending upon whether the associated Frobenius system is strong or weak. Briefly stated, our primary results of this chapter will be:

Theorem 2.1.2. *Let $f : \mathbb{N} \to k$ be a function such that $f(1) = n$. Then f is a Frobenius function iff:*

1. *$f(1) = n$ is a positive integer.*

2. *f possesses a recurrence relation of degree n.*

3. *f admits a recurrence relation of degree strictly less than n*

(OR)

f has a minimal-order recurrence of degree precisely n and there exist constants $\gamma_1, ..., \gamma_n \in k$ such that:

   (a) $f(0)e_n = e_{n-1}$

   (b) $f(i) = p^{i-1}$ *for all* $i \geq 2$

Where e_i is the i^{th} elementary symmetric polynomial in the γ_i and p_i is the i^{th} power symmetric polynomial in the γ_i.

To close the chapter, we will apply the machinery of Section 2.3 to answer the so-called "Vacuum Hypothesis", which asks whether the Frobenius algebra $Z(S^1) = A$ associated to a certain 2-D TQFT Z may be naturally identified with the algebra of 2-D surfaces that have S^1 as their boundary (modulo surfaces that are "evaluated similarly" by Z). This hypothesis will prove to be true iff the associated Frobenius system is strong, giving some topological credence to the algebraic strong/weak distinction that we develop in Section 2.3:

Theorem 2.1.3. *Let Z be a 2-D TQFT with associated Frobenius system (A, ε), so that $Z(S^1) = A$. If (A, ε) is a strong Frobenius system, then it is Frobenius equivalent to the algebra of orientable 2-D surfaces M such that $\partial M = S^1$, modulo surfaces that are "evaluated equivalently" by Z. If (A, ε) is a weak Frobenius system, then that same algebra of 2-D surfaces is isomorphic to some proper sub-algebra of A (which may or may not admit the natural Frobenius structure).*

2.2 "Genus Reduction"

In Subsection 1.2.3 we introduced a series of local relations for "marked" cobordisms, which identified surfaces that were "evaluated similarly" by the 2-D TQFT in question. One of those relations was a neck-cutting relation, which allowed us to compress along a compression disk and replace a component of the effected cobordism by a direct sum of cobordisms. Here we consider the case where the "top" and "bottom" on the right side of the neck-cutting relation are of the same component, so that neck-cutting amounts to eliminating a 1-handle on the given component. All of this is done with respect to a general 2-D TQFT, not the specific TQFT underlying Khovanov homology that we treated in Subsection 1.2.3

The first step here is to notice that a 1-handle can be represented as a composition of the "cup" cobordism S_0^1 with the "upside-down pants" cobordism S_1^2 and then the "pants" cobordism S_2^1. After applying the 2-D TQFT Z, this composition of cobordisms corresponds to the k-linear map $m \circ \Delta \circ u : k \to A$, which we identify with the element $m(\Delta(1)) \in A$ via the image of 1 (see Figure 2.1). The effect of removing a 1-handle, after passing to the associated Frobenius system (A, ε), then amounts to multiplication by the element $m(\Delta(1)) = g \in A$. We henceforth refer to this element as the **genus reduction term** of the Frobenius system (A, ε).

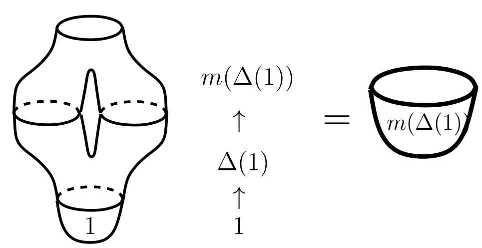


Figure 2.1: A handle

Kock alternatively refers to $g = \omega$ as the handle element of A in several exercises ([15], pgs. 128-131). We choose the terminology of "genus reduction term" to emphasize its geometric meaning for closed 2-D surfaces. For a choice of dual-basis

(x_i, y_i) we always have $g = m(\Delta(1)) = m(\sum_i x_i \otimes y_i) = \sum_i x_i y_i$. In Lemma 2.2.2 we will see that this summation is invariant under change in dual-bases.

For closed cobordisms, this implies that a torus decorated with $a \in A$ evaluates to a sphere marked by $ag \in A$. Since a sphere marked by $a \in A$ is evaluated by Z to $\varepsilon(a)$, we have that a unmarked torus evaluates to $\varepsilon(g) = \varepsilon(\sum_i x_i y_i) = \sum_i(\varepsilon(x_i y_i) = \sum_i(1) = n \in k$ for any 2-D TQFT Z. Repeated genus reduction similarly shows that the (unmarked) oriented genus-i surface Σ_i evaluates to $\varepsilon(g^i) \in k$ for any 2-D TQFT Z (see Figure 2.2). In the language of Frobenius functions, this means that $f_Z(i) = \varepsilon(g^i)$ for the function $f = f_Z$ induced by the TQFT Z. These observations will be the starting point for our investigation of Frobenius functions in Section 2.3.

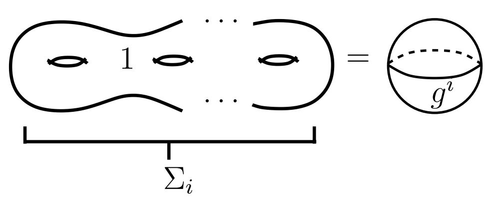


Figure 2.2: Repeated neck-cutting down to a sphere

For the remainder of this section we prove a number of foundational results involving the genus-reduction term g, which will be useful both in this chapter and in Chapter 3. We begin by citing the following well-known theorem involving Frobenius systems, whose proof may be found in [15] or [9]:

Lemma 2.2.1. *Let $(A, \varepsilon, (x_i, y_i))$ and $(A, \tilde{\varepsilon}, (\tilde{x}_i, \tilde{y}_i))$ be two Frobenius systems with* *the same underlying algebra A. Up to change of dual-bases, we have* $(A, \tilde{\varepsilon}, (\tilde{x}_i, \tilde{y}_i)) = (A, \varepsilon(d\_), (x_i, d^{-1}y_i))$ *for some invertible* $d \in A$*. Therefore, there is a bijection between (equivlance classes of) Frobenius systems over* A *and invertible* $d \in A$*.*

One immediate consequence of Lemma 2.2.1 is that, over a fixed algebra A, two Frobenius systems are equivalent iff they have identical Frobenius forms $\varepsilon \equiv \tilde{\varepsilon}$. The next lemma shows that g is unaffected by changes in dual-bases:

Lemma 2.2.2. *Let* $(A, \varepsilon, (x_i, y_i))$ *and* $(A, \varepsilon, (\tilde{x}_i, \tilde{y}_i))$ *be two Frobenius systems that differ only via the choice of dual-bases, and denote their genus-reduction terms by* g *and* $\tilde{g}$*, respectively. Then* $g = \tilde{g}$*.*

Proof. We argued in Subsection 1.2.2 that $\varepsilon \equiv \tilde{\varepsilon}$ iff $\sum(x_i \otimes y_i) = \sum(\tilde{x}_i \otimes \tilde{y}_i) \in A \otimes A$. The well-definedness of the map $A \otimes A \to A,\ a \otimes b \mapsto ab$ then ensures that $g = \sum x_i y_i = \sum \tilde{x}_i \tilde{y}_i = \tilde{g}$. □

The import of Lemma 2.2.1 to this thesis is the following corollary:

Corollary 2.2.3. *Let* $(A, \varepsilon, (x_i, y_i))$ *and* $(A, \tilde{\varepsilon}, (\tilde{x}_i, \tilde{y}_i))$ *be two Frobenius systems with the same underlying algebra* A*, and with genus-reduction terms given by* g *and* $\tilde{g}$ *(respectively). Then* $\tilde{g} = d^{-1}g$ *for some invertible* $d \in A$*.*

Proof. By Lemma 2.2.1, after a possible change of dual-bases we have $(A, \tilde{\varepsilon}, (\tilde{x}_i, \tilde{y}_i)) = (A, \varepsilon(d\_), (x_i, d^{-1}y_i))$ for some invertible $d \in A$. By Lemma 2.2.2, change of dual-bases leaves $\tilde{g}$ unchanged, so by comparing $(A, \varepsilon(d\_), (x_i, d^{-1}y_i))$ with $(A, \varepsilon, (x_i, y_i))$ we have $\tilde{g} = \sum \tilde{x}_i \tilde{y}_i = \sum x_i d^{-1} y_i = d^{-1}g$. □

Corollary 2.2.3 clearly extends to the result that $\tilde{g}^i = (d^{-i})g^i$ for all integers $i \geq 1$. Sadly, this corollary tells us very little about how the $\varepsilon(g^i)$ relate to the $\tilde{\varepsilon}(\tilde{g}^i)$, since the invertible $d \in A$ above need not be in k and hence can't be pulled out of the argument for ε. Later in this chapter, this fact will make the determination of Frobenius functions a highly non-trivial manner. However, do note that $g^i = 0$ (respectively $g^i \neq 0$) does imply that $\tilde{g}^i = 0$ ($\tilde{g}^i \neq 0$), since the unit d cannot be a zero divisor. This final fact will be recalled in Chapter 3, where it will place a crucial role in our primary theorem about "universal sl(n) Frobenius extensions".

Our final lemma states that the genus-reduction term is respected by Frobenius equivalence. When the underlying algebra A of two equivalent Frobenius systems is the same, this is an immediate consequence of Lemma 2.2.1 and Corollary 2.2.3. We now prove the more general case, where we have a Frobenius equivalence between systems with possibly distinct underlying algebras A and $\tilde{A}$ (although obviously where we still require $A \cong \tilde{A}$ as algebras).

Lemma 2.2.4. *Let $(A, \varepsilon, (x_i, y_i))$ and $(\tilde{A}, \tilde{\varepsilon}, (\tilde{x}_i, \tilde{y}_i))$ be Frobenius equivalent via $\phi : A \to \tilde{A}$. If g and $\tilde{g}$ are the respective genus-reduction terms of these two systems, then $\phi(g) = \tilde{g}$.*

Proof. By definition we have $\varepsilon = \tilde{\varepsilon} \circ \phi$. Lemma 2.2.2 also tells us that the genus-reduction term is invariant under change of dual-bases, so we may assume WLOG that $(\tilde{x}_i, \tilde{y}_i)$ has been chosen such that $\varepsilon(x_i) = \tilde{x}_i$ for all i. If λ and $\tilde{\lambda}$ are the Frobenius matrices for these systems, we then have $\tilde{\lambda} = [[\tilde{\varepsilon}(\tilde{x}_i\tilde{x}_j)]] = [[\tilde{\varepsilon}(\phi(x_i)\phi(x_j))]] = [[\tilde{\varepsilon}(\phi(x_ix_j))]] = [[\varepsilon(x_ix_j)]] = \lambda$. As the Frobenius matrix is directly used to determine the second half of any dual-bases, if $y_i = \sum_m c_{im} x_m$ in the first system ($c_{im} \in k$) then we have the same constants in the expressions $\tilde{y}_i = \sum_m c_{im} \tilde{x}_m$. It follows that our genus-reduction terms are $g = \sum_i x_i y_i = \sum_{i,m} c_{im} x_i x_m$ and $\tilde{g} = \sum_i \tilde{x}_i \tilde{y}_i = \sum_{i,m} c_{im} \tilde{x}_i \tilde{x}_m$. We may then conclude that $\phi(g) = \tilde{g}$. $\square$

2.3 Strong & Weak Frobenius Systems

Once again, let $f_Z : \mathbb{N} \to k$ be the Frobenius function associated to the TQFT Z, and let $(A, \varepsilon, (x_i, y_i))$ be the associated Frobenius system. In Subsection 2.2 we noted how $f_Z(i) = \varepsilon(g^i)$ for all $i \geq 0$, where $g = \sum_i x_i y_i$ is the genus-reduction term of A. We begin by noting that the dual-bases condition always gives $f_Z(1) = \varepsilon(g) = \sum_i \varepsilon(x_i y_i) = \sum_i 1 = n$, where $n \geq 1$ is the rank of A. It follows that $f_Z(1)$ must be a positive integer for any Frobenius function f_Z. Higher values of f_Z resist a similarly immediate interpretation, and in fact may be extremely difficulty to compute. The purpose of this subsection is to develop a quick set of necessary conditions for any map $f : \mathbb{N} \to k$ to be a Frobenius function $f = f_Z$. This discussion directly prompts the most important algebraic distinction of this entire chapter: the difference between strong Frobenius systems and weak Frobenius systems.

So take any map $f : \mathbb{N} \to k$ with $f(1) = n$, n a positive integer. If f were to be a Frobenius function, any associated Frobenius system $(A, \varepsilon, (x_i, y_i))$ would necessarily have an algebra of rank n. As such, the first $n+1$ powers of the genus-reduction term $\{1, g, ..., g^n\}$ would have to be linearly dependent over k. We may then write g^k as a linearly combination of $1, ..., g^{k-1}$ for some $1 \leq k \leq n$, and after multiplying both sides of the dependence equation by g^{n-k} we have $g^n = \alpha_1 g^{n-1} + ... + \alpha_n$ for some constants $\alpha_i \in k$. Applying any associated Frobenius form ε to both sides of that equation gives $f(n) = \alpha_1 f(n-1) + ... + \alpha_n f(0)$. Similarly, we may multiply both sides of the original equation by any power g^i and then apply ε to give $f(i) = \alpha_1 f(i-1) + ... + \alpha_n f(i-n)$ for all $i \geq n$. The conclusion- for f to be a Frobenius function it must admit a recurrence relation of degree n:

Lemma 2.3.1. *Let $f : \mathbb{N} \to k$ be any map. For f to be a Frobenius function, we require $f(1) = n$ to be a positive integer and for f to admit a recurrence relation of degree n.*

From this point forward we will refer to maps that satisfy Lemma 2.3.1 as **(rank-n) candidate functions**, where the rank refers to $f(1) = n$.

Before we apply this definition in Subsection 2.3.1, we pause to recap some important facts about recurrence relations. As alluded to above, a rank-n recurrence relation is one where each entry (beyond the first n entries) is determined by the previous n previous entries in an equivalent fashion. Thus we have a rule of the form $f(i) = \alpha_1 f(i-1) + ... + \alpha_n f(i-n)$ for all $i \geq n$, where $\alpha_i \in k$ are the recurrence coefficients. To any rank-n recurrence relation one may assign a degree-n characteristic polynomial $p(t) \in k[t]$. For the arbitrary rank-n recurrence above we have $p(t) = t^n - \alpha_1 t^{n-1} - ... - \alpha_n$.

One extremely important fact to remember is that a single function $f : \mathbb{N} \to k$ may admit many recurrence relations, including recurrences of different orders. A standard result shows that any recurrence relation of minimal order (i.e.- the smallest order admitted by the map f) is necessarily unique. The characteristic polynomial $m(t)$ of that unique minimal-order recurrence is known as the minimal polynomial of the recurrence, and $m(t)$ necessarily divides the characteristic polynomial $p(t)$ of any recurrence relation admitted by f. When we say THE recurrence relation of f, we mean the minimal order recurrence relation.

2.3.1 Strong & Weak Candidate Functions

Generalizing the procedure above, for a rank-n Frobenius system (A, ε) suppose that $\{1, g, ..., g^{m-1}\}$ is the largest linearly independent set consisting of powers of g. By definition we have $g^m = \beta_1 g^{m-1} + ... + \beta_m$ for some constants $\beta_i \in k$, from which we can follow a similar line of reasoning as above to give the degree-m recurrence relation $f(i) = \beta_1 f(i-1) + ... + \beta_m f(i-m)$ for all $i \geq m$. There are two significant cases here: $m = n$ and $m < n$.

If $m = n$, then $\{1, g, ..., g^{m-1}\}$ serves as a basis for A. The associated Frobenius matrix is then $\lambda = [[\varepsilon(g^{i-1}g^{j-1})]] = [[f_Z(i+j-2)]]$, where f_Z is the corresponding Frobenius function. Via the nondegeneracy of ε this matrix λ is necessarily invertible, a condition that is equivalent to f_Z lacking a recurrence relation of degree less than n. It follows that f_Z has a minimal order recurrence relation of degree precisely n iff $\{1, g, ..., g^{m-1}\}$ is a k-linear basis of A. We henceforth refer to such a Frobenius system as a **strong Frobenius system**, and to its associated Frobenius function as a **strong Frobenius function**. Similarly, a rank-n candidate function with a minimal order recurrence relation of degree precisely n is referred to as a **strong (rank-n)**

candidate function. By definition, the set of all strong Frobenius functions are contained in the set of all strong candidate functions. Non-strong Frobenius systems/functions/candidate functions are alternatively referred to as **weak Frobenius systems/functions/candidate functions**, and we have a similar inclusion of weak Frobenius functions within weak candidate functions. Notice that, since the g^i are invariant under changes of dual-basis (Lemma 2.2.2), being a strong/weak Frobenius system is dependent solely upon A and ε.

Now consider the situation where $m < n$, which corresponds to the aforementioned weak case. We already know that the associated Frobenius function f_Z admits a recurrence relation of degree m, but saying more than this becomes a bit more complicated in the weak case. As $\{1, g, ..., g^{m-1}\}$ is no longer a valid basis for A, we can no longer apply the nondegeneracy of ε as we did in the previous paragraph. In particular, if we take the subalgebra $G = \langle g^i \rangle \subset A$ generated by the powers of g, in the weak case the restricted Frobenius form $\varepsilon|_G$ may be degenerate. This corresponds to the situation where the $m \times m$ submatrix $[[f_Z(i+j-2]]$ is singular, which implies that f_Z admits a recurrence relation of degree strictly less than m. Hence, the best that we can do here is to assert that, if $\{1, g, ..., g^{m-1}\}$ is the largest such linearly independent set, then f_Z has a minimal order recurrence relation of degree AT MOST m. This nuisance, where the restriction $\varepsilon|_G$ need not be a nondegenerate Frobenius form over G, will weaken one of our results concerning the Vacuum Hypothesis in Section 2.6. Luckily, this phenomenon won't similarly effect our investigation of weak Frobenius functions in Sections 2.4 and 2.5.

The following proposition summarizes our results of this subsection:

Proposition 2.3.2. *Let $f_Z : \mathbb{N} \to k$ be a Frobenius function with $f(1) = n$. f_Z is associated with a strong Frobenius system iff f_Z has a minimal-order recurrence relation of degree precisely n. If f_Z is associated with a weak Frobenius system such that $\{1, g, ..., g^{m-1}\}$ is the largest such linearly-independent set, then f_Z has a minimal-order recurrence relation of degree at most m.*

2.3.2 Examples

In this subsection we introduce several examples of strong and weak Frobenius systems, with a mind towards the universal $sl(n)$ Frobenius systems that underlie the popular class of $sl(n)$ link invariants. Several of these examples rely upon computational machinery developed for [7], and which will be proven in Chapter 3.

Example 2.3.3. *In [13], Khovanov investigated "universal $sl(2)$ Frobenius systems", which have underlying algebra $A = \mathbb{C}[x]/(p(x))$, where $p(x) = x^2 - ax - b$ for some $a, b \in \mathbb{C}$, and have Frobenius form that is defined over the basis $\{1, x\}$ by $\varepsilon(1) = 0$, $\varepsilon(x) = 1$. The genus-reduction term of any such system is $g = 2x - a \notin \mathbb{C}$, which means that $\{1, g\}$ is a linearly-independent basis. It follows that any such Frobenius system is strong. Specific examples of these strong $sl(2)$ systems include the Frobenius system underlying Khovanov's original categorification of the Jones polynomial [11], where $p(x) = x^2$, as well as the rank-2 system treated by Lee [16], where $p(x) = x^2 - 1$.*

Example 2.3.4. *More generally, any rank-2 Frobenius system such that $\varepsilon(1) = 0$ is strong. This follows from the fact that $f(0) = 0$ and $f(1) = 2$ in such systems,* *making a degree-1 recurrence impossible.*

Example 2.3.5. *In [12], Khovanov introduced a* $sl(3)$ *link homology theory that relied upon the rank-3 Frobenius system with algebra* $A = \mathbb{C}[x]/(x^3)$*, and with trace defined on the standard basis* $\{1, x, x^2\}$ *by* $\varepsilon(1) = \varepsilon(x) = 0$*,* $\varepsilon(x^2) = 1$*. This system has genus-reduction term* $g = 3x^2$*, so that* $g^2 = 9x^4 = 0$ *in* A*. The associated Frobenius function* f_Z *is then* $f_Z(1) = 3$ *and* $f_Z(i) = 0$ *for* $i \neq 1$*, which has a degree-2 recurrence given by* $f_Z(i) = 0f_Z(i-1) + 0f_Z(i-2)$ *for all* $i \geq 2$*. It follows that this Frobenius system it weak.*

Example 2.3.6. *More generally, in Chapter 3 we examine "universal* $sl(n)$ *Frobenius systems", which have underlying algebra* $A = \mathbb{C}[x]/(p(x))$ *for some monic degree-*n *polynomial* $p(x) \in C[x]$ *and a Frobenius form that is given on the standard basis* $\{1, x, ..., x^{n-1}\}$ *by* $\varepsilon(x^{n-1}) = 1, \varepsilon(x^i) = 0$ *(for* $i < n-1$*). In that chapter we will see that* $g^i = 0$ *in* A *for all* $i \geq 2$ *iff every root of* $p(x)$ *had multiplicity of at least* 2*. In this "all roots repeated" case, we then have a degree-*2 *recurrence relation on* f_Z *given by* $f_Z(i) = 0f_Z(i-1) + 0f_Z(i-2)$ *for all* $i \geq 2$*. Since* $f(1) = n \neq 0$*, a degree-*1 *recurrence relation is impossible in this case, and the degree-two recurrence above is the unique minimal order recurrence on* f_Z*. For* A *of rank* $n \geq 3$*, it follows that* (A, ε) *is a weak Frobenius system whenever every root of* $p(x)$ *is repeated.*

Example 2.3.7. *Here we consider the relation of strong Frobenius systems to Frobenius systems over a semisimple algebra. In the* $sl(n)$ *situation of Example 2.3.6, semisimple is obviously equivalent to* $p(x)$ *having* n *distinct roots. The repeated root condition referenced in that example may then seem to suggest a compatibility between* *the strong and semisimple cases, but even for $sl(n)$ Frobenius systems neither inclusion holds. From Example 2.3.3, any $sl(2)$ system such that $p(x) = (x-\gamma)^2$ is strong but not semisimple. Conversely, the $sl(3)$ system with $p(x) = x^3 - x$ is semisimple but not strong: here we have $g = 3x^2 - 1$ and hence $g^2 = g + 2$, giving a Frobenius function with degree-2 recurrence relation. In fact, it can be shown that every $sl(3)$ system with three "evenly spaced" roots defines a weak Frobenius system.*

2.3.3 Strong Frobenius Systems

We now restrict our attention to strong Frobenius systems and give an extremely succinct description of their structure. So let (A, ε) be a strong rank-n Frobenius system with genus-reduction term g. By definition, $\{1, g, ..., g^{n-1}\}$ is a k-linear basis for A, and the associated Frobenius function f_Z has a minimal order recurrence relation of degree precisely n. Suppose that this recurrence relation is given by $f_Z(i) = \alpha_1 f_Z(i-1) + ... + \alpha_n f_Z(i-n)$ for all $i \geq n$, giving the recurrence a minimal polynomial of $p(t) = t^n - \alpha_1 t^{n-1} - ... - \alpha_n \in k[t]$.

Since $\{1, g, ..., g^{n-1}\}$ is a basis for A, we may always perform a change of basis and rewrite our original Frobenius system in terms of this basis. Since our new basis is cyclic, as a ring A is generated by the single element g. The powers of g also have a single generating relation given by setting the characteristic polynomial of the recurrence relation equal to zero: $p(g) = g^n - \alpha_1 g^{n-1} - ... - \alpha_n = 0$. Thus, no matter the initial presentation of our algebra A, after this change of basis we may rewrite it in the form $A = k[g]/(p(g))$. The Frobenius form for A in terms of this new basis is necessarily given by $\varepsilon(g^i) = f_Z(i)$, while the genus-reduction term of A is still g (by Lemma 2.2.2). Recall once again that the nondegeneracy of this ε is equivalent to the invertibility of $\lambda = [[f_Z(i+j-2)]]$, which is guaranteed due to the lack of a lower-order recurrence relation. This proves the following proposition:

Proposition 2.3.8. *Let (A, ε) be a strong Frobenius system, whose associated Frobenius function f_Z has a minimal (rank-n) recurrence relation with minimal polynomial $p(t)$. Then (A, ε) is Frobenius equivalent to the Frobenius system with $A = k[g]/(p(g))$ and $\varepsilon(g^i) = f_Z(i)$.*

We henceforth refer to the system of Proposition 2.3.8 as the **"intuitive" Frobenius system** associated to f_Z, since it explicitly exhibits the associated recurrence relation and initial conditions. One quick consequence of this Proposition is the uniquness of strong Frobenius functions, an extremely nice property that does not carry over to the weak case:

Theorem 2.3.9. *Two strong Frobenius systems are Frobenius equivalent iff they generate identical Frobenius functions. Moreover, if a strong candidate function is a Frobenius function, it is a Frobenius function for precisely one equivalence class of strong Frobenius systems.*

Proof. The first result follows from the fact that every strong Frobenius system is equivalent to its intuitive Frobenius system. The (somewhat tautological) second statement is due to the fact that strong candidate functions cannot give rise to weak Frobenius systems. $\square$

In Section 2.4 we will show that not every strong candidate function is a strong Frobenius function, with our necessary and sufficient conditions resulting directly from a consideration of any associated "intuitive" Frobenius system. As such, not every system of the form $A = k[t]/(p(t))$, $\varepsilon(t^i) = f(i)$ will be the intuitive system of some strong Frobenius system, even if $deg(p(t)) = f(1) = n$. In particular, the intuitive system may lack proper initial conditions $\varepsilon(t^i)$ to guarantee that the genus-reduction term is $g = t$. Also notice that it is perfectly possible for a weak Frobenius system to have an algebra of the form $A = k[t]/(p(t))$, just so long as the associated Frobenius form doesn't give the proper initial conditions $\varepsilon(t^i) = f_Z(i)$.

2.3.4 Weak Frobenius Systems

Now let (A, ε) be a weak rank-n Frobenius system. If the associated weak Frobenius function $f_Z : \mathbb{N} \to k$ has a minimal recurrence relation of order m, where $n - m = d \geq 1$, we say that the system has a **degeneracy** of d. The primary challenge here is that there may be many (possibly non-equivalent) "intuitive" Frobenius systems associated to f_Z, as we need to append as many as d "freely chosen" generators u_i to complete the intuitive basis that includes $\{1, g, g^2, ...\}$. The resulting intuitive presentations may not be equivalent to ones of the form $A = k[t]/(p(t))$, as they require a number of relations involving any additional generators u_i. Luckily, as we will see in Section 2.5, every weak candidate will be a weak Frobenius function via a fairly direct argument, allowing us to avoid analysis of the underlying algebras.

One important property of weak Frobenius systems that we will address here is the lack of a weak equivalent to Theorem 2.3.9. Non Frobenius-equivalent systems producing the same weak Frobenius function are actually fairly easy to find, as illustrated by the large family of $sl(n)$ systems in the following example:

Example 2.3.10. *Consider the weak* $sl(4)$ *Frobenius systems with* $p(x) = (x-a)^2(x-b)^2$ *and* $\tilde{p}(x) = (x-a)^4$*, where* $a \neq b$*. As* p *and* $\tilde{p}$ *contain no multiplicity one roots, from what we will prove in Chapter 3 we know that both of these systems have* $f_Z(1) = 4$ *and* $f_Z(i) = 0$ *if* $i \neq 1$*. Hence these two systems generate identical weak Frobenius functions of degeneracy* 2*, but the Frobenius systems are not Frobenius equivalent because their underlying algebras are not isomorphic. A similar argument shows that, for any* $n \geq 4$*, the weak Frobenius function given by* $f_Z(1) = n$ *and* $f_Z(i) = 0$ *(*$i \neq 1$*) can be generated by non-equivalent Frobenius* $sl(n)$ *systems that have distinct root structures for* $p(x)$*.*

In Section 2.5, when we show that every weak candidate function is a Frobenius function, we will use only a single particularly nice choice for the associated "intuitive" system. This begs the question of whether every weak Frobenius function may be realized by at least two non-equivalent Frobenius systems. This would give a particularly nice modification of Theorem 2.3.9 stating that a Frobenius function is "uniquely induced" iff it is a strong Frobenius function. Unfortunately, this result proves to be false, as exhibited by the following weak Frobenius function that is associated with only a single equivalence class of weak Frobenius systems.

Example 2.3.11. *Consider the weak rank-*2 *Frobenius function defined by* $f(i) = 2\alpha^{i-1}$*, so that we have a degree-*1 *recurrence with minimal polynomial* $m(t) = t - \alpha$*.*

Any weak Frobenius system yielding this Frobenius function is Frobenius equivalent to an "intuitive" system of the following form:

$$A = k[t,u]/(t-\alpha, u^2 - a_1 u - a_2)$$

$$\varepsilon(1) = \tfrac{2}{\alpha},\ \varepsilon(u) = s$$

Where $a_1, a_2, s \in k$ *are constants that guarantee a genus-reduction term of* $g = t = \alpha$. *Our approach here is to determine what such constants are necessary to guarantee* $g = t$, *and see what restrictions that places upon the given Frobenius systems. Taking the obvious basis of* $\{1, u\}$, *the associated Frobenius matrix is:*

$$\lambda = \begin{bmatrix} \frac{2}{\alpha} & s \\ s & a_1 s + \frac{2a_2}{\alpha} \end{bmatrix}$$

Which has $det(\lambda) = \frac{2a_1 s}{\alpha} + \frac{4a_2}{\alpha^2} - s^2$ *and inverse:*

$$\lambda^{-1} = \tfrac{1}{det(\lambda)} \begin{bmatrix} a_1 s + \frac{2a_2}{\alpha} & -s \\ -s & \frac{2}{\alpha} \end{bmatrix}$$

From this we calculate a genus-reduction term of $g = \frac{1}{det(\lambda)}[(a_1 s + \frac{4a_2}{\alpha}) + (\frac{2a_1}{\alpha} - 2s)u]$. *Enforcing* $g = t = \alpha$ *we need* $\frac{2a_1}{\alpha} - 2s = 0$, *which forces* $a_1 = \alpha s$. *We then have* $det(\lambda) = s^2 + \frac{4a_2}{\alpha^2}$, *so as we're assuming that* λ *is invertible we can't have* $det(\lambda) = 0 \Rightarrow a_2 = -(\frac{s\alpha}{2})^2$. *This precisely corresponds to when* $u^2 - a_1 u - a_2 = (u - \frac{\alpha s}{2})^2$ *factors as a perfect square. The conclusion- any "intuitive" Frobenius system associated to our Frobenius function must be of the form:*

$$A = k[u]/((u-\gamma_1)(u-\gamma_2))$$

$$\varepsilon(1) = \tfrac{2}{\alpha},\ \varepsilon(u) = \tfrac{a_1}{\alpha} = \tfrac{\gamma_1 + \gamma_2}{\alpha}$$

where $\gamma_1 \neq \gamma_2$ are constants. Any algebra of the form above is isomorphic via the Chinese Remainder Theorem, and the direct linear dependence of $\varepsilon(u)$ upon γ_1, γ_2 ensures that the given algebra isomorphism respects Frobenius form. It follows that any two Frobenius systems generating the given f must be Frobenius equivalent.

2.4 Strong Frobenius Functions

We are now ready to definitively characterize which functions $f : \mathbb{N} \to k$ are the Frobenius function $f = f_Z$ of some 2-D TQFT Z. As the techniques used to answer this question differ significantly in the strong and weak cases, we treat them separately. Necessary and sufficient conditions for strong candidate functions to be strong Frobenius functions are addressed in this section; necessary and sufficient conditions for weak candidate functions to be weak Frobenius functions are developed in Section 2.5.

So let $f : \mathbb{N} \to k$ be a strong rank-n candidate function with initial conditions $f(0) = c_0,\ f(1) = c_1 = n,\ ...,\ f(n-1) = c_{n-1}$ and minimal-order recurrence relation $f(m) = \alpha_1 f(m-1) + ... + \alpha_n f(m-n)$ (for all $m \geq n$). Let $p(t) = t^n - \alpha_1 t^{n-1} - ... - \alpha_n$ denote the associated (minimal) characteristic polynomial for this recurrence.

Now if f were to be a strong Frobenius function, it would need to be the Frobenius function induced by a strong Frobenius system. Our starting point for determining whether f is Frobenius is then Theorem 2.3.8, which associates to every strong Frobenius system an "intuitive" system with identical Frobenius function. For our given f we then consider the "intuitive" system $(\hat{A}, \hat{\varepsilon})$ defined by $\hat{A} = k[t]/(p(t))$

and $\hat{\varepsilon}(t^i) = c_i$. Recall that, since f lacks a recurrence relation of degree less than n, the given trace map $\hat{\varepsilon}$ is always non-degenerate and hence this system is in fact Frobenius (regardless of whether f actually proves to be a Frobenius function). To recover f as the Frobenius function of this "intuitive" Frobenius system, we need $g = t$. This condition is actually the only thing that may prevent f from being a strong Frobenius function:

Lemma 2.4.1. *Let f be a strong rank-n candidate function whose recurrence relation has characteristic polynomial $p(t) = t^n - \alpha_1 t^{n-1} - \ldots - \alpha_n$. Then f is a strong Frobenius function iff the "intuitive" Frobenius system with $\hat{A} = k[t]/(p(t))$ and $\hat{\varepsilon}(t^i) = f(i)$ has genus-reduction term $g = t$*

Proof. ($\Leftarrow$) Immediate. ($\Rightarrow$) Assume f is realized by the strong system (A, ε). After a change of basis, (A, ε) is Frobenius equivalent to the "intuitive" system above. If $g \neq t$ we contradict that f was the Frobenius function of (A, ε). $\square$

Although the condition provided by Lemma 2.4.1 is extremely elegant, it isn't particularly useful. The problem with this condition is that dual-bases (and hence genus-reduction terms) can be extremely difficult to compute, as one needs to invert the $n \times n$ Frobenius matrix in order to find the second basis. The rest of this subsection is devoted towards finding an equivalent set of necessary and sufficient conditions that are more tractable (and combinatorially interesting!). The trick here is to work with a second basis $\{s_0, \ldots, s_{n-1}\}$ that is dual to $\{1, t, \ldots, t^{n-1}\}$, without explicitly calculating the elements s_i. The result below, a slight restatement of Lemma 2.4.1, is the first step in this approach:

Lemma 2.4.2. *Let f be a strong rank-n candidate function, as defined above, and consider the "intuitive" Frobenius system $\hat{A} = k[t]/(p(t))$, $\hat{\varepsilon}(t^i) = f(i)$. Then f is a strong Frobenius function iff $\hat{\varepsilon}(gt^k) = f(k+1)$ for $k = 0, 1, ..., n-1$.*

Proof. ($\Rightarrow$) Let f be strong Frobenius. By Lemma 2.4.1 we have that $g = t$ in the associated "intuitive" system, giving $\hat{\varepsilon}(gt^k) = \hat{\varepsilon}(g^{k+1}) = f(k+1)$ for all $k \geq 0$.
($\Leftarrow$) By definition of the Frobenius form in the "intuitive" system, $f(k+1) = \hat{\varepsilon}(t^{k+1})$ for all $k \geq 0$. If $\hat{\varepsilon}(gt^k) = f(k+1)$ for $k = 0, 1, ..., n-1$ we then have $\hat{\varepsilon}(gt^k) = \hat{\varepsilon}(t^{k+1}) \Leftrightarrow \hat{\varepsilon}((g-t)t^k) = 0$ for $k = 0, 1, ..., n-1$. As $\{1, t, ..., t^{n-1}\}$ forms a basis for $\hat{A}$, nondegeneracy of $\hat{\varepsilon}$ forces $g = t$. $\square$

Temporarily denote $\phi_k = \hat{\varepsilon}(gt^{k-1})$. Lemma 2.4.2 then states that f is strong Frobenius iff $f(k) = c_k = \phi_k$ for $k = 1, ..., n$. The power of this Lemma comes only when we assert $\{s_0, ..., s_n\}$ to be the dual-basis to $\{1, t, ..., t^{n-1}\}$. Using only that $\hat{\varepsilon}(s_i t^i) = \delta_{ij}$ and $s_0 + s_1 t + ... + s_{n-1}t^{n-1} = g$, we may explicitly calculate the conditions $c_k = \phi_k$ for small values of n. The results for $n = 2, 3, 4, 5$ are presented below. In each case, the $c_n = \phi_n$ condition has been simplified using the recurrence relation for the given f.

$\underline{n=2}$	$\underline{n=3}$	$\underline{n=4}$	$\underline{n=5}$
$c_1 = 2$	$c_1 = 3$	$c_1 = 4$	$c_1 = 5$
$\alpha_1 = -\alpha_2 c_0$	$c_2 = \alpha_1$	$c_2 = \alpha_1$	$c_2 = \alpha_1$
	$\alpha_2 = -\alpha_3 c_0$	$c_3 = \alpha_1^2 + 2\alpha_2$	$c_3 = \alpha_1^2 + 2\alpha_2$
		$\alpha_3 = -\alpha_4 c_0$	$c_4 = \alpha_1^3 + 3\alpha_1\alpha_2 + 3\alpha_3$
			$\alpha_4 = -\alpha_5 c_0$

Some obvious patterns can be observed here, suggesting that it may be possible to obtain a formula for arbitrary $n \geq 2$ (perhaps one that involves ordered partitions in the subscripts of the α_i). Unfortunately, pathologies begin to appear for higher n, and our usage of the recurrence coefficients α_i proves to be a suboptimal choice. What does hold for all $n \geq 2$ is the following recursive definition for the ϕ_k:

Lemma 2.4.3. *Let f be a strong rank-n candidate function, as defined above, and let $(\hat{A}, \hat{\varepsilon})$ be the associated "intuitive" system. Then f is a strong Frobenius function iff $f(i) = c_i = \phi_i$ for $k = 1, 2, ..., n$, where the ϕ_i are defined recursively as:*

$$\phi_1 = n \quad \phi_2 = \alpha_1$$

$$\phi_k = \alpha_1\phi_{k-1} + \alpha_2\phi_{k-2} + ... + \alpha_{k-2}\phi_2 + (k-1)\alpha_{k-1}$$

Proof. That $\phi_1 = \hat{\varepsilon}(g) = n$ has already been established. For $k > 1$, let $\{s_0, ..., s_{n-1}\}$ be the basis dual to $\{1, ..., t^{n-1}\}$. Using $\hat{\varepsilon}(s_i t^i) = \delta_{ij}$ and $g = s_0 + s_1 t + ... s_{n-1}t^{n-1}$:

$$\phi_k = \hat{\varepsilon}(gt^{k-1}) = \hat{\varepsilon}(s_0 t^{k-1} + s_1 t^k + ... + s_{n-1}t^{n+k-2})$$

$$= 0 + ... + 0 + \hat{\varepsilon}(s_{n-k+1}t^n) + ... + \hat{\varepsilon}(s_{n-k+(k-1)}t^{n+k-2})$$

The $k = 2$ case then follows easily:

$$\phi_2 = \hat{\varepsilon}(s_{n-1}t^n) = \hat{\varepsilon}(s_{n-1}(\alpha_1 t^{n-1} + ... + \alpha_n)) = \alpha_1$$

Thus we turn to $k \geq 2$. With the notation $\phi_{k,j} = \hat{\varepsilon}(s_{n-k+j}t^{n+j-1})$, when $i \geq 2$ the expression for ϕ_k reduces to $\phi_k = \phi_{k,1} + ... + \phi_{k,k-1}$. First we claim that:

$$\phi_{k,1} = \alpha_{k-1}$$

$$\phi_{k,2} = \alpha_1\phi_{k-1,1} + \alpha_{k-1}$$

$$\vdots$$

$$\phi_{k,k-1} = \alpha_1\phi_{k-1,k-2} + \alpha_2\phi_{k-2,k-3} + ... + \alpha_{k-2}\phi_{2,1} + \alpha_{k-1}$$

These expressions follow from repeated application of $t^n = \alpha_1 t^{n-1} + ... + \alpha_n$ and the dual-basis condition:

$$\phi_{k,1} = \hat{\varepsilon}(s_{n-k+1}t^n) = \hat{\varepsilon}(s_{n-(k-1)}(\alpha_1 t^{n-1} + ... + \alpha_n)) = \alpha_{k-1}$$

$$\vdots$$

$$\phi_{k,j} = \hat{\varepsilon}(s_{n-k+j}(\alpha_1 t^{n+j-2} + ... + \alpha_n t^{j-1}))$$

$$= \alpha_1\hat{\varepsilon}(s_{n-k+j}t^{n+j-2}) + ... + \alpha_{j-1}\hat{\varepsilon}(s_{n-k+j}t^n) + \hat{\varepsilon}(s_{n-k+j}(\alpha_j t^{n-1} + ... + \alpha_n t^{j-1}))$$

$$= \alpha_1\phi_{k-1,j-1} + ... + \alpha_{j-1}\phi_{k-(j-1),1} + \alpha_{k-1}$$

$$\vdots$$

As $\phi_k = \phi_{k,1} + ... + \phi_{k,k-1}$, we sum the given expressions for the $\phi_{k,j}$ to give:

$$\phi_k = (k-1)\alpha_{k-1} + \alpha_1(\phi_{k-1,1} + ... + \phi_{k-1,k-2}) + \alpha_{k-2}(\phi_{2,1})$$

$$= (k-1)\alpha_{k-1} + \alpha_1\phi_{k-1} + ... + \alpha_{k-2}\phi_2$$

□

As will occur with our $sl(n)$ Frobenius extensions in Chapter 3 (which are of similar form to our "intuitive" systems, in that they have a single generating relation involving one generator), things become far nicer when the relation $p(t) = t^n - \alpha_1 t^{n-1} - ... - \alpha_n$ is fully factored[1] over k as $p(t) = (t - \gamma_1)...(t - \gamma_n)$. With this notation we have $\alpha_i = (-1)^{i-1} e_i$, where e_i is the i$^{\text{th}}$ elementary symmetric polynomial in the roots γ_k: $e_i = \sum_{1 \leq k_1 < ... < k_i \leq n} \gamma_{k_1} \gamma_{k_2} ... \gamma_{k_i}$.

Now let p_i denote the i$^{\text{th}}$ power symmetric polynomial in the γ_k, so that $p_i = \sum_k \gamma_k^i$. We recall Newton's relations for symmetric polynomials:

$$p_1 = e_1$$

$$p_2 = e_1 p_1 - 2e_2$$

$$p_3 = e_1 p_2 - e_2 p_1 + 3e_3$$

$$p_4 = e_1 p_3 - e_2 p_2 + e_3 p_1 - 4e_4$$

$$\vdots$$

Rewritten in terms of the e_i, the recursive definition of the ϕ_k from Lemma 2.4.3 miraculously transforms into Newton's relations:

$$\phi_2 = \alpha_1 = e_1 = p_1$$

$$\phi_3 = \alpha_1 \phi_2 + 2\alpha_2 = e_1 p_1 + 2(-e_2) = p_2$$

$$\phi_4 = \alpha_1 \phi_3 + \alpha_2 \phi_2 + 3\alpha_3 = e_1 p_2 + (-e_2) p_1 + 3e_3 = p_3$$

$$\vdots$$

[1] This is the only point where we require that k be an algebraically closed field.

We have proven the following (particularly elegant) characterization of strong Frobenius functions:

Lemma 2.4.4. *Let $f : \mathbb{N} \to k$ be a function with $f(1) = n$ (n a positive integer) and a minimal order recurrence relation of degree precisely n. If the given recurrence has (minimal) characteristic polynomial $p(t) = t^n - \alpha_1 t^{n-1} - ... - \alpha_n = (t - \gamma_1)...(t - \gamma_n)$ (where $\alpha_i, \gamma_i \in k$), then f is a Frobenius function iff:*

$$f(1) = n$$

$$f(2) = \gamma_1^1 + ... + \gamma_n^1$$

$$\vdots$$

$$f(n) = \gamma_1^{n-1} + ... + \gamma_n^{n-1}$$

Pause to note that, if all of the γ_i are nonzero (or if we take $0^0 = 1$), then $f(1) = n = p_0 = \gamma_1^0 + ... + \gamma_n^0$, making the above result even more tidy.

As one final refinement of our necessary and sufficient conditions for strong Frobenius functions, we look to restate Lemma 2.4.4 in such a way that we don't require knowledge of the underlying recurrence relation. This actually proves to be another straightforward application of Newton's relations, as long as we consider how those relations apply for $i > n$ (as the i^{th} elementary symmetric polynomial e_i doesn't exist when i is greater than the number of variables n). In standard fashion, this is accomplished by setting $e_i = 0$ when $i > n$, so that the equations for the power symmetric polynomials eventually stabilize to $p_i = e_1 p_{i-1} - e - 2p_{i-2} + ... + (-1)^{n-1} e_n p_{i-n}$ when $i > n$.

Once again recalling that our recurrence coefficients are related to the elementary symmetric polynomials in the γ_i by $\alpha_i = (-1)^{i-1}e_i$, and noting how closely the characteristic polynomial $p(t) = t^n - e_1t^{n-1} + e_2t^{n-2} - ...(-1)^{n-1}e_n$ resembles our "higher-order" Newton's relations after this substitution, we are led to the following:

Theorem 2.4.5. *Let $f : \mathbb{N} \to k$ be a function with $f(1) = n$ (n a positive integer) and a minimal order recurrence relation of degree precisely n. Then f is a Frobenius function iff there exist $\gamma_1, ..., \gamma_n \in k$ such that $f(i) = p_{i-1}$ for all $i > 1$ and we have $f(0)e_n = e_{n-1}$, where e_i is the i^{th} elementary symmetric polynomial and p_i is the i^{th} power symmetric polynomial in the γ_i.*

Proof. ($\Rightarrow$) Let f be a strong rank-n Frobenius function whose recurrence relation has minimal polynomial $p(t) = (t-\gamma_1)...(t-\gamma_n)$. By Lemma 2.4.4 we have $f(i) = \gamma_1^{i-1} + ... + \gamma_n^{i-1}$ for $1 \leq i \leq n$. We use our recurrence relation to check:

$$f(n+1) = e_1f(n) - e_2f(n-1) + ... + (-1)^{n-1}e_nf(1)$$
$$= e_1p_{n-1} - e_2p_{n-2} + ... + (-1)^{n-1}ne_n = p_n$$

Where the last equality comes from the "standard" Newton's relations. Looking to use the "higher-order" Newton's relations, now we inductively assume that our condition is satisfied for all $i \leq m$ ($m \geq n$). , our recurrence relation gives:

$$f(m+1) = e_1f(m) - e_2f(m-1) + ... + (-1)^{n-1}e_nf(m-n)$$
$$= e_1p_{m-1} - e_2p_{m-2} + ... + (-1)^{n-1}e_np_{m-n-1} = p_m.$$

($\Leftarrow$) Now assume that there exist $\gamma_1, ..., \gamma_n \in k$ such that $f(i) = p_{i-1}$ for all $i \geq 1$, as well as that $f(0)e_n = e_{n-1}$. By Lemma 2.4.4, we only need to show that f is in possession of a degree-n recurrence relation with minimal polynomial $p(t) = (t - \gamma_1)...(t - \gamma_n)$. If $m > n - 1$, the "higher-order" Newton's relations give:

$$f(m) = p_{m-1} = e_1 p_{m-2} - e_2 p_{m-3} + ... + (-1)^{n-1} e_n p_{m-n-1}$$

$$= e_1 f(m-1) - e_2 f(m-2) + ... + (-1)^{n-1} e_n f(m-n)$$

For $m = n - 1$ we use the "standard" Newton's relations:

$$f(n+1) = p_n = e_1 p_{n-1} - e_2 p_{n-2} + ... + (-1)^{n-1} e_n(n)$$

$$= e_1 f(n-2) - e_2 f(n-3) + ... + (-1)^{n-1} e_n f(1)$$

For $m = n$ we require the mysterious additional condition that $f(0)e_n = e_{n-1} \Leftrightarrow (-1)^{n-2} e_{n-1} + (-1)^{n-1} e_n f(0) = 0$, as then:

$$f(n) = p_{n-1} = e_1 p_{n-2} - e_2 p_{n-3} + ... + (-1)^{n-2} e_{n-1}(n-1)$$

$$= e_1 p_{n-2} - e_2 p_{n-3} + ... + (-1)^{n-2} e_{n-1} n + (-1)^{n-1} e_n f(0)$$

$$= e_1 f(n-1) - e_2 f(n-2) + ... + (-1)^{n-1} e_n f(0)$$

Combining these cases, we see that f has a degree-n recurrence with minimal polynomial $p(t) = t^n - e_1 t^{n-1} + e_2 t^{n-2} - ... + (-1)^{n-1} e_n = (t - \gamma_1)...(t - \gamma_n)$. □

Theorem 2.4.5 tells us that a strong rank-n Frobenius function is completely determined by a choice of n constants $\gamma_1, ...\gamma_n \in k$. We henceforth refer to as these γ_i as the **roots of the strong Frobenius function** f; if Z is the 2-D TQFT such that $f = f_Z$, we refer to these γ_i as the **roots of the TQFT** Z.

Notice how the curious additional condition that $f(0)e_n = e_{n-1} \Rightarrow -c_0\alpha_n = -\alpha_{n-1}$ corresponds with the final condition that we already explicitly calculated for the $n = 2, 3, 4, 5$ cases. In the situation where $e_n \neq 0$ we also have that $e_{n-1}/e_n = p_{-1}$, as long as we define $p_{-1} = \gamma_1^{-1} + ... + \gamma_n^{-1}$, putting this condition into alignment with our requirements upon the later $f(i)$. This added condition on $f(0)$ is significant in that it vastly limits the number of rank-n Frobenius functions that may arise for any specific choice of n roots. In fact, except for the case where $e_n = e_{n-1}$ (corresponding to when 0 is a root of multiplicity at least 2), there is at most one rank-n Frobenius function for any choice of n roots, up to permutation of those roots. The only thing that can keep a choice of roots from yielding a single Frobenius function is also the condition $e_{n-1}f(0) = e_n$, which fails for any choice of $f(0)$ when $\gamma_i = 0$ for precisely one i:

Corollary 2.4.6. *Take n (not necessarily distinct) constants $\gamma_1, ...\gamma_n \in k$. There exists a strong Frobenius system with roots $\gamma_1, ..., \gamma_n$ iff we don't have $\gamma_i = 0$ for precisely one constant γ_i.*

Proof. ($\Leftarrow$) First assume that 0 is a root of multiplicity precisely 1, say $\gamma_1 = 0$ and $\gamma_i \neq 0$ for $i > 1$. Then $e_n = 0$ while $e_{n-1} = \gamma_2 ... \gamma_n \neq 0$. No matter our choice of $f(0)$, it is then impossible to fulfill the condition $f(0)e_n = e_{n-1}$ demanded by Theorem 2.4.5. Hence no strong Frobenius function exists with these roots.

($\Rightarrow$) Here we treat two separate cases. First assume that 0 does not appear as a root. Then $e_n \neq 0$ and we may always choose $f(0) = e_{n-1}/e_n$ to satisfy Theorem 2.4.5. Alternatively assume that 0 is a repeated root. Then $e_n = 0$ and $e_{n-1} = 0$, so that any choice of $f(0)$ satisfies the hypotheses of Theorem 2.4.5 and gives a strong Frobenius system. $\square$

Recall that equivalence classes of strong Frobenius systems are in bijective correspondence with strong Frobenius functions. This observation yields another quick corollary of Theorem 2.4.5:

Corollary 2.4.7. *Take n (not necessarily distinct) constants $\gamma_1, ...\gamma_n \in k$ such that $\gamma_i \neq 0$ for all i. Up to permutation of the γ_i, there exists precisely one equivalence class of strong Frobenius systems whose roots are $\gamma_1, ..., \gamma_n$. If $\gamma_i = 0$ for two or more roots, there alternatively exists infinitely many equivalence classes of strong Frobenius systems whose roots are $\gamma_1, ...\gamma_n$.*

2.5 Weak Frobenius Functions

Giving a precise classification of weak Frobenius functions proves to be far less interesting than the strong case. To that end, let $f : \mathbb{N} \to k$ be a weak rank-n candidate function with initial conditions $f(0) = c_0, f(1) = c_1 = n, f(n-d-1) = c_{n-d-1}$ and minimal order recurrence relation of degree $n-d$ $(1 \leq d < n)$ given by $f(m) = \beta_1 f(m-1) + ... + \beta_{n-d} f(m-(n-d))$ for all $m \geq n-d$. Recall that the number d is referred to as the degeneracy of f. We will independently address two separate cases:

1) "maximum degeneracy" weak candidate functions with $d = n-1$

2) weak candidate functions with $d < n-1$

In both cases we will see that all such weak candidate functions are in fact weak Frobenius functions.

2.5.1 Weak Frobenius Functions of Degeneracy $d = n - 1$

Here we have a minimal-order recurrence relation of degree 1, given by $f(m) = \beta f(m-1)$ for all $m \geq 1$ (where $\beta \neq 0$). As we must have $f(1) = n$, the recurrence coefficient (and hence all of f) is forced by the value $f(0) = c_0 = \frac{n}{\beta}$. Explicitly, $f(m) = \beta^m c_0 = \frac{n^m}{c_0^{m-1}}$.

The reason that this situation requires special treatment is because the genus-reduction term $g = t$ appears as a constant in any associated "intuitive" Frobenius system (owing to the relation $p(t) = t - \beta = 0$ provided by the recurrence). The algebra underlying any "intuitive" system then requires a full complement of $n-1$ ring generators $u_1, ..., u_{n-1}$ apart from t. With these things in mind, consider the following system with ordered basis $\{1, u_1, ..., u_{n-1}\}$:

$$\hat{A} = k[u_1, ..., u_{n-1}]/(u_i^2 - \beta, ..., u_i u_j)$$

$$\hat{\varepsilon}(1) = \tfrac{n}{\beta} = c_0,\ \hat{\varepsilon}(u_i) = 0$$

(where the generating relations in the algebra, $(n-1)^2$ in number, run over all i, j and where $i \neq j$ when both indices appear together).

The Frobenius matrix of this system is the diagonal matrix $\hat{\lambda} = diag(\frac{n}{\beta}, n, ..., n)$, with its invertibility ensuring that $\hat{\varepsilon}$ is in fact a non-degenerate Frobenius form. From $\hat{\lambda}^{-1} = diag(\frac{\beta}{n}, \frac{1}{n}, ..., \frac{1}{n})$ we may calculate the dual-basis $\{\frac{\beta}{n}, \frac{u_1}{n}, ..., \frac{u_{n-1}}{n}\}$. The resulting genus-reduction term is then $g = \frac{\beta}{n} + \frac{1}{n}u_1^2 + ... + \frac{1}{n}u_{n-1}^2 = \beta$. This yields $\hat{\varepsilon}(g^m) = \hat{\varepsilon}(\beta^m) = \beta^m c_0$, proving the following:

Proposition 2.5.1. *Let f be a weak rank-n candidate function with degree-1 recurrence relation $f(m) = \beta f(m-1)$ (for all $m \geq 1$). Then f is the Frobenius function* *generated by the weak Frobenius system:*

$$\hat{A} = k[u_1, ..., u_{n-1}]/(u_i^2 - \beta, ..., u_i u_j)$$

$$\hat{\varepsilon}(1) = \tfrac{n}{\beta} = c_0,\ \hat{\varepsilon}(u_i) = 0$$

2.5.2 Weak Frobenius Functions of Degeneracy $d < n-1$

The procedure here is similar to the $d = n-1$ case but is significantly more involved. Now we have a minimal-order recurrence relation of degree $n - d > 1$, with characteristic polynomial $p(t) = t^{n-d} - \beta_1 t^{n-d-1} - ... - \beta_{n-d}$. Any associated "intuitive" system requires d additional generators $u_1, ..., u_d$ to complete the basis of which $\{1, t, ..., t^{n-d-1}\}$ is a part. We begin by considering the relatively general system below with ordered basis $\{1, t, ..., t^{n-d-1}, u_1, ..., u_d\}$:

$$\hat{A} = k[t, u_1, ..., u_d]/(t^{n-d} - \beta_1 t^{n-d-1} - ... - \beta_{n-d},\ u_i t,\ u_i u_j,$$

$$u_i^2 - a_{i,0} - a_{i,1} t - ... - a_{i,n-d-1} t^{n-d-1})$$

$$\hat{\varepsilon}(t^k) = f(k) = c_k,\ \hat{\varepsilon}(u_i) = 0$$

Once again, the generating relations in $\hat{A}$ run over all possible i, j (with $i \neq j$ when relevant). The structure constants $a_{i,j}$ in the relations for the u_i^2 have yet to be determined, and will provide us with the additional degrees of freedom necessary to produce a genus-reduction term of $g = t$.

The relations $u_i t = 0$, $u_i u_j = 0$ make the resulting Frobenius matrix $\hat{\lambda}$ block-diagonal, with a single $(n-d) \times (n-d)$ block corresponding to the "intuitive" basis elements $\{1, t, ..., t^{n-d-1}\}$ and a string of 1×1 blocks (d in total) for the u_i. Note that the "intuitive" $(n-d)\times(n-d)$ block is necessarily invertible, as f lacks a recurrence relation of degree less than $n-d$.

As a condition upon the $a_{i,j}$, we enforce $\hat{\varepsilon}(u_i^2)=1$ for all i. This choice ensures that $\hat{\lambda}$ is invertible and hence that $\hat{\varepsilon}$ is a non-degenerate Frobenius form. Inverting $\hat{\lambda}$ and calculating the genus-reduction term then gives $g=b_0+b_1t+...+b_{n-d-1}t^{n-d-1}+u_1^2+...+u_d^2$, where the $b_i\in k$ are constants that come from collecting terms within the "intuitive" part of the basis (as $\hat{\lambda}$ is block diagonal, the dual-basis partners s_i of the t^i are always a linear combination of merely the $\{1,t,...,t^{n-d-1}\}$). As in Section 2.4, the trick here is to avoid explicitly determining the s_i and hence also the b_i.

With a slight abuse of notation, temporarily denote $\tilde{g}=b_0+b_1t+...+b_{n-d-1}t^{n-d-1}$ to be the "intuitive" part of g, so that $g=\tilde{g}+u_1^2+...+u_d^2$. The dual-basis condition yields $\hat{\varepsilon}(\tilde{g})=n-d$. Directly calculating $\hat{\varepsilon}(\tilde{g})$, we then get the condition that $b_0c_0+b_1c_1+...+b_{n-d-1}c_{n-d-1}=n-d$. Finally enforcing our conditions that $\hat{\varepsilon}(u_i^2)=1$, we also require that $a_{i,0}c_0+a_{i,1}c_1+...+a_{n-d-1}c_{n-d-1}=1$ for all i.

When calculating $\hat{\varepsilon}(g)$, to achieve the desired genus-reduction term of $g=t$ we need $b_1+a_{1,1}+...+a_{d,1}=1$ and $b_j+a_{1,j}+...+a_{d,j}=0$ whenever $j\neq 1$. To satisfy the latter equations, we have enough degrees of freedom with the $a_{i,j}$ to make the painfully easy choice of $a_{i,j}=\frac{-b_j}{d}$ for all i,j with $j\neq 1$. Via the dependencies of the previous paragraph, this forces $a_{i,1}=\frac{1-b_1}{d}$ for all i. Thus, all that's left to check is whether these required values for the $a_{a,1}$ combine to satisfy the first equation. Thankfully, $b_1+a_{1,1}+...+a_{d,1}=b_1+\frac{1-b_1}{d}+...+\frac{1-b_1}{d}=b_1+(1-b_1)=1$. We have proven the following counterpart to Proposition 2.5.1:

Proposition 2.5.2. *Let f be a weak rank-n candidate function with minimal-order degree-$(n-d)$ recurrence relation $(d < n-1)$ given by $f(m) = \beta_1 f(m-1) + ... + \beta_{n-d} f(m-n+d)$ (for all $m \geq n-d$). Then f is the Frobenius function generated by a weak Frobenius system of the form:*

$$\hat{A} = k[t, u_1, ..., u_d]/(t^{n-d} - \beta_1 t^{n-d-1} - ... - \beta_{n-d},\ u_i t,\ u_i u_j,$$

$$u_i^2 - a_{i,0} - a_{i,1}t - ... - a_{i,n-d-1}t^{n-d-1})$$

$$\hat{\varepsilon}(t^k) = f(k) = c_k,\ \hat{\varepsilon}(u_i) = 0$$

Where the structure constants $a_{i,1} = \frac{1-b_1}{d}$ and $a_{i,j} = \frac{-b_j}{d}$ $(j \neq 1)$ are defined as above.

Combining Propositions 2.5.1 and 2.5.2, we have the following theorem that completely categorizes weak Frobenius functions:

Theorem 2.5.3. *Let $f : \mathbb{N} \to k$ be a function with $f(1) = n$ a positive integer $(n \geq 2)$ and a minimal order recurrence relation of degree strictly less than n. Then f is necessarily the Frobenius function of some 2-D TQFT Z.*

Comparing Theorem 2.5.3 with the more specific requirements for strong Frobenius functions from Theorem 2.4.5, one may ask whether the two results are compatible. In other words, if we modify Theorem 2.4.5 so that "minimal order recurrence relation of degree precisely n" reads "recurrence relation of degree n", do we then have a complete categorization of all Frobenius functions?

Sadly, this assertion proves to be false. The counterexample below is motivated via the fact that Theorem 2.4.5 requires only that f has some recurrence relation of degree n (not necessarily minimal). As such, one may alternatively consider whether a modification of Lemma 2.4.4 to include non-minimal degree n recurrence relations incorporates all weak Frobenius functions. This shift is useful because the minimal polynomial $m(t)$ for the minimal order recurrence relation on a function f necessarily divides the characteristic polynomial $p(t)$ for any recurrence on f.

Example 2.5.4. *Consider the weak rank-3 Frobenius function given by $f(i) = 3i$ for all $i \geq 0$. This function has a minimal order recurrence relation of degree 2, with minimal polynomial $m(t) = t^2 - 2t + 1 = (t-1)^2$. It follows that any degree 3 recurrence on f must have a characteristic polynomial of the form $p(t) = (t-1)^2(t-\gamma)$ for some $\gamma \in \mathbb{C}$. If f were to satisfy our modified version of Lemma 2.4.4, we would then require that $f(2) = 6 = 1 + 1 + \gamma = 2 + \gamma$ and $f(3) = 9 = 1^2 + 1^2 + \gamma^2 = 2 + \gamma^2$, an impossibility for any $\gamma \in \mathbb{C}$. Therefore, f can't satisfy a modified Theorem 2.4.5.*

2.6 Vacuum Hypothesis

In this section we present an interesting topological application of the algebraic machinery from Section 2.3 in the so-called "Vacuum Hypothesis":

Hypothesis 2.6.1. *Let Z be a 2-D TQFT with $Z(S^1) = A$. Then the Frobenius algebra A is equivalent to the algebra of orientable 2-D surfaces M such that $\delta M = S^1$, modulo surfaces that are "evaluated equivalently" by Z*

As we will see in Theorem 2.6.2, this proposition is true iff the associated Frobenius system is strong, lending additional weight to the strong/weak dichotomy as a signficant topological distinction.

As suggested by Hypothesis 2.6.1, for this one section we focus upon oriented

2-manifolds M such that $\partial M = S^1$. Any such M is diffeomorphic to s_i for some $i \geq 0$, where s_i is a (positively-oriented) copy of Σ_i with a single 2-disc removed. $M = s_i$ may be interpreted as a "handled cup" cobordism from the empty 1-manifold to S^1, so that applying our TQFT Z gives the k-linear map $Z(s_i) : k \to A$. The map $Z(s_i)$ is completely determined by where it sends $1 \in k$, allowing us to identify $Z(s_i)$ with the element $Z(s_i)(1) \in A$.

Now form the k-vector space B with countably-generated basis $\langle s_i \rangle$. B can be made into a k-algebra via the product $s_i s_j = s_{i+j}$, which may be interpreted geometrically via the "pants" cobordism (see Figure 2.3). If we define s_j^* to be s_j when it is alternatively viewed as a "handled cap" cobordism from S^1 to the empty 1-manifold, $s_j^* \circ s_i$ is the closed orientable surface of genus $i + j$. For any i, j we then have a map $Z(s_j^*) \circ Z(s_i) = Z(s_j^* \circ s_i) = Z(\Sigma_{i+j})$, which we identify with $\varepsilon(g^{i+j}) \in k$ via the same reasoning that we used in Section 2.2. For $j = 0$, this map induces a functional $\tilde{\varepsilon} : B \to k$ via $\tilde{\varepsilon}(s_i) = \varepsilon(g^i)$ (see Figure 2.4). We wish for this $\tilde{\varepsilon}$ to define a nondegenerate Frobenius form, making B into a Frobenius algebra, but before we consider that question we must pass to an appropriate quotient space.

Define a k-linear map $\phi_A : B \to A$ by $\phi_A(s_i) = g^i$, noticing that $\varepsilon(\phi_A(s_i)) = Z(s_i)(1)$. ϕ_A extends to a k-algebra morphism via $\phi_A(s_i s_j) = \phi_A(s_{i+j}) = g^{i+j} = g^i g^j = \phi_A(s_i)\phi_A(s_j)$. It is clear that ϕ_A is surjective iff the powers $\{1, g, g^2, ...\}$ span A as a k-vector space. Using our definition from Section 2.3, it follows that ϕ_A is onto iff (A, ε) is a strong Frobenius system. As such, we temporarily restrict our attention to strong Frobenius systems.

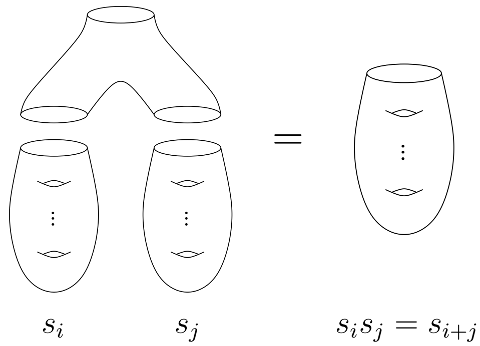


Figure 2.3: Multiplication in B

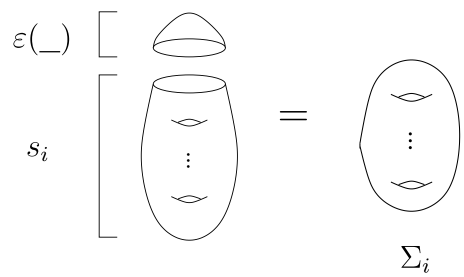


Figure 2.4: k-linear Functional in B

So assume that (A, ε) is a strong Frobenius system of rank n, and consider the kernel $I = ker(\phi_A)$. For any $m = \sum_i a_i s_i \in B$ (where $a_i \in k$), $m \in I$ iff $\phi_A(m) = \sum a_i g^i = 0$. Utilizing the nondegeneracy of ε and that $\{1, g, ..., g^{n-1}\}$ form a k-linear basis, this condition is equivalent to $\varepsilon(g^j \sum a_i g^i) = 0$ for all $j \geq 0$, which is equivalent to the fact that $Z(s_j^* \circ m) = Z(s_j^*) \circ Z(m)$ is the zero map for all $j \geq 0$. It follows that $m \in I$ iff every way of "closing off" the surface evaluates to zero via Z, so that the quotient algebra B/I identifies surfaces that "evaluate similarly" via Z. As (A, ε) was strong and ϕ_A was surjective, we then have $B/I \cong A$ as algebras. We also have $\varepsilon(\phi_A(s_i)) = \varepsilon(g^i) = \tilde{\varepsilon}(s_i)$, giving an equivalence of Frobenius structures. Since

ϕ_A is an isomorphism when the domain is taken to be B/I, this same equation also shows that $\tilde{\varepsilon}$ is a nondegenerate Frobenius form for B/I. This completes the primary proof of this section:

Theorem 2.6.2 (Vacuum Hypothesis)**.** *Let (A, ε) be a strong Frobenius system and let Z be the associated 2-D TQFT. Then A is Frobenius equivalent to the algebra of orientable 2-D surfaces M such that $\partial M = S^1$, modulo surfaces that are "evaluated equivalently" by Z.*

Now consider the case where (A, ε) is a weak Frobenius system. As the subalgebra $G = \langle g^i \rangle$ generated by the powers of g is no longer all of A, we alternatively have that $B/I \cong G \subsetneq A$ as algebras. It follows that the Vacuum Hypothesis holds iff the associated Frobenius system is strong; if the associated Frobenius system is weak, the best we can say is that our algebra of orientable 2-D surfaces is Frobenius equivalent to some proper subalgebra of A.

Unfortunately, there is an additional difficulty that arises when attempting to apply the Vacuum Hypothesis in the weak case. The problem here is that ε, which was assumed to be nondegenerate over all of A, may not be a nondegenerate Frobenius form $\varepsilon|_G$ when restricted to G. In Section 2.3 we noted how this corresponded to the case where the associated Frobenius function possessed a recurrence relation of degree less than m, where $\{1, g, ..., g^{m-1}\}$ was the largest such linearly independent set. This nuisance prevents us from extending the algebra equivalence $B/I \cong G$ to a Frobenius equivalence, because G doesn't necessarily possesses the Frobenius structure required in the proof of Theorem 2.6.2 (it may still possess some other Frobenius structure).

CHAPTER 3
UNIVERSAL $SL(N)$ SKEIN MODULES

In Subsection 1.2.3 we introduced Bar-Natan's marked cobordism category $\mathbf{2Cob}_A/l$ and described how those cobordisms could be embedded within a 3-manifold to produce a skein module diffeomorphism invariant. All of this way done with respect to the 2-D TQFT whose corresponding rank-2 Frobenius system is given below:

$$A = \mathbb{C}[x]/(x^2)$$

$$\varepsilon(1) = 0,\ \varepsilon(x) = 1$$

This particular TQFT was important in that it provides the algebraic structure underlying the boundary maps in Khovanov's link homology theory. In the original work of Khovanov [11] and Bar-Natan [3], the underlying algebraic structure was cast in the more general setting of Frobenius extensions, where the field $\mathbb{C}$ is replaced by the commutative ring $\mathbb{Z}$. We then have the ring extension $R \hookrightarrow A$:

$$R = \mathbb{Z}$$

$$A = \mathbb{Z}[x]/(x^2)$$

$$\varepsilon(1) = 0,\ \varepsilon(x) = 1$$

This approach presents more technical difficulties, but one may still prove a generalization of Theorem 1.2.7 that gives an equivalence of categories between Frobenius extensions $R \hookrightarrow A$ (R commutative with 1) and TQFTs over R. In Section 3.1 will introduce the framework for Frobenius extensions that we will use throughout this chapter (although, apart from several proofs, the casual reader will recognize little need for this extra machinery and is safe in replacing R with $\mathbb{C}$).

In the time since Khovanov introduced his aforementioned homology theory [11], a number of related Frobenius systems have been utilized to produce link homology theories. These include the rank-2 theory of Lee [16], the "universal" $sl(2)$ theories of Khovanov [13], the rank-3 $sl(3)$ theory of Khovanov [12], the "universal $sl(3)$ theory of Mackaay and Vaz [18], and the $sl(n)$ systems of Khovanov and Rozansky [14]. To pick a few illustrating examples, the Frobenius systems underlying the boundary maps in those theories are shown below for the "universal" $sl(2)$ case, "universal" $sl(3)$ case, and the $sl(n)$ case:

$$R = \mathbb{Z}[a,b], \text{ where } a,b \in \mathbb{C}$$

$$A = \mathbb{Z}[a,b][x]/(x^2 - ax - b)$$

$$\varepsilon(1) = 0 \;\; \varepsilon(x) = 1$$

$$R = \mathbb{Z}[a,b,c], \text{ where } a,b,c \in \mathbb{C}$$

$$A = \mathbb{Z}[a,b,c][x]/(x^3 - ax^2 - bx - c)$$

$$\varepsilon(1) = \varepsilon(x) = 0 \;\; \varepsilon(x^2) = 1$$

$$R = \mathbb{Z}$$

$$A = \mathbb{Z}[x]/(x^n)$$

$$\varepsilon(x^i) = \delta_{i,(n-1)}, \text{ for } 0 \geq i \geq n-1$$

The first major goal of this chapter, presented in Section 3.2, will be to generalize Bar-Natan's marked cobordism category to the class of "universal" $sl(n)$ TQFTs.

These "universal" $sl(n)$ TQFTs have associated Frobenius extensions (henceforth referred to as universal $sl(n)$ Frobenius extensions) of the form presented below. As a result, their marked cobordism categories directly generalize those of the specific systems mentioned above, and they would form the first step in developing a "universal" rank-n foam category **Foam** that underlies the link homology theories in papers such as [18] and [5].

$$R = \mathbb{Z}[a_1, ..., a_n], \text{ where } a_1, ..., a_n \in \mathbb{C}$$

$$A = \mathbb{Z}[a_1, ..., a_n][x]/(p(x)), \text{ where } p(x) = x^n - a_1x^{n-1} - ... - a_n$$

$$\varepsilon(x^i) = \delta_{i,(n-1)}, \text{ for } 0 \leq i \leq n-1$$

In Subection 3.2.1 we will present our primary theorem about the marked cobordism categories associated with $sl(n)$ Frobenius extensions, which gives us an extremely nice condition for when higher-genus closed surfaces evaluate to zero via the associated TQFT. Shown below, this theorem is the primary piece linking the diffeomorphism invariant results of Chapter 2 (and the genus-reduction term g) with universal $sl(n)$ skein modules, as it gives a fairly distinct categorization of the diffeomorphism invariants that may come from $sl(n)$ systems. In fact, this theorem has already been cited in some of the examples from Subsection 2.3.2.

Theorem 3.0.3. *Let* $R = \mathbb{Z}[a_1, ..., a_n] \hookrightarrow A = \mathbb{Z}[a_1, ..., a_n][x]/(p(x))$ *be a* $sl(n)$ *Frobenius extension of rank* $n \geq 2$. *If every root of* $p(x)$ *is a repeated root, then* $g^i = 0 \in A$ *for all* $i \geq 2$. *Otherwise* $g^i \neq 0 \in A$ *for all* $i \geq 2$.

We close this chapter with Section 3.3, which applies the results of Section 3.2 towards the development of the related skein module invariants. Our primary theorem from that section, which directly generalizes one of the central results from [1], is the following.

Theorem 3.0.4. *Let M be an irreducible 3-manifold, and let $K_A(M)$ denote the universal $sl(n)$ skein module of M induced by the rank-n Frobenius extension $R \hookrightarrow A$ defined above. If every root of $p(x)$ is repeated, then the unmarked incompressible surfaces in $K_A(M)$ are linearly independent over $R = \mathbb{Z}[a_1, ..., a_n]$.*

3.1 Frobenius Extensions

In this section we introduce the notion of Frobenius extensions, which are a direct generalization of commutative Frobenius algebra in that they replace the underlying base field k with an arbitrary commutative ring R (always with 1). As such, we will essentially be working with a commutative ring A that admits a compatible R-module structure as well as a non-degenerate trace map $\varepsilon : A \to R$ (when this latter notion is suitable defined). To prove things in their greatest generality, we will instead approach this definition from an angle that makes use of homological algebra and category theory. Our approach here mimics that taken by Khovanov in [13], while many of the definitions follow from Kadison [9].

We begin with a ring extension $\iota : R \hookrightarrow A$ of commutative rings with 1 such that $\iota(1) = 1$. The map ι makes A into an R-bimodule via $r * a = \iota(r)a = a\iota(r) = a * r$, where transpoition simply denotes ring multiplication in A. We may then define the standard restriction functor $R : \textbf{A-mod} \to \textbf{R-mod}$ by $r * a = \iota(r) * m$ for all elements m of our A-module M. This prompts our most fundamental definition of a Frobenius extension:

Definition 3.1.1. *A ring extension $\iota : R \hookrightarrow A$ of commutative rings with 1 (where $\iota(1) = 1$) is a Frobenius extension iff the restriction functor R : **A-mod** $\to$ **R-mod** has a two-sided adjoint.*

In order to ensure the desired equivalence between Frobenius extensions of R and TQFTs over R, for the rest of this section we furthermore assume that A is finitely-generated and projective as an R-module.

As is well-known from homological algebra, the left and right adjoints of the restriction functor are the induction functor $T : M_R \mapsto (M \otimes_R A)_A$ and coinduction functor $H : M_R \mapsto (Hom_R(A, M))_A$, respectively. In light of the definition above, our ring extension is then Frobenius iff these two functors $T, H : \textbf{R-mod} \to \textbf{A-mod}$ are isomorphic. For the particular choices of $M = A$ and $M = R$, this functor isomorphism specializes to A-linear isomorphisms $End_R(A) \cong A \otimes_R A$ and $A^* \cong A$. For A finite projective, the latter isomorphism prompts the equivalent definition of a Frobenius extension $R \hookrightarrow A$ as a ring extension such that A is self-dual as a R-module.

When A is taken to be a finite projective R-module, yet another equivalent definition of a Frobenius extension is a ring extension $R \hookrightarrow A$ such that A has a coassociative, cocommutative A-bimodule comultiplication map $\Delta : A \to A \otimes_R A$ as well as an R-linear counit map $\varepsilon : A \to R$. The equivalence of these different definitions is thoroughly proven in [9], although that text treats the even more general case where R and A aren't necessarily commutative. That proof is largely analogous to the equivalence relating the different definitions of commutative Frobenius algebra introduced in Subection 1.2.2. The generality of our original definition for Frobenius extension also allows the following definition [9], which will be the foremost definition of Frobenius extension utilized in this Chapter.

Definition 3.1.2. *A* ***Frobenius extension*** *is a finitely-generated, projective ring extension $R \hookrightarrow A$ of commutative rings (with 1) in possession of a non-degenerate R-linear trace map $\varepsilon : A \to R$ and a collection of tuples $(x_i, y_i) \in A \times A$ such that $a = \sum_i x_i \varepsilon(y_i a) = \sum_i \varepsilon(a x_i) y_i$ for all $a \in A$.*

Analogously to the situation with commutative Frobenius algebras, the map ε is known as the Frobenius form and is directly identified with the counit map mentioned in the previous paragraph. Nondegeneracy of ε once again means that there are no (principal) ideals in the nullspace of ε. The set of tuples (x_i, y_i) constitute our dual-bases, and satisfy the familiar condition that $\varepsilon(x_i y_j) = \delta_{i,j}$. Here we choose to directly specify R as part of the Frobenius system, and write $(R, A, \varepsilon, (x_i, y_i))$ or (R, A, ε) when the choice of dual-bases is irrelevant.

Many of the additional properties of commutative Frobenius algebras that we discussed in Subsection 1.2.2 carry over to Frobenius extensions, including the R-module isomorphism $A \otimes A \cong End(A)$ and the so-called ε-multiplication on $A \otimes A$: $(a \otimes b)(a' \otimes b') = a\varepsilon(ba') \otimes b' = a \otimes \varepsilon(ba')b'$. More importantly, our basic results about the genus-reduction term g in Chapter 2 (Lemma 2.2.1, Lemma 2.2.2, Corollary 2.2.3, Lemma 2.2.4) directly generalize to Frobenius extensions.

One fact that is slightly different for Frobenius extension is the notion of Frobenius equivalence. When the underlying algebras $A = \tilde{A}$ are the same, Kadison and others define (R, A, ε) and $(R, A, \tilde{\varepsilon})$ to be Frobenius equivalent iff $\varepsilon = \tilde{\varepsilon}$, which occurs iff $\sum(x_i \otimes y_i) = \sum(\tilde{x}_i \otimes \tilde{y}_i)$ for any choice of dual-bases. Luckily, the generalization of Lemma 2.2.1 to Frobenius extensions quickly shows that this notion is equivalent to our original version of Frobenius equivalence. For us, a Frobenius equivalence will alaways be an equivalence of ring extensions along with the equality $\varepsilon = \tilde{\varepsilon} \circ \phi$.

3.2 Universal $sl(n)$ Frobenius Extensions

As noted in the introduction to this chapter, we focus upon rank-n Frobenius extensions (R, A, ε) of the form $A = R[x]/(p(x))$ for some monic degree-n polynomial $p(x) \in R[x]$, and with Frobenius form defined on the standard basis $\{1, x, ..., x^{n-1}\}$ by $\varepsilon(x^i) = \delta_{i,(n-1)}$. Throughout the rest of this chapter we will use the notation $p(x) = x^n - a_1 x^{n-1} - ... - a_n$. We henceforth refer to such extensions as **universal $sl(n)$ Frobenius extensions**. Versions of many of the results from this particular section originally appeared in [4], albeit occasionally in less general form.

In the spirit of Khovanov we choose to work over the commutative ring $R = \mathbb{Z}[a_1, ..., a_n] \subset \mathbb{C}$, adjoining "just enough" to $\mathbb{Z}$ to ensure that our defining polynomials $p(x)$ lie in $R[x]$. This will increase the difficulty of several proofs, as we will frequently need to pass to larger rings $\tilde{R}$ containing R and then argue that our desired results descend back down to R. For this reason, the less enthusiastic reader is encouraged to replace $\mathbb{Z}[a_1, ..., a_n]$ with the "easy choice" of $\mathbb{C}$, negating the need for the more general algebraic machinery of Section 3.1 but producing comparable results.

Our first step towards understanding universal $sl(n)$ Frobenius extensions is the following lemma, which gives a general dual-basis with respect to the standard basis $\{1, x, ..., x^{n-1}\}$:

Lemma 3.2.1. *Let (R, A, ε) be a universal $sl(n)$ Frobenius extension. With respect to the standard basis $\{1, x, ..., x^{n-1}\}$, (R, A, ε) has dual-bases given by:*

$$\begin{gathered}\{(x^{n-1}, 1),\\ (x^{n-2}, x - a_1),\\ (x^{n-3}, x^2 - a_1 x - a_2),\\ \vdots\\ (1, x^{n-1} - a_1 x^{n-2} - \ldots - a_{n-2}x - a_{n-1})\}\end{gathered}$$

Proof. Let $\lambda = [[\varepsilon(x^{i+j-2})]]$ be the (invertible) Frobenius matrix with respect to the basis $\{1, x, ..., x^{n-1}\}$. Recall that we may find the dual basis $\{y_0, ..., y_{n-1}\}$ via the matrix equation $\lambda Y = X$, where $X = [1\ x\ ...\ x^{n-1}]^T$ and $Y = [y_0\ ...\ y_{n-1}]^T$. Direct computation shows that, after reducing modulo $p(x)$, the inverse of λ is:

$$\lambda^{-1} = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \ldots & -a_1 & 1 \\ -a_{n-2} & \ldots & \ldots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_1 & 1 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & \ldots & 0 \end{bmatrix}$$

$\square$

In the rank $n = 2$ case, the dual-bases above specializes to the dual-bases $\{(x, 1), (1, x-a_1)\}$ given in [13]. For $n = 3$ we have the same dual-bases $\{(x^2, 1), (x, x-a_1), (1, x^2 - a_1x - a_2)\}$ as from [18].

With a generalized dual-bases in hand, we are now ready to generalize Bar-Natan's marked cobordism category $\mathbf{2Cob}_A/l$ from Subsection 1.2.3. Now we are working in the R-linear category whose objects are cobordisms $Q \in \mathbf{2Cob}_A$ with components marked by elements of A. These cobordisms are subject to the three sets of local relations l below, which one again take the form of sphere relations, a "dot reduction" relation, and a neck-cutting relation. Notice that, in this general rank $n \geq 2$ case, Bar-Natan's usage of a dot to denote $x \in A$ becomes rather unwieldly. As such, we abandon the dot notation and directly write the actual elements of A upon our cobordism components.

1) Sphere Relations: As in Subsection 1.2.3, the sphere relations follow directly from our definition of the Frobenius form. Here we have $\varepsilon(x^{n-1}) = 1$ and $\varepsilon(x^i) = 0$ for $0 \leq i \leq n-2$, which give the $n-1$ relations of Figure 3.1.

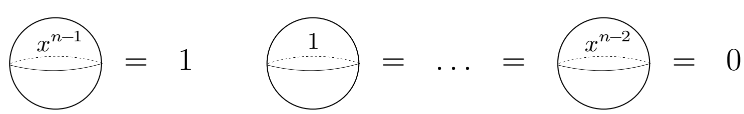


Figure 3.1: Universal $sl(n)$ Sphere Relations

2) "Dot Reduction" Relation: Our algebra A has a single generating relation in $p(x) = x^n - a_1x^{n-1} - ... - a_n = 0$, so that $x^n = a_1x^{n-1} - ... - a_n$. Thus, due to the R-linearity of $\mathbf{2Cob}_A$, all allowable ways to decorate cobordism components are generated by the single "dot-reduction" relation of Figure 3.2.

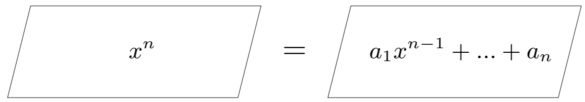


Figure 3.2: Universal $sl(n)$ "Dot Reduction" Relation

3) Neck-Cutting Relation: In Lemma 3.2.1 we derived dual-bases for our universal $sl(n)$ Frobenius extensions. Via the same reasoning as from Subsection 1.2.3, the dual-bases elements appear in pairs on the right-hand side of the neck-cutting relation. In Figure 3.3 below, we have utilized the R-linearity of $\mathbf{2Cob}_A$ and regrouped terms on the right-side in terms of their coefficients $a_i \in R$.

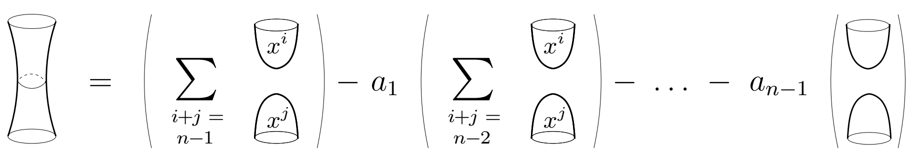


Figure 3.3: Universal $sl(n)$ Neck-Cutting Relation

3.2.1 Neck-Cutting in $sl(n)$ Frobenius Extensions

In this subsection we examine the effect of neck-cutting in universal $sl(n)$ Frobenius extensions. When applied to closed cobordisms, this allows us to detemine how the 2-D TQFT associated with the given extension evaluates closed, oriented 2-D surfaces. This represents the most explicit link between our examinations in Chapter 3 and the diffeomorphism invariants discussion from Chapter 2.

Our first result gives the genus-reduction term g for a universal $sl(n)$ Frobenius extension:

Lemma 3.2.2. *Let (R, A, ε) be a universal $sl(n)$ Frobenius extension, where $A = R[x]/(p(x))$ for the degree-n monic polynomial $p(x) = x^n - a_1x^{n-1} - ... - a_n \in R[x]$. The genus-reduction term of (R, A, ε) is $g = p'(x)$, the derivative of $p(x)$.*

Proof. Recalling that $g = \sum_i x_i y_i$ for any dual-bases (x_i, y_i), this follows directly from our calculation of the universal $sl(n)$ dual-bases in Lemma 3.2.1. Namely:

$$\begin{aligned} g &= (x^{n-1})(1) + (x^{n-2})(x - a_1) + ... + (1)(x^{n-1} - a_1x^{n-2} - ... - a_{n-1}) \\ &= nx^{n-1} - (n-1)a_1x^{n-2} - (n-2)a_2x^{n-3} - ... - a_{n-1} \end{aligned}$$

□

Although writing $p(x) = x^n - a_1x^{n-1} - ... - a_n$ was useful to show that $g = p'(x)$, for most of our purposes it will be much more desirable to fully factor $p(x)$ over $\mathbb{C}$. We write $p(x) = \prod_{i=1}^{n}(x + \alpha_i)$, so that we may relate the original coefficients a_i to the elementary symmetric polynomials of $\{\alpha_1, ..., \alpha_n\}$ as $a_k = -e_k$.

Notice how this technique mirrors our approach to strong Frobenius systems in Section 2.4 (systems whose underlying algebra could always be rewritten so that it had a single generating relation, as is the case here with $p(x) = 0$). However, there are several significant differences in this chapter. First off, here we make the (primarily aesthetic) choice of factoring so that the α_i are actually the negatives of our roots. This allows us to assert that $a_k = -e_k$ for all $1 \leq k \leq n$, thus avoiding the introduction of alternating $(-1)^k$ terms in our calculations.

More importantly, in this chapter we do not pass entirely to the algebraic closure $\mathbb{C}$. This causes some difficulty in that our roots α_i may not all lie in our base ring $R = \mathbb{Z}[a_1, ..., a_n] = \mathbb{Z}[e_1, ..., e_n]$, although we will always have the inclusion $R \subseteq \tilde{R} = R[\alpha_1, ..., \alpha_n]$. As a result, some of our proofs will require us to temporarily pass to the "larger" Frobenius extension $\tilde{R} \hookrightarrow \tilde{R}[x]/(p(x))$ (with identical Frobenius form and dual-bases). With a little extra work, all our results will then descend back down to become valid proofs in the original Frobenius system.

In hopes of motivating our upcoming theorems, we stop to analyze the effect of neck-cutting in the relatively simple $n = 2$ case. The universal $sl(2)$ Frobenius extension has $p(x) = x^2 - a_1x - a_2 = (x+\alpha_1)(x+\alpha_2)$, giving $g = 2x - a_1 = 2x + (\alpha_1 + \alpha_2)$ by Lemma 3.2.2. Here we have:

$$g^2 = (2x - a_1)^2 = 4x^2 - 4a_1x + a_1^2 = 4(a_1x + a_2) - 4a_1x + a_1^2$$
$$= 4a_2 + a_1^2 = -4\alpha_1\alpha_2 + (-\alpha_1 - \alpha_2)^2 = \alpha_1^2 - 2\alpha_1\alpha_2 + \alpha_2^2 = (\alpha_1 - \alpha_2)^2$$

It follows that $g^2 = 0 \in A$ (and hence $g^i = 0 \in A$ for all $i \geq 2$) iff the two roots of $p(x)$ coincide. Theorems 3.2.4 and 3.2.6 will provide a direct generalization of this result to all $n \geq 2$.

In the $n = 2$ case we also have the peculiar phenomenon that $g^2 = (\alpha_1 - \alpha_2)^2 \in R$ is a constant, no matter our choice of $p(x)$. This fact makes it very easy to explicitly determine all powers of g. Namely, $g^{2i} = (4a_2 + a_1^2)^i = (\alpha_1 - \alpha_2)^{2i}$ and

$g^{2i+1} = (4a_2 + a_1^2)^i(2x - a_1) = (\alpha_1 - \alpha_2)^{2i}(2x + \alpha_1 + \alpha_2)$ for all $i \geq 0$. Multiplying those equations by x also gives $x * g^2 i = (4a_2 + a_1^2)^i x = (\alpha_1 - \alpha_2)^{2i} x$ and $x * g^{2i+1} = (4a_2 + a_1^2)^i(2x^2 - a_1 x) = (4a_2 + a_1^2)^i(a_1 x + 2a_2) = (\alpha_1 - \alpha_2)^{2i}(-\alpha_1 x - \alpha_2 x - 2\alpha_1\alpha_2)$ for all $i \geq 0$

Recall from Chapter 2 that a closed oriented surface of genus-i that is marked by $a \in A$ evaluates to $\varepsilon(ag^i) \in R$ via the TQFT Z associated to the given Frobenius system. If we denote an unmarked closed, oriented surface of genus-i by Σ_i and that same surface marked by x as $\dot{\Sigma}_i$, we can explicitly show how the associated TQFT evaluates any marked, closed cobordism:

$$Z(\Sigma_{2i}) = \varepsilon(g^{2i}) = 0$$
$$Z(\Sigma_{2i+1}) = \varepsilon(g^{2i+1}) = 2(4a_2 + a_1^2)^i = 2(\alpha_1 - \alpha_2)^{2i}$$
$$Z(\dot{\Sigma}_{2i}) = \varepsilon(x * g^{2i}) = (4a_2 + a_1^2)^i = (\alpha_1 - \alpha_2)^{2i}$$
$$Z(\dot{\Sigma}_{2i+1}) = \varepsilon(x * g^{2i+1}) = (4a_2 + a_1^2)^i a_1 = -(\alpha_1 - \alpha_2)^{2i}(\alpha_1 + \alpha_2)$$

Unlike our observation about repeated roots above, this result does not extend to arbitrary $n \geq 2$. In particular, for higher n it is difficult to guarantee that $g^i \in R$ is a constant for any $i > 0$. See Example 3.2.3 below for an especially pathological $sl(3)$ Frobenius extension. Via brute force, one can still hope to calculate closed cobordism evaluations in universal $sl(n)$ Frobenius extensions via linear algebra; Subsection 3.2.2 is dedicated towards investigating this approach.

Example 3.2.3. *Consider the $sl(3)$ Frobenius extension with $p(x) = x^3 - x$ (this is actually the same problematic rank-3 system that already provided us with a valuable* *counterexample in Example 2.3.7). Here we have* $g = 3x^2 - 1$ *and* $g^2 = (3x^2-1)^2 = 9x^4 - 6x^2 + 1 = 3x^2 + 1 = g + 2$. *As* $g^i = g^{i-1} + 2g^{i-2}$ *for all* $i \geq 2$, *we may prove by induction on* i *that* $g^i = ag + b$ $(a, b \in R,\ a \neq 0)$ *for all* $i \geq 2$. *As* g *contains an* x^2 *term, every expression of the form* $ag + b$ *with* $a \neq 0$ *contains a nonzero* x^2 *term. Thus* $g^i \notin R$ *for all* $i \geq 0$.

We now return to the promised rank-n analogue of the $g^2 = 0$ iff $\alpha_1 = \alpha_2$ situation above for $n = 2$. The appropriate generalization is that $g^2 = 0$ iff every root of $p(x)$ is a repeated root- a theorem whose two implications will be presented separately as Theorems 3.2.4 and 3.2.6. The right-to-left implication of this result proves to be the more immediate of the two directions:

Theorem 3.2.4. *Let* (R, A, ε) *be the* $sl(n)$ *Frobenius extension with* $p(x) = x^n - a_1x^{n-1} - \ldots - a_n = \prod_{i=1}^{n}(x + \alpha_i)$. *If every* $\alpha_i \in R$ *is a repeated root, then* $g^2 = 0$ *in* A.

Proof. We temporarily pass to the "larger" Frobenius extension with $\tilde{R} = \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$, $\tilde{A} = R[x]/(p(x))$, and first show that $g^2 = 0$ in $\tilde{A}$. So let $p(x)$ be defined as above, and assume that every root α_i is of multiplicity ≥ 2. By Lemma 3.2.2 we have $g = p'(x)$, which allows us to use the product rule for derivates to give $g = \sum_{i=1}^{n} \frac{p(x)}{(x+\alpha_i)}$. Hence $g^2 = \sum_{i,j=1}^{n} \frac{p(x)^2}{(x+\alpha_i)(x+\alpha_j)}$. As every α_i is a repeated root, $(x+\alpha_i)(x+\alpha_j)$ divides $p(x)$ for all i, j (even when $i = j$). Thus $g^2 = p(x)\sum_{i,j=1}^{n} \frac{p(x)}{(x+\alpha_i)(x+\alpha_j)} = p(x)f(x) \in \tilde{R}[x]$ and $p(x)$ divides g^2 in $\tilde{R}[x]$. It follows that $g^2 = 0$ in $\tilde{A}$.

It is only left to show that the above implies $g^2 = 0$ in A when we descend back down to the original Frobenius extension. To this end, define $e_k^{\alpha_i\alpha_j}$ to be the k$^{\text{th}}$ elementary symmetric polynomial in the $n-2$ roots of $p(x)$ apart from α_i and

α_j. Expanding the summation $f(x) = \sum_{i,j=1}^{n} \frac{p(x)}{(x+\alpha_i)(x+\alpha_j)}$, the coefficient of x^k from each term in the summation is $e_{n-2-k}^{\alpha_i\alpha_j}$. The complete coefficient of x^k for $f(x)$ is then $c_k = \sum_{i,j=1}^{n} e_{n-2-q}^{\alpha_i\alpha_j} \in Z[\alpha_1, ..., \alpha_n]$. As the c_k sum over all $1 \leq i, j \leq n$, each is a symmetric polynomial in all n of the roots $\{\alpha_1, ...\alpha_n\}$. By the Fundamental Theorem of Symmetric Functions we know that the elementary symmetric polynomials in any fixed set of variables generate the set of all symmetric polynomials in those roots. It follows that we actually have $c_k \in \mathbb{Z}[e_1, ..., e_n] = \mathbb{Z}[a_1, ..., a_n]$ for all k. Thus $f(x) \in R[x]$ and $p(x)$ divides g^2 in $R[x]$, showing that $g^2 = 0$ in A. □

The import of Theorem 3.2.4 is that, in universal $sl(n)$ Frobenius extensions whose generating polynomials $p(x)$ have no multiplicity-one roots, we immediately know that all surfaces (marked or unmarked) of genus $i \geq 2$ evaluate to zero via the associated TQFT. More generally, any cobordism component admitting multiple non-separating compressions must evaluate to zero via the associated TQFT. This final insight will prove particularly useful in Section 3.3, when we look to give presentations of universal $sl(n)$ skein modules embedded within an arbitrary 3-manifold.

For the rest of this subsection we work towards the converse of Theorem 3.2.4 and draw a few quick corollaries. The significantly more involved approach required by this direction of the theorem results from the fact that, although the summation from the proof of Theorem 3.2.4 may be easily reduced to $\sum \frac{p(x)^2}{(x+\alpha_i)^2}$ (the summation being over the non-repeated roots α_i of $p(x)$), it is surprisingly difficult to show that this remaining summation is nonzero in A.

As was required in the proof of Theorem 3.2.4, we begin by passing to the "larger" Frobenius extension with $\tilde{R} = \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$ and $\tilde{A} = \tilde{R}[x]/(p(x))$ to ensure that all of the roots of $p(x)$ are in the base ring. The difference here is that we will need to prove several full lemmas in this larger extension, which runs against the tradition (begun by Khovanov) that we adjoin "just enough" to $\mathbb{Z}$ when defining our Frobenius extensions. Our approach is justified by the fact that our enlarged Frobenius extension $\tilde{R} \hookrightarrow \tilde{A}$ has the same Frobenius form and dual-bases as the original extension $R \hookrightarrow A$, thus producing an identical genus-reduction term $\tilde{g} = g$ that is being reduced modulo the same polynomial $p(x)$.

Our reliance upon the enlarged Frobenius extension is explained by the following lemma, which is a direct application of the Chinese Remainder Theorem:

Lemma 3.2.5. *Let $\tilde{R} = \mathbb{Z}[\alpha_1, ..., \alpha_n]$ and $\tilde{A} = \tilde{R}[x]/(p(x))$. Consider the "enlarged" $sl(n)$ Frobenius extension $(\tilde{R}, \tilde{A}, \varepsilon)$ with $p(x) = \prod_{i=1}^{n}(x+\alpha_i) = \prod_{i=1}^{m}(x+\alpha_i)^{k_i}$, where in the second product we have fully grouped like roots. This Frobenius extension is Frobenius equivalent to the Frobenius extension $(\tilde{R}, \hat{A}, \hat{\varepsilon})$ defined as:*

$$\hat{A} = \tilde{R}[x]/((x+\alpha_1)^{k_1}) \times \ldots \times \tilde{R}[x]/((x+\alpha_m)^{k_m})$$

$$\hat{\varepsilon}(x^{n-1}, \ldots, x^{n-1}) = 1$$

$$\hat{\varepsilon}(x^{i}, \ldots, x^{i}) = 0 \text{ (for } 0 \le i \le n-2\text{)}$$

Proof. The $\tilde{R}$-linear isomorphism $\phi : \tilde{A} \to \hat{A}$ underlying the Frobenius equivalence is a direct consequence of the Chinese Remainder Theorem, and is given by $\phi(a) = (a, \ldots, a)$. That $\tilde{A} = \hat{A} \circ \phi$ is immediate from the definition of ϕ. Note how $\tilde{A}$ containing no non-trivial ideals in its null-space ensures the same about $\hat{A}$, making $\hat{A}$ a valid Frobenius form. □

Although Lemma 3.2.5 is nice in that it allows us to pass to an equivalent Frobenius extension in which our original algebra has been decomposed into simpler pieces, it suffers from the fact that the Frobenius structure emplaced on $\hat{A}$ isn't the "natural" one. In particular, we would like to put a Frobenius structure on $\hat{A}$ that brings together the $sl(n)$ Frobenius structures on each of the coordinates. This is necessary because we have done none of the prerequisite work towards determining the dual-bases (and hence the genus-reduction term) for more "exotic" Frobenius structures. The "natural" Frobenius structure that we want for $\tilde{R} \hookrightarrow \hat{A}$ is the following:

- Basis that brings together the bases of each of the coordinates:

$$\{(1,0,\ldots,0),\ldots,(x^{k_1-1},0,\ldots,0),(0,1,0,\ldots,0),\ldots,\ldots,(0,\ldots,0,x^{k_m-1})\}$$

- Frobenius form ε' defined on that basis so that in the i$^{\text{th}}$ coordinate it corresponds with the Frobenius form on the projection $\pi_i(\hat{A})$:

$$\varepsilon'(x^{k_1-1},0,\ldots,0)=\ldots=\varepsilon'(0,\ldots,0,x^{k_m-1})=1$$
$$\varepsilon'(u)=0 \text{ for every other element } u \text{ of the basis}$$

This "natural" Frobenius structure is desirable in that its Frobenius matrix λ' is block diagonal, with one block for each distinct root α_i of $p(x)$ (one for each coordinate of $\hat{A}$). If α_i is a root of multiplicity 1, its corresponding block is the 1×1 constant matrix [[1]]. If α_i is a root of multiplicity $k_i \geq 2$, its corresponding block is $k_i \times k_i$ and has entries identical to the Frobeius matrix λ_i of the $sl(k_i)$ Frobenius extension with $R_i = \mathbb{Z}[\alpha_i]$, $A_i = R_i[x]/(p_i(x))$, and generating polynomial

$p_i(x) = (x - \alpha_i)^{k_i}$. As each coordiante defines a valid $sl(k_i)$ Frobenius extension (or, in the case of $k_i = 1$, is always nonzero), each of the blocks is invertible. It follows that λ' is invertible, ensuring that ε' is in fact a nondegenerate Frobenius form.

The inverse $(\lambda')^{-1}$ is also block diagonal, with blocks [[1]] or $(\lambda_i)^{-1}$. This allows us to compute the dual-bases of our Frobenius extension in terms of the dual-bases on the coordinates. If $y_{i,j}$ is the dual-basis companion of x^{j-1} for the $sl(k_i)$ Frobenius extension $R_i \hookrightarrow A_i$ in the i$^{\text{th}}$ coordinate, our dual basis is:

$$\{(1, 0, \ldots, 0), (y_{1,1}, 0, \ldots, 0)\}, \ldots, \{(x^{k_1-1}, 0, \ldots, 0), (y_{1,k_1}, 0, \ldots, 0)\},$$

$$\{(0, 1, 0, \ldots, 0), (0, y_{2,1}, 0, \ldots, 0)\}, \ldots, \ldots, \{(0, \ldots, 0, x^{k_m-1}), (0, \ldots, 0, y_{m,k_m})\}$$

The genus-reduction term for our "natural" Frobenius structure on $\hat{A}$ is then $g' = (g_1, ..., g_m)$, where g_i is the genus-reduction term for the $sl(k_i)$ Frobenius extension $R_i \hookrightarrow A_i$ (and $g_i = 1$ in the coordinates corresponding to multiplicity 1 roots α_i). We also have $(g')^i = (g_1^i, ..., g_m^i)$ for all $i \geq 1$. By Theorem 3.2.4 we have that $g_j^2 = 0$ iff $k_j \geq 2$, so that $(g')^2 = (g_1^2, ..., g_m^2) = (0, ..., 0) = 0$ iff every root of $p(x)$ has multiplicity at least 2. We are finally ready for the converse of Theorem 3.2.4:

Theorem 3.2.6. *Let (R, A, ε) be the $sl(n)$ Frobenius extension with $p(x) = x^n - a_1x^{n-1} - ... - a_n = \prod_{i=1}^n (x + \alpha_i)$. If $p(x)$ has at least one root of multiplicity precisely 1, then $g^2 \neq 0$ in A.*

Proof. Let $p(x) = \prod_{i=1}^m (x + \alpha_i)^{k_i}$, where we have completely grouped like roots, and assume without loss of generality that $k_1 = 1$. As in the proof of Theorem 3.2.4, we pass to the "larger" Frobenius extension $\tilde{R} \hookrightarrow \tilde{A}$ and first show that $\tilde{g} = g \neq 0$ in $\tilde{A}$.

So take the "enlarged" Frobenius extension $(\tilde{R}, \tilde{A}, \tilde{\varepsilon})$. By Lemma 3.2.5, this extension is Frobenius equivalent to the "product" Frobenius extension $(\tilde{R}, \hat{A}, \hat{\varepsilon})$. By the preceding paragraphs, there exists a second Frobenius structure $(\tilde{R}, \hat{A}, \varepsilon')$ over $\tilde{R} \hookrightarrow \hat{A}$ such that $\hat{g}^2 \neq 0$. By Lemma 2.2.3, $\tilde{g}^2 = (d^{-1})^2 \hat{g}^2$ for some invertible $d^{-1} \in A$, which ensures that $\tilde{g}^2 \neq 0$ in $\tilde{A}$. As such, there cannot exist $\tilde{f}(x) \in \tilde{R}[x]$ so that $p(x)\tilde{f}(x) = \tilde{g} \in \tilde{A}$. It immediately follows that there cannot exit $f(x) \in R[x] \subseteq \tilde{R}[x]$ so that $p(x)f(x) = \tilde{g} = g \in A$, guaranteeing that $g^2 \neq 0$ over our original Frobenius extension (R, A, ε). □

Given the commentary concerning $\hat{g}^i$ (for $i \geq 2$) that directly precedes Theorem 3.2.6, an equivalent argument shows that $g^i \neq 0$ in A if $p(x)$ has at least one root of multiplicity precisely 1. Combining Theorems 3.2.4 and 3.2.6 with this observation gives a particularly useful corollary:

Corollary 3.2.7. *Let (R, A, ε) be the $sl(n)$ Frobenius extension ($n \geq 2$) with $p(x) = x^n - a_1 x^{n-1} - \ldots - a_n = \prod_{i=1}^{n}(x + \alpha_i)$. If every root α_i of $p(x)$ is a repeated root, then $g^i = 0 \in A$ for all $i \geq 2$. Otherwise, $g^i \neq 0 \in A$ for all $i \geq 2$.*

This corollary imples that the 2-D TQFTs associated to $sl(n)$ Frobenius extensions whose $p(x)$ have multiplicity 1 roots may become extremely complicated, at least when it comes to the evaluation of closed 2-dimensional cobordisms. These TQFTs may have closed, orientable 2-D surfaces of arbitrarily high genus that evaluate to nonzero elements of R via the Frobenius form ε, a fact that has the potential to greatly complicate the skein modules associated with such TQFTs. On the other hand, for the TQFTs associated with $p(x)$ such that every root is repeated, closed orientable surfaces of genus ≥ 2 always evaluate to zero. As such, these TQFTs yield skein modules that are far more tractable, and in Section 3.3 we will be able to prove interesting results concerning these particular skein modules.

3.2.2 The Genus-Reduction Matrix

In Sebsection 3.2.1 we presented an explicit solution of how the 2-D TQFT associated to a universal $sl(2)$ Frobenius system evaluated marked surfaces of arbitrary genus. This was tractable because $g^2 \in A$ was actually a constant in the base ring R, and thus could be pulled out of the argument of the Frobenius form ε. We briefly noted that our technique did not extend to higher n, as we didn't necessarily have $g^i \in R$ for any $i \geq 2$. In this subsection, we alternatively approach the problem of evaluating closed, marked cobordisms using linear algebra. It should be noted that the results of this section stand apart from the rest of Chapter 3, and are interesting primarily in that they reveal a further indebtedness of 2-D topological quantum field theory to elementary symmetric polynomials.

Utilizing the same notation as previous sections, take the $sl(n)$ Frobenius extension with $A = R[x]/(p(x))$ for $p(x) = x^n - a_1x^{n-1} - ... - a_n$. Earlier in Section 3.1, we noted how the genus-reduction term g for this Frobenius extension was $g = p'(x) = nx^{n-1} - (n-1)a_1x^{n-2} - ... - a_{n-1}$. The technique of this subsection is to interpret multiplication by g as an R-linear operator from A to A. Always working with the standard basis $\{1, x, \ldots, x^{n-1}\}$, this operator takes the form of an $n \times n$

genus-reduction matrix $G_n \in Mat_n(R)$.

Pause to note that, with our chosen basis, the first column of G_n directly corresponds to the coefficients of g. As the Frobenius form ε for any $sl(n)$ Frobenius extension "picks off" the coefficient of x^{n-1}, how the associated TQFT actually evaluates closed surfaces relates to the entries in the bottom row of G_n (or $(G_n)^k$ for some $k \geq 1$). In particular, a genus-k surface decorated with x^j $(0 \leq j \leq n-1)$ always corresponds to the $(n, j+1)$ entry of $(G_n)^k$.

Our first step in categorizing G_n for an arbitrary $sl(n)$ Frobenius extension $(n \geq 2)$ is a recursive formula for the entries of G_n:

Lemma 3.2.8. *The entries of $G_n = [[g_{i,j}]]$ (any $n \geq 2$) are recursively defined as:*

$$\begin{array}{ll} g_{i,j} = -ia_{n-i} & \textit{(for } j = 1,\ i < n\textit{)} \\ g_{i,j} = n & \textit{(for } j = 1,\ i = n\textit{)} \\ g_{i,j} = a_{n-i+1}g_{n,j-1} + g_{i-1,j-1} & \textit{(for } j > 1,\ i > 1\textit{)} \\ g_{i,j} = a_{n-i+1}g_{n,j-1} & \textit{(for } j > 1,\ i = 1\textit{)} \end{array}$$

Proof. The first two lines are a direct consequence of the equation $g = p'(x)$ that we derived at the beginning of Subsection 3.2.1. As for the last two lines, note that moving from the j$^{\text{th}}$ column of G_n to the (j+1)$^{\text{th}}$ column of G_n amounts to multiplication by $x \in A$. Reducing modulo $p(x) = x^n - a_1x^{n-1} - ... - a_n$ we have:

$$x(g_{i,j-1} + g_{2,j-1}x + ... + g_{n-1,j-1}x^{n-2} + g_{n,j-1}x^{n-1})$$

$$= g_{1,j-1}x + g_{2,j-1}x^2 + ... + g_{n-1,j-1}x^{n-1} + g_{n,j-1}x^n$$

$$= g_{1,j-1}x + g_{2,j-1}x^2 + ... + g_{n-1,j-1}x^{n-1} + g_{n,j-1}(a_1x^{n-1} + a_2x^{n-2} + ... + a_n)$$

$$= (a_ng_{n,j-1}) + (a_{n-1}g_{n,j-1} + g_{1,j-1})x + ... + (a_1g_{2,j-1} + g_{n-1,j-1})x^{n-1}$$

□

Lemma 3.2.8 may be used to quickly produce G_n for small values of n:

$$G_2 = \begin{bmatrix} -a_1 & 2a_2 \\ 2 & a_1 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} -a_2 & 3a_3 & a_1a_3 \\ -2a_1 & 2a_2 & a_1a_2 + 3a_3 \\ 3 & a_1 & a_1^2 + 2a_2 \end{bmatrix}$$

$$G_4 = \begin{bmatrix} -a_3 & 4a_4 & a_1a_4 & a_1^2a_4 + 2a_2a_4 \\ -2a_2 & 3a_3 & a_1a_3 + 4a_4 & a_1^2a_3 + 2a_2a_3 + a_1a_4 \\ -3a_1 & 2a_2 & a_1a_2 + 3a_3 & a_1^2a_2 + 2a_2^2 + a_1a_3 + 4a_4 \\ 4 & a_1 & a_1^2 + 2a_2 & a_1^3 + 3a_1a_2 + 3a_3 \end{bmatrix}$$

Note that, for $n = 2$, we have:

$$(G_2)^2 = \begin{bmatrix} a_1^2 + 4a_2 & 0 \\ 0 & a_1^2 + 4a_2 \end{bmatrix} = (a_1^2 + 4a_2)\ E_2$$

This final matrix equation allows use to explicitly calculate $(G_2)^i$ for all $i \geq 1$. This directly correlates to how we were able to explicitly determine the evaluation of closed, marked cobordisms for universal $sl(2)$ Frobenius systems in Subsection 3.2.1.

$$(G_2)^{2k} = \begin{bmatrix} (a_1^2+4a_2)^k & 0 \\ 0 & (a_1^2+4a_2)^k \end{bmatrix}$$

$$(G_2)^{2k+1} = \begin{bmatrix} -a_1(a_1^2+4a_2)^k & 2a_2(a_1^2+4a_2)^k \\ 2(a_1^2+4a_2)^k & a_1(a_1^2+4a_2)^k \end{bmatrix}$$

In Subsection 3.2.1 we gained a considerable amount of traction by fully factoring $p(x)$ over $\mathbb{C}$ as $p(x) = x^n - a_1x^{n-1} - ... - a_n = \prod_{i=1}^{n}(x+\alpha_i)$. Via this factorization, our original coefficient a_k was equal to the negative of the k$^{\text{th}}$ elementary symmetric polynomial e_k in the roots $\{\alpha_1, ..., \alpha_n\}$. Here we examine how that same factorization effects our genus-reduction matrices G_n, but first we need to standardize notation for more general sorts of symmetric polynomials.

For the collection of n variables $\{\alpha_1, ...\alpha_n\}$, let $m_{(k_1...k_n)}$ denote the sum of all monomimals of the form $\alpha_{i_1}^{k_1}...\alpha_{i_n}^{k_n}$ $(i_1 \neq ... \neq i_n)$. These $m_{(k_1...k_n)}$ are known as the monomial symmetric polynomials, and contain as a special case the elementary symmetric polynomials $e_k = m_{(1^k0^{n-k})} = m_{(1^k)}$ (where 1^k denotes k consecutive ones, and we follow tradition by dropping all 0 indices). The monomial symmetric polynomials also include the power symmetric polynomials $p_k = m_{(k^1)} = \alpha_1^k + ... + \alpha_n^k$.

For small n, rewriting G_n in terms of the roots α_i begins to suggest a general pattern involving the monomial symmetric polynomials, a pattern that "stabilizes" following the first two columns of the given matrix:

$$G_2 = \begin{bmatrix} m_{(1^1)} & -2m_{(1^2)} \\ 2 & -m_{(1^1)} \end{bmatrix}$$

$$G_3 = \begin{bmatrix} m_{(1^2)} & -3m_{(1^3)} & m_{(2^1 1^2)} \\ 2m_{(1^1)} & -2m_{(1^2)} & m_{(2^1 1^1)} \\ 3 & -m_{(1^1)} & m_{(2^1)} \end{bmatrix}$$

$$G_4 = \begin{bmatrix} m_{(1^3)} & -4m_{(1^4)} & m_{(2^1 1^3)} & -m_{(3^1 1^3)} \\ 2m_{(1^2)} & -3m_{(1^3)} & m_{(2^1 1^2)} & -m_{(3^1 1^2)} \\ 3m_{(1^1)} & -2m_{(1^2)} & m_{(2^1 1^1)} & -m_{(3^1 1^1)} \\ 4 & -m_{(1^1)} & m_{(2^1)} & -m_{(3^1)} \end{bmatrix}$$

Before proving a general form for all G_n $(n \geq 2)$ involving the monomial symmetric functions, we require the following technical lemma involving products of elementary and symmetric polynomials. The proofs of these equations are directly verifiable and left to the reader.

Lemma 3.2.9. *Let $p_a = m_{(a^1)}$ and $e_b = m_{(1^b)}$ be monomial symmetric polynomials in n variables. Then:*

1. $p_a e_b = m_{((a+1)^1 1^{n-1})}$ *(for $b = n$)*

2. $p_a e_b = m_{(2^1 1^{b-1})} + (b+1)m_{(1^{b+1})}$ *(for $a = 1$ and $b < n$)*

3. $p_a e_b = m_{((a+1)^1 1^{b-1})} + m_{(a^1 1^b)}$ *(for $a > 1$ and $b < n$)*

Proposition 3.2.10. *For all $n \geq 2$, we have universal genus-reduction matrix $G_n =$*

$$\begin{bmatrix} m_{(1^{n-1})} & -nm_{(1^n)} & m_{(2^1 1^{n-1})} & -m_{(3^1 1^{n-1})} & \dots & (-1)^{n-1} m_{((n-1)^1 1^{n-1})} \\ 2m_{(1^{n-2})} & -(n-1)m_{(1^{n-1})} & m_{(2^1 1^{n-2})} & -m_{(3^1 1^{n-2})} & \dots & (-1)^{n-1} m_{((n-1)^1 1^{n-2})} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (n-1)m_{(1^1)} & -2m_{(1^2)} & m_{(2^1 1^1)} & -m_{(3^1 1^1)} & \dots & (-1)^{n-1} m_{((n-1)^1 1^1)} \\ n & -m_{(1^1)} & m_{(2^1)} & -m_{(3^1)} & \dots & (-1)^{n-1} m_{((n-1)^1)} \end{bmatrix}$$

Proof. We present a general proof that works for all $n \geq 2$. In light of the recursive relations from Lemma 3.2.8, we proceed by induction on the columns of G_n. This induction has two base-steps, for columns $j = 1$ and $j = 2$; the inductive step covers the columns $j \geq 3$ for which we noted a "stablization" of the pattern in the $n = 2, 3, 4$ cases explicitly presented above.

$\underline{j = 1}$: Directly from Lemma 3.2.8 we have:

$$g_{n,1} = n$$

$$g_{i,1} = -ia_{n-i} = ie_{n-i} = im_{(1^{n-i})} \text{ (for } i < n)$$

$\underline{j = 2}$: Once again directly from Lemma 3.2.8 we have:

$$g_{1,2} = a_n g_{n,1} = na_n = -ne_n = -nm_{(1^1)}$$

$$g_{i,2} = a_{n-i+1} g_{n,1} + g_{i-1,1} = -ne_{n-i+1} + (i-1)e_{n-i+1} = -(n-i+1)e_{n-i+1} \text{ (for } i > 1)$$

<u>Induction base step for $j \geq 3$</u>: First, note that $g_{n,2} = -p_1$. By Lemma 3.2.9(1):

$$g_{1,3} = a_n g_{n,2} = -e_n(-p_1) = m_{(2^1 1^{n-1})}$$

By Lemma 3.2.9(2), for $i > 1$ we also have:

$$g_{i,3} = a_{n-i+1}g_{n,2} + g_{i-1,2} = -e_{n-i+1}(-p_1) - (n-i+2)e_{n-i+2}$$

$$= m_{(2^1 1^{n-i})} + (n-i+2)m_{(1^{n-i+2})} - (n-i+2)m_{(1^{n-i+2})} = m_{(2^1 1^{n-i})}$$

Inductive step for $j \geq 3$: Assume that the pattern holds for column k, we show that it holds for column $k+1$. Here we begin by noting that $g_{n,k} = (-1)^{k-1}p_{k-1}$. By Lemma 3.2.9(1) we have:

$$g_{1,k+1} = a_n g_{n,k} = -e_n(-1)^{k-1}p_{k-1} = (-1)^k m_{(k^1 1^{n-1})}$$

By Lemma 3.2.9(3) we also have:

$$g_{i,k+1} = a_{n-i+1}g_{n,k} + g_{i-1,k} = -e_{n-i+1}(-1)^{k-1}p_{k-1} + (-1)^{k-1}m_{((k-1)^1 1^{n-i+1})}$$

$$= (-1)^k m_{(k^1 1^{n-i})} + (-1)^k m_{((k-1)^1, 1^{n-i+1})} + (-1)^{k-1} m_{((k-1)^1 1^{n-i+1})} = (-1)^k m_{(k^1 1^{n-i})}$$

□

3.3 Universal $sl(n)$ Skein Modules

As noted in Subsection 1.2.3, the notion of a skein module dates back to the work of Asaeda and Frohman [1]. They studied the free module of isotopy classes of marked 2-D cobordisms in a 3-manifold, subject to relations identifying surfaces that evaluated similarly by the 2-D TQFT underlying Khovanov homology (thus corresponding to Bar-Natan's original marked cobordism category). Their work was extended by Kaiser [10], who rigorously investigated how to obtain skein modules from an arbitrary Frobenius extension. In Kaiser's notation, the skein module of the three-manifold M arising from the extension $F = (R, A, \varepsilon)$ is denoted $C(F, M)$.

In this section we give our major result about skein modules arising from the universal $sl(n)$ Frobenius extensions $R \hookrightarrow A$ of Section 3.2. Here we continue to utilize the notation $A = R[a_1, ..., a_n][x]/(p(x))$, where $p(x) = x^n - a_1x^{n-1} - ... - a_n$. In a slight modification of Kaiser's original notation, we define the skein module of M that arises from such a Frobenius extension as $K_A(M) = C(F, M)$. Before we examine these skein modules, we pause to recall some basic definitions from 3-manifold topology.

Definition 3.3.1. *A three-manifold M is **irreducible** if every two-sphere S^2 embedded in M bounds a three-ball B^3.*

Definition 3.3.2. *A curve γ on a 2-D surface S is an **inessential curve** if it bounds a disk on S. Otherwise, the curve is said to be an **essential curve**.*

Definition 3.3.3. *Let S be a 2-D surface embedded within the 3-manifold M. S is a **compressible surface** if there exists an essential curve $\gamma \subset S$ bounding a 2-D disc $D \subset M$ (a compression disc) such that $S \cap D = \delta D$. If $S \subset M$ contains no such curves, then S is an **incompressible surface**.*

The compressibiliity of a surface is especially important in the study of skein modules, due to the presence of the neck-cutting relation. Namely, a surface is compressible iff it admits a neck-cutting such that neither the "top" nor the "bottom" on the right side of the neck-cutting relation is a sphere component.

3.3.1 Linear Independence of Unmarked Surfaces

One of the primary results from [1] was the following:

Theorem 3.3.4. *Let M be an irreducible 3-manifold, and let (R, A, ε) be the $sl(2)$ Frobenius extension with $a_1 = a_2 = 0$ (so that $p(x) = x^2$). Then the unmarked, incompressible surfaces in $K_A(M)$ are linearly independent over $R = \mathbb{Z}$.*

This result immediately breaks down when you generalize to the universal $sl(n)$ case, even in rank $n = 2$. As suggested by Corollary 3.2.7, things stay nice precisely when the roots of $p(x)$ are repeated. Theorem 3.3.5 below is a direct generalization of the proof to Theorem 3.3.4 to all $sl(n)$ Frobenius extensions whose $p(x)$ has no multiplicity 1 roots:

Theorem 3.3.5. *Let M be an irreducible 3-manifold, and let (R, A, ε) be an $sl(n)$ Frobenius extension such that every root of $p(x)$ is repeated. Then the unmarked, incompressible surfaces in $K_A(M)$ are linearly independent over $R = \mathbb{Z}[a_1, ..., a_n]$.*

Proof. Let $p(x) = \prod_{i=1}^{m}(x + \alpha_i)^{k_i}$, where we have fully grouped like roots so that $k_i > 1$ for all i. We show that the unmarked, incompressible surfaces are linearly independent over $\tilde{R} = \mathbb{Z}[\alpha_1, ..., \alpha_m]$, immediately implying that there are linearly-indpendent over $R \subseteq \tilde{R}$.

Let F be any unmarked, incompressible surface in M. For each such F we look to define a $\tilde{R}$-linear functional λ_F so that $\lambda_F(F) = 1$ and $\lambda_F(\tilde{F}) = 0$ for any other unmarked, incompressible surface $\tilde{F} \subset M$. The existence of such a functional for each such F implies the linear independence of unmarked, incompressible surfaces.

So fix a root $\alpha_i = \alpha$ of $p(x)$, any choice of root will give a suitable family of linear functionals. We then define λ_F over all 2-D surfaces $S \subset M$:

- $\lambda_F(S) = (-\alpha)^k \prod_\sigma \varepsilon(S_\sigma) \prod_\tau \varepsilon(T_\tau)$ for S a disjoint union of F, marked by x^k, with (marked or unmarked) spheres S_σ and (marked or unmarked) compressible tori T_τ

- $\lambda_F(S) = \prod_\sigma \varepsilon(S_\sigma) \prod_\tau \varepsilon(T_\tau)$ for S a disjoint union of (marked or unmarked) spheres S_σ and (marked or unmarked) compressible tori T_τ

- $\lambda_F(S) = 0$ for all other surfaces $S \subset M$

It remains to be shown that λ_F, as defined, respects the local skein relations as presented in Section 3.2. But first we pause to note that, since M is an irreducible 3-manifold, all compressible tori T_τ compress to spheres that bound balls, regardless of the specific compression disc chosen.

That λ_F respects both the sphere relations and the "dot reduction" relation follows directly from the definition. In particular, the only components apart from F that yield nonzero results (i.e.- the closed 2-D surfaces) are evaluated precisely as proscribed by the associated 2-D TQFT. Since disjoint union corresponds to multiplication after the application of that 2-D TQFT, these evaluations behave properly when appearing in as a disjoint union with F.

To see that λ_F respects the neck-cutting relation, note that $-\alpha_1 = \alpha$ being a repeated root implies that $-\alpha$ is a root of the genus-reduction term $g = p'(x)$. By definition, all unmarked compressible surfaces that aren't compressible tori evaluate to zero. That compressible tori respect the neck-cutting relation is a immediate consequence of the definition in terms of the Frobenius form ε. Otherwise, the left side of the neck-cutting relation is an unmarked compressible surface that evaluates to zero, so we just need to show that the right side of the neck-cutting relation is also zero in this case. The only situation here that doesn't immediately evaluate to zero is when the neck-cutting yields F. As the original surface was unmarked, we then have a copy of F marked by $g = p'(x)$. Be definition of λ_F, this surface evaluates to $p'(-\alpha) = 0$. □

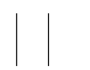
Note that, although the linear functionals λ_F from above only required a single repeated root $\alpha_i = \alpha$, the proof still required that every root be repeated. Otherwise, as implied by Corollary 3.2.7, we would need to consider how our linear functionals evaluate compressible, closed surfaces of arbitrarily high genus.

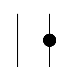
A direct converse for Theorem 3.3.5 has proven elusive, and is an active area of research. Theorem 3.3.6 below does provide an explicit counterexample to the linear independence of incompressible, unmarked surfaces in the universal $sl(2)$ case when $p(x)$ has two distinct roots. Explicit counterexamples for rank $n \geq 3$ are currently lacking due to the overwhelming computational complexity they would require, but we suspect that a similar argument may be adapted for general $n \geq 2$. We note that the earliest version of Theorem 3.3.6, as originally presented in [4], was primarily the work of Boerner.

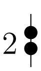
Theorem 3.3.6. *Let $M = S^1 \times S^1 \times S^1$, which is an irreducible 3-manifold, and let (R, A, ε) be a $sl(2)$ Frobenius extension with $p(x) = x^2 - a_1 x - a_2 = (x+\alpha)(x+\beta)$. The unmarked, incompressible surfaces in $K_A(M)$ are linearly independent iff $\alpha = \beta$ (or equivalently, iff $4a_2 + a_1^2 = 0$).*

Proof. The linear independence of unmarked, incompressible surfaces in the repeated root case follows directly from Theorem 3.3.5. We demonstrate a linear dependence of unmarked, incompressible surfaces when $\alpha \neq \beta$, but first we need to introduce some new notation.

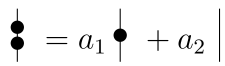
We work in $M = S^1 \times S^1 \times S^1 = T^2 \times S^1$, which fibers over S^1. For each fiber of M over S^1 one may embed an incompressible torus. Thus, the incompressible surfaces we may embed within M include (disjoint) parallel collections of such incompressible tori. We graphically represent these incompressible surfaces via parallel "strands", with one vertical strand for each incompressible tori. Since we are working within the marked cobordism category, we may write constants $r \in R$ in front of the strands and use a dot to denote a strand marked by $x \in A$. For example

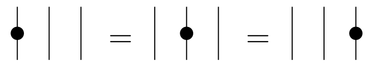
Three parallel incompressible tori: 

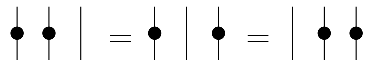
Two parallel incompressible tori, one marked by x: 

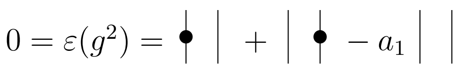
An incompressible tori marked by $2x^2$: 2 

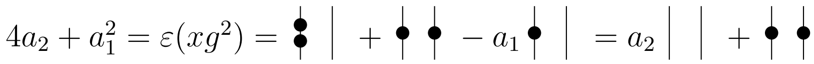
The underlying Frobenius system yields the following relations on our incompressible surfaces, which directly follow from the relations introduced in Section 3.2 for universal $sl(n)$ Frobenius extensions. #1 below is simply the "dot-reduction" relation, while #2 and #3 are due to the fact that M fibers over a circle (in these two relations, there can be no additional strands to the ones shown). #4 and #5 follow from the neck-cutting relation. In #4, two unmarked parallel tori are tubed together to produce a compressible genus-2 surface, and then that tube is cut on the right side of the equation. #5 is similar except that one of the original tori was marked by x, such that we have a marked genus-2 surface after tubing (here we use relation #1 to simplify the right side).

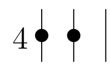
1. 

2. 

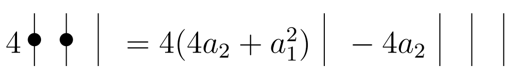
3. 

4. 

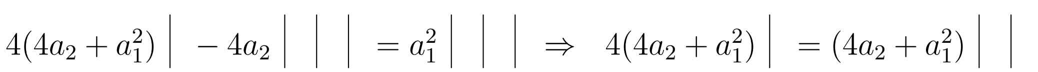
5. 

Now consider the marked incompressible surface . Applying relation #3, followed by relations #4, #2, and #4 gives:

Alternatively, applying relation #5 to the same original surface gives:



Comparing these results, we have:



Which gives a linear dependence of unmarked, incompressible surfaces over R if $4a_2 + a_1^2 \neq 0$, or equivalently if $\alpha \neq \beta$. $\square$

A slightly more general version of Theorem 3.3.6 produces a linear dependence of unmarked, incompressible surfaces in any 3-manifold M that fibers over a circle, so long as $p(x)$ has two distinct roots in the $sl(2)$ Frobenius extension.

REFERENCES

[1] Marta Asaeda and Charles Frohman. A note on the Bar-Natan skein module. *Internat. J. Math*, 10:1225–1243, 2007.

[2] Michael Atiyah. Topological quantum field theories. *Inst. Hautes Etudes Sci. Publ. Math.*, 68:175–186, 1989.

[3] Dror Bar-Natan. Khovanov's homology for tangles and cobordisms. *GEOM.TOPOL.*, 9:1443, 2005.

[4] Jeffrey Boerner and Paul Drube. Generalized skein modules of surfaces. *arXiv: 1009.3207*.

[5] Carmen Caprau. The universal sl(2) cohomology via webs and foams. *Topology Appl.*, 156, no. 9:1684–1702, 2009.

[6] Robbert Dijkgraaf. *A geometric approach to two dimensional conformal field theory*. Ph.D. thesis, University of Utrecht, Utrecht, 1989.

[7] Paul Drube. TQFT invariants. *(in preparation)*.

[8] V. F. R. Jones. Hecke algebra representations of braid groups and link polynomials. *Ann. of Math*, 126:335–388, 1987.

[9] Lars Kadison. *New examples of Frobenius extensions*. American Mathematical Society, 1999.

[10] Uwe Kaiser. Frobenius algebras and skein modules of surfaces in 3-manifolds. *Algebraic Topology - Old and New*, Volume 85:59–81, 2009.

[11] Mikhail Khovanov. A categorification of the Jones polynomial. *DUKE MATH.J*, 3:359, 2000.

[12] Mikhail Khovanov. sl(3) link homology. *Algebra. Geom. Topol.*, 4:1045–1081, 2004.

[13] Mikhail Khovanov. Link homology and Frobenius extensions. *Fund. Math.*, 190:179–190, 2006.

[14] Mikhail Khovanov and Lev Rozansky. Matrix factorizations and link homology. *Fundamenta Mathematicae*, arXiv:math:QA/0401268.

[15] Joachim Kock. *Frobenius algebras and 2D topological quantum field theories.* Cambridge University Press, 2003.

[16] Eun Soo Lee. An endomorphism of the Khovanov invariant. *Adv. Math.*, 197(2):554–586, 2005.

[17] Jacob Lurie. On the classification of topological field theories. *Current developments in Mathematics*, pages 129–280, 2008.

[18] Marco Mackaay and Pedro Vaz. The universal sl_3-link homology. *Algebr. Geom. Topol.*, 7:1135–1169, 2007.

[19] Saunders MacLane. *Categories for the working mathematician.* Springer Graduate texts in mathematics 5, Springer Verlag 1971.