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On commutativity of unbounded operators in Hilbert space

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ON COMMUTATIVITY OF UNBOUNDED OPERATORS IN HILBERT SPACE

by

Feng Tian

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

May 2011

Thesis Supervisor: Professor Palle Jorgensen

ABSTRACT

We study several unbounded operators with view to extending von Neumann's theory of deficiency indices for single Hermitian operators with dense domain in Hilbert space. If the operators are non-commuting, the problems are difficult, but special cases may be understood with the use of representation theory. We will further study the partial derivative operators in the coordinate directions on the L^2 space on various covering surfaces of the punctured plane. The operators are defined on the common dense domain of C^∞ functions with compact support, and they separately are essentially selfadjoint, but the unique selfadjoint extensions will be non-commuting. This problem is of a geometric flavor, and we study an index formulation for its solution.

The applications include the study of vector fields, the theory of Dirichlet problems for second order partial differential operators (PDOs), Sturm-Liouville problems, H. Weyl's limit-point/limit-circle theory, Schrödinger equations, and more.

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Thesis Supervisor

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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To my family

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ABSTRACT

We study several unbounded operators with view to extending von Neumann's theory of deficiency indices for single Hermitian operators with dense domain in Hilbert space. If the operators are non-commuting, the problems are difficult, but special cases may be understood with the use of representation theory. We will further study the partial derivative operators in the coordinate directions on the L^2 space on various covering surfaces of the punctured plane. The operators are defined on the common dense domain of C^∞ functions with compact support, and they separately are essentially selfadjoint, but the unique selfadjoint extensions will be non-commuting. This problem is of a geometric flavor, and we study an index formulation for its solution.

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CHAPTER 1
INTRODUCTION

At the foundation of quantum mechanics, we have such notions as states, observables, expectation values. Quantum mechanical states become vectors in Hilbert space, and observables are represented by selfadjoint operators. These fundamental concepts in quantum theory came from early attempts by physicists at making sense of spectral lines in early atomic experiments. With the introduction of the Spectral Theorem for selfadjoint operators and the work of J. von Neumann, these notions from physics now acquire mathematical precision. For example the spectral resolution for a particular selfadjoint operator offers the measures referred to in the notion of "expected value" of an observable measured in a state. Operators from quantum physics such as position and momentum do not commute, and this is at the root of Heisenberg's uncertainty principle. But there are still many commuting families, and one then seeks a simultaneous diagonalization. In mathematical terms, we speak of a common spectral resolution for the commuting selfadjoint operators. Since the important operators are unbounded, one is faced with two notions "formally selfadjoint," and selfadjoint for a single operator, say A. Initially this distinction was poorly understood, but von Neumann resolved it with his introduction of deficiency-spaces and deficiency-indices. If the closure of A is selfadjoint, we say that A is essentially selfadjoint; and this corresponds to von Neumann indices $(0, 0)$. Now von Neumann's theory applies to a single operator A, formally selfadjoint (also called symmetric or Hermitian) with a dense domain $\mathscr{D}$ in a fixed Hilbert space $\mathscr{H}$. But von Neumann's theory does not apply to more than one operator. If for example, we have just two Hermitian operators defined and commuting on a common dense domain $\mathscr{D}$ in $\mathscr{H}$, then there is nothing available (like deficiency indices) to account for whether the two operators might, or might not, have commuting extensions; where "commuting" now refers to existence of a common spectral resolution. Even if the two operators, commuting on a common domain $\mathscr{D}$, are known to both be essentially selfadjoint this does not mean that they have a common resolution (informally, that they are simultaneously diagonalizable.) With a common spectral resolution, if one exists, we say that the two operators are strongly commuting. The question is important because it is the strongly commuting operators that offer well-defined expectation values in the sense of quantum mechanical measurements. Our thesis is concerned with "what can go wrong." We show that the answer entails invariants for particular Riemann surfaces, and new notions of index. We show a link between this new index and a counting number, more precisely, the counting of sheets in a multiple-cover Riemann surface.

In this thesis, we study several unbounded operators with view to extending von Neumann's theory of deficiency-indices. In the case of a single Hermitian operator A with dense domain in a Hilbert space, von Neumann proved that the possibility of selfadjoint extensions is decided by a pair of index numbers (m, n). In the abstract theory, von Neumann further proved that all values of the two numbers m and n (deficiency indices) are possible; but a given Hermitian operator A admits selfadjoint extensions if and only if $m = n$. Still there are remaining unsolved problems: For example if $m = n = 2$, there is no classification; similarly the case of indices $(1, 2)$ is not well understood in the sense of classification; and for higher indices the situation is even harder.

The applications here include bounded domains Ω in $\mathbb{R}^n$, and the partial derivative operators in the coordinate directions. For Hilbert space we may take $L^2(\Omega)$, and for common dense domain the C^∞ functions with compact support in Ω. In this case, all the partial derivative operators in the coordinate directions separately have indices (∞, ∞), but commuting selfadjoint extensions may or may not exist. If it exits, one obtains n commuting one-parameter unitary groups, and so a unitary representation of the abelian group $(\mathbb{R}^n, +)$ on $L^2(\Omega)$. The representation acts locally as translation in Ω. A choice of mutually commuting selfadjoint extensions amounts to certain boundary condtions imposed on Ω. For example if $n = 2$, and Ω is a disk or a planar triangle, then the two Hermitian partial derivative operators do not have commuting selfadjoint extensions. This problem is of a spectral theoretic flavor. The pioneering work in this direction is due to Fuglede [21]. For the extensive results on this problem, we refer to the papers [24], [35], [46], and the survey [15].

Special cases of the problem can be understood with the use of representation theory. Our motivation is the famous example of Nelson [32]. We will further consider the L^2 space on various covering surfaces $\tilde{M}$ of the punctured plane, and the partial derivative operators in the coordinate directions defined on the common dense domain $\mathscr{C}_c^\infty(\tilde{M})$. The derivative operators separately are essentially selfadjoint, but the unique selfadjoint extensions will be non-commuting. This problem is of a geometric flavor, and we will be studying an index formulation for its solution.

In Chapter 2, we review the basic theory of unbounded operators in Hilbert space. We emphasize Stone's paper on characteristic matrix [50], and illustrate some of its applications in normal operators. This is a very useful tool in operator theory, but has long been overlooked in the literature.

In Chapter 3, we review the basic theory of $*$-algebras and their representations. R.T. Powers showed that every unitary representation of a Lie group U induces a selfadjoint representation dU of the enveloping algebra on the the Gårding space. Conversely, if ρ is a representation of the enveloping algebra, we study conditions so that $\rho = dU$, i.e. ρ is derived from some unitary representation of the Lie group. For general notions, we refer to [25], [35], [36] and [46].

In Chapter 4, we consider a system of n Hermitian operators, commuting on a common invariant dense domain in a Hilbert space, separately essentially selfadjoint, and we ask when do they have mutually commuting selfadjoint extensions? We study an index theory for such systems, and formulate the solution in the settings of representations of $*$-algebras. For $n = 2$, such systems has been extensively studied in a series of papers [41], [40], [42], [43], [44], [45]. The methods used there is based on resolvents of the operators. We take a different approach and the emphasis is on the link between geometry of the manifolds and spectrum of the Nelson-Laplace operator on $L^2(\tilde{M})$.

CHAPTER 2
UNBOUNDED OPERATORS IN HILBERT SPACES

In this chapter, we review the basic theory of unbounded operators in Hilbert space. This theory has been developed in many textbooks and monographs. For general notions, we refer to [13, 14], [6], [38]. For von Neumann algebras, we refer to [28, 29]. The most succinct treatment on spectral theory and spectral multiplicities can be found in E. Nelson's famous lecture notes [33]. For applications in quantum mechanics, we also refer to the lecture notes by W. Arveson [5].

2.1 Preliminaries

In this section, we recall some definitions in the theory of unbounded operators. The idea of characteristic matrix was developed in the beautiful paper of M.S. Stone [50]. This is a very useful tool in operator theory, but has long time been overlooked in the literature. For some of its recent applications, we refer to [27][8].

2.1.1 Domain, Graph

Let $\mathscr{H}$ be a complex Hilbert space. An operator A is a linear mapping whose domain $\mathscr{D}(A)$ and range $\mathscr{R}(A)$ are subspaces in $\mathscr{H}$. The kernel $\mathscr{N}(A)$ of A consists of all $a \in \mathscr{D}(A)$ such that $Aa = 0$. A is uniquely determined by its graph

$$\mathscr{G}(A) = \{(a, Aa) : a \in \mathscr{D}(A)\} \tag{2.1}$$

in $\mathscr{H} \oplus \mathscr{H}$, carrying the graph inner product and norm

$$\langle b, a\rangle_A = \langle b, a\rangle + \langle Ab, Aa\rangle \tag{2.2}$$

$$\|a\|_A = \sqrt{\langle a, a\rangle_A} \tag{2.3}$$

for all $a, b \in \mathscr{D}(A)$. In general, a subspace $\mathscr{K}$ in $\mathscr{H} \oplus \mathscr{H}$ is an operator graph if and only if $(0, a) \in \mathscr{K}$ implies $a = 0$.

Let A, B be two operators. B is an extension of A, denoted by $A \subset B$, if $\mathscr{G}(A) \subset \mathscr{G}(B)$. A is closable if $\overline{\mathscr{G}(A)}$ is the graph of an operator $\bar{A}$, namely, the closure of A. A is closed if $A = \bar{A}$.

If A is closed, a dense subspace $\mathscr{D}$ in $\mathscr{H}$ is said to be a core of A if $\overline{A\big|_{\mathscr{D}}} = A$.

Proposition 2.1. *The following are equivalent.*

1. $A = \bar{A}$.
2. $\mathscr{G}(A) = \overline{\mathscr{G}(A)}$.
3. $\mathscr{D}(A)$ *is a Hilbert space with respect to* $\langle\cdot,\cdot\rangle_A$.
4. *If* (a_n, Aa_n) *is a sequence in* $\mathscr{G}(A)$ *such that* $(a_n, Aa_n) \to (a, b)$, *then* $(a, b) \in \mathscr{G}(A)$. *In particular,* $b = Aa$.

2.1.2 Adjoint Operators

Let A be an operator in a Hilbert space $\mathscr{H}$. $\mathscr{G}(A)^{\perp}$ consists of $(-b^*, b)$ such that $(-b^*, b) \perp \mathscr{G}(A)$ in $\mathscr{H} \oplus \mathscr{H}$.

Proposition 2.2. *The following are equivalent.*

1. $\mathscr{D}(A)$ *is dense in* H.

2. $(b, 0) \perp \mathscr{G}(A) \Longrightarrow b = 0.$

3. *If* $(b, -b^*) \perp \mathscr{G}(A)$, *the map* $b \mapsto b^*$ *is well-defined.*

If any of the conditions is satisfied, $A^* : b \mapsto b^*$ defines an operator, called the adjoint of A, such that

$$\langle b, Aa \rangle = \langle A^* b, a \rangle \tag{2.4}$$

for all $a \in \mathscr{D}(A)$. $\mathscr{G}(A)^\perp$ is the inverted graph of A^*. The adjoints are only defined for operators with dense domains in $\mathscr{H}$.

For unbounded operators, $(AB)^* = B^* A^*$ does not hold in general. The situation is better if one of them is bounded.

Theorem 2.3 (Theorem 13.2 [39])**.** *If* S, T, ST *are densely defined operators then* $(ST)^* \supset T^* S^*$. *If, in addition,* S *is bounded then* $(ST)^* = T^* S^*$.

The next theorem follows directly from the definition of the adjoint operators.

Theorem 2.4. *If* A *is densely defined then* $\mathscr{H} = \overline{\mathscr{R}(A)} \oplus \mathscr{N}(A^*)$.

Finally, we recall some definitions:

- A is selfadjoint if $A = A^*$.
- A is essentially selfadjoint if $\bar{A} = A^*$.
- A is normal if $A^* A = A A^*$.
- A is regular if $\mathscr{D}(A)$ is dense and it is closed.

2.1.3 Characteristic Matrix

The method of characteristic matrix was developed by M.S. Stone's [50]. It is extremely useful in operator theory, but has long been overlooked in the literature. We recall some of its applications in normal operators.

Let A be an operator in a Hilbert space $\mathscr{H}$. Let $P = (P_{ij})$ be the projection from $\mathscr{H} \oplus \mathscr{H}$ onto $\overline{\mathscr{G}(A)}$. The 2×2 operator matrix (P_{ij}) of bounded operators in $\mathscr{H}$ is called the characteristic matrix of A.

Since $P^2 = P^* = P$, the following identities hold

$$P_{ij}^* = P_{ji} \tag{2.5}$$

$$\sum_k P_{ik} P_{kj} = P_{ij} \tag{2.6}$$

In particular, P_{11}, P_{22} are selfadjoint.

Theorem 2.5. *Let $P = (P_{ij})$ be the projection from $\mathscr{H} \oplus \mathscr{H}$ onto a closed subspace $\mathscr{K}$. The following are equivalent.*

1. *$\mathscr{K}$ is an operator graph.*
2. $\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} 0 \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ a \end{bmatrix} \Longrightarrow a = 0.$
3. $P_{12}a = 0$, $P_{22}a = 0 \Longrightarrow a = 0.$

If any of these conditions is satisfied, let A be the operator whose graph is equal to $\mathscr{K}$ then

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} P_{11}a + P_{12}b \\ P_{21}a + P_{22}b \end{bmatrix} \in \mathscr{G}(A)$$

that is,

$$A : (P_{11}a + P_{12}b) \mapsto P_{21}a + P_{22}b. \tag{2.7}$$

In particular,

$$AP_{11} = P_{21} \tag{2.8}$$

$$AP_{12} = P_{22} \tag{2.9}$$

Proof. Setting $v := (a, b)$, then $v \in \mathscr{K}$ if and only if $Pv = v$. The theorem follows from this. $\square$

The next theorem describes the adjoint operators.

Theorem 2.6. *Let A be an operator with characteristic matrix $P = (P_{ij})$. The following are equivalent.*

1. *$\mathscr{D}(A)$ is dense in $\mathscr{H}$.*
2. $\begin{bmatrix} b \\ 0 \end{bmatrix} \perp \mathscr{G}(A) = 0 \Longrightarrow b = 0.$
3. *For all $(-b^*, b) \in \mathscr{G}(A)^\perp$, the map $A^* : b \mapsto b^*$ is a well-defined operator.*
4. $\begin{bmatrix} 1 - P_{11} & -P_{12} \\ -P_{21} & 1 - P_{22} \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} \Longrightarrow b = 0.$
5. $P_{11}b = 0,\ P_{21}b = 0 \Longrightarrow b = 0.$

If any of the above conditions is satisfied, then

$$\begin{bmatrix} 1 - P_{11} & -P_{12} \\ -P_{21} & 1 - P_{22} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (1 - P_{11})a - P_{12}b \\ (1 - P_{22})b - P_{21}a \end{bmatrix} \in \mathscr{G}(A)^\perp$$

that is,

$$A^* : P_{21}a - (1 - P_{22})b \mapsto (1 - P_{11})a - P_{12}b. \tag{2.10}$$

In particular,

$$A^*P_{21} = 1 - P_{11} \tag{2.11}$$

$$A^*(1 - P_{22}) = P_{12}. \tag{2.12}$$

Proof. The projection from $\mathscr{H} \oplus \mathscr{H}$ onto $\mathscr{G}(A)^\perp$ is $1 - P$. The theorem follows from this and Proposition 2.2. $\square$

Theorem 2.7. *Let A be a regular operator with characteristic matrix $P = (P_{ij})$.*

1. *The matrix entries P_{ij} are given by*

$$\begin{array}{rclcrcl} P_{11} & = & (1+A^*A)^{-1} & & P_{12} & = & A^*(1+AA^*)^{-1} \\ P_{21} & = & A(1+A^*A)^{-1} & & P_{22} & = & AA^*(1+AA^*)^{-1} \end{array} \tag{2.13}$$

2. $1 - P_{22} = (1 + AA^*)^{-1}$.

3. $1 + A^*A$, $1 + AA^*$ *are selfadjoint operators.*

4. *The following containments hold*

$$A^*(1+AA^*)^{-1} \supset (1+A^*A)^{-1}A^* \tag{2.14}$$

$$A(1+A^*A)^{-1} \supset (1+AA^*)^{-1}A \tag{2.15}$$

Proof. From $AP_{11} = P_{21}$ and $A^*P_{21} = 1 - P_{11}$, it follows that

$$(1 + A^*A)P_{11} = 1.$$

That is, $1 + A^*A$ is a Hermitian extension of P_{11}^{-1}. By (2.5), P_{11} is selfadjoint and so is its inverse. Therefore, $1 + A^*A$ is equal to P_{11}^{-1}, or

$$P_{11} = (1 + A^*A)^{-1}.$$

This also yields $P_{21} = AP_{11} = A(1+A^*A)^{-1}$. From $AP_{12} = P_{22}$ and $A^*(1-P_{22}) = P_{12}$, it follows that

$$(1 + AA^*)(1 - P_{22}) = 1.$$

A similar argument shows that

$$1 - P_{22} = (1 + AA^*)^{-1}.$$

Therefore, $P_{12} = A^*(1 - P_{22}) = A^*(1 + AA^*)^{-1}$ and $P_{22} = AP_{12} = AA^*(1 + AA^*)^{-1}$. This proves 1, 2 and 3.

Note that

$$P_{12} = P_{21}^* = (AP_{11})^* \supset P_{11}A^*$$

this yields (2.14). Similarly,

$$P_{21} = P_{12}^* = (A^*(1 - P_{22}))^* \supset (1 - P_{22})A$$

gives (2.15). Thus 4 holds. $\square$

2.1.4 Commutants

Let A, B be operators in a Hilbert space $\mathscr{H}$, and suppose B is bounded. B is said to commute (strongly) with A if $BA \subset AB$.

Lemma 2.8. *B commutes with A if and only if B commutes with $\bar{A}$ (assuming $\overline{A}$ exists).*

Proof. Suppose $BA \subset AB$, we check that $B\overline{A} \subset \overline{A}B$. The converse is trivial. For $(a, \overline{A}a) \in \mathscr{G}(\overline{A})$, choose a sequence $(a_n, Aa_n) \in \mathscr{G}(A)$ such that $(a_n, Aa_n) \to (a, \overline{A}a)$. By assumption, $(Ba_n, ABa_n) = (Ba_n, BAa_n) \in \mathscr{G}(A)$. Thus, $(Ba_n, ABa_n) \to (Ba, B\overline{A}a) \in \mathscr{G}(\overline{A})$. That is, $Ba \in \mathscr{D}(\overline{A})$ and $\overline{A}Ba = B\overline{A}a$. $\square$

Lemma 2.9. *Let A be a closed operator with characteristic matrix $P = (P_{ij})$. Let B be a bounded operator, and*

$$Q_B := \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}.$$

1. *B commutes with $A \Leftrightarrow B$ leaves $\mathscr{G}(A)$ invariant $\Leftrightarrow Q_B P = P Q_B P$.*

2. *B commutes with $P_{ij} \Leftrightarrow Q_B P = P Q_B \Leftrightarrow Q_{B^*} P = P Q_{B^*} \Leftrightarrow B^*$ commutes with P_{ij}.*

3. *If B, B^* commute with A, then B, B^* commute with P_{ij}.*

Proof. Obvious. □

A closed operator is said to be affiliated with a Von Neumann algebra $\mathfrak{M}$ if it commutes with every unitary operator in $\mathfrak{M}'$. By Theorem 4.1.7 in [28], every operator in $\mathfrak{M}'$ can be written as a finite linear combination of unitary operators in $\mathfrak{M}'$. Thus, A is affiliated with $\mathfrak{M}$ if and only if A commutes with every operator in $\mathfrak{M}'$.

Theorem 2.10. *Let A be a closed operator with characteristic matrix $P = (P_{ij})$. Let $\mathfrak{M}$ be a Von Neumann algebra, and*

$$Q_B := \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, B \in \mathfrak{M}'.$$

The following are equivalent:

1. *A is affiliated with $\mathfrak{M}$.*

2. *$P Q_B = Q_B P$, for all $B \in \mathfrak{M}'$.*

3. *$P_{ij} \in \mathfrak{M}$.*

4. If $\mathscr{D}(A)$ is dense, then A^* is affiliated with $\mathfrak{M}$.

Proof. Notice that $\mathfrak{M}$ is selfadjoint. The equivalence of $1, 2, 3$ is a direct consequence of Lemma 2.9.

$P^\perp := 1 - P$ is the projection onto the inverted graph of A^*, should the latter exists. $PQ_B = Q_B P$ if and only if $P^\perp Q_B = Q_B P^\perp$. Thus, 1 is equivalent to 4. $\square$

2.1.5 Normal Operators

As a first application of Stone's characteristic matrix, we give a new proof to following theorem concerning operators of the form A^*A.

Theorem 2.11 (Von Neumann)**.** *If A is a regular operator in a Hilbert space $\mathscr{H}$, then A^*A is selfadjoint and $\mathscr{D}(A^*A)$ is a core of A. In particular, $\mathscr{D}(A^*A)$ is dense in $\mathscr{H}$.*

Proof. By Theorem 2.7, A^*A is selfadjoint. Let $\mathscr{D} := \mathscr{D}(A^*A)$, and $(a, Aa) \in \mathscr{G}(A)$ such that $(a, Aa) \perp \mathscr{G}(A\big|_{\mathscr{D}})$. That is, for all b in $\mathscr{D}$,

$$\langle a, b\rangle + \langle Aa, Ab\rangle = \langle a, (1 + A^*A)b\rangle = 0.$$

By Theorem 2.7, $1 + A^*A = P_{11}^{-1}$. Since P_{11} is a bounded operator, $\mathscr{R}(1 + A^*A) = \mathscr{D}(P_{11}) = \mathscr{H}$. It follows that $a \perp \mathscr{H}$, and so $a = 0$. $\square$

Theorem 2.12 (Von Neumann)**.** *Let A be a regular operator in a Hilbert space $\mathscr{H}$. Then A is normal if and only if $\mathscr{D}(A) = \mathscr{D}(A^*)$ and $\|Aa\| = \|A^*a\|$, for all $a \in \mathscr{D}(A)$.*

Proof. Suppose A is normal, and let $\mathscr{D} := \mathscr{D}(A^*A) = \mathscr{D}(AA^*)$. Then $\|Aa\| = \|A^*a\|$ for all $a \in \mathscr{D}$, and so $\mathscr{D}(\overline{A\big|_{\mathscr{D}}}) = \mathscr{D}(\overline{A^*\big|_{\mathscr{D}}})$. By Theorem 2.11, $\overline{A\big|_{\mathscr{D}}} = A$ and $\overline{A^*\big|_{\mathscr{D}}} = A^*$.

It follows that $\mathscr{D}(A) = \mathscr{D}(A^*)$ and $\|Aa\| = \|A^*a\|$, for all $a \in \mathscr{D}(A)$.

Conversely, the map $Aa \mapsto A^*a$, $a \in \mathscr{D}(A)$, extends uniquely to a partial isometry V with initial space $\overline{\mathscr{R}(A)}$ and final space $\overline{\mathscr{R}(A^*)}$, such that $A^* = VA$. By Theorem 2.3, $A = A^*V^*$. Then $A^*A = A^*(V^*V)A = (A^*V^*)(VA) = AA^*$. Thus, A is normal.

$\square$

The following theorem is due to M.S. Stone.

Theorem 2.13 (M.S. Stone)**.** *Let A be a regular operator in a Hilbert space $\mathscr{H}$. Let $P = (P_{ij})$ be the characteristic matrix of A. The following are equivalent.*

1. *A is normal.*
2. *P_{ij} are mutually commuting.*
3. *A is affiliated with an abelian Von Neumann algebra.*

Remark 2.14. For the equivalence of 1 and 2, we refer to the original paper of Stone. The most interesting part is $1 \Leftrightarrow 3$. The idea of characteristic matrix gives rise to an elegant proof without reference to the spectral theorem.

Proof of Theorem 2.13. Assuming $1 \Leftrightarrow 2$, we prove that $1 \Leftrightarrow 3$.

Suppose A is normal, i.e. P_{ij} are mutually commuting. Then A is affiliated with the abelian Von Neumann algebra $\{P_{ij}\}''$. For if $B \in \{P_{ij}\}'$, then B commutes P_{ij}, and so B commutes with A by Lemma 2.9.

Conversely, if A is affiliated with an abelian Von Neumann algebra $\mathfrak{M}$, then $P_{ij} \in \mathfrak{M}$ by Theorem 2.10. This shows that P_{ij} are mutually commuting, and A is normal. $\square$

2.1.6 Functional Calculus

We recall the spectral theorem of a single selfadjoint operator.

Theorem 2.15 (multiplication operator)**.** *Let A be a selfadjoint operator on a Hilbert space $\mathscr{H}$. Then there is a measure space (M, μ) and a unitary operator $U : \mathscr{H} \to L^2(M, \mu)$, such that UAU^* is a multiplication operator by a real measurable function $\hat{A}$ on M. The domain $\mathscr{D}(A)$ of A consists of all $u \in \mathscr{H}$, such that*

$$\int_M \left(1 + |\hat{A}|^2(x)\right) |\hat{u}(x)|^2 d\mu(x) < \infty$$

where $\hat{v} := Uv$, for all $v \in \mathscr{H}$.

Since A is in general unbounded, its spectrum is an unbounded closed subset of $\mathbb{R}$. For all Borel function ψ on $\mathbb{R}$, define the operator

$$\psi(A) := U^* \psi(\hat{A}) U$$

with domain $\mathscr{D}(\psi(A))$ consisting of all $u \in \mathscr{H}$, such that

$$\int_M \left(1 + |\psi(\hat{A})|^2(x)\right) |\hat{u}(x)|^2 d\mu(x) < \infty$$

In particular,

$$E : \omega \mapsto U^* \chi_\omega(\hat{A}) U$$

is a well-defined bounded operator for all ω in the Borel σ-algebra $\mathscr{B}$ of $\mathbb{R}$. Since $\chi_w = \bar{\chi}_w = \chi_w^2$, $E(\omega)$ is a selfadjoint projection. $E(\cdot)$ is a projection-valued measure, i.e. a homomorphism from $\mathscr{B}$ into the lattice of projections in $\mathscr{H}$.

Theorem 2.16 (PVM)**.** *Let A be a selfadjoint operator on $\mathscr{H}$. There is a unique projection-valued measure $E(d\lambda)$ defined on the Borel σ-algebra of $\mathbb{R}$ such that*

$$A = \int_{\mathbb{R}} \lambda E(d\lambda).$$

Moreover, for all Borel functiona ψ on $\mathbb{R}$,

$$\psi(A) = \int_{\mathbb{R}} \psi(\lambda) E(d\lambda).$$

$\mathscr{D}(\psi(A))$ consists of all $u \in \mathscr{H}$, such that

$$\int_{\mathbb{R}} (1 + |\psi(\lambda)|^2)\|E(d\lambda)u\|^2 < \infty$$

Diagonalizing a family of bounded selfadjoint operators may be formulated in the settings of commutative C^*-algebras. By the structure theorem of Gelfand and Naimark, every commutative C^*-algebra containing the identity element is isomorphic to the algebra $\mathscr{C}(X)$ of continuous functions on a compact Hausdorff space X, and X is unique up to homeomorphism. The classification of all the representations of $\mathscr{C}(X)$ may be understood using the idea of σ-measures (square densities). It also leads to the multiplicity theory of selfadjoint operators. The best treatment on this subject can be found in [33]. We also refer to [4].

2.1.7 Polar Decomposition

Let A be a regular operator in a Hilbert space $\mathscr{H}$. By Theorem 2.11, A^*A is a positive selfadjoint operator and it has a unique positive square root $|A| := \sqrt{A^*A}$.

Theorem 2.17.

1. *$\sqrt{A^*A}$ is the unique positive selfadjoint operator T satisfying $\mathscr{D}(T) = \mathscr{D}(A)$, and $\|Ta\| = \|Aa\|$ for all $a \in \mathscr{D}(A)$.*

2. *$\mathscr{N}(|A|) = \mathscr{N}(A)$, $\overline{\mathscr{R}(|A|)} = \overline{\mathscr{R}(A^*)}$.*

Proof. Suppose $T = \sqrt{A^*A}$, i.e. $T^*T = A^*A$. Let $\mathscr{D} := D(T^*T) = \mathscr{D}(A^*A)$. By Theorem 2.11, $\mathscr{D}$ is a core of both T and A. Moreover, $\|Ta\| = \|Aa\|$, for all $a \in \mathscr{D}$. We conclude from this norm identity that $\mathscr{D}(T) = \mathscr{D}(A)$ and $\|Ta\| = \|Aa\|$, for all $a \in \mathscr{D}(A)$.

Conversely, suppose T has the desired properties. For all $a \in \mathscr{D}(A) = \mathscr{D}(T)$, and $b \in \mathscr{D}(A^*A)$,

$$\langle Tb, Ta \rangle = \langle Ab, Aa \rangle = \langle A^*Ab, a \rangle$$

This implies that $Tb \in \mathscr{D}(T^*) = \mathscr{D}(T)$, $T^2 b = A^*Ab$, for all $b \in \mathscr{D}(A^*A)$. That is, T^2 is a selfadjoint extension of A^*A. Since A^*A is selfadjoint, $T^2 = A^*A$.

The second part follows from Theorem 2.4. $\square$

Consequently, the map $|A|a \mapsto Aa$ extends to a unique partial isometry V with initial space $\overline{\mathscr{R}(A^*)}$ and final space $\overline{\mathscr{R}(A)}$, such that

$$A = V|A|. \tag{2.16}$$

Equation (2.16) is called the polar decomposition of A. It is clear that such decomposition is unique.

Taking adjoints in (2.16) yields $A^* = |A|V^*$, so that

$$AA^* = VA^*AV^* \tag{2.17}$$

Restrict AA^* to $\overline{\mathscr{R}(A)}$, and restrict A^*A restricted to $\overline{\mathscr{R}(A^*)}$. Then the two restrictions are unitarily equivalent. It follows that A^*A, AA^* have the same spectrum, aside from possibly the point 0.

By (2.17), $|A^*| = V|A|V^* = VA^*$, where $|A^*| = \sqrt{AA^*}$. Apply V^* on both sides gives

$$A^* = V^*|A^*|. \tag{2.18}$$

By uniqueness, (2.18) is the polar decomposition of A^*.

Theorem 2.18. *A is affiliated with a Von Neumann algebra $\mathfrak{M}$ if and only if $|A|$ is affiliated with $\mathfrak{M}$ and $V \in \mathfrak{M}$.*

Proof. Let U be a unitary operator in $\mathfrak{M}'$. The operator UAU^* has polar decomposition

$$UAU^* = (UVU^*)(U|A|U^*).$$

By uniqueness, $A = UAU^*$ if and only if $V = UVU^*$, $|A| = U|A|U^*$. Since U is arbitrary, we conclude that $V \in \mathfrak{M}$, and A is affiliated with $\mathfrak{M}$. □

2.2 Extensions of Hermitian Operators

von Neumann's index theory gives a complete classification of extensions of single Hermitian unbounded operators with dense domain in a given Hilbert space. The theory may be adapted to Hermitian representations of $*$-algebras [26].

2.2.1 von Neumann's Index Theory

Let A be a densely defined Hermitian operator on a Hilbert space $\mathscr{H}$, i.e. $A \subset A^*$. If B is any Hermitian extension of A, then

$$A \subset B \subset B^* \subset A^*. \tag{2.19}$$

Since the adjoint operator A^* is closed, there is no loss of generality to assume that A is closed and only consider its closed extensions.

The relation in (2.19) suggests a detailed analysis in $\mathscr{D}(A^*)\backslash\mathscr{D}(A)$. Since the domain of A is dense, the usual structural analysis in $\mathscr{H}$ (orthogonal decomposition, etc.) is not applicable. However, this structure is brought out naturally when $\mathscr{D}(A^*)$ is identified with the operator graph $\mathscr{G}(A^*)$ in $\mathscr{H} \oplus \mathscr{H}$. Under this identification, $\mathscr{D}(A)$ is a closed subspace in the Hilbert space $\mathscr{D}(A^*)$, and

$$\mathscr{D}(A^*) = \mathscr{D}(A) \oplus (\mathscr{D}(A^*)\backslash\mathscr{D}(A)). \tag{2.20}$$

The question of extending A amounts to a further decomposition

$$\mathscr{D}(A^*)\backslash\mathscr{D}(A) = S \oplus K \tag{2.21}$$

in such a way that

$$\mathscr{D}(\tilde{A}) := \mathscr{D}(A) \oplus S \tag{2.22}$$

$$\tilde{A} := A^*\big|_{\mathscr{D}(\tilde{A})}. \tag{2.23}$$

defines a Hermitian operator $\tilde{A}$. This is true if and only if S is symmetric, in the sense that

$$\langle A^* g, f\rangle - \langle g, A^* f\rangle = 0 \tag{2.24}$$

for all $f, g \in S$. By the polarization identity, (2.24) is equivalent to

$$\langle f, A^* f\rangle \in \mathbb{R} \tag{2.25}$$

for all $f \in S$, Thus, there is a bijection between (closed) Hermitian extensions of A and (closed) symmetric subspaces in $\mathscr{D}(A^*)\backslash\mathscr{D}(A)$.

If $A^*\varphi = z\varphi$, $\Im(z) \neq 0$, then φ does not belong to the domain of any Hermitian extension $\tilde{A} \supset A$. Otherwise, $\tilde{A}\varphi = A^*\varphi = z\varphi$ and $\langle \varphi, \tilde{A}\varphi\rangle \notin \mathbb{R}$. This observation is in fact ruling out the "wrong" eigenvalues of $\tilde{A}$, which is supposed to be Hermitian. Theorem 2.19 shows that A is selfadjoint if and only if all the "wrong" eigenvalues are excluded.

A complete characterization of Hermitian extensions of a given Hermitian operator is due to von Neumann.

Theorem 2.19. *Let A be a densely defined closed Hermitian operator on $\mathscr{H}$. A is selfadjoint if and only if there exists z, $\Im(z) \neq 0$, such that $\mathscr{N}(A^* - z) = \mathscr{N}(A^* - \overline{z}) = 0$.*

Proof. We prove that $\mathscr{N}(A^* - z) = \mathscr{N}(A^* - \overline{z}) = 0$, $\Im(z) \neq 0$, implies that $A = A^*$. The hypothesis is equivalent to $\mathscr{R}(A - z) = \mathscr{R}(A - \overline{z}) = \mathscr{H}$. Let $g \in \mathscr{D}(A^*)$. For all $f \in \mathscr{D}(A)$,

$$\langle g, (A - z)f\rangle = \langle (A^* - \overline{z})g, f\rangle = \langle (A - \overline{z})g_0, f\rangle = \langle g_0, (A - z)f\rangle$$

where $(A^* - \overline{z})g = (A - \overline{z})g_0$, for some $g_0 \in \mathscr{D}(A)$. Thus $\langle g - g_0, (A - z)f\rangle = 0$ for all $f \in \mathscr{D}(A)$. Since $\mathscr{R}(A - z) = \mathscr{H}$, then $g - g_0 = 0$, so that $g = g_0 \in \mathscr{D}(A)$. This shows that $\mathscr{D}(A^*) \subset \mathscr{D}(A)$. $\square$

Definition 2.20. Suppose A is a densely defined Hermitian operator in $\mathscr{H}$. The closed subspaces

$$\mathscr{D}_+ := \{f \in \mathscr{D}(A^*) : A^*f = if\} \tag{2.26}$$

$$\mathscr{D}_- := \{f \in \mathscr{D}(A^*) : A^*f = -if\} \tag{2.27}$$

are called the deficiency spaces of A, and their dimensions

$$d_+ := \dim \mathscr{D}_+ \tag{2.28}$$

$$d_- := \dim \mathscr{D}_- \tag{2.29}$$

are called the deficiency indices of A.

Theorem 2.21 (von Neumann)**.** *Let A be a densely defined closed Hermitian operator on $\mathscr{H}$. Then*

$$\mathscr{D}(A^*) = \mathscr{D}(A) \oplus \mathscr{D}_+ \oplus \mathscr{D}_-. \tag{2.30}$$

Remark 2.22. $\mathscr{D}(A^*)$ is identified with its graph $\mathscr{G}(A^*)$, carrying the graph inner product and the graph norm. Thus, $\mathscr{D}(A^*)$ is a Hilbert space.

Proof of Theorem 2.21. Since A is a closed operator, it follows that $\mathscr{D}(A)$, identified with $\mathscr{G}(A)$, is a closed subspace in $\mathscr{D}(A^*)$. Notice that $\mathscr{D}_\pm$ are closed subspaces in $\mathscr{H}$, since $\mathscr{D}_\pm = \mathscr{R}(A \mp i)^\perp$. For all $f \in \mathscr{D}_\pm$,

$$\|f\|^2_{A^*} = \|f\|^2 + \|A^*f\| = 2\|f\|$$

and this implies that $\mathscr{D}_\pm$, identified with $\mathscr{G}(A^*\big|_{\mathscr{D}_\pm})$, are also closed subspaces in $\mathscr{D}(A^*)$.

For all $f \in \mathscr{D}(A)$ and $f_+ \in \mathscr{D}_+$,

$$\begin{aligned}
\langle f, f_+\rangle_{A^*} &= \langle f, f_+\rangle + \langle A^* f, A^* f_+\rangle \\
&= \langle if, if_+\rangle + \langle Af, if_+\rangle \\
&= i\langle (A+i)f, f_+\rangle \\
&= i\langle f, (A^* - i)f_+\rangle \\
&= 0
\end{aligned}$$

thus $\mathscr{D}(A) \perp \mathscr{D}_+$. Similar computations show that $\mathscr{D}(A)$ and $\mathscr{D}_\pm$ are mutually orthogonal in $\mathscr{D}(A^*)$.

Let $g \in \mathscr{D}(A^*)$. The decomposition $\mathscr{H} = \mathscr{R}(A+i) \oplus \mathscr{D}_+$ shows that

$$(A^* + i)g = (A+i)f + 2if_+$$

for some $f \in \mathscr{D}(A)$, and $f_+ \in \mathscr{D}_+$. Since $A^*(g-f-f_+) = -i(g-f-f_+)$, it follows that $f_- := g-f-f_+ \in \mathscr{D}_-$, and $g = f+f_++f_-$. Therefore, $\mathscr{D}(A^*) = \mathscr{D}(A) \oplus \mathscr{D}_+ \oplus \mathscr{D}_-$. $\square$

Corollary 2.23. *Let A be a densely defined Hermitian operator on $\mathscr{H}$.*

1. *A is maximally Hermitian if and only if one of the deficiency indices is 0.*

2. *A has a selfadjoint extension if and only if $d_+ = d_- \neq 0$.*

3. *$\bar{A}$ is selfadjoint if and only if $d_+ = d_- = 0$.*

It remains to describe the closed symmetric subspaces S in $\mathscr{D}_+ \oplus \mathscr{D}_-$. See

(2.24) and (2.25). Let $f = f_+ + f_-$, where $f_\pm \in \mathscr{D}_\pm$, then

$$\begin{aligned}
\langle f, A^* f\rangle &= \langle f_+ + f_-, A^*(f_+ + f_-)\rangle \\
&= \langle f_+ + f_-, i(f_+ - f_-)\rangle \\
&= i\langle f_+ + f_-, f_+ - f_-\rangle \\
&= i(\|f_+\|^2 - \|f_-\|^2) - i(\langle f_+, f_-\rangle - \overline{\langle f_+, f_-\rangle}).
\end{aligned} \tag{2.31}$$

Thus $\langle f, A^* f\rangle \in \mathbb{R}$ if and only if $\|f_+\| = \|f_-\|$. It follows that S is identified with the graph of an isometric mapping from a closed subspace in $\mathscr{D}_+$ onto a closed subspace in $\mathscr{D}_-$.

Theorem 2.24 (von Neumann)**.** *Let A be a densely defined closed Hermitian operator on $\mathscr{H}$.*

1. *The Hermitian extensions of A are indexed by partial isometries U with initial space in $\mathscr{D}_+$ and final space in $\mathscr{D}_-$.*
2. *Given U as in part 1, the Hermitian extension $\tilde{A}_U \supset A$ is given by*

$$\mathscr{D}(\tilde{A}_U) := \{f + f_+ + Uf_+ : f \in \mathscr{D}(A), f_+ \in \mathscr{D}_+\} \tag{2.32}$$

$$\tilde{A}_U(f + f_+ + Uf_+) := Af + if_+ - iUf_+. \tag{2.33}$$

There is a simple criterion to test whether a Hermitian operator has equal deficiency indices.

Definition 2.25. An operator J is called a conjugation if it is conjugate linear, $J^2 = 1$, and $\langle Jg, Jf\rangle = \langle f, g\rangle$, for all $f, g \in \mathscr{H}$.

Theorem 2.26 (von Neumann)**.** *Let A be a densely defined closed Hermitian operator on $\mathscr{H}$. Suppose $AJ = JA$, where J is a conjugation, then $d_+ = d_-$. In particular, A has selfajoint extensions.*

Proof. Let $f \in \mathscr{D}(A)$, $g \in \mathscr{D}(A^*)$, then

$$\langle Jg, Af\rangle = \langle JAf, g\rangle = \langle AJf, g\rangle = \langle Jf, A^*g\rangle = \langle JA^*g, f\rangle. \tag{2.34}$$

So, the map $f \mapsto \langle Jg, Af\rangle$ is continuous and $Jg \in \mathscr{D}(A^*)$. It follows that $J\mathscr{D}(A^*) \subset \mathscr{D}(A^*)$. In fact, $J\mathscr{D}(A^*) = \mathscr{D}(A^*)$, as $J^2 = 1$. We may also deduce that $JA^* = A^*J$.

For all $f_+ \in \mathscr{D}_+$,

$$A^*(Jf_+) = J(A^*f_+) = J(if_+) = -i(Jf_+)$$

so that $J\mathscr{D}_+ \subset \mathscr{D}_-$. Similarly, $J\mathscr{D}_- \subset \mathscr{D}_+$. Thus $J\mathscr{D}_+ = \mathscr{D}_-$. Since J preserves norm, $\mathscr{D}_+$ and $\mathscr{D}_-$ have the same dimension. □

In applications to differential equations, it is convenient to characterize self-adjoint extensions using boundary conditions. The idea of boundary spaces shows up naturally in this setting. For recent developments, we refer to [11][22]. A slightly modified version can be found in [10].

Definition 2.27. Let A be a densely defined, closed, Hermitian operator in $\mathscr{H}$. Suppose A has equal deficiency indices (n, n). A boundary space for A is a triple $(\mathscr{H}_b, \rho_1, \rho_2)$ consisting of a Hilbert space $\mathscr{H}_b$ and two linear maps $\rho_1, \rho_2 : \mathscr{D}(A^*) \to \mathscr{H}_b$, such that the images of ρ_1, ρ_2 are dense in $\mathscr{H}_b$, and

$$\langle g, A^*f\rangle - \langle A^*g, f\rangle = c\,[\langle\rho_1(g), \rho_1(f)\rangle_b - \langle\rho_2(g), \rho_2(f)\rangle_b] \tag{2.35}$$

for all $f, g \in \mathscr{D}(A^*)$, where c is a nonzero constant.

Remark 2.28. This definition is motivated by the boundary form

$$\langle g, A^* f\rangle - \langle A^* g, f\rangle = 2i\,[\langle g_+, f_+\rangle - \langle g_-, f_-\rangle] \tag{2.36}$$

for all $f, g \in \mathscr{D}(A^*)$, which follows immediately from (2.31). For example, let V be a partial isometry with initial space $\mathscr{D}_-$ and final space $\mathscr{D}_+$. Choose

$$\begin{aligned} \mathscr{H}_b &:= \mathscr{D}_+ \\ \rho_1(f_0 + f_+ + f_-) &:= f_+ \\ \rho_2(f_0 + f_+ + f_-) &:= V f_- \end{aligned}$$

for any $f = f_0 \oplus f_+ \oplus f_-$ in $\mathscr{D}(A^*)$. The triple $(\mathscr{H}_b, \rho_1, \rho_2)$ is a boundary space for A. In this special case, ρ_1, ρ_2 are surjective. It is clear that the choice of a boundary triple is not unique. In applications, $\mathscr{H}_b$ is usually chosen to have the same dimension as $\mathscr{D}_\pm$.

Theorem 2.24 can now be restated as follows.

Theorem 2.29. *Let A be a densely defiend, closed, Hermitian operator in $\mathscr{H}$. Suppose A has equal deficiency indices. Let $(\mathscr{H}_b, \rho_1, \rho_2)$ be a boundary triple for A. Then the selfadjoint extensions of A are indexed by unitary operators U on $\mathscr{H}_b$. Given U, the selfadjoint extension $\tilde{A}_U \supset A$ is given by*

$$\mathscr{D}(\tilde{A}_U) := \{f \in \mathscr{D}(A^*) : U\rho_1(f) = \rho_2(f)\} \tag{2.37}$$

$$\tilde{A}_U f := A^* f, f \in \mathscr{D}(\tilde{A}_U). \tag{2.38}$$

Remark 2.30. Some variants arise more naturally in the boundary value problems of differential equations. In [11][22], $(\mathscr{H}_b, \Gamma_1, \Gamma_2)$ is defined to satisfy

$$\langle g, A^* f\rangle - \langle A^* g, f\rangle = c' \left[\langle \Gamma_2(g), \Gamma_1(f)\rangle_b - \langle \Gamma_1(g), \Gamma_2(f)\rangle_b\right] \tag{2.39}$$

for all $f, g \in \mathscr{D}(A^*)$, where c' is some nonzero constant. The connection between (2.35) and (2.39) can be seen by setting

$$\rho_1 := \Gamma_1 - i\Gamma_2 \tag{2.40}$$

$$\rho_2 := \Gamma_1 + i\Gamma_2 \tag{2.41}$$

so that

$$\langle \rho_1(g), \rho_1(f)\rangle - \langle \rho_2(g), \rho_2(f)\rangle = 2i \left[\langle \Gamma_2(g), \Gamma_1(f)\rangle - \langle \Gamma_1(g), \Gamma_2(f)\rangle\right].$$

Under this formulation, the family of selfadjoint extensions $\tilde{A}_U \supset A$ is again indexed by unitary operators on $\mathscr{H}_b$, such that

$$\mathscr{D}(\tilde{A}_U) := \{f \in \mathscr{D}(A^*) : (U-1)\Gamma_1(f) = i(U+1)\Gamma_2(f)\} \tag{2.42}$$

$$\tilde{A}_U f := A^* f, f \in \mathscr{D}(\tilde{A}_U). \tag{2.43}$$

2.2.2 Friedrichs Extension

We recall some basic ideas in the theory of rigged Hilbert space. These will be used to construct certain selfadjoint extension of semibounded Hermitian operators with dense domains on a Hilbert space.

Let $\mathscr{H}_0$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle_0$ and norm $\|\cdot\|_0$. Suppose $\mathscr{H}_1$ is a dense subspace in $\mathscr{H}_0$, and itself is a Hilbert space with inner product $\langle\cdot,\cdot\rangle_1$

and norm $\|\cdot\|_1$. Further, assume that

$$\|x\|_1 \geq \|x\|_0, x \in \mathscr{H}_1 \tag{2.44}$$

It follows that the inclusion map

$$i : \mathscr{H}_1 \hookrightarrow \mathscr{H}_0 \tag{2.45}$$

is continuous and has a dense image.

Let $\mathscr{H}_{-1}$ be the Banach space of bounded conjugate linear functionals l on $\mathscr{H}_1$, whose norm is given by

$$\|l\|_{-1} = \sup\{|l(x)| : x \in \mathscr{H}_1, \|x\|_1 = 1\}. \tag{2.46}$$

By Riesz's theorem, there is an isometric bijection $J : \mathscr{H}_1 \to \mathscr{H}_{-1}$ which turns $\mathscr{H}_{-1}$ into a Hilbert space carrying the inner product

$$\langle l_2, l_1\rangle_{-1} = \langle J^{-1}l_2, J^{-1}l_1\rangle_1 \tag{2.47}$$

for all $l_1, l_2 \in \mathscr{H}_{-1}$. Thus, J is a unitary operator.

There is a natural injection of $\mathscr{H}_0$ into $\mathscr{H}_{-1}$, by

$$f \mapsto \langle\cdot, f\rangle_0, f \in \mathscr{H}_0 \tag{2.48}$$

For if $f \in \mathscr{H}_0$ and $x \in \mathscr{H}_1$, then

$$|\langle x, f\rangle_0| \leq \|x\|_0\|f\|_0 \leq \|x\|_1\|f\|_0 \tag{2.49}$$

which implies that $\langle\cdot, f\rangle_0 \in \mathscr{H}_{-1}$, and

$$\|f\|_0 \geq \|\langle\cdot, f\rangle_0\|_{-1}, f \in \mathscr{H}_0 \tag{2.50}$$

Therefore,

$$\langle x, f\rangle_0 = \langle x, J^{-1}\langle\cdot, f\rangle_0\rangle_1, x \in \mathscr{H}_1. \tag{2.51}$$

Lemma 2.31. *The image of $\mathscr{H}_1$ (so is the image of $\mathscr{H}_0$) under the injection (2.48) is dense in $\mathscr{H}_{-1}$.*

Proof. For $l \in \mathscr{H}_{-1}$, $x \in \mathscr{H}_1$,

$$\langle l, \langle\cdot, x\rangle_0\rangle_{-1} = \langle J^{-1}l, J^{-1}\langle\cdot, x\rangle_0\rangle_1 = \langle J^{-1}l, x\rangle_0$$

and the last step follows from (2.51). Thus, $l \perp \{\langle\cdot, x\rangle_0 : x \in \mathscr{H}_1\}$ if and only if $\langle J^{-1}l, x\rangle_0 = 0$, for all $x \in \mathscr{H}_1$. Since $\mathscr{H}_1$ is dense in $\mathscr{H}_0$, it follows that $J^{-1}l = 0$. Since J is unitary, this implies that $l = 0$. □

We identify $\mathscr{H}_0$ as a subspace in $\mathscr{H}_{-1}$, i.e. identity f with $\langle\cdot, f\rangle_0$, and obtain the triple of Hilbert spaces

$$\mathscr{H}_1 \hookrightarrow \mathscr{H}_0 \hookrightarrow \mathscr{H}_{-1}. \tag{2.52}$$

The following are immediate:

1. All the embeddings in (2.52) are continuous with dense images. Recall the first is the identity map in (2.45), and the second is given by (2.48).

2. Under the identification of x and $\langle\cdot, x\rangle_0$, $x \in \mathscr{H}_1$, (2.50) is equivalent to $\|x\|_0 \geq \|x\|_{-1}$. Combined with (2.46), this yields

$$\|x\|_1 \geq \|x\|_0 \geq \|x\|_{-1}, x \in \mathscr{H}_1 \tag{2.53}$$

3. By Lemma 2.31, $\mathscr{H}_{-1}$ is the completion of $\mathscr{H}_0$ with respect to the $\|\cdot\|_{-1}$-norm. Recall that

$$\langle g, f\rangle_{-1} = \langle J^{-1}g, J^{-1}f\rangle_0, f, g \in \mathscr{H}_0.$$

4. The canonical bilinear form on $\mathscr{H}_1 \times \mathscr{H}_{-1}$ is given by

$$\langle x, l\rangle := \langle x, J^{-1}l\rangle_1 = \langle Jx, l\rangle_{-1} \tag{2.54}$$

for all $x \in \mathscr{H}_1$ and $l \in \mathscr{H}_{-1}$. If, in addition, $l \in \mathscr{H}_0$, then

$$\langle x, J^{-1}l\rangle_1 = \langle x, l\rangle_0 \tag{2.55}$$

Thus, (2.54) is a continuous extension of the inner product on $\mathscr{H}_0$.

Theorem 2.32. *Let $\mathscr{H}_1 \hookrightarrow \mathscr{H}_0 \hookrightarrow \mathscr{H}_{-1}$ be the triple in (2.52). Define $B : \mathscr{H}_0 \to \mathscr{H}_0$ by*

$$Bf = J^{-1}f. \tag{2.56}$$

The following statements hold.

1. B is invertible.

2. $0 \leq B \leq 1$. In particular, B is a bounded selfadjoint operator on $\mathscr{H}_0$.

Remark 2.33. The precise meaning of (2.56) is

$$f \mapsto (i \circ J^{-1})(\langle \cdot, f\rangle_0).$$

Proof of Theorem 2.32. 1. By (2.55), $\langle x, Bf\rangle_1 = \langle x, f\rangle_0$, for all $f \in \mathscr{H}_0$, $x \in \mathscr{H}_1$. If $Bf = 0$, then $\langle x, f\rangle_0 = 0$, for all $x \in \mathscr{H}_1$. This implies that $f = 0$, since $\mathscr{H}_1$ is dense in $\mathscr{H}_0$. Thus, B is invertible.

2. Setting $x := Bf$, then $\langle Bf, Bf\rangle_1 = \langle Bf, f\rangle_0 \geq 0$, so that $B \geq 0$. Moreover, the estimate

$$\|Bf\|_0 \leq \|Bf\|_1 = \|J^{-1}f\|_1 = \|f\|_{-1} \leq \|f\|_0$$

shows that $B \leq 1$. Since B is positive and bounded, it follows that B is selfadjoint. □

Theorem 2.34. *Let* $A := B^{-1}$*, so that* $\mathscr{D}(A) = \mathscr{R}(B)$*,* $\mathscr{R}(A) = \mathscr{H}_0$*. Then*

1. $A = A^*$*,* $A \geq 1$*;*

2. *For all* $x \in \mathscr{D}(A)$*,* $y \in \mathscr{H}_1$*,* $\langle y, x\rangle_1 = \langle y, Ax\rangle_0$*;*

3. $\mathscr{D}(A)$ *is dense in* $\mathscr{H}_1$*;*

4. $\mathscr{H}_1 = \mathscr{D}(A^{1/2})$*;*

5. $\langle y, x\rangle_1 = \langle A^{1/2}y, A^{1/2}x\rangle_0$*, for all* $x, y \in \mathscr{H}_1$*;*

6. $\langle g, f\rangle_{-1} = \langle A^{-1/2}g, A^{-1/2}f\rangle_0$*, for all* $f, g \in \mathscr{H}_0$*;*

7. $\|Ax\|_{-1} = \|x\|_1$*, for all* $x \in \mathscr{D}(A)$*.*

Moreover, the first three conditions determine $\mathscr{H}_1$ *and* A *uniquely.*

Remark 2.35. The last part says that the map $J_0 : x \mapsto Ax$ is norm-preserving from $\mathscr{D}(A) \subset \mathscr{H}_1$ onto $\mathscr{R}(A) = \mathscr{H}_0 \subset \mathscr{H}_{-1}$. By density argument, J_0 extends to a unique unitary operator $\tilde{J}_0 : \mathscr{H}_1 \to \mathscr{H}_{-1}$. It follows readily that $\tilde{J}_0 = J$.

Proof of Theorem 2.34.

Parts 1, 2 follow directly from the definition. For 3, since $\mathscr{H}_0$ is dense in $\mathscr{H}_{-1}$, therefore, $J^{-1}\mathscr{H}_0$ is dense in $J^{-1}\mathscr{H}_{-1}$, i.e. $\mathscr{D}(A)$ is dense in $\mathscr{H}_1$.

From part 2, we derive that

$$\|x\|_1^2 = \langle A^{1/2}x, A^{1/2}x\rangle_0, x \in \mathscr{D}(A).$$

Since $A \geq 1$, the $\|\cdot\|_1$-norm is equivalent to the graph norm $\|\cdot\|_{A^{1/2}}$. Moreover, $\mathscr{D}(A)$ is dense in both $\mathscr{H}_1$ and $\mathscr{D}(A^{1/2})$. Consequently, the completions of $\mathscr{D}(A)$ with respect to $\|\cdot\|_1$ and $\|\cdot\|_{A^{1/2}}$ coincide. This proves part 4. The other statements are consequences of part 4.

We proceed to show that conditions $1, 2, 3$ determine $\mathscr{H}_1$ and A uniquely. Suppose we start with another embedding

$$\mathscr{H}_{1'} \hookrightarrow \mathscr{H}_0$$

such that $\mathscr{D}(A)$ is dense in $\mathscr{H}_{1'}$, and $\langle y, x\rangle_{1'} = \langle y, Ax\rangle_0$, for all $x \in \mathscr{D}(A)$, $y \in \mathscr{H}_{1'}$. Then $\langle y, x\rangle_{1'} = \langle y, x\rangle_1$, for all $x, y \in \mathscr{D}(A)$, i.e. the inner products agree on a common dense domain. It follows that $\mathscr{H}_{1'} = \mathscr{H}_1$.

On the other hand, suppose there is another selfadjoint operator $\tilde{A}$, such that $\mathscr{D}(\tilde{A})$ is dense in $\mathscr{H}_1$, and $\langle y, x\rangle_1 = \langle y, \tilde{A}x\rangle_0$, for all $x \in \mathscr{D}(\tilde{A})$, $y \in \mathscr{H}_1$. Then

$$\langle y, \tilde{A}x\rangle = \langle y, x\rangle_1 = \langle A^{1/2}y, A^{1/2}x\rangle_0$$

for all $x \in \mathscr{D}(\tilde{A})$, $y \in \mathscr{H}_1$. It follows that $A^{1/2}x \in \mathscr{D}(A^{1/2})$, so that $x \in \mathscr{D}(A)$ and $\tilde{A}x = Ax$. That is, $\tilde{A} \subset A$. Similarly, $A \subset \tilde{A}$, and so $\tilde{A} = A$. $\square$

Let A be a densely defined Hermitian operator in $\mathscr{H}_0$, and suppose that A is bounded below, i.e. for some constant $C > 0$,

$$\langle f, Af\rangle_0 \geq C\langle f, f\rangle_0$$

for all $f \in \mathscr{D}(A)$. The lower bound m_A of A is given by

$$m_A := \inf\{\langle f, Af\rangle_0 : f \in \mathscr{D}(A), \|f\|_0 = 1\}$$

There is no loss of generality to assume $C = 1$.

Introduce the inner product and norm on $\mathscr{D}(A)$, by

$$\begin{aligned} \langle y, x\rangle_1 &:= \langle y, Ax\rangle_0 \\ \|x\|_1 &:= \sqrt{\langle x, x\rangle_1} \end{aligned}$$

for all $x, y \in \mathscr{D}(A)$. Let $\mathscr{H}_1$ be the completion of $\mathscr{D}(A)$ with respect to $\|\cdot\|_1$. Lemma 2.36 shows that $\mathscr{H}_1$ is a Hilbert space, and it is continuously embedded into $\mathscr{H}_0$. Thus, there exists a triple of Hilbert spaces

$$\mathscr{H}_1 \hookrightarrow \mathscr{H}_0 \hookrightarrow \mathscr{H}_{-1}$$

as constructed before.

Lemma 2.36. *The norms $\|\cdot\|_1$ and $\|\cdot\|_0$ are topologically consistant. That is, $\mathscr{H}_1$ is continuousely embedded, via the identity map, into $\mathscr{H}_0$ as a subspace. In particular, $\|x\|_1 \geq \|x\|_0$, for all $x \in \mathscr{H}_1$.*

Proof. By assumption, $A \geq 1$, and this implies that $\|x\|_1 \geq \|x\|_0$, for all $x \in \mathscr{D}(A)$. The norm estimate passes to the completion. Thus, the identity map i from $\mathscr{H}_1$ into $\mathscr{H}_0$ is continuous. It remains to check that i is injective.

Let x_n be a sequence in $\mathscr{D}(A)$, such that it is Cauchy in $\mathscr{H}_1$. The norm estimate implies that x_n is also Cauchy in $\mathscr{H}_0$. Suppose $\|x_n\|_0 \to 0$. By assumption, A is closable, so that $(x_n, Ax_n) \to (0, 0) \in \overline{\mathscr{G}(A)}$. Thus,

$$\|x_n\|_1^2 = \langle x_n, Ax_n \rangle_0 \to 0.$$

This proves the lemma. □

Theorem 2.37 (Friedrichs)**.** *Let A be a densely defined Hermitian operator in a Hilbert space $\mathscr{H}_0$. Suppose $A \geq 1$. Then A has a selfadjoint extension $\tilde{A}$, such that $m_A = m_{\tilde{A}}$.*

Proof. Given A, construct $\mathscr{H}_1$ as before. By Theorem 2.34, there is a unique selfadjoint operator $\tilde{A}$, such that $\mathscr{D}(\tilde{A})$ is dense in $\mathscr{H}_1$, and

$$\langle y, x\rangle_1 = \langle y, \tilde{A}x\rangle_0,\ x \in \mathscr{D}(\tilde{A}), y \in \mathscr{H}_1. \tag{2.57}$$

If, in addition, $x \in \mathscr{D}(A)$, then

$$\langle y, Ax\rangle_0 = \langle y, \tilde{A}x\rangle_0,\ x \in \mathscr{D}(A), y \in \mathscr{H}_1 \tag{2.58}$$

from which we conclude that $A \subset \tilde{A}$.

By (2.58), $m_A \geq m_{\tilde{A}}$. Since $\|x\|_1^2 \geq m_A\|x\|_0^2$, for all $x \in \mathscr{D}(A)$, it follows that $\|x\|_1^2 \geq m_A\|x\|_0^2$, for all $x \in \mathscr{H}_1$. That is,

$$\langle \tilde{A}^{1/2}x, \tilde{A}^{1/2}x\rangle \geq m_A\langle x, x\rangle, x \in \mathscr{D}(\tilde{A}^{1/2}).$$

Thus,

$$\langle x, \tilde{A}x\rangle \geq m_A\langle x, x\rangle, x \in \mathscr{D}(\tilde{A})$$

and so $m_{\tilde{A}} \geq m_A$. □

2.3 Sturm-Liouville Problem

von Neumann's index theory on extension of Hermitian operators in Hilbert space has found a very important application in the classical Sturm-Liouville problem. We briefly review some basic facts in this theory, with an emphasis on the selfadjoint boundary conditions. For the operator theoretic approach to the subject, we refer to [51], [30], [2], [37], and the recent surveys [54], [18]. A specific example used extensively in later chapters is the Bessel differential operator defined on the half-line.

2.3.1 Differential Operators

We consider the modern form of the Sturm-Liouville differential equation specified as following:

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \tag{2.59}$$

1. $x \in (a, b)$, $-\infty \leq a < b \leq \infty$
2. $p, q, w : (a, b) \to \mathbb{R}$, and $p^{-1}, q, w \in L^1_{loc}(a, b)$
3. $w(x) > 0$, a.e. in (a, b) with respect to the Lebesgue measure
4. $\lambda \in \mathbb{C}$

Definition 2.38. Let M be the differential expression such that

1. $\mathscr{D}(M) = \{f : (a, b) \to \mathbb{C} : f, Mf \in AC_{loc}(a, b)\}$
2. $(Mf)(x) = -(p(x)f'(x))' + q(x)f(x)$, for all $f \in \mathscr{D}(M)$

Clearly, $\mathscr{D}(M)$ is the largest space of functions on which M has a natural meaning. Various operators associated with M will be considered on the Hilbert space $L^2((a, b), w)$, consisting of f such that

$$\int_a^b |f(x)|^2 w(x)dx < \infty.$$

Definition 2.39. Let T_0 be the pre-minimum operator with

1. $\mathscr{D}(T_0) = \mathscr{C}_c^\infty(a, b)$;
2. $T_0 f = w^{-1}Mf$, for all $f \in \mathscr{D}(T_0)$.

T_0 is a densely defined and Hermitian in $L^2(a, b)$. Let T be the closure of T_0. T is called the minimum operator, and its adjoint T^* is called the maximum operator.

Theorem 2.40. *Let T be the minimum operaotr, that is $T = \overline{T}_0$. Then*

1. $\mathscr{D}(T^*) = \{f \in \mathscr{D}(M) : f, w^{-1}Mf \in L^2(a,b)\}$;

2. $T^*f = w^{-1}Mf$, *for all* $f \in \mathscr{D}(T^*)$.

Remark 2.41. $\mathscr{D}(T^*)$ is the largest subspace in $L^2(a,b)$, where M has a natural meaning.

Let $f, g \in \mathscr{D}(M)$. The Lagrange bracket $[\cdot,\cdot]$ is defined by

$$[f,g](x) := f(x)\overline{(Mg)(x)} - (Mf)(x)\overline{g(x)} \tag{2.60}$$

Given any compact interval $[\alpha,\beta] \subset (a,b)$, the Green's identity holds

$$[f,g](\beta) - [f,g](\alpha) = \int_\alpha^\beta \overline{g(x)}(Mf)(x)dx - \int_\alpha^\beta \overline{(Mg)(x)}f(x)dx. \tag{2.61}$$

If, in addition, $f, g \in \mathscr{D}(T^*)$, then both limits

$$\begin{aligned} [f,g](a) &:= \lim_{x\to a+}[f,g](x) \\ [f,g](b) &:= \lim_{x\to b-}[f,g](x) \end{aligned}$$

exist and are finite.

Theorem 2.42. *Let T be the minimum operator, that is $T = \overline{T}_0$. Then*

1. $\mathscr{D}(T) = \{f \in \mathscr{D}(T^*) : [f,g](b) - [f,g](a) = 0, \forall g \in \mathscr{D}(T^*)\}$;

2. $Tf = w^{-1}Mf$, *for all* $f \in \mathscr{D}(T)$.

Proof. Recall that $T = T^{**} \subset T^*$, so all the operators are restrictions of T^*. It suffices to verify that $\mathscr{D}(T^{**})$ is as specified in the theorem. $f \in \mathscr{D}(T^{**})$ if and only if

$$\langle f, T^*g\rangle = \langle T^{**}f, g\rangle = \langle Tf, g\rangle, \forall g \in \mathscr{D}(T^*).$$

That is,

$$\langle f, T^*g\rangle - \langle T^*f, g\rangle = 0, \forall g \in \mathscr{D}(T^*).$$

The theorem follows from Green's identity. □

Weyl's 1910 paper played a fundamental role in the development of the Sturm-Liouville theory. Weyl classified the problem into two types.

Theorem 2.43 (Weyl)**.** *Let $c \in (a, b)$. Suppose for some λ_0, every solution f of $Mf = \lambda_0 f$ belongs to $L^2((a, c])$. Then for all λ, every solution f of $Mf = \lambda f$ belongs to $L^2((a, c])$. Similar results hold over the interval $[c, b)$.*

By Weyl's theorem, the differential expression M may classified into two types at the endpoint a (similar at b):

1. If every solution f to the equation $Mf = \lambda f$ belongs to $L^2((a, c])$, M is said to be of limit-circle (l.c.) type at a;
2. Otherwise, M is said to be of limit-point (l.p.) type at a.

Therefore, there are four cases for M:

1. l.p. at both a and b;
2. l.c. at a, l.p. at b;
3. l.p. at a, l.c. at b;
4. l.c. at both a and b.

Note that 2 and 3 can be treated in the same way, with a change of variable $x \mapsto -x$.

Stone's remarkable book [51] provided a detailed analysis of the Sturm-Liouville problem as an illustration of von Neumann's index theory on extension of Hermitian operators in Hilbert space. Even though Stone's book treated a special case of the problem (2.59) with $w(x) \equiv 1$, $x \in (a, b)$, it is not hard to incorporate the general weight function $w(x)$ with additional modifications to some technical details.

Recall that the assumption on the coefficients in (2.59) implies that the minimum operator T commutes with complex conjugation. It follows that T has deficiency indices (m, m), $0 \leq m \leq 2$, and it has selfadjoint extensions. Since any Hermitian extension of T is a restriction of the adjoint T^*, it suffices to specify the domains of the various extensions.

Further analysis of the Hermitian extensions of T would require Weyl's classification of the endpoints. The main results are summarized below.

Theorem 2.44.

1. *The deficiency indices of T is $(2, 2)$ if and only if M is limit-circle type at both endpoints. If $\tilde{T}$ be any selfadjoint extension of T, then the resolvent $(\tilde{T} - \lambda)^{-1}$, $\Im(\lambda) \neq 0$, is a Hilbert-Schmidt integral operator; the point spectrum of $\tilde{T}$ is countablely infinite having no finite accumulation points; all eigenvalues have multiplicity no more than two.*

2. *The deficiency indices of T is $(1, 1)$ if and only if M is limit-point type at one endpoint and limit-circle type at the other endpoint. A selfadjoint extension $\tilde{T}$ of T may have eigenvalues but only of multiplicity one; the continuous spectrum need not be empty.*

3. *The deficiency indices of T is $(0, 0)$ if and only if M is limit-point type at both endpoints. A selfadjoint extension $\tilde{T}$ of T may have eigenvalues but only of*

multiplicity one; the continuous spectrum may not be empty.

2.3.2 Selfadjoint Boundary Conditions

We collect some facts concerning separated selfadjoint boundary conditions. To incorporate applications in later chapters, we restrict to the l.c./l.p. type of the classification, i.e. the endpoint a is limit-circle type, and the endpoint b is limit-point type.

By Theorem 2.44, the assumption on the classification of the endpoints amounts to the minimum operator T having deficiency indices $(1,1)$. von Neumann's index theory shows that T has a one-parameter family of selfadjoint extensions.

Theorem 2.45. *Let M be l.c./l.p. so that T has deficiency indices $(1,1)$. Let $\mathscr{D}_{\pm}(T)$ be the deficiency spaces of T, spanned by the unit vectors $\phi_{\pm} \in \mathscr{D}(T^*)$, satisfying $T^*\phi_{\pm} = \pm i\phi$. Then the family of selfadjoint extensions of T is characterized by*

$$\mathscr{D}(\tilde{T}_\theta) = \{f + c\phi_+ + ce^{i\theta}\phi_- : f \in \mathscr{D}(T), c \in \mathbb{C}, \theta \in [0, 2\pi)\},$$

$$\tilde{T}_\theta(f + c\phi_\nu + ce^{i\theta}\phi_\nu) = Tf + ic\phi_+ - ice^{i\theta}\phi_-.$$

In fact, all selfadjoint domains in the l.c./l.p. classification may be described using the boundary condition at the limit-circle endpoint.

Lemma 2.46. *The endpoint a (or b) is l.p. type if and only if $[f,g](a) = 0$, for all $f, g \in \mathscr{D}(T^*)$.*

Proof. We refer to the elegant treatment in [2], Appendix II. □

Theorem 2.47. *Let M be l.c./l.p. and let $\phi_{\pm}$ and $\tilde{T}_\theta$ as given before. Then*

1. $\mathscr{D}(T) = \{f \in \mathscr{D}(T^*) : [f, \phi_+](0) = [f, \phi_-](a) = 0\}$

2. $\mathscr{D}(\tilde{T}_\theta) = \{f \in \mathscr{D}(T^*) : [f, \phi_\theta](a) = 0\}$, *where* $\phi_\theta := \phi_+ + e^{i\theta}\phi_-$, *and* $\theta \in [0, 2\pi)$.

Proof. By Lemma 2.46 and Theorem 2.42, the only boundary restriction for $f \in \mathscr{D}(T)$ is on the limit-circle endpoint a, i.e. $f \in \mathscr{D}(T)$ if and only if $[f, g](a) = 0$ for all $g \in \mathscr{D}(T^*)$.

If $f \in \mathscr{D}(T)$, it is clear that $[f, \varphi_\pm](a) = 0$. On the other hand, any $f \in \mathscr{D}(T^*)$ has the form

$$f = \varphi_0 + c_+\varphi_+ + c_-\varphi_-$$

for some $\varphi_0 \in \mathscr{D}(T)$ and $c_1, c_2 \in \mathbb{C}$. Thus, $[f, \varphi_\pm](a) = 0$ implies that $f = \varphi_0$, so that $f \in \mathscr{D}(T)$. This proves part 1. The proof of part 2 is similar. □

We will need the following bracket decomposition.

Lemma 2.48. *Let* $f, g, u, v \in \mathscr{D}(T^*)$. *Suppose* $[u, v](c) = 1$, *for some* $c \in [a, b]$. *Then*

$$[f, g](c) = [f, \overline{u}](c)[\overline{g}, v](c) - [f, v](c)[\overline{g}, \overline{u}](c).$$

Proof. Direct computation shows that

$$\begin{aligned}
[f, g] &= (\bar{g}, \bar{g}') \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f \\ f' \end{pmatrix} \\
&= (\bar{g}, \bar{g}') \begin{pmatrix} u' & \overline{v}' \\ -u & -\overline{v} \end{pmatrix} \begin{pmatrix} -\overline{v}' & \overline{v} \\ u' & -u \end{pmatrix} \begin{bmatrix} f \\ f' \end{bmatrix} \\
&= ([\overline{g}, \overline{u}], [\overline{g}, v]) \begin{pmatrix} -[f, v] \\ [f, \overline{u}] \end{pmatrix} \\
&= [f, \overline{u}][\overline{g}, v] - [f, v][\overline{g}, \overline{u}].
\end{aligned}$$

□

Theorem 2.49. *Let M be l.c./l.p. Let $c \in (a,b)$, $\lambda \in \mathbb{R}$. Suppose u, v are non-trivial real solutions to the equation $ly = \lambda y$ on (a,c), normalized to satisfy $[u,v](t) = 1$, for all $t \in (a,c)$. The family of selfadjoint extensions of T is given by*

$$\begin{aligned}\mathscr{D}(\tilde{T}_{c_1,c_2}) = &\{f \in \mathscr{D}(T^*) : c_1[f,u](a) + c_2[f,v](a) = 0 \\ &c_1, c_2 \in \mathbb{R}, c_1^2 + c_2^2 \neq 0\}\end{aligned}$$

Equivalently,

$$\mathscr{D}(\tilde{T}_\alpha) = \{f \in \mathscr{D}(T^*) : [f, \phi_\alpha](a) = 0, \alpha \in \mathbb{R}\}$$

where $\phi_\alpha := \alpha u + v$, and setting $\phi_\infty := u$. If, in addition, u is the principal solution, then

$$\mathscr{D}(\tilde{T}_\infty) = \{f \in \mathscr{D}(T^*) : [f,u](0) = 0\} \tag{2.62}$$

gives rise to the Friedrichs extension.

Proof. Let $\tilde{T}_\theta$ be a selfadjoint extension of T as in Theorem 2.47. We show $\mathscr{D}(\tilde{T}_\theta)$ may be characterized as $\mathscr{D}(\tilde{T}_{c_1,c_2})$ for some $c_1, c_2 \in \mathbb{R}$. Note that by Naimark's patching lemma, u, v extends to functions in $\mathscr{D}(T^*)$. For all $f, \phi \in \mathscr{D}(T^*)$,

$$\begin{aligned}[f,\phi] &= [\bar{\phi}, \overline{\phi'}] \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f \\ f' \end{bmatrix} \\ &= [\bar{\phi}, \overline{\phi'}] \begin{bmatrix} u' & v' \\ -u & -v \end{bmatrix} \begin{bmatrix} -v' & v \\ u' & -u \end{bmatrix} \begin{bmatrix} f \\ f' \end{bmatrix} \\ &= [f,u][\overline{\phi'}, v] - [f,v][\bar{\phi}, u]. \end{aligned} \tag{2.63}$$

By Theorem 2.47, $f \in \mathscr{D}(\tilde{T}_\theta)$ if and only if $[f, \phi_+ + e^{i\theta}\phi_-](a) = 0$. The result follows by setting $\phi = \phi_{\nu,+} + e^{i\theta}\phi_{\nu,-}$.

To see $\mathscr{D}(\tilde{T}_{c_1,c_2})$ gives rise to all selfadjoint extensions, we refer to [54], Theorem 10.4.5. □

Remark 2.50. In fact, the decomposition in (2.63) is valid for all real-valued pairs u, v in $\mathscr{D}(T^*)$, satisfying $[u, v] = 1$ over $(a, c) \subset (a, b)$. The conditions on u, v can be further relaxed, see [54].

2.3.3 Example: Bessel Differential Operator

As an example of the general theory, we consider the classical Bessel differential operator defined on the half-line. Properties of this operator is well known, see for example [52][2]. The selfadjoint extension problem can be seen as a special case of a class of Schrödinger operators considered in [9], also see [3]. Its Friedrichs extension is studied in the more recent paper [17].

Definition 2.51. Let l be the Bessel differential operator of order ν on $(0, \infty)$, i.e.

$$\mathscr{D}(l) = \{f : (a, b) \to \mathbb{C} : f, lf \in AC_{loc}(0, \infty)\} \tag{2.64}$$

$$lf = -\frac{d^2 f}{dx^2} + \frac{\nu^2 - 1/4}{x^2} f. \tag{2.65}$$

Throughout, we restrict to $\nu \in [0, 1)$. In this case, the endpoint ∞ is strong limit-point and Dirichlet; and the endpoint 0 is limit-circle non-oscillatory, except for the regular case when $\nu = 1/2$, which can be treated as limit-circle non-oscillatory. A detailed classification of singularities can be found in [16] and the reference therein.

We consider operators associated with l_0 in the Hilbert space $L^2(0, \infty)$. Let

$$h_\nu := \overline{l\big|_{\mathscr{C}_c^\infty(0,\infty)}} \tag{2.66}$$

be the minimum operator, and h_ν^* the adjoint of h_ν. From the general theory, we have

$$\mathscr{D}(h_\nu^*) = \{f \in \mathscr{D}(l) : f, lf \in L^2(0,\infty)\} \tag{2.67}$$

$$\mathscr{D}(h_\nu) = \{f \in \mathscr{D}(h_\nu^*) : [f,g](0) = 0, \forall g \in \mathscr{D}(h_\nu^*)\} \tag{2.68}$$

Notice that only boundary conditions on the limit-circle endpoint is needed, since for $\nu \in [0,1]$, the endpoint ∞ is of limit-point type.

If $\nu \geq 1$, the operator h_ν is selfadjoint. For $\nu \in [0,1)$, h_ν has deficiency indices $(1,1)$. Let

$$\mathscr{D}_\pm(h_\nu) = span\{\phi_{\nu,\pm}\} \tag{2.69}$$

be the deficiency spaces such that

$$h_\nu^* \phi_{\nu,\pm} = \pm i \phi_{\nu,\pm}. \tag{2.70}$$

The defect vectors are given by

$$\phi_{\nu,+}(x) = x^{1/2} H_\nu^{(1)}(x\sqrt{i}) \tag{2.71}$$

$$\phi_{\nu,-}(x) = x^{1/2} H_\nu^{(2)}(x\sqrt{-i}) \tag{2.72}$$

where $H_\nu^{(1)}$, $H_\nu^{(2)}$ are the Hankel functions of order ν [53][2].

By von Neumann's theory on extensions of Hermitian operators, h_ν has a one-paramter family of selfadjoint extensions:

$$\mathscr{D}(\tilde{h}_{\nu,\theta}) = \{f + c\phi_{\nu,+} + ce^{i\theta}\phi_{\nu,-} : f \in \mathscr{D}(h_\nu), c \in \mathbb{C}, \theta \in [0,2\pi)\}, \tag{2.73}$$

$$\tilde{h}_{\nu,\theta}(f + c\phi_{\nu,+} + ce^{i\theta}\phi_{\nu,-}) = h_\nu f + ic\phi_{\nu,+} - ice^{i\theta}\phi_{\nu,-}. \tag{2.74}$$

In applications, it is convenient to specify the selfadjoint domains using boundary conditions at the limit-circle endpoint.

Corollary 2.52. *Let $\nu \in [0, 1)$. Suppose u, v is a fundamental system of solutions to the homogeneous equation $ly = 0$, such that $[u, v] = 1$ and let u be the principal solution. The family of selfadjoint extensions of h_ν is characterized by*

$$\mathscr{D}(\tilde{h}_{\nu,\alpha}) = \{f \in \mathscr{D}(h_\nu^*) : [f, \phi_\alpha](0) = 0, -\infty < \alpha \leq \infty\} \tag{2.75}$$

where $\phi_\alpha := \alpha u + v$, setting $\phi_\infty := u$. Moreover, $\alpha = \infty$ amounts to the Friedrichs extension.

Further analysis of the selfadjoint boundary conditions at the limit-circle endpoint can be found as a special case of a class of Schrödinger operators [9]

$$-\frac{d^2}{dr^2} + \frac{\lambda(\lambda - 1)}{r^2} + \frac{\gamma}{r} + \frac{\beta}{r^s} + W(r), r > 0 \tag{2.76}$$

where $W \in L^\infty(0, \infty)$ real-valud, $\lambda \in [1/2, 3/2)$, $\beta, \gamma \in \mathbb{R}$, $s \in (0, 2)$. It reduces to (2.65), when $\lambda = \nu + 1/2$, $\gamma = \beta = W = 0$. See also [3].

Theorem 2.53. *Let $\nu \in [0, 1)$. Suppose u, v is a fundamental system of solutions to the homogeneous equation $ly = 0$, such that $[u, v] = 1$ and let u be the principal solution. Then the family of selfadjoint extensions of h_ν is characterized by*

$$\mathscr{D}(\tilde{h}_{\nu,\alpha}) = \{f \in \mathscr{D}(h_\nu^*) : \alpha f_0 = f_1, -\infty < \alpha \leq \infty\} \tag{2.77}$$

where

$$f_0 := \lim_{x \to 0+} f(x)/v(x) \tag{2.78}$$

$$f_1 := \lim_{x \to 0+} (f(x) - f_0 v(x))/u(x) \tag{2.79}$$

Moreover, $\alpha = \infty$, i.e. $f_0 = 0$, amounts to the Friedrichs extension.

Proof. Any $f \in \mathscr{D}(h_\nu^*)$ can be written as

$$f(x) = c_1 u + c_2 v + v \int_{x_0}^{x} u\,(h_\nu^* f) - u \int_{x_0}^{x} v\,(h_\nu^* f)$$

for some $x_0 > 0$. Direct computation shows that

$$[f, \alpha u + v](x) = c_1 - \alpha c_2 + \alpha \int_{x_0}^{x} u\,(h_\nu^* f) - \int_{x_0}^{x} v\,(h_\nu^* f)\,.$$

It follows that $f \in \mathscr{D}(\tilde{h}_{\nu,\alpha})$ if and only if

$$c_1 - \int_{x_0}^{0} v\,(h_\nu^* f) = \alpha(c_2 + \int_{x_0}^{0} u\,(h_\nu^* f)).$$

But

$$f/v \sim f_0 := c_2 + \int_{x_0}^{0} u\,(h_\nu^* f)$$

$$\begin{aligned} f/u &= c_1 - \int_{x_0}^{x} v\,(h_\nu^* f) + (v/u)\left(c_2 + \int_{x_0}^{x} u\,(h_\nu^* f)\right) \\ &\sim c_1 - \int_{x_0}^{0} v\,(h_\nu^* f) + v f_0/u \end{aligned}$$

that is,

$$(f - v f_0)/u \sim c_1 - \int_{x_0}^{0} v\,(h_\nu^* f)\,.$$

This proves the first part. The second part is immediate. $\square$

Remark 2.54. We record a fundamental system of solutions to the homogeneous equation $ly = 0$.

- $\nu = 0$, $u = \sqrt{x}$, $v = \sqrt{x}\log x$
- $\nu = 1/2$, $u = x$, $v = -1$

- $\nu \in (0, 1/2) \cup (1/2, \infty)$, $u = x^{\nu+1/2}$, $v = -x^{-\nu+1/2}/2\nu$.

In all cases, $[u, v] = 1$ and u is the principal solution.

Corollary 2.55. *Consider the family of selfadjoint extensions of h_ν characterized by (2.77). Let $f \in \mathscr{D}(h_\nu^*)$. Then (2.78) and (2.79) are given by*

1. $\nu = 0$,

$$\begin{aligned} f_0 &= \lim_{x \to 0+} f(x) / \left(\sqrt{x} \log x \right) \\ f_1 &= \lim_{x \to 0+} \left(f(x) - f_0 \sqrt{x} \log x \right) / \sqrt{x} \end{aligned}$$

2. $\nu \in (0, 1/2) \cup (1/2, 1)$

$$\begin{aligned} f_0 &= \lim_{x \to 0+} f(x) / x^{-\nu+1/2} \\ f_1 &= \lim_{x \to 0+} (f(x) - f_0 x^{-\nu+1/2}) / x^{\nu+1/2} \end{aligned}$$

3. $\nu = 1/2$ *(the endpoint 0 is reguar)*

$$\mathscr{D}(\tilde{h}_{\nu,\alpha}) = \{ f \in \mathscr{D}(h_\nu^*) : \alpha f(0) + f'(0) = 0, -\infty < \alpha \leq \infty \}.$$

CHAPTER 3
*-ALGEBRAS AND REPRESENTATIONS

In this chapter, we review the basic theory of $*$-algebras and their representations. These algebras are to be represented as operators acting on a common invariant domain in some Hilbert space, and in general, such operators are unbounded. In the special case when the algebra has one generator, the theory reduces to the study of a single unbounded operator.

The main application is when the $*$-algebra comes from the universal enveloping algebra of a Lie algebra. Starting with a unitary representation of a Lie group, one derives a representation of the Lie algebra, and the derived representation extends uniquely to a selfadjoint representation of the enveloping algebra in the sense of R.T. Powers [35]. We will study the converse problem. That is, starting with a selfadjoint representation ρ of the enveloping algebra $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, we study the conditions under which ρ is derived from some unitary representation U of the Lie group, i.e. $\rho = dU$. Elements of the form $dU(x)$ include important quantum mechanical operators.

For the general notions, we refer to [25], also see [35, 36] and [46].

3.1 Preliminaries

The notion of densely defined Hermitian operators has a direct generalization in the setting of representation of $*$-algebras.

3.1.1 Domain, Hermitian Representations

A $*$-algebra $\mathfrak{A}$ is a complex algebra with an involution, $x \mapsto x^*$, $x \in \mathfrak{A}$, i.e. a period-2, conjugate linear, anti-automorphism. Throughout, we always assume that $\mathfrak{A}$ contains identity. A representation of $\mathfrak{A}$ is a homomorphism

$$\rho : \mathfrak{A} \to End(\mathscr{D}(\rho)) \tag{3.1}$$

from $\mathfrak{A}$ into endomorphisms over a dense subspace $\mathscr{D}(\rho)$ in a Hilbert space $\mathscr{H}$. $\mathscr{D}(\rho)$ is called the domain of ρ. It carries the projective topology induced by the family of semi-norms

$$\|a\|_F := \sum_{x \in F} \|\rho(x)a\|, \ a \in \mathscr{D}(\rho) \tag{3.2}$$

where F runs through all finite subsets of $\mathfrak{A}$. ρ is said to be closed if $\mathscr{D}(\rho)$ is complete with respect to this topology.

For two representations ρ_1, ρ_2 of $\mathfrak{A}$ on the same Hilbert space, ρ_2 is said to be an extension of ρ_1, denoted by $\rho_1 \subset \rho_2$, if $\rho_1(x) \subset \rho_2(x)$ for all $x \in \mathfrak{A}$.

A representation ρ of $\mathfrak{A}$ is called Hermitian if $\rho(x) \subset \rho(x^*)^*$, for all $x \in \mathfrak{A}$. That is, for all $a, b \in \mathscr{D}(\rho)$, and $x \in \mathfrak{A}$,

$$\langle b, \rho(x)a \rangle = \langle \rho(x^*)b, a \rangle \tag{3.3}$$

Every Hermitian representation ρ is associated with its adjoint representation ρ^*, given by

$$\mathscr{D}(\rho^*) := \bigcap_{x \in \mathfrak{A}} \mathscr{D}(\rho(x)^*) \tag{3.4}$$

$$\rho^*(x) := \rho(x^*)^*\big|_{\mathscr{D}(\rho^*)} \tag{3.5}$$

Thus, ρ is Hermitian if and only if $\rho \subset \rho^*$. By definition, $\mathscr{D}(\rho^*)$ contains $\mathscr{D}(\rho)$, and so it is dense in $\mathscr{H}$. The fact that ρ^* defines a representation is due to the following lemma.

Lemma 3.1. *$\mathscr{D}(\rho^*)$ is dense in $\mathscr{H}$ if and only if ρ^* is a representation of $\mathfrak{A}$.*

Proof. If ρ^* is a representation of $\mathfrak{A}$, then $\mathscr{D}(\rho^*)$ is dense in $\mathscr{H}$ by definition. Conversely, suppose $\mathscr{D}(\rho^*)$ is dense in $\mathscr{H}$. For all $a \in \mathscr{D}(\rho^*)$, $b \in \mathscr{D}(\rho)$ and $x, y \in \mathfrak{A}$,

$$\begin{aligned}
\langle \rho(x^*)b, \rho^*(y)a \rangle &= \langle \rho(x^*)b, \rho(y^*)^*a \rangle \\
&= \langle \rho(y^*)\rho(x^*)b, a \rangle \\
&= \langle \rho(y^*x^*)b, a \rangle \\
&= \langle b, \rho(y^*x^*)^*a \rangle \\
&= \langle b, \rho^*(xy)a \rangle.
\end{aligned}$$

That is,

$$\begin{aligned}
\rho^*(y)a &\in \mathscr{D}(\rho(x^*)^*) \\
\langle b, \rho^*(x)\rho^*(y)a \rangle &= \langle b, \rho^*(xy)a \rangle.
\end{aligned}$$

Since x, y, a, b were arbitrary, we conclude that

$$\begin{aligned}
\rho^*(y) &\in End(\mathscr{D}(\rho^*)) \\
\rho^*(xy) &= \rho^*(x)\rho^*(y)
\end{aligned}$$

Therefore, ρ^* is a representation of $\mathfrak{A}$. $\square$

Lemma 3.2. *ρ^* is a closed representation.*

Proof. Let (a_s) be a Cauchy net in $\mathscr{D}(\rho^*)$. Then $\lim_s a_s = a$, for some a. For all $x \in \mathfrak{A}$, $\rho(x)^*$ is a closed operator and $a_s \in \mathscr{D}(\rho(x)^*)$, it follows that $\lim_s \rho(x)^* a_s = \rho(x)^* a$. Thus, $a \in \mathscr{D}(\rho(x)^*)$, for all $x \in \mathfrak{A}$. This shows that $a \in \mathscr{D}(\rho^*)$. □

Every Hermitian representation ρ has a unique closure $\overline{\rho}$. We record the following theorem whose proof can be found in [35].

Theorem 3.3. *If ρ is a Hermitian representation of a $*$-algebra $\mathfrak{A}$, then ρ has a minimum closed extension $\bar{\rho}$, given by*

$$\begin{aligned} \mathscr{D}(\bar{\rho}) &:= \bigcap_{x \in \mathfrak{A}} \mathscr{D}(\overline{\rho(x)}) && (3.6) \\ \bar{\rho}(x) &:= \rho(x^*)^* \big|_{\mathscr{D}(\bar{\rho})} && (3.7) \end{aligned}$$

Moreover,

$$\rho \subset \bar{\rho} \subset \rho^*. \tag{3.8}$$

3.1.2 Selfadjoint Representations

The notion of selfadjoint representation is an extension of the notion of selfadjoint operator in the theory of single unbounded operators. Selfadjoint representations have important applications in the theory of unitary representation of Lie group and quantum field theory.

Let ρ be a Hermitian representation of a $*$-algebra $\mathfrak{A}$. ρ is called essentially selfadjoint if $\bar{\rho} = \rho^*$, and selfadjoint if $\rho = \rho^*$.

Just as a selfadjoint operator is maximally Hermitian (having no proper Hermitian extensions), a selfadjoint representation is maximally Hermitian. This means no boundary conditions have been overlooked in physical problems.

Lemma 3.4. *Let ρ be a Hermitian representation of a $*$-algebra $\mathfrak{A}$.*

1. *ρ is selfadjoint if and only if $\mathscr{D}(\rho) = \mathscr{D}(\rho^*)$.*

2. *If ρ is selfadjoint, it is maximally Hermitian.*

Proof. By assumption, $\rho \subset \rho^*$. Thus $\rho = \rho^*$ if and only if $\mathscr{D}(\rho) = \mathscr{D}(\rho^*)$. On the other hand, let π be a Hermitian representation of $\mathfrak{A}$, such that $\rho \subset \pi$, then

$$\rho \subset \pi \subset \pi^* \subset \rho^*.$$

Thus $\rho = \rho^*$ implies that $\pi = \rho$. That is, ρ has no proper Hermitian extensions. $\square$

The following result for checking selfadjointness is due to Powers. Its higher dimensional analogue is not true, as first demonstrated by Nelson [32]. We also refer to section 5 in [35].

Theorem 3.5 ([35])**.** *Let $\mathfrak{A}$ be the free commutative $*$-algebra on a single Hermitian generator x. A Hermitian representation ρ of $\mathfrak{A}$ is essentially selfadjoint if and only if $\rho(x)^n$, $n \in \mathbb{N}$, is essentially selfadjoint.*

Proof. ([25]) For an operator T in $\mathscr{H}$, we introduce the notion

$$\mathscr{D}^\infty(T) := \bigcap_{n=1}^{\infty} \mathscr{D}(T^n).$$

Let $A := \rho(x)$. Note that

$$\begin{aligned} \mathscr{D}(\overline{\rho}) &= \mathscr{D}^\infty(\overline{A}) \\ \mathscr{D}(\rho^*) &= \mathscr{D}^\infty(A^*) \end{aligned}$$

since elements in $\mathfrak{A}$ are polynomials in the x variable with complex coefficients.

Suppose $\overline{A^n}$ is selfadjoint for all $n \in \mathbb{N}$. Then $\mathscr{D}^\infty(\overline{A}) = \mathscr{D}^\infty(A^*)$, and so $\mathscr{D}(\bar{\rho}) = \mathscr{D}(\rho^*)$. This implies that $\bar{\rho} = \rho^*$, by Lemma 3.4.

Conversely, suppose $\bar{\rho} = \rho^*$. Let $a \in \mathscr{D}(A^*)$, such that $A^* a = \pm ia$. Then

$$\begin{aligned} a &\in \mathscr{D}^\infty(A^*) \\ (A^*)^n a &= (\pm i)^n a, \forall n \in \mathbb{N}. \end{aligned}$$

By assumption, $\mathscr{D}^\infty(\overline{A}) = \mathscr{D}^\infty(A^*)$, thus $a \in \mathscr{D}(\overline{A})$ and $\overline{A}a = \pm ia$. Since $\overline{A}$ is a Hermitian operator, it has no imaginary eigenvalues, and so $a = 0$. That is, the deficiency spaces of $\overline{A}$ are trivial. Therefore, $\overline{A}$ is selfadjoint by von Neumann's theory on selfadjoint extensions.

It remains to show that $\overline{\rho(x)^n}$, $n > 1$, is selfadjoint. Let ρ_0 be the restriction of ρ to the subalgebra $\mathfrak{A}_0$ generated by x^n, such that $\mathscr{D}(\rho_0) = \mathscr{D}(\rho)$. Let $B := \rho(x)^n = \rho(x^n)$. Observe that

$$\begin{aligned} \mathscr{D}(\overline{\rho_0}) &= \mathscr{D}(\overline{\rho}) \\ \mathscr{D}(\rho_0^*) &= \mathscr{D}(\rho^*) \end{aligned}$$

therefore, $\overline{\rho_0} = \rho_0^*$. The previous argument shows that $\overline{B}$ is selfadjoint. $\square$

3.1.3 Commutants

Let ρ be a Hermitian representation of a $*$-algebra $\mathfrak{A}$ on a Hilbert space $\mathscr{H}$. The commutant $\mathscr{M}$ of $\rho(\mathfrak{A})$ consists of all bounded operators B in $\mathscr{H}$, such that B commutes weakly with $\rho(x)$, $x \in \mathfrak{A}$.

Lemma 3.6. *The following are equivalent.*

1. $B \in \mathscr{M}$.

2. $B\rho(x) \subset \rho^*(x)B$, *for all* $x \in \mathfrak{A}$.

3. $\langle b, B\rho(x)a\rangle = \langle \rho(x^*)b, Ba\rangle$, *for all* $a, b \in \mathscr{D}(\rho)$, $x \in \mathfrak{A}$.

If ρ *is selfadjoint, the above conditions are equivalent to* $B\rho(x) \subset \rho(x)B$, *for all* $x \in \mathfrak{A}$. *That is,* B *commutes strongly with* $\rho(x)$, $x \in \mathfrak{A}$. *In that case, strong and weak commutativity coincide.*

Proof. It all follows from the definitions.

□

Lemma 3.7. $\mathscr{M}$ *is weak* $*$ *closed and selfadjoint.*

Proof. The fact that $\mathscr{M}$ is weak $*$ closed follows directly from the definition. Let B in $\mathscr{M}$, then

$$\begin{aligned} \langle b, B^*\rho(x)a\rangle &= \langle Bb, \rho(x)a\rangle \\ &= \langle B\rho(x^*)b, a\rangle \\ &= \langle \rho(x^*)b, B^*a\rangle \end{aligned}$$

for all $a, b \in \mathscr{D}(\rho)$, $x \in \mathfrak{A}$. Thus $B^* \in \mathscr{M}$. Conversely, $B^* \in \mathscr{M}$ implies that $B^{**} = B \in \mathscr{M}$. □

The commutant $\mathscr{M}$ of a Hermitian representation of a $*$-algebra may fail to be closed under multiplication. If, in addition, ρ is selfadjoint, then $\mathscr{M}$ is multiplicative and thus a von Neumann algebra.

Theorem 3.8. *Let ρ be a selfadjoint representation of a $*$-algebra $\mathfrak{A}$ on a Hilbert space $\mathscr{H}$. Then the commutant $\mathscr{M}$ of $\rho(\mathfrak{A})$ is a von Neumann algebra.*

Proof. It remains to check that $\mathscr{M}$ is closed under multiplication. Let $B_1, B_2 \in \mathscr{M}$, then

$$\begin{aligned} B_1 B_2 \rho(x) &\subset B_1 \rho^*(x) B_2 \\ &= B_1 \rho(x) B_2 \\ &\subset \rho^*(x) B_1 B_2 \\ &= \rho(x) B_1 B_2 \end{aligned}$$

for all $x \in \mathfrak{A}$. Therefore, $B_1 B_2 \in \mathscr{M}$. □

Theorem 3.9 ([35])**.** *Let ρ be a selfadjoint representation of a $*$-algebra $\mathfrak{A}$ on a Hilbert space $\mathscr{H}$, and let $\mathscr{M}$ be the commutant of $\rho(\mathfrak{A})$. There is a bijection between the lattice of projections in $\mathscr{M}$ and the lattice of reducing closed subspaces of $\mathscr{H}$. Moreover, ρ restricts to a selfadjoint representation on each of its closed reducing subspaces.*

Proof. Let P be a projection in $\mathscr{M}$. If ρ is selfadjoint, then $B\rho(x) \subset \rho(x)B$, for all $x \in \mathfrak{A}$. It follows that $P\mathscr{H}$ is a reducing closed subspace. Conversely, suppose $\mathscr{K}$ is a reducing closed subspace. Let P be the projection onto $\mathscr{K}$. Then

$$P\rho(x)a = \rho(x)Pa = \rho^*(x)Pa$$

for all $a \in \mathscr{D}(\rho)$, $x \in \mathfrak{A}$. That is, $P \in \mathscr{M}$. Finally, ρ restricts to a representation $\rho_P := \rho\big|_{P\mathscr{H}}$ on $P\mathscr{H}$ with $\mathscr{D}(\rho_P) = P\mathscr{D}(\rho)$. □

3.1.4 Derived Representations

Let U be a unitary representation of a Lie group G. One may differentiate U along various directions in the Lie algebra $\mathfrak{g}$, and derive a representation dU of the universal enveloping algebra $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$, that is, the algebra of polynomials in the elements of $\mathfrak{g}$ modulo the commutation relations of $\mathfrak{g}$. The derived representation is selfadjoint. The converse problem is more interesting. Let ρ be a selfadjoint representation of $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$, does there exist a unitary representation U of G, such that $\rho = dU$? If this is true, we say ρ is integrable or exact. The answer is negative in general. We will study examples of non-integrable representations in the next chapter.

Let U be a unitary representation of a Lie group G with Lie algebra $\mathfrak{g}$. Throughout, strong continuity will always be assumed. For all $x \in \mathfrak{g}$,

$$U(\exp(tx)), t \in \mathbb{R} \tag{3.9}$$

is a stronlgy continuous one-parameter unitary group. By Stone's theorem, it has an infinitesimal generator, formally given by

$$dU(x) := \frac{d}{dt}\Big|_{t=0} U(\exp(tx)). \tag{3.10}$$

Here, $\exp : \mathfrak{g} \to G$ is the exponential mapping. $\mathscr{D}(dU(x))$ consists of all $a \in \mathscr{H}$, such that

$$\lim_{t \to 0} \frac{U(\exp(tx)) - 1}{t} a \tag{3.11}$$

exists. $dU(x)$ is skew-adjoint in the sense that

$$\langle b, dU(x)a \rangle = -\langle dU(x)b, a \rangle \tag{3.12}$$

for all $a, b \in \mathscr{D}(dU(x))$, $x \in \mathfrak{g}$.

Recall that $\mathscr{D}^\infty(U)$ denotes the space of C^∞-vectors for U consisting of all $a \in \mathscr{H}$, such that

$$g \mapsto U(g)a \tag{3.13}$$

is a smooth mapping from G into $\mathscr{H}$. The Gårding space $\mathscr{D}_G(U)$ of U is the linear span of the vectors

$$U(\varphi)a := \int_G \varphi(g)U(g)a\, dg \tag{3.14}$$

for all $\varphi \in \mathscr{C}_c^\infty(G)$, $a \in \mathscr{H}$, and dg is a left-invariant Haar measure on G.

Theorem 3.10. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and U a representation of G on a Hilbert space $\mathscr{H}$.*

1. *$\mathscr{D}_G(U)$ is dense in $\mathscr{H}$.*

2. *For all $g \in G$, $\varphi \in \mathscr{C}_c^\infty(G)$ and $a \in \mathscr{H}$,*

$$U(g)U(\varphi)a = U(\varphi(g^{-1}\cdot))a \tag{3.15}$$

3. *For all $\varphi \in \mathscr{C}_c^\infty(G)$, $a \in \mathscr{H}$ and $x \in \mathfrak{g}$,*

$$dU(x)U(\varphi)a = U(\tilde{x}\varphi)a \tag{3.16}$$

where $\tilde{x}$ is the right-invariant vector field, given by

$$(\tilde{x}\varphi)(g) := \frac{d}{dt}\Big|_{t=0}\varphi(\exp(-tx)g) \tag{3.17}$$

4. *The map*

$$dU : \mathfrak{g} \to End(\mathscr{D}_G(U)) \tag{3.18}$$

is a homomorphism.

Setting $1^* = 1$, $x^* = -x$, for all $x \in \mathfrak{g}$. The $*$ operation extends uniquely to an involution on the enveloping algebra. Restrict the operators $dU(x)$, $x \in \mathfrak{g}$, to $\mathscr{D}_G(U)$. Then (3.12) reads

$$\langle b, dU(x)a\rangle = \langle dU(x^*)b, a\rangle \tag{3.19}$$

for all $a, b \in \mathscr{D}_G(U)$. Combined with (3.18), dU may be viewed as a Hermitian representation of $\mathfrak{g}$, and it extends uniquely to a Hermitian representaiton

$$dU : \mathfrak{A}_{\mathbb{C}}(\mathfrak{g}) \to End(\mathscr{D}_G(U)) \tag{3.20}$$

of the enveloping algebra by its universal property. (3.20) is called the derived representation of U, and its domain is equal to the Gårding space $\mathscr{D}_G(U)$.

Proof of Theorem 3.10.

1. Choose an approximation of identity $\varphi_\epsilon \in \mathscr{C}_c^\infty(G)$. That is, $\varphi_\epsilon \geq 0$, $\int \varphi_\epsilon(g)dg = 1$, and the support of φ_ϵ shrinks to the identity element in G, as $\epsilon \to 0+$. For all $a \in \mathscr{H}$,

$$\begin{aligned} \|U(\varphi_\epsilon)a - a\| &= \|\int \varphi_\epsilon(g)(U(g) - 1)a\, dg\| \\ &\leq \sup_{g \in support(\varphi_\epsilon)} \|(U(g) - 1)a\|. \end{aligned}$$

   Thus, $\mathscr{D}_G(U)$ is dense in $\mathscr{H}$.

2. Direct computation shows that

$$\begin{aligned} U(g)U(\varphi)a &= \int_G \varphi(h)U(g)U(h)a\,dh \\ &= \int_G \varphi(h)U(gh)a\,dh \\ &= \int_G \varphi(g^{-1}h)U(h)a\,dh \\ &= U(\varphi(g^{-1}\cdot))a \end{aligned}$$

using the fact that dg is a left-invariant Haar measure on G.

3. By (3.15),

$$\begin{aligned} dU(x)U(\varphi)a &= \frac{d}{dt}\Big|_{t=0} U(\exp(tx))U(\varphi)a \\ &= \frac{d}{dt}\Big|_{t=0} U(\varphi(\exp(-tx)\cdot))a \\ &= U(\tilde{x}\varphi)a \end{aligned}$$

This yields (3.16) and (3.17).

4. By (3.16), $dU(x) \in End(\mathscr{D}_G(U))$ for all $x \in \mathfrak{g}$. It is clear that dU is linear. For all $a, b \in \mathscr{D}_G(U)$ and $x, y \in \mathfrak{g}$,

$$\begin{aligned} &\langle b, dU([x,y])a\rangle \\ = \;& \frac{d}{ds}\frac{d}{dt}\Big|_{s,t=0}\langle b, U\left(\exp(tx)\exp(sy)\exp(-tx)\right)a\rangle \\ = \;& \frac{d}{ds}\frac{d}{dt}\Big|_{s,t=0}\langle b, U\left(\exp(tx)\right)U\left(\exp(sy)\right)U\left(\exp(-tx)\right)a\rangle \\ = \;& \langle b, [dU(x), dU(y)]a\rangle \end{aligned}$$

Thus, dU has the representation property

$$dU([x,y]) = [dU(x), dU(y)] \tag{3.21}$$

which holds on $\mathscr{D}_G(U)$.

$\square$

It follows from (3.17) that elements in the enveloping algebra are identified with right (left)-invariant analytic partial differential operators on G, and $\mathfrak{g}$ is the Lie algebra of all right (left)-invariant vector fields on G. The 'right' or 'left' depends on the choice of a Haar measure in the definition of the Gårding vectors.

By Theorem 3.10, $\mathscr{D}_G(U)$ is invariant under the group actions, and $\mathscr{D}_G(U) \subset \mathscr{D}^\infty(U)$. If $\mathscr{D}^\infty(U)$ is equipped with the projective topology induced by the operators $\overline{dU(x)}$, $x \in \mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$, the following theorem shows that $\mathscr{D}_G(U)$ is dense in $\mathscr{D}^\infty(U)$.

Theorem 3.11 (Theorem 10.1.14 [46]). *Let U be a unitary representation of a Lie group G on a Hilbert space $\mathscr{H}$. Suppose $\mathscr{D}$ is a dense subspace in $\mathscr{H}$, contained in $\mathscr{D}^\infty(U)$, and invariant under $U(g)$, $g \in G_0$, where G_0 is the connected component in G containing the identity element. Then $\mathscr{D}$ is dense in $\mathscr{D}^\infty(U)$.*

Proof. Let $a \in \mathscr{D}^\infty(U)$. For all $x \in \mathfrak{g}$,

$$\begin{aligned} dU(Ad_g(x))a &= \frac{d}{dt}\Big|_{t=0} U(e^{t\cdot Ad_g(x)})a \\ &= \frac{d}{dt}\Big|_{t=0} U(ge^{tx}g^{-1})a \\ &= \frac{d}{dt}\Big|_{t=0} U(g)U(e^{tx})U(g^{-1})a \\ &= U(g)dU(x)U(g^{-1})a. \end{aligned}$$

Thus,

$$\begin{aligned} \|dU(x)U(g)a\| &= \|U(g)dU(Ad_{g^{-1}}(x))a\| \\ &= \|dU(Ad_{g^{-1}}(x))a\| \end{aligned}$$

and this extends to all $x \in \mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$. It follows that the map $a \mapsto U(g)a$ is continuous, from $\mathscr{D}^\infty(U)$ into $\mathscr{D}^\infty(U)$. Therefore, we may assume $\mathscr{D}$ is closed in $\mathscr{D}^\infty(U)$.

Let $b \in \mathscr{D}$, $\varphi \in \mathscr{C}_c^\infty(G_0)$. Then $U(g)b \in \mathscr{D}$, for all $g \in G_0$. Note that $dU(x)$, $x \in \mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$, is continuous in $\mathscr{D}^\infty(U)$, closable in $\mathscr{H}$, and

$$dU(x)U(\varphi)b = \int_{G_0} \varphi(g) dU(x)U(g)b\, dg.$$

That is, the Riemann sum of $U(\varphi)b$ converges in $\mathscr{D}^\infty(U)$. Since $\mathscr{D}$ is closed in $\mathscr{D}^\infty(U)$, we conclude that $U(\varphi)b \in \mathscr{D}$. Since $\mathscr{D}$ is dense in $\mathscr{D}^\infty(U)$, it follows that $U(\varphi)a \in \mathscr{D}$, for all $a \in \mathscr{D}^\infty(U)$. A standard approximation shows that every $a \in \mathscr{D}^\infty(U)$ is the limit in of a sequence $U(\varphi_n)a \in \mathscr{D}$. This shows that $\mathscr{D}$ is dense in $\mathscr{D}^\infty(U)$. $\square$

In fact, the two spaces coincide due to Dixmier and Malliavin. In particular, every C^∞-vector can be written as a finite linear combination of Gårding vectors.

Theorem 3.12 ([12]). *Let π be a continuous representation of a Lie group G on a Fréchet space E, then the Gårding space coincides with the space of C^∞-vectors.*

Corollary 3.13. *Let dU be the derived representation in (3.20). If $\mathscr{D}$ is a dense subspace in $\mathscr{H}$, such that $\mathscr{D} \subset \mathscr{D}_G(U)$ and $\mathscr{D}$ is invariant under $U(g)$, $g \in G$, then*

$$\overline{dU(x)\big|_{\mathscr{D}}} = \overline{dU(x)}, x \in \mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$$

i.e. $\mathscr{D}$ is a core of $dU(x)$, for all $x \in \mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$.

The fact that the derived representation is selfadjoint is due to Powers. Its proof is based on characterizing C^∞-vectors using elliptic elements in the enveloping algebra, combined with Theorem 3.12. We need the following technical lemma.

Lemma 3.14. *Let U be a unitary representation of a Lie group G on a Hilbert space $\mathscr{H}$. Let $a \in \mathscr{H}$. The following are equivalent.*

1. *The map $g \mapsto U(g)a$ is smooth.*

2. *The map $g \mapsto \langle b, U(g)a \rangle$ is smooth, for all $b \in \mathscr{H}$.*

3. *The map $g \to \langle U(g^{-1})b, a \rangle$ is smooth, for all $b \in \mathscr{H}$.*

Theorem 3.15. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and enveloping algebra $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$. Suppose U is a unitary representation of G on a Hilbert space $\mathscr{H}$. Let x be an elliptic element in $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$, then*

$$\mathscr{D}^{\infty}(U) = \bigcap_{n=1}^{\infty} \mathscr{D}((dU(x)^*)^n) \tag{3.22}$$

Proof. Recall that $\mathscr{D}(dU) = \mathscr{D}_G(U) = \mathscr{D}^{\infty}(U)$ by Theorem 3.12. Since dU is Hermitian, $\mathscr{D}(dU) \subset \mathscr{D}(dU^*)$, and

$$\mathscr{D}(dU^*) \subset \bigcap_{n=1}^{\infty} \mathscr{D}((dU(x)^*)^n).$$

Thus, $\mathscr{D}^{\infty}(U)$ is contained in the right side of (3.22).

Conversely, let $b \in \mathscr{D}((dU(x)^n)^*)$, $n = 1, 2, 3, \ldots$. We show that $b \in \mathscr{D}^{\infty}(U)$. Let $a \in \mathscr{H}$, and define

$$\begin{aligned} f(g) &:= \langle b, U(g)a \rangle \\ f_n(g) &:= \langle (dU(x)^*)^n b, U(g)a \rangle. \end{aligned}$$

Then for all $\varphi \in C_c^\infty(G)$,

$$\begin{aligned}\int_G \varphi(g) f_n(g) &= \langle (dU(x)^*)^n b, U(\varphi)a\rangle \\ &= \langle b, dU(x)^n U(\varphi)a\rangle \\ &= \langle b, U(\tilde{x}^n \varphi)a\rangle \\ &= \int_G (\tilde{x}^n\varphi)(g) f(g) dg.\end{aligned}$$

That is, f is the weak solution to the system of elliptic partial differential equations

$$\tilde{x}^n f = f_n$$

Since f_n is continuous for all $n = 1, 2, 3, \ldots$, by elliptic regularity, it follows that f is C^∞. Since a was arbitrary, we conclude that $b \in \mathscr{D}^\infty(U)$, by Lemma 3.14. □

Theorem 3.16 ([36]). *Let G be a Lie group, $\mathfrak{g}$ the Lie algebra and $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$ the enveloping algebra. Let U be a unitary representation of G. Then dU is a selfadjoint representation of $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$ with domain equal to the Gårding space $\mathscr{D}_G(U)$.*

Proof. Let x be any elliptic element in $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$. By Theorem 3.15,

$$\mathscr{D}(dU^*) \subset \bigcap_{n=1}^{\infty} \mathscr{D}((dU(x)^*)^n) = \mathscr{D}^\infty(U)$$

But $\mathscr{D}^\infty(U) = \mathscr{D}_G(U)$, by Theorem 3.12. Thus, $\mathscr{D}(dU^*) \subset \mathscr{D}(dU)$. The converse is also true, as dU is a Hermitian representation. Therefore, $\mathscr{D}(dU) = \mathscr{D}(dU^*)$. It follows that dU is a selfadjoint representation by Lemma 3.4. □

3.2 Operators in the Enveloping Algebras

Let U be a unitary representation of a Lie group G with Lie algebra $\mathfrak{g}$ and enveloping algebra $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$. The derived representation dU is selfadjoint on the Gårding space for U. In applications, certain elements in the enveloping algebra have physical interpretations. For Hermitian elements $x \in \mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$, it is desirable that $dU(x)$ is essentially selfadjoint. If not, an appropriate selfadjoint extension has to be made, since the latter have spectral resolutions and correspond to physical observables. In this section, we review some fundamental results in these developments.

3.2.1 Central Elements

The earliest results in this direction goes back to I.E. Segal. He showed that $dU(x)$ is essentially selfadjoint for all x in the center of the enveloping algebra [47].

First, we recall the following well-known result in the theory of unitary representations of Lie groups.

Lemma 3.17. *Let G be a Lie group, $\mathfrak{g}$ the Lie algebra, and $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$ the enveloping algebra. Let G_0 be the connected component of the identity element in G. Let U be a unitary representation of G on a Hilbert space $\mathscr{H}$. Then $U(G_0)' = dU(\mathfrak{A}_{\mathbb{C}}(\mathfrak{g}))'$.*

Proof. Let $B \in U(G_0)'$. For all $a, b \in \mathscr{D}_G(U)$, $x \in \mathfrak{g}$,

$$\begin{aligned}
\langle b, BdU(x)a\rangle &= \frac{d}{dt}\Big|_{t=0}\langle b, BU(\exp(tx))a\rangle \\
&= \frac{d}{dt}\Big|_{t=0}\langle b, U(\exp(tx))Ba\rangle \\
&= \frac{d}{dt}\Big|_{t=0}\langle U(\exp(-tx))b, Ba\rangle \\
&= \langle dU(x^*)b, Ba\rangle.
\end{aligned}$$

This obviously extends to all x in the enveloping algebra. Therefore, $B \in dU(\mathfrak{A}_{\mathbb{C}}(\mathfrak{g}))'$.

Conversely, suppose $B \in dU(\mathfrak{A}_{\mathbb{C}}(\mathfrak{g}))'$. Let $b \in \mathscr{D}_G(U)$, $x \in \mathfrak{g}$. Define

$$f(t) := U(\exp(tx))BU(\exp(1-t)x)b.$$

f is continuously differentiable and

$$f'(t) = U(\exp(tx))[dU(x), B]U(\exp(1-t)x)b.$$

Note that $U(\exp(\mathfrak{g}))$ leave $\mathscr{D}_G(U)$ invariant, so that $U(\exp(1-t)x)b \in \mathscr{D}_G(U)$. By assumption, $[dU(x), B]U(\exp(1-t)x)b = 0$, it follows that $f'(t) = 0$, and so $f(0) = f(1)$, i.e.

$$BU(\exp(x))b = U(\exp(x))Bb.$$

Since $\mathscr{D}_G(U)$ is dense in $\mathscr{H}$, and $B, U(\exp(x))$ are bounded operators, we conclude that B commutes with $U(\exp(x))$, for all $x \in \mathfrak{g}$. Since G_0 is generated by $\exp(\mathfrak{g})$, it follows that B commutes with $U(g)$, for all $g \in G_0$. That is, $B \in U(G_0)'$. $\square$

Theorem 3.18 (I.E. Segal)**.** *Let G be a Lie group, $\mathfrak{g}$ the Lie algebra and $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$ the enveloping algebra. Let $\mathfrak{Z}$ be the center of $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$. For all $w \in \mathfrak{Z}$, $dU(w)$ is essentially normal on the Gårding space $\mathscr{D}_G(U)$.*

Proof. For $w \in \mathfrak{Z}$, let $N := \overline{dU(w)}$. We show that N is affiliated with the abelian von Neumann algebra

$$\mathscr{M} := U(G_0)' \cap U(G_0)''.$$

Recall that dU is a selfadjoint representation, so that strong and weak commutants coincide, see Lemma 3.6. Since $dU(w)$ commutes strongly with $dU(\mathfrak{A}_{\mathbb{C}}(\mathfrak{g}))'$, and this passes to the closure N of $dU(w)$, by Lemma 2.8. In view of Lemma 3.17, N commutes strongly with $U(G_0)'$.

It remains to show that N commutes strongly with $U(G_0)''$. Since $U(G_0)''$ is the weak-closure of $U(G_0)$, it suffices to check that N commutes strongly with $U(G_0)$. This can be done using a similar argument as in the proof of the second part of Lemma 3.17.

By Theorem 2.13, N is a normal operator. □

Remark 3.19. As a special case, if $w = w^*$, then $dU(w)$ is essentially selfadjoint. The theorem and its special case was originally proved by Segal [47, 48]. The proof using Stone's characteristic matrix is due to Jorgensen, see Chapter 6 of [25]. An interesting history of this problem can be found in [23].

3.2.2 Second Order Elements

The importance of elliptic elements in the enveloping algebras was pointed out by Nelson and Stinespring. These elements had played a key role at various places of the theory.

To proceed, we shall need the following results on strongly commuting selfadjoint operators. Recall that two selfadjoint operators are said to be strongly commuting if their spectral projections commute.

Lemma 3.20. *Let $\mathscr{D}$ be a dense domain in $\mathscr{H}$. Suppose $A, B \in End(\mathscr{D})$ and $B \subset A^*$. If AB or BA is essentially selfadjoint then $\overline{B} = A^*$.*

Proof. Let $(a, A^*a) \in \mathscr{G}(A^*)$, such that $(a, A^*a) \perp \mathscr{G}(B)$. That is, for all $b \in \mathscr{D}$,

$$\langle a, b\rangle + \langle A^*a, Bb\rangle = 0. \tag{3.23}$$

Suppose AB is essentially selfadjoint. By (3.23), $\langle a, (1 + AB)b\rangle = 0$. Since AB is positive on $\mathscr{D}$, $\mathscr{R}(1 + AB)$ is dense in $\mathscr{H}$. This implies $a = 0$.

On the other hand, if BA is essentially selfadjoint, we may substitute Ab for b in (3.23), and get $\langle A^*a, (1 + BA)b\rangle = 0$. A similar argument as before shows that $\mathscr{R}(1 + BA)$ is dense in $\mathscr{H}$, and so $A^*a = 0$. By (3.23), $\langle a, b\rangle = 0$ for all $b \in \mathscr{D}$. This implies $a = 0$. □

Theorem 3.21. *Let $\mathscr{D}$ be a dense domain in a Hilbert space $\mathscr{H}$. Suppose $A_1, A_2 \in End(\mathscr{D})$, A_1, A_2 Hermitian, and $A_1A_2a = A_2A_1a$, for all $a \in \mathscr{D}$. Let $N := A_1 + iA_2$, $N^+ := A_1 - iA_2$. Then*

1. *$\overline{N}$ is normal if and only if $\overline{N^+} = N^*$. (Equivalently, $(N^+)^* = \overline{N}$, by taking adjoints.)*
2. *If $\overline{N}$ is normal, then $\overline{A_1}$, $\overline{A_2}$ are strongly commuting selfadjoint operators.*
3. *If N^+N is essentially selfadjoint, then $\overline{N}$ is normal and $\overline{N^+N} = \overline{N}\,\overline{N}^* = \overline{N}^*\overline{N}$.*

Proof. Note that all the operators are defined on the common invariant domain $\mathscr{D}$. A_1, A_2 commute on $\mathscr{D}$ if and only if $\mathscr{D}(N) \subset \mathscr{D}(N^*)$ and $\|Na\| = \|N^*a\|$, for all $a \in \mathscr{D}$.

1. By definition, $N^+ = N^*\big|_{\mathscr{D}}$. For all $a \in \mathscr{D}$, $\|Na\| = \|N^+a\|$, and so $\mathscr{D}(\overline{N^+}) = \mathscr{D}(\overline{N})$. The statement follows from Theorem 2.12.

2. Suppose $\overline{N}$ is normal, then $\overline{N} = N_1 + iN_2$, where N_1, N_2 are strongly commuting selfadjoint operators, given by

$$N_1 = \frac{\overline{\overline{N} + N^*}}{2}, \; N_2 = \frac{\overline{\overline{N} - N^*}}{2i}$$

By definition, $A_k = N_k\big|_{\mathscr{D}}$, $N_k = \overline{N_k\big|_{\mathscr{D}(\overline{N})}}$, for $k = 1, 2$. Claim: $\overline{A_k} = N_k$, i.e. $\mathscr{D}$ is a core of N_k. It suffices to check $\mathscr{D}(\overline{N}) \subset \mathscr{D}(\overline{A_k})$, which follows from the norm estimate

$$\|A_k a\| \leq \|Na\|, a \in \mathscr{D}.$$

Therefore, $\overline{A_1}$, $\overline{A_2}$ are strongly commuting selfadjoint operators.

3. Suppose N^+N is essentially selfadjoint. Since $N^+, N \in End(\mathscr{D})$, $N^+ \subset N^*$, and $N^+Na = NN^+a$ for all $a \in \mathscr{D}$, it follows that $\overline{N^+} = N^*$, by Lemma 3.20. By part 1, $\overline{N}$ is normal. Write $\overline{N} = N_1 + iN_2$ as above. Then

$$\overline{N}\,\overline{N}^* = \overline{N}^*\overline{N} = N_1^2 + N_2^2 \supset \overline{A_1^2 + A_2^2} = \overline{N^+N}$$

Since $\overline{N^+N}$ is selfadjoint, it has no proper selfadjoint extensions. Thus, all terms are equal.

$\square$

Theorem 3.22 ([34])**.** *Let U be a unitary representation of a Lie group G with Lie algebra $\mathfrak{g}$. Let x, y be elements in the enveloping algebra $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$.*

1. *If x is elliptic, then $\overline{dU(x^*)} = dU(x)^*$.*

2. *Suppose x is elliptic and y^*y commutes with x, then $dU(y^*y)$ is essentially selfadjoint and $\overline{dU(y^*)} = dU(y)^*$. If, in addition, $y^*y = yy^*$, then $dU(y)$ is essentially normal.*

Remark 3.23. For the proof, we refer to the original paper by Nelson.

1. If $dU(y^*y)$ is essentially selfadjoint, then $\overline{dU(y^*)} = dU(y)^*$ by Lemma 3.20. In particular, for $y = y^*$, $dU(y)$ is essentially selfadjoint.

2. Write $y = y_1 + iy_2$, where y_1, y_2 are Hermitian elements. $y^*y = yy^*$ if and only if $y_1y_2 = y_2y_1$. If any of these conditions is satisfied, $dU(y)$ is essentially normal by Theorem 3.21. That is,

$$\overline{dU(y_1 - iy_2)} = dU(y_1 + iy_2)^*$$

   Also, $\overline{dU(y_1)}$, $\overline{dU(y_2)}$ are strongly commuting selfadjoint operators.

We record some important implications.

Corollary 3.24 (Segal's result on central elements)**.** *Let y be an element in the center of $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$. Then $dU(y)$ is essentially normal.*

Corollary 3.25. *If G is abelian, then $\overline{dU(y^*)} = dU(y)^*$ for all y in $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$.*

Remark 3.26. In both cases, x may be chosen as any elliptic element in $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$. For example, choose the Nelson-Laplace operator

$$x := x_1^2 + \cdots + x_d^2$$

where $\{x_1, \ldots, x_d\}$ is a basis of $\mathfrak{g}$.

Corollary 3.27. *Let $x \in \mathfrak{g}$, p a complex polynomial in one free variable, then*

$$\overline{dU(p(x)^*)} = dU(p(x))^*.$$

Proof. Let $U_1(t) := U(\exp(tx))$, $t \in \mathbb{R}$. By Stone's theorem, $U_1(t) = \exp(tX)$, where X is a skew-adjoint operator defined on its natural domain. Let s be a basis for the Lie algebra of $\mathbb{R}$, then

$$\overline{dU_1(p(s)^*)} = dU_1(p(s))^* \tag{3.24}$$

by Corollary 3.25.

Clearly, $dU \subset dU_1$ (defined on the respective Gårding spaces), and $U_1(g)$ leaves $\mathscr{D}_G(U)$ invariant, for all $g \in G$. By Theorem 3.11, $\mathscr{D}_G(U)$ is a core of $\overline{dU_1(p(s)^*)}$. That is,

$$\overline{dU(p(x)^*)} = \overline{dU_1(p(s)^*)\big|_{\mathscr{D}_G(U)}} = \overline{dU_1(p(s)^*)}. \tag{3.25}$$

Combine (3.24) and (3.25), it follows that

$$\overline{dU(p(x)^*)} = dU_1(p(s))^* \supset dU(p(s))^*.$$

Since dU is a Hermitian representation, the converse containment also holds. This proves the corollary. $\square$

Finally, we give another characterization of C^∞-vectors.

Corollary 3.28. *Let $x_1, \ldots, x_d$ be a basis of $\mathfrak{g}$, and $k \in \{1, \ldots, d\}$.*

1. $X_k^n = \overline{dU(x_k^n)}$, $n \in \mathbb{N}$.
2. *Let* $\mathscr{C}^\infty(X_k) := \bigcap_{n=1}^\infty \mathscr{D}(X_k^n)$ *. Then*

$$\mathscr{D}^\infty(U) = \bigcap_{k=1}^d \mathscr{C}^\infty(X_k) = \bigcap_{k=1}^d \mathscr{C}^\infty(\overline{dU(x_k)}).$$

Proof. Part 1 is immediate. Thus,

$$\bigcap_{k=1}^{d} \mathscr{C}^{\infty}(X_k) = \bigcap_{k=1}^{d} \mathscr{C}^{\infty}(\overline{dU(x_k)}).$$

Let $\triangle = x_1^2 + \cdots + x_d^2$ be the Nelson-Laplace operator. By Theorem 3.15

$$\bigcap_{k=1}^{d} \mathscr{C}^{\infty}(X_k) \subset \bigcap_{k=1}^{d} \mathscr{C}^{\infty}(dU(\triangle)^*) = \mathscr{D}^{\infty}(U).$$

The converse containment is trivial. This proves the corollary. □

3.3 Integrable Representations

R.T. Powers proved that every unitary representation of a Lie group U induces a selfadjoint representation dU of the enveloping algebra on the the Gårding space. By a theorem of Dixmier and Malliavin, the Gårding space is precisely the space of C^{∞}-vectors. In this section, we study the converse problem. If ρ is a representation of the enveloping algebra, is it possible to reconstruct a unitary representation U of the Lie group, so that $\rho = dU$? If this can be done, ρ is said to be integrable or exact. A necessary condition is that the domain of ρ has to be maximal in certain sense, as it would be the Gårding space of U, if the latter can be recovered. The answer to the converse problem is no in general. In the next chapter, we will study examples of non-integrable representations under the commutative settings.

Known techniques to reconstruct U from a selfadjoint representation of the enveloping algebra include the analytic vector method of Nelson, complete positivity condition of Arveson-Powers, and the perpurbation method. The latter two methods were developed in the monograph [27]. Moreover, the heat kernel method developed more recently in [8] applies to the most general integrability problem in the setting of Banach spaces.

3.3.1 Analytic Vectors

We review the idea of analytic vectors, and study the integrability problem in the setting of unitary representations.

Let A be an operator in a Hilbert $\mathscr{H}$. An element a in $\mathscr{D}(A)$ is called an analytic vector of A, if there exists $t > 0$, such that

$$\sum_{n=0}^{\infty} \frac{\|A^n a\|}{n!} t^n < \infty \tag{3.26}$$

Implicitly, $a \in \mathscr{D}(A^n)$, for $n = 1, 2, 3 \ldots$. Obviously, (3.26) is equivalent to

$$\|A^n \varphi\| < n! M^n, n \in \mathbb{N} \tag{3.27}$$

for some $M > 0$.

Lemma 3.29. *Let A be a selfadjoint operator. Let a be an analytic vector for A, and t as in (3.26). Then*

$$e^{zA} a = \sum_{n=0}^{\infty} \frac{z^n}{n!} A^n a. \tag{3.28}$$

for all $z \in \mathbb{C}$, such that $|z| \leq t$. Moreover, the unitary group e^{isA}, $s \in \mathbb{R}$, may be analytically continued to e^{izA} for $z \in \mathbb{C}$ and $|\Im(z)| < t$.

Proof. Let $A = \int \lambda E(d\lambda)$ be the spectral decomposition of A. (3.28) is equivalent to

$$\int \sum_{n=0}^{\infty} \frac{z^n}{n!} x^n E(dx) a = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int x^n E(dx) a \tag{3.29}$$

Implicitly, $a \in \mathscr{D}(e^{zA})$. We check that switching the order of summation and integration is valid, using Fubini's theorem. This follows from the estimate

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{|z|^n}{n!} \int |x|^n \|E(dx)a\| &\leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!} \left(\int |x|^{2n} \|E(dx)a\| \right)^{1/2} \|a\| \\
&= \|a\| \sum_{n=0}^{\infty} \frac{|z|^n}{n!} \|A^n a\| \\
&\leq \|a\| \sum_{n=0}^{\infty} \frac{t^n}{n!} \|A^n a\| < \infty.
\end{aligned}$$

The second part is obvious. $\square$

Theorem 3.30. *Let A be a densely defined Hermitian operator. If A has a dense set of analytic vectors, then A is essentially selfadjoint.*

Proof. Suppose $A^* b = ib$, for some $b \in \mathscr{D}(A^*)$. Let a be an analytic vector of A, and t as in (3.26).

First, we assume that A has equal deficiency indices, thus A has non-trivial selfadjoint extensions. Let $\tilde{A} \supset A$ be a selfadjoint extension of A. Then a is also an analytic vector for $\tilde{A}$. By Lemma 3.29, for $|\Im(z)| \leq t$,

$$\begin{aligned}
\langle b, e^{iz\tilde{A}} a \rangle &= \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \langle b, \tilde{A}^n a \rangle \\
&= \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \langle (A^*)^n b, a \rangle \\
&= \sum_{n=0}^{\infty} \frac{z^n}{n!} \langle b, a \rangle
\end{aligned}$$

That is, $\langle b, e^{iz\tilde{A}} a \rangle = \langle b, a \rangle e^z$, for all $|\Im(z)| \leq t$. Restrict to $z \in \mathbb{R}$, since $e^{iz\tilde{A}}$ is a unitary operator, $\langle b, e^{iz\tilde{A}} a \rangle$ is bounded. But $\langle b, a \rangle e^z$ is not, unless $\langle b, a \rangle = 0$. Since the set of analytic vectors of A is dense in $\mathscr{H}$, it follows that $b = 0$. Simiarly, $A^*b = -ib$ implies $b = 0$. Thus, the deficiency spaces of A are trivial, and A is essentially selfadjoint.

If the deficiency indices of A are not equal, we may apply the trick by setting $A_1 := A \oplus (-A)$ acting on the Hilbert space $\mathscr{H} \oplus \mathscr{H}$. Then A_1 has equal deficiency indices, and A_1 is essentially selfadjoint. It follows that A is essentially selfadjoint. For more details of this method, we refer to [2]. □

For an algebra $\mathfrak{A}$ of operators on a Hilbert space $\mathscr{H}$, $a \in \mathscr{H}$ is an analytic vector of $\mathfrak{A}$ if it is an analytic vector of every element in $\mathfrak{A}$. Usually $\mathfrak{A}$ has a finite generating set $\mathfrak{S}_0$, then a is an analytic vector of $\mathfrak{A}$ if and only if there exits some $t > 0$, such that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sup\{\|X_1 \cdots X_n a\| : X_1, \cdots, X_n \in \mathfrak{S}_0\} < \infty \tag{3.30}$$

Implicitly, a is assumed to be in the domains of all the operators involved. Condition (3.30) is equivalent to

$$\|X_1 \cdots X_n a\| \leq n! M^n, n \in \mathbb{N} \tag{3.31}$$

for arbitrary $X_1, \cdots, X_n \in \mathfrak{S}_0$, and M is some positive constant.

Let $L \in \mathfrak{A}$. Under certain conditions, the set of analytic vectors of L is contained in that of the algebra $\mathfrak{A}$. In that case, we say L analytically dominates $\mathfrak{A}$. As an illustration, let $X \in \mathfrak{A}$, such that $\|Xa\| \leq \|La\|$. If the algebra is abelian then $\|X^n a\| \leq \|L^n a\|$, it follows that L analytically dominates X. For non-abelian algebras, we have

$$\begin{aligned}\|X^2a\| &\leq \|LXa\| \\ &\leq \|XLa\| + \|ad(X)(L)a\| \\ &\leq \|L^2a\| + \|ad(X)(L)a\|.\end{aligned}$$

More generally,

$$X^nL = \sum_{k=0}^{n} \binom{n}{k} (ad(X))^k (L)X^{n-k}$$

where $ad(X)(L) := XL - LX$. Thus, an estimate on the numbers $\|ad^n(X)(L)a\|$, $n = 1, 2, 3 \ldots$, is required.

In fact, for all our applications in later sections, the abelian case is sufficient. We record here some general results on the non-abelian algebras.

Theorem 3.31. *Let $\mathfrak{A}$, $\mathfrak{S}_0$, L as before. Suppose*

1. *$\|Xa\| \leq \|La\|$, for all $X \in \mathfrak{S}_0$;*
2. *$\|ad(X_1) \cdots ad(X_n)(L)a\| < n!\|La\|$, for arbitrary $X_1, \ldots, X_n \in \mathfrak{S}_0$, $n \in \mathbb{N}$.*

Then L analytically dominates $\mathfrak{A}$.

Let $\mathfrak{g}$ be a Lie algebra with a basis $\{x_1, \ldots, x_d\}$. Recall the enveloping algebra $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$ is the polynomial algebra generated by $\{1, x_1, \ldots, x_d\}$ modulo the commutation relations of $\mathfrak{g}$. Let ρ be a $*$-representation of the $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$ on a Hilbert space $\mathscr{H}$, and $\triangle := x_1^2 + \cdots + x_d^2$.

Theorem 3.32. *Let $L := \rho(1 - \triangle)$, and $a \in \mathscr{D}(\rho)$. Then*

1. *There is $\alpha > 0$, such that*

$$\begin{aligned} \|Xa\| &\leq \alpha\|La\| \\ \|adX_1 \cdots adX_n(L)a\| &\leq \alpha^n\|L^n a\| \end{aligned}$$

for arbitrary $X, X_1, \ldots, X_n \in \{\rho(x_1), \ldots, \rho(x_d)\}$, $n \in \mathbb{N}$.

2. *L analytically dominates the algebra $\rho(\mathfrak{A}_{\mathbb{C}}(\mathfrak{g}))$.*

For abelian algebras, the theorem is trivial. For the proof of the general case, we refer to the original paper of Nelson [32]. It suffices to say that Theorem 3.31 combined with part one of Theorem 3.32 implies that L dominates $\rho(\mathfrak{A}_{\mathbb{C}}(\mathfrak{g}))$ analytically.

Let $\mathscr{D}^w(\rho)$ be the space of all analytic vectors of the algebra $\rho(\mathfrak{A}_{\mathbb{C}}(\mathfrak{g}))$. We proceed to show that for derived representations, i.e. $\rho = dU$, $\mathscr{D}^w(\rho)$ is dense in $\mathscr{H}$.

Theorem 3.33. *Let G be a Lie group, $\mathfrak{g}$ the Lie algebra and $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$ the enveloping algebra. Let U be a unitary representation of G on a Hilbert space $\mathscr{H}$. Then $\mathscr{D}^w(dU)$ is dense in $\mathscr{H}$.*

Proof. Let $L := \overline{dU(1 - \triangle)}$. By Theorem 3.22, L is selfadjoint with spectral decomposition

$$L = \int \lambda E(d\lambda).$$

Then $E_0 := \cup E(0,t)\mathscr{H}$ is a dense set of analytic vectors of L. By Theorem 3.32, $E_0 \subset \mathscr{D}^w(dU)$. Moreover, $E_0 \subset \mathscr{C}^\infty(L)$ and the latter is equal to $\mathscr{D}(dU)$ by Theorem 3.15. Thus $E_0 \subset \mathscr{D}(dU)$. This proves the theorem. $\square$

3.3.2 Integrability

Let G be a Lie group, $\mathfrak{g}$ the Lie algebra and $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$ the enveloping algebra. Throughout, $\mathfrak{g}$ is assumed to be finite dimensional with a basis $\{x_1, \ldots, x_d\}$. We also assume that G is simply connected. That is, G is generated by $\exp(\mathfrak{g})$, where exp is the exponential map from $\mathfrak{g}$ into G.

Let $\rho : \mathfrak{A}_{\mathbb{C}}(\mathfrak{g}) \to End(\mathscr{D}(\rho))$ be a Hermitian representation on a Hilbert space $\mathscr{H}$. If ρ is derived from a unitary representation U of G on $\mathscr{H}$, then $\mathscr{D}(\rho)$ contains a dense set of analytic vectors for the algebra $\rho(\mathfrak{A}_{\mathbb{C}}(\mathfrak{g}))$, see Theorem 3.33. The converse also holds. The following variants of Nelson's results are taken from [46].

Theorem 3.34. *Suppose $\mathscr{D}^w(\rho)$ is dense in $\mathscr{H}$, then there exists a unique unitary representation U of G, such that*

$$\overline{\rho(x_k)} = \overline{dU(x_k)} \tag{3.32}$$

for all $k = 1, \ldots, d$.

Remark 3.35.

1. By definition, $\mathscr{D}(dU) = \mathscr{D}_G(U)$. Corollary 3.27 shows that $\overline{dU(x_k)}$, $k = 1, \ldots d$, are skew-adjoint operators, and

$$U(\exp(tx_k)) = \exp(t\overline{dU(x_k)}).$$

   Since G is generated by $\exp(\mathfrak{g})$, it follows that (3.32) determines U uniquely.

2. Note that

$$\mathscr{D}(\rho) \subset \bigcap_{k=1}^{d} \mathscr{C}(\overline{dU(x_k)})$$

and the right side is equal to $\mathscr{D}(dU)$, by Corollary 3.28. It follows that $\rho \subset dU$. Since dU is selfadjoint, then $dU \subset \rho^*$. It is straightforward to check that ρ^* is selfadjoint, therefore, $\rho^* = dU$.

3. If, in addition, ρ is selfadjoint, then ρ is integrable.

Since $\rho(1 - \triangle)$ dominates the algebra $\rho(\mathfrak{A}_{\mathbb{C}}(\mathfrak{g}))$ analytically, it is expected to obtain conditions on integrability from the Nelson-Laplace operator.

Theorem 3.36. *Suppose $\rho(\triangle)$ is essentially selfadjoint, then (3.32) is true.*

We summarize the main results in the following theorem.

Theorem 3.37. *The following are equivalent.*

1. *ρ is integrable.*
2. *ρ is selfadjoint, and $\mathscr{D}^w(\rho)$ is dense in $\mathscr{H}$.*
3. *ρ is selfadjoint, and $\rho(\triangle)$ is essentially selfadjoint.*

CHAPTER 4
NON-INTEGRABLE REPRESENTATIONS

Motivated by the famous example of Nelson [32], we consider a system of n Hermitian operators, commuting on a common invariant dense domain in a Hilbert space $\mathscr{H}$, separately essentially selfadjoint, and we ask when do they have mutually commuting spectral projections? We study an index theory for such systems in view to extend von Neumann's extension theory for single Hermitian operators. The solution is formulated in the settings of representations of $*$-algebras. When adapted to the Nelson-type examples on various covering surfaces $\tilde{M}$ of the punctured plane, the index yields a natural link between geometry of the manifolds and spectrum of the Nelson-Laplace operator on $L^2(\tilde{M})$.

4.1 Nelson's Example

E. Nelson constructed a striking example of two essentially selfadjoint operators commuting on a common invariant domain in a separable Hilbert space, without having commuting spectral projections [32]. It raised the question on integrability of Lie algebras in general. That is, whether a selfadjoint representation of the enveloping algebra of a Lie algebra can be integrated to a unitary representation of the corresponding Lie group. Nelson's example shows that the question is intimately related to the geometry of the underlying manifold, and cannot be answered by algebraic method alone.

4.1.1 Covering Surfaces of the Punctured Plane

Let $(\tilde{M}, \pi)$ be the N-covering surface of the punctured plane $M \simeq \mathbb{R}^2 \backslash \{0\}$, and p the projection from $\tilde{M}$ onto M. Each fiber is isomorphic to the cyclic group of order N. The universal covering, $N = \infty$, corresponds to the Riemann surface of the complex log function, and each fiber is isomorphic to a copy of the integers.

For all $m \in \tilde{M}$, there exists an open neighborhood U of m, such that

$$p_U := p\big|_U \tag{4.1}$$

is a homeomorphism from U onto $p(U)$. The family $\{(U, p_U)\}$ forms an open cover of $\tilde{M}$.

For all $f \in \mathscr{C}_c(U)$, the map

$$f \mapsto \int_{p(U)} f \circ p_U^{-1}\, d\mu \tag{4.2}$$

is a positive linear functional, where μ is the two-dimensional Lebesgue measure. By Riesz's theorem, there exists a unique regular Borel measure λ_U, such that

$$\int_U f d\lambda_U = \int_{p(U)} f \circ p_U^{-1}\, d\mu. \tag{4.3}$$

Let λ be the Riemannian measure on $\tilde{M}$, uniquely determined by

$$\lambda\big|_U := \lambda_U. \tag{4.4}$$

Let U_i be a locally finite coordinate cover of $\tilde{M}$, such that $\overline{U}_i$ is compact, and choose a smooth partition of unity h_i subordinate to U_i. For all $f \in \mathscr{C}_c(\tilde{M})$,

$$\int f d\lambda = \sum_i \int_{U_i} h_i f\, d\lambda_{U_i}. \tag{4.5}$$

and the summation is taken over a finite set.

The Sobolev space $H^s(\tilde{M})$, $s \in \mathbb{R}$, consists of ψ such that $\psi\big|_U \in H^s_{loc}(U)$, i.e.

$$\psi \circ p_U^{-1} \in H^s_{loc}(p(U)) \tag{4.6}$$

for all chart (U, p_U).

We will need to parameterize $\tilde{M}$ using polar coordinates. For $N < \infty$, let q be the homeomorphism from $\tilde{M}$ onto

$$\{(r, \theta); r \in \mathbb{R}_+, \theta \in [0, 2\pi N)\}. \tag{4.7}$$

For $N = \infty$, (4.7) is replaced by

$$\{(r, \theta) : r \in \mathbb{R}_+, \theta \in \mathbb{R}\}. \tag{4.8}$$

In both cases, $\tilde{M}$ is covered by a single chart $(\tilde{M}, q)$, and

$$d\lambda = r dr d\theta. \tag{4.9}$$

Switching coordinates

$$p_U \circ q^{-1} : q(U \cap \tilde{M}) \to p_U(U \cap \tilde{M}) \tag{4.10}$$

is given by the standard mapping

$$x = r\cos\theta \tag{4.11}$$

$$y = r\sin\theta \tag{4.12}$$

Every sheet U_{θ_0} in $\tilde{M}$ is parameterized by

$$U_{\theta_0} := q^{-1}(\{(r, \theta) : r \in \mathbb{R}_+, \theta \in [\theta_0, \theta_0 + 2\pi), \theta_0 \in \mathbb{R}\}). \tag{4.13}$$

For $N < \infty$, it is understood that $\theta \in [\theta_0, \theta_0 + 2\pi) \bmod 2\pi N$. Thus, $p(U_{\theta_0})$ is a copy of $\mathbb{R}^2 \backslash \{0\}$ with a branch cut along some radial direction θ_0; form the local Hilbert space

$$L^2(U_{\theta_0}) \simeq L^2(\mathbb{R}^2). \tag{4.14}$$

4.1.2 Local Translations

Let $m \in \tilde{M}$. Choose a coordinate chart (U, π_U) of m. Let

$$T^{(1)}(m) \;:=\; \sup\{|t| : p_U(m) + (t, 0) \in p(U), t \in \mathbb{R}\} \tag{4.15}$$

$$T^{(2)}(m) \;:=\; \sup\{|t| : p_U(m) + (0, t) \in p(U), t \in \mathbb{R}\} \tag{4.16}$$

and define local translations

$$\tau_U^{(1)}(s)(m) \;:=\; p_U^{-1}\left(p_U(m) + (s, 0)\right) \tag{4.17}$$

$$\tau_U^{(2)}(t)(m) \;:=\; p_U^{-1}\left(p_U(m) + (0, t)\right) \tag{4.18}$$

for $|s| < T^{(1)}(m)$, $|t| < T^{(2)}(m)$. Locally, the two translations commute. That is,

$$\tau_U^{(1)}(s)\left(\tau_U^{(2)}(t)(m)\right) = \tau_U^{(2)}(t)\left(\tau_U^{(1)}(s)(m)\right) \tag{4.19}$$

provided that s, t are small so that points stay in U. $\tau_U^{(k)}$ extends uniquely to the translation group $\tau^{(k)}$ on $\tilde{M}$, given by

$$\tau^{(k)}\big|_U := \tau_U^{(k)} \tag{4.20}$$

for $k = 1, 2$. In general, however,

$$\tau^{(1)} \circ \tau^{(2)} \neq \tau^{(2)} \circ \tau^{(1)}. \tag{4.21}$$

Remark 4.1. To see what happens, imagine the logarithmic Riemann surface as a two-way spiral parking ramp. One driving in the ramp would end up at a level up or down by following different drive ways!

4.1.3 Translation Groups in $L^2(\tilde{M}, \lambda)$

Let $k = 1, 2$. Define

$$U_k(t)f(m) := f(\tau^{(k)}(t)(m)), t \in \mathbb{R} \tag{4.22}$$

for all $f \in L^2(\tilde{M} \backslash E_k, \lambda)$, where

$$E_1 := \pi^{-1}(\mathbb{R} \times \{0\}) \tag{4.23}$$

$$E_2 := \pi^{-1}(\{0\} \times \mathbb{R}) \tag{4.24}$$

Since E_k is a measure zero set, U_k extend uniquely to a strongly continuous unitary group on $L^2(\tilde{M}, \lambda)$. By Stone's theorem,

$$U_k(t) = e^{itX_k} \tag{4.25}$$

where X_k is the unique selfadjoint generator, and $\mathscr{D}(X_k)$ consists of all $f \in L^2(\tilde{M}, \lambda)$, such that

$$\lim_{t \to 0} \frac{U_k(t) - 1}{t} f$$

exists. It follows that

$$X_k = \frac{1}{i} \frac{\partial}{\partial x_k} \Big|_{\mathscr{D}(X_k)}. \tag{4.26}$$

Note that (4.19) and (4.21) implies that U_1, U_2 commute locally, but in general,

$$U_1(s)U_2(t) \neq U_2(t)U_1(s). \tag{4.27}$$

We proceed to show that X_1, X_2 do not commute strongly. That is, their spectral projections are non-commuting. This is a result of (4.27) and Theorem 4.2.

Theorem 4.2. *Suppose A, B are selfadjoint operators on $\mathscr{H}$. The following are equivalent.*

1. *A, B strongly commute.*

2. *The unitary groups e^{isA}, e^{itB} commute, $s, t \in \mathbb{R}$.*

Remark 4.3. $1 \Rightarrow 2$ follows from the spectral theorem. The other implication is a well-known result in the representation theory of Lie groups. We sketch the proof for the abelian group $(\mathbb{R}, +)$ as a special case. An elegant argument using representation theory of C^*-algebras can be found in Chapter 3, [5].

Proof of Theorem 4.2. Suppose the unitary groups commute, then

$$U(s,t) := e^{isA}e^{itB}$$

is a unitary representation of $(\mathbb{R}^2, +)$. If z_1, z_2 is a basis of the Lie algebra, the universal enveloping is identified with $\mathbb{C}[z_1, z_2]$, with an involution determined by $1^* = 1$, $z_k^* = -z_k$, for $k = 1, 2$. Then

$$\begin{aligned} dU(z_1) &= iA\big|_{\mathscr{D}} \\ dU(z_2) &= iB\big|_{\mathscr{D}} \end{aligned}$$

where $\mathscr{D}$ is the Gårding space for U. In particular, the operator

$$dU(-iz_1 + z_2) = A + iB\big|_{\mathscr{D}}$$

is affiliated with the abelian Von Neumann algebra

$$\{e^{isA}, e^{itB} : s, t \in \mathbb{R}\}''.$$

By Stone's theorem, $A + iB\big|_{\mathscr{D}}$ is essentially normal, or equivalently, $\overline{A\big|_{\mathscr{D}}}$ and $\overline{B\big|_{\mathscr{D}}}$ are strongly commuting selfadjoint operators. Since A, B are selfadjoint, it follows that $A = \overline{A\big|_{\mathscr{D}}}$, $B = \overline{B\big|_{\mathscr{D}}}$, and so A, B strongly commute. □

4.1.4 Non-Commuting Operators

We will consider restrictions of X_1, X_2 to the subspace $\mathscr{C}_c^\infty(\tilde{M})$. In fact, $\mathscr{C}_c^\infty(\tilde{M})$ is a core of both operators.

Lemma 4.4. *$\mathscr{C}_c^\infty(\tilde{M})$ is dense in $L^2(\tilde{M}, \lambda)$.*

Proof. Let $f \in L^2(\tilde{M}, \lambda)$. If $f \perp \mathscr{C}_c^\infty(\tilde{M})$ then $f \perp \mathscr{C}_c^\infty(U)$, for all coordinate charts (U, π_U). Since $\mathscr{C}_c^\infty(U)$ is dense in $L^2(U, \lambda_U)$, it follows that $f\big|_U = 0$. Therefore, $f = 0$. □

Lemma 4.5. *Let A be a selfadjoint operator in a Hilbert space $\mathscr{H}$. Let $U(t) = e^{itA}$, $t \in \mathbb{R}$, be the strongly continuous one-parameter unitary group generated by A. Suppose $\mathscr{D}$ is a dense subspace in $\mathscr{H}$, $\mathscr{D} \subset \mathscr{D}(A)$, and $U(t)\mathscr{D} \subset \mathscr{D}$, for all $t \in \mathbb{R}$. Then $\overline{A\big|_{\mathscr{D}}} = A$. In particular, $A\big|_{\mathscr{D}}$ is essentially selfadjoint.*

Proof. Let $B := A\big|_{\mathscr{D}}$. Suppose $B^*b = ib$, for some $b \in \mathscr{D}(B^*)$. Then for all $a \in \mathscr{D}$,

$$\begin{aligned}\frac{d}{dt}\langle b, U(t)a\rangle &= \langle b, iAU(t)a\rangle,\ (\mathscr{D} \subset \mathscr{D}(A)) \\ &= \langle b, iBU(t)a\rangle,\ (U(t)\mathscr{D} \subset \mathscr{D}) \\ &= i\langle B^*b, U(t)a\rangle \\ &= \langle b, U(t)a\rangle.\end{aligned}$$

That is, the function $f(t) := \langle b, U(t)a\rangle$ satisfies the differential equation $f' = f$. It follows $f(t) = f(0)e^t$, and so

$$\langle U(-t)b, a\rangle = \langle e^t b, a\rangle$$

for all $a \in \mathscr{D}$, $t \in \mathbb{R}$. Since $\mathscr{D}$ is dense in $\mathscr{H}$, this implies that $U(t)b = e^{-t}b$. Since $U(t)$ is unitary,

$$\|U(t)b\| = \|b\| = e^{-t}\|b\|$$

for all $t \in \mathbb{R}$. Therefore, $\|b\| = 0$ and $b = 0$.

Similarly, $B^*b = -ib$, $b \in \mathscr{D}(B^*)$, implies that $b = 0$. Therefore, the deficiency index of B is $(0, 0)$. By Von Neumann's index theory of selfadjoint extensions, B is essentially selfadjoint. Since $B \subset A$ and $A = A^*$, it follows that $\overline{B} = A$. □

Theorem 4.6. *Let X_k, $k = 1, 2$, as given in (4.26). Then*

$$\overline{X_k\big|_{\mathscr{C}_c^\infty(\tilde{M})}} = X_k. \tag{4.28}$$

Proof. The proof of Lemma 4.4 shows that $\mathscr{C}_c^\infty(\tilde{M}\backslash E_k)$ is dense in $L^2(\tilde{M}, \lambda)$. See

(4.23), (4.24) for the definition of E_k. Moreover,

$$X_k\big|_{\mathscr{C}_c^\infty(\tilde{M}\backslash E_k)} \subset X_k\big|_{\mathscr{C}_c^\infty(\tilde{M})} \subset X_k$$

and $U_k(t)$ leaves $\mathscr{C}_c^\infty(\tilde{M}\backslash E_k)$ invariant. By Lemma 4.5,

$$\overline{X_k\big|_{\mathscr{C}_c^\infty(\tilde{M}\backslash E_k)}} = \overline{X_k\big|_{\mathscr{C}_c^\infty(\tilde{M})}} = X_k.$$

□

In summary, setting $\mathscr{D} = \mathscr{C}_c^\infty(\tilde{M})$, $A_1 := X_1\big|_{\mathscr{D}}$, $A_2 := X_2\big|_{\mathscr{D}}$, i.e.

$$\begin{aligned} A_1 &= \frac{1}{i}\frac{\partial}{\partial x_1}\Big|_{\mathscr{D}} \\ A_2 &= \frac{1}{i}\frac{\partial}{\partial x_2}\Big|_{\mathscr{D}} \end{aligned}$$

The system $A_1, A_2, \mathscr{D}$ has the following properties:

1. $A_1, A_2 \in End(\mathscr{D})$;

2. $A_1A_2\varphi = A_2A_1\varphi$, for all $\varphi \in \mathscr{D}$;

3. A_1, A_2 are essentially selfadjoint;

4. $\overline{A}_1$, $\overline{A}_2$ are not strongly commuting.

Interestingly, with the first three properties, it is reasonable to expect the unique selfadjoint extensions commute. However, these conditions are still not sufficient.

4.2 Commutativity

Motivated by Nelson's example, we consider a system of n Hermitian operators, commuting on a common invariant dense domain in a Hilbert space, separately essentially selfadjoint, and we ask when do they have mutually commuting spectral projections? We study an index theory for such systems in view to extend von Neumann's extension theory for single Hermitian operators. Adapted to Nelson's example, this index naturally brings out the geometry of the covering surfaces.

4.2.1 An Index Theory

The question may be formulated in the setting of representations of $*$-algebras. Recall that for the abelian group $(\mathbb{R}^n, +)$, let $z_1, \ldots, z_n$ be a basis of the Lie algebra $\mathfrak{g}$, then universal enveloping algebra $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$ is the free abelian algebra on the skew-Hermitian generators $z_1, \ldots, z_n$. The involution on $\mathfrak{A}_{\mathbb{C}}(\mathfrak{g})$ is determined by $1^* = 1$, $z_k^* = -z_k$, for $k = 1, \ldots, n$.

Theorem 4.7. *Let $\mathscr{H}$ be a Hilbert space, $\mathscr{D}_0$ a dense subspace in $\mathscr{H}$. Let $A_1, \ldots, A_n$ be a system of operators on $\mathscr{H}$ satisfying*

1. *$\mathscr{D}(A_k) = \mathscr{D}_0$;*
2. *$A_k \in End(\mathscr{D}_0)$;*
3. *$A_k A_l \varphi = A_l A_k \varphi$, for all $\varphi \in \mathscr{D}_0$;*
4. *A_k is essentially selfadjoint.*

Let $z_1, \ldots, z_n$ be a basis of an n-dimensional abelian Lie algebra, $\mathfrak{A}$ the universal enveloping algebra. Setting $\rho(z_k) := iA_k$, and extend ρ to a Hermitian representation of $\mathfrak{A}$ on the domain $\mathscr{D}_0$. Setting

$$\mathscr{D} := \bigcap \mathscr{D}(\overline{A}_{k_1}^{l_1} \cdots \overline{A}_{k_n}^{l_n}) \tag{4.29}$$

where $k_1, \ldots, k_n \in \{1, \ldots, n\}$, $l_1, \ldots, l_n \in \mathbb{N}$. Then $\mathscr{D}(\rho^*) = \mathscr{D}$, and ρ^* is a selfadjoint *representation.*

Proof. Note that $\mathscr{D}_0 \subset \mathscr{D}$, so that $\mathscr{D}$ is dense. Obviously, $\mathscr{D}(\rho^*) \subset \mathscr{D}$. Conversely, elements in $\mathfrak{A}$ are linear combinations of monomials $z_{k_1}^{l_1} \cdots z_{k_n}^{l_n}$, and so

$$\begin{aligned}
\mathscr{D}(\rho^*) &= \bigcap \mathscr{D}((\rho(z_{k_1}^{l_1}) \cdots \rho(z_{k_n}^{l_n}))^*) \\
&\supset \bigcap \mathscr{D}(\rho(z_{k_n}^{l_n})^* \cdots \rho(z_{k_1}^{l_1})^*) \\
&= \bigcap \mathscr{D}((A_{k_n}^{l_n})^* \cdots (A_{k_1}^{l_1})^*) \\
&= \bigcap \mathscr{D}(\overline{A}_{k_n}^{l_n} \cdots \overline{A}_{k_1}^{l_1}) \\
&= \mathscr{D}
\end{aligned}$$

Thus, $\mathscr{D}(\rho^*) = \mathscr{D}$.

By definition, $\rho^*(z_k) = i\overline{A}_k\big|_{\mathscr{D}(\rho^*)}$, $k = 1, \ldots, n$, it follows that ρ^* is a Hermitian representation, i.e. $\rho^* \subset \rho^{**}$. Since $\rho^{**} \subset \rho^*$ always holds, therefore $\rho^* = \rho^{**}$. □

Theorem 4.8. *The following are equivalent.*

1. ρ^* *is integrable.*
2. $\rho^*(z_1^2 + \cdots + z_n^2)$ *is essentially selfadjoint on* $\mathscr{D}(\rho^*)$*.*
3. *The operators* $\overline{\rho^*(-iz_k)}$, $k = 1, \ldots, n$, *are mutually strongly commuting.*

Proof. This follows from general theory on $*$-representations. □

Corollary 4.9. *Let* $A_1, \ldots, A_n, \mathscr{D}_0$ *and* $\mathscr{D}$ *as in Theorem 4.7. The following are equivalent.*

1. $\overline{A}_1, \ldots, \overline{A}_n$ *have mutually commuting spectral projections.*
2. $(A_1^2)^* + \cdots + (A_n^2)^*$ *is essentially selfadjoint on* $\mathscr{D}$*.*

By Corollary 4.9, the question on the commutativity of $A_1, \ldots, A_n$ on $\mathscr{D}_0$ translates to the essential selfadjointness of

$$L := (A_1^2)^* + \cdots + (A_n^2)^*\big|_{\mathscr{D}}. \tag{4.30}$$

Notice that L is semibounded ($L \geq 0$), so by von Neumann's theory on extension of operators, it has equal deficiency indices. To classify the family of its selfadjoint extensions, it suffices to characterize one deficiency space. This leads naturally to an index for the system.

Definition 4.10. Let $A_1, \ldots, A_n, \mathscr{D}_0$ as before, and $\mathscr{D}, L$ as in (4.29) and (4.30). The defect number of the system $(A_1, \ldots, A_n, \mathscr{D}_0)$ is the dimension of the closed subspace

$$\mathscr{D}_{-1}(L) := \{\psi \in \mathscr{D}(L^*) : L^*\psi = -\psi\}$$

Remark 4.11. It is essential to formuate the index on $\mathscr{D}$, as oppose to $\mathscr{D}_0$. This is expected, as $\mathscr{D}$ is the maximal common invariant dense domain such that the system has all the desired properties. We illustrate with an example where $\mathscr{D}_0 \subsetneq \mathscr{D}$, and the index breaks down on $\mathscr{D}_0$. It also shows that the Nelson-type example does not exist on the punctured plane itself, and one has to go to the covering surfaces.

4.2.2 An Example in $L^2(\mathbb{R}^2)$

Consider the Hilbert space $L^2(\mathbb{R}^2)$, and the unitary groups

$$U_1(s)f(x_1, x_2) := f(x_1 + s, x_2) \tag{4.31}$$

$$U_2(t)f(x_1, x_2) := f(x_1, x_2 + t) \tag{4.32}$$

for $s, t \in \mathbb{R}$, and all $f \in L^2(\mathbb{R}^2)$. By Stone's theorem,

$$U_1(s) = e^{isX_1} \tag{4.33}$$

$$U_2(t) = e^{itX_2} \tag{4.34}$$

where X_1, X_2 are the selfadjoint operators, and

$$X_1 = \frac{1}{i}\frac{\partial}{\partial x_1}\Big|_{\mathscr{D}(X_1)} \tag{4.35}$$

$$X_2 = \frac{1}{i}\frac{\partial}{\partial x_2}\Big|_{\mathscr{D}(X_2)} \tag{4.36}$$

both defined on their natural domains. Since U_1, U_2 commute, X_1, X_2 are strongly commuting. Moreover,

$$U(s,t) := U_1(s)U_2(t) \tag{4.37}$$

is a unitary representation of the abelian group $(\mathbb{R}^2, +)$ on $L^2(\mathbb{R}^2)$. Setting

$$A_1 := \frac{1}{i}\frac{\partial}{\partial x_1}\Big|_{\mathscr{C}_c^\infty(\mathbb{R}^2\backslash\{0\})} \tag{4.38}$$

$$A_2 := \frac{1}{i}\frac{\partial}{\partial x_2}\Big|_{\mathscr{C}_c^\infty(\mathbb{R}^2\backslash\{0\})} \tag{4.39}$$

and let

$$\mathscr{D} := \bigcap \mathscr{D}(\overline{A}_{k_1}^{l_1}\overline{A}_{k_2}^{l_2}) \tag{4.40}$$

where $k_1, k_2 \in \{1, 2\}$, and $l_1, l_2 \in \mathbb{N}$. Then,

1. $A_1, A_2 \in End(\mathscr{C}_c^\infty(\mathbb{R}^2\backslash\{0\}))$;
2. $A_1 A_2 \varphi = A_2 A_1 \varphi$, for all $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^2\backslash\{0\})$;
3. $\overline{A}_1 = X_1$, $\overline{A}_2 = X_2$. In particular, A_1, A_2 are essentially selfadjoint. See the proof of Theorem 4.6.

Lemma 4.12. *$\mathscr{D}$ is equal to the space $\mathscr{D}^\infty(U)$ of C^∞-vectors of U. Moreover, $\psi \in \mathscr{D}$ if and only if $\psi \in \mathscr{C}^\infty(\mathbb{R}^2)$ with all the derivatives in $L^2(\mathbb{R}^2)$. That is,*

$$\mathscr{D} = \mathscr{D}^\infty(U) = \bigcap_{k=0}^{\infty} H^k(\mathbb{R}^2) \tag{4.41}$$

where $H^k(\mathbb{R}^2)$ is the k^{th} Sobolev space on $\mathbb{R}^2$.

Proof. Notice that $\psi \in \mathscr{D}$ if and only if the map

$$(s,t) \mapsto U(s,t)\psi = e^{is\overline{A}_1} e^{it\overline{A}_2}\psi$$

is C^∞ from $(\mathbb{R}^2, +)$ into $L^2(\mathbb{R}^2)$, i.e. ψ is a C^∞-vector for U. In fact,

$$(-i\partial/\partial x_1)^{l_1} (-i\partial/\partial x_2)^{l_2}\, U(s,t)\psi = e^{is\overline{A}_1} e^{it\overline{A}_2} \overline{A}_1^{l_1} \overline{A}_2^{l_2} \psi.$$

This proves part 1.

If ψ has the stated properties then integration by parts shows that $\psi \in \mathscr{D}$. Conversely, let $\psi \in \mathscr{D}$, then for all $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned}
& \langle \psi, (-i\partial/\partial x_1)^{l_1} (-i\partial/\partial x_2)^{l_2} \varphi \rangle \\
= \;& (-i\partial/\partial x_1)^{l_1} (-i\partial/\partial x_2)^{l_2} \big|_{s,t=0} \langle \psi, U(s,t)\varphi \rangle \\
= \;& (-i\partial/\partial x_1)^{l_1} (-i\partial/\partial x_2)^{l_2} \big|_{s,t=0} \langle U(-s,-t)\psi, \varphi \rangle \\
= \;& \langle \overline{A}_1^{l_1} \overline{A}_2^{l_2} \psi, \varphi \rangle
\end{aligned}$$

Thus, ψ has distributional derivatives in $L^2(\mathbb{R}^2)$ of arbitrary order. By the Sobolev embedding theorem, $\psi \in C^\infty(\mathbb{R}^2)$ with all the (classical) derivatives in $L^2(\mathbb{R}^2)$. This proves part 2. $\square$

We will consider restrictions of $X_1^2 + X_2^2$ to $\mathscr{D}$ and $\mathscr{C}_c^\infty(\mathbb{R}^2\backslash\{0\})$ respectively. By Lemma 4.12 and the definitions of X_1, X_2, these are precisely

$$-\triangle\big|_{\mathscr{D}^\infty(U)},\ -\triangle\big|_{\mathscr{C}_c^\infty(\mathbb{R}^2\backslash\{0\})}$$

where $\triangle$ is the formal Laplace operator in $\mathbb{R}^2$. The first operator is essentially selfadjoint, a fact that holds in the more general setting of representations of Lie groups, see Theorem 3.22 A direct argument is given in Lemma 4.13. In any case, this agrees with the fact that $\overline{A}_1, \overline{A}_2$ are strongly commuting selfadjoint operators. The second operator, however, is not essentially selfadjoint, see Lemma 4.14. Consequently, our index theory should be formulated on $\mathscr{D}$ as opposed to $\mathscr{C}_c^\infty(\mathbb{R}^2\backslash\{0\})$.

Lemma 4.13. $-\triangle\big|_{\mathscr{C}_c^\infty(\mathbb{R})}$ *is essentially selfadjoint. Thus,* $-\triangle\big|_{\mathscr{D}^\infty(U)}$ *is essentially selfadjoint.*

Proof. Let $L := -\triangle\big|_{\mathscr{C}_c^\infty(\mathbb{R})}$. We show that L has deficiency indices $(0,0)$. Since $L \geq 0$, it suffices to check that

$$\mathscr{D}_{-1}(L) := \{\psi \in \mathscr{D}(L^*) : L^*\psi = -\psi\}$$

is trivial. Let $\psi \in \mathscr{D}_{-1}(L)$, then

$$\langle \psi, (1-\triangle)\varphi\rangle = 0$$

for all $\varphi \in \mathscr{C}_c^\infty(\mathbb{R})$. Therefore,

$$(1-\triangle)\psi \equiv 0$$

in the sense of distributions in $\mathscr{C}_c^\infty(\mathbb{R})'$. But Fourier transform yields

$$(1+|\xi|^2)\hat{\psi}(\xi) \equiv 0$$

so that $\hat{\psi} \equiv 0$, and $\psi \equiv 0$. Thus, $\mathscr{D}_{-1}(L)$ is trivial. $\square$

Lemma 4.14. $-\triangle\big|_{\mathscr{C}_c^\infty(\mathbb{R}^2\backslash\{0,0\})}$ *has deficiency indices* $(1,1)$.

Proof. Let $L := -\triangle\big|_{\mathscr{C}_c^\infty(\mathbb{R}^2\backslash\{0,0\})}$. Suppose $\psi \in \mathscr{D}(L^*)$, such that $L^*\psi = -\psi$. Then

$$\langle \psi, (1-\triangle)\varphi \rangle = 0$$

for all $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^2\backslash\{0,0\})$. Therefore,

$$(1-\triangle)\psi \equiv 0$$

in the sense of distributions in $\mathscr{C}_c^\infty(\mathbb{R}^2\backslash\{0,0\})'$. Notice that the Fourier transform trick does not apply in this case, for ψ might break up at $(0,0)$, and $(1-\triangle)\psi$ is supported at $(0,0)$, i.e. it has the form

$$(1-\triangle)\psi = \sum_{finite} c_k \delta_0^{(k)}$$

By elliptic regularity, $\psi \in \mathscr{C}^\infty(\mathbb{R}^2\backslash\{0\})$. The problem is whether there exists an L^2-solution. Indeed, there is a unique L^2-solution, represented by the Hankel functions. It follows that L_1 has deficiency indices $(1,1)$. Since we will use these functions extensively in the examples on the covering surfaces, we leave out the details here. $\square$

4.2.3 Connection to Point Interactions

The subject of point interactions has been extensively studied in [3]. These models are also known in the literatures as “zero range potentials”, “delta interactions”, “contact interactions” or “solvable models” in the sense that parameters involved can be explicitly determined.

Specifically, we consider the quantum mechanical system

$$-\triangle + \sum_{finite} c_a \delta_a \tag{4.42}$$

where $\triangle$ denotes the selfadjoint Laplacian in the Hilbert space $L^2(\mathbb{R}^d)$. The Dirac delta-type potential models potentials concentrated on a finite subset of $\mathbb{R}^d$. E. Nelson was the first to study point interactions as limits of potentials with supports shrink to a point [20]. Various methods to the problem is also considered in [55].

The basic idea is to interpret (4.42) as a selfadjoint operator defined on an appropriately chosen domain. Once the Hamiltonians are well defined and understood, they may be used to construct more realistic interactions. As an illustration, consider the one-point interaction at the origin. Mathematically, the Hamiltonian is defined by

$$L := -\triangle\big|_{\mathscr{C}_c^\infty(\mathbb{R}^d\backslash\{0\})}.$$

The questoin is whether L is essentially selfadjoint; if it is not, how to select a suitable selfadjoint extension. Note that L commutes with complex conjugation, and so it has equal deficiency indices. Recall that $-\triangle$ is the free Hamiltonian defined on $H^2(\mathbb{R}^d)$. Under the Fourier transform, $-\triangle$ is unitarily equivalent to the operator of multiplication by $|\xi|^2$. The following formulations of the question are all equivalent:

1. L is essentially selfadjoint.

2. $\mathscr{C}_c^\infty(\mathbb{R}^d\backslash\{0\})$ is a core of $-\triangle$.

3. $\mathscr{C}_c^\infty(\mathbb{R}^d\backslash\{0\})$ is dense in $H^2(\mathbb{R}^d)$.

Theorem 4.15 ([19] page 33). *$\mathscr{C}_c^\infty(\mathbb{R}^d\backslash\{0\})$ is dense in $H^k(\mathbb{R}^d)$ if and only if $d \geq 2k$.*

Proof. Let $\psi \in H^{-k}(\mathbb{R}^d)$ such that ψ vanishes on $\mathscr{C}_c^\infty(\mathbb{R}^d \backslash \{0\})$. Thus, ψ is a tempered distribution supported at the origin. By the structure theory of distributions,

$$\psi = \sum_{finite} c_k \delta_0^{(k)}$$

and its Fourier transform $\hat{\psi}$ is a polynomial. Also,

$$\|\psi\|^2_{H^{-2}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\hat{\psi}(\xi)|(1+|\xi|^{2k})^{-1} d\xi < \infty.$$

Consequently, $\hat{\psi} \equiv 0$ (i.e. $\psi \equiv 0$) if and only if

$$\int_{\mathbb{R}^d} (1+|\xi|^{2k})^{-1} d\xi = \infty.$$

Passing to polar coordinates, the above equation is equivalent to

$$\int_0^\infty \frac{1}{r^{2k-d+1}} dr = \infty$$

which holds if and only if $d \geq 2k$. □

Corollary 4.16. *$\mathscr{C}_c^\infty(\mathbb{R}^d \backslash \{0\})$ is dense in $H^2(\mathbb{R}^d)$ if and only if $d \geq 4$.*

We record the following results in connection to the example in the previous section.

Theorem 4.17 ([3, 7])**.** *Let (n,n) be the deficiency indices of L. For $d = 1$, $n = 2$; for $d = 2, 3$, $n = 1$; for $d \geq 4$, $n = 0$.*

More generally, we may consider whether the operator $-\triangle\big|_{\mathscr{C}_c^\infty(\mathbb{R}^d \backslash \Gamma)}$ is essentially selfadjoint in $L^2(\mathbb{R}^d)$, when Γ is a closed subset in $\mathbb{R}^d$ with Lebesgue measure zero. For this development, we refer to Section 7 of [7].

4.3 The Nelson-Laplace Operator

In this section, we adapt the index theory to the Nelson-type examples on various covering surfaces of the punctured plane. It turns out that the index is solvable, and it naturally brings out the geometry of the manifold.

Recall that $\tilde{M}$ is the N-covering surface of $\mathbb{R}^2\backslash\{0\}$, and it carries the Riemannian measure λ, such that the restriction of λ to every coordinate neighborhood is the two-dimensional Lebesgue measure. Form the Hilbert space $L^2(\tilde{M},\lambda)$, and let

$$A_1 = \frac{1}{i}\frac{\partial}{\partial x_1}\Big|_{\mathscr{C}_c^\infty(\tilde{M})} \tag{4.43}$$

$$A_2 = \frac{1}{i}\frac{\partial}{\partial x_2}\Big|_{\mathscr{C}_c^\infty(\tilde{M})} \tag{4.44}$$

The special case $N = 1$ has been treated in Section 4.2.2. We summerize the properties of the two partial derivative operators below.

1. $A_1, A_2 \in End(\mathscr{C}_c^\infty(\tilde{M}))$.
2. $A_1A_2\varphi = A_2A_1\varphi$, for all $\varphi \in \mathscr{C}_c^\infty(\tilde{M})$.
3. A_1, A_2 are essentially selfadjoint. $e^{is\overline{A}_1}, e^{it\overline{A}_2}$ are the unitary translation groups along coordinate directions in $L^2(\tilde{M},\lambda)$.
4. The unitary groups $e^{is\overline{A}_1}, e^{it\overline{A}_2}$ commute locally. That is, for all $m \in \tilde{M}$, there is a coordinate neighborhood U of m, and $\epsilon > 0$, such that

$$e^{is\overline{A}_1}e^{it\overline{A}_2}\varphi = e^{it\overline{A}_2}e^{is\overline{A}_1}\varphi$$

   for all $\varphi \in \mathscr{C}_c^\infty(U)$, and all $|s|, |t| < \epsilon$.
5. Globally, $e^{is\overline{A}_1}, e^{it\overline{A}_2}$ do not commute unless in the special case $N = 1$, see Section 4.2.2. Equivalently, $\overline{A}_1, \overline{A}_2$ do not have commuting spectral projections.

For more details, please refer to Section 4.1.

Lemma 4.18. *Let*

$$\mathscr{D} := \bigcap \mathscr{D}(\overline{A}_{k_1}^{l_1} \overline{A}_{k_2}^{l_2}) \tag{4.45}$$

where $k_1, k_2 \in \{1, 2\}$*,* $l_1, l_2 \in \mathbb{N}$*. Then* $\psi \in \mathscr{D}$ *if and only if* ψ *has distributional derivatives in* $L^2(\tilde{M}, \lambda)$ *for arbitrary orders. That is,*

$$\mathscr{D} = \bigcap_{k=0}^{\infty} H^k(\tilde{M}) \tag{4.46}$$

Proof. This follows directly from the definition of the Sobolev spaces on $\tilde{M}$. $\square$

Let $\triangle$ be the formal two-dimensional Laplace operator. By Theorem 4.7 and its corollaries, the operators $\overline{A}_1, \overline{A}_2$ are strongly commuting if and only if

$$L := \overline{-\triangle\big|_{\mathscr{D}}} \tag{4.47}$$

is selfadjoint in $L^2(\tilde{M}, \lambda)$. Since L is semibounded, it has deficiency indices (n, n). So, it suffices to characterize one deficiency space of L, i.e.

$$\mathscr{D}_{-1}(L) := \{\psi \in \mathscr{D}(L^*) : L^*\psi = -\psi\}. \tag{4.48}$$

See Section 4.2 for more details. Notice that $L^*\psi = -\psi$ if and only if $\langle \psi, (1 - \triangle)\varphi \rangle = 0$, for all $\varphi \in \mathscr{C}_c^\infty(\tilde{M})$. By elliptic regularity,

$$\psi \in \mathscr{C}^\infty(\tilde{M}) \cap L^2(\tilde{M}).$$

Remark 4.19. There is a distinction between L and

$$L' := \overline{-\triangle\big|_{\mathscr{C}_c^\infty(\tilde{M})}}.$$

For $\tilde{M} = \mathbb{R}^2\backslash\{0\}$, L' serves as a model for one-point interaction on $\mathbb{R}^2$ [3]. L' has deficiency indices $(1,1)$, and it has a one-parameter family of selfadjoint extensions. $\mathscr{D}$ as in (4.45) is the space of C^∞ functions on $\mathbb{R}^2$ with all the derivatives in $L^2(\mathbb{R}^2)$, see Lemma 4.12. L is the free selfadjoint Hamiltonian, and it turns out to be the Friedrichs extension of L'. These facts will be recovered below.

Lifting to the covering surfaces is more subtle. In the following, we explicitly compute the deficiency indices of L, and characterize its deficiency spaces. It is convenient to divide the problem into two cases, i.e. $N < \infty$ and $N = \infty$.

4.3.1 N-Covering Surface ($N < \infty$)

Following [49], we show that the classical spherical harmonic analysis on the plane may be carried over to the covering surface of the punctured plane. This follows from the fact that $\tilde{M}$ is covered by a single coordinate chart $(\tilde{M}, q)$ under polar coordinates, see (4.7)-(4.14) for definitions.

Define the unitary operator $S : L^2(\tilde{M}, \lambda) \to L^2(\mathbb{R}^2\backslash\{0\})$ by

$$(Sf)(r,\theta) := \sqrt{N} f(r, N\theta). \tag{4.49}$$

The Fourier transform on $L^2(\tilde{M})$ is defined by

$$\tilde{\mathcal{F}} := S^* \mathcal{F} S \tag{4.50}$$

where $\mathcal{F}$ denotes the two-dimensional Fourier transform. Specifically,

$$\tilde{\mathcal{F}} f(r', \theta') = \frac{1}{2\pi N} \int_0^{2\pi N} \int_0^\infty f(r,\theta) e^{-ir'r\cos((\theta'-\theta)/N)} r dr d\theta \tag{4.51}$$

for all $f \in L^2(\tilde{M})$.

Theorem 4.20. *$L^2(\tilde{M}, \lambda)$ has the following decomposition*

$$\begin{aligned} L^2(\tilde{M}) &= \sum_{k\in\mathbb{Z}} \oplus \mathscr{H}_{k/N} && (4.52)\\ \mathscr{H}_{k/N} &:= L^2(\mathbb{R}_+, rdr) \otimes span\{e^{ik\theta/N}\}. && (4.53) \end{aligned}$$

Moreover, $\mathscr{H}_{k/N}$ is invariant under $\tilde{\mathcal{F}}$, for all $k \in \mathbb{Z}$.

Proof. Consider the punctured plane $\mathbb{R}^2\backslash\{0\}$, parameterized using polar coordinates. Fourier series expansion in the θ variable yields the decomposition (Chapter IV [49])

$$L^2(\mathbb{R}^2\backslash\{0\}) = \sum_{k\in\mathbb{Z}} \oplus \mathscr{H}_k$$

where $\mathscr{H}_k = L^2(\mathbb{R}_+, rdr) \otimes span\{e^{ik\theta}\}$, and $\mathscr{H}_k$ is invariant under $\mathcal{F}$ for all $k \in \mathbb{Z}$. Thus,

$$L^2(\tilde{M}, \lambda) = S^* L^2(\mathbb{R}^2\backslash\{0\}) = \sum_{k\in\mathbb{Z}} \oplus S^* \mathscr{H}_k.$$

Setting $\mathscr{H}_{k/N} := S^*\mathscr{H}_k$. It is invariant under $\tilde{\mathcal{F}}$, since

$$\mathcal{F}\mathscr{H}_{k/N} = S^*\mathcal{F}S\mathscr{H}_{k/N} = S^*\mathcal{F}\mathscr{H}_k = S^*\mathscr{H}_k = \mathscr{H}_{k/N}.$$

□

Theorem 4.21. *Let $f \in L^2(\tilde{M}, \lambda)$ and suppose f belongs to $\mathscr{H}_{k/N}$, i.e. $f(r, \theta) = f_0(r)e^{ik\theta/N}$ for some $f_0 \in L^2(\mathbb{R}_+, rdr)$. Then*

$$\begin{aligned} (\tilde{\mathcal{F}}f)(r', \theta') &= F_0(r')e^{ik\theta'/N}\\ F_0(r') &= \sqrt{2\pi} i^k \int_0^\infty f_0(r) J_k(r'r) rdr \end{aligned}$$

where J_k is the Bessel function of order k.

Proof. Notice that $f_0(r)e^{ik\theta} \in L^2(\mathbb{R}\backslash\{0\})$, and its Fourier transform is given by

$$\begin{aligned}
\mathcal{F}\left(f_0(r)e^{ik\theta}\right) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \int_0^{2\pi} f_0(r)e^{ik\theta}e^{-ir'r\cos(\theta'-\theta)}rdrd\theta \\
&= \frac{1}{\sqrt{2\pi}}e^{ik\theta'}\int_0^\infty f_0(r)\left(\int_0^{2\pi} e^{ik\theta}e^{-ir'r\cos\theta}d\theta\right)rdr \\
&= \frac{1}{\sqrt{2\pi}}e^{ik\theta'}\int_0^\infty f_0(r)\left(2\pi i^k J_k(r'r)\right)rdr \\
&= \sqrt{2\pi}i^k e^{ik\theta'}\int_0^\infty f_0(r)J_k(r'r)rdr \\
&= F_0(r')e^{ik\theta'}
\end{aligned}$$

By definition,

$$\begin{aligned}
\tilde{\mathcal{F}}f &= S^*\mathcal{F}S\left(f_0(r)e^{ik\theta/N}\right) \\
&= \sqrt{N}S^*\mathcal{F}\left(f_0(r)e^{ik\theta}\right) \\
&= \sqrt{N}S^*\left(F_0(r')e^{ik\theta'}\right) \\
&= F_0(r')e^{ik\theta'/N}.
\end{aligned}$$

□

Under the decomposition (4.52), the formal two-dimensional Laplace operator takes the form,

$$-\triangle = \sum_{k\in\mathbb{Z}}\oplus\left(-\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)+\frac{(k/N)^2}{r^2}\right)\otimes 1 \tag{4.54}$$

Define $W : L^2(\mathbb{R}_+, rdr) \to L^2(\mathbb{R}_+, dr)$ by

$$Wf(r) := r^{1/2}f(r) \tag{4.55}$$

Then

$$\tilde{W} := \sum_{k\in\mathbb{Z}}\oplus(W\otimes 1) \tag{4.56}$$

is a unitary operator on $L^2(\tilde{M}, \lambda)$, such that

$$\tilde{W}(-\triangle)\tilde{W}^* = \sum_{k\in\mathbb{Z}} \oplus \left(l_{k/N} \otimes 1\right) \tag{4.57}$$

$$l_{k/N} := -\frac{d^2}{dr^2} + \frac{(k/N)^2 - 1/4}{r^2} \tag{4.58}$$

Note that (4.58) is the formal Bessel differential operator of order k/N on the half-line $\mathbb{R}_+$, see the example in Section 2.3.3. Setting

$$\mathscr{D}_0 := \bigcap_{k=0}^{\infty} H^k(0,\infty) \tag{4.59}$$

it follows that

$$\tilde{W}(-\triangle\big|_{\mathscr{D}})\tilde{W}^* = \sum_{k\in\mathbb{Z}} \oplus \left(h_{k/N} \otimes 1\right) \tag{4.60}$$

$$h_{k/N} := l_{k/N}\big|_{\mathscr{D}_0}, k \in \mathbb{Z}. \tag{4.61}$$

To proceed, we need some facts on the formal Bessel differential operator l_ν of order ν acting on the Hilbert space $L^2(\mathbb{R}_+, dr)$, where dr is the Lebesgue measure. The details can be found in Section2.3.3 and the references therein.

1. The operator $l_\nu\big|_{\mathscr{C}_c^\infty(\mathbb{R}_+)}$ is essentially selfadjoint if and only if $|\nu| \geq 1$. Since $\mathscr{C}_c^\infty(\mathbb{R}_+) \subset \mathscr{D}_0$, it follows that $h_{k/N}$ is essentially selfadjoint, for $k = \pm N, \pm(N+1), \ldots$. In view of equations (4.60) and (4.61), the operator $L = -\triangle\big|_{\mathscr{D}}$ only defects in the components

$$h_0, h_{\pm 1/N}, \cdots, h_{\pm(N-1)/N}$$

2. For $|\nu| < 1$, the operator $l_\nu\big|_{\mathscr{C}_c^\infty(\mathbb{R}_+)}$ has deficiency indices $(1,1)$, and the defect vectors $\phi_\pm$ satisfying

$$\left(l_\nu\big|_{\mathscr{C}_c^\infty(\mathbb{R}_+)}\right)^* \phi_{\nu,\pm} = \pm i\phi_{\nu,\pm}$$

are given by

$$\begin{aligned}\phi_{\nu,+}(r) &= r^{1/2} H_\nu^{(1)}(r\sqrt{+i}) \\ \phi_{\nu,-}(r) &= r^{1/2} H_\nu^{(2)}(r\sqrt{-i})\end{aligned}$$

where $H_\nu^{(1)}$, $H_\nu^{(2)}$ are the Hankel functions of order ν [53][2].

3. The n^{th} derivative of Hankel functions (first or second kind) are given by

$$\begin{aligned}\left(\frac{d}{dz}\right)^n H_\nu(z) &= \frac{1}{2^k}\{H_{\nu-n}(z) - \begin{pmatrix} n \\ 1 \end{pmatrix} H_{\nu-n+2}(z) \\ &+ \begin{pmatrix} n \\ 2 \end{pmatrix} H_{\nu-n+4}(z) - \cdots + (-1)^n H_{\nu+n}(z)\}\end{aligned}$$

See formula (9.1.31) of [1]. Thus,

$$\left(\frac{d}{dr}\right)^n \phi_{\nu,\pm}(r) \notin L^2(\mathbb{R}_+, dr), n = 1, 2, 3, \ldots$$

That is,

$$\phi_{\nu,\pm} \in \mathscr{C}^\infty(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$$

but all the derivatives of $\phi_{\nu,\pm}$ are not in $L^2(\mathbb{R}_+)$.

4. By parts 2 and 3, $\phi_{\nu,\pm} \notin \mathscr{D}_0$. It follows that $\phi_{\nu,\pm}$ are also the defect vectors of the operators $h_{k/N}$, for $k = 0, \pm 1, \ldots, \pm(N-1)/N$.

5. $L = -\triangle\big|_{\mathscr{D}}$ has deficiency indices $(2N-1, 2N-1)$.

We summarize the main results of this section in the theorem below.

Theorem 4.22. *Let $\tilde{M}$ be the N-covering surface ($N < \infty$) of the punctured plane. Let $L = -\triangle\big|_{\mathscr{D}}$ be the Nelson-Laplace operator in (4.47) and $\mathscr{D}$ as in (4.46).*

1. *L is unitarily equivalent to $\sum_{k\in\mathbb{Z}} \oplus \big(h_{k/N} \otimes 1\big)$, where $h_{k/N}$ is the formal Bessel differential operator on the half-line restricted to the domain $\cap_{k=0}^{\infty} H^k(\mathbb{R}_+)$ in the Hilbert space $L^2(\mathbb{R}_+, dr)$.*

2. *For $|k| < N$, $h_{k/N}$ has deficiency indices $(1,1)$. For $|k| \geq N$, $h_{k/N}$ is essentially selfadjoint.*

3. *Let $\mathscr{D}_\pm(L)$ be the deficiency spaces of L. Then*

$$\begin{aligned} \mathscr{D}_+(L) &= span\{r^{1/2} H_\nu^{(1)}(r\sqrt{+i}) e^{i(k/N)\theta} : k = 0, \pm 1, \ldots, \pm(N-1)\} \\ \mathscr{D}_-(L) &= span\{r^{1/2} H_\nu^{(2)}(r\sqrt{-i}) e^{i(k/N)\theta} : k = 0, \pm 1, \ldots, \pm(N-1)\} \end{aligned}$$

   Consequently, L has deficiency indices $(2N-1, 2N-1)$.

4. *The family of selfadjoint extensions of L is indexed by partial isometries V with initial space $\mathscr{D}_+(L)$ and final space $\mathscr{D}_-(L)$. Given V, a selfadjoint extension $\tilde{L}_V \supset L$ is specified by*

$$\begin{aligned} \mathscr{D}(\tilde{L}_V) &= \{\varphi + \varphi_+ + V\varphi_+ : \varphi \in \mathscr{D}, \varphi_+ \in \mathscr{D}_+(L)\} \\ \tilde{L}_V(\varphi + \varphi_+ + V\varphi_+) &= L\varphi + i\varphi_+ - iV\varphi_+. \end{aligned}$$

We single out a sub-family of selfadjoint extensions that are obtained from extensions of each components $h_{k/N}$ in the Hilbert space $\mathscr{H}_{k/N}$.

Theorem 4.23. *Let $\alpha = (\alpha_{k/N})$, $-\infty < \alpha_{k/N} \leq \infty$, be a multi-index for $k = 0, \pm 1, \ldots, \pm(N-1)$. There is a sub-family of selfadjoint extension $\tilde{L}_\alpha \supset L$ such that*

$$\tilde{W}\tilde{L}_\alpha\tilde{W}^* = \left(\sum_{|k|<N} \oplus \left(\tilde{h}_{k/N,\alpha_{k/N}} \otimes 1\right)\right) \oplus \left(\sum_{|k|\geq N} \oplus \left(\overline{h}_{k/N} \otimes 1\right)\right)$$

where $\tilde{h}_{k/N,\alpha_{k/N}}$ *is the one-parameter family of selfadjoint extensions of* $h_{k/N}$ *in* $\mathscr{H}_{k/N}$*, and*

$$\mathscr{D}(\tilde{h}_{k/N,\alpha_{k/N}}) = \{f \in \mathscr{D}(h^*_{k/N}) : \alpha_{k/N} f_0 = f_1\}$$

f_0, f_1 *are given by*

1. $k = 0$,

$$\begin{aligned} f_0 &= \lim_{r\to 0+} f(r)/\left(\sqrt{r}\log r\right) \\ f_1 &= \lim_{r\to 0+} \left(f(r) - f_0\sqrt{r}\log r\right)/\sqrt{r} \end{aligned}$$

2. $|k/N| \in (0, 1/2) \cup (1/2, 1)$

$$\begin{aligned} f_0 &= \lim_{r\to 0+} f(r)/r^{-k/N+1/2} \\ f_1 &= \lim_{r\to 0+} (f(r) - f_0 r^{-k/N+1/2})/r^{k/N+1/2} \end{aligned}$$

3. $|k/N| = 1/2$ *(the endpoint* 0 *is regular)*

$$\mathscr{D}(\tilde{h}_{\pm 1/2,\alpha_{\pm 1/2}}) = \{f \in \mathscr{D}(h^*_{\pm 1/2}) : \alpha_{\pm 1/2} f(0) + f'(0) = 0\}.$$

Moreover, $\tilde{L}_\infty$ *is the Friedrichs extension.*

Proof. This follows from Theorem 2.53 and its corollary. For details, please see Section 2.3.3. □

We make a few comments on the defect vectors. Since the Nelson-Laplace operator $L = -\triangle\big|_{\mathscr{D}}$ is semibounded, it suffices to characterize one particular deficiency space, i.e. the closed subspace

$$\mathscr{D}_{-1}(L) = \{\psi \in \mathscr{D}(L^*) : L^*\psi = -\psi\}.$$

$\psi \in \mathscr{D}_{-1}(L)$ if and only if

$$(1 - \triangle)\psi = 0 \tag{4.62}$$

in the sense of distribution. By elliptic regularity, $\psi \in \mathscr{C}^\infty(\tilde{M}) \cap L^2(\tilde{M})$. See also equation (4.48) and the discussion there. Turns out it is more convenient to work with $\mathscr{D}_{-1}(L)$, and a simple change of variable will yield the deficiency spaces $\mathscr{D}_\pm(L)$.

Locally, (4.62) always has solutions of the form

$$e^{c_1 x_1 + c_2 x_2} \tag{4.63}$$

with $c_1, c_2 \in \mathbb{C}$, $c_1^2 + c_2^2 = 1$. The question is how to piece together these local solutions and obtain an L^2 solution on the covering surface. Formally, the answer would take the form

$$\sum_k \int e^{c_1 x_1 + c_2 x_2} d\mu_k \tag{4.64}$$

where k accounts for the winding number, and μ_k is a Borel measure supported on the set $\{(c_1, c_2) \in \mathbb{C}^2 : c_1^2 + c_2^2\}$.

Passing to polar coordinates and apply the decomposition in (4.60), we obtain all the product solutions to (4.62), given by

$$K_{k/N}(r) e^{\pm i\theta(k/N)} \tag{4.65}$$

for $k = 0, \ldots, N-1$. Here, $K_\nu(z)$ denotes the modified Bessel function of the second kind of order ν [53].

Remark 4.24. From (4.65), it follows that what accounts for the non commutativity of the selfadjoint operators $\overline{A}_1, \overline{A}_2$, see (4.43) and (4.44), is the appearance of the phase factors $e^{\pm i\theta(k/N)}$. This is a unique feature of the covering surface. For $N = 1$, there is no phase factor, and $\overline{A}_1, \overline{A}_2$ are strongly commuting.

Linear combinations of (4.65) yields

$$r^{1/2} K_0(r) \tag{4.66}$$

$$r^{1/2} K_{k/N}(r) \cos(\theta(k/N)) \tag{4.67}$$

$$r^{1/2} K_{k/N}(r) \sin(\theta(k/N)) \tag{4.68}$$

for $k = 1, \ldots, N-1$. These are all the linearly independent product solutions to (4.62), up to multiplicative constants; and they span $\mathscr{D}_{-1}(L)$. In particular, for $N = 1$, i.e. $\tilde{M} = \mathbb{R}^2 \backslash \{0\}$, there is a unique L^2 solution $r^{1/2} K_0(r)$.

The connection to equation (4.64) can be seen from the integral representations of $K_\nu(z)$. Recall that [53, 31]

$$K_{k/N}(r) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-r\cosh(t) - t(k/N)} dt. \tag{4.69}$$

The t variable may be analytically continued to $t + i\theta$, so that

$$\begin{aligned}
K_{k/N}(r) &= \frac{1}{2} \int_{-\infty}^{\infty} e^{-r\cosh(t+i\theta) - (t+i\theta)(k/N)} dt \\
&= \frac{1}{2} e^{-i\theta(k/N)} \int_{-\infty}^{\infty} e^{-r\cosh(t+i\theta) - t(k/N)} dt \\
&= \frac{1}{2} e^{-i\theta(k/N)} \int_{-\infty}^{\infty} e^{-r(\cosh(t)\cos(\theta) + i\sinh(t)\sin(\theta))} e^{-t(k/N)} dt \\
&= \frac{1}{2} e^{-i\theta(k/N)} \int_{-\infty}^{\infty} e^{-\cosh(t)x_1 - i\sinh(t)x_2} e^{-t(k/N)} dt.
\end{aligned}$$

The last line is precisely of the form (4.64).

Remark 4.25. To obtain the deficiency spaces $\mathscr{D}_{\pm}(L)$, we use the fact that

$$H^{(1)}_{k/N}(r\sqrt{+i}) = -i\frac{2}{\pi}e^{-i\pi(k/N)/2}K_{k/N}(r) \tag{4.70}$$

$$H^{(2)}_{k/N}(r\sqrt{-i}) = +i\frac{2}{\pi}e^{+i\pi(k/N)/2}K_{k/N}(r) \tag{4.71}$$

4.3.2 N-Covering Surface ($N = \infty$)

In this section, we consider the infinite covering surface of the punctured plane, i.e. the Riemann surface of the complex log function. As in the finite case, $\tilde{M}$ is covered by a single chart $\{(\tilde{M}, q)\}$ in the polar coordinates, only the angle variable takes values in $\mathbb{R}$, instead of $[0, 2\pi N)$. For definitions, please see Section 4.1.

In contrast to Theorem 4.20, the change in the θ variable yields a direct integral decomposition. In fact, for all $f \in L^2(\tilde{M}, \lambda)$, Fourier transform in the θ variable yields

$$\hat{f}(r,\xi) = \frac{1}{\sqrt{2\pi}}\int f(r,\theta)e^{-i\xi\theta}d\theta \tag{4.72}$$

$$f(r,\theta) = \frac{1}{\sqrt{2\pi}}\int \hat{f}(r,\xi)e^{i\xi\theta}d\xi \tag{4.73}$$

and the map $f \mapsto \hat{f}$ is unitary, i.e.

$$\int\int |f(r,\theta)|^2 r dr d\theta = \int\int |\hat{f}(r,\xi)|^2 r dr d\xi < \infty. \tag{4.74}$$

This leads to the decomposition:

Theorem 4.26. *Let $\tilde{M}$ be the Riemann surface of the complex log function. Then*

$$L^2(\tilde{M}, \lambda) = \int_{\mathbb{R}}^{\oplus} \mathscr{H}_\xi \, d\xi \tag{4.75}$$

$$\mathscr{H}_\xi := L^2(\mathbb{R}_+, rdr) \otimes span\{e^{i\xi\theta}\} \tag{4.76}$$

Under the decomposition (4.75), the formal two-dimensional Laplace operator takes the form,

$$-\triangle = \int^{\oplus} \left(-\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) + \frac{\xi^2}{r^2}\right) \otimes 1 \tag{4.77}$$

Let $W : L^2(\mathbb{R}_+, rdr) \to L^2(\mathbb{R}_+, dr)$ by $Wf(r) := r^{1/2}f(r)$ as in (4.55). Then

$$\tilde{W} := \int^{\oplus} W \otimes 1 \tag{4.78}$$

is a unitary operator on $L^2(\tilde{M}, \lambda)$, such that

$$\tilde{W}(-\triangle)\tilde{W}^* = \int^{\oplus} l_\xi \otimes 1 \tag{4.79}$$

$$l_\xi := -\frac{d^2}{dr^2} + \frac{\xi^2 - 1/4}{r^2} \tag{4.80}$$

Setting

$$\mathscr{D}_0 := \bigcap_{k=0}^{\infty} H^k(0, \infty) \tag{4.81}$$

it follows that

$$\tilde{W}(-\triangle\big|_{\mathscr{D}})\tilde{W}^* = \int^{\oplus} h_\xi \otimes 1 \tag{4.82}$$

$$h_\xi := l_\xi\big|_{\mathscr{D}_0} \tag{4.83}$$

Lemma 4.27. *Let K_ν be the Macdonald function of order ν, and suppose $\nu \in (-1, 1)$. Then*

$$\int_0^\infty |K_\nu(z)|^2 z dz = \frac{1}{2}\frac{\pi\nu}{\sin \pi\nu}. \tag{4.84}$$

Proof. By Nicholson's integral representation of the product of the Macdonald functions with arbitrary complex orders ([53], page 440),

$$K_\mu(z)K_\nu(z) = 2\int_0^\infty K_{\mu+\nu}(2z\cosh t)\cosh((\mu-\nu)t)dt. \tag{4.85}$$

Setting $\mu = \nu$, since K_ν is real-valued, it follows that

$$|K_\nu(z)|^2 = 2\int_0^\infty K_{2\nu}(2z\cosh t)dt \tag{4.86}$$

Recall also the identity ([53], page 388, equation (8))

$$\int_0^\infty K_\nu(z)z^{\beta-1}dz = 2^{\beta-2}\Gamma\left(\frac{\beta+\nu}{2}\right)\Gamma\left(\frac{\beta-\nu}{2}\right), \ \Re(\beta) > |\Re(\nu)|. \tag{4.87}$$

For $\nu \in (-1,1)$, $\beta = 2$, (4.87) leads to

$$\begin{aligned} \int_0^\infty K_{2\nu}(2z\cosh t)z\,dz &= \frac{1}{(2\cosh t)^2}\Gamma\left(\frac{2+2\nu}{2}\right)\Gamma\left(\frac{2-2\nu}{2}\right) \\ &= \frac{1}{(2\cosh t)^2}\Gamma(1+\nu)\Gamma(1-\nu) \\ &= \frac{1}{(2\cosh t)^2}\nu\Gamma(\nu)\Gamma(1-\nu) \\ &= \frac{1}{(2\cosh t)^2}\left(\frac{\pi\nu}{\sin\pi\nu}\right) \end{aligned} \tag{4.88}$$

The last step follows from the identity

$$\Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin\pi\nu} \tag{4.89}$$

for the Gamma function Γ.

Apply (4.86) and (4.88), and switch the order of integration, we get

$$\begin{aligned} \int_0^\infty |K_\nu(z)|^2 z dz &= 2\int_0^\infty \left(\int_0^\infty K_{2\nu}(2z\cosh t)dt\right) zdz \\ &= 2\int_0^\infty \left(\int_0^\infty K_{2\nu}(2z\cosh t)zdz\right) dt \\ &= 2\left(\frac{\pi\nu}{\sin\pi\nu}\right)\int_0^\infty \frac{1}{(2\cosh t)^2}dt \\ &= \frac{1}{2}\frac{\pi\nu}{\sin\pi\nu}\int_0^\infty \frac{1}{\cosh^2 t}dt \\ &= \frac{1}{2}\frac{\pi\nu}{\sin\pi\nu}\cdot\lim_{s\to\infty}\left(\tanh(t)\big|_0^s\right) \\ &= \frac{1}{2}\frac{\pi\nu}{\sin\pi\nu}. \end{aligned}$$

It remains to justify the switch of order of integration in the computation. This is based on the following estimates: [53]

1. as $z \to \infty$,

$$K_\nu(z) \sim z^{-1/2}e^{-z}$$

2. as $z \to 0$,

$$\begin{aligned} K_\nu(z) &\sim \frac{1}{2}\Gamma(\nu)(\frac{z}{2})^{-\nu},\ \nu \neq 0 \\ K_0(z) &\sim -\ln z \end{aligned}$$

For $z, t \to \infty$,

$$K_{2\nu}(2z\cosh t) \sim (ze^t)^{-1/2}e^{-ze^t}$$

so for $a, b \to \infty$,

$$\begin{aligned} \int_a^\infty \int_b^\infty |K_{2\nu}(2z\cosh t)|\, zdzdt &\sim \int_a^\infty \int_b^\infty (ze^t)^{-1/2}e^{-ze^t} zdzdt \\ &= \int_a^\infty \int_b^\infty z^{1/2}e^{-t/2}e^{-ze^t}dzdt \\ &< \int_a^\infty \int_b^\infty z^{1/2}e^{-t/2}e^{-z}dzdt \\ &= \int_a^\infty z^{1/2}e^{-z}dz \int_b^\infty e^{-t/2}dt \\ &< \infty. \end{aligned}$$

For $z, t \to 0$, and $\nu = 0$

$$K_0(2z\cosh t) \sim -\ln z + \ln\cosh(t) \sim -\ln z$$

Since $z\ln z \to 0$, as $z \to 0$, so if $a, b \to 0$,

$$\int_0^a \int_0^b |K_{2\nu}(2z\cosh t)|\, zdzdt \sim -\int\int \ln z\; zdzdt < \infty.$$

For $t, z \to 0$, and $\nu \neq 0$,

$$K_{2\nu}(2z\cosh t) \sim \frac{1}{2}\Gamma(2\nu)(z\cosh t)^{-\nu} \sim z^{-\nu}$$

so for $a, b \to 0$,

$$\begin{aligned}\int_0^a \int_0^b |K_{2\nu}(2z\cosh t)|\, zdzdt &\sim \int_0^a \int_0^b z^{-\nu}\, zdzdt \\ &= \int_0^a \int_0^b z^{1-\nu} dzdt < \infty\end{aligned}$$

since by assumption, $\nu \in (-1, 1)$. We conclude that

$$\int_0^\infty \int_0^\infty |K_{2\nu}(2z\cosh t)|\, zdzdt < \infty$$

and Fubini's theorem applies. This finishes the proof of the lemma. □

Theorem 4.28. *Let* $L = \overline{-\triangle\big|_{\mathscr{D}}}$ *be the Nelson-Laplace operator, and* $\mathscr{D}_{-1}(L)$ *the deficiency space corresponding to the eigenvalue* $\lambda = -1$*, see (4.48). Then* $\psi \in \mathscr{D}_{-1}(L)$ *if and only if there is a Lebesgue measurable function* g *supported in* $(-1, 1)$ *and satisfies*

$$\int_{-\infty}^{\infty} |g(\xi)|^2 \frac{1}{2}\frac{\pi\xi}{\sin\pi\xi} d\xi < \infty \tag{4.90}$$

such that

$$\psi(r, \theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) K_\xi(r) e^{i\xi\theta} d\xi. \tag{4.91}$$

Proof. In view of equation (4.82), $\psi \in \mathscr{D}_{-1}(L^*)$ if and only if $\hat{\psi} := \tilde{W}\psi$ satisfies

$$\int^{\oplus} (h_\xi^* \otimes 1)\hat{\psi} e^{i\xi\theta} = -\hat{\psi} e^{i\xi\theta}$$

That is,

$$\left(-\frac{d^2}{dr^2} + \frac{\xi^2 - 1/4}{r^2}\right)\hat{\psi}(r, \xi) = -\hat{\psi}(r, \xi) \tag{4.92}$$

for all $\xi \in \mathbb{R}$.

Fix ξ. The left side of (4.92) is the Bessel differential operator on the half-line $(0,\infty)$. From previous discussions, the unique solution in $L^2(\mathbb{R}_+, dr)$, up to a multiplicative constant depending on ξ, is given by

$$r^{1/2}K_\xi(r), |\xi| < 1. \tag{4.93}$$

Here, K_ξ is the Macdonald function of order ξ. For $|\xi| > 1$, there is no L^2 solution. For details, please see Section 2.3.3 and we also refer to [2].

Thus $\psi \in \mathscr{D}_{-1}(L)$ if and only if

$$\hat{\psi}(r,\xi) = g(\xi)r^{1/2}K_\xi(r) \tag{4.94}$$

for some measurable function g supported in $(-1,1)$, such that

$$\int\int |g(\xi)K_\xi(r)|^2 r dr d\xi < \infty. \tag{4.95}$$

By Lemma 4.27, condition (4.95) is equivalent to

$$\begin{aligned}\int\int |g(\xi)K_\xi(r)|^2 r dr d\xi &= \int |g(\xi)|^2 \left(\int |K_\xi(r)|^2 r dr\right) d\xi \\ &= \int |g(\xi)|^2 \frac{1}{2}\frac{\pi\xi}{\sin \pi\xi} d\xi < \infty\end{aligned}$$

This proves (4.90). Finally, $\psi = \tilde{W}\hat{\psi}$, so that

$$\psi(r,\theta) = \frac{1}{\sqrt{2\pi}} \int g(\xi)K_\xi(r)e^{i\xi\theta} d\xi$$

which is (4.91). □

Remark 4.29. The change from the compact group T to the non-compact group $\mathbb{R}$ in the θ variable accounts for some difficulties in representing the solutions. The classical

Fourier-Bessel series that appears in solving the Helmholtz equation is replaced by the Fourier-Bessel transform.

Corollary 4.30. *Define* $L^2((-1,1), d\mu)$ *with respect to*

$$d\mu := \frac{1}{2}\frac{\pi\xi}{\sin\pi\xi}d\xi$$

where $d\xi$ *is the Lebesgue measure. The map*

$$g \mapsto \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\xi)K_\xi(r)e^{i\xi\theta}d\xi$$

is a unitary operator from $L^2((-1,1), d\mu)$ *onto* $\mathscr{D}_{-1}(L)$.

Corollary 4.31. *The Nelson-Laplace operator* L *has deficiency indices* (∞, ∞).

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