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Fully nonlinear flows and Hessian equations on compact Kahler manifolds

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FULLY NONLINEAR FLOWS AND HESSIAN EQUATIONS ON COMPACT KÄHLER MANIFOLDS

by

Mijia Lai

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

May 2011

Thesis Supervisors: Assistant Professor Hao Fang
Professor Lihe Wang

ABSTRACT

In this thesis, we will study a class of fully nonlinear flows on Kähler manifolds. This family of flows generalizes the previously studied J-flow. We use the quotients of elementary symmetric polynomials or log of them to construct the flow. We obtain a necessary and sufficient condition in terms of positivity of certain cohomology class to guarantee the convergence of the flow. The corresponding limit metric gives rise to a critical metric satisfying a Hessian type equation on the manifold. We shall also discuss several geometric applications of our main result.

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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To my dearest daughter,
Emma (Wuxia)

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ABSTRACT

In this thesis, we will study a class of fully nonlinear flows on Kähler manifolds. This family of flows generalizes the previously studied J-flow. We use the quotients of elementary symmetric polynomials or log of them to construct the flow. We obtain a necessary and sufficient condition in terms of positivity of certain cohomology class to guarantee the convergence of the flow. The corresponding limit metric gives rise to a critical metric satisfying a Hessian type equation on the manifold. We shall also discuss several geometric applications of our main result.

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CHAPTER 1
INTRODUCTION

Complex Monge-Ampère equations arise naturally in the study of Kähler geometry. A classical example is the Calabi conjecture. It asks for the existence of Kähler-Einstein metrics on compact Kähler manifolds with definite first Chern Classes. A Kähler metric is called Kähler-Einstein if its Ricci form is a constant multiple of itself, i.e., $Ric(\omega) = \lambda\omega$. Since $Ric(\omega)$ represents the first Chern class $c_1(M)$, hence a necessary condition for a Kähler manifold admitting Kähler-Einstein metrics is that $c_1(M)$ is definite, i.e., positive, negative or vanishing. The search for Kähler-Einstein metric within a fixed cohomology class can be reduced to a complex Monge-Ampère equation on the manifold. In his famous resolution of Calabi conjecture [Yau], Yau solved the corresponding Monge-Ampère equations on compact Kähler manifolds with $c_1(M) < 0$ or $c_1(M) = 0$. Aubin [Au] independently solved the Calabi conjecture in the case $c_1(M) < 0$.

Their approach is the so-called continuity method. Briefly speaking, they embed the target equation into a continuous family of equations with a starting point admitting an obvious solution, then use an open and closed argument to prove all the equations in the family are solvable. The openness follows from the implicit function theorem in Banach spaces and the closeness is proved by a priori estimates.

Kähler manifolds with positive first Chern classes are called Fano manifolds in algebraic geometry. The question of the existence of Kähler-Einstein metrics on such manifolds is much more delicate. There are several obstructions to the existence, e.g., the Futaki invariant. It is conjectured by Yau that the existence of Kähler-Einstein metrics on Fano manifolds should be equivalent to some notion of stability in the sense of geometric invariant theory. Tian first unveiled this deep relation. In [T2], he first provided an analytical criterion which is equivalent to the existence of Kähler-Einstein metrics on Fano manifolds, namely the properness of F-functional. Moreover, he derived that the existence of Kähler-Einstien metrics implies a notion of stability which he called (weak) K-stable. Later Donaldson [D4, D6] refined this notion of stability. The most recent progress on this subject is announced in [D7].

Another interesting example where complex Monge-Ampère equations arise is in the study of the space of Kähler metrics in a fixed Kähler class. Donaldson [D3], Mabuchi [Ma2] and Semmes [Se] independently discovered that one can define a Weil-Peterson type metric on such space which turns it formally into an infinite dimensional symmetric space with nonpositive sectional curvature. It was observed by Semmes [Se] that the geodesic equation in this setting can be reduced to a homogenous complex Monge-Ampère equation in one dimension higher. See [Ch1, CT2, CS] for the development along this direction.

Monge-Ampère equation falls into a special category of fully nonlinear equations: the Hessian equations. The definition of Hessian equations is as follows: denote σ_k the elementary symmetric polynomial of n-variables. For an $n \times n$ matrix A, $\sigma_k(A)$ evaluates σ_k at the eigenvalues of A. With this notation, Hessian equations in general domains are of the form: $\sigma_k(D^2u) = f$. Therefore Hessian equations include Possion equations ($k = 1$) and Monge-Ampère equations ($k = n$) as two special cases. Hessian equations in general domains have been intensively studied since the seminal works of Caffarelli et al. [CNS1, CKNS, CNS2], see also the lecture notes [Wang]. The corresponding Hessian equations on manifolds have received considerable study, e.g., in the generalized Yamabe problem. Readers are referred to [Cha] and [Gu]. Since the important role played by the complex Monge-Ampére equations in Kähler geometry, it is desirable to study its generalizations, namely the Hessian equations on general Kähler manifolds. Hou [Hou] has considered a class of Hessian equations on Kähler manifolds.

Geometric flows provide a natural way to deform metrics to canonical ones. One classical example is the Kähler-Ricci flow. Using Kähler-Ricci flow, Cao [Cao] reproved the Calabi conjecture when the manifold has negative or vanishing first Chern class. In the Fano case, Chen and Tian [CT2] proved the convergence of Kähler-Ricci flow on Kähler-Einstein manifolds under the assumption that the initial metric has nonnegative bisectional curvature. Perelman has made ground-breaking work [P], where by using his famous W-functional, he showed that scalar curvature, diameter and normalized Ricci potential are all uniformly bounded along Kähler-Ricci flow. Tian and Zhu [TZ] generalized this to Kähler-Ricci solitons. Recently much interest was devoted to studying the Kähler-Ricci flow on Kähler manifolds with indefinite first Chern class, see e.g., [ST1, ST2, ST3].

Another example of geometric flow is the so-called J-flow. It is introduced by Donaldson [D1] in the setting of moment map and by Chen [Ch2, Ch3] as a gradient flow of J-functional, which is a term in the Mabuchi energy [Ma1]. In a series of papers [Wein1, Wein2, SW2], Song and Weinkove provide a necessary and sufficient condition for J-flow to converge to a critical metric which satisfying a

Hessian equation.

Methodologically speaking, geometric flows are a parabolic version of continuity method. However, they exhibit much more preferable geometric properties as the flow is defined explicitly using geometric quantities. In other words, geometric behavior is more explicit along the geometric flows.

Motivated by these previous results, we study a class of fully nonlinear geometric flows on Kähler manifolds which includes J-flow as a special case. We give a necessary and sufficient condition for the flow to converge to the critical equation which is of Hessian type. In the definition of the flow, we have used the quotient of elementary symmetric polynomials or logarithm of such quotient. Such construction has appeared in the study of the evolution hypersurfaces in Euclidean spaces which is an attempt to generalize mean curvature flow, inverse mean curvature flow, etc. See e.g. [G, U, An1].

The rest of the thesis is organized as follows. In Chapter 2, we give preliminary facts on Kähler geometry and elementary symmetric polynomials. In Chapter 3, we state and prove the main theorems. In Chapter 4, we discuss some geometric applications of the main theorem and list some future problems. In the appendixes, we provide complete proofs of some algebraic propositions.

CHAPTER 2
PRELIMINARIES

2.1 Kähler geometry

In this section, we introduce basic knowledge for Kähler geometry. We adopt the approach from real manifolds. Let (M, g) be a $2n$-dimensional Riemannian manifold. An almost complex structure J is an endomorphism of TM satisfying $J^2 = -Id$.

Extending J $\mathbb{C}$-linearly to $TM \otimes \mathbb{C}$, at each point $p \in M$, there is a canonical decomposition of $T_pM \otimes \mathbb{C}$:

$$T_pM \otimes \mathbb{C} = T_p^{1,0}M \oplus T_p^{0,1}M$$

into eigen-spaces of J corresponding eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively.

J is called integrable if M is also a complex manifold such that $J|_{T_p^{1,0}M}$ corresponds to the complex multiplication on complex tangent plane at p.

The famous Newlander-Nirenberg theorem states that J is integrable if and only if the Nijenhuis tensor $N(J) : TM \times TM \to TM$:

$$N(J)(u, v) = [u, v] + J[Ju, v] + J[u, Jv] - [Ju, Jv]$$

vanishes. If J is an integrable almost complex structure, we simply call it a complex structure.

Now let (M, g, J) be a $2n$-dimensional Riemannian manifold, J is a complex structure, J is compatible with g if

$$g(u, v) = g(Ju, Jv).$$

Then we call M a hermitian manifold. We associate with it a 2-form ω_g defined by

$$\omega_g(u,v) = g(Ju,v),$$

and call it the associated Kähler form.

(M,g,J) is called a Kähler manifold if it is a hermitian manifold and the associated Kähler form ω_g is closed. An equivalent condition is that J is parallel with respect to the Levi-Civita connection of M, i.e., $\nabla J = 0$.

If (M,g,J) is a Kähler manifold, then in local coordinate chart $(z_1,\cdots,z_n)$ with $z_i = x_i + \sqrt{-1}y_i$, we have

$$dz_i = dx_i + \sqrt{-1}dy_i, \quad d\bar{z}_i = dx_i - \sqrt{-1}dy_i$$

$$\frac{\partial}{\partial z_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}).$$

Under this coordinate chart J is

$$J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}, \quad J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i},$$

thus $J(\frac{\partial}{\partial z_i}) = \sqrt{-1}\frac{\partial}{\partial z_i}$, and

$$T_pM = Span\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\};$$

$$T_p^{1,0}M = Span\{\frac{\partial}{\partial z_i}\}, \quad T_p^{0,1}M = Span\{\frac{\partial}{\partial \bar{z}_i}\}.$$

We call $T_p^{1,0}M$ the holomorphic tangent plane and $T_p^{0,1}M$ the anti-holomorphic tangent plane. Since J is compatible with g, it follows g is nonzero if and only if two tangent vectors are of different type. Hence g induces an Hermitian metric h on $T_p^{1,0}M$ by $h(u,v) = g(u,\bar{v})$.

In local coordinates, we can write ω_g as

$$\omega_g = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

where $g_{i\bar{j}} = g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})$.

Let ∇ be the Levi-Civita connection of (M, g), extending it $\mathbb{C}$-linearly. By the fact that $\nabla J = 0$, it follows that the only non-vanishing Christoffel symbols are Γ_{ij}^k and $\Gamma_{\bar{i}\bar{j}}^{\bar{k}}$, i.e.,

$$\nabla_{\frac{\partial}{\partial z_i}} \frac{\partial}{\partial z_j} = \Gamma_{ij}^k \frac{\partial}{\partial z_k}, \quad \nabla_{\frac{\partial}{\partial \bar{z}_i}} \frac{\partial}{\partial \bar{z}_j} = \Gamma_{\bar{i}\bar{j}}^{\bar{k}} \frac{\partial}{\partial \bar{z}_k}.$$

Since the connection is compatible with the metric, it follows that

$$\Gamma_{ij}^k = g^{k\bar{l}} \frac{\partial g_{j\bar{l}}}{\partial z_i},$$

where $g^{k\bar{l}}$ is the inverse of $g_{i\bar{j}}$, i.e., $g_{i\bar{j}} g^{k\bar{j}} = \delta_{ik}$.

Similarly, we extend $\mathbb{C}$-linearly the curvature operator, $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$. By the fact $\nabla J = 0$, we get that the only non-vanishing components of R viewed as a $(0,4)$ tensor are

$$R_{i\bar{j}k\bar{l}} = g(R(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l}) = -\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial \bar{z}_j} \frac{\partial g_{p\bar{l}}}{\partial z_i}.$$

Taking the trace we get Ricci tensor, which is only non-vanishing for different types of tangent vectors and compatible with J, thus we can similarly define a 2-form associated to it.

$$Ric(\frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial z_k}) = tr(R(\cdot, \frac{\partial}{\partial \bar{z}_j})\frac{\partial}{\partial z_k}) = -\frac{\partial \Gamma_{ki}^i}{\partial \bar{z}_j} = -\frac{\partial^2 \log det(g_{p\bar{q}})}{\partial \bar{z}_j \partial z_k},$$

hence the corresponding Ricci form is

$$Ric(\omega) = -\partial\bar{\partial}\log det(g_{i\bar{j}}).$$

If we denote

$$R(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})\frac{\partial}{\partial z_k} = R^l_{i\bar{j}k}\frac{\partial}{\partial z_l}, \quad R(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})\frac{\partial}{\partial \bar{z}_k} = R^{\bar{l}}_{i\bar{j}\bar{k}}\frac{\partial}{\partial \bar{z}_l},$$

we have

$$R_{\bar{j}k} = R^i_{i\bar{j}k}, \text{or} \quad R_{i\bar{k}} = R^{\bar{j}}_{i\bar{j}k},$$

and

$$R_{i\bar{j}k\bar{l}} = R^m_{i\bar{j}k}g_{m\bar{l}}.$$

The first Bianchi identity gives the following commutative property of $R_{i\bar{j}k\bar{l}}$:

$$R_{i\bar{j}k\bar{l}} = R_{i\bar{l}k\bar{j}} = R_{k\bar{j}i\bar{l}} = R_{k\bar{l}i\bar{j}}.$$

Second Bianchi identity is

$$R_{i\bar{j}k\bar{l},m} = R_{i\bar{j}m\bar{l},k}, \text{and}\, R_{i\bar{j}k\bar{l},\bar{m}} = R_{i\bar{j}k\bar{m},\bar{l}}.$$

The Ricci identity in Kähler case is simpler as the adjacent indexes of the same type commute.

For a vector field of type $(1,0)$, say $X = X^\alpha\frac{\partial}{\partial z_\alpha}$, Ricci identity reads

$$X^\alpha_{,i\bar{j}} - X^\alpha_{,\bar{j}i} = R^\alpha_{i\bar{j}\beta}X_\beta.$$

For a one form of type $(1,0)$, say $Y = Y_\alpha dz_\alpha$, the Ricci identity reads

$$Y_{\alpha,i\bar{j}} - Y_{\alpha,\bar{j}i} = -R^\beta_{i\bar{j}\alpha}Y_\beta.$$

Applying this commutative rule, we get the following formula for a $(2,0)$ tensor $T = T_{\alpha\bar{\beta}}$:

$$T_{\alpha\bar{\beta},i\bar{j}} - T_{i\bar{j},\alpha\bar{\beta}} = R^{l}_{i\bar{\beta}\alpha}T_{l\bar{j}} + R^{\bar{l}}_{i\bar{\beta}\bar{j}}T_{\alpha\bar{l}} - R^{l}_{i\bar{j}\alpha}T_{l\bar{\beta}} - R^{\bar{l}}_{i\bar{j}\bar{\beta}}T_{l\bar{\beta}}.$$

Finally, we define the scalar curvature and bisectional curvature.

Scalar curvature is the trace of Ricci tensor, i.e., $R = g^{i\bar{j}}R_{i\bar{j}}$.

Pick an orthonormal basis $e_i, J(e_i)$ with respect to g, we set $u_i = \frac{1}{\sqrt{2}}(e_i - \sqrt{-1}J(e_i)), \bar{u}_i = \frac{1}{\sqrt{2}}(e_i + \sqrt{-1}J(e_i))$. They form a unitary basis. Then

$$\begin{aligned} R(u_i, \bar{u}_i) &= \sum_j R(u_i, \bar{u}_i, u_j, \bar{u}_j) \\ &= \sum_j R(e_i, e_j, e_j, e_i) + R(e_i, J(e_j), J(e_j), e_i) = R(e_i, e_i). \end{aligned}$$

We have used first Bianchi identity above. It follows $R = \sum R(u_i, \bar{u}_i) = \sum R(e_i, e_i) = \frac{1}{2}\sum R(e_i, e_i) + R(J(e_i), J(e_i))$, i.e., the scalar curvature R on Kähler manifold is half of the scalar curvature of the underlining real manifold.

From above computation, we see that

$$R(u_i, \bar{u}_i, u_j, \bar{u}_j) = R(e_i, e_j, e_j, e_i) + R(e_i, J(e_j), J(e_j), e_i),$$

$R(u_i, \bar{u}_i, u_j, \bar{u}_j)$ is called the bisectional curvature, as it is the sum of two real sectional curvature.

2.2 Elementary symmetric polynomials

In this section, we list several algebraic results on elementary symmetric polynomials. Hessian equations are defined by elementary symmetric polynomials:

$$S_k(D^2u) = \sigma_k(\lambda_1, \cdots, \lambda_n) = f, \tag{2.1}$$

where

$$\sigma_k(\lambda) = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k},$$

and $\lambda_1, \cdots, \lambda_n$ are eigenvalues of $D^2 u$.

We see immediately from the definition that for $k = 1$, (2.1) is a Possion equation, and for $k = n$ (2.1) becomes a Monge-Ampére equation.

To study admissible solutions of the Hessian equations, we need to introduce the Γ_k cones:

$$\Gamma_k = \{(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n | \sigma_j(\lambda) > 0, \forall j = 1, \cdots, k\}.$$

They are open symmetric cones in $\mathbb{R}^n$, with vertex at the origin. Clearly we have the nested property:

$$\Gamma_n \subset \Gamma_{n-1} \subset \cdots \subset \Gamma_1,$$

with Γ_n being the positive cone:

$$\Gamma_n = \{\lambda \in \mathbb{R}^n | \lambda_j > 0, \forall j = 1, \cdots, n\}.$$

We call u is k-admissible if the set of eigenvalues of $D^2 u$ is in Γ_k.

For Hessian equations, we have the following important properties:

Proposition *2.2.1.*

- $S_k(D^2 u) = f > 0$ is elliptic if u is k-admissible.
- $S_k^{\frac{1}{k}}(D^2 u)$ is concave.

This proposition follows from the following general fact on fully nonlinear equations of the form:

$$F(D^2 u) = f(\lambda_1, \cdots, \lambda_n) = 0,$$

where $\lambda = (\lambda_1, \cdots, \lambda_n)$ are eigenvalues of $D^2 u$.

***Proposition** 2.2.2.*

- If $f_{\lambda_i} > 0, \forall i$, then F is elliptic, i.e., $F^{ij} := \frac{\partial F}{\partial u_{ij}}$ is positive definite.
- If f is concave in λ, then F is concave in the space of symmetric matrices.

Readers are referred to [Sp] for a nice exposition. One can also easily compute the derivative of F when $D^2 u$ is diagonal. See Theorem 1.4 in [Sp].

The fact that $\sigma_k^{1/k}(\lambda)$ is concave follows from Gårding's beautiful theory on hyperbolic polynomials.

Following [Ga], a homogenous polynomial $P(\lambda)$ of degree k defined in $\mathbb{R}^n$ is called hyperbolic with respect to a direction $a \in \mathbb{R}^n$, if $\forall \lambda \in \mathbb{R}^n$, the polynomial

$$P(\lambda + ta)$$

has exactly k real roots.

Necessarily $P(a) \neq 0$. Assume $P(a) > 0$. It is easy to see that

$$Q(\lambda) = \sum a_i \frac{\partial P}{\partial \lambda_i}$$

is hyperbolic at a if P is.

Typical example is $P(\lambda) = \sigma_n(\lambda) = \prod \lambda_i$. P is hyperbolic at $a = (1, 1, \cdots, 1)$. It follows $\sigma_k(\lambda)$ is also hyperbolic at a.

Let $\Gamma(P, a)$ be the connected component in $\mathbb{R}^n$ of $P > 0$ which contains a. Then P is hyperbolic at $\forall b \in \Gamma(P, a)$.

It is then proved that $P^{1/k}(\lambda)$ is a concave function on $\Gamma(P, a)$. Indeed, Gårding proved a very general inequality which is equivalent to the concavity. In the case $P = \sigma_k$, it reads as follows:

Proposition *2.2.3 (Gårding's inequality).* For $\lambda, \mu \in \Gamma_k$,

$$\sum_{i=1}^{n} \mu_i \sigma_{k-1}(\lambda|i) \geq k\sigma_k^{\frac{1}{k}}(\mu)\sigma_k(\lambda)^{1-1/k},$$

where $\sigma_{k-1}(\lambda|i) := \sigma_{k-1}(\lambda)|_{\lambda_i=0}$.

Next we collect some algebraic identities and inequalities for $\sigma_k(\lambda)$. Assume $\lambda \in \Gamma_k$, arrange $\lambda = (\lambda_1, \cdots, \lambda_n)$ in the descending order, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

$$\begin{aligned}
\sigma_k(\lambda) &= \sigma_k(\lambda|i) + \lambda_i \sigma_{k-1}(\lambda|i), \\
\sum_i \sigma_k(\lambda|i) &= (n-k)\sigma_k(\lambda), \\
\sum_i \lambda_i \sigma_{k-1}(\lambda|i) &= k\sigma_k(\lambda), \\
\lambda_k &\geq 0, \\
\sigma_{k-1}(\lambda|n) &\geq \sigma_{k-1}(\lambda|n-1) \geq \cdots \geq \sigma_{k-1}(\lambda|1) > 0.
\end{aligned}$$

We set $H_k(\lambda) = \frac{\sigma_k(\lambda)}{\binom{n}{k}}$ to be the normalized elementary symmetric polynomials. We have

Proposition *2.2.4 (generalized Newton-Maclaurin inequalities).* For $\lambda \in \Gamma_k$,

$$\left(\frac{H_k(\lambda)}{H_l(\lambda)}\right)^{\frac{1}{k-l}} \leq \left(\frac{H_s(\lambda)}{H_t(\lambda)}\right)^{\frac{1}{s-t}},$$

for $0 \leq l \leq k$, $0 \leq t \leq s$, $k \geq s$ and $l \geq t$.

Finally, a few trickier inequalities:

$$\begin{aligned}
\lambda_1 \sigma_{k-1}(\lambda|1) &\geq C\sigma_k(\lambda), \\
\sigma_{k-1}(\lambda|k) &\geq C\sum_{i=1}^{n} \sigma_{k-1}(\lambda|i), \\
\sigma_{k-1}(\lambda|k) &\geq C\sigma_{k-1}(\lambda), \\
\prod_i \sigma_k(\lambda|i) &\geq C(\sigma_k(\lambda))^{n(k-1)/k},
\end{aligned}$$

C differs from line to line.

CHAPTER 3
MAIN RESULT

This chapter contains the main result of this thesis. We study a class of fully nonlinear flows on a compact Kähler manifolds. The flow is defined on the space of Kähler metrics in a fixed Kähler class. The construction of the flow uses the quotient of elementary symmetric polynomials or logarithm of them. Our flow includes J-flow as a special case. We shall obtain a necessary and sufficient condition in terms of the positivity of certain cohomology class, which guarantees the convergence of the flow. We study the limit metric, which solves a Hessian type equation on the manifold. The organization of this chapter is as following: in section 3.1, we state the main theorem, in section 3.2, we prove a technical lemma. In section 3.3, using the technical lemma in section 3.2, we prove the partial C^2 estimates. In section 3.4, we prove the convergence of the flow. In the last section, we discuss a variant of the flow (3.5).

3.1 A class of fully nonlinear flow

Let (M, ω) be a Kähler manifold. Let χ be another Kähler form on M. We define the corresponding space of Kähler metrics in a fixed class and the space of Kähler potentials as:

$$\mathcal{K}_{[\chi]} = \{\chi' |\, \chi' \in [\chi], \chi' > 0\}. \tag{3.1}$$

$$\mathcal{P}_\chi = \{\varphi \in C^\infty(M) |\, \chi_\varphi := \chi + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi > 0\}. \tag{3.2}$$

By the $\partial\bar{\partial}$-lemma, $\mathcal{K}_{[\chi]}$ can be identified with $\mathcal{P}_\chi$ modulo the addition of constants.

For a fixed integer k, $1 \le k \le n$, we introduce the following notation:

$$\sigma_k(\chi) = \binom{n}{k} \frac{\chi^k \wedge \omega^{n-k}}{\omega^n}. \tag{3.3}$$

It is easy to see $\sigma_k(\chi)$ is the k-th elementary symmetric polynomial evaluating at the eigenvalues of χ with respect to ω.

Let

$$c_k = \frac{\int_M \chi^{n-k} \wedge \omega^k}{\int_M \chi^n} = \frac{[\chi]^{n-k} \cdot [\omega]^k}{[\chi]^n}, \quad c'_k = \frac{\int_M \sigma_{n-k}(\chi)}{\int_M \sigma_n(\chi)} = \binom{n}{k} c_k, \tag{3.4}$$

be two topological constants.

Define the following fully nonlinear flow in $\mathcal{P}_\chi$:

$$\begin{cases} \frac{\partial}{\partial t}\varphi & = c_k'^{\frac{1}{k}} - \left(\frac{\sigma_{n-k}(\chi_\varphi)}{\sigma_n(\chi_\varphi)}\right)^{\frac{1}{k}}, \\ \varphi(0) & = 0 \end{cases} \tag{3.5}$$

Clearly, the stationary point of this flow is a Kähler metric $\tilde{\chi} \in \mathcal{K}_{[\chi]}$ satisfying:

$$\tilde{\chi}^{n-k} \wedge \omega^k = c_k \tilde{\chi}^n. \tag{3.6}$$

In the case $k = 1$, our flow is same as the J-flow. Song and Weinkove [SW2] gave a necessary and sufficient condition for the J-flow existing globally and converging to a critical metric solving (3.6).

We shall provide a similar necessary and sufficient condition for flow (3.5) to converge to critical metric solving (3.6), which we state as a cone condition. For a given Kähler manifold (M, ω), for $k \ne n$, we define $\mathcal{C}_k = \mathcal{C}_k(\omega)$ as

$$\mathcal{C}_k(\omega) = \{[\chi] > 0,\ \exists \chi' \in [\chi], \text{s.t.}\, nc_k \chi'^{n-1} > (n-k)\chi'^{n-k-1} \wedge \omega^k\}. \tag{3.7}$$

The condition in (3.7) means that $nc_k\chi'^{n-1} - (n-k)\chi'^{n-k-1} \wedge \omega^k$ is a positive $(n-1, n-1)$ form. By the definition, it follows $\mathcal{C}_k(\omega)$ is an affine cone in $H^{1,1}(M, \mathbb{C}) \cap H^2(M, \mathbb{R})$. $\mathcal{C}_k$ is non-empty as the condition in (3.7) holds trivially for ω. For $k = n$, we take $\mathcal{C}_n$ to be the entire Kähler cone of M. Note that we always treat this case separately in the proof.

One can check that the cone condition (3.7) is a necessary condition for (3.6) admitting a solution. Indeed, let $\tilde{\chi}$ be the solution to (3.6), in local coordinates, $\tilde{\chi}$ satisfies

$$\sigma_k(\tilde{\chi}^{-1}) = c'_k, \tag{3.8}$$

where $\tilde{\chi}^{-1}$ is locally the inverse matrix of $\tilde{\chi}$. Since all eigenvalues of $\tilde{\chi}^{-1}$ are positive, then we must have

$$\sigma_k(\tilde{\chi}^{-1}|i) < c'_k. \tag{3.9}$$

This condition is equivalent to the cone condition (3.7).

Proposition *3.1.1.* For $k < n$, $\chi' \in \mathcal{C}_k$ is equivalent to

$$\sigma_k(\chi'^{-1}|j) = \frac{\sigma_{n-k-1}(\chi'|j)}{\sigma_{n-1}(\chi'|j)} < c'_k, \tag{3.10}$$

for any $j \in \{1, \cdots, n\}$, where $(\chi'|j)$ denotes the matrix obtained by deleting the j-th column and j-th row of χ'.

Proof. Assume $\chi' \in \mathcal{C}_k$. By (3.7), for any given integer $1 \leq j \leq n$, the coefficient of the $(n-1, n-1)$ form $\Pi_{i=1 i\neq j}^{n} dz^i d\bar{z}^i$ in $c_k\chi'^{n-1} - \frac{n-k}{n}\omega^k \wedge \chi'^{n-k-1}$ should be positive; that is,

$$(n-1)!c_k\sigma_{n-1}(\chi'|j) - \frac{n-k}{n}k!(n-k-1)!\sigma_{n-k-1}(\chi'|j) > 0.$$

Dividing both sides by $\frac{n-k}{n}k!(n-k-1)!\sigma_{n-1}(\chi'|j)$, one obtains

$$\frac{\sigma_{n-k-1}(\chi'|j)}{\sigma_{n-1}(\chi'|j)} < c'_k.$$

A similar computation yields that (3.10) implies $\chi' \in \mathcal{C}_k$.

$\square$

Now we state the main theorem.

Theorem 3.1. *Let (M, ω) be a compact Kähler manifold, let $\chi_0 \in [\chi]$ be another Kähler metric. k is an integer, $1 \le k \le n$. If $[\chi] \in \mathcal{C}_k$, then the flow (3.5) with initial value χ_0 has long time existence, and it converges to a metric $\tilde{\chi} \in [\chi]$ which is the unique solution of (3.6).*

3.2 A technical lemma

In this section, we set up the notation and prove a technical lemma. By scaling, we may assume $c_k = \frac{\int \chi_0^{n-k} \wedge \omega^k}{\int \chi_0^n} = 1$. Then $c'_k = \binom{n}{k}$ which we denote by c when no confusion arises.

In local coordinates, we write

$$\begin{aligned}
\omega &= \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge dz^{\bar{j}}, \\
\chi_0 &= \frac{\sqrt{-1}}{2} \chi_{0i\bar{j}} dz^i \wedge dz^{\bar{j}}, \\
\chi' &= \frac{\sqrt{-1}}{2} \chi'_{i\bar{j}} dz^i \wedge dz^{\bar{j}}, \\
\chi_\varphi &= \frac{\sqrt{-1}}{2} (\chi_{0i\bar{j}} + \varphi_{i\bar{j}}) dz^i \wedge dz^{\bar{j}}, \\
\chi_{i\bar{j}} &= \chi_{0i\bar{j}} + \varphi_{i\bar{j}}.
\end{aligned}$$

It is also convenient to use χ_0, χ', χ_φ to denote the corresponding Hermitian matrices when no confusion arises. At the point of interest, we always choose a normal coordinate of ω such that χ_φ is diagonal, i.e., $\chi_\varphi = \mathrm{diag}(\chi_1, \cdots, \chi_n)$ and $\chi_1 \geq \chi_2 \geq \cdots \geq \chi_n$.

For a Hermitian matrix $A = (a_{i\bar{j}})_{n\times n}$, define

$$F(A) := -[\frac{\sigma_{n-k}(A)}{\sigma_n(A)}]^{1/k} = -\sigma_k^{\frac{1}{k}}(A^{-1}).$$

We can rewrite the flow (3.5) as:

$$\begin{cases} \frac{\partial}{\partial t}\varphi & = F(\chi_\varphi \cdot \omega^{-1}) - F(c_k'), \\ \varphi(0) & = 0, \end{cases} \tag{3.11}$$

where $\chi_\varphi \cdot \omega^{-1}$ should regarded locally as a matrix or globally as a section of $T^{1,0}(M) \wedge \Omega^{0,1}(M)$.

We explore some structure properties of F. Let $g(\lambda) = \sigma_k(\lambda)^{1/k}$. We have the following properties for g:

Proposition *3.2.1.* For $\lambda \in \Gamma_n$, we have

1. Ellipticity: $g_i > 0$, $\forall i$.

2. Concavity: $g_{ij} \leq 0$.

3. Weak concavity: $g_{ij} + \frac{g_i}{\lambda_j}\delta_{ij} \geq 0$.

Proof. Ellipticity and concavity were discussed in Chapter 2. The proof of weak concavity will be given in the appendix B.

$\square$

Let $f(\lambda_1, \cdots, \lambda_n) = -g(\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_n})$.

Proposition *3.2.2.* Let f be given as above, for $\lambda \in \Gamma_n$, f satisfies:

1. Ellipticity: $f_i > 0$, $\forall i$;

2. Concavity: $f_{ij} \leq 0$;

3. Strong concavity, i.e., $f_{ij} + \frac{f_i}{\lambda_j}\delta_{ij} \leq 0$.

Proof. By definition, we have

$$f_i = g_i \frac{1}{\lambda_i^2}, \tag{3.12}$$

$$\begin{aligned} f_{ij} &= -g_{ij}\frac{1}{\lambda_i^2}\frac{1}{\lambda_j^2} - \frac{2g_i}{\lambda_i^3}\delta_{ij} \\ &= -\frac{1}{\lambda_i^2}\frac{1}{\lambda_j^2}(g_{ij} + \frac{2g_i}{1/\lambda_i}\delta_{ij}) \leq 0. \end{aligned} \tag{3.13}$$

$$f_{ij} + \frac{f_i}{\lambda_i}\delta_{ij} = -\frac{1}{\lambda_i^2\lambda_j^2}(g_{ij} + \frac{g_i}{1/\lambda_i}\delta_{ij}) \leq 0. \tag{3.14}$$

□

Since $F(A) = f(e(A))$, where $e(A)$ are eigenvalues of A. By Proposition 2.2.2, we deduce that F is elliptic, concave, and strong concave. Note F is also homogenous of degree -1, thus

$$-F(A) = \sum_{i,j} F^{i\bar{j}}(A)a_{i\bar{j}}.$$

We need some algebraic lemmas.

Lemma *3.2.* Let $I = (i_1, i_2, \cdots, i_k)$ be an index set, denote its complement in $(1, 2, \cdots, n)$ by $\bar{I}$. We shall order $\bar{I}$ so that $(I, \bar{I})$ is an even permutation of $(1, 2, \cdots, n)$. For A, an $n \times n$ positive hermitian matrix, let A_I be the principle minor $(a_{i\bar{j}})_{i,j\in I}$. Then

$$\det(A) \leq \det(A_I)\det(A_{\bar{I}}).$$

We will give a detailed proof in the appendix A. The following corollary is a direct consequence of Lemma A.1.

Corollary *3.2.1.* Let A be as above. Then $\det(A) \leq \prod_{i=1}^{n} a_{i\bar{i}}$.

Then we can prove following:

Lemma *3.3.* Let $A = (a_{i\bar{j}})$ be a positive Hermitian matrix. Denote $\tilde{A} = (a_{i\bar{j}}\delta_{i\bar{j}})$ to be the matrix containing only the diagonal terms of A. Then

$$\sigma_k(\tilde{A}^{-1}) \leq \sigma_k(A^{-1}). \tag{3.15}$$

Proof. By Corollary 3.2.1, we have

$$\frac{1}{\det(\tilde{A})} \leq \frac{1}{\det(A)},$$

thus Lemma 3.3 holds for $k = n$. For general k, we have

$$\begin{aligned}
\sigma_k(\tilde{A}^{-1}) &= \sum_{|I|=k,(i_1,i_2,\cdots,i_k)\in I} \frac{1}{a_{i_1\bar{i}_1}} \frac{1}{a_{i_2\bar{i}_2}} \cdots \frac{1}{a_{i_k\bar{i}_k}} \\
&\leq \sum_{|I|=k} \frac{1}{\det(A_I)} \leq \sum_{|I|=k} \frac{\det(A_{\bar{I}})}{\det(A)} = \frac{\sigma_{n-k}(A)}{\sigma_n(A)} = \sigma_k(A^{-1}).
\end{aligned}$$

□

Finally, we are in position to prove a technical lemma which is crucial in proving partial C^2 estimates.

Theorem 3.4. *Let $M, \omega, \chi \in [\chi]$ be given as above. Assume that $k < n$ and $[\chi] \in \mathcal{C}_k$. Let $\chi' \in [\chi]$ be a Kähler form satisfying the condition of $\mathcal{C}_k$. Assuming $C_1 \leq \frac{\sigma_{n-k}(\chi)}{\sigma_n(\chi)} \leq$*

C_2, for two universal constants C_1 and C_2. Then there exists a universal constant N, depending only on the given geometric data, such that, if $\frac{\chi_1}{\chi_n} \geq N$ then there exists $\epsilon > 0$ such that

$$(1-\epsilon)\sum_{i=1}^{n} F^{i\bar{i}}(\chi)\chi'_{i\bar{i}} \geq c^{-\frac{1}{k}}\sigma_k^{\frac{2}{k}}(\chi^{-1}). \tag{3.16}$$

Proof. We first notice that in the case $\chi_n \ll 1$, (3.16) follows easily. Notice χ' is a fixed Kähler form, so there is a constant $\lambda > 0$ such that

$$\chi' \geq \lambda\omega.$$

Thus,

$$\begin{aligned} \sum_{i=1}^{n} F^{i\bar{i}}(\chi)\chi'_{i\bar{i}} &\geq \lambda\sum_{i=1}^{n} F^{i\bar{i}}(\chi) \qquad (3.17)\\ &= \lambda\frac{1}{k}\sigma_k^{1/k-1}(\chi^{-1})\sum_{i=1}^{n}\sigma_{k-1}(\chi^{-1}|i)\frac{1}{\chi_i^2}\\ &\geq \lambda\frac{1}{k}\sigma_k^{1/k-1}(\chi^{-1})\sigma_{k-1}(\chi^{-1}|n)\frac{1}{\chi_n^2}. \end{aligned}$$

We claim $\sigma_{k-1}(\chi^{-1}|n)\frac{1}{\chi_n}$ is bounded from below. Indeed, $\sigma_{k-1}(\chi^{-1}|n)\frac{1}{\chi_n}$ is the largest term among $\sigma_{k-1}(\chi^{-1}|i)\frac{1}{\chi_i}$ since χ_n is the smallest among χ_i, $1 \leq i \leq n$. Thus,

$$\sigma_{k-1}(\chi^{-1}|n)\frac{1}{\chi_n} \geq 1/n\sum_{i=1}^{n}\sigma_{k-1}(\chi^{-1}|i)\frac{1}{\chi_i} = k/n\sigma_k(\chi^{-1}). \tag{3.18}$$

Now if $\chi_n < \delta = \lambda(C_1c)^{1/k}$, (3.16) follows from (3.17) and (3.18). So we just need to consider the case $\chi_n \geq \delta$. Recall Gårding's inequality: for $\mu, \nu \in \Gamma_n$,

$$\frac{1}{k}\sum_{j=1}^{n}\nu_j\frac{\partial}{\partial\mu_j}\sigma_k(\mu) \geq \sigma_k^{1/k}(\nu)\sigma_k^{1-1/k}(\mu).$$

We have for the matrix $B = \mathrm{diag}(\frac{\chi'_{1\bar{1}}}{\chi_1^2}, \cdots, \frac{\chi'_{n\bar{n}}}{\chi_n^2}) = \chi^{-1}\tilde{\chi}'\chi^{-1}$,

$$\begin{aligned}\sum_{i=1}^{n} F^{i\bar{i}}(\chi)\chi'_{i\bar{i}} &= \sigma_k^{1-1/k}\frac{1}{k}\sum_{i=1}^{n}\sigma_{k-1}(\chi^{-1}|i)\frac{\chi'_{i\bar{i}}}{\chi_i^2} \\ &\geq \sigma_k^{1-1/k}(\chi^{-1})\sigma_k^{1-1/k}(\chi^{-1})\sigma_k^{1/k}(B) \\ &= \sigma_k^{1/k}(B).\end{aligned} \tag{3.19}$$

Comparing with (3.16), it is suffice to show

$$c^{1/k}\sigma_k^{1/k}(B) \geq (1+\theta)\sigma_k^{2/k}(\chi^{-1}), \quad \text{for some } \theta > 0. \tag{3.20}$$

By Proposition 3.1.1, since M is compact, there exists a universal constant $\eta > 0$ such that

$$\sigma_k((\chi'|1)^{-1}) \leq \binom{n}{k} - \eta = c - \eta. \tag{3.21}$$

Then we have

$$\begin{aligned}c^{1/k}\sigma_k^{1/k}(B) &\geq (\frac{c}{c-\eta})^{1/k}\sigma_k^{1/k}((\chi'|1)^{-1})\sigma_k^{1/k}(B) \\ &\geq (\frac{c}{c-\eta})^{1/k}\sigma_k^{1/k}((\tilde{\chi'}|1)^{-1})\sigma_k^{1/k}(B) \\ &\geq (\frac{c}{c-\eta})^{1/k}\sigma_k^{1/k}((\tilde{\chi'}|1)^{-1})\sigma_k^{1/k}(B|1) \\ &\geq (\frac{c}{c-\eta})^{1/k}\sigma_k^{2/k}(\chi^{-1}|1).\end{aligned} \tag{3.22}$$

We explain the second and last inequalities in (3.22). Apply Lemma 3.3 to the matrix $(\chi'|1)$, we have

$$\sigma_k((\chi'|1)^{-1}) \geq \sigma_k((\tilde{\chi'}|1)^{-1}). \tag{3.23}$$

Recall that $B = \chi^{-1}\tilde{\chi}'\chi^{-1}$, then Cauchy-Schwarz inequality yields

$$\sigma_k(\chi^{-1}\tilde{\chi}'\chi^{-1})\sigma_k((\tilde{\chi'}|1)^{-1}) \geq \sigma_k^2(\chi^{-1}|1). \tag{3.24}$$

Now suppose $\chi_1 \geq N\chi_n$, and $\chi_n \geq \delta$. Then

$$\begin{aligned} \frac{\sigma_k(\chi^{-1}|1)}{\sigma_k(\chi^{-1})} &= 1 - \frac{1/\chi_1 \sigma_{k-1}(\chi^{-1}|1)}{\sigma_k(\chi^{-1})} \\ &\geq 1 - \frac{1/\chi_1 \frac{\binom{n-1}{k-1}}{\delta^k}}{\sigma_k(\chi^{-1})} \geq 1 - \frac{\binom{n-1}{k-1}}{C_1 N \delta^k}. \end{aligned} \tag{3.25}$$

Combining (3.20), (3.22), and (3.25), for θ sufficient small, a positive number

$$N = \frac{\binom{n-1}{k-1}}{C_1 \delta^k} \frac{1}{1 - (1+\theta)^{1/2}(c-\eta)/c}$$

will work.

□

3.3 Partial C^2 estimates

In this section, we apply the maximum principle to obtain an estimate on the mixed second order derivatives of φ.

First we establish the short time existence of the flow. Differentiating (3.5) with respect to t yields

$$\frac{\partial}{\partial t}(\frac{\partial \varphi}{\partial t}) = F^{i\bar{j}}(\chi)\partial_i \partial_{\bar{j}}(\frac{\partial \varphi}{\partial t}). \tag{3.26}$$

Since F is elliptic, standard theory for parabolic equation guarantees the short time existence. By the maximum principle, $\frac{\partial \varphi}{\partial t}$ achieves extremal values at $t = 0$, i.e.

$$\min_{t=0} \frac{\partial \varphi}{\partial t} \leq \frac{\partial \varphi}{\partial t} \leq \max_{t=0} \frac{\partial \varphi}{\partial t}, \tag{3.27}$$

which implies

$$\inf_M \frac{\sigma_{n-k}}{\sigma_n}(\chi_0) \leq \frac{\sigma_{n-k}}{\sigma_n}(\chi_\varphi) \leq \sup_M \frac{\sigma_{n-k}}{\sigma_n}(\chi_0). \tag{3.28}$$

Consequently, $\chi_\varphi > 0$ when the flow exists because if one of the eigenvalue reaches 0, $\frac{\sigma_{n-k}}{\sigma_n}(\chi_\varphi)$ blows up.

Now we state the partial C^2 estimates:

Theorem 3.5. *Let M, ω, and $\chi_0 \in [\chi]$ as above. k is an integer, $1 \leq k \leq n$. Suppose $[\chi] \in \mathcal{C}_k$, i.e., there exists $\chi' \in [\chi]$ such that:*

$$\chi'^{n-1} - \frac{n-k}{n}\omega^k \wedge \chi'^{n-k-1} > 0.$$

Let φ be a solution of (3.5) on $[0, T)$. Then there exist universal constants $A > 0$, $C > 0$, depending only on the initial data and independent of T, such that for any time $t \geq 0$,

$$\|\partial\bar{\partial}\varphi\|_{C^0} \leq Ce^{A(\varphi - \inf_{M\times[0,t]}\varphi)}.$$

Proof. By hypothesis, there exists $\phi \in \mathcal{P}_{\chi_0}$, such that $\chi' = \chi_0 + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\phi$, then $\chi_\varphi = \chi' + \frac{\sqrt{-1}}{2}\partial\bar{\partial}(\varphi - \phi)$. Consider the function

$$G(x, t, \xi) := \log(\chi_{i\bar{j}}\xi^i\xi^{\bar{j}}) - A(\varphi - \phi),$$

for $x \in M$, and $\xi \in \mathbf{T}_x^{(1,0)}M$, $g_{i\bar{j}}\xi^i\xi^{\bar{j}} = 1$. A is a constant to be determined. For a fixed time t, we can assume G attains maximum at $(x_0, t_0) \in M \times [0, t]$, along the direction ξ_0. Choose normal coordinates of ω at x_0, so that $\xi_0 = \frac{\partial}{\partial z_1}$ and $(\chi_{i\bar{j}})$ is diagonal at x_0. By the definition of G, it is easy to see that $\chi_{1\bar{1}} = \chi_1$ is the largest eigenvalue of $\{\chi_{i\bar{j}}\}$ at x_0. Without loss of generality, we can assume $t_0 > 0$. Thus, locally, we consider $H := \log\chi_{1\bar{1}} - A(\varphi - \phi)$ instead, which also attains maximum at (x_0, t_0), with $H(x_0, t_0) = G(x_0, t_0)$. We compute the evolution of H, i.e., the quantity $\frac{\partial H}{\partial t} - F^{i\bar{j}}H_{i\bar{j}}$. Then at (x_0, t_0), we have

$$\frac{\partial H}{\partial t} = \frac{\chi_{1\bar{1},t}}{\chi_{1\bar{1}}} - A\frac{\partial\varphi}{\partial t}, \tag{3.29}$$

$$H_{i\bar{i}} = \frac{\chi_{1\bar{1},i\bar{i}}}{\chi_{1\bar{1}}} - \frac{|\chi_{1\bar{1},i}|^2}{\chi_{1\bar{1}}^2} - A(\varphi_{i\bar{i}} - \phi_{i\bar{i}}). \tag{3.30}$$

Take two derivatives along $\frac{\partial}{\partial z_1}$ direction to the equation (3.5), we have

$$\chi_{1\bar{1},t} = (\frac{\partial \varphi}{\partial t})_{1\bar{1}} = \sum_{i=1}^{n} F^{i\bar{i}} \chi_{i\bar{i},1\bar{1}} + \sum_{1 \leq i,j,k,l \leq n} F^{i\bar{j},k\bar{l}} \chi_{i\bar{j},1} \chi_{k\bar{l},\bar{1}}. \tag{3.31}$$

By (3.5),(3.29),(3.30),(3.31) we have at (x_0, t_0)

$$\begin{aligned}
& \frac{\partial H}{\partial t} - \sum_{i=1}^{n} F^{i\bar{i}} H_{i\bar{i}} \\
= \; & \frac{1}{\chi_{1\bar{1}}} (\sum_{i=1}^{n} F^{i\bar{i}} \chi_{i\bar{i},1\bar{1}} + \sum_{1 \leq i,j,k,l \leq n} F^{i\bar{j},k\bar{l}} \chi_{i\bar{j},1} \chi_{k\bar{l},\bar{1}}) - A \frac{\partial \varphi}{\partial t} - \sum_{i=1}^{n} F^{i\bar{i}} H_{i\bar{i}} \\
= \; & \frac{1}{\chi_{1\bar{1}}} \sum_{i=1}^{n} F^{i\bar{i}} (\chi_{i\bar{i},1\bar{1}} - \chi_{1\bar{1},i\bar{i}}) - A \frac{\partial \varphi}{\partial t} + A \sum_{i=1}^{n} F^{i\bar{i}} (\varphi_{i\bar{i}} - \phi_{i\bar{i}}) + B \\
= \; & \frac{1}{\chi_{1\bar{1}}} \sum_{i=1}^{n} F^{i\bar{i}} (\chi_{i\bar{i},1\bar{1}} - \chi_{1\bar{1},i\bar{i}}) - A(c^{\frac{1}{k}} + F) + A \sum_{i=1}^{n} F^{i\bar{i}} (\chi'_{i\bar{i}} + \varphi_{i\bar{i}} - \phi_{i\bar{i}}) \\
& - A \sum_{i=1}^{n} F^{i\bar{i}} \chi'_{i\bar{i}} + B \\
= \; & \frac{1}{\chi_{1\bar{1}}} \sum_{i=1}^{n} F^{i\bar{i}} (\chi_{i\bar{i},1\bar{1}} - \chi_{1\bar{1},i\bar{i}}) - A c^{\frac{1}{k}} - 2AF - A \sum_{i=1}^{n} F^{i\bar{i}} \chi'_{i\bar{i}} + B,
\end{aligned}$$

where

$$B = \frac{1}{\chi_{1\bar{1}}} \sum_{1 \leq i,j,k,l \leq n} F^{i\bar{j},k\bar{l}} \chi_{i\bar{j},1} \chi_{k\bar{l},\bar{1}} + \sum_{i=1}^{n} F^{i\bar{i}} \frac{|\chi_{1\bar{1},i}|^2}{\chi_{1\bar{1}}^2},$$

includes all the third order derivative terms.

We claim that $B \leq 0$, the proof of which will be given later. By the maximum principle, $\frac{\partial H}{\partial t} - \sum_{i=1}^{n} F^{i\bar{i}} H_{i\bar{i}} \geq 0$ at (x_0, t_0), thus

$$\frac{1}{\chi_{1\bar{1}}} \sum_{i=1}^{n} F^{i\bar{i}} (\chi_{i\bar{i},1\bar{1}} - \chi_{1\bar{1},i\bar{i}}) - A c^{\frac{1}{k}} - 2AF - A \sum_{i=1}^{n} F^{i\bar{i}} \chi'_{i\bar{i}} \geq 0,$$

i.e.,

$$\begin{aligned}\frac{1}{\chi_{1\bar{1}}}\sum_{i=1}^{n}F^{i\bar{i}}(\chi_{i\bar{i},1\bar{1}}-\chi_{1\bar{1},i\bar{i}}) &\geq A\sum_{i=1}^{n}F^{i\bar{i}}\chi'_{i\bar{i}}+Ac^{\frac{1}{k}}+2AF \\ &\geq A\sum_{i=1}^{n}F^{i\bar{i}}\chi'_{i\bar{i}}-Ac^{-\frac{1}{k}}F^{2} \\ &= A\sum_{i=1}^{n}F^{i\bar{i}}\chi'_{i\bar{i}}-Ac^{-\frac{1}{k}}\sigma_k^{\frac{2}{k}}(\chi^{-1}).\end{aligned}\tag{3.32}$$

By Ricci identity, we have

$$\chi_{1\bar{1},i\bar{i}}=\chi_{i\bar{i},1\bar{1}}+\chi_{1\bar{1}}R_{1\bar{1},i\bar{i}}-\chi_{i\bar{i}}R_{i\bar{i},1\bar{1}},$$

so the left hand side of (3.32) can be simplified as follows

$$\begin{aligned}\frac{1}{\chi_{1\bar{1}}}\sum_{i=1}^{n}F^{i\bar{i}}(\chi_{i\bar{i},1\bar{1}}-\chi_{1\bar{1},i\bar{i}}) &= \frac{1}{\chi_{1\bar{1}}}\sum_{i=1}^{n}F^{i\bar{i}}(\chi_{i\bar{i}}R_{i\bar{i}1\bar{1}}-\chi_{1\bar{1}}R_{1\bar{1}i\bar{i}}) \\ &= \frac{1}{\chi_{1\bar{1}}}\sum_{i=1}^{n}F^{i\bar{i}}\chi_{i\bar{i}}R_{i\bar{i}1\bar{1}}-\frac{1}{\chi_{1\bar{1}}}\sum_{i=1}^{n}F^{i\bar{i}}\chi_{1\bar{1}}R_{1\bar{1}i\bar{i}} \\ &\leq \frac{-C_1F}{\chi_{1\bar{1}}}-\sum_{i=1}^{n}F^{i\bar{i}}R_{1\bar{1}i\bar{i}} \\ &\leq \frac{C_0}{\chi_{1\bar{1}}}+C_2\sum_{i=1}^{n}F^{i\bar{i}},\end{aligned}\tag{3.33}$$

where $C_1=\max\{1,\sup_{i,j}\{R_{i\bar{i}j\bar{j}}\}\}$, $-C_2=\min\{-1,\inf_{i,j}\{R_{i\bar{i}j\bar{j}}\}\}$ are upper and lower bound of holomorphic bisectional curvature of M, and $C_0=C_1\sup_M[-F(\chi_0)]$. All constants here are positive.

Let $\chi_1\geq\cdots\geq\chi_n$ be the eigenvalues of χ with respect to ω at x_0. Our goal is to get a uniform upper bound for $\chi_1=\chi_{1\bar{1}}$.

If $k<n$, we have two cases:

Case 1. $\frac{\chi_1}{\chi_n}\leq N$, where N is the constant in Theorem 3.4. From (3.28), it follows that there exists a constant C_3 such that

$$C_3 \leq \sigma_k(\chi^{-1}) \leq \frac{\binom{n}{k}}{\chi_n^k},$$

from which we get an upper bound for χ_n:

$$\chi_n \leq (\frac{\binom{n}{k}}{C_3})^{\frac{1}{k}}.$$

Hence

$$\chi_1 \leq N\chi_n \leq C,$$

for some uniform constant C.

Case 2. $\frac{\chi_1}{\chi_n} \geq N$. Then by Theorem 3.4, there exists $\epsilon > 0$ such that

$$\sum_{i=1}^{n} F^{i\bar{i}}\chi'_{i\bar{i}} - c^{-\frac{1}{k}}\sigma_k^{\frac{2}{k}}(\chi^{-1}) \geq \epsilon \sum_{i=1}^{n} F^{i\bar{i}}\chi'_{i\bar{i}}. \tag{3.34}$$

Since χ' is fixed and M is compact, there exists $\gamma > 0$, such that

$$\epsilon \sum_{i=1}^{n} F^{i\bar{i}}\chi'_{i\bar{i}} \geq \gamma \sum_{i=1}^{n} F^{i\bar{i}}. \tag{3.35}$$

Combining (3.32),(3.33),(3.34) and (3.35), we get

$$\frac{C_0}{\chi_1} + C_2 \sum_{i=1}^{n} F^{i\bar{i}} \geq A\gamma \sum_{i=1}^{n} F^{i\bar{i}}. \tag{3.36}$$

Since $\gamma > 0$, we can choose A so that $A\gamma - C_2 = 1$. Hence,

$$\frac{C_0}{\chi_1} \geq \sum_{i=1}^{n} F^{i\bar{i}}. \tag{3.37}$$

Apply Gårding's inequality, Cauchy inequality and (3.28), we have

$$\begin{aligned} \sum_{j=1}^{n} F^{j\bar{j}} &= \sum_{j=1}^{n} \frac{1}{k}\sigma_k^{\frac{1}{k}-1}(\chi^{-1})\sigma_{k-1}(\chi^{-1}|j)\frac{1}{\chi_j^2} \\ &\geq \sigma_k^{\frac{1}{k}-1}(\chi^{-1})\sigma_k^{1-\frac{1}{k}}(\chi^{-1})\sigma_k^{\frac{1}{k}}(\chi^{-2}) \\ &\geq \frac{\sigma_k^{\frac{2}{k}}(\chi^{-1})}{\binom{n}{k}} \geq \frac{C_3^{\frac{2}{k}}}{\binom{n}{k}}. \end{aligned} \tag{3.38}$$

Combine (3.37) and (3.38), we have

$$\chi_1 \leq C,$$

for some universal constant C.

For $k = n$, notice in this case $c = 1$. Then

$$\sum_{i=1}^{n} F^{i\bar{i}} = \frac{1}{n}\sigma_n^{-\frac{1}{n}}(\chi)\sum_{i=1}^{n}\frac{1}{\chi_i}. \tag{3.39}$$

By (3.28), there exists two positive constants C_4 and C_5, such that

$$C_4 \leq \sigma_n^{-\frac{1}{n}}(\chi) \leq C_5. \tag{3.40}$$

Now we can proceed directly from (3.32) and (3.33), namely:

$$A + 2AF + A\sum_{i=1}^{n} F^{i\bar{i}}\chi'_{i\bar{i}} \leq \frac{C_0}{\chi_1} + C_1\sum_{i=1}^{n} F^{i\bar{i}}. \tag{3.41}$$

Assume $\chi'_{i\bar{i}} \geq \epsilon_o > 0$. Using (3.40) it follows that

$$A - 2A\sigma_n^{-\frac{1}{n}}(\chi) + \frac{A\epsilon_o}{n}\sigma_n^{-\frac{1}{n}}(\chi)\sum_{i=1}^{n}\frac{1}{\chi_i} \leq \frac{C_0}{\chi_1} + C_6\sum_{i=1}^{n}\frac{1}{\chi_i} \leq C_7\sum_{i=1}^{n}\frac{1}{\chi_i}. \tag{3.42}$$

Applying (3.40) again, we get

$$(\frac{A\epsilon_o}{n}C_4 - C_7)\sum_{i=1}^{n}\frac{1}{\chi_i} \leq 2AC_5. \tag{3.43}$$

Now if we take A such that $\frac{A\epsilon_o}{n}C_4 - C_7 = 1$, then from (3.43), we have

$$\sum_{i=1}^{n}\frac{1}{\chi_i} \leq C_8.$$

Since $\chi_i > 0$

$$\chi_i \geq C_8{}^{-1}, \tag{3.44}$$

Combining (3.40) and (3.44), it follows

$$\chi_1 = \frac{(\Pi_{i=2}^n \chi_i^{-1})}{\sigma_n(\chi^{-1})} \leq C_8^{n-1}/C_4^n = C, \tag{3.45}$$

for a uniform constant C.

Finally, we prove the claim:

$$B = \frac{1}{\chi_{1\bar{1}}} \sum_{i,j,k,l} F^{i\bar{j},k\bar{l}} \chi_{i\bar{j},1} \chi_{k\bar{l},\bar{1}} + \sum_i F^{i\bar{i}} \frac{|\chi_{1\bar{1},i}|^2}{\chi_{1\bar{1}}^2} \leq 0.$$

Case 1. $k < n$.

$F^{i\bar{j},k\bar{l}}$ is nontrivial if and only if $i = j, k = l$ or $i = l, k = j$. Indeed, we have for $i \neq j$

$$\begin{aligned} F^{i\bar{j},j\bar{i}} &= \frac{1}{k}(\frac{\sigma_{n-k}(\chi)}{\sigma_n(\chi)})^{\frac{1}{k}-1}(\frac{\sigma_n \sigma_{n-k-2}(\chi|i,j) - \sigma_{n-k}\sigma_{n-2}(\chi|i,j)}{\sigma_n^2}) \\ &= -\frac{1}{k}(\frac{\sigma_{n-k}(\chi)}{\sigma_n(\chi)})^{\frac{1}{k}-1}(\frac{\chi_i \sigma_{n-k-1}(\chi|i,j) + \chi_j \sigma_{n-k-1}(\chi_j|i,j) + \chi_i \chi_j \sigma_{n-k-2}(\chi|i,j)}{\sigma_n^2}) \\ &< 0. \end{aligned} \tag{3.46}$$

We divide B into three groups:

The first group:

$$X = \frac{1}{\chi_{1\bar{1}}}(\sum_{1 \leq i,j \leq n} F^{i\bar{i},j\bar{j}} \chi_{i\bar{i},1} \chi_{j\bar{j},\bar{1}}) + F^{1\bar{1}} \frac{|\chi_{1\bar{1},1}|^2}{\chi_{1\bar{1}}^2} \leq 0.$$

Non-positivity of X follows from the Proposition 3.2.2.

Second group:

$$Y = \frac{1}{\chi_{1\bar{1}}} \sum_{i=2}^n F^{i\bar{1},1\bar{i}} \chi_{i\bar{1},1} \chi_{1\bar{i},\bar{1}} + \sum_{i=2}^n F^{i\bar{i}} \frac{|\chi_{1\bar{1},i}|^2}{\chi_{1\bar{1}}^2} \leq 0.$$

By Ricci identity we have:

$$\chi_{i\bar{j},k} = \chi_{k\bar{j},i}, \chi_{i\bar{j},\bar{k}} = \chi_{i\bar{k},\bar{j}}.$$

It is suffice to show

$$\chi_{1\bar{1}}F^{j\bar{1},1\bar{j}} + F^{j\bar{j}} \leq 0.$$

After factoring common factor $\frac{1}{k\sigma_n^2(\chi)}(\frac{\sigma_{n-k}(\chi)}{\sigma_n(\chi)})^{1/k}$, we are left to show

$$\begin{aligned}&\chi_1[\sigma_n(\chi)\sigma_{n-k-2}(\chi|1,j) - \sigma_{n-k}(\chi)\sigma_{n-2}(\chi|1,j)]\\ &+\sigma_{n-k}(\chi)\sigma_{n-1}(\chi|j) - \sigma_{n-k-1}(\chi|j)\sigma_n(\chi) \leq 0.\end{aligned} \tag{3.47}$$

Use the identity $\sigma_k(\chi) = \sigma_k(\chi|1) + \chi_1\sigma_{k-1}(\chi|1)$, we have

$$\begin{aligned}& \chi_1[\sigma_n(\chi)\sigma_{n-k-2}(\chi|1,j) - \sigma_{n-k}(\chi)\sigma_{n-2}(\chi|1,j)]\\ + \ & \sigma_{n-k}(\chi)\sigma_{n-1}(\chi|j) - \sigma_{n-k-1}(\chi|j)\sigma_n(\chi)\\ = \ & \sigma_n(\chi)[\chi_1\sigma_{n-k-1}(\chi|j) - \sigma_{n-k-1}(\chi|j)] - \sigma_{n-k}(\chi)[\chi_1\sigma_{n-2}(\chi|1,j) - \sigma_{n-1}(\chi|j)]\\ = \ & -\sigma_n(\chi)\sigma_{n-k-1}(\chi|1,j) \leq 0.\end{aligned}$$

The rest terms form the third group:

$$Z = \frac{1}{\chi_{1\bar{1}}}\sum_{1\leq i\leq n, 2\leq j\leq n, i\neq j} F^{i\bar{j},j\bar{i}}\chi_{i\bar{j},1}\chi_{j\bar{i},\bar{1}} \leq 0.$$

By (3.46), each term in Z is negative.

To sum up, we have

$$B = X + Y + Z \leq 0.$$

Case 2. $k = n$.

If we use the convention $\sigma_{-1}(\chi) = 0$, the computation above is still valid.

In summary, for all $1 \leq k \leq n$, there exists a uniform constant C, such that $\chi_1 \leq C$. Back in the definition of G, we have

$$\log(\chi_{i\bar{j}}) - A(\varphi - \phi) \leq \log(\chi_1(x_0)) - A(\varphi(x_0) - \phi(x_0)), \tag{3.48}$$

so

$$\log(\chi_{i\bar{j}}) \leq \log C - A\varphi(x_0) + A\varphi + C'.$$

Exponentiate both sides, we get the desired estimate.

$\square$

From the partial C^2 estimates, we can conclude that the flow (3.5) has long time existence.

3.4 Convergence of the flow

In this section, we will prove a uniform C^0 bound for the oscillation of φ_t. To do so, we introduce a set of functionals on $\mathcal{P}_{\chi_0}$. After obtaining C^0 estimate of oscillation of φ_t, all the arguments in [SW2] can be applied verbatim to get the convergence of the flow.

We define the following functionals: fix χ_0 as reference metric, for any $\phi \in \mathcal{P}_{\chi_0}$, define

$$\mathcal{F}_{k,\chi_0}(\phi) = \mathcal{F}_k(\phi) = \int_0^1 \int_M \dot{\phi}_t \chi_{\phi_t}^k \wedge \omega^{n-k} dt, \tag{3.49}$$

where ϕ_t is an arbitrary smooth path in $\mathcal{P}_{\chi_0}$ connecting 0 and ϕ, and $\dot{\phi}_t$ denotes time derivative. One can show the definition is independent of the choice of the path ϕ_t. Taking derivative of (3.49), we have

$$\frac{d}{dt}\mathcal{F}_k(\phi_t) = \int_M \dot{\phi}_t \chi_{\phi_t}^k \wedge \omega^{n-k}. \tag{3.50}$$

Define

$$\mathcal{F}_{k,n}(\phi) = \mathcal{F}_k(\phi) - c_{n-k}\mathcal{F}_n(\phi). \tag{3.51}$$

By (3.50), we get

$$\frac{d}{dt}\mathcal{F}_{n-k,n}(\phi) = \int_M \dot{\phi}_t(\chi_{\phi_t}^{n-k} \wedge \omega^k - c_k\chi_{\phi_t}^n). \tag{3.52}$$

Thus the Euler-Lagrange equation of $\mathcal{F}_{n-k,n}$ is exactly the critical equation (3.6):

$$\chi_\phi^{n-k} \wedge \omega^k = c_k\chi_\phi^n. \tag{3.53}$$

Regarding the second derivative of $\mathcal{F}_{n-k,n}$, we choose a path ϕ_t and use (3.49) and (3.50) to get:

$$\begin{aligned}\frac{d^2\mathcal{F}_{n-k,n}(\phi_t)}{dt^2} &= \int_M \ddot{\phi}_t(\chi_\phi^{n-k} \wedge \omega^k - c_k\chi_\phi^n) \\ &\quad + \int_M \dot{\phi}_t\partial\bar{\partial}\dot{\phi}_t((n-k)\chi_\phi^{n-k-1} \wedge \omega^k - nc_k\chi_\phi^n) \\ &= \int_M \ddot{\phi}_t(\chi_\phi^{n-k} \wedge \omega^k - c_k\chi_\phi^{n-1}) \\ &\quad + \int_M \partial\dot{\phi}_t\bar{\partial}\dot{\phi}_t(nc_k\chi_\phi^{n-1} - (n-k)\chi_\phi^{n-k-1} \wedge \omega^k).\end{aligned} \tag{3.54}$$

Thus we observe the following

Theorem 3.6. *The solution of (3.6) is unique if it exists.*

Proof. Suppose we have two distinct critical metrics χ_{ϕ_0} and χ_{ϕ_1} in the same class $[\chi]$. Consider an affine path $\phi_t = (1-t)\phi_0 + t\phi_1$, $t \in [0,1]$. ϕ_0 and ϕ_1 being critical points are equivalent to

$$F(\chi_{\phi_0}) = F(\chi_{\phi_1}) = -c_k'^{1/k}.$$

Recall that $F = -(\frac{\sigma_{n-k}}{\sigma_n})^{1/k}$ is concave, thus

$$F(\chi_{\phi_t}) \geq (1-t)F(\chi_{\phi_0}) + tF(\chi_{\phi_1}) = -c_k'^{1/k},$$

Consequently, we have, in local coordinates,

$$\sigma_k(\chi_{\phi_t}^{-1}) \leq c_k'.$$

By Proposition 3.1.1, it follows

$$nc_k\chi_{\phi_t}^{n-1} - (n-k)\chi_{\phi_t}^{n-k-1} \wedge \omega^k > 0.$$

Thus according to (3.54) and the facts $\dot{\phi}_t = \phi_1 - \phi_0$, $\ddot{\phi}_t = 0$, we conclude that $\mathcal{F}_{n-k,n}(\phi_t)$ is a smooth convex function with vanishing first derivatives at two end points, so it must be a constant. Moreover,

$$\frac{d^2\mathcal{F}_{n-k,n}(\phi_t)}{dt^2} = 0$$

implies $\dot{\phi}_t = \phi_1 - \phi_0$ is constant, hence $\chi_{\phi_0} = \chi_{\phi_1}$.

□

Next, we establish some monotonicity properties of the functionals.

Proposition *3.4.1.* The functional $\mathcal{F}_{n-k,n}$ is decreasing along the flow (3.5).

Proof. We write (3.5) as

$$\dot{\varphi}_t = c_k'^{1/k} + F.$$

Then direct computation shows:

$$\begin{aligned} \frac{d}{dt}\mathcal{F}_{n-k,n}(\varphi_t) &= \int_M \dot{\varphi}_t(\chi_\varphi^{n-k} \wedge \omega^k - c_k\chi_\varphi^n) \\ &= \frac{1}{\binom{n}{k}}\int_M (c_k'^{1/k} + F)(F^k - c_k')\chi_\varphi^n \leq 0. \end{aligned} \tag{3.55}$$

The integrand is of the form $(a^{1/k} - b^{1/k})(b-a)$ which is clearly non-positive.

□

Corollary *3.4.1.* Assume the convergence of the flow, i.e., the existence of the solution of (3.6), then the global minimum of $\mathcal{F}_{n-k,n}$ is realized by the critical metric.

Concerning C^0 estimate, we need another monotonicity:

Proposition *3.4.2.* Let $\mathcal{F}_{n-k}$ be defined as above, φ_t be the solution of flow (3.5), then

$$\frac{d\mathcal{F}_{n-k}(\varphi_t)}{dt} \leq 0,$$

i.e., $\mathcal{F}_{n-k}(\varphi_t)$ decreases along the flow. In particular, $\mathcal{F}_{n-k}(\varphi_t) \leq 0$ for all $t > 0$.

Proof. First we make following observation:

$$\begin{aligned} \int_M \sigma_{n-k} dv &= \int_M (\frac{\sigma_{n-k}}{(\sigma_n)^{\frac{1}{k+1}}})(\sigma_n)^{\frac{1}{k+1}} dv \\ &\leq [\int_M (\frac{\sigma_{n-k}}{(\sigma_n)^{\frac{1}{k+1}}})^{\frac{1+k}{k}} dv]^{\frac{k}{k+1}} (\int_M \sigma_n dv)^{\frac{1}{1+k}} \\ &= (\int_M \frac{(\sigma_{n-k})^{\frac{1+k}{k}}}{(\sigma_n)^{\frac{1}{k}}} dv)^{\frac{k}{k+1}} (\int_M \sigma_n dv)^{\frac{1}{1+k}}, \end{aligned} \tag{3.56}$$

where $dv = \frac{\omega^n}{n!}$, so $\sigma_{n-k}(\chi)dv = \frac{\binom{n}{k}}{n!}\chi^{n-k} \wedge \omega^k$. Therefore (3.56) yields

$$\int_M (\frac{\sigma_{n-k}}{\sigma_n})^{\frac{1}{k}} \chi^{n-k} \wedge \omega^k \geq c^{\frac{1}{k}} \int_M \chi^{n-k} \wedge \omega^k. \tag{3.57}$$

Now we compute $\frac{d}{dt}\mathcal{F}_{n-k}(\varphi_t)$ by choosing the path given by the flow then

$$\begin{aligned} \frac{d}{dt}\mathcal{F}_{n-k}(\varphi_t) &= \int_M \dot{\varphi}_t \chi_{\varphi_t}^{n-k} \wedge \omega^k \\ &= \int_M [c_k'^{1/k} + F]\chi_{\varphi_t}^{n-k} \wedge \omega^k \\ &= \int_M c_k'^{1/k} \chi_{\varphi_t}^{n-k} \wedge \omega^k - \int_M (\frac{\sigma_{n-k}}{\sigma_n})^{1/k} \chi_{\varphi_t}^{n-k} \wedge \omega^k \leq 0. \end{aligned} \tag{3.58}$$

$\square$

From Proposition 3.4.2, we know $\mathcal{F}_{n-k}(\varphi_t) \leq 0$. Since the definition of $\mathcal{F}_{n-k}$ is independent of the choice of the path, we now choose the path $\gamma(s) = s\varphi_t$ to compute $\mathcal{F}_{n-k}(\varphi_t)$.

$$\begin{aligned}
\mathcal{F}_{n-k}(\varphi_t) &= \int_0^1 \int_M \varphi_t \chi_{s\varphi_t}^{n-k} \wedge \omega^k ds \\
&= \int_0^1 \int_M \varphi_t (s\chi_{\varphi_t} + (1-t)\chi_{_0})^{n-k} \wedge \omega^k ds \\
&= \sum_{l=0}^{n-k} \int_0^1 \binom{n-k}{l} s^l (1-s)^{n-k-l} ds \int_M \varphi_t \chi_{\varphi_t}^l \wedge \chi_{_0}^{n-k-l} \wedge \omega^k \leq 0.
\end{aligned}$$

So at time t, we may write in short $\mathcal{F}_{n-k}(\varphi_t) = \int_M \varphi_t d\mu_t$. Now we are in the position to prove following:

Theorem 3.7. *Let φ_t be a solution of (3.5) on $[0, \infty)$. Then there exists a universal constant C, such that*

$$\| \sup \varphi_t - \inf \varphi_t \|_{C^0} \leq C.$$

Proof. It is suffice to show a uniform lower bound of $\inf \tilde{\varphi}_t$, where $\tilde{\varphi}_t = \varphi_t - \sup_M \varphi_t$. Following [SW2], we prove this by contradiction. If such a lower bound does not exist, then we can choose a sequence of times $t_i \to \infty$ such that

- $\inf_M \tilde{\varphi}_{t_i} = \inf_{t \in [0, t_i]} \inf_M \tilde{\varphi}_t$
- $\inf_M \tilde{\varphi}_{t_i} \to -\infty$

Set $B = A/(1-\delta)$ where A is the constant in Theorem 3.5, and let δ be a small positive constant to be determined later. Let $u = e^{-B\tilde{\varphi}_{t_i}}$. We apply Lemma 3.3, Lemma 3.4 of [SW2], there is a constant c' independent of u, such that

$$\|u\|_{C^0} \leq C'\|u\|_\delta.$$

Since $u = e^{-B\tilde{\varphi_{t_i}}}$ and $\tilde{\varphi_{t_i}}$ satisfies $\sup_M \tilde{\varphi_{t_i}} = 0$ and

$$\chi_{0k\bar{l}} + (\tilde{\varphi_{t_i}})_{k\bar{l}} = \chi_{k\bar{l}} > 0,$$

we can apply Proposition 2.1 of [T1] to get a bound on $\|u\|_\delta$ for δ small enough. This gives the uniform C^0 estimate of $\tilde{\varphi}_t$.

$\square$

Proof of Theorem 3.1. So far we have got the uniform C^0 estimate for oscillation of φ_t, in order to get convergence we have to normalize φ_t, namely let

$$\hat{\varphi}_t = \varphi_t - \frac{\mathcal{F}_{n-k}(\varphi_t)}{\int_M d\mu_t}.$$

Then $\hat{\varphi}_t$ takes value zero somewhere, by Theorem 3.7, $\|\hat{\varphi}_t\|_{C^0} \leq C$. With this choice of normalization, we see the partial C^2 estimate is actually uniform. By Theorem 3.5

$$\|\partial\bar{\partial}\hat{\varphi}_t\|_{C^0} = \|\partial\bar{\partial}\varphi_t\|_{C^0} \leq Ae^{c(\varphi_t - \inf_{M\times[0,t]}\varphi_t)}.$$

For the exponent, we have

$$\begin{aligned} \varphi_t - \inf_{M\times[0,t]}\varphi_t &= \hat{\varphi}_t + \frac{\mathcal{F}_{n-k}(\varphi_t)}{\int_M d\mu_t} - \inf_{M\times[0,t]}(\hat{\varphi}_t + \frac{\mathcal{F}_{n-k}(\varphi_t)}{\int_M d\mu_t}) \\ &\leq \hat{\varphi}_t + \frac{\mathcal{F}_{n-k}(\varphi_t)}{\int_M d\mu_t} - \inf_{M\times[0,t]}\hat{\varphi}_t - \inf_{M\times[0,t]}\frac{\mathcal{F}_{n-k}(\varphi_t)}{\int_M d\mu_t} \\ &= \hat{\varphi}_t - \inf_{M\times[0,t]}\hat{\varphi}_t + \frac{\mathcal{F}_{n-k}(\varphi_t)}{\int_M d\mu_t} - \inf_{M\times[0,t]}\frac{\mathcal{F}_{n-k}(\varphi_t)}{\int_M d\mu_t} \\ &= \hat{\varphi}_t - \inf_{M\times[0,t]}\hat{\varphi}_t \leq 2C. \end{aligned} \tag{3.59}$$

Last equality follows from Proposition 3.4.2 and the fact $\int_M d\mu_t$ is independent of t. Hence we have the uniform C^2 estimate for $\hat{\varphi}_t$. Then we can apply the Evans-Krylov theory and Schauder estimates to get uniform C^∞ estimates for $\hat{\varphi}_t$. To show the convergence without passing to a subsequence, one can follow the methods in [Cao],[Wein2] verbatim.

□

Remark. In the case $k = n$, our theorem gives an alternative flow proof of *Calabi Conjecture* in the case $c_1(M) = 0$. We restate it here.

Theorem 3.8. *Let (M, ω) be a Kähler manifold. Assume $\chi \in [\chi]$ is another Kähler form. Let $c = \frac{\int_M \omega^n}{\int_M \chi^n}$, then the flow*

$$\begin{cases} \frac{\partial}{\partial t}\varphi &= c^{1/n} - (\frac{1}{\sigma_n(\chi_\varphi)})^{\frac{1}{n}}, \\ \varphi(0) &= 0 \end{cases} \tag{3.60}$$

has long time existence and converges to the unique critical metric $\tilde{\chi} \in [\chi]$ satisfying

$$c\tilde{\chi}^n = \omega^n. \tag{3.61}$$

In contrast to Cao's Kähler-Ricci flow [Cao], we use $(\det \chi^n)^{1/n}$ instead of $\log(\det \chi^n)$.

3.5 A variant of (3.5) using logarithm

At the end of last section, we briefly compared the construction of our flow (3.5) with Kähler-Ricci flow. In this section, we will study a variant of (3.5) by using logarithm. It is defined as follows

$$\begin{cases} \frac{\partial}{\partial t}\varphi &= \log \frac{\sigma_n(\chi_\varphi)}{\sigma_{n-k}(\chi_\varphi)} + \log c'_k, \\ \varphi(0) &= 0 \end{cases} \tag{3.62}$$

For this flow, we still have the convergence result:

Theorem 3.9. *Let (M, ω) be a compact Kähler manifold. Let constants c_k and c'_k be defined as above. Suppose $[\chi] \in \mathcal{C}_k$, let $\chi_0 \in [\chi]$ be the metric satisfying*

$$nc_k\chi_0^{n-1} - (n-k)\chi_0^{n-k-1} \wedge \omega^k > 0. \tag{3.63}$$

Then the flow (3.62) with initial value χ_0 has long time existence and converges to the critical metric $\tilde{\chi} \in [\chi]$ satisfying

$$c_k\tilde{\chi}^n = \tilde{\chi}^{n-k} \wedge \omega^k. \tag{3.64}$$

We will first list algebraic structure properties of $F = \log \frac{\sigma_n}{\sigma_{n-k}}$ which are key to the partial C^2 estimates.

Proposition *3.5.1.* Let $g = \log \sigma_k(\lambda)$, $\lambda \in \Gamma_n$, then we have the following:

1. g is elliptic, i.e., $g_i > 0$, $\forall i$;
2. g is concave, i.e., $g_{ij} \leq 0$;
3. Weak concavity, i.e., $g_{ij} + \frac{g_i}{\lambda_j}\delta_{ij} \geq 0$.

Proof. Recall that $h = \sigma_k^{1/k}$ has the same properties. Now $g = \frac{1}{k}\log h$, for simplicity we can ignore the constant $\frac{1}{k}$. We have

$$g_i = \frac{h_i}{h} > 0, \tag{3.65}$$

thus g has ellipticity property. For concavity, taking two directional derivatives along e, we have

$$g_{ee} = \frac{h_{ee}}{h} - \frac{h_e^2}{h^2} \leq 0. \tag{3.66}$$

We will give the the proof of weak concavity in the appendix B.

□

Let $f(\lambda_1, \cdots, \lambda_n) = -g(\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_n})$. Then $F(\chi_\varphi) = f(\chi)$, where χ denotes the eigenvalues of χ_φ with respect to ω. By proposition 3.5.1, we have

Proposition *3.5.2.* For f be given as above, it satisfies:

1. f is elliptic, i.e., $f_i > 0$, $\forall i$;
2. f is concave, i.e., $f_{ij} \leq 0$;
3. Strong concavity, i.e., $f_{ij} + \frac{f_i}{\lambda_j}\delta_{ij} \leq 0$.

Proof. By definition, we have

$$f_i = g_i \frac{1}{\lambda_i^2}, \tag{3.67}$$

$$\begin{aligned} f_{ij} &= -g_{ij}\frac{1}{\lambda_i^2}\frac{1}{\lambda_j^2} - \frac{2g_i}{\lambda_i^3}\delta_{ij} \\ &= -\frac{1}{\lambda_i^2}\frac{1}{\lambda_j^2}(g_{ij} + \frac{2g_i}{1/\lambda_i}\delta_{ij}) \leq 0. \end{aligned} \tag{3.68}$$

$$f_{ij} + \frac{f_i}{\lambda_i}\delta_{ij} = -\frac{1}{\lambda_i^2\lambda_j^2}(g_{ij} + \frac{g_i}{1/\lambda_i}\delta_{ij}) \leq 0. \tag{3.69}$$

□

For this F, we have the following homogeneity property:

$$F(\lambda\chi) = k\log\lambda + F(\chi), \forall\lambda. \tag{3.70}$$

Taking derivative with respect to λ, letting $\lambda = 1$, it follows that

$$F^{i\bar{j}}\chi_{i\bar{j}} = k. \tag{3.71}$$

Proof of the Theorem 3.9. By scaling, we may assume $c_k' = 1$. And we only treat $k \neq n$ case, as $k = n$ reduces to Kähler-Ricci flow [Cao].

Step 1: Short time existence and the preservation of the Kähler form.

Taking time derivative of the flow (3.62), we have

$$\frac{\partial}{\partial t}\frac{\partial \varphi}{\partial t} = F^{i\bar{j}}(\frac{\partial \varphi}{\partial t})_{i\bar{j}}. \tag{3.72}$$

By proposition 3.5.2, the above equation is strictly parabolic for $\frac{\partial \varphi}{\partial t}$, consequently by maximum principle we have

$$\min \frac{\partial \varphi}{\partial t}|_{t=0} \leq \frac{\partial \varphi}{\partial t} \leq \max \frac{\partial \varphi}{\partial t}|_{t=0}. \tag{3.73}$$

It follows that there exists two universal positive constants C_1 and C_2 such that

$$C_1 \leq \frac{\sigma_n(\chi_\varphi)}{\sigma_{n-k}(\chi_\varphi)} \leq C_2, \tag{3.74}$$

consequently, χ_φ remains Kähler.

Step 2: partial C^2 estimates.

We will consider the same auxiliary function $G = \log \chi_{1\bar{1}} - A\varphi$. Computing the evolution of G, we have

$$\begin{aligned} 0 &\geq F^{i\bar{i}}G_{i\bar{i}} - G_t \qquad (3.75)\\ &= F^{i\bar{i}}\frac{(\chi_{1\bar{1},i\bar{i}} - \chi_{i\bar{i},1\bar{1}})}{\chi_{1\bar{1}}} - AF^{i\bar{i}}\varphi_{i\bar{i}} + AF - B. \end{aligned}$$

We collect terms involving third order derivatives and denote them by B, i.e.,

$$B = F^{i\bar{i}}\frac{\chi_{1\bar{1},i}\chi_{1\bar{1},\bar{i}}}{\chi_{1\bar{1}}^2} + F^{i\bar{j},k\bar{l}}\frac{\chi_{i\bar{j},1}\chi_{k\bar{l},\bar{1}}}{\chi_{1\bar{1}}}. \tag{3.76}$$

We claim $B \leq 0$ whose proof will be given at the end. Then following (3.75) and by (3.71), we get

$$\frac{C_1 k}{\chi_{1\bar{1}}} \geq -C_2 \sum_{i=1}^{n} F^{i\bar{i}} - Ak + AF^{i\bar{i}}\chi_{0i\bar{i}} + AF. \tag{3.77}$$

Corresponding to the Theorem 3.4, we have a similar lemma.

Lemma *3.10 (technical lemma in the proof).* If $[\chi] \in \mathcal{C}_k$, and let χ_0 be the form satisfying the cone condition. Then there exists N, such that if $\frac{\chi_1}{\chi_n} \geq N$ then there exists $\epsilon > 0$ such that

$$(1-\epsilon)\sum_{i=1}^{n} F^{i\bar{i}}\chi_{0i\bar{i}} \geq k\sigma_k^{1/k}(\chi^{-1}). \tag{3.78}$$

Proof. We let $G = -(\frac{\sigma_{n-k}}{\sigma_n})^{1/k}$ which is same as F in Theorem 3.4. Then $F = -k\log(-G)$. Combine (3.19) and (3.20), there exists N, so that if $\frac{\chi_1}{\chi_n} \geq N$ then there exists $\epsilon > 0$ such that

$$(1-\epsilon)\sum_{i=1}^{n} G^{i\bar{i}}\chi_{0i\bar{i}} \geq \sigma_k^{2/k}(\chi^{-1}). \tag{3.79}$$

Since $F^{i\bar{i}} = \frac{-k}{-G} - G^{i\bar{i}}$, it follows that

$$\begin{aligned}(1-\epsilon)\sum_{i=1}^{n} F^{i\bar{i}}\chi_{0i\bar{i}} &= (1-\epsilon) - \frac{k}{G}\sum_{i=1}^{n} G^{i\bar{i}}\chi_{0i\bar{i}} \\ &\geq k\sigma_k^{1/k}(\chi^{-1}).\end{aligned}$$

$\square$

Applying this lemma (3.77) becomes

$$\frac{C_1 k}{\chi_{1\bar{1}}} \geq (A\epsilon\lambda - C_2)\sum_{i=1}^{n} F^{i\bar{i}} + A(F - k + k\sigma_k^{1/k}(\chi^{-1})). \tag{3.80}$$

It is elementary to show $F-k+k\sigma_k^{1/k}(\chi^{-1})\geq 0$. Indeed, let $a=\sigma_k(\chi^{-1})$, then $F-k+k\sigma_k^{1/k}(\chi^{-1})=ka^{1/k}-\log a-k$ which is nonnegative by elementary calculus. Thus taking $A=\frac{1+C_2}{\epsilon\lambda}$, we get

$$\frac{C_1 k}{\chi_{1\bar{1}}}\geq\sum_{i=1}^n F^{i\bar{i}}. \tag{3.81}$$

Claim: $\sum F^{i\bar{i}}$ is bounded from below.

$$\begin{aligned}\sum_{i=1}^n F^{i\bar{i}}&=\sum_{i=1}^n\frac{\sigma_{n-1,i}}{\sigma_n}-\frac{\sigma_{n-k-1,i}}{\sigma_{n-k}} \qquad (3.82)\\ &=\sum_{i=1}^n\frac{\sigma_{n-1}}{\sigma_n}-\frac{(k+1)\sigma_{n-k-1}}{\sigma_{n-k}}\geq\frac{k}{n}\sigma_1(\lambda_i^{-1})\geq c\frac{k}{n}\sigma_k^{1/k}(\lambda_i^{-1})\geq C.\end{aligned}$$

We have used the Newtow-Maclaurin inequality and (3.74). Together with (3.81), we get an upper bound for $\chi_{1\bar{1}}$ in the case $\frac{\chi_1}{\chi_n}\geq N$. In the case $\frac{\chi_1}{\chi_n}\leq N$, first we have $\chi_i\leq N\chi_n$, $\forall i$. Then by (3.74), we conclude

$$C\leq\sigma_k(\chi^{-1})\leq c\frac{1}{\chi_n^k}, \tag{3.83}$$

from which we obtain an upper bound for χ_n, thus $\chi_1\leq N\chi_n$ is bounded from above.

We prove the claim that $B\leq 0$. As previously, we shall divide terms in B into three groups:

The first group:

$$X=\frac{1}{\chi_{1\bar{1}}}\Big(\sum_{1\leq i,j\leq n}f^{i\bar{i},j\bar{j}}\chi_{i\bar{i},1}\chi_{j\bar{j},\bar{1}}\Big)+F^{1\bar{1}}\frac{|\chi_{1\bar{1},1}|^2}{\chi_{1\bar{1}}^2}\leq 0,$$

which follows directly from the strong concavity of F.

The second group:

$$Y=\frac{1}{\chi_{1\bar{1}}}\sum_{i=2}^n F^{i\bar{1},1\bar{i}}\chi_{i\bar{1},1}\chi_{1\bar{i},\bar{1}}+\sum_{i=2}^n F^{i\bar{i}}\frac{|\chi_{1\bar{1},i}|^2}{\chi_{1\bar{1}}^2}\leq 0.$$

Since by Ricci identity, we have

$$\chi_{i\bar{1},1} = \chi_{1\bar{1},i}, \text{and} \quad \chi_{1\bar{i},\bar{1}} = \chi_{1\bar{1},\bar{i}}.$$

Therefore, it is suffice to prove

$$\chi_{1\bar{1}} F^{i\bar{1},1\bar{i}} + F^{i\bar{i}} \leq 0, \forall i.$$

We have the following direct computation:

$$\begin{aligned}
\chi_{1\bar{1}} F^{i\bar{1},1\bar{i}} + F^{i\bar{i}} &= \chi_{1\bar{1}}\left(-\frac{\sigma_{n-2}(\chi|1,i)}{\sigma_n} + \frac{\sigma_{n-k-2}(\chi|1,i)}{\sigma_{n-k}}\right) \\
&+ \frac{\sigma_{n-1}(\chi|i)}{\sigma_n} - \frac{\sigma_{n-k-1}(\chi|i)}{\sigma_{n-k}} \\
&= -\frac{\sigma_{n-k-1}(\chi|1,i)}{\sigma_{n-k}} \leq 0.
\end{aligned}$$

The last group:

$$Z = \frac{1}{\chi_{1\bar{1}}}\left(\sum_{2 \leq i,j \leq n} F^{i\bar{j},j\bar{i}} \chi_{i\bar{j},1} \chi_{j\bar{i},1}\right) \leq 0.$$

This follows from the fact $F^{i\bar{j},j\bar{i}} \leq 0$. Since

$$\begin{aligned}
F^{i\bar{j},j\bar{i}} &= -\frac{\sigma_{n-2}(\chi|i,j)}{\sigma_n} + \frac{\sigma_{n-k-2}(\chi|i,j)}{\sigma_{n-k}} \\
&= \frac{\sigma_{n-2}(\chi|i,j)(\sigma_{n-k-2}(\chi|i,j)\chi_i\chi_j - \sigma_{n-k}(\chi))}{\sigma_n \sigma_{n-k}} \leq 0.
\end{aligned}$$

Step 3: C^0 estimates.

We still consider $\mathcal{F}_{n-k,n}$. We have similar monotonic properties:

Proposition *3.5.3.* The functional $\mathcal{F}_{n-k,n}$ is decreasing along the flow (3.62).

Proof.

$$\frac{d}{dt}\mathcal{F}_{n-k,n} = \frac{1}{\binom{n}{k}} \int_M (\log \sigma_n - \log \sigma_{n-k})(\sigma_{n-k} - \sigma_n)\omega^n \leq 0. \tag{3.84}$$

Because the integrand is of the form $(a-b)(\log b - \log a)$, which is clearly non-positive.

$\square$

***Proposition** 3.5.4.* The functional $\mathcal{F}_{n-k}$ is decreasing along the flow (3.62).

Proof. By Jenson's inequality, we have

$$\begin{aligned} \frac{d}{dt}\frac{1}{\int_M \chi^{n-k}\wedge\omega^k}\mathcal{F}_{n-k} &= \frac{1}{\int_M \chi^{n-k}\wedge\omega^k}\int_M \log\frac{\sigma_n}{\sigma_{n-k}}\chi^{n-k}\wedge\omega^k \\ &\leq \log\int_M \frac{\sigma_n}{\sigma_{n-k}}\frac{\chi^{n-k}\wedge\omega^k}{\int_M \chi^{n-k}\wedge\omega^k} \\ &= \log\frac{\int_M \chi^n}{\binom{n}{k}\int_M \chi^{n-k}\wedge\omega^k} = \log 1 = 0. \end{aligned} \tag{3.85}$$

$\square$

Now by the partial C^2 estimates, we can follow Theorem 3.7 to get a uniform bound for the oscillation of φ_t. Assume $\mathcal{F}_{n-k}(\varphi_t) = \int_M \varphi_t\mu_t$, we normalize φ_t as

$$\hat{\varphi}_t = \varphi_t - \frac{\mathcal{F}_{n-k}(\varphi_t)}{\int_M \mu_t}, \tag{3.86}$$

then by proposition 3.5.4, it follows that $\hat{\varphi}_t$ has uniform estimate up to second order. By Evans-Krylov and Shauder estimates, we get uniform bound on C^∞ norm. Consequently, we have convergence of the flow and the critical metric satisfying (3.64).

$\square$

CHAPTER 4
FURTHER DISCUSSIONS

4.1 Applications to the product of Kähler manifolds

In this section, we shall apply the main theorem to the product of Kähler manifolds. As a result, we give a partial solution to Chen's question (Question 4 of [Ch2]) on a class of mixed Hessian equations on Kähler manifolds.

Let us consider a product Kähler manifold, $(M, \omega_0) \times (N, \omega_1)$, where N is a one-dimensional compact Kähler manifold. Let $\chi_0 \in [\chi]$ be another Kähler metric on M. We shall denote, by abuse of notation, $\omega_0 + \omega_1$ the corresponding metric on product manifold, i.e., $\omega_0 + \omega_1 = \pi_0^*(\omega_0) + \pi_1^*(\omega_1)$. We interpret $\chi_0 + \omega_1$ similarly. Now we fix $\omega_0 + \omega_1$ as the metric on $M \times N$. Consider $\chi_0 + a\omega_1$ as another metric on $M \times N$, $a > 0$.

Follow the same notation in previous sections, we define:

$$\sigma_{n+1-k}(\chi_0 + a\omega_1) = \binom{n+1}{k} \frac{(\chi_0 + a\omega_1)^{n+1-k} \wedge (\omega_0 + \omega_1)^k}{(\omega_0 + \omega_1)^{n+1}}. \tag{4.1}$$

Recall the definition of c_k and c'_k:

$$c_k = \frac{\int_M \chi^{n-k} \wedge \omega_0^k}{\int_M \chi_0^n}, \quad c'_k = \frac{\sigma_{n-k}(\chi)}{\sigma_n(\chi)} = \binom{n}{k} c_k. \tag{4.2}$$

Let

$$c'_{k,a} = c'_k + \frac{1}{a} c'_{k-1}. \tag{4.3}$$

We define the flow in $\mathcal{K}_{[\chi]}$:

$$\begin{cases} \frac{d}{dt}\varphi &= c'^{\frac{1}{k}}_{k,a} - \left(\frac{\sigma_{n+1-k}(\chi_\varphi + a\omega_1)}{\sigma_{n+1}(\chi_\varphi + a\omega_1)}\right)^{\frac{1}{k}}, \\ \varphi(0) &= 0 \end{cases} \tag{4.4}$$

It follows the critical equation of the flow is

$$c'_{k,a}\sigma_{n+1}(\tilde{\chi} + a\omega_1) = \sigma_{n+1-k}(\tilde{\chi} + a\omega_1). \tag{4.5}$$

Since locally the metric $\chi_\varphi + a\omega_1$ regarded as a matrix is $\begin{pmatrix} \chi_{i\bar{j}} & 0 \\ 0 & a \end{pmatrix}$, one of the eigenvalues with respect to $\omega_0 + \omega_1$ is a. Thus it follows

$$\sigma_{n+1}(\tilde{\chi} + a\omega_1) = a\sigma_n(\tilde{\chi}), \tag{4.6}$$

$$\sigma_{n+1-k}(\tilde{\chi} + a\omega_1) = a\sigma_{n-k}(\tilde{\chi}) + \sigma_{n-k+1}(\tilde{\chi}). \tag{4.7}$$

Therefore (4.5) can be reduced to an equation purely on M, i.e.,

$$c'_{k,a}\sigma_n(\tilde{\chi}) = \sigma_{n-k}(\tilde{\chi}) + \frac{1}{a}\sigma_{n+1-k}(\tilde{\chi}). \tag{4.8}$$

In terms of forms, (4.8) becomes

$$c'_{k,a}\tilde{\chi}^n = \binom{n}{k}\tilde{\chi}^{n-k} \wedge \omega_0^k + \frac{1}{a}\binom{n}{k-1}\tilde{\chi}^{n-k+1} \wedge \omega_0^{k-1} \tag{4.9}$$

In order to show the convergence of the flow (4.4), we follow the same strategy as in the proof of the main theorem. The key point is to prove the partial C^2 estimates. Again since the eigenvalue along the N direction is constant a, we can assume the largest eigenvalue of $\chi_\varphi + a\omega_1$ corresponds to an eigenvector tangential to M. Hence to realize partial C^2 estimates, we only need to impose the condition

$$\sigma_k((\chi + a\omega_1)^{-1}|i) < c'_{k,a}, \forall 1 \leq i \leq n. \tag{4.10}$$

In terms of form condition, it is

$$c'_{k,a}n\chi^{n-1} > \binom{n}{k}(n-k)\chi^{n-k-1} \wedge \omega_0^k + \frac{1}{a}\binom{n}{k-1}(n-k+1)\chi^{n-k} \wedge \omega_0^{k-1}. \tag{4.11}$$

In summary, we have the following theorem:

Theorem 4.1. *Let (M,ω) be a compact Kähler manifold. k is a fixed integer, $1 \le k \le n$. Let α, β be two constants satisfying $\alpha c_k + \beta c_{k-1} = 1$. Assume the Kähler class $[\chi]$ contains a Kähler form χ' satisfying:*

$$n\chi'^{n-1} > \alpha(n-k)\chi'^{n-k-1} \wedge \omega^k + \beta(n-k+1)\chi^{n-k} \wedge \omega^{k-1}. \tag{4.12}$$

then there exists a unique critical metric $\tilde{\chi}$ satisfying

$$\tilde{\chi}^n = \alpha\tilde{\chi}^{n-k} \wedge \omega^k + \beta\tilde{\chi}^{n-k+1} \wedge \omega^{k-1}. \tag{4.13}$$

Based on the known result, we can refine Chen's problem into the following:

Conjecture 1. *For fixed q, $0 \le q \le n$, and for any given $\alpha = (\alpha_0, \cdots, \alpha_p) \in \mathbb{R}^{p+1}$,$p \le n-q$ $\alpha_i > 0$, $0 \le i \le p$, define*

$$\begin{aligned}
c_\alpha &= c_{k,\alpha,[\omega],[\chi]} = \sum_{i=0}^{p} c_{i+q}\alpha_i, \\
\mathcal{F}_{\alpha,n}(\chi_0,\chi) &= \sum_{i=0}^{p} \alpha_i \mathcal{F}_{i+q,n}(\chi_0,\chi), \\
\mathcal{C}_\alpha(\omega) &= \{[\chi] \in \mathcal{H}^+,\ \exists \chi' \in [\chi],\ \text{such that} \\
&\qquad c_\alpha n\chi'^{n-1} > \sum_{i=0}^{p} \alpha_i(n-i-q)\chi'^{n-i-q-1} \wedge \omega^{i+q}\}.
\end{aligned}$$

Then

$$c_\alpha \chi_\varphi^n = \sum_{i=0}^{p} \alpha_i \chi_\varphi^{i+q} \wedge \omega^{n-i-q}, \tag{4.14}$$

has a unique smooth solution if and only if $[\chi] \in \mathcal{C}_\alpha(\omega)$; in this case, $\mathcal{F}_{\alpha,n}(\chi_0,\chi)$ obtains minimal at the given solution.

Use the same method we can verify Conjecture 1 under some additional conditions on α_i's. We consider $M \times C_1 \times C_2 \cdots \times C_p$, where C_i are all algebraic curves.

Set ω_i be Kähler forms on C_i. For $a_i > 0$ set

$$\tilde{\chi_0} = \chi_0 + \sum_{i=1}^{p} a_i \omega_i, \quad \tilde{\omega} = \sum_{i=0}^{n} \omega_i.$$

Follow the method above one can solve

$$c\sigma_{n+p}(\tilde{\chi}) = \sigma_{n+p-k}(\tilde{\chi}), \text{on } \tilde{M} := M \times C_1 \times C_2 \cdots \times C_p, \tag{4.15}$$

where c is the constant satisfying

$$c = \frac{\int_{\tilde{M}} \sigma_{n+p-k}(\tilde{\chi})}{\int_{\tilde{M}} \sigma_{n+p}(\tilde{\chi})}.$$

Similarly, one reduces (4.24) to an equation on M. According to the relationship of k, n, and p, there will be four cases which we state as a theorem.

Theorem 4.2. *Let M, ω, and $[\chi]$ be as above. Γ_p is the positive cone in $\mathbb{R}^p$. Conjecture 1 holds for the following special equations:*

1. *For $p \geq k$ and $n > k$,*

$$c\chi^n = \beta_0 \chi^n + \beta_1 \chi^{n-1} \wedge \omega + \cdots + \beta_k \chi^{n-k} \wedge \omega^k, c = \sum_{i=0}^{k} \beta_i c_i,$$

   for which we require the existence of a $b = (b_1, b_2, \cdots, b_p) \in \Gamma_p$ such that $\beta_i = \sigma_{k-i}(b)\binom{n}{i}, i = 0, 1, \cdots k$;

2. *For $p < k < n$,*

$$c\chi^n = \beta_0 \chi^{n+p-k} \wedge \omega^{k-p} + \beta_1 \chi^{n+p-k-1} \wedge \omega^{k-p+1} + \cdots + \beta_p \chi^{n-k} \wedge \omega^k, c = \sum_{i=0}^{p} \beta_i c_{k-p+i},$$

   for which we require the existence of a $b = (b_1, b_2, \cdots, b_p) \in \Gamma_p$ such that $\beta_i = \sigma_{p-i}(b)\binom{n}{k-p+i}, i = 0, 1, \cdots p$;

3. *For $p \geq k \geq n$,*

$$c\chi^n = \beta_0\chi^n + \beta_1\chi^{n-1} \wedge \omega + \cdots + \beta_n\omega^n, c = \sum_{i=0}^{n} \beta_i c_i,$$

for which we require the existence of a $b = (b_1, b_2, \cdots, b_p) \in \Gamma_p$ such that $\beta_i = \sigma_{k-i}(b)\binom{n}{i}, i = 0, 1, \cdots, n$;

4. *For $k > p$ and $k \geq n$,*

$$c\chi^n = \beta_0\chi^{n+p-k} \wedge \omega^{k-p} + \beta_1\chi^{n+p-k-1} \wedge \omega^{k-p+1} + \cdots + \beta_{n+p-k}\omega^n, c = \sum_{i=0}^{n+p-k} \beta_i c_{k-p+i},$$

where we require there exist some $b = (b_1, b_2, \cdots, b_p) \in \Gamma_p$ such that $\beta_i = \sigma_{p-i}(b)\binom{n}{k-p+i}, i = 0, 1, \cdots, n + p - k$.

4.2 Study of $\mathcal{F}$-functionals

In this Chapter, we shall explore some relation of the functional $\mathcal{F}_{k,n}$ with the energy functionals E_k first introduced in the study of Kähler-Ricci flow [CT1]. We will first recall some well-known functionals in the analytical study of Kähler-Einstein metrics on Fano manifolds.

Assume M is Fano, i.e., first Chern class $c_1(M)$ is positive. Fix a Kähler metric $\omega \in c_1(M)$, then both $Ric(\omega)$ and ω are in the same class, thus there exists a so-called Ricci potential h_ω satisfying

$$Ric(\omega) - \omega = \partial\bar{\partial}h_\omega, \quad \int_M e^{h_\omega}\omega^n = \int_M \omega^n.$$

The existence of Kähler-Einstein metric in $c_1(M)$ can be reduced to the following complex Monge-Ampère equation:

$$(\omega + \partial\bar{\partial}\psi)^n = e^{h_\omega - \psi}\omega^n. \tag{4.16}$$

Indeed, applying $-\partial\bar{\partial}\log$ both sides, we get $Ric(\omega_\psi) = \omega_\psi$.

Unlike the cases $c_1(M) < 0$ and $c_1(M) = 0$, a priori estimate of C^0 norm of ψ breaks down in Fano case. There are obstructions to the existence of Kähler-Einstein metrics on Fano manifolds, such as Futaki invariants. Tian [T2] initiates a variational approach by studying various functionals. We briefly introduce his work.

All the functionals will be defined on the space of Kähler potentials of a fixed Kähler metric ω:

$$\mathcal{P}_\omega := \{\varphi \in C^\infty(M) \quad | \quad \omega + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\varphi > 0\}.$$

$\mathcal{J}$-functional is defined as:

$$\mathcal{J}_\omega(\varphi) = \frac{1}{V}\sum_{i=0}^{n-1}\frac{i+1}{n+1}\int_M \partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^i \wedge \omega_\varphi^{n-1-i}, \tag{4.17}$$

where $V = \int_M \omega^n = [\omega]^n([M])$.

Easy computation shows that first variation of $\mathcal{J}$-functional is:

$$\delta\mathcal{J}_\omega(\varphi) = \frac{1}{V}\int_M \delta\varphi(\omega^n - \omega_\varphi^n), \tag{4.18}$$

where δ denotes infinitesimal variation.

F-functional is defined as follows:

$$F_\omega(\varphi) = \mathcal{J}_\omega(\varphi) - \frac{1}{V}\int_M \varphi\omega^n - \log(\frac{1}{V}\int_M e^{h_\omega - \varphi}\omega^n). \tag{4.19}$$

Two important properties of F are:

1. Invariant under addition of constant: $F_\omega(\psi) = F_\omega(\psi + c)$, thus it descends to a functional on the space of Kähler metrics $\mathcal{K}_\omega = \mathcal{P}_\omega/\mathbb{R}$.

2. Cocycle condition: for $\omega' = \omega + \partial\bar{\partial}\varphi$,

$$F_\omega(\psi) = F_\omega(\varphi) + F_{\omega'}(\psi - \varphi).$$

It also follows from (4.18) that F has (4.16) as its Euler-Lagrange equation.

In [T2], Tian gave the following analytic criterion for the existence of Kähler-Einstein metrics on Fano manifolds.

We say F_ω is proper on $\mathcal{P}_\omega$, if there exists an increasing function $\rho : \mathbb{R} \to [c, \infty)$ satisfying $\lim_{x\to\infty} \rho(x) = \infty$ such that for any $\psi \in \mathcal{P}_\omega$,

$$F_\omega(\psi) \geq \rho(\mathcal{J}_\omega(\psi)).$$

Theorem 4.3 (Tian). *Let M be a Fano manifold with no non-trivial holomorphic vector fields, then M has a Kähler-Einstein metric if and only if F_ω is proper on $\mathcal{P}_\omega$.*

Mabuchi introduced the so-called Mabuchi's energy [Ma1], which plays an important role in the analytic study of canonical metrics on Kähler manifolds.

It can be defined via its first variation:

$$\delta\nu_\omega(\psi) = -\frac{1}{V}\int_M \delta\psi(R - \underline{R})\omega_\psi^n, \tag{4.20}$$

where R and $\underline{R}$ denote the scalar curvature and the average of scalar curvature respectively. One can check this defines a close one form. Hence one can integrate along any curve in $\mathcal{P}_\omega$ to get a general formula for the Mabuchi's energy $\nu_\omega(\psi)$:

$$\nu_\omega(\psi) = -\frac{1}{V}\int_0^1\int_M \dot{\psi}_t(R_t - \underline{R_t})\omega_{\psi_t}^n dt, \tag{4.21}$$

where ψ_t is a smooth path in $\mathcal{P}_\omega$ connecting 0 and ψ. We have the following relationship between Mabuchi energy and F-functional.

Proposition *4.2.1.* Assume M is a Fano manifold, with $\omega \in c_1(M)$. Denote h to be the Ricci potential of ω, i.e., $Ric(\omega) - \omega = \partial\bar{\partial}h$. Then we have for any $\psi \in \mathcal{P}_\omega$,

$$\nu_\omega(\psi) \geq F_\omega(\psi) + \frac{1}{V}\int_M h\omega^n.$$

Proof. Assume that

$$Ric(\omega_\psi) - \omega_\psi = \partial\bar{\partial}h_\psi, \tag{4.22}$$

and

$$f = \log\frac{\omega_\psi^n}{\omega^n}. \tag{4.23}$$

Then it follows that

$$\omega_\psi^n = e^f\omega^n. \tag{4.24}$$

Applying $-\partial\bar{\partial}\log$ both sides, together with (4.22) we get,

$$h_\psi = h - f - \psi. \tag{4.25}$$

We compute the derivative of $\int_M h_\psi\omega_\psi^n$ (for simplicity we shall ignore the constant $\frac{1}{V}$ in front of every term):

$$\begin{aligned} \frac{d}{dt}(\int_M h_\psi\omega_\psi^n) &= -\int_M \dot{\psi}\omega_\psi^n - \int_M \dot{f}\omega_\psi^n + \int_M h_\psi n\omega_\psi^{n-1}\wedge\partial\bar{\partial}\dot{\psi} \\ &= -\int_M \dot{\psi}\omega_\psi^n - \int_M \Delta_{\omega_\psi}\dot{\psi} + \int_M \dot{\psi}(Ric(\omega_\psi) - \omega_\psi)n\omega_\psi^{n-1} \\ &= -\int_M \dot{\psi}\omega_\psi^n + \int_M \dot{\psi}(R_\psi - \underline{R_\psi})\omega_\psi^n. \end{aligned} \tag{4.26}$$

Comparing with (4.18) and (4.20), we get

$$(-\int_M h_\psi\omega_\psi^n)' = \nu_\omega' - \mathcal{J}_\omega' + (\int_M \psi\omega^n)'. \tag{4.27}$$

Hence integrating and with the initial value $\nu_\omega(0) = 0$, $\mathcal{J}_\omega(0) = 0$, we get

$$\nu_\omega(\psi) = \mathcal{J}_\omega(\psi) - \frac{1}{V}\int_M \psi\omega^n + \frac{1}{V}\int_M f\omega_\psi^n + \frac{1}{V}\int_M h(\omega^n - \omega_\psi^n) + \frac{1}{V}\int_M \psi\omega_\psi^n. \quad (4.28)$$

By (4.19) and (4.24) we get

$$\begin{aligned}
\nu_\omega(\psi) &= F_\omega(\psi) + \log(\frac{1}{V}\int_M e^{h-\psi}\omega^n) + \frac{1}{V}\int_M f\omega_\psi^n + \frac{1}{V}\int_M h(\omega^n - \omega_\psi^n) + \frac{1}{V}\int_M \psi\omega_\psi^n \\
&= F_\omega(\psi) + \log(\frac{1}{V}\int_M e^{h-\psi-f}\omega_\psi^n) + \frac{1}{V}\int_M f\omega_\psi^n + \frac{1}{V}\int_M h(\omega^n - \omega_\psi^n) + \frac{1}{V}\int_M \psi\omega_\psi^n \\
&\geq F_\omega(\psi) + \frac{1}{V}\int_M (h - \psi - f)\omega_\psi^n + \frac{1}{V}\int_M f\omega_\psi^n + \frac{1}{V}\int_M h(\omega^n - \omega_\psi^n) + \frac{1}{V}\int_M \psi\omega_\psi^n \\
&= F_\omega(\psi) + \frac{1}{V}\int_M \psi\omega_\psi^n.
\end{aligned}$$

□

From (4.20), it follows the critical points of ν_ω are metrics with constant scalar curvature (cscK metrics). The question of the existence of a cscK metric in a given Kähler class is a difficult and interesting problem and is expected to be equivalent to a notion of stability in the sense of geometric invariant theory. The behavior of Mabuchi energy plays a central role in this question. It is known that if there exists a Kähler-Einstein metric in a class $[\chi]$ then the Mabuchi energy is bounded below on that class [BM, T3]. Moreover, Chen and Tian [CT3] have shown that the existence of a cscK metric in any class implies the lower boundedness of Mabuchi energy. More generally, Tian [T3] has conjectured that the existence of a cscK metric is equivalent to the properness of the Mabuchi energy. The behavior of Mabuchi energy is shown to be related with stability. A lower bound on the Mabuchi energy implies the corresponding class is K-semistable [T2]. Conversely, Donaldson [D6] showed that for toric surfaces, K-stability implies the lower boundedness of the Mabuchi energy.

Energy functionals E_k, $0 \leq k \leq n$ are introduced in [CT1, CT2]. They are generalizations of Mabuchi energy, with E_0 being precisely the Mabuchi energy. They can also be described in terms of Deligne paring. The behavior of energy functionals are closely related with F-functional. It has been shown that properness of F-functional, properness of Mabuchi functional, and properness of E_1 are all equivalent and they are all equivalent to the existence of Kähler-Einstein metric. Rubinstein [Ru] proved the lowerboundedness of E_k are all equivalent on the set of metrics with positive Ricci curvature in a given class. It is widely believed properness of these functionals is related to stability, while lowerboundedness is related to semi-stability.

In the following, we use the functional $\mathcal{F}_{k,n}$ in Chapter 3 and Yau's theorem on *Calabi conjecture* to establish some formulae for E_k.

Following [SW1], E_k functionals are defined as

$$\begin{aligned} E_{k,\omega}(\phi) =& \frac{k+1}{V} \int_0^1 \int_M \Delta_{\phi_t} \dot{\phi}_t Ric(\omega_{\phi_t})^k \wedge \omega_{\phi_t}^{n-k} dt \\ & - \frac{n-k}{V} \int_0^1 \int_M \dot{\phi}_t (Ric(\omega_{\phi_t})^{k+1} - \mu_k \omega_{\phi}^{k+1}) \wedge \omega_{\phi_t}^{n-k-1} dt, \end{aligned} \tag{4.29}$$

where

$$\mu_k = \frac{\int_M Ric(\omega)^{k+1} \wedge \omega^{n-k-1}}{\int_M \omega^n} = \frac{c_1(M)^{k+1} \cdot [\omega] n-k-1}{[\omega]^n}.$$

From the definition we see that E_0 is same as the Mabuchi energy ν_ω.

Following [SW2], we have

Proposition *4.2.2.* Let M be a Kähler manifold and $c_1(M) < 0$. Mabuchi energy ν_χ is proper on the classes $[\chi] \in \cup_{\omega>0 \in [-c_1(M)]} \mathcal{C}_{n-1}(\omega)$.

Proof. We derive a formulae relating ν_χ and $\mathcal{F}_{n-1,n}$. We have

$$\mu = \frac{c_1(M)\cdot[\chi]^{n-1}}{[\chi]^n} = -\frac{[\omega]\cdot[\chi]^{n-1}}{[\chi]^n} = -c_{n-1}.$$

By Yau's theorem on Calabi conjecture, for $-\omega \in c_1(M)$, there exist a unique $\chi_0 \in [\chi]$ such that

$$Ric(\chi_0) = -\omega.$$

We fix χ_0 as the reference metric in $[\chi]$, then we have

$$\nu_{\chi_0}(\phi) = -\int_0^1\int_M \dot\phi(Ric(\chi_\phi)\wedge\chi_\phi^{n-1} - \mu\chi_\phi^n)dt, \tag{4.30}$$

$$\mathcal{F}_{n-1,n} = \int_0^1\int_M \dot\phi(\chi_\phi^{n-1}\wedge\omega + \mu\chi_\phi^n)dt. \tag{4.31}$$

We compute

$$\begin{aligned}\frac{d}{dt}\frac{1}{n}\int_M \log(\frac{\chi_\phi^n}{\chi_0^n})\chi_\phi^n &= \int_M \dot\phi(Ric(\chi_0) - Ric(\chi_\phi))\chi_\phi^{n-1}\\ &= \int_M \dot\phi(-\omega - Ric(\chi_\phi))\chi_\phi^{n-1} \\ &= -\frac{d}{dt}\mathcal{F}_{n-1,n} + \frac{d}{dt}\nu_{\chi_0}.\end{aligned} \tag{4.32}$$

Hence we have

$$\nu_{\chi_0} = \mathcal{F}_{n-1,n} + \frac{1}{n}\int_M \log(\frac{\chi_\phi^n}{\chi_0^n})\chi_\phi^n. \tag{4.33}$$

By Corollary 3.4.1, if $[\chi]\in\mathcal{C}_{n-1}(\omega)$ for some ω, then $\mathcal{F}_{n-1,n}$ is bounded from below. The second term is proper by Lemma 4.1 of [SW2].

Using the same idea, we derive two formulae relating Energy functionals E_k and $\mathcal{F}_{k,n}$. First we define a family of A_k functionals. For fixed k, let

$$A_k = \int_M \log\frac{\chi_\phi^n}{\chi_0^n}\chi_\phi^{n-k}\wedge\sum_{i=0}^k Ric(\chi_\phi)^{k-i}\wedge\omega^i. \tag{4.34}$$

Then the first variation of A_k is

$$\begin{aligned}\frac{d}{dt}A_k =&\frac{d}{dt}\int_M \log\frac{\chi_\phi^n}{\chi_0^n}\chi_\phi^{n-k}\wedge\sum_{i=0}^{k} Ric(\chi_\phi)^{k-i}\wedge\omega^i \\ =&\sum_{i=1}^{k}[\int_M \Delta_\phi\dot\phi\chi_\phi^{n-k}\wedge Ric(\chi_\phi)^{k-i}\wedge\omega^i \\ &+(n-k)\int_M \dot\phi(Ric(\chi_0)-Ric(\chi_\phi))\chi_\phi^{n-k-1}\wedge Ric(\chi_\phi)^{k-i}\wedge\omega^i \\ &+(k-i)\int_M \Delta_\phi\dot\phi(Ric(\chi_\phi)-Ric(\chi_0))\chi_\phi^{n-k}\wedge Ric(\chi_\phi)^{k-i-1}\wedge\omega^i].\end{aligned} \tag{4.35}$$

If $\omega\in c_1(M)$, then by Yau's theorem on Calabi conjecture, there exists χ_0 such that $Ric(\chi_0)=\omega$. Then

$$\begin{aligned}\frac{d}{dt}A_k =&\sum_{i=0}^{k}[(k-i+1)\int_M \Delta_\phi\dot\phi\chi_\phi^{n-k}\wedge Ric(\chi_\phi)^{k-i}\wedge\omega^i \\ &-(k-i)\int_M \Delta_\phi\dot\phi\chi_\phi^{n-k}\wedge Ric(\chi_\phi)^{k-i-1}\wedge\omega^{i+1} \\ &-(n-k)\int_M \dot\phi\chi_\phi^{n-k-1}\wedge Ric(\chi_\phi)^{k-i+1}\wedge\omega^i \\ &+(n-k)\int_M \dot\phi\chi_\phi^{n-k-1}\wedge Ric(\chi_{phi})^{k-i}\wedge\omega^{i+1}] \\ =&(k+1)\int_M \Delta_\phi\dot\phi\chi_\phi^{n-k}\wedge Ric(\chi_\phi)^{k}-(n-k)\int_M \dot\phi\chi_\phi^{n-k-1}\wedge Ric(\chi_\phi)^{k+1} \\ &+(n-k)\int_M \dot\phi\chi_\phi^{n-k-1}\wedge\omega^{k+1}.\end{aligned} \tag{4.36}$$

Compare with (4.29) and the definition of $\mathcal{F}_{n-k-1,n}$, we have

$$(\frac{n-k}{V}\mathcal{F}_{n-k-1,n}+E_k)'=(\frac{1}{V}A_k)'. \tag{4.37}$$

Hence we get the following formula.

Proposition *4.2.3.* Let (M,ω) be a Fano manifold, i.e., $c_1(M)>0$. Assume $\omega\in c_1(M)$. Let $[\chi]$ be another Kähler class. Let χ_0 be the Kähler metric in $[\chi]$ satisfying

$Ric(\chi_0) = \omega$. Let $\mathcal{F}$ and A_k be defined as above. Then we have the following formula for energy functionals E_k in $[\chi]$ with χ_0 as the reference metric:

$$E_k = \frac{1}{V} A_k - \frac{n-k}{V} \mathcal{F}_{n-k-1,n}. \tag{4.38}$$

In the case $c_1(M) < 0$, we have the following

Proposition *4.2.4.* Let (M, ω) be a compact Kähler manifold with $\omega \in -c_1(M)$. Let $[\chi]$ be another Kähler class. Let χ_0 be the Kähler metric in $[\chi]$ satisfying $Ric(\chi_0) = -\omega$. Define B_k as follows:

$$B_k = \int_M \log \frac{\chi_\phi^n}{\chi_0^n} \chi_\phi^{n-k} \wedge \sum_{i=0}^{k} (-1)^i Ric(\chi_\phi)^{k-i} \wedge \omega^i. \tag{4.39}$$

Then we have following formula for E_k in the class $[\chi]$ with χ_0 as reference metric.

$$E_k = (-1)^k \frac{n-k}{V} \mathcal{F}_{n-k-1,n} + \frac{1}{V} B_k. \tag{4.40}$$

Proof. The first variation of B_k is

$$\begin{aligned}
\frac{d}{dt} B_k =& \frac{d}{dt} \int_M \log \frac{\chi_\phi^n}{\chi_0^n} \chi_\phi^{n-k} \wedge \sum_{i=0}^{k} (-1)^i Ric(\chi_\phi)^{k-i} \wedge \omega^i \qquad (4.41)\\
=& \sum_{i=1}^{k} (-1)^i \Big[\int_M \Delta_\phi \dot{\phi} \chi_\phi^{n-k} \wedge Ric(\chi_\phi)^{k-i} \wedge \omega^i \\
&+ (n-k) \int_M \dot{\phi} (Ric(\chi_0) - Ric(\chi_\phi)) \chi_\phi^{n-k-1} \wedge Ric(\chi_\phi)^{k-i} \wedge \omega^i \\
&+ (k-i) \int_M \Delta_\phi \dot{\phi} (Ric(\chi_\phi) - Ric(\chi_0)) \chi_\phi^{n-k} \wedge Ric(\chi_\phi)^{k-i-1} \wedge \omega^i \Big].
\end{aligned}$$

Plugging $Ric(\chi_0) = -\omega$, we get

$$\begin{aligned}
\frac{d}{dt}B_k =& \sum_{i=0}^{k}(-1)^i[(k-i+1)\int_M \Delta_\phi\dot{\phi}\chi_\phi^{n-k}\wedge Ric(\chi_\phi)^{k-i}\wedge\omega^i \\
&- (k-i)\int_M \Delta_\phi\dot{\phi}\chi_\phi^{n-k}\wedge Ric(\chi_\phi)^{k-i-1}\wedge\omega^{i+1} \\
&- (n-k)\int_M \dot{\phi}\chi_\phi^{n-k-1}\wedge Ric(\chi_\phi)^{k-i+1}\wedge\omega^i \\
&+ (n-k)\int_M \dot{\phi}\chi_\phi^{n-k-1}\wedge Ric(\chi_{phi})^{k-i}\wedge\omega^{i+1}] \\
=&(k+1)\int_M \Delta_\phi\dot{\phi}\chi_\phi^{n-k}\wedge Ric(\chi_\phi)^k - (n-k)\int_M \dot{\phi}\chi_\phi^{n-k-1}\wedge Ric(\chi_\phi)^{k+1} \\
&- (n-k)(-1)^k\int_M \dot{\phi}\chi_\phi^{n-k-1}\wedge\omega^{k+1}.
\end{aligned} \tag{4.42}$$

Since $[\omega] = -c_1(M)$, thus

$$\mu = \frac{c_1^{k+1}\cdot[\chi]^{n-k-1}}{[\chi]^n} = (-1)^{k+1}\frac{[\omega]^{k+1}\cdot[\chi]^{n-k-1}}{[\chi]^n} = (-1)^{k+1}c_{k+1}. \tag{4.43}$$

Compare with (4.29) and the definition of $\mathcal{F}_{n-k-1,n}$, we have

$$(E_k - (-1)^k\frac{n-k}{V}\mathcal{F}_{n-k-1,n})' = B_k'. \tag{4.44}$$

Thus the conclusion follows. Notice that for $k = 0$, the above is same as (4.33).

□

4.3 Future problems

We end the thesis by listing a few future problems.

Question 1 (Study of cone $\mathcal{C}_k(\omega)$). It is interesting to study the cone $\mathcal{C}_k(\omega)$ further. By the definition, $\mathcal{C}_k$ is clear an open affine cone. If $\dim H^2(M) > 1$, by the open condition, it follows that $\mathcal{C}_k(\omega)$ contain a small neighborhood of $[\omega]$ for every k. Other than these properties, we pose the following interesting questions:

1. Is $\mathcal{C}_k(\omega)$ a convex cone $\forall k$? If not, give counterexamples. If $\mathcal{C}_k(\omega)$ is not convex in general, then can we expect that $\bigcup_{\omega \text{ is Kähler} \in [\omega]} \mathcal{C}_k(\omega)$ to be convex?

2. Is there a nested property for $\mathcal{C}_k$, i.e., $\mathcal{C}_1 \subseteq \mathcal{C}_2 \cdots \subseteq \mathcal{C}_n$?

These questions are interesting on their own and also may shed some lights on the study of structure of Kähler cones.

Question 2. Comparing the definitions of two flows (3.5) and (3.62), we have used two concave functions $(\cdot)^{1/k}$ and $\log(\cdot)$. These functions guarantee the structure conditions, i.e., ellipticity, concavity and weak(strong) concavity. Thus for the sake of convergence, we expect there exists a large family of flows which will converge to the critical metrics, at least for the interpolation of $(\cdot)^{1/k}$ and $\log(\cdot)$. We remark that it seems that concavity of log is the optimal one. As for $k = n$, $f = \log \sigma_n$ satisfies $f_{ij} + \frac{f_i}{\lambda_j}\delta_{ij} = 0$.

The understanding of such structure could possibly yield a solution to Chen's question (Conjecture 1) on general Hessian equations on Kähler manifolds.

Question 3 (Singular case). In [SW2], Song and Weinkove also study the flow (3.5) in the case when $[\chi]$ is on the boundary of $\mathcal{C}_k(\omega)$. They obtained C^∞ convergence away from a divisor. We shall study the corresponding problem for general k in the future.

Question 4 (Examples of explicit solutions). One could look at some examples of explicit solutions to the critical equation (3.6). One possible approach is the famous Calabi ansatz.

Its construction is as follows: Let ω_{FS} be the Fubini-Study metric on $\mathbb{P}^n$. Let h be the hermitian metric on $\mathcal{O}(-1)$ such that $Ric(h) = -\omega_{FS}$. The induced hermitian metric h_E on $E = \mathcal{O}(-1)^{\oplus(m+1)}$ is given by $h_E = h^{\oplus(m+1)}$. Under local trivialization of E, we write $\rho = (1+|z|^2)|\xi|^2, \xi = (\xi_1, \cdots, \xi_n)$, the norm squared function on E. Then

$$\omega = a\omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u(\rho)$$

will be a Kähler metric on the projective compactification of $\mathbb{P}(\mathbb{C} \oplus E)$ under proper conditions. One can reduce (3.6) to an ODE on u. However the ODE is still non trivial to analyze.

Question 5 (Analytical inequality). A by-product of Tian's analytical criterion for the existence of Kähler-Einstein metrics is the generalized Moser-Trudinger inequalities [T2], [PSSW]. We can explore some useful inequalities from the lowerboundedness of $\mathcal{F}_{k,n}$ for classes $[\chi] \in \mathcal{C}_k(\omega)$.

APPENDIX A
PROOF OF LEMMA 3.20

***Lemma** A.1.* Let $I = (i_1, i_2, \cdots, i_k)$ be an index set, denote its complement in $(1, 2, \cdots, n)$ by $\bar{I}$. We shall order $\bar{I}$ so that $(I, \bar{I})$ is an even permutation of $(1, 2, \cdots, n)$. For A, an $n \times n$ positive hermitian matrix, let A_I be the principle minor $(a_{i\bar{j}})_{i,j \in I}$. Then

$$\det(A) \leq \det(A_I)\det(A_{\bar{I}}).$$

Proof. Rearrange A if necessary we may write A as

$$A = \begin{bmatrix} A_I & M \\ M' & A_{\bar{I}} \end{bmatrix}. \tag{A.1}$$

By

$$\begin{bmatrix} Id & 0 \\ -M'A_I^{-1} & Id \end{bmatrix} \begin{bmatrix} A_I & M \\ M' & A_{\bar{I}} \end{bmatrix} = \begin{bmatrix} A_I & M \\ 0 & A_{\bar{I}} - M'A_I M \end{bmatrix}, \tag{A.2}$$

one obtains

$$\det(A) = \det(A_I)\det(A_{\bar{I}} - M'A_I M) \leq \det(A_I)\det(A_{\bar{I}}),$$

where M' means the conjugate transpose matrix of M. The last inequality follows from the fact that $M'A_I M$ is positive definite.

$\square$

APPENDIX B
PROOF OF WEAK CONCAVITY

In this appendix, we present the proof of weak concavity of $g_1 = \sigma_k^{1/k}$, and $g_2 = \log \sigma_k$. For the convenience of readers, we state it as a proposition here:

Proposition *B.0.1.* Let $g_1(\lambda) = \sigma_k^{\frac{1}{k}}(\lambda)$, $g_2 = \log \sigma_k(\lambda)$ and assume $\lambda \in \Gamma_n$, then both g_1 and g_2 satisfies the weak concavity for the corresponding Hessian matrix, i.e.,

$$(g_{ij} + \frac{g_i}{\lambda_j}\delta_{ij}) \geq 0.$$

Proof. Step 1.

Consider $h := \sigma_k(\lambda_1^{1/k}, \cdots, \lambda_n^{1/k})$. For simplicity, we denote $\lambda^{1/k} = (\lambda_1^{1/k}, \cdots, \lambda_n^{1/k})$. We claim

$$h_{ij} + \frac{h_i}{\lambda_j}\delta_{ij} \geq 0.$$

Direct computation shows that:

$$\begin{aligned} h_i &= \frac{1}{k}\sigma_{k-1}(\lambda^{\frac{1}{k}}|i)\lambda_i^{\frac{1}{k}-1}, && \text{(B.1)} \\ h_{ij} &= \frac{1}{k^2}\sigma_{k-2}(\lambda^{\frac{1}{k}}|i,j)\lambda_i^{\frac{1}{k}-1}\lambda_j^{\frac{1}{k}-1}(1-\delta_{ij}) + \frac{1}{k}(\frac{1}{k}-1)\sigma_{k-1}(\lambda^{\frac{1}{k}}|i)\lambda_j^{\frac{1}{k}-2}\delta_{ij}. && \text{(B.2)} \end{aligned}$$

Introduce the following notation: for $I = (i_1, i_2, \cdots i_l)$ an arbitrary index set, let $\sigma_{k;I} := \sum_{|I|=k} \lambda_I \sigma_{k-l}(\lambda|I)$,where $\lambda_I = \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_l}$ and $\sigma_k(\lambda|I) = \sigma_k(\lambda)|_{\lambda_{i_1}=0,\cdot,\lambda_{i_l}=0}$. Following this notation, we can rewrite (B.1), (B.2) as:

$$\begin{aligned} h_i &= \frac{\sigma_{k;i}}{k\lambda_i}, && \text{(B.3)} \\ h_{ij} &= \frac{\sigma_{k;i,j}}{k^2\lambda_i\lambda_i}, \quad \text{for } i \neq j, && \text{(B.4)} \\ h_{ii} &= \frac{1}{k}(\frac{1}{k}-1)\frac{\sigma_{k;i}}{\lambda_i^2}. && \text{(B.5)} \end{aligned}$$

Thus $h_{ij} + \frac{h_i}{\lambda_j}\delta_{ij}$ equals:

$$\begin{bmatrix} \frac{\sigma_{k;1}}{k^2\lambda_1^2} & \frac{\sigma_{k;1,2}}{k^2\lambda_1\lambda_2} & \cdot & \cdot & \frac{\sigma_{k;1,n}}{k^2\lambda_1\lambda_n} \\ \frac{\sigma_{k;1,2}}{k^2\lambda_1\lambda_2} & \frac{\sigma_{k;2}}{k^2\lambda_2^2} & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \frac{\sigma_{k;1,n}}{k^2\lambda_1\lambda_n} & \cdot & & & \frac{\sigma_{k;n}}{k^2\lambda_n^2} \end{bmatrix}. \tag{B.6}$$

Then it is suffice to show that

$$A := \begin{bmatrix} \sigma_{k;1} & \sigma_{k;1,2} & \cdot & & \sigma_{k;1,n} \\ \sigma_{k;1,2} & \sigma_{k;2} & & & \cdot \\ \cdot & & \cdot & & \\ & & & \cdot & \\ \sigma_{k;1,n} & & & & \sigma_{k;n} \end{bmatrix}$$

is nonnegative. For an index set I, Let E_I be the matrix with 1 in the ij-th entry whenever $i, j \in I$, and 0 elsewhere. It is clear that E_I is nonnegative. Moreover,

$$A = \sum_{|I|=k} \lambda_I E_I \geq 0. \tag{B.7}$$

Thus

$$h_{ij} + \frac{h_i}{\lambda_j}\delta_{ij} \geq 0.$$

Step 2.

We claim

$$h_{ij} + \frac{h_i}{\lambda_j}\delta_{ij} - \frac{h_i h_j}{h} \geq 0.$$

We use a nice trick due to Andrews [An2]. Since h is homogenous of degree 1, $h_i\lambda_i = h$. Differentiate both sides, we get $h_{ij}\lambda_i = 0$. Consequently,

$$(h_{ij} + \frac{h_i}{\lambda_j}\delta_{ij} - \frac{h_i h_j}{h})\lambda_i\lambda_j = 0, \tag{B.8}$$

i.e., λ is a null vector. In order to show $h_{ij} + \frac{h_i}{\lambda_j}\delta_{ij} - \frac{h_i h_j}{h} \geq 0$, it is left to check the non-negativeness on a subspace which is transversal to the null vector $\lambda = (\lambda_1, \cdots, \lambda_n)$.

Since $h_i\lambda_i = h$, thus we see that the subspace defined by $\{\xi | h_i\xi_i = 0\}$ is transversal to λ. On this subspace, we simply have $(h_{ij} + \frac{h_i}{\lambda_j}\delta_{ij} - \frac{h_ih_j}{h})\xi_i\xi_j = (h_{ij} + \frac{h_i}{\lambda_j}\delta_{ij})\xi_i\xi_j$, which is nonnegative from step 1.

Step 3.

- $g = g_1 = \sigma_k(\lambda)^{1/k}$.

$g(\lambda_1, \cdots \lambda_n) = h^{\frac{1}{k}}(\lambda_1^k, \cdots, \lambda_n^k)$, a simple computation shows that

$$g_{ij} + \frac{g_i}{\lambda_j}\delta_{ij} = kh^{\frac{1}{k}-1}(\lambda)\lambda_i^{1-\frac{1}{k}}\lambda_j^{1-\frac{1}{k}}[h_{ij} + \frac{h_i}{\lambda_j}\delta_{ij} - \frac{k-1}{k}\frac{h_ih_i}{h}], \tag{B.9}$$

where $\lambda_i^k = \lambda_i$. Thus,

$$h_{ij} + \frac{h_i}{\lambda_j}\delta_{ij} - \frac{k-1}{k}\frac{h_ih_i}{h} \geq h_{ij} + \frac{h_i}{\lambda_j}\delta_{ij} - \frac{h_ih_i}{h} \geq 0. \tag{B.10}$$

Remark. The conclusion of the Proposition B.0.1 holds for $g = \sigma_k^\beta(\lambda)$, with $\beta > 0$.

proof of the remark. Similarly, we have that $g = h^\beta(\lambda^k)$, thus

$$g_i = k\beta h^{\beta-1}h_i\lambda_i^{k-1}, \tag{B.11}$$

$$\begin{aligned} g_{ij} = {} & k^2\beta(\beta-1)h^{\beta-2}h_ih_j\lambda_i^{k-1}\lambda_j^{k-1} + k^2\beta h^{\beta-1}h_{ij}\lambda_i^{k-1}\lambda_j^{k-1} \\ & + k(k-1)\beta h^{\beta-1}h_i\lambda_i^{k-2}\delta_{ij}. \end{aligned} \tag{B.12}$$

Thus,

$$g_{ij} + \frac{g_i}{\lambda_j}\delta_{ij} = k^2\beta h^{\beta-1}\lambda_i^{k-1}\lambda_j^{k-1}(h_{ij} + (\beta-1)\frac{h_ih_j}{h} + \frac{h_i}{\lambda_j^k}\delta) \geq 0,$$

as long as $\beta > 0$.

- $g = g_2 = \log \sigma_k(\lambda)$

  In this case, $g = \log h(\lambda_1^k, \cdots, \lambda_n^k)$, direct computation shows that

$$g_{ij} + \frac{g_i}{\lambda_j}\delta_{ij} = \frac{k^2 \lambda_i^{k-1} \lambda_j^{k-1}}{h}(h_{ij} + \frac{h_i}{\lambda_j^k} - \frac{h_i h_j}{h}) \geq 0. \tag{B.13}$$

$\square$

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