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Extremal sextic truncated moment problems

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EXTREMAL SEXTIC TRUNCATED MOMENT PROBLEMS

by

Seongouk Yoo

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
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Thesis Supervisor: Professor Raúl Curto

ABSTRACT

Inverse problems naturally occur in many branches of science and mathematics. An inverse problem entails finding the values of one or more parameters using the values obtained from observed data. A typical example of an inverse problem is the inversion of the Radon transform. Here a function (for example of two variables) is deduced from its integrals along all possible lines. This problem is intimately connected with image reconstruction for X-ray computerized tomography.

Moment problems are a special class of inverse problems. While the classical theory of moments dates back to the beginning of the 20th century, the systematic study of *truncated* moment problems began only a few years ago. In this dissertation we will first survey the elementary theory of truncated moment problems, and then focus on those problems with cubic column relations.

For a degree $2n$ real d-dimensional multisequence $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in Z_+^d, |i| \leq 2n}$ to have a representing measure μ, it is necessary for the *associated moment matrix* $\mathcal{M}(n)$ to be positive semidefinite, and for the *algebraic variety* associated to β, $\mathcal{V}_\beta$, to satisfy rank $\mathcal{M}(n) \leq$ card $\mathcal{V}_\beta$ as well as the following *consistency condition*: if a polynomial $p(x) \equiv \sum_{|i| \leq 2n} a_i x^i$ vanishes on $\mathcal{V}_\beta$, then $\Lambda(p) := \sum_{|i| \leq 2n} a_i \beta_i = 0$. In 2005, Professor Raúl Curto collaborated with L. Fialkow and M. Möller to prove that for the *extremal* case (rank $\mathcal{M}(n) =$ card $\mathcal{V}_\beta$), positivity and consistency are sufficient for the existence of a (unique, rank $\mathcal{M}(n)$-atomic) representing measure.

In joint work with Professor Raúl Curto we have considered cubic column re-

lations in $M(3)$ of the form (in complex notation) $Z^3 = itZ + u\bar{Z}$, where u and t are real numbers. For (u, t) in the interior of a real cone, we prove that the algebraic variety $\mathcal{V}_\beta$ consists of exactly 7 points, and we then apply the above mentioned solution of the extremal moment problem to obtain a necessary and sufficient condition for the existence of a representing measure. This requires a new representation theorem for sextic polynomials in Z and $\bar{Z}$ which vanish in the 7-point set $\mathcal{V}_\beta$. Our proof of this representation theorem relies on two successive applications of the Fundamental Theorem of Linear Algebra. Finally, we use the Division Algorithm from algebraic geometry to extend this result to other situations involving cubic column relations.

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the May 2011 graduation.

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CHAPTER 1
INTRODUCTION

Inverse problems naturally occur in many branches of science and mathematics. An inverse problem entails finding the values of one or more parameters using the values obtained from observed data. A typical example of an inverse problem is the inversion of the Radon transform. Here a function (for example of two variables) is deduced from its integrals along all possible lines. This problem is intimately connected with image reconstruction for X-ray computerized tomography.

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For a degree $2n$ real d-dimensional multisequence $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in Z_+^d, |i| \leq 2n}$ to have a representing measure μ, it is necessary for the *associated moment matrix* $\mathcal{M}(n)$ to be positive semidefinite, and for the *algebraic variety* associated to β, $\mathcal{V}_\beta$, to satisfy rank $\mathcal{M}(n) \leq$ card $\mathcal{V}_\beta$ as well as the following *consistency condition*: if a polynomial $p(x) \equiv \sum_{|i| \leq 2n} a_i x^i$ vanishes on $\mathcal{V}_\beta$, then $p(\beta) := \sum_{|i| \leq 2n} a_i \beta_i = 0$. In 2005, Professor Raúl Curto collaborated with L. Fialkow and M. Möller to prove that for the *extremal* case (rank $\mathcal{M}(n) =$ card $\mathcal{V}_\beta$), positivity and consistency are sufficient for the existence of a (unique, rank $\mathcal{M}(n)$-atomic) representing measure.

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To describe our results, we need to start from the beginning, with the standard setup for truncated moment problems. We defer to Chapter 2 the basic mathematical material needed to fully understand the topics in this dissertation. There, we present the notation and terminology, and also some of the basic results that constitute our toolbox. These basic facts come from the areas of functional analysis, algebraic geometry and the theory of moment matrices.

Let $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in \mathbb{Z}_+^d, |i| \le 2n}$ denote a real 2-dimensional multisequence of degree $2n$. The *truncated moment problem* for β concerns the existence of a positive Borel measure μ, supported in $\mathbb{R}^2$, such that

$$\beta_{i,j} = \int_{\mathbb{R}^2} x^i y^j \, d\mu, \qquad i + j \le 2n. \tag{1.0.1}$$

A measure μ as in (1.0.1) is a *representing measure* for β. The truncated moment problem is more general than the classical *full moment problem* (cf. [2], [1], [30], [35], [32] and [38]). Indeed, a result of J. Stochel [37] shows that a full moment sequence

$\beta^{(\infty)}$ has a representing measure supported in a prescribed closed set $K \subseteq \mathbb{R}^2$ if and only if for each n, $\beta^{(2n)}$ has a representing measure supported in K.

In [9] and [22], R. Curto and L. Fialkow succeeded in obtaining a complete solution to the truncated moment problem in case the interpolating measure has compact support in the real line; the main contribution there consisted in bringing to light the notion of *recursiveness*, which was central to their analysis. As we move into several variables, the interpolating measure must be allowed to have support away from the line; one instance of particular interest, associated with 2-variable weighted shifts, is the case of compact support in the complex plane, which we label as the truncated complex moment problem (TCMP).

Let μ be a positive Borel measure on $\mathbf{C}$, assume that $\mathbf{C}[z, \bar{z}] \subseteq L^1(\mu)$ and define $\gamma_{ij} := \int \bar{z}^i z^j d\mu(z, \bar{z})$. Given $p \in \mathbf{C}[z, \bar{z}]$, $p(z, \bar{z}) = \sum_{ij} a_{ij} \bar{z}^i z^j$, we have

$$0 \leq \int \mid p(z, \bar{z}) \mid^2 d\mu(z, \bar{z}) = \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \int \bar{z}^{i+\ell} z^{j+k} d\mu(\bar{z}, z) = \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \gamma_{i+\ell, j+k}. \tag{1.0.2}$$

Observe that $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$ for all i, j. To understand the matricial positivity associated with $\gamma := \{\gamma_{ij}\}$, we introduce the following lexicographic order on the rows and columns of infinite matrices: *1*, $Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, Z^3, \bar{Z}Z^2, \bar{Z}^2 Z, \bar{Z}^3, \ldots$, e.g., the first column is labeled *1*, the second column is labeled Z, the third $\bar{Z}$, the fourth Z^2, et cetera; this order corresponds to the graded homogeneous decomposition of $\mathbf{C}[z, \bar{z}]$. For $m, n \geq 0$ let $M[m, n]$ be the $(m+1) \times (n+1)$ block of Toeplitz form whose first row has entries given by $\gamma_{mn}, \gamma_{m+1,n-1}, \ldots, \gamma_{m+n,0}$ and whose first column has entries given by $\gamma_{mn}, \gamma_{m-1,n+1}, \ldots, \gamma_{0,n+m}$ (as a consequence, the lower right-hand corner of $M[m,n]$ is γ_{nm}). The matrix $M \equiv M(\gamma)$ is then built as follows:

$$M := \begin{pmatrix} M[0,0] & M[0,1] & M[0,2] & \cdots \\ M[1,0] & M[1,1] & M[1,2] & \cdots \\ M[2,0] & M[2,1] & M[2,2] & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

It is now not hard to see that the above mentioned positivity (1.0.2) is equivalent to the condition $M \geq 0$, as a quadratic form on $\mathbf{C}^{\omega}$. Suppose now that we are just given a double-indexed sequence $\gamma \equiv \{\gamma_{ij}\}$ subject to the constraints $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$ for all i, j. The classical (complex) *full* moment problem asks for necessary and sufficient conditions on the sequence γ to guarantee the existence of a positive Borel measure μ which interpolates γ, i.e.,

$$\int \bar{z}^i z^j \, d\mu(z, \bar{z}) = \gamma_{ij} \qquad (i, j \geq 0). \tag{1.0.3}$$

An obvious necessary condition is then $M \geq 0$; this corresponds to the positivity of the Riesz functional $\Lambda_\gamma(p) := \sum_{ij} a_{ij}\gamma_{ij}$ on the cone Σ^2 generated by polynomials of the form $p\bar{p}$. If K is a closed subset of $\mathbf{C}$, the Riesz-Haviland Criterion states that γ admits a representing measure supported on K if and only if $\Lambda_\gamma(p) \geq 0$ for every polynomial p which is nonnegative on K.

With γ, Λ_γ, M and K as above, suppose there exists a polynomial q such that $K = K_q := \{z \in \mathbf{C} : q(z, \bar{z}) \geq 0\}$. In the presence of a representing measure μ supported on K, the inequality $L_q(p\bar{p}) := L(qp\bar{p}) = \int_K qp\bar{p} \geq 0$ (all $p \in \mathbf{C}[z, \bar{z}]$) must hold, in addition to $L(p\bar{p}) \geq 0$ (all $p \in \mathbf{C}[z, \bar{z}]$). Therefore both conditions are necessary for the existence of a representing measure supported in K. K. Schmüdgen established in [34, Theorem 1] that for K_q compact these two conditions are indeed sufficient, and this is the case also for compact sets K which are *semi-algebraic*, that is, obtained as the intersection of a finite family of K_q's. (For related results, see [34, Corollary 3], [3], [32], [38].)

The *truncated* complex moment problem (TCMP) corresponds to the case when only an *initial segment* of γ is known. Our approach to TCMP follows the strategy we employed to solve the *real* TMP [9]. Indeed, part of the overall strategy can still be carried out, and concrete conditions can be found in a number of fundamental cases.

What must be used now is a combination of a few revealing examples (cf. [10, Chapter 6], [11, Sections 2, 3, 4, and Appendix], [14]) and the interplay between $M(n)$ and $M(n)_q$, a new associated matrix we have introduced in [14].

Theorem 1.0.1. [14, Theorem 1.1] Let $M(n) \geq 0$ and suppose $\deg q = 2k$ or $2k-1$. There exists rank $M(n)$-atomic representing measure supported in K_q if and only if there is some flat extension $M(n+1)$ for which $M_q(n+k) \geq 0$. In this case, there exists such a representing measure having exactly rank $M(n) -$ rank $M_q(n+k)$ atoms in $\mathcal{Z}(q) := \{z \in \mathbf{C} : q(z, \bar{z}) = 0\}$.

M_q keeps track of the location of the support, and this in turn can be used to establish additional constraints when searching for representing measures.

Let $\mathcal{P} \equiv \mathbb{R}[x, y]$ denote the space of real valued 2-variable polynomials, and for $k \geq 1$, let $\mathcal{P}_k \equiv \mathbb{R}_k[x, y]$ denote the subspace of $\mathcal{P}$ consisting of polynomials p with $\deg p \leq k$. Corresponding to β we have the *Riesz functional* $\Lambda \equiv \Lambda_\beta : \mathcal{P}_{2n} \rightarrow$

$\mathbb{R}$, which associates to an element p of $\mathcal{P}_{2n}$, $p(x,y) \equiv \sum_{0 \le i+j \le 2n} a_{ij} x^i y^j$, the value $\Lambda(p) := \sum_{0 \le i+j \le 2n} a_{ij} \beta_{ij}$; of course, in the presence of a representing measure μ, we have $\Lambda(p) = \int p \, d\mu$. In the sequel, $\hat{p}$ denotes the coefficient vector (a_{ij}) of p.

Following [10], we associate to β the *moment matrix* $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$, with rows and columns $X^i Y^j$ indexed by the monomials of $\mathcal{P}_n$ in degree-lexicographic order; for example, with $n = 2$, the columns of $\mathcal{M}(2)$ are denoted as $\mathit{1}, X, Y, X^2, XY, Y^2$. The entry in row $X^i Y^j$, column $X^k Y^\ell$ of $\mathcal{M}(n)$ is $\beta_{(i+k, j+\ell)}$, so $\mathcal{M}(n)$ is a real symmetric matrix characterized by

$$\langle \mathcal{M}(n)\hat{p}, \hat{q} \rangle = \Lambda(pq) \;\; (p, q \in \mathcal{P}_n). \tag{1.0.4}$$

If μ is a representing measure for β, then $\langle \mathcal{M}(n)\hat{p}, \hat{p} \rangle = \Lambda(p^2) = \int p^2 d\mu \ge 0$; since $\mathcal{M}(n)$ is real symmetric, it follows that $\mathcal{M}(n)$ is positive semidefinite (in symbols, $\mathcal{M}(n) \ge 0$). The *algebraic variety* of β (or of $\mathcal{M}(n)(\beta)$) is defined by

$$\mathcal{V} \equiv \mathcal{V}_\beta := \bigcap_{p \in \mathcal{P}_n, \hat{p} \in \ker \mathcal{M}(n)} \mathcal{Z}(p),$$

where $\mathcal{Z}(p) := \{(x,y) \in \mathbb{R}^2 : p(x,y) = 0\}$. Each element of $\ker\ \mathcal{M}(n)$ is of the form $\hat{p}$ for some $p \in \mathcal{P}_n$, and corresponds to a column dependence relation that we denote by $p(X,Y) = 0$. If β admits a representing measure μ, then $p \in \mathcal{P}_n$ satisfies $p(X,Y) = 0$ if and only if $\operatorname{supp}\ \mu \subseteq \mathcal{Z}(p)$ [10, Proposition 3.1]. Thus $\operatorname{supp}\ \mu \subseteq \mathcal{V}$, and it follows from [11, (1.7)] that $r := \operatorname{rank}\ \mathcal{M}(n)$ and $v := \operatorname{card}\ \mathcal{V}$ satisfy $r \le \operatorname{card}\ \operatorname{supp}\ \mu \le v$. Further, in this case, if $p, q, pq \in \mathcal{P}_n$ and $p(X) = 0$ in the column space of $\mathcal{M}(n)$, then $(pq)(X) = 0$. To summarize the preceding discussion, we have the following basic necessary conditions for the existence of a representing measure for $\beta^{(2n)}$:

$$\text{(Positivity)} \quad \mathcal{M}(n) \geq 0; \tag{1.0.5}$$

$$\text{(Recursiveness)} \quad p, q, pq \in \mathcal{P}_n,\ p(X) = 0 \Longrightarrow (pq)(X) = 0; \tag{1.0.6}$$

$$\text{(Variety Condition)} \quad r \leq v, \text{ i.e., } \operatorname{rank}\ \mathcal{M}(n) \leq \operatorname{card}\ \mathcal{V}_\beta. \tag{1.0.7}$$

It was shown in [12, Section 4] that there exists $\mathcal{M}(3) > 0$ (positive definite) for which β has no representing measure. Since an invertible moment matrix satisfies (1.0.6) and (1.0.7) vacuously, it follows that in general (1.0.5)-(1.0.7) are not sufficient for the existence of representing measures. Nevertheless, it is of interest to identify cases where (1.0.5)-(1.0.7) are sufficient, because these conditions are "concrete." Indeed, only elementary linear algebra is needed to compute $r \equiv \operatorname{rank}\ \mathcal{M}(n)$ and to check positivity and recursivenesss, and for moderate values of n, mathematical software can be used to estimate $v \equiv \operatorname{card}\ \mathcal{V}_\beta$.

In [8], R. Curto and L. Fialkow showed that on the real line $\mathbb{R}$ (the *truncated Hamburger Moment Problem*), positivity and recursiveness of the associated Hankel matrix are sufficient for the existence of a representing measure supported in $\mathbb{R}$. In $\mathbb{R}^2$ and when $\deg\ p \leq 2$, the results of [15], [16] and [17] together show that $\beta^{(2n)}$ has a representing measure supported in the curve $p(x, y) = 0$ if and only if $\mathcal{M}(n)$ has a column dependence relation $p(X, Y) = 0$ and (1.0.5)-(1.0.7) hold. Further, in the *truncated complex moment problem*, a planar complex multisequence $\gamma^{(2n)}$ admits a representing measure supported in the (finite) variety $z^k - q(z, \bar{z}) = 0$ (where *deg* $q < k \leq [n/2] + 1$) if and only if the associated complex moment matrix $M(n)(\gamma)$

is positive and recursively generated, and has a column relation $Z^k = q(Z, \bar{Z})$. The preceding results motivate the study of cubic column relations, to which we will devote this dissertation.

In [20], R. Curto, L. Fialkow and M. Möller initiated the study of extremal moment problems, that is, the case when $r := \text{rank } \mathcal{M}(n)(\beta)$ and $v \equiv \text{card } \mathcal{V}_\beta$ satisfy $r = v$. In this dissertation we will focus on the planar $\mathcal{M}(3)$ moment problem with $\mathcal{M}(2) > 0$. For the special case of a column relation $Y = X^3$ (so that representing measures are necessarily supported in the curve $y = x^3$), a detailed study was carried out in [20]. In such a setting, the extremal case can occur only with $r = v = 7$ or $r = v = 8$. For the former case, it was shown in [20, Section 4] that positivity and recursiveness are sufficient for representing measures. But in the case $r = v = 8$, a new condition, Consistency, is needed, as Recursiveness won't do:

$$\text{(Consistency)} \qquad p \in \mathcal{P}_{2n},\ p|_{\mathcal{V}_\beta} = 0 \Longrightarrow \Lambda_\beta(p) = 0. \tag{1.0.8}$$

The main result in [20] is:

Theorem 1.0.2. Let $d \geq 1$. For $\beta \equiv \beta^{(2n)}$ extremal, i.e., $r = v$, the following are equivalent:

(i) β has a representing measure;

(ii) β has a unique representing measure, which is rank $\mathcal{M}(n)$-atomic;

(iii) $\mathcal{M}(n)$ is positive semidefinite and β is consistent.

We use this result to study cubic column relations with associated finite algebraic variety. In Chapter 5, we will discuss a case of sextic moment problems with a cubic harmonic column relation, $Z^3 = itZ + u\bar{Z}$ $(t, u \in \mathbb{R})$. It will be shown that the polynomial has symmetric seven zeros for certain values of t and u. In fact, this problem turns out to be extremal and a solution will be presented by checking consistency of the moment matrix after acquiring a representation theorem of polynomials vanishing on the algebraic variety $\mathcal{V}$, namely $\mathcal{I}_{\mathcal{V}}$.

In Chapter 6, sextic moment problems with more general cubic column relations in real form will be studied. Our focus is on the extremal cases and a modest application for non-extremal cases will be suggested. Since $\mathcal{M}(3)$ has 10 columns, we might think there are three types of extremal cases: $r = v = 7$, $r = v = 8$, and $r = v = 9$, but the last one does not happen due to the fact that only one column relation leads to an infinite algebraic variety. While, for the harmonic case, a representation theorem of $\mathcal{I}_{\mathcal{V}}$ is achieved by dimensional analysis between spaces, division algorithm for two variable is used instead for general cases. The set $\mathcal{I}_{\mathcal{V}}$ behaves like an ideal and contains polynomials in the column relations and a few more obtained from solving Vandermonde equations.

CHAPTER 2
NOTATION AND PRELIMINARIES

In order to present our results, we need to introduce a number of concepts and ideas. We also need to describe some foundational results which will enable us to build the proofs of our main results. We organize the basic terminology and examples into three main areas: functional analysis, algebraic geometry, and moment matrices. In each case, we will try to present the material in a way as accessible as possible. Throughout this chapter, we assume that the reader is familiar with the material in a graduate course in real analysis, and in a theoretical course in linear algebra, including the notion of positive semi-definiteness for matrices. Although we will discuss some introductory material from real algebraic geometry, we only require basic knowledge of the vector space of polynomials in one or more real or complex variables; this knowledge is actually presented in standard courses in linear algebra. I used the following books to prepare this chapter, and I am grateful to have learned many concepts and mathematical results from: R. Bix, *Conics and Cubics*, J. B. Conway, *A Course in Functional Analysis*, D. Cox, J. Little and D. O'Shea, *Ideals, Varieties and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, G. B. Folland, *Real Analysis : Modern Techniques and Their Applications*, P. R. Halmos, *A Hilbert Space Problem Book*, R. Horn and C. Johnson, *Matrix Analysis*, E. Kreyszig, *Introductory Functional Analysis with Applications*, H. Royden, *Real Analysis*, and W. Rudin, *Real and Complex Analysis*.

The first part of Chapter 2 deals with Basic Results from Functional Analysis.

2.1 Finite Dimensional Hilbert Spaces

2.1.1 Vector Spaces

Suppose $\mathbb{F}$ is either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. A *vector space* over $\mathbb{F}$ is a set V closed under addition $V \times V \to V$ and scalar multiplication $\mathbb{F} \times V \to V$ and satisfying the following properties:

(i) Additive identity: There exists an element $0 \in V$ such that $0 + v = v$ for all $v \in V$;

(ii) Additive inverse: For every $v \in V$, there exists an element $w \in V$ such that $v + w = 0$;

(iii) Multiplicative identity: $1v = v$ for all $v \in V$;

(iv) Commutativity: $u + v = v + u$ for all $u, v \in V$;

(v) Associativity: $(u + v) + w = u + (v + w)$ and $(\alpha\beta)v = \alpha(\beta v)$ for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{F}$;

(vi) Distributivity: $\alpha(u + v) = \alpha u + \alpha v$ and $(\alpha + \beta)u = \alpha u + \beta u$ for all $u, v \in V$ and $\alpha, \beta \in \mathbb{F}$.

A vector space over $\mathbb{R}$ is called a real vector space and a vector space over $\mathbb{C}$ is called a complex vector space. The elements in a vector space are called vectors.

2.1.2 Matrices

A *matrix* is a rectangular array of objects. An item in a matrix is called an *entry* and the size of matrix is denoted as $m \times n$, where m is the number of rows and n is the number of columns.

Matrices have been used in a variety of areas; they are a key tool in linear algebra. They are particularly helpful to represent mappings between vector spaces (which we will define later.) Analyzing the mappings has intrinsic interest and also allows us to investigate the spaces more concretely.

We may also consider matrices with infinitely many rows and/or columns. Through proper indexing of rows and columns, any entry in the infinite matrix can be well-defined. The basic operations of addition, subtraction, scalar multiplication and transposition, which are well known for finite matrices, can be extended to infinite matrices. One use of infinite matrices is to describe operators on Hilbert spaces, which can be thought of as natural generalizations of the notion of Euclidean space.

2.1.3 Gaussian Elimination and Rank of a Matrix

Through a sequence of three fundamental operations (so-called *elementary operation*), it is possible to convert any matrix into a simple and unique form. The entire sequence of row operations is called *row reduction*. Focusing on rows, the operations are as follows: (i) Interchanging of two rows, (ii) Multiplication of a row by a nonzero scalar, and (iii) Addition of a scalar multiple of one row to another row.

Let $M_{m,n}(\mathbb{C})$ be the set of all matrices of size $m \times n$ with complex number entries. To each $A \in M_{m,n}(\mathbb{C})$, after a sequence of elementary operations, we may associate a canonical form, called the *row-reduced echelon form* of A. The form satisfies all of the following:

(i) Each nonzero row has 1 as the first nonzero entry;

(ii) All other entries in the column with the leading entry 1 (called a *pivot*) are zero;

(iii) The leading 1's of a nonzero row is always strictly to the right of the leading coefficient of the row above it;

(iv) Any rows with all zero entries appear at the bottom of the matrix.

For example, let $A := \begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 7 & 0 \\ 2 & 1 & 5 & 8 \end{pmatrix}$. Then its row-reduced echelon form is $\begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

2.1.4 Rank of a Matrix

There are a few important nonnegative integers associated with each matrix $A \in M_{m,n}(\mathbb{C})$, one of which is the *rank*, denoted as rank A. It is the largest number of linearly independent columns of A; it is also the largest number of linearly independent rows of A. The rank of a matrix is unchanged under row operations. Thus, to evaluate the rank of a matrix, we may use the above mentioned row reduction, and then calculate the rank of the reduced matrix by inspection. In other words, rank A

is equal to the rank of the row-reduced echelon form of A that is the number of nonzero rows in the row-reduce form of the matrix.

2.1.5 Row Reduction with *Mathematica*

Since we will use matrices of relatively large size, we require assistance from a computer. Many of the examples, and parts of the proofs of some results in this dissertation were attained by using calculations with the software tool *Mathematica* *[40]*. One advantage with *Mathematica* (as compared to other software packages) is that it handles symbolic calculations well, and it remains within the smallest subfield of $\mathbb{C}$ generated by all the coefficients present in a calculation. However, care must be exercised when dealing with abstract quantities, as the following example illustrates.

Example 2.1.3. Recall that for $A \in M_{m,m}(\mathbb{C})$, rank $A = m$ if and only if the determinant of A is nonzero. Let

$$A := \begin{pmatrix} a & b & b \\ 0 & c & d \\ 0 & e & f \end{pmatrix}$$

In this case, *Mathematica* will show rank $A = 3$ when we type `MatrixRank[A]`). However, det $A = a(cf - de)$ and hence if $a = 0$ or $cf = de$, then rank A will be less than 3. What happens is the *Mathematica* assumes (implicitly) that any character different from the character 0 is nonzero.

Therefore, when we wish to obtain the row reduction of a matrix using *Mathematica* we must resort to a step-by-step approach; alternatively, we could write down an algorithm and add it to the *Mathematica* toolbox.

2.1.6 Linear Transformations Between Vector Spaces

Let V, W be vector spaces over $\mathbb{C}$. A *linear transformation* (or *operator*) $T : V \to W$ is a function satisfying

$$T(v+w) = T(v) + T(w), \qquad T(\alpha v) = \alpha T(v).$$

for all $v, w \in V$ and $\alpha \in \mathbb{C}$. An example of linear transformation is operator of the definite integration T_I from the set of all continuous function on $[a, b]$, denoted as $C[a, b]$ to $\mathbb{R}$ defined by

$$T_I(f) = \int_a^b f(x)dx$$

for $f \in C[a, b]$.

2.1.7 Linear Functionals on Vector Spaces

A *linear functional* is a linear transformation from a vector space to the set of complex numbers.

Example 2.1.4. The above mentioned operator T_I of definite integration is also a functional on the set of polynomials of degree less than or equal to n, denoted as $\mathcal{P}_n$. Let $x_0, \ldots, x_n$ be $n+1$ distinct points in $[a, b]$. Then it is known that for all $f \in \mathcal{P}_n$ there are coefficients $w_0, \ldots, w_n$ for which

$$T_I(f) = \sum_{k=0}^{n} w_k f(x_k),$$

where the right hand side is called a *quadrature formula*. The points x_k's are called *nodes*, and the constants w_k's are called *weights*. We will see that a representing measure of a moment sequence has a similar form.

2.1.8 Fundamental Theorem of Linear Algebra

The Fundamental Theorem of Linear Algebra states that the sum of the rank and the dimension of the kernel of a matrix is the same as the number of columns of the matrix. We can apply this result to linear transformations as well. Let V and W be vector spaces over a field and let $T : V \to W$ be a linear transformation. Then we have rank $T + \ker T = \dim V$.

2.1.9 Inner Products

A Hilbert space is a generalization of the classical finite-dimensional Euclidean space, in which we retain the structure of the inner product and with that the notion of orthogonality. A Hilbert space can have finite or infinite dimension. First and foremost, a Hilbert space is a vector space. But in addition, a Hilbert space carries an inner product, which we now define. This inner product allows us to study geometric properties of the space.

Definition 2.1.5. Let V be a vector space over $\mathbb{C}$. A *semi-inner product* on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that for all $\alpha, \beta \in \mathbb{C}$, and $u, v \in V$, the following are satisfied:

(i) $\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle$;

(ii) $\langle u, \alpha v + \beta w\rangle = \bar{\alpha}\langle u, v\rangle + \bar{\beta}\langle u, w\rangle$;

(iii) $\langle u, u\rangle \geq 0$;

(iv) $\langle u, v\rangle = \overline{\langle v, u\rangle}$.

Note that property (i) with $\alpha = 0$ implies that $\langle 0, v\rangle = \langle \alpha \cdot 0, v\rangle = \alpha\langle 0, v\rangle = 0$ for all $v \in V$. Similarly, we see $\langle u, 0\rangle = 0$ for all $u \in V$.

An *inner product* on V is a semi-inner product such that if $\langle u, u\rangle = 0$, then $u = 0$. Further, an inner product on V defines a *norm* on V given by

$$||u|| := \sqrt{\langle u, u\rangle}.$$

Example 2.1.6. Let l^2 be the set of all functions $x : \mathbb{N} \to \mathbb{C}$ such that $x(i)$ is zero for all but a countable number of i and $\sum_{i=1}^{\infty} |x(i)|^2 < \infty$. For $x, y \in l^2$, if we set $\langle x, y\rangle := \sum_{i=1}^{\infty} x(i)\overline{y(i)}$, then this defines an inner product on l^2.

Definition 2.1.7. An element x of an inner product X is said to be *orthogonal* to an element $y \in X$ if $\langle x, y\rangle = 0$.

Example 2.1.8. Let $\mathbb{T}$ be the unit circle in the Euclidean plane. The integral of a function $f(z)$ on $\mathbb{T}$ is defined by $\int f(z)dz := \frac{1}{2\pi i}\int_{-\pi}^{\pi} f(e^{i\theta})e^{i\theta}d\theta$ with $z := e^{i\theta} = \cos\theta + i\sin\theta$. Also, let $L^2(\mathbb{T}) := \{f : \mathbb{T} \to \mathbb{C}|$ continuous and $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta < \infty\}$. Then we can verify that any element in $L^2(\mathbb{T})$ may be written as a linear combination of analytic monomials z^n's and the vectors z^n's are mutually orthogonal. In fact, with the inner product $\langle p, q\rangle := \int p\bar{q}dz$,any $f \in L^2(\mathbb{T})$ can be written as

$$f = \sum_{k=-\infty}^{\infty} \langle f, z^k\rangle z^k.$$

2.1.10 Positivity of Matrices

One important necessary condition for the existence of a representing measure of a moment sequence is the positive semi-definiteness of the associated moment matrix. We give a formal definition of this notion.

A *Hermitian matrix* is a square matrix which is invariant under conjugate transpose. An $n \times n$ Hermitian matrix A with complex number entries is said to be *positive semidefinite* (or *positive* for short) if

$$u^* A u \geq 0 \qquad \text{for all nonzero } u^T \in \mathbb{C}^n;$$

(here $*$ means conjugate transpose, and u^T indicates that we regard vectors in $\mathbb{C}^n$ as column vectors). We write $A \geq 0$. If the inequality is strict, then A is said to be *positive definite* (or *strictly positive*); denoted as $A > 0$. Thus strict positivity implies that $A \geq 0$ and A is invertible, that is, the determinant of A is nonzero.

A good example of a positive matrix is the Hilbert matrix, whose entries are of the form:

$$h_{ij} = \frac{1}{i+j-1}.$$

$(i, j \geq 1)$. We see that the $n \times n$ Hilbert matrix is Hankel:

$$H = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{pmatrix}.$$

From the perspective of moment problems, the Hilbert matrix can be viewed as derived from the integral

$$h_{ij} = \int_0^1 x^{i+j-2}\, dx,$$

which is a truncated version of the Hamburger moment problem. We will prove that the Hilbert matrix is positive semidefinite in Subsection 2.8.1.

2.1.11 Inner Products Generated by Positive Matrices

Using a positive matrix $A \in M_{m,m}(\mathbb{C})$, it is possible to define a semi-inner product associated with A. Let $u^T, v^T \in \mathbb{C}^m$. Define $\langle u, v\rangle_A := \langle Au, v\rangle$. It is straightforward to show that $\langle \cdot, \cdot\rangle_A$ satisfies (i) and (ii) in Definition 2.1.5. For (iii), $\langle u, u\rangle_A = \langle Au, u\rangle = u^*Au \geq 0$ and for (iv), $\langle u, v\rangle_A = \langle Au, v\rangle = v^*Au = \overline{v^T A^T \bar{u}} = \overline{(\bar{u}^T Av)^T} = \overline{u^*Av} = \overline{\langle Av, u\rangle} = \langle v, u\rangle_A$. Lastly, we note that it is not an inner product since $\langle u, u\rangle_A = 0$ for possibly nonzero vector $u \in \ker\ A$. However, if A is invertible, then we do have a genuine inner product.

2.2 Infinite Dimensional Hilbert Spaces

2.2.1 Hilbert spaces

An *inner product space* is a vector space over a field equipped with an inner product $\langle \cdot, \cdot\rangle$. A *Hilbert space* is a complete inner product space $\mathcal{H}$ in the metric defined by $d(x, y) := ||x-y||$. More importantly, we note that every Hilbert space has an orthonormal (orthogonal and unit-norm) basis. We have already seen examples of Hilbert spaces like l^2 and $L^2(\mathbb{T})$.

2.2.2 Bounded Operators on Hilbert Space

Proposition 2.2.9. [6, Proposition II.1.1] Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $T : \mathcal{H} \to \mathcal{K}$ a linear transformation. The following are equivalent:

(i) T is continuous;

(ii) T is continuous at 0;

(iii) T is continuous at some point;

(iv) There is a constant $c > 0$ such that $||Tx|| \leq c||h||$ for all $x \in \mathcal{H}$.

We set $||T|| := \sup \{||Tx|| : x \in \mathcal{H}, ||x|| \leq 1\}$ and call it the *norm* of T; a linear transformation with finite norm is called a *bounded operator.*

Example 2.2.10. A matrix $A \in M_{m,n}(\mathbb{C})$ defines a operator $T : \mathbb{C}^n \to \mathbb{C}^m$ by means of $y = Ax$, where $x = (x_i) \in \mathbb{C}^n$ and $y = (y_i) \in \mathbb{C}^m$. With the norm $||x|| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$, it can be shown that T is bounded.

2.2.3 Operator Matrices

A matrix on a finite dimensional Hilbert space is a very neat example of a bounded operator. Our aim now is to discuss how to decompose any operator as a block matrix. Given an orthonormal basis, a Hilbert space can be expressed as the direct sum of one-dimensional subspaces. Thus, our discussion below extends to more general direct sums of Hilbert spaces.

Let $\mathcal{H}$ be a Hilbert space, with an orthogonal decomposition as $\mathcal{H} \equiv \mathcal{H}_1 \oplus \mathcal{H}_2$.

A bounded linear operator $T \in \mathcal{L}(\mathcal{H})$ admits a decomposition as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

where $T_{ij} : \mathcal{H}_j \longrightarrow \mathcal{H}_i$ is bounded and linear $(i, j = 1, 2)$. We say that T admits an 2×2 operator matrix representation, and we call the T_{ij} operator entries. Of special interest to us is the case when $\mathcal{H}$ is finite dimensional. If dim $\mathcal{H}_1 = m$ and dim $\mathcal{H}_2 = n$, then T_{11} can be identified with an $m \times m$ matrix, T_{22} with an $n \times n$ matrix, and T_{12} (resp. T_{21}) with a rectangular matrix of size $m \times n$ (resp. $n \times m$). For the purposes of checking whether an $N \times N$ matrix T is positive semi-definite, we will often split T as a 2×2 operator matrix, by selecting the first m rows and columns and forming a submatrix, denoted T_{11}, and then by denoting the remaining blocks as T_{12}, T_{21} and T_{22}.

2.3 Measures

2.3.1 Borel Measures

A measure is a map from a σ-algebra to the set of complex numbers. We first consider the families of sets used as domains of measures. Suppose X is a nonempty set. An *algebra* over X is a nonempty collection of subsets of X that is closed under complements and finite unions. A σ-*algebra* is an algebra which is closed under countable unions. Here are examples of σ-algebras: the family consisting only of the empty set and the X (the minimal σ-algebra over X), the power set of X (the maximal σ-algebra over X), and the collection of subsets of X which are countable or whose complements are countable (This is the σ-algebras generated by the singleton subsets of X.)

If X is a topological space, then the *Borel σ-algebra* is defined as the *σ-algebra* generated by all open sets in X and is denoted by $\mathcal{B}_X$. We call the members in the family *Borel sets*. We thus note that $\mathcal{B}_X$ contains all closed sets, and also countable intersections and unions of closed or open sets. For instance, $\mathcal{B}_{\mathbb{R}}$ contains intervals of the form: (a,b), $[a,b]$, $(a,b]$, $(-\infty,b)$, $[a,\infty)$, and so on.

We next define a *measure* on a σ-algebra $\mathfrak{M}$ over a set X as a function $\mu : \mathfrak{M} \to [0,\infty]$ such that

(i) $\mu(\varnothing) = 0$;

(ii) $\mu\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$ for a sequence $\{E_i\}_{i=1}^{\infty}$of mutually disjoint sets in X (this property is called *countable additivity*.)

A measure whose domain is the Borel σ-algebra $\mathcal{B}_X$ is called *Borel measure* on X. The triple $(X,\mathfrak{M},\mu)$ is said to be a *measure space*.

We can summarize the basic properties of measures in the following theorem.

Theorem 2.3.11. [25, 1.8 Theorem] Let $(X,\mathfrak{M},\mu)$ be a measure space.

(i) (*Monotonocity*) If $E, F \in \mathfrak{M}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$;

(ii) (*Subadditivity*) If $\{E_i\}_{i=1}^{\infty} \subseteq \mathfrak{M}$, then $\mu\left(\cup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$;

(iii) (*Continuity from below*) If $\{E_i\}_{i=1}^{\infty} \subseteq \mathfrak{M}$ and $E_1 \subseteq E_2 \subseteq \cdots$, then $\mu\left(\cup_{i=1}^{\infty} E_i\right) = \lim_{i\to\infty} \mu(E_i)$;

(iv) (*Continuity from above*) If $\{E_i\}_{i=1}^{\infty} \subseteq \mathfrak{M}$ and $E_1 \supseteq E_2 \supseteq \cdots$, then $\mu\left(\cap_{i=1}^{\infty} E_i\right) = \lim_{i\to\infty} \mu(E_i)$;

If a set has measure zero, then the set is called a *null* set. By subadditivity, any countable union of null sets is a null set, which means there are many sets of measure zero other than the empty set.

We should examine a few more examples of measures that are practically important with trivial nature. Let X be a nonempty set, $\mathfrak{M}$ be the power set of X. Suppose $f : X \to [0, \infty]$ is a function. Then a measure $\mu \equiv \mu_f$ on $\mathfrak{M}$ is defined by $\mu(E) \equiv \mu_f(E) := \sum_{x\in E} f(x)$ associated to f. Two special cases are crucial: If $f(x) = 1$ for all x, then μ is called *counting measure*; and if for some $x_0 \in X$,

$$f(x) := \begin{cases} 1 & \text{if } x = x_0; \\ 0 & \text{if } x \neq x_0, \end{cases}$$

then μ is called the *point mass* or *Dirac measure* at x_0. Finally, we can discuss the most important measure on $\mathbb{R}$ as another special case. Let $X = \mathbb{R}$ and $f(x) = x$. Then in particular the measure of an interval is simply its length. The measure associated to f on $\mathcal{B}_{\mathbb{R}}$ is said to be *Lebesque measure.*

2.3.2 Linear Functionals from Integrals

We now introduce a rough description of integration on a general measure space. Let $(X, \mathfrak{M}, \mu)$ be a measure space and f be a complex-valued function on X. Then the sets in $\mathfrak{M}$ are called *measurable sets* and the function f is called a *measurable function* if $f^{-1}(E)$ is a measurable set in $\mathfrak{M}$ for any $E \in \mathcal{B}_{\mathbb{C}}$.

If $E \subseteq X$, the *characteristic function* χ_E of E is defined by

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E; \\ 0 & \text{if } x \notin E. \end{cases}$$

A *simple function* on X is a finite linear combination of characteristic functions with complex coefficients. An important fact is that any measurable function is approximated by a sequence of simple functions; explicitly, for any measure function $f : X \to \mathbb{C}$, there is a sequence $\{\varphi_n\}$ of simple functions such that $0 \leq |\varphi_1| \leq |\varphi_2| \leq \cdots \leq |f|$, and $|\varphi_n| \to f$ pointwise.

The *integral* of a simple function $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$ with respect to μ is defined by

$$\int \varphi d\mu := \sum_{i=1}^n a_i \mu(E_i).$$

Since we have an approximation of f as above, the following definition makes sense:

$$\int f d\mu := \sup \left\{ \int \varphi d\mu : 0 \leq |\varphi| \leq |f| \text{ and } \varphi \text{ is a simple function} \right\}.$$

Now, we can now discuss more general L^p spaces.

Example 2.3.12. Let $(X, \mathfrak{M}, \mu)$ be a measure space, and assume that $\mu(X) < \infty$. Let $L^2(\mu) \equiv L^2(X, \mathfrak{M}, \mu) := \left\{ f \mid f : X \to \mathbb{C} \text{ is measurable and } \sqrt{\int |f|^2} < \infty \right\}$. If f and $g \in L^2(\mu)$, then Hölder's inequality implies $f\bar{g} \in L^1(\mu) \subseteq L^2(\mu)$. If we set

$$\langle f, g \rangle := \int f \bar{g} d\mu,$$

then this defines an inner product on $L^2(\mu)$.

Also, we can consider "positivity" of a linear functional defined by the integral. A linear functional L is said to be *positive*, denoted as $L \geq 0$, if for any f satisfying $f(X) \geq 0$, $L(f) \geq 0$.

Proposition 2.3.13. Let μ be a measure space on a locally compact set X, and let $L_\mu : C(X) \to \mathbb{C}$ be given by $L_\mu(f) = \int f d\mu$, where $C(X)$ is the set of all continuous function on X. Then $L_\mu \geq 0$.

We finish this section with deriving a linear functional from integrals. Given a measure μ and a complex-valued function on X, a linear functional is defined as follows:

$$\varphi(f) := \int f d\mu$$

and we note that $|\varphi(f)| = \left|\int f d\mu\right| \leq \int |f| d\mu = ||f||$, where $||\cdot||$ is the norm in $L^1(\mu)$. Conversely, the Riesz representation theorem states that to each bounded linear functional φ on a Hilbert space $L^2(X)$, there corresponds a vector $g \in L^2(X)$ such that $\varphi(f) := \langle f, g\rangle = \int f\bar{g} d\mu$ for all $f \in L^2(X)$. Furthermore, on a locally compact Hausdorff space X we may have another version of the Riesz representation theorem. Suppose $L \geq 0$ on $C_c(X)$, the collection of all continuous functions on X whose support is compact. Then there exists a σ-algebra $\mathfrak{M}$ in X which contains all Borel set in X, and there is a unique measure μ on $\mathfrak{M}$ such that

$$L(f) := \int_X f d\mu \qquad \text{for every } f \in C_c(X).$$

2.3.3 Finitely Atomic Borel Measures

In this section, we will discuss a measure that is a type of representing measure for a moment sequence. Let $(X, \mathfrak{M}, \mu)$ be a measure space. A set $E \in \mathfrak{M}$ is called an *atom* if $\mu(E) > 0$ and for any measurable set $F \subseteq E$, $\mu(F) = 0$ or $\mu(E \setminus F) = 0$. In the Lebesgue measure space, we can show that if E is an atom, then E contains a singleton atom with the same measure. In this case, a singleton atom is described as a "point mass".

Finally, a *finitely atomic Borel measure* in the Euclidean space is defined as

$$\mu := \sum_{k=1}^{l} \rho_k \delta_{w_k},$$

where the ρ_k's are called *densities* and the w_k's are atoms of the measure. For example, if $\{(x_i, y_i)\}_{k=1}^{\ell}$ is the set of atoms, then

$$\int x^i y^j \, d\mu = \sum_{k=1}^{\ell} \rho_k x_k^i y_k^j;$$

in due time we will see that moments will be represented by these integrals if the truncated moment sequence admits a measure.

2.3.4 Weak* Density in the Space of Measures

In this section, we will see any probability measure is a weak* limit of a discrete (or atomic) measure. Let μ be a measure on a compact space X. For $x \in X$, let δ_x be the *point mass* at x as

$$\delta_x(E) = \begin{cases} 1 & \text{if } x \in E; \\ 0 & \text{if } x \notin E. \end{cases}$$

For $x : x_1, \ldots, x_k \in X$, pick $\rho :\ \rho_1, \ldots, \rho_k > 0$ satisfying $\rho_1 + \cdots + \rho_k = 1$, let

$$\nu_{x,\rho} := \sum_{i=1}^{k} \rho_i \delta_{x_i}. \tag{2.3.1}$$

Then $\nu_{x,\rho}$ is called a *probability measure* on X. Note that the extreme points of $\mathfrak{P}(X) := \{\mu :$ measures on X satisfying $\mu(X) = 1\}$ consist of $\{\delta_x\}_{x \in X}$. The Krein-Millman theorem says that the weak* closure of the convex hull of extreme points of $\mathfrak{P}(X)$ is the same as $\mathfrak{P}(X)$. Thus, every probability measure μ is the weak* limit of a net of convex combinations of δ's; explicitly,

$$\mu = \text{w*-}\lim_{\alpha} \sum_{i=1}^{k(\alpha)} \rho_i^{(\alpha)} \delta_{x_i^{(\alpha)}},$$

where $\left\{ \sum_{i=1}^{k(\alpha)} \rho_i^{(\alpha)} \delta_{x_i^{(\alpha)}} \right\}_\alpha$ is a suitable net. A famous example is:

Example 2.3.14. Let λ be the Lebesgue measure on $[0, 1]$. Then it is well known that in this case the Lebesgue and Riemann integral are identical on continuous functions. If $\left\{ x_i^{(n)} \right\}$ is a partition of $[0, 1]$, then

$$\int_0^1 f d\lambda = \lim_{n \to \infty} \sum_{i=1}^{k} f\left(x_i^{(n)}\right)\left(x_i^{(n)} - x_{i-1}^{(n)}\right); \qquad f \in C[0, 1].$$

The second part of Chapter 2 deals with Basic Results from Algebraic Geometry.

2.4 Polynomials

We are familiar with polynomials in one or two variables, which are enough for the purpose of this dissertation because we focus on truncated moment problems of one-dimensional complex or two-dimensional real cases. However, R. Curto and

L. Fialkow have found several results with multivariable versions. For further study, we thus need to discuss polynomials and the polynomial space in the multivariable setting.

Let k be an arbitrary field and $x_1, \ldots, x_n$ be indeterminates. A *monomial* is a product of non-negative powers of $x_1, \ldots, x_n$; that is, $x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where $\alpha_i \geq 0$ for $i = 1, \ldots, n$. Let $\alpha := (\alpha_1, \ldots, \alpha_n)$. Then $|\alpha| := \alpha_1 + \cdots + \alpha_n$ is called the *total degree* of the monomial and we write $x^\alpha := x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

We now define a *polynomial* f in $x_1, \ldots, x_n$ as a finite linear combination with coefficients in k; $f = \sum_\alpha a_\alpha x^\alpha$, where a_α is in k. The *total degree* of f is the maximum $|\alpha|$ such that the coefficient α_n is nonzero, and is denoted as deg f. In the next two sections, we examine two types of polynomials in two indeterminates.

2.4.1 Polynomials in Two Real Variables

Let us define the set of polynomials in two real indeterminates with real coefficients:

$$\mathbb{R}[x, y] := \left\{ \sum_{0 \leq i,j} a_{ij} x^i y^j : a_{ij} \in \mathbb{R} \right\}.$$

For instance, $x^3 + xy^2 + y^2 - 1$ in $\mathbb{R}[x, y]$. This set is a real vector space and infinite dimensional. Polynomials in $\mathbb{R}[x, y]$ will be used to show column relations of a real moment matrix.

2.4.2 Polynomials in z and $\bar{z}$

The set of polynomials in z and $\bar{z}$ with complex coefficients denotes:

$$\mathbb{C}[z,\bar{z}] := \left\{ \sum_{0\le i,j} a_{ij}\bar{z}^i z^j : a_{ij} \in \mathbb{C} \right\}.$$

Similarly, this set is a complex vector space. Setting $z := x + iy$ and $\bar{z} := x - iy$, any polynomial in $\mathbb{C}[z,\bar{z}]$ can be rewritten with real and imaginary parts, which are two-variable polynomials in $\mathbb{R}[x,y]$. For example, $z^2 + 2\bar{z} \in \mathbb{C}[z,\bar{z}]$ is converted into $(x^2 - y^2 + 2x) + 2y(x-1)i$. Due to this conversion, we see $\mathbb{R}[x,y]$ and $\mathbb{C}[z,\bar{z}]$ can be regarded as equivalent, at least from the perspective of vector spaces.

2.4.3 Spaces of Polynomials

As seen above sections, polynomials of a certain degree form a vector space and we can therefore discuss dimension, linear independence, and special subsets of the space.

2.4.4 Lagrange Interpolation

This section is devoted to reviewing a well-known result that will be used in the proof of a representation theorem of polynomials vanishing on finite points. Lagrange's interpolation method is a simple and smart way, in the one-variable case, to build the unique polynomial exactly passing through given points. Explicitly, given $V := \{z_1, \ldots, z_s\} \subseteq \mathbb{C}$, define *Lagrange interpolation polynomial* $\ell_j(t)$ as

$$\ell_j(z) := \frac{\displaystyle\prod_{i=1, i\ne j}^{s} (z - z_i)}{\displaystyle\prod_{i=1, i\ne j}^{s} (z_j - z_i)} \qquad (1 \le j \le s).$$

Then any function $f : V \to \mathbb{C}$ satisfies $f(z) = \sum_{j=1}^{s} f(z_j)\ell_j(z)$. The notion we need in this thesis is that $\ell_j(z_i) = \delta_{ij}$, the Kronecker delta.

2.5 Ideals and Zero Sets

2.5.1 The Polynomial Space $\mathcal{P}_n$

In our work we will often use two-variable polynomials in $\mathbb{R}[x, y]$ and $\mathbb{C}[z, \bar{z}]$. Since they are equivalent, we will often abuse the notation, and switch freely from real to complex and vice versa. Let us denote $\mathcal{P}$ as $\mathbb{R}[x, y]$ or $\mathbb{C}[z, \bar{z}]$. Furthermore, the polynomial space $\mathcal{P}_n$ is defined as the set of all polynomials in $\mathbb{R}[x, y]$ or $\mathbb{C}[z, \bar{z}]$ of degree less than or equal to n.

2.5.2 Ideals in $\mathbb{C}[z, \bar{z}]$ and $\mathbb{R}[x, y]$

In this section, we discuss a special subset of $\mathcal{P}_n$. We will have in later chapters solutions of truncated moment problems obtained by analyzing subsets of $\mathcal{P}_n$ called an *ideal*-like sets. Here we first present the definition of an ideal.

Definition 2.5.15. If a subset $\mathcal{I} \in k[x_1, \ldots, x_n]$ satisfies:

(i) $0 \in \mathcal{I}$;

(ii) if $f, g \in \mathcal{I}$, then $f + g \in \mathcal{I}$;

(iii) if $f \in \mathcal{I}$ and $h \in k[x_1, \ldots, x_n]$, then $hf \in \mathcal{I}$,

then it is called an *ideal*.

Example 2.5.16. We consider a set of polynomials. For $f_1, \dots, f_s \in k[x_1, \dots, x_n]$, we set

$$\langle f_1, \dots, f_s \rangle := \left\{ \sum_{i=1}^{s} h_i f_i : h_1, \dots, h_s \in k[x_1, \dots, x_n] \right\}.$$

It is crucial to see that $\langle f_1, \dots, f_s \rangle$ is an ideal, which is called the *ideal generated by* $f_1, \dots, f_s$.

We then have an amazing fact:

Theorem 2.5.17. Hilbert Basis Theorem[7] Every ideal in $k[x_1, \dots, x_n]$ is generated by a finite number of polynomials.

While it is a strong result, in general it is difficult to determine the generating set. Another aspect to remark is that an ideal may be represented with many different groups of polynomials, as we show below.

Lemma 2.5.18. [7] Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal, and $f_1, \dots, f_s \in k[x_1, \dots, x_n]$. Then

$$f_1, \dots, f_s \in I \iff \langle f_1, \dots, f_s \rangle \subseteq I.$$

This lemma helps to show one ideal is contained in another.

Example 2.5.19. We want to show two ideals generated by different polynomials are identical: $\langle 2x^2 + 3y^2 - 11, x^2 - y^2 - 3 \rangle = \langle x^2 - 4, y^2 - 1 \rangle$. Since $2(x^2 - 4) + 3(y^2 - 1) = 2x^2 + 3y^2 - 11$, it follows $2x^2 + 3y^2 - 11 \in \langle x^2 - 4, y^2 - 1 \rangle$. By Lemma 2.5.18, we conclude that $\langle 2x^2 + 3y^2 - 11, x^2 - y^2 - 3 \rangle \subseteq \langle 2x^2 + 3y^2 - 11 \rangle \subseteq \langle x^2 - 4, y^2 - 1 \rangle$. The equation $(x+y+1)(2x^2+3y^2-11) + (-2x-2y-2)(x^2-y^2-3) = (5x+5y+5)(y^2-1)$ proves the other inclusion.

Also, we can consider ideals generated by sets. For example, if $f \in k[x_1, \ldots, x_n]$ and $V := \{(x_1, \ldots, x_n) \in k^n : f(x_1, \ldots, x_n) = 0\}$, then it is easy prove that $\{g \in k[x_1, \ldots, x_n] : g|_V = 0\}$ is an ideal and by the Hilbert Basis Theorem it must be generated by a finite number of polynomial obviously including f. For some cases, we can find those polynomial explicitly; this is what we do in proving the main results in this dissertation.

2.5.3 Zero Sets

Let k be an algebraically closed field and let f be a function in the ring $k[x_1, \ldots, x_n]$. Then a *zero set* of f is the set of all points in k^n where f vanishes, denoted $\mathcal{Z}(f)$. If $f(x, y) \in \mathbb{R}[x, y]$, then the algebraic curve of f is simply the zero set. However, the complex case is more subtle. For $p(z, \bar{z}) \in \mathbb{C}[z, \bar{z}]$, the zero set could be an algebraic curve or a set with a finite number of points. We should rewrite the complex polynomial as real and imaginary parts and then find the intersection of the zero sets of these two real polynomials.

Example 2.5.20. Let $z = x + iy$ for $x, y \in \mathbb{R}$. First, consider $p(z, \bar{z}) = \bar{z} + iz$. After decomposition, we get $p(x + iy, x - iy) = (1 + i)(x - y)$ and its zero set is the set of complex numbers of the form $x + ix$, which is the line $y = x$ in $\mathbb{R}^2$. Note that the cardinality of the zero set is infinite.

Second, one of real and imaginary parts could be the zero polynomial. This is another kind of an example with an infinite zero set. If $q(z, \bar{z}) = (1 - i)z + (1 + i)\bar{z}$, then $q(x + iy, x - iy) = 2(x + y)$, whose zero set is all the points on the line $y = -x$.

Third, consider $r(z, \bar{z}) = z^2 + 2\bar{z} + 1$. We similarly see

$$r(x + iy, x - iy) = (1 + 2x + x^2 - y^2) + 2y(x - 1)i.$$

To find the zero set, the following system of polynomials should be solved:

$$\begin{cases} 1 + 2x + x^2 - y^2 = 0; \\ 2y(x - 1) = 0. \end{cases}$$

Indeed, the first polynomial is a hyperbola and the second is two lines one of which is tangent to the hyperbola. The intersection of the two is $(-1, 0), (1, -2), (1, 2)$, or $-1, 1 - 2i, 1 + 2i$ as complex numbers. It is an example of a complex polynomial with the finite zero set.

2.5.4 Algebraic Curves

An *algebraic curve* is a subset of ordered pairs in $\mathbb{R}^2$ satisfying the polynomial equation $f(x, y) = 0$. Observe that the polynomial f can be viewed as an implicit function in $\mathbb{R}^2$ and that without taking off our hand from the paper we can draw the graph of the polynomials of any higher degrees. In this section, we refer to any polynomial as in two variables.

First, we distinguish uninteresting cases. If the algebraic curve of a quadratic polynomial is one of two lines, one line doubled, one point, or empty set, it is said to be *degenerate*. If an algebraic curve of degree two is not degenerate, we call it a *conic*. It is known that all conics are classified into ellipses, hyperbolas, and parabolas.

In addition, a *cubic* is an algebraic curve defined by a polynomial equation of degree three. Newton categorized cubics into 72 species. Six more species were found later, and hence there are a total of 78 species of cubics.

2.5.5 Bézout's Theorem

Theorem 2.5.21 (Bézout, 1779)**.** Let $f, g \in \mathbb{R}[x, y]$ be relatively prime polynomials. If deg $f = m$ and deg $g = n$, then the system of two algebraic equations $f(x, y) = 0$ and $g(x, y) = 0$ has at most mn simultaneous solutions.

In the later chapters, we will often discuss intersection of a few cubic polynomials and, roughly speaking, Bézout's theorem says we can have at most nine points. Those points are important because they will be in the support of the measure, if a measure exists. We may want to apply the theorem to complex cases because the zero set of a complex polynomial is the intersection of the zero sets of real and imaginary parts. As we have seen Example 2.5.20, Bézout's theorem cannot be adopted for all cases. Nevertheless, we may have a test to guarantee a finite zero set of harmonic polynomials.

Theorem 2.5.22. [39] If $p(z) := f(z) - \overline{g(z)}$ is a harmonic polynomial of degree n satisfying $\lim_{|z|\to\infty} |p(z)| = \infty$, then $p(z)$ has at most n^2 zeros.

Furthermore, there is an example of $p(z)$ with different n^2 zeros. To see how the theorem can be applied, we use the polynomials in Example 2.5.20.

Example 2.5.23. For $p(z, \bar{z}) = \bar{z} + iz$, we see $|p(x + iy, x - iy)| = |(1 + i)(x - y)|$ $= \sqrt{2}|x - y|$ has no limit as $|z| \equiv \sqrt{x^2 + y^2} \to \infty$ and hence Theorem 2.5.22 is not applicable. On the other hand, the norm of $r(z, \bar{z}) = z^2 + 2\bar{z} + 1$ is infinity as

$|z| \to \infty$. For,

$$|z^2 + 2\bar{z} + 1| \geq \left||z^2 + 2\bar{z}| - 1\right| \to \infty$$

since

$$\begin{aligned} |z^2 + 2\bar{z}| &\geq \left||z^2| - |2\bar{z}|\right| \\ &\geq ||z|^2 - 2|z|| \\ &= |z| \cdot ||z| - 2| \to \infty \end{aligned}$$

as $|z| \to \infty$. By Theorem 2.5.22, card $\mathcal{Z}(r) \leq 4$. As we showed in Example 2.5.20, card $\mathcal{Z}(r) = 3$.

2.5.6 Lexicographic Order on Monomials

The division algorithm in several variables turns out to be a key tool to prove one of the main results in this dissertation. In order to build the algorithm, the notion of ordering of terms in polynomials is precedent.

We want to compare any pair of monomials in a clear manner, and so we should be able to arrange all terms in ascending or descending order. This requires our ordering to be *total*, which means for every pairs of monomials $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$, only one of the following

$$x^\alpha > x^\beta, \qquad x^\alpha = x^\beta, \qquad x^\alpha < x^\beta$$

must be true. Here is the formal definition.

Definition 2.5.24. If a relation $>$ on the set of monomials $x^\alpha \in k[x_1, \ldots, x_n]$ for $\alpha \in \mathbb{Z}^n_{\geq 0}$ satisfies:

(i) $>$ is a total ordering;

(ii) If $x^\alpha > x^\beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $x^\alpha \cdot x^\gamma > x^\beta \cdot x^\gamma$;

(iii) Every nonempty subset of monomials has a smallest element under $>$ (this property is called well-ordering),

it is called a *monomial ordering* on $k[x_1, \ldots, x_n]$.

The most well-known example of an ordering is *lexicographic* order (or *lex* order for short).

Definition 2.5.25 (Lexicographic Order)**.** For $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, we say $x^\alpha >_{lex} x^\beta$ if $\alpha \neq \beta$ and the left-most nonzero entry of $\alpha - \beta$ is positive.

For example, in the three-variable case $xy^2 >_{lex} y^2z^4$ since $\alpha - \beta = (1, 0, -4)$ and $x^3y^2z >_{lex} x^3y^2$ since $\alpha - \beta = (0, 0, 1)$.

Observe that we have $n!$ nonequivalent lexicographic orderings with a particular order of the variables. Here are examples of two-dimensional cases:

(a) Real case: with $x < y$, the ordering appears as $1 < x < y < x^2 < xy < y^2 < x^3 < x^2y < xy^2 < y^3 < \cdots$.

(b) Complex case: with $z < \bar{z}$, the ordering appears as $1 < z < \bar{z} < z^2 < \bar{z}z < \bar{z}^2 < z^3 < \bar{z}z^2 < \bar{z}^2z < \bar{z}^3 < \cdots$.

We finish this section with another terminology about the monomial ordering.

Definition 2.5.26. The *leading term*, denoted by LT, is the maximal monomial in a polynomial under the given monomial order.

For example, under the lex order with $x < y$, we see LT $(1 + x - 2xy + x^3 + 2x^2y) = 2x^2y$.

2.5.7 Resultants

In the study of truncated moment problems, it is essential to identify the zero set of two or more polynomials in two variables. We may use elimination theory to solve a system of polynomials [7]; using the theory, we cancel out all other variables in the given system and in turn we will have a polynomial with only one variable whose zeros give candidates of the zeros of the polynomial system. This means, theoretically, we have a way to solve a system of several variable polynomials of higher degree. However, the calculation would not be pleasant; we might need help from a computer and in most cases we would be able to get only approximations due to Galois's Theorem (that is, quintic equations, sextic equations, and so on, may not have closed forms for their solutions).

We now present the notion of resultant that has a key role in elimination theory.

Definition 2.5.27. Suppose $f, g \in k[x]$ (f, g may contain other variables) are polynomials of degree $\ell > 0$ and $m > 0$, respectively. Write them in descending order as:

$$\begin{aligned} f &= a_0 x^\ell + \cdots + a_\ell, \qquad a_0 \neq 0, \\ g &= b_0 x^m + \cdots + b_m, \qquad b_0 \neq 0. \end{aligned}$$

Then the *Sylvester matrix* of f and g with respect to x, denoted $\mathrm{Syl}(f, g, x)$, is the following $(\ell+m)\times(\ell+m)$ matrix:

$$\operatorname{Syl}(f,g,x)=\begin{pmatrix} a_0 & & & & b_0 & & & \\ a_1 & a_0 & & & b_1 & b_0 & & \\ a_2 & a_1 & \ddots & & b_2 & b_1 & \ddots & \\ \vdots & & \ddots & a_0 & \vdots & & \ddots & b_0 \\ & \vdots & & a_1 & & \vdots & & b_1 \\ a_\ell & & & & b_m & & & \\ & a_\ell & & \vdots & & b_m & & \vdots \\ & & \ddots & & & & \ddots & \\ & & & a_\ell & & & & b_m \end{pmatrix},$$

where the empty spaces are filled with zeros. The *resultant* of f and g with respect to x, denoted $\operatorname{Res}(f,g,x)$, is the determinant of $\operatorname{Syl}(f,g,x)$.

Example 2.5.28. We try to solve the system of two quadratic polynomials: $f(x,y)=-12-8x+x^2-y+2xy=0$ and $g(x,y)=(x-1)^2+y^2-25=0$. Regard as f and g in $\mathbb{R}[x]$ and rewrite $f(x,y)=x^2+(2y-8)x+(-y-12)$ and $g(x,y)=x^2-2x+(y^2-24)$. Then

$$\operatorname{Syl}(f,g,x)=\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2y-8 & 1 & -2 & 1 \\ -y-12 & 2y-8 & y^2-24 & -2 \\ 0 & -y-12 & 0 & y^2-24 \end{pmatrix},$$

and its determinant, $\operatorname{Res}(f,g,x)=(y-4)(5y+24)(y-3)^2$. Putting in $y=4$, $y=-24/5$, and $y=3$ into f or g gives the solution of the system: $(4,4)$, $(-2/5,-24/5)$, $(-3,3)$, and $(5,3)$.

The third part of Chapter 2 deals with the Basic Theory of Moment Matrices.

2.6 Structure

2.6.1 Structure of Complex Moment Matrices

Given a complex moment sequence $\gamma \equiv \gamma^{(2n)}$, we define an associated moment matrix $M(n)(\gamma)$. In the sequel, some necessary conditions for the existence of a representing measure and the support of the measure will be obtained by analyzing the moment matrix.

Let us introduce some notation. For $m \in \mathbb{Z}_+$, we write $M_m(\mathbb{C})$ as the set of all $m \times m$ complex matrices. For $n \geq 0$, let $m \equiv m(n) := \frac{(n+1)(n+2)}{2}$. For $A \in M_{m(n)}(\mathbb{C})$, the successive rows and columns of A are denoted by the accompanying lexicographic polynomial ordering: $\mathit{1}, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \ldots, Z^n, \bar{Z}Z^{n-1}, \ldots, \bar{Z}^{n-1}Z, \bar{Z}^n$. For $0 \leq i+j \leq n$, $0 \leq \ell + k \leq n$, the entry of A in row $\bar{Z}^\ell Z^k$ and column $\bar{Z}^i Z^j$ is denoted by $A_{(\ell,k)(i,j)}$. (For notational convenience, in the inequality $0 \leq i+j \leq n$ we always implicitly assume that $i, j \geq 0$.)

Given $\gamma \equiv \gamma^{(2n)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,2n}, \ldots, \gamma_{2n,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, we first define a block matrix

$$M[i,j] := \begin{pmatrix} \gamma_{i,j} & \gamma_{i+1,j-1} & \cdots & \cdots & \gamma_{i+j,0} \\ \gamma_{i-1,j+1} & \gamma_{i,j} & \cdots & \cdots & \gamma_{i+j-1,1} \\ \gamma_{i-2,j+2} & \gamma_{i-1,j+1} & \cdots & \cdots & \gamma_{i+j-2,2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \gamma_{0,j+i} & \gamma_{1,j+i-1} & \cdots & \cdots & \gamma_{i,j} \end{pmatrix}_{(i+1)\times(j+1)},$$

where $M[i,j]$ has the Toeplitz-like property in which entries on each diagonal are constant. Finally, the *(complex) moment matrix* $M(n) \equiv M(n)(\gamma) \in M_m(\mathbb{C})$ is defined as:

$$M(n) \equiv M(n)(\gamma) := \begin{pmatrix} M[0,0] & M[0,1] & \cdots & M[0,n] \\ M[1,0] & M[1,1] & \cdots & M[1,n] \\ \vdots & \vdots & \ddots & \vdots \\ M[n,0] & M[n,1] & \cdots & M[n,n] \end{pmatrix}.$$

Note that $M(n)$ is self-adjoint and hence we can discuss the positivity of the moment matrix. As the main results of this dissertation are about sextic moment sequences, we need to see the associated moment matrix explicitly:

$$M(3)(\gamma) = \left(\begin{array}{c|cc|ccc|cccc} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} & \gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \hline \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} & \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \hline \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} & \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{50} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{41} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{05} & \gamma_{14} & \gamma_{23} & \gamma_{32} \\ \hline \gamma_{30} & \gamma_{31} & \gamma_{40} & \gamma_{32} & \gamma_{41} & \gamma_{50} & \gamma_{33} & \gamma_{42} & \gamma_{51} & \gamma_{60} \\ \gamma_{21} & \gamma_{22} & \gamma_{31} & \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{24} & \gamma_{33} & \gamma_{42} & \gamma_{51} \\ \gamma_{12} & \gamma_{13} & \gamma_{22} & \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{15} & \gamma_{24} & \gamma_{33} & \gamma_{42} \\ \gamma_{03} & \gamma_{04} & \gamma_{13} & \gamma_{05} & \gamma_{14} & \gamma_{23} & \gamma_{06} & \gamma_{15} & \gamma_{24} & \gamma_{33} \end{array}\right).$$

2.6.2 Structure of Real Moment Matrices

We define a real moment matrix in a similar fashion. Now the successive rows and columns are denoted by the accompanying lexicographic polynomial ordering: *1*, $X, Y, X^2, XY, Y^2, \ldots, X^n, X^{n-1}Y, \ldots, XY^{n-1}, Y^n$,. For a given $\beta \equiv \beta(2n)$: $\beta_{00}, \beta_{10}, \beta_{01}, \ldots, \beta_{2n,0}, \ldots, \beta_{0,2n}$ with $\beta_{00} > 0$, we first denote a block matrix

$$N[i,j] := \begin{pmatrix} \beta_{i+j,0} & \beta_{i+j-1,1} & \cdots & \cdots & \beta_{i,j} \\ \beta_{i+j-1,1} & \beta_{i+j-2,2} & \cdots & \cdots & \beta_{i-1,j+1} \\ \beta_{i+j-2,2} & \beta_{i+j-3,3} & \cdots & \cdots & \beta_{i-2,j+2} \\ \vdots & \vdots & \iddots & \iddots & \vdots \\ \beta_{i,j} & \beta_{i-1,j+1} & \cdots & \cdots & \beta_{0,j+i} \end{pmatrix}_{(i+1)\times(j+1)},$$

where $N[i,j]$ has the Hankel-like property and its entries are constant on each diagonal to the left-lower corner. Finally, the *(real) moment matrix* $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta) \in M_m(\mathbb{C})$ is defined as:

$$\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta) := \begin{pmatrix} N[0,0] & N[0,1] & \cdots & N[0,n] \\ N[1,0] & N[1,1] & \cdots & N[1,n] \\ \vdots & \vdots & \ddots & \vdots \\ N[n,0] & N[n,1] & \cdots & N[n,n] \end{pmatrix}.$$

Note that $\mathcal{M}(n)(\beta)$ is self-adjoint as well. If $n = 3$, then

$$\mathcal{M}(3)(\beta) = \left(\begin{array}{c|cc|ccc|cccc} \beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{03} \\ \hline \beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{40} & \beta_{31} & \beta_{22} & \beta_{13} \\ \beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} & \beta_{31} & \beta_{22} & \beta_{13} & \beta_{04} \\ \hline \beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} & \beta_{50} & \beta_{41} & \beta_{32} & \beta_{23} \\ \beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} & \beta_{41} & \beta_{32} & \beta_{23} & \beta_{14} \\ \beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} & \beta_{32} & \beta_{23} & \beta_{14} & \beta_{05} \\ \hline \beta_{30} & \beta_{40} & \beta_{31} & \beta_{50} & \beta_{41} & \beta_{32} & \beta_{60} & \beta_{51} & \beta_{42} & \beta_{33} \\ \beta_{21} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{23} & \beta_{51} & \beta_{42} & \beta_{33} & \beta_{24} \\ \beta_{12} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{14} & \beta_{42} & \beta_{33} & \beta_{24} & \beta_{15} \\ \beta_{03} & \beta_{13} & \beta_{04} & \beta_{23} & \beta_{14} & \beta_{05} & \beta_{33} & \beta_{24} & \beta_{15} & \beta_{06} \end{array}\right)$$

2.6.3 Degree-One Transformations

We introduce transformations (called degree-one transformations) through which we can make a given moment problem simpler. For $a, b, c \in \mathbb{C}$, $|b| \neq |c|$, let $\varphi(z) := a + bz + c\bar{z}$ $(z \in \mathbb{C})$. Given $\gamma^{(2n)}$, define a new moment sequence $\tilde{\gamma}^{(2n)}$ by $\tilde{\gamma}_{ij} := \Lambda_\gamma(\bar{\varphi}^i \varphi^j)$ $(0 \leq i + j \leq 2n)$, where Λ_γ denotes the Riesz functional associated with γ. We can easily verify that if $\Phi(z, \bar{z}) := \Big(\varphi(z), \overline{\varphi(z)}\Big)$, then $\Lambda_{\tilde{\gamma}}(p) = \Lambda_\gamma(p \circ \Phi)$ for every $p \in \mathcal{P}_n$. (Note that for $p(z, \bar{z}) \equiv \sum a_{ij} \bar{z}^i z^j$, $(p \circ \Phi)(z, \bar{z}) = p\Big(\varphi(z), \overline{\varphi(z)}\Big) \equiv \sum a_{ij} \overline{\varphi(z)}^i \varphi(z)^j$.)

Proposition 2.6.29. [15, Proposition 1.7] (Invariance under degree-one transformations.) Let $M(n)$ and $\tilde{M}(n)$ be the moment matrices associated with γ and $\tilde{\gamma}$, and let $J\hat{p} := \widehat{p \circ \Phi}$ $(p \in \mathcal{P}_n)$. Then the following are true:

(i) $\tilde{M}(n) = J^* M(n) J$;

(ii) J is invertible;

(iii) $\tilde{M}(n) \geq 0 \Leftrightarrow M(n) \geq 0$;

(iv) rank $\tilde{M}(n) =$ rank $M(n)$;

(v) The formula $\mu = \tilde{\mu} \circ \Phi$ establishes a one-to-one correspondence between the sets of representing measures for γ and $\tilde{\gamma}$, which preserves measure class and cardinality of the support; moreover, $\varphi(\text{supp } \mu) = \text{supp } \tilde{\mu}$;

(vi) $M(n)$ admits a flat extension if and only if $\tilde{M}(n)$ admits a flat extension;

(vii) For $p \in \mathcal{P}_n$, $p\left(\tilde{Z}, \bar{\tilde{Z}}\right) = J^* \left((p \circ \Phi)\left(Z, \bar{Z}\right)\right)$.

Explicitly, if $n = 3$, the transition matrix J looks like:

$$J = \begin{pmatrix} 1 & a & \bar{a} & a^2 & a\bar{a} & \bar{a}^2 & a^3 & a^2\bar{a} & a\bar{a}^2 & \bar{a}^3 \\ 0 & b & \bar{c} & 2ab & \bar{a}b + a\bar{c} & 2\bar{a}\bar{c} & 3a^2b & 2a\bar{a}b + a^2\bar{c} & \bar{a}^2b + 2a\bar{a}\bar{c} & 3\bar{a}^2\bar{c} \\ 0 & c & \bar{b} & 2ac & a\bar{b} + \bar{a}c & 2\bar{a}\bar{b} & 3a^2c & a^2\bar{b} + 2a\bar{a}c & 2a\bar{a}\bar{b} + \bar{a}^2c & 3\bar{a}^2\bar{b} \\ 0 & 0 & 0 & b^2 & b\bar{c} & \bar{c}^2 & 3ab^2 & \bar{a}b^2 + 2ab\bar{c} & 2\bar{a}b\bar{c} + a\bar{c}^2 & 3\bar{a}\bar{c}^2 \\ 0 & 0 & 0 & 2bc & b\bar{b} + c\bar{c} & 2\bar{b}\bar{c} & 6abc & s & t & 6\bar{a}\bar{b}\bar{c} \\ 0 & 0 & 0 & c^2 & \bar{b}c & \bar{b}^2 & 3ac^2 & 2a\bar{b}c + \bar{a}c^2 & a\bar{b}^2 + 2\bar{a}\bar{b}c & 3\bar{a}\bar{b}^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & b^3 & b^2\bar{c} & b\bar{c}^2 & \bar{c}^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3b^2c & b^2\bar{b} + 2bc\bar{c} & 2b\bar{b}\bar{c} + c\bar{c}^2 & 3\bar{b}\bar{c}^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3bc^2 & 2b\bar{b}c + c^2\bar{c} & b\bar{b}^2 + 2\bar{b}c\bar{c} & 3\bar{b}^2\bar{c} \\ 0 & 0 & 0 & 0 & 0 & 0 & c^3 & \bar{b}c^2 & \bar{b}^2c & \bar{b}^3 \end{pmatrix},$$

where $s := 2ab\bar{b} + 2\bar{a}bc + 2ac\bar{c}$ and $t := 2\bar{a}b\bar{b} + 2a\bar{b}\bar{c} + 2\bar{a}c\bar{c}$. Here is how to attain new column relations under a transformation.

Example 2.6.30. Suppose $M(n)$ has a column relation $AZ + B\bar{Z} + Z^3 = 0$ for $A, B \in \mathbb{C}$. Consider a degree-one transformation $\varphi(z) = c\bar{z}$ and we next try to see the transformed column relation in $\tilde{M}(n)$. If $p(z,\bar{z}) = z$, then from (vii) we have $\tilde{Z} = J^*(c\bar{Z}) \Rightarrow \bar{Z} = \frac{1}{c}(J^*)^{-1}\tilde{Z}$. Similarly, $\tilde{\bar{Z}} = J^*\left(\overline{c\bar{Z}}\right) = J^*(\bar{c}Z) \Rightarrow Z = \frac{1}{\bar{c}}(J^*)^{-1}\tilde{\bar{Z}}$ and $\tilde{\bar{Z}}^3 = J^*\left(\overline{(c\bar{Z})^3}\right) = J^*(\bar{c}^3Z^3) \Rightarrow Z^3 = \frac{1}{\bar{c}^3}(J^*)^{-1}\tilde{\bar{Z}}^3$. Thus,

$$\begin{aligned} AZ + B\bar{Z} + Z^3 = 0 \quad &\Rightarrow \quad A \cdot \frac{1}{\bar{c}}(J^*)^{-1}\tilde{\bar{Z}} + B \cdot \frac{1}{c}(J^*)^{-1}\tilde{Z} + \frac{1}{\bar{c}^3}(J^*)^{-1}\tilde{\bar{Z}}^3 = 0 \\ &\Rightarrow \quad (J^*)^{-1}\left[\frac{A}{\bar{c}}\tilde{\bar{Z}} + \frac{B}{c}\tilde{Z} + \frac{1}{\bar{c}^3}\tilde{\bar{Z}}^3\right] = 0 \\ &\Rightarrow \quad \frac{A}{\bar{c}}\tilde{\bar{Z}} + \frac{B}{c}\tilde{Z} + \frac{1}{\bar{c}^3}\tilde{\bar{Z}}^3 = 0. \end{aligned}$$

We also have these transformations in the real version. The transformation $\varphi(z) := a + bz + c\bar{z}$ $(z \in \mathbb{C})$ can be written as a linear mapping on the plane. Let $z := x + iy$ for $x, y \in \mathbb{R}$ and $a = a_1 + ia_2, b = b_1 + ib_2, c = c_1 + ic_2$ for $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$. Then the mapping can be viewed as

$$(x, y) \mapsto (a_1 + (b_1 + c_1)x + (-b_2 + c_2)y, a_2 + (b_2 + c_2)x + (b_1 - c_1)y)$$

with $b_1^2 + b_2^2 \neq c_1^2 + c_2^2$, which implies the degree-one transformations is an affine transformation on the $\mathbb{R}^2$. Conversely, any affine transformation $x' = t_1x + t_2y + t_3$ and $y' = s_1x + s_2y + s_3$ with $t_1s_2 \neq t_2s_1$ is also a degree-one transformation through $b_1 := \frac{t_1+s_2}{2}$, $b_2 := \frac{-t_2+s_1}{2}$, $c_1 := \frac{t_1-s_2}{2}$, and $c_2 := \frac{t_2+s_1}{2}$.

2.6.4 Equivalence of Real and Complex Moment Matrices

We now establish the equivalence between truncated real moment problems and truncated complex moment problems through a unitary transformation. We should find the transition matrix L which associates $M(n)(\gamma) \equiv M_\gamma$ and $\mathcal{M}(n)(\beta) \equiv \mathcal{M}_\beta$, that is, $M(n)(\gamma) = L^* \mathcal{M}_\beta L$. In order to build L, let $\psi(x, y) := z \equiv x + iy$ and let $\Psi(x, y) := (z, \bar{z})$. Exactly as in description in section 2.6.3, we have $L_\gamma(p) = L_\beta(p \circ \Psi)$, so that $L\hat{p} := \widehat{p \circ \Psi}$. The matrix L has a direct sum decomposition $\bigoplus_{k=0}^{n} L_k$, where the columns of L_k are coefficients vectors acting on monomials of total degree k. For instance, $L_0 = (1)$, $L_1 = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$,

$$L_2 = \begin{pmatrix} 1 & 1 & 1 \\ 2i & 0 & -2i \\ -1 & 1 & -1 \end{pmatrix}, \text{ and } L_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3i & i & -i & -3i \\ -3 & 1 & 1 & -3 \\ -i & i & -i & i \end{pmatrix}.$$

We now have the equivalence of the truncated complex moment problem with the truncated real moment problem.

Proposition 2.6.31. [15, Proposition 1.12] Let M_γ and $\mathcal{M}_\beta$ be the moment matrices associated with γ and β, and define L by $L\hat{p} := \widehat{p \circ \Psi}$ $(p \in \mathcal{P}_n)$.

(i) $M_\gamma = L^* \mathcal{M}_\beta L$.

(ii) L is invertible.

(iii) $M_\gamma \geq 0 \Leftrightarrow \mathcal{M}_\beta \geq 0$.

(iv) rank $M_\gamma =$ rank $\mathcal{M}_\beta$.

(v) The formula $\mu_{\mathbb{R}} = \mu \circ \Psi$ establishes a one-to-one correspondence between the sets of representing measures for β and γ, which preserves measure class and cardinality of the support; moreover, $\psi(\text{supp}\ \mu_{\mathbb{R}}) = \text{supp}\ \mu$.

(vi) M_γ admits a flat extension if and only if $\mathcal{M}_\beta$ admits a flat extension.

(vii) For $p \in \mathcal{P}_n$, $p\left(Z, \bar{Z}\right) = L^*((p \circ \Psi)(X, Y))$.

Example 2.6.32. Consider the moment matrix

$$M(3)(\gamma) := \begin{pmatrix} 1 & 0 & 0 & \frac{11i}{14} & \frac{13}{14} & -\frac{11i}{14} & 0 & 0 & 0 & 0 \\ 0 & \frac{13}{14} & -\frac{11i}{14} & 0 & 0 & 0 & \frac{7i}{8} & \frac{59}{56} & -\frac{7i}{8} & -\frac{23}{56} \\ 0 & \frac{11i}{14} & \frac{13}{14} & 0 & 0 & 0 & -\frac{23}{56} & \frac{7i}{8} & \frac{59}{56} & -\frac{7i}{8} \\ -\frac{11i}{14} & 0 & 0 & \frac{59}{56} & -\frac{7i}{8} & -\frac{23}{56} & 0 & 0 & 0 & 0 \\ \frac{13}{14} & 0 & 0 & \frac{7i}{8} & \frac{59}{56} & -\frac{7i}{8} & 0 & 0 & 0 & 0 \\ \frac{11i}{14} & 0 & 0 & -\frac{23}{56} & \frac{7i}{8} & \frac{59}{56} & 0 & 0 & 0 & 0 \\ 0 & -\frac{7i}{8} & -\frac{23}{56} & 0 & 0 & 0 & \frac{277}{224} & -\frac{227i}{224} & -\frac{97}{224} & -\frac{61i}{224} \\ 0 & \frac{59}{56} & -\frac{7i}{8} & 0 & 0 & 0 & \frac{227i}{224} & \frac{277}{224} & -\frac{227i}{224} & -\frac{97}{224} \\ 0 & \frac{7i}{8} & \frac{59}{56} & 0 & 0 & 0 & -\frac{97}{224} & \frac{227i}{224} & \frac{277}{224} & -\frac{227i}{224} \\ 0 & -\frac{23}{56} & \frac{7i}{8} & 0 & 0 & 0 & \frac{61i}{224} & -\frac{97}{224} & \frac{227i}{224} & \frac{277}{224} \end{pmatrix}$$

A straightforward application of the formula $M(3)(\gamma) = L^*\mathcal{M}(3)(\beta)L$ shows that

$$\begin{pmatrix}
1 & 0 & 0 & \frac{11}{7} & \frac{13}{7} & \frac{11}{7} & 0 & 0 & 0 & 0 \\
0 & \frac{13}{7} & \frac{11}{7} & 0 & 0 & 0 & \frac{7}{2} & \frac{59}{14} & \frac{7}{2} & \frac{23}{14} \\
0 & \frac{11}{7} & \frac{13}{7} & 0 & 0 & 0 & \frac{23}{14} & \frac{7}{2} & \frac{59}{14} & \frac{7}{2} \\
\frac{11}{7} & 0 & 0 & \frac{59}{14} & \frac{7}{2} & \frac{23}{14} & 0 & 0 & 0 & 0 \\
\frac{13}{7} & 0 & 0 & \frac{7}{2} & \frac{59}{14} & \frac{7}{2} & 0 & 0 & 0 & 0 \\
\frac{11}{7} & 0 & 0 & \frac{23}{14} & \frac{7}{2} & \frac{59}{14} & 0 & 0 & 0 & 0 \\
0 & \frac{7}{2} & \frac{23}{14} & 0 & 0 & 0 & \frac{277}{28} & \frac{227}{28} & \frac{97}{28} & -\frac{61}{28} \\
0 & \frac{59}{14} & \frac{7}{2} & 0 & 0 & 0 & \frac{227}{28} & \frac{277}{28} & \frac{227}{28} & \frac{97}{28} \\
0 & \frac{7}{2} & \frac{59}{14} & 0 & 0 & 0 & \frac{97}{28} & \frac{227}{28} & \frac{277}{28} & \frac{227}{28} \\
0 & \frac{23}{14} & \frac{7}{2} & 0 & 0 & 0 & -\frac{61}{28} & \frac{97}{28} & \frac{227}{28} & \frac{277}{28}
\end{pmatrix}$$

In Chapter 5, we will examine the column relations in detail and show both moment sequences admit 7-atomic representing measures.

2.7 Linear Functionals

2.7.1 Column Relations and Functional Calculus

We now examine how the column relations are written as polynomials in $Z, \bar{Z}$.

Example 2.7.33. Consider a moment sequence $\gamma^{(4)}$:

$$\begin{array}{lll}
\gamma_{00} = 4, & \gamma_{01} = \bar{\gamma}_{10} = -1 - i, & \\
\gamma_{02} = \bar{\gamma}_{20} = 6, & \gamma_{11} = 20, & \\
\gamma_{03} = \bar{\gamma}_{30} = -13 + 13i, & \gamma_{12} = \bar{\gamma}_{21} = -5 - 5i, & \\
\gamma_{04} = \bar{\gamma}_{40} = -28 + 48i, & \gamma_{13} = \bar{\gamma}_{31} = 30, & \gamma_{22} = 100.
\end{array}$$

Then we may build $M(2)(\gamma)$:

$$\begin{pmatrix} 4 & -1-i & -1+i & 6 & 20 & 6 \\ -1+i & 20 & 6 & -5-5i & -5+5i & -13-13i \\ -1-i & 6 & 20 & -13+13i & -5-5i & -5+5i \\ 6 & -5+5i & -13-13i & 100 & 30 & -28-48i \\ 20 & -5-5i & -5+5i & 30 & 100 & 30 \\ 6 & -13+13i & -5-5i & -28+48i & 30 & 100 \end{pmatrix}.$$

After Gauss elimination, we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 5 & 3+i \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{5}-\frac{7i}{5} \\ 0 & 0 & 1 & 0 & 0 & -1+i \\ 0 & 0 & 0 & 1 & 0 & -\frac{4}{5}-\frac{3i}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and see that $M(2)$ has two column relations; the fifth column is written as $\bar{Z}Z = 5 \cdot \mathit{1}$ (here $\mathit{1}$ denotes the first column) and the sixth as $\bar{Z}^2 = (3+i)\,\mathit{1} + \left(-\frac{1}{5} - \frac{7}{5}i\right) Z$ $+(-1+i)\bar{Z} + \left(-\frac{4}{5} - \frac{3}{5}i\right) Z^2$.

We refer to these column relations as complex polynomials (so-called “functional calculus”), which will play an important role in our quest to find a representing measure.

2.7.2 Inner Products

Associated to a Linear Functional

For computational ease, we need a way to write moments using monomials and an inner product defined on the polynomial space. Thus, we first define a sesquilinear form on $\mathcal{P}_n$, the polynomial space with all complex polynomials in $z, \bar{z}$ of degree at most n. As a first step, for $p \in \sum_{0\le i+j\le 0} a_{ij}\bar{z}^i z^j \in \mathcal{P}_n$ we denote $\hat{p}$ as the coefficient vector $(a_{00}\ a_{01}\ a_{10}\ \cdots\ a_{0,n}\ \cdots\ a_{n,0})^T$ and a sesquilinear form $\langle\cdot,\cdot\rangle_{M(n)(\gamma)}$ on $\mathcal{P}_n$ is defined by

$$\langle p, q\rangle_{M(n)} := \langle M(n)\hat{p}, \hat{q}\rangle \qquad (p, q \in \mathcal{P}_n).$$

In particular, $\langle \bar{z}^i z^j, \bar{z}^k z^\ell\rangle_{M(n)} = M(n)_{(k,\ell)(i,j)} = \gamma_{i+k,j+\ell}$. Note that since $M(n)$ is self-adjoint, it follows that $\langle\cdot,\cdot\rangle_{M(n)}$ is hermitian, that is,

$$\langle p, q\rangle_{M(n)} = \overline{\langle q, p\rangle}_{M(n)}.$$

For the real case, for $p \in \sum_{0\le i+j\le 0} a_{ij}\bar{x}^i y^j \in \mathcal{P}_n$ $(a_{ij} \in \mathbb{R})$ we denote $\hat{p}$ as the coefficient vector $(a_{00}\ a_{10}\ a_{01}\ \cdots\ a_{n,0}\ \cdots\ a_{0,n})^T$ and a sesquilinear form $\langle\cdot,\cdot\rangle_{\mathcal{M}(n)(\beta)}$ on $\mathcal{P}_n$ is defined by

$$\langle p, q\rangle_{\mathcal{M}(n)} := \langle \mathcal{M}(n)\hat{p}, \hat{q}\rangle \qquad (p, q \in \mathcal{P}_n).$$

Particularly, $\langle x^i y^j, x^k y^\ell\rangle_{\mathcal{M}(n)} = \beta_{i+k,j+\ell}$.

Furthermore, the sesquilinear form in the complex case satisfies:

Theorem 2.7.34. [10, Theorem 2.1]

(i) $\langle 1, 1\rangle_{\mathcal{M}(n)} > 0$;

(ii) $\langle p, q\rangle_{M(n)} = \langle \bar{q}, \bar{p}\rangle_{M(n)}$ for $p, q \in \mathcal{P}_n$ (*symmetric*);

(iii) $\langle zp, q\rangle_{M(n)} = \langle p, \bar{z}q\rangle_{M(n)}$ for $p, q \in \mathcal{P}_{n-1}$ (*Hankel-type*);

(iv) $\langle \bar{z}p, q\rangle_{M(n)} = \langle p, zq\rangle_{M(n)}$ for $p, q \in \mathcal{P}_{n-1}$;

(v) $\langle zp, zq\rangle_{M(n)} = \langle \bar{z}p, \bar{z}q\rangle_{M(n)}$ for $p, q \in \mathcal{P}_{n-1}$.

The last statement (v) is said to represent the *normality* of the moment matrix.

2.7.3 The Riesz Functional

We next define a linear functional on the polynomial space, whose values are linear combinations of the terms in the moment sequence. Corresponding to γ we have the *Riesz functional* $\Lambda \equiv \Lambda_\gamma : \mathcal{P}_{2n} \to \mathbb{C}$ defined as follows: for $p(z, \bar{z}) \equiv \sum_{0 \le i+j \le 0} a_{ij}\bar{z}^i z^j \in \mathcal{P}_{2n}$, $\Lambda(p) := \sum_{0 \le i+j \le 0} a_{ij}\gamma_{ij}$. Obviously, if there is a representing measure μ for γ, then $\Lambda(p) = \int p \, d\mu$ and self-adjointness of $M(n)$ is characterized by

$$\langle M(n)\hat{p}, \hat{q}\rangle = \langle p, q\rangle_{M(n)} = \Lambda(p\bar{q}) \quad (p, q \in \mathcal{P}_n).$$

In the presence of a measure, we have $\langle M(n)\hat{p}, \hat{p}\rangle = \Lambda(p^2) = \int p^2 \, d\mu \ge 0$, i.e., $M(n)$ is positive, which is an important necessary condition for the existence of a measure.

For the real case, corresponding to β we define the Riesz functional $\Lambda \equiv \Lambda_\beta : \mathcal{P}_{2n} \to \mathbb{R}$ as follows: for a polynomial $p(x, y) \equiv \sum_{0 \le i+j \le 0} a_{ij}x^i y^j \in \mathcal{P}_{2n}$, $\Lambda(p) := \sum_{0 \le i+j \le 0} a_{ij}\beta_{ij}$. In the presence of a measure, $\Lambda(p) = \int p \, d\mu$ and we have the following identity:

$$\langle \mathcal{M}(n)\hat{p}, \hat{q}\rangle = \langle p, q\rangle_{\mathcal{M}}(n) = \Lambda(pq) \quad (p, q \in \mathcal{P}_n).$$

2.7.4 Algebraic Variety of a Moment Matrix

A way to find candidates for atoms of a representing measure is about to be introduced. The atoms of a measure are obviously in the support of the measure and it is known that the support is contained in the *algebraic variety*, which is the intersection of all the zero sets of polynomials derived from the column relations of the moment matrix. The formal definition is as follows: The algebraic variety of a moment sequence is

$$\mathcal{V} \equiv \mathcal{V}(M(n)) := \bigcap_{p\in\mathcal{P}_n, \hat{p}\in\ker\, M(n)} \mathcal{Z}(p),$$

where $\mathcal{Z}(p)$ is the zero set of the polynomial. With a real moment matrix, we can immediately compute the algebraic variety from the column relations just after Gaussian elimination. On the other hand, we need another step with a complex moment matrix. Let us consider Example 2.7.33 again:

Example 2.7.35. The column relations of $M(2)$ are $p_1(Z,\bar{Z}) := \bar{Z}Z - 5\cdot \mathit{1} = 0$ and $p_2(Z,\bar{Z}) := \bar{Z}^2 - \left[(3+i)\,\mathit{1} + \frac{-1-7i}{5}Z + (-1+i)\bar{Z} + \frac{-4-3i}{5}Z^2\right] = 0$ (for convenience, here 0 denotes a zero vector of size 6×1.) Next, we should rewrite the two complex polynomials with real and imaginary parts and hence we dealt with more polynomials:

$$\begin{aligned}
\mathrm{Re}\, p_1 &= -5 + x^2 + y^2;\\
\mathrm{Im}\, p_1 &= 0;\\
\mathrm{Re}\, p_2 &= -3 + \frac{6x}{5} + \frac{9x^2}{5} - \frac{12y}{5} - \frac{6xy}{5} - \frac{9y^2}{5};\\
\mathrm{Im}\, p_2 &= -1 + \frac{2x}{5} + \frac{3x^2}{5} - \frac{4y}{5} - \frac{2xy}{5} - \frac{3y^2}{5}.
\end{aligned}$$

Set these polynomials equal to zero and solve the system of polynomial equations. We then get the algebraic variety, $\mathcal{V} = \{1 - 2i, 2 + i, -2 - i, -2 + i\}$.

The cardinality of the algebraic variety is important to us in concerning moment problems because it controls a necessary condition for the existence of a representing measure. Finally, we should notice that since it is cardinality, a multiple root is counted only once.

2.7.5 Positivity of Linear Functionals

For a closed set K in $\mathbb{R}^2$, we say that a linear functional L is *K-positive* if whenever $p \in \mathcal{P}_{2n}$ and $p|_K \geq 0$, then $L(p) \geq 0$. In [19], R. Curto and L. Fialkow proved an analogue of the Riesz-Haviland theorem, that $\beta^{(2n)}$ admits a representing measure which is supported in K if and only if the Riesz functional Λ_β admits a K-positive linear extension $\tilde{\Lambda}_\beta : \mathcal{P}_{2n+2} \to \mathbb{R}$. Thus, we know that positivity of the Riesz functional is the key solution to the moment problem.

While we have an abstract solution, it is not easy to check positivity of a functional. Since the positivity of $\mathcal{M}(n)$ is a necessary condition (but not sufficient) for the existence of a measure, at least we may want to use the information that $\Lambda(p^2) \geq 0$ for $p \in \mathcal{P}_n$ (said to be *square positive*) if and only if $\mathcal{M}(n)$ is positive semidefinite. However, square positivity of a functional obviously does not imply positivity, which may give rise to need for more stronger necessary conditions. Furthermore, we have no general structure theorem for K-positive polynomial (even for $\mathbb{R}^2$), so we have trouble in checking K-positivity of the Riesz functional.

For singular quartic moment problems, R. Curto and L. Fialkow presented a complete solution in 2002 based on building a rank-preserving extension, and L. Fialkow and J. Nie recently showed that if $\mathcal{M}(2)(\beta) > 0$, then β has a representing measure, using an argument which relies on the extension of a positive linear functional. However, it is still unknown how to get an explicit representation of the measure.

2.8 Positivity

2.8.1 Choleski's Algorithm

In this section, we will discuss technology for checking positivity of a matrix, the so-called *Choleski's algorithm*, or *Sylvester's criterion* or *the nested determinant test*. Let us start with a basic terminology. A *principal minor* of a square matrix A is the determinant of the upper left corner of A. The criterion says a Hermitian matrix A is positive definite if and only if all of the principal minors are positive. Now we can prove the positivity of the Hilbert matrix.

Example 2.8.36. A calculation shows determinant of the $n \times n$ Hilbert matrix is:

$$\det(H) = \frac{c_n^4}{c_{2n}} > 0, \qquad \text{where } c_n = \prod_{i=1}^{n-1} i^{n-i} = \prod_{i=1}^{n-1} i!.$$

According to the criterion, we know H is positive definite.

For positive semidefinite matrices, it is necessary that all principal minors be non-negative. This, however, is not sufficient, as can be seen in the following example.

Example 2.8.37. Consider the sequence of matrices:

$$A(2) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad A(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad A(4) = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & a \end{pmatrix}.$$

A simple calculation show $A(3)$ is positive semidefinite with $\det A(3) = 0$ and the determinant of the extension $A(4)$ is zero for any real number a. But, for $x = (0, 0, 0, -1)$ we have $x^* A(4) x = a$ that can be chosen as negative. We will see later (Example 2.8.39) that $A(4) \geq 0$ if and only if $a \geq 1$.

The most important aspect of solving moment problems is to build a rank-preserving extension of the associated moment matrix while preserving positive semidefiniteness. But the above example shows Sylvester's criterion might not be valid to build a positive semidefinite extension. In detail, suppose $A(n)$ is a positive definite matrix of size $n \times n$. If an extension $A(n+1)$ of size $(n+1) \times (n+1)$ has determinant zero, then we can prove that $A(n+1)$ is positive semidefinite. However, for the next extension with $\det A(n+2) = 0$, we cannot say $A(n+2)$ of size $(n+2) \times (n+2)$ is positive semidefinite. A good remedy for this situation is coming in the next section.

2.8.2 Smul'jan's Theorem

We introduce the extension problem for positive matrices. For $k, \ell \in \mathbb{Z}_+$, let $A \in M_k(\mathbb{C}), A = A^*, B \in M_{k,\ell}(\mathbb{C}), C \in M_\ell(\mathbb{C})$; the following matrix of the form

$$\tilde{A} \equiv \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

is an *extension* of A. (Please note that this notion is different from the usual notion of extension for Hilbert space operators.) Incidentally, we mention in passing that Iohvidov discusses the general theory of extensions for Hankel and Toeplitz matrices.

Theorem 2.8.38. [36] $\tilde{A} \geq 0$ if and only if $A \geq 0$ and there exists $W \in M_{k,\ell}(\mathbb{C})$ such that $B = AW$ and $C \geq W^*AW$. Moreover, rank $\tilde{A} =$ rank A if and only if $C = W^*AW$.

Conversely, if $A \geq 0$ and any extension $\tilde{A}$ satisfies rank $\tilde{A} =$ rank A, then $\tilde{A}$ is positive. Most of all, the condition $\operatorname{Ran} B \subseteq \operatorname{Ran} A$ should be prescribed at first. Let us revisit one of the examples in the previous section.

Example 2.8.39. We want to identify which values of a make the extension $A(4)$ positive.

$$A(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad A(4) = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & a \end{pmatrix}.$$

Denote the fourth column up to the third row of $A(4)$ as $B := (1\ 1\ 1)^T$ and $C := (a)$.

After Gauss elimination of $[A(4)]_3$, we know $A(3)\,(0\;\; -t+1\;t)^T = B$ for any $t \in \mathbb{C}$. Fix $t=0$ and set $W = (0\;1\;0)^T$. Then $C - W^* A(3) W = a - 1 \geq 0 \iff A(4) \geq 0$.

In the solutions of the quadratic and quartic moment problems, Smul'jan's Theorem played an important role in finding a positive extension of the associated moment matrix.

2.8.3 Vandermonde Matrices

We will build a matrix whose structure is very similar to that of a given moment matrix after reduction. It is also closely related to a necessary condition for the existence of a representing measure. For a finite algebraic variety $\mathcal{V}(\gamma) = \{z_1, \cdots, z_r\}$, define the *generalized Vandermonde matrix* $W^{(n)}(\gamma)$ as

$$W^{(n)}(\gamma) = \begin{pmatrix} 1 & z_1 & \bar{z}_1 & {z_1}^2 & \bar{z}_1 z_1 & \bar{z}_1^2 & \cdots & {z_1}^n & \bar{z}_1 z_1^{n-1} & \cdots & \bar{z}_1^{n-1} z_1 & \bar{z}_1^n \\ 1 & z_2 & \bar{z}_2 & {z_2}^2 & \bar{z}_2 z_2 & \bar{z}_2^2 & \cdots & {z_2}^n & \bar{z}_2 z_2^{n-1} & \cdots & \bar{z}_2^{n-1} z_2 & \bar{z}_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & z_r & \bar{z}_r & {z_r}^2 & \bar{z}_r z_r & \bar{z}_r^2 & \cdots & {z_r}^n & \bar{z}_r z_r^{n-1} & \cdots & \bar{z}_r^{n-1} z_r & \bar{z}_r^n \end{pmatrix}$$

The matrix is simply evaluation of points in the variety according to the monomials arranged lexicographically. Let $W_{\mathcal{B}}^{(n)}(\gamma)$ be the compression matrix of $W^{(n)}(\gamma)$ to a basis $\mathcal{B}$.

Similarly, for a real version, let an algebraic variety for $\mathcal{M}(n)(\beta)$ be $\mathcal{V}(\beta) = \{(x_1, y_1), \cdots, (x_s, y_s)\}$ and denote the generalized Vandermonde matrix

$W^{(n)}(\beta)$ as

$$W(\beta) = \begin{pmatrix} 1 & x_1 & y_1 & {x_1}^2 & y_1x_1 & {y_1}^2 & \cdots & {x_1}^n & x_1^{n-1}y_1 & \cdots & x_1y_1^{n-1} & y_1^n \\ 1 & x_2 & y_2 & {x_2}^2 & y_2x_2 & {y_2}^2 & \cdots & {x_2}^n & x_2^{n-1}y_2 & \cdots & x_2y_2^{n-1} & y_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_s & y_s & {x_s}^2 & y_sx_s & {y_s}^2 & \cdots & {x_s}^n & x_s^{n-1}y_s & \cdots & x_sy_s^{n-1} & y_s^n \end{pmatrix}$$

Now let $W_{\mathcal{D}}$ be the compression matrix of $W^{(n)}(\gamma)$ to a basis $\mathcal{D}$.

Finally, we remark that it can be verified that after Gauss elimination, the left-upper corners of the moment matrix and the reduced form of the generalized Vandermonde matrix must be the same in the presence of a measure.

CHAPTER 3
THE TRUNCATED MOMENT PROBLEM

This chapter is designed to serve as a brief introduction to truncated moment problems, to the fundamental necessary conditions for the existence of a representing measure, and to the basic properties of moment problems. There are two versions of 2-dimensional moment problems: real and complex. As seen in Proposition 2.6.31, the two problems are in turn equivalent. The so-called functional calculus for the columns of the moment matrix helps us understand properties of moment problems.

3.1 Definition of Truncated Moment Problems

Given $\gamma \equiv \gamma^{(2n)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,2n}, \ldots, \gamma_{2n,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, the *truncated complex moment problem* entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu \qquad (0 \leq i + j \leq 2n);$$

μ is called a representing measure for γ.

On the other hand, to define the truncated moment problem in the real case, let $\beta \equiv \beta^{(2n)} : \beta_{00}, \beta_{10}, \beta_{01}, \ldots, \beta_{2n,0}, \ldots, \beta_{0,2n}$ denote a real 2-dimensional multisequence of degree $2n$. The *truncated real moment problem* for β concerns the existence of a positive measure μ supported in $\mathbb{R}^2$ such that

$$\beta_{ij} = \int_{\mathbb{R}^2} x^i y^j d\mu \qquad (0 \leq i + j \leq 2n);$$

μ is called a representing measure for β.

3.2 Necessary Conditions

We are ready to state the basic necessary conditions for the existence of a representing measure. As stated in Chapter 1, for truncated moment problems of low degree, a suitable combination of the necessary conditions is often sufficient to guarantee the existence of a representing measure. Since the necessary conditions are very concrete, they lend themselves well to numerical and symbolic tests.

3.2.1 Positivity

The first necessary condition is established by studying the matrix positivity induced by a representing measure. Let μ be a positive Borel measure on $\mathbb{C}$. For $i, j \geq 0$ and $0 \leq i + j \leq 2n$ define the (i, j)-*power moment* by $\gamma_{ij} := \int \bar{z}^i z^j d\mu(z, \bar{z})$. Then given $p \in \mathcal{P}_n$, $p(z, \overline{z}) \equiv \sum_{0 \leq i+j \leq n} a_{ij} \bar{z}^i z^j$, we have

$$\begin{aligned} 0 &\leq \int |\, p(z,\overline{z})\,|^2 \, d\mu(z,\overline{z}) \\ &= \sum_{ijk\ell} a_{ij}\bar{a}_{k\ell} \int \overline{z}^{i+\ell} z^{j+k} d\mu(z,\overline{z}) \\ &= \sum_{ijk\ell} a_{ij}\bar{a}_{k\ell}\gamma_{i+\ell,j+k}. \end{aligned} \tag{3.2.1}$$

Using the moment matrix $\mathcal{M}(n)(\gamma)$ associated with $\gamma := \{\gamma_{ij}\}$, we can understand this "matricial" positivity from the fact that γ satisfies (3.2.1) if and only if $\mathcal{M}(n)(\gamma)$ is positive semidefinite. This requires listing the columns of $\mathcal{M}(n)(\gamma)$ using the lexicographic ordered of polynomials in Z and $\bar{Z}$, that is, the first column is denoted *1*, the second column Z, the third $\bar{Z}$, and then Z^2, $\bar{Z}Z$, $\bar{Z}^2$, and so on. It is rather straightforward to verify that (3.2.1) is equivalent to the positive semi-definiteness of

$\mathcal{M}(n)(\gamma)$. We do not run the calculation here, but instead refer the reader to the monograph [10].

3.2.2 Cardinality Condition

The next result will help us find the location of the support of a representing measure. For $p \in \mathcal{P}_n$, $p(z, \bar{z}) := \sum a_{ij}\bar{z}^i z^j$, let

$$p(Z, \bar{Z}) := \sum a_{ij}\bar{Z}^i Z^j \in \mathcal{C}_{\mathcal{M}(n)}, \tag{3.2.2}$$

where $\mathcal{C}_{\mathcal{M}(n)}$ denotes the column space of $\mathcal{M}(n)$. This is what we call the *functional calculus*, which associates a vector in the column space of $\mathcal{M}(n)(\gamma)$ to each polynomial in z and $\bar{z}$ of degree at most n.

Proposition 3.2.40. [10, Proposition 3.1] Suppose μ is a representing measure for $\gamma \equiv \gamma^{(2n)}$. Then for $p \in \mathcal{P}_n$,

$$\operatorname{supp} \mu \subseteq \mathcal{Z}(p) \iff p(Z, \bar{Z}) = 0. \tag{3.2.3}$$

Next, we observe that (3.2.1) leads to $\mathcal{P}_n \subseteq L^2(\mu)$. For $p \in \mathcal{P}_n$, its continuity on supp μ implies that $p = 0$ if and only if $p|_{\operatorname{supp} \mu} \equiv 0$. Through the following proposition, we reach the cardinality condition. We need to define mappings in advance as follows:

$$\begin{aligned}
\psi :\ \mathcal{C}_{\mathcal{M}(n)} &\to L^2(\mu) & \text{by} \qquad & \psi(p(Z, \bar{Z})) := p(z, \bar{z});\\
\rho :\ \mathcal{C}_{\mathcal{M}(n)} &\to \mathcal{P}_n|_{\operatorname{supp} \mu} & \text{by} \qquad & \rho(p(Z, \bar{Z})) := p(z, \bar{z})|_{\operatorname{supp} \mu};\\
\iota : \mathcal{P}_n|_{\operatorname{supp} \mu} &\to L^2(\mu) & \text{by} \qquad & \iota(p(z, \bar{z})|_{\operatorname{supp} \mu}) := p(z, \bar{z}).
\end{aligned}$$

Proposition 3.2.41. [10, Proposition 3.3] Suppose μ is a representing measure for $\gamma \equiv \gamma^{(2n)}$.

(i) ψ, ρ, and ι are well-defined, linear, and injective;

(ii) ρ is an isomorphism and $\psi = \iota \circ \rho$;

(iii) If $f, g, fg \in \mathcal{P}_n$, then $\psi((fg)(Z, \bar{Z})) = \psi(f(Z, \bar{Z}))\psi(g(Z, \bar{Z}))$;

(iv) If $f \in \mathcal{P}_n$, then $\psi(\bar{f}(Z, \bar{Z})) = \overline{\psi(f(Z, \bar{Z}))}$.

That is, ψ is multiplicative and compatible with conjugation. Applying Proposition 3.2.40 and 3.2.41, we can prove the following results.

Corollary 3.2.42. [10, Corollary 3.4] Suppose μ is a representing measure for γ. Then the following holds:

$$f, g, fg \in \mathcal{P}_n,\ f(Z, \bar{Z}) = 0 \Longrightarrow (fg)(Z, \bar{Z}) = 0. \tag{3.2.4}$$

When (3.2.4) holds, γ (or $\mathcal{M}(n)(\gamma)$) is said to be *recursively generated*, which is regarded as another necessary condition for the existence of a measure. If a moment matrix is invertible, then there are no column relations, and we thus regard this case as vacuously satisfying the recursive condition. It follows that every invertible moment matrix is automatically recursively generated. From Proposition 3.2.41 (i), we see that rank $\mathcal{M}(n) = \dim \mathcal{C}_{\mathcal{M}(n)} \leq \dim L^2(\mu)$. Further, we can prove that if μ is a finitely atomic measure on $\mathbb{C}$ with $k :=$ card supp μ, then $\{1, z, \ldots, z^{k-1}\}$ is a basis for $L^2(\mu)$. Once these two results are combined, we get

Corollary 3.2.43. [10, Corollary 3.7] If μ is a representing measure for γ, then rank $\mathcal{M}(n) \leq$ card supp μ.

Finally, together with Proposition 3.2.40, we have the variety condition: Let $\mathcal{V}$ be the algebraic variety of $\mathcal{M}(n)$. Then

$$\text{rank } \mathcal{M}(n) \leq \text{card } \mathcal{V}. \tag{3.2.5}$$

Recall that any representing measure of a truncated moment sequence, if it exists, is finitely atomic (see [19]). Consequently, the candidates for atoms must all come from the algebraic variety, obtained as the intersection of the zero sets of all polynomials associated to column relations in $\mathcal{M}(n)(\gamma)$.

3.2.3 Weak Consistency and Consistency

Since the necessary conditions we had so far are not sufficient in general, we should investigate stronger conditions. The first is called weak consistency; let $\mathcal{V} \equiv \mathcal{V}_\gamma$ be the algebraic variety of a moment sequence γ. Then γ is said to be *Weakly Consistent* if it satisfies the following property:

$$p \in \mathcal{P}_n, \ p|_{\mathcal{V}} \equiv 0 \Longrightarrow p(Z, \bar{Z}) = 0. \tag{3.2.6}$$

Note that if γ has a representing measure μ, then γ is Weakly Consistent. For, if $p \in \mathcal{P}_n$ and $p|_{\mathcal{V}} \equiv 0$, then we know $p|_{\text{supp}\,\mu} \equiv 0$ because supp $\mu \subseteq \mathcal{V}$. From Proposition 3.2.40, we conclude that $p(Z, \bar{Z}) = 0$.

Similarly, we can show that if γ is Weakly Consistent, then $M(n)(\gamma)$ is recursively generated. Next, we define a much stronger condition; γ is said to be

Consistent if the following is satisfied:

$$p \in \mathcal{P}_{2n},\ p|_{\mathcal{V}} \equiv 0 \Longrightarrow p(Z, \bar{Z}) = 0. \tag{3.2.7}$$

Clearly, consistency implies weak consistency and we summarize:

$$\begin{aligned} \gamma \text{ Consistent} \quad &\Longrightarrow \quad \gamma \text{ Weakly Consistent} \\ &\Longrightarrow \quad M(n)(\gamma) \text{ is recursively generated.} \end{aligned}$$

The following lemma will show that Consistency is a strong condition. It already produces a linear combination of point masses, even though it may allow for some negative densities.

Lemma 3.2.44. [20, Lemma 2.3] Let $L : \mathcal{P}_{2n} \to \mathbb{C}$ be a linear functional and let $\mathcal{V} \subseteq \mathbb{C}$. Then the following are equivalent:

(i) There exist $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ and there exist $w_1, \ldots, w_m \in \mathcal{V}$ such that $L(p) = \sum_{i=1}^{m} \alpha_i p(w_i)$ $(p \in \mathcal{P}_{2n})$.

(ii) If $p \in \mathcal{P}_{2n}$ and $p|_{\mathcal{V}} \equiv 0$, then $L(p) = 0$.

In [20], the following question was asked:

> Suppose $\mathcal{M}(n)(\gamma)$ is positive and singular. If γ satisfies the variety condition and is Consistent, does γ admit a representing measure?

The answer is, a bit surprisingly, no. In [21, Example 3.2], L. Fialkow presented a real moment matrix $\mathcal{M}(3)(\beta)$ with only one column relation $Y = X^3$, which means that $\mathcal{M}(3)(\beta)$ has an infinite algebraic variety. Moreover, in [21, Lemma 3.1], L. Fialkow also showed that $\mathcal{M}(3)$ is consistent. However, $\mathcal{M}(3)$ does not admit a representing measure.

3.3 Properties of the Truncated Moment Problem

In this section, we discuss several results examining the structure of positive moment matrices. For $A \in M_{k+1}(\mathbb{C})$ and $0 \le \ell \le k$,

$$A := \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0\ell} & \cdots & a_{0k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{\ell 0} & a_{\ell 1} & \cdots & a_{\ell\ell} & \cdots & a_{\ell k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k0} & a_{k1} & \cdots & a_{k\ell} & \cdots & a_{kk} \end{pmatrix} \equiv \left(v_0 \middle| v_1 \middle| \cdots \middle| v_k \right),$$

where v_i denotes the $(i+1)$-th column of A. Let $A(\ell)$ be the upper-left corner restriction of A up to $(\ell+1)$-th row and column. Set $v(i,\ell)$ as $(i+1)$-th columns of $A(\ell)$.

Proposition 3.3.45 (Extension Principle)**.** [22, Proposition 2.4] Let $A \in M_{k+1}(\mathbb{C})$ and $A \ge 0$. If there exit p $(0 \le p \le k)$ and scalars $c_0, \ldots, c_p$ such that $c_0 v(0,p) + \cdots + c_p v(p,p) = 0$, then $c_0 v_0 + \cdots + c_p v_p = 0$.

Roughly speaking, the Extension Principle says if a compression of a positive matrix has a column relation, the same column relation must appear in the original matrix as well. In a certain sense, the structure of a positive matrix seems to be quite formidable, in that there is strong rigidity throughout.

Lemma 3.3.46. [10, Lemma 3.10 - 3.13] Let $M(n)$ be a moment matrix and let $p \in \mathcal{P}_n$ and $q \in \mathcal{P}_{n-2}$.

(i) If $p(Z, \bar{Z}) = 0$, then $\bar{p}(Z, \bar{Z}) = 0$;

(ii) If $M(n) \geq 0$ and $q(Z, \bar{Z}) = 0$, then $(zq)(Z, \bar{Z}) = 0$ and $(\bar{z}q)(Z, \bar{Z}) = 0$;

(iii) If $M(n) \geq 0$, $p(Z, \bar{Z}) = 0$, $r, s \geq 0$ satisfies $r + s + \deg\ p \leq n - 1$, then $(\bar{z}^r z^s p)(Z, \bar{Z}) = 0$.

After applying Lemma 3.3.46 (iii) repeatedly, the structure theorem for positive moment matrices is delivered.

Theorem 3.3.47 (Structure Theorem)**.** [10, Theorem 3.14] Let $M(n) \geq 0$. Then the following is true:

$$f, g, fg \in \mathcal{P}_{n-1},\ f(Z, \bar{Z}) = 0 \Longrightarrow (fg)(Z, \bar{Z}) = 0. \tag{3.3.1}$$

Basically, the structure theorem states that when we find a column relation associated to a polynomial p of relatively low degree (i.e., degree less than n), every multiple of p by monomials must give rise to another column relation.

Example 3.3.48. Consider $M(3)$

$$\begin{pmatrix} 5 & 3 & 3 & \gamma_{12} & 9 & \gamma_{21} & 9 & 9 & 9 & 9 \\ 3 & 9 & \gamma_{21} & 9 & 9 & 9 & \gamma_{13} & 25 & \gamma_{31} & 9 \\ 3 & \gamma_{12} & 9 & 9 & 9 & 9 & 9 & \gamma_{13} & 25 & \gamma_{31} \\ \gamma_{21} & 9 & 9 & 25 & \gamma_{31} & 9 & 33 & 33 & 33 & 33 \\ 9 & 9 & 9 & \gamma_{13} & 25 & \gamma_{31} & 33 & 33 & 33 & 33 \\ \gamma_{12} & 9 & 9 & 9 & \gamma_{13} & 25 & 33 & 33 & 33 & 33 \\ 9 & \gamma_{31} & 9 & 33 & 33 & 33 & 81 & \gamma_{42} & 49 & \gamma_{24} \\ 9 & 25 & \gamma_{31} & 33 & 33 & 33 & \gamma_{24} & 81 & \gamma_{42} & 49 \\ 9 & \gamma_{13} & 25 & 33 & 33 & 33 & 49 & \gamma_{24} & 81 & \gamma_{42} \\ 9 & 9 & \gamma_{13} & 33 & 33 & 33 & \gamma_{42} & 49 & \gamma_{24} & 81 \end{pmatrix},$$

where $\gamma_{12} = \bar{\gamma}_{21} := 5 + 4i$, $\gamma_{13} = \bar{\gamma}_{31} := 17 + 8i$, and $\gamma_{24} = \bar{\gamma}_{42} := 65 + 16i$. Row reduction shows that the leftmost column relation is $p(Z, \bar{Z}) := -iZ^2 - (1-i)\bar{Z}Z + \bar{Z}^2 = 0$, and we can check that its conjugate, $i\bar{Z}^2 - (1+i)\bar{Z}Z + Z^2 = 0$, is another column relation. For a higher order, let us try to multiply by z. Here, the functional calculus must be applied because it is not possible to multiply columns in a matrix. We first pass from $p(Z, \bar{Z})$ to the polynomial $p(z, \bar{z}) := -iz^2 - (1-i)\bar{z}z + \bar{z}^2$ and multiply by z: $zp(z, \bar{z}) := -iz^3 - (1-i)\bar{z}z^2 + \bar{z}^2 z$. Again, the polynomial $zp(z, \bar{z})$ is converted back into a column relation $-iZ^3 - (1-i)\bar{Z}Z^2 + \bar{Z}^2 Z$, which is inherent in $M(3)$. Lastly, we remark that the last column relation is not seen by the row-reduced echelon form of $M(3)$. In fact, the column relation with the leading term $\bar{Z}^2 Z$ in the row-reduced echelon form of $M(3)$ is $(2+2i)Z - 2i\bar{Z} - (3/2 + (3i)/2)Z^2 - (3/2 - (3i)/2)\bar{Z}Z + \bar{Z}^2 Z = 0$. This phenomenon illustrates the difference between polynomials in $\mathcal{P}_n$ and polynomials obtained from the functional calculus. Regardless, both polynomials are likely to stay in a certain (ideal-like) structure.

We finish this chapter with a quick preview of how we plan to build a positive extension of $M(n)$. Using Theorem 3.3.47, it can be verified that recursiveness in (3.2.4) is a necessary condition to build a positive extension $M(n+1)$ of $M(n)$. This means there is no freedom for some higher order column relation in $M(n+1)$, whose leading term is the multiple of the leading term of a column relation in $M(n)$. Therefore, when we build a positive extension, it is essential to keep the column relations in $M(n)$ and to maintain positivity.

CHAPTER 4
GENERAL RESULTS ON THE TRUNCATED MOMENT PROBLEM

In this chapter we present the solution of TMP in the four well-known instances in which flat extensions can be built: the case of TMP's of flat data type; the case of analytic columns given in terms of polynomials of low degree; the singular quartic MP; and the extremal MP. Each case has a conceptually different approach, but all four cases are based on the premise that to obtain a representing measure a flat extension must be built at some point.

4.1 Truncated Moment Problems of Flat Data Type

Given a complex moment sequence $\gamma \equiv \gamma^{(2n)}$, for $0 \leq i, j \leq n$ we define the $(i+1) \times (j+1)$ matrix $M[i,j]$ whose entries are the moments of order $i+j$:

$$M[i,j] := \begin{pmatrix} \gamma_{i,j} & \gamma_{i+1,j-1} & \cdots & \cdots & \gamma_{i+j,0} \\ \gamma_{i-1,j+1} & \gamma_{i,j} & \cdots & \cdots & \gamma_{i+j-1,1} \\ \gamma_{i-2,j+2} & \gamma_{i-1,j+1} & \cdots & \cdots & \gamma_{i+j-2,2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \gamma_{0,j+i} & \gamma_{1,j+i-1} & \cdots & \cdots & \gamma_{i,j} \end{pmatrix}_{(i+1)\times(j+1)} . \tag{4.1.1}$$

According to the equation $\langle M(n)\hat{p}, \hat{q}\rangle = \langle p, q\rangle_{M(n)} = \Lambda(pq)$ $(p, q \in \mathcal{P}_n)$, we have seen that $M(n)(\gamma)$ admits a block decomposition $M(n) = (M[i,j])_{0 \leq i,j \leq n}$.

Once "new moments" of degree $2n+1$ for a prospective representing measure are given, it is possible to define blocks $M[0, n+1], \ldots, M[n-1, n+1]$ via (4.1.1).

For $i = 0, \ldots, n$ let $M[i, n+1]$ denote the corresponding moment matrix block given by (4.1.1), and let

$$B(n) := \begin{pmatrix} M[0, n+1] \\ \vdots \\ M[n-1, n+1] \\ M[n, n+1] \end{pmatrix}.$$

Similarly, given a moment matrix block $C(n)$ of the form $M[n+1, n+1]$ (corresponding to "new moments" of degree $2n+2$), we can build the moment matrix *extension* $M(n+1)$ via the block decomposition

$$M(n+1) = \begin{pmatrix} M(n) & B(n) \\ B(n)^* & C(n) \end{pmatrix}. \tag{4.1.2}$$

It follows from a theorem of Smul'jan (Theorem 2.8.38) that $M(n+1)$ is positive if and only if (i) $M \geq 0$, (ii) there exists a matrix W such that $B(n) = M(n)W$, and (iii) $C(n) \geq W^*M(n)W$ (since $M(n)$ is Hermitian, $W^*M(n)W$ is independent of W provided $B(n) = M(n)W$). Note also that if $M(n) \geq 0$, then rank $M(n) =$ rank $M(n+1)$ if and only if $C = W^*M(n)W$; conversely, if $M(n) \geq 0$ and there exists W such that $B(n) = M(n)W$ and $C = W^*M(n)W$, then $M(n) \geq 0$ and rank $M(n+1) =$ rank $M(n)$.

A block matrix $M(n+1)$ as in (4.1.2) is an *extension* of $M(n)$, and it is said to be a *flat extension* if rank $M(n+1) =$ rank $M(n)$. If we choose a block $B(n)$ satisfying $B(n) = M(n)W$ and $C = W^*M(n)W$ for some matrix W, then we can build a flat extension of a positive matrix $M(n)$. The following theorem shows that the existence of a flat extension enables us to find a representing measure and that there is a technology to ensure that the measure is finitely atomic.

Theorem 4.1.49. (Flat Extension Theorem) [10, Remark 3.15, Theorem 5.4, Corollary 5.12, Theorem 5.13, and Corollary 5.15] [13, Lemma 1.9] [22] Suppose $M(n)(\gamma)$ is positive and admits a flat extension $M(n+1)$, so that $Z^{n+1} = p\left(Z, \bar{Z}\right)$ in $\mathcal{C}_{M(n+1)}$ for some $p \in \mathcal{P}_n$. Then there exist *unique* successive flat (positive, recursively generated) moment matrix extensions $M(n+2)$, $M(n+3)$, $\ldots$, which are determined by the relations

$$Z^{n+k} = \left(z^{k-1}p\right)\left(Z, \bar{Z}\right) \in \mathcal{C}_{M(n+k)} \quad (k \geq 2). \tag{4.1.3}$$

Let $r := \operatorname{rank} M(n)$. There exist unique scalars $a_0, \ldots, a_{r-1}$ such that in $\mathcal{C}_{M(r)}$,

$$Z^r = a_0 \mathit{1} + \cdots + a_{r-1} Z^{r-1}.$$

The *characteristic polynomial* $g_\gamma(z) \equiv z^r - (a_0 + \cdots + a_{r-1} z^{r-1})$ has r distinct roots, $z_0, \ldots, z_{r-1}$, and γ has a rank $M(n)$-atomic minimal representing measure of the form

$$\nu = \nu\left[M(n+1)\right] = \sum \rho_i \delta_{z_i},$$

where the densities $\rho_i > 0$ are determined by the Vandermonde equation

$$V(z_0, \ldots, z_{r-1})(\rho_0, \ldots, \rho_{r-1})^t = (\gamma_{00}, \ldots, \gamma_{0,r-1})^t.$$

The measure $\nu\left[M(n+1)\right]$ is the unique representing measure for $\gamma^{(2n+2)}$, and is also the unique representing measure for $M(\infty)$.

This theorem is the most concrete criterion for existence of a measure so far. We can attempt to solve a higher order moment problem using a software tool *Mathematica* via symbolic calculation, but building an extension requires more parameters for new moments in the $B(n)$-block. It cannot be implemented in full generality because of memory overflow. However, in later sections, solutions of quadratic and singular quartic moment problems are established by applying Theorem 4.1.49.

4.2 Flat Extension for $Z^k = p_{k-1}(z, \bar{z})$

When we study a singular moment problem, it is natural to consider the case that the left-most column in k-th order is linearly dependent on lower order columns; a column relation is of the form $Z^k = p(Z, \bar{Z})$ for some $p \in \mathcal{P}_{k-1}$. R. Curto and L. Fialkow proved that if $k \leq \left[\frac{n}{2}\right] + 1$, then there is a unique flat extension $M(n+1)$.

Theorem 4.2.50. [12, Theorem 3.1] Suppose $M(n)$ is positive and recursively generated. If $1 \leq k \leq \left[\frac{n}{2}\right] + 1$ and $Z^k = p(Z, \bar{Z})$ for some $p \in \mathcal{P}_{k-1}$, then $M(n)$ admits a unique flat extension $M(n+1)$.

To consider the case $\left[\frac{n}{2}\right] + 1 \leq n$ with the same form of column relation, let $p(z, \bar{z}) \equiv \sum_{0 \leq i+j \leq k-1} a_{ij} \bar{z}^i z^j$. In the process of building a flat extension $M(n+1)$, it follows from recursiveness that the column relation $Z^{n+1} = \sum a_{ij} \bar{Z}^i Z^{n-k+j+1}$ must be in $M(n+1)$, which uniquely determines the block matrix $M[n, n+1]$, in turn the whole $B(n)$-block (because the block is Toeplitz, new $(n+1)$-order moments are linear combinations of lower degree moments.) If the range of the resulting $B(n)$-block is not in $\mathcal{C}_{M(n)}$, then $M(n)$ does not admit a flat extension $\mathcal{C}_{M(n+1)}$. Otherwise, there is W such that $B(n) = M(n)W$. By checking $W^* M(n) W$ is Toeplitz, we can see that $M(n+1)$ is a flat extension of $M(n)$.

4.3 Quadratic Moment Problem

In this section and next, existence criteria for some lower degree moment problems are presented. We first consider the quadratic moment problem with data $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}$. Let us denote $M(1)$ the corresponding moment matrix, and let $r := \text{rank}\, M(1)$.

Theorem 4.3.51. [10, Theorem 6.1] The following are equivalent:

(i) γ has a representing measure;

(ii) γ has an r-atomic representing measure;

(iii) $M(1) \geq 0$.

We can notice that the basic necessary condition, positivity of $M(1)$, is also sufficient in the case of the quadratic moment problem, but not beyond this case. We summarize more detailed results according to the rank of $M(1)$, as in [10]:

(i) If $r = 1$, then there exists a unique representing measure;

(ii) If $r = 2$, then the 2-atomic representing measures are parameterized by a line;

(iii) If $r = 3$, then the set of 3-atomic representing measures contains a subordinate parametrization by a circle.

We conclude this section with a remark about the case $r = 3$ or, equivalently, the case of invertible $M(1)$. This occasion provides concrete evidence that strict positivity of a moment matrix may imply the existence of a representing measure. Our interest in pursuing this idea may take us to higher degree moment problems. Recently, this idea turned out to be true for the quartic moment problem; by L. Fialkow and J. Nie in [24] proved that a positive invertible $\mathcal{M}(2)$ always admits a representing measure. But of course we know that the idea cannot be extended further, into the sextic moment problem; indeed, an example of an invertible $M(3)$ with no representing measure was found in [12].

4.4 Singular Quartic Moment Problem

The complete solution of the singular quartic moment problem was discovered by R. Curto and L. Fialkow in [15]. We begin with a general solution of $M(n)$ whose third column is linearly dependent.

Theorem 4.4.52. [12, Theorem 2.1] Let $M(n) \geq 0$ with a column relation $\bar{Z} = \alpha 1 + \beta Z$. Assume that $M(n)$ is recursively generated. Then $M(n)$ admits a flat extension $M(n+1)$.

We want to study a complex moment sequence with 15 terms:

$$\gamma \equiv \gamma^4 : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \dots, \gamma_{04}, \gamma_{13}, \gamma_{22}, \gamma_{31}, \gamma_{40}.$$

Let $M(2)$ denote the corresponding moment matrix, and let $r := \text{rank}\, M(2)$.

First of all, we classify singular quartic moment problems according to the basis of the associated moment matrix. Recall two important properties of the moment problem:

First, for $p \in \mathcal{P}_n$,

$$p(Z, \bar{Z}) = 0 \Longrightarrow \bar{p}(Z, \bar{Z}) = 0; \tag{4.4.1}$$

Second, suppose μ is a representing measure for γ. Then the following holds (recursiveness):

$$f, g, fg \in \mathcal{P}_n,\ f(Z, \bar{Z}) = 0 \Longrightarrow (fg)(Z, \bar{Z}) = 0. \tag{4.4.2}$$

Without loss of generality, we may assume that the first column is linearly independent and if the second column Z is linearly dependent, it follows from above two properties that all other columns in $M(n)$ are linearly dependent and hence that $M(n)$ admits a flat extension $M(n+1)$, which solves this case. In addition, in view of Theorems 4.2.50 and 4.4.52, if $\bar{Z}$ is linearly dependent or Z^2 is a linear combination of the first three columns, then we have a positive solution. Therefore, from now on we may assume that $M(2)$ is positive and that $\{\mathit{1}, Z, \bar{Z}, Z^2\}$ is independent in $\mathcal{C}_{M(2)}$. Then there are only three singular quartic cases according to the basis of $M(2)$: (I) $\{\mathit{1}, Z, \bar{Z}, Z^2\}$, (II) $\{\mathit{1}, Z, \bar{Z}, Z^2, \bar{Z}^2\}$, and (III) $\{\mathit{1}, Z, \bar{Z}, Z^2, \bar{Z}Z\}$. The case (I) is solved by the following Theorem 4.4.53 and 4.4.54, the case (II) is covered by Theorem 4.4.53. Finally, Theorem 4.4.55 solves the case (III).

Theorem 4.4.53. [20, Theorem 1.2] Suppose $M(2) \geq 0$, $\left\{\mathit{1}, Z, \bar{Z}, Z^2\right\}$ is independent in $\mathcal{C}_{M(2)}$ and $\bar{Z}Z \in \left\langle \mathit{1}, Z, \bar{Z} \right\rangle$. Then $\bar{Z}Z = A\mathit{1} + BZ + \bar{B}\bar{Z}$ in $\mathcal{C}_{M(2)}$ satisfying $A + |B|^2 > 0$, and $\gamma^{(4)}$ admits a rank $M(2)$-atomic (minimal) representing measure. Moreover, each representing measure is supported in the circle $C_\gamma = \left\{ z \in \mathbb{C} : \left|z - \bar{B}\right|^2 = A + |B|^2 \right\}$. If $\left\{\mathit{1}, Z, \bar{Z}, Z^2\right\}$ is a basis for $\mathcal{C}_{M(2)}$, then there exists a unique representing measure, which is 4-atomic. Otherwise, $\left\{\mathit{1}, Z, \bar{Z}, Z^2, \bar{Z}^2\right\}$ is a basis for $\mathcal{C}_{M(2)}$, and there exist infinitely many flat extensions, each corresponding to a distinct 5-atomic (minimal) representing measure.

The next theorem is about the case with a column relation $\bar{Z}Z = A\mathit{1} + BZ + C\bar{Z} + DZ^2$, $D \neq 0$. Using the property (4.4.1), the moment matrix must have another column relation $\bar{Z}Z = \bar{A}\mathit{1} + \bar{B}\bar{Z} + \bar{C}Z + \bar{D}\bar{Z}^2$, which shows $\bar{Z}^2$ is dependent, too. Thus, the rank of $M(2)$ is 4 that is the same as the number of atoms of a representing measure.

Theorem 4.4.54. [20, Theorem 1.3] Suppose $M\,(2) \geq 0$, $\left\{\mathit{1}, Z, \bar{Z}, Z^2\right\}$ is independent in $\mathcal{C}_{M(2)}$, and $\bar{Z}Z = A\mathit{1} + BZ + C\bar{Z} + DZ^2$, $D \neq 0$. The following are equivalent:

(i) $\gamma^{(4)}$ admits a finitely atomic representing measure;

(ii) $\gamma^{(4)}$ admits a 4-atomic (minimal) representing measure;

(iii) $M\,(2)$ admits a flat extension $M\,(3)$;

(iv) $M\,(2)$ admits a recursively generated extension $M\,(3) \geq 0$;

(v) there exists $\gamma_{23} \in \mathbb{C}$ such that

$$\bar{\gamma}_{23} = A\gamma_{21} + B\gamma_{22} + C\gamma_{31} + D\gamma_{23}.$$

In [15], it was shown that if $\left\{\mathit{1}, Z, \bar{Z}, Z^2, \bar{Z}Z\right\}$ is a basis for $\mathcal{C}_{M(2)}$, then the associated algebraic variety $\mathcal{V}(\gamma)$ is the zero set of a real quadratic equation in $x :=$ Re $[z]$ and $y :=$ Im $[z]$. After this conversion, it is possible by Propositions 2.6.29

and 2.6.31 to reduce the case (III) to subcases corresponding to the following four algebraic curves of degree 2: (i) $y = x^2$; (ii) $yx = 1$; (iii) $yx = 0$; and (iv) $x^2 + y^2 = 1$.

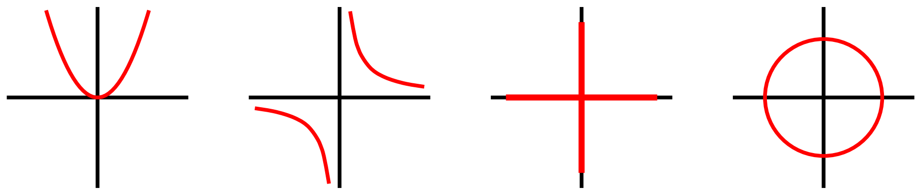


Figure 4.1: Every nontrivial conic can be brought into a canonical conic by a degree-one transformation.

Using the flat data result together with the preceding arguments, the following result can be established.

Theorem 4.4.55. [15, Theorem 1.5] Let $\gamma^{(4)}$ be given, and assume $M(2) \geq 0$ and that $\left\{\mathit{1}, Z, \bar{Z}, Z^2, \bar{Z}Z\right\}$ is a basis for $\mathcal{C}_{M(2)}$. Then $\gamma^{(4)}$ admits a representing measure μ. Moreover, it is possible to find μ with card supp $\mu =$ rank $M\,(2)$, except in some cases when $\mathcal{V}\left(\gamma^{(4)}\right)$ is a pair of intersecting lines, in which cases there exist μ with card supp $\mu \leq 6$.

Consequently, we have complete solutions of singular quartic moment problem. Once we know a moment sequence has a representing measure, we may write the measure as finitely atomic by applying the technology in Theorem 4.1.49.

4.5 Extremal Moment Problem

As we mentioned earlier, there is a nonsingular $M(3)$ that does not admit a representing measure. This example led to the search for a stronger necessary condition, Consistency instead of recursiveness since invertibility of a moment matrix always implies recursiveness. In [20], for the *extremal* case, when $r \equiv \text{rank } \mathcal{M}(n)(\beta)$ and $v \equiv \text{card } \mathcal{V}_\beta$ satisfy $r = v$, it was shown that Consistency is sufficient as well. The results were obtained in the case of the real version of TMP. The authors also discovered a solution of $\mathcal{M}(3)$ with a cubic column relation $Y = X^3$, in which the associated algebraic variety is finite. We will discuss more general results under the cubic column relation $Y = X^3$ in Chapter 5.

Suppose now that a real moment sequence β is extremal i.e., $r = v$ and $\mathcal{V} := \{w_1, w_2, \ldots, w_r\}$. Let $\mathcal{B} := \{p_1(X,Y), \ldots, p_r(X,Y)\}$ denote a basis for $\mathcal{C}_{\mathcal{M}(n)}$. Recall that if β is Weakly Consistent, then the compression of the generalized Vandermonde matrix $W_\mathcal{B}$, as defined in (2.8.1), is invertible, and we define the signed measure $\mu_\mathcal{B}$ by the equation $\mu_\mathcal{B} = \sum_{i=1}^{r} \rho_i \delta_{w_i}$ and

$$(\rho_1, \ldots, \rho_r)^T = W_\mathcal{B}^{-1} \left(\Lambda_\beta(p_1), \ldots, \Lambda_\beta(p_r)\right)^T, \tag{4.5.1}$$

where Λ_β is the Riesz functional on $\mathcal{P}_n$. Lastly, we say a signed Borel measure ν *interpolating* for β if $\int p d\nu = \Lambda_\beta(p)$ $(p \in \mathcal{P}_{2n})$. The main result of the extremal case is the following.

Theorem 4.5.56. [20, Theorem 2.8] For $\beta \equiv \beta^{(2n)}$ extremal, the following are equivalent:

(i) β has a representing measure;

(ii) β has a unique representing measure, which is rank $\mathcal{M}(n)$-atomic;

(iii) for some (respectively, for every) basis $\mathcal{B}$ of $\mathcal{C}_{\mathcal{M}(n)}$, $W_{\mathcal{B}}$ is invertible and $\mu_{\mathcal{B}}$ is a representing measure for β;

(iv) $\mathcal{M}(n) \geq 0$ and for some (respectively, for every) basis $\mathcal{B}$ of $\mathcal{C}_{\mathcal{M}(n)}$, $W_{\mathcal{B}}$ is invertible and $\mu_{\mathcal{B}}$ is an interpolating measure for β.

If the points in the algebraic variety are known exactly, then row reduction of the generalized Vandermonde matrix of degree $2n$ allows us to check whether β is Consistent or not. Another approach for checking Consistency comes from Theorem 4.5.56 (iv); since $W_{\mathcal{B}}$ is nonsingular, we have interpolation of all the moments up to degree n with $\mu_{\mathcal{B}}$ and hence β admits a representing measure if and only if the densities obtained in (4.5.1) are positive and $\mu_{\mathcal{B}}$ interpolates all the remaining higher degree moments. Due to this approach, the next results were obtained.

Theorem 4.5.57 (Theorem 1.5)**.** [20] Suppose $\mathcal{M}(3) \geq 0$ satisfies $Y = X^3$.

(i) If $r \leq v \leq 7$, $\beta^{(6)}$ has a representing measure if and only if $\mathcal{M}(3)$ is recursively generated.

(ii) If $r \leq v = 8$, $\beta^{(6)}$ has a representing measure if and only if β is Consistent.

In [10], R. Curto, L. Fialkow, and M. Möller found an example of an extremal $\mathcal{M}(3)$ that is Weakly Consistent but not Consistent and hence does not have a representing measure. We will quote this example in Chapter 6.

CHAPTER 5
CUBIC COLUMN RELATIONS

5.1 First Approach

In Chapter 4, we saw that a complete solution of the singular quartic moment problem was obtained by R. Curto and L. Fialkow. Furthermore, it was shown that any nonsingular quartic moment problem has a solution even though it is not known how to represent it concretely. We thus may focus on a singular $M(3)$ whose submatrix $M(2)$ is invertible. It is natural to study, as a first step, the case when the 7th column is linearly dependent on the previous six columns. Such column relation looks like $Z^3 = p(Z, \bar{Z})$, where $p \in \mathcal{P}_2$. Recall

Theorem 5.1.58. [12]. If $M(n)$ admits a column relation of the form $Z^k = p_{k-1}(Z, \bar{Z})$ $(1 \leq k \leq \left[\frac{n}{2}\right] + 1$ and deg $p_{k-1} \leq k-1)$, then $M(n)$ admits a flat extension $M(n+1)$, and therefore a representing measure.

To discuss the column relation $Z^3 = p(Z, \bar{Z})$, we have to take $k = 3$ and Theorem 5.1.58 can be applied to $M(n)$ for $n \geq 4$. Therefore, we must build the extension $M(4)$ of $M(3)$ while keeping the column relation $Z^3 = p_2(Z, \bar{Z})$. However, it is nontrivial to check that the C block of $M(4)$ is Toeplitz.

Example 5.1.59. Let us consider $M(3)(\gamma)$ with the column relation $Z^3 = 2iZ + \frac{5}{4}\bar{Z}$. We will see that the computer does not handle even the specific case when we assign some numerical moments. Let

$$\gamma_{00} = 1, \qquad \gamma_{01} = \bar{\gamma}_{10} = 0, \qquad \gamma_{11} = 1,$$

$$\gamma_{02} = \bar{\gamma}_{20} = a + bi, \qquad \gamma_{12} = \bar{\gamma}_{21} = c + di, \qquad \gamma_{22} = e,$$

where $a, b, c, d, e \in \mathbb{R}$. The column relation determines all the remaining moments:

$$\begin{array}{ll} \gamma_{03} = \bar{\gamma}_{30} = 0, & \gamma_{04} = \bar{\gamma}_{40} = \frac{5-8b}{4} + 2ai, \\ \gamma_{13} = \bar{\gamma}_{31} = \frac{5a}{4} + \frac{8-5b}{4} i, & \gamma_{05} = \bar{\gamma}_{50} = \frac{5c}{4} + \frac{5d}{4} i, \\ \gamma_{14} = \bar{\gamma}_{41} = \frac{5c-8d}{4} + \frac{8c-5d}{4} i, & \gamma_{23} = \bar{\gamma}_{32} = 2d + 2ci, \\ \gamma_{06} = \bar{\gamma}_{60} = -\frac{39a}{16} + \frac{80-89b}{16} i, & \gamma_{15} = \bar{\gamma}_{51} = \frac{-16+10b+5e}{4} + \frac{5a}{2} i, \\ \gamma_{24} = \bar{\gamma}_{42} = \frac{25a}{16} + \frac{-40+25b+32e}{16} i, & \gamma_{33} = \frac{89}{16} - 5b. \end{array}$$

Next we try to build the B-block of the extension matrix $M(4)$. First of all, the new moments $\gamma_{07}, \gamma_{16}, \ldots, \gamma_{70}$ must be parameterized or represented by the lower order moments. Since $M(4)$ must be recursively generated, we have a column relation $Z^4 = 2iZ^2 + \frac{5}{4}\bar{Z}Z$ and so all moments of degree seven in the B-block must satisfy:

$$\begin{aligned} \text{Im } \gamma_{07} &= \tfrac{80c-25d}{16} + \left(\tfrac{-25c+80d}{16} + \text{Re } \gamma_{07}\right) i, \\ \text{Im } \gamma_{16} &= 5c - 4d + (4c - 5d + \text{Re } \gamma_{16}) i, \\ \text{Im } \gamma_{25} &= \tfrac{-5c+8d}{16} + \left(\tfrac{8c-5d}{2} + \text{Re } \gamma_{25}\right) i, \\ \gamma_{34} &= \bar{\gamma}_{43} = \tfrac{89c-40d}{16} + \tfrac{-40c+89d}{16} i. \end{aligned}$$

Observe that we need three more parameters and should find a matrix W such that $M(3)W = B$ which requires Gaussian elimination of a matrix of size 10×15 (in practice, the calculation can be reduced to a 7×11 matrix). It is a very complicated symbolic calculation, and we have been unable to complete it because of memory overflow.

5.2 Second Approach

Observe first that it is generally very difficult to find the case of maximum valence of complex polynomials. Recently, some examples were found in [39] and one of them has the form $Z^3 = p(Z, \bar{Z})$ with nine points in the zero set. However, it was shown that cubic harmonic polynomials $q(z, \bar{z}) := f(z) - \overline{z}$, with deg $q = 3$ have seven or fewer zeros. Our first aim here is to study this type of cubic harmonic polynomials focusing on the given cubic harmonic column relation; we will discuss more general cubic polynomials in Chapter 6. We now present the general result about the zero set.

Theorem 5.2.60. [29] If deg $p = n \geq 1$, then

$$\text{card } \mathcal{Z}\left(f(z) - \bar{z}\right) \leq 3n - 2.$$

In the case when deg $f(z) = 3$, we have card $\mathcal{Z}(f(z) - \overline{z}) \leq 7$. Furthermore, it follows from the normality of the moment matrix that $\bar{Z}^3 = \bar{p}(Z, \bar{Z})$ must be another column relation. Due to the variety condition, we make sure that the rank of $M(3)$ is less than or equal to seven which means we should be able to identify one more column relation.

There is a way to consider simpler column relations. Using a degree-one transformation of the form $w = z + b/3$, we can transform a cubic $z^3 + bz^2 + cz + d$ into $w^3 + \tilde{c}w + \tilde{d}$; without loss of generality, we can always assume that the quadratic term in the analytic piece is absent. That is, it suffices to study column relations like $Z^3 = A\mathit{1} + BZ + C\bar{Z}$, where $A, B, C \in \mathbb{C}$.

5.3 Harmonic Polynomials with Seven Symmetric Points

Making sure that the zero set of cubic harmonic polynomials has seven points is not easy. A possible approach is to let the zero set K have symmetry conditions. The idea is that whenever symmetries occur, the presence of one point in the zero set immediately guarantees the presence of the associated symmetric point. For example, for a polynomial of the form $Z^3 = A\mathit{1} + BZ + C\bar{Z}$, we want K to satisfy $z \in K \Rightarrow -z \in K$. Then $A = 0$ which implies that $0 \in K$. Another natural symmetry would be about the line $y = x$ whose complex form is $z = i\bar{z}$. This condition requires that $B \in i\mathbb{R}$ and $C \in \mathbb{R}$. Finally, the column relation looks like $Z^3 = itZ + u\bar{Z}$, with $t, u \in \mathbb{R}$. Under these symmetries, if we can find only two nonzero points, one on the lines $y = x$ or $y = -x$ and the other outside that line, then we will have seven points in the zero set.

5.4 The Algebraic Variety of a Cubic Harmonic Polynomial

We thus consider the cubic harmonic polynomial of the form $q_7(z, \bar{z}) := z^3 - itz - u\bar{z}$ and identify when the zero set of the polynomial has exactly different seven points.

Proposition 5.4.61. For $0 < u < |t| < 2u$, card $\mathcal{Z}(q_7) = 7$. In fact,

$$\text{for } t > 0, \quad \mathcal{Z}(q_7) = \{0, p + iq, q + ip, -p - iq, -q - ip, r + ir, -r - ir\};$$

$$\text{for } t < 0, \quad \mathcal{Z}(q_7) = \{0, p + iq, q + ip, -p - iq, -q - ip, r - ir, -r + ir\},$$

where $p, q, r > 0$, $t = 4p\,q$, $p^2 + q^2 = u$, and $r^2 = \frac{t-u}{2}$ $(t > 0)$ or $r^2 = \frac{-t-u}{2}$ $(t < 0)$.

5.5 The Resultant of Two Real Polynomials

To prove Proposition 5.4.61, q_7 is decomposed into real and imaginary parts; $\mathrm{Re}\,(q_7) = x^3 - 3xy^2 + ty - ux$ and $\mathrm{Im}\,(q_7) = -y^3 + 3x^2y - tx + uy$. The symmetry about the line $y = x$ between two polynomials allows three kinds of common zeros:

Case 1. On the line $y = x$;

Case 2. On the line $y = -x$;

Case 3. Outside of both lines.

In Case 3, one point in the zero set automatically determines the other three zeros $-z$, $i\bar{z}$, and $-i\bar{z}$. Further, it follows from Bézout's Theorem that in Case 1 and Case 2 we can find at most three zeros including the origin because the zeros come from the intersection between a line and an irreducible cubic in $\mathbb{R}^2$. Lastly, we observe that these two cases cannot happen at the same time because the cubic harmonic polynomial q_7 has at most seven zeros. Consequently, we must find three zeros on either $y = x$ or $y = -x$ and four zeros outside of both lines. Pick a zero $p + iq$ outside of both lines in quadrant I. Since $p + iq$ is a common zero of the two real polynomials $\mathrm{Re}\,q_7$ and $\mathrm{Im}\,q_7$, t and u can be expressed in terms of p and q; $t = 4p\,q$ and $u = p^2 + q^2$. It is worth noting that u is always nonnegative. Next, we try to find conditions to guarantee seven zeros. We now calculate Resultant(Re q_7, Im q_7, y),

$$\det\begin{pmatrix} -3x & t & x^3-ux & 0 & 0 \\ 0 & -3x & t & x^3-ux & 0 \\ 0 & 0 & -3x & t & x^3-ux \\ -1 & 0 & 3x^2+u & -tx & 0 \\ 0 & -1 & 0 & 3x^2+u & -tx \end{pmatrix}$$

$$= x\left(2x^2+u-t\right)\left(2x^2+u+t\right)\left(16x^4-16x^2u+t^2\right).$$

First, if $q=0$, then $t=0$ and the resultant becomes $16x^3\left(2x^2+u\right)^2\left(x^2-u\right)$, in which case the card $\mathcal{Z}(q_7)$ is less than seven. Thus, we need another condition that q is different from zero. Here is a summary:

(1) One zero (the origin) comes from the factor x;

(2) Two zeros come from the factors $2x^2+u-t$ or $2x^2+u+t$;

(3) Four zeros outside of both lines come from the factor $16x^4-16x^2u+t^2$.

It is clearly sufficient to prove the result for the case of $t>0$ (i.e., $q>0$), in (2), with the factor $2x^2+u-t$. Thus, the condition $u<t$ is essential to have three points on $y=x$ and we need to investigate (3); set $16x^4-16x^2u+t^2=0$. Then

$$x^2=\frac{2u\pm\sqrt{4u^2-t^2}}{4},$$

where the right hand side is always positive under the second necessary condition $4u^2-t^2>0$.

Combining these two conditions, we present a cone on (t, u)-plane, in which q_7 has exactly seven zeros.

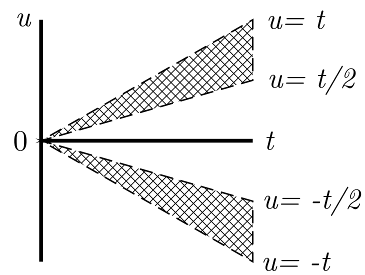


Figure 5.1: (t, u)-cone where card $\mathcal{Z}(q_7) = 7$.

The following example shows what the zero set of q_7 looks like.

Example 5.5.62. Consider a cubic harmonic polynomial $q_7(z) := z^3 - 2iz - \frac{5}{4}\bar{z} = 0$. We can rewrite $q_7(z)$ as $-\frac{5}{4}x + 2y + x^3 - 3xy^2 + i\left(-2x + \frac{5}{4}y + 3x^2y - y^3\right) = 0$. Note that the real and imaginary parts are symmetric with respect to the line $y = x$. By computing resultant we can eliminate y from both polynomials and get $\frac{1}{4}x(-1+x)(1+x)(-1+2x)(1+2x)\left(-3+8x^2\right)\left(13+8x^2\right) = 0$, which gives candidates for zeros. By a simple test, we see that the zero set has the seven points $z_1 = 0$, $z_2 = 1+\frac{1}{2}i$, $z_3 = \frac{1}{2}+i$, $z_4 = -1-\frac{1}{2}i$, $z_5 = -\frac{1}{2}-i$, $z_6 = \frac{\sqrt{6}}{4}+\frac{\sqrt{6}}{4}i$, and $z_7 = -\frac{\sqrt{6}}{4}-\frac{\sqrt{6}}{4}i$.

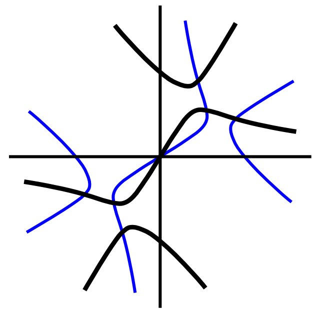


Figure 5.2: Re q_7 (in blue) and Im q_7 (in black).

Having the maximum number of zeros in $\mathcal{Z}(q_7)$ is crucial for us to solve a sextic truncated moment problem with a column relation of the form $q_7(Z, \bar{Z}) = 0$. If $M(2)$ is invertible, three alternatives arise: $M(2)$ admits a flat extension, the problem has no solution, or the problem is necessarily extremal. In detail, let $r_2 = \text{rank } M(2)$, $r_3 = \text{rank } M(3)$ and $v_3 = \text{card } \mathcal{V}_{M(3)}$. The following are the only possible cases:

Table 5.1: Possible cases when $r_2 = 6$

r_2	r_3	v_3	
6	6	6 or 7	$M(3)$ is a flat extension of $M(2)$
6	7	< 7	no representing measure
6	7	7	extremal

The case $(r_2, r_3, v_3) = (6, 6, 7)$ does not seem to be interesting but we should mention that it cannot happen because of the following theorem

Theorem 5.5.63. [18, Theorem 2.3] If $\mathcal{M}(n) \geq 0$ admits a flat extension $\mathcal{M}(n+1)$, then rank $\mathcal{M}(n) = \mathcal{V}(\mathcal{M}(n+1))$ and $\mathcal{V}(\mathcal{M}(n+1))$ forms the support of the unique representing measure ν for $\mathcal{M}(n+1)$.

As a consequence, the only case to be covered is $(r_3, v_3) = (7, 7)$. Since it is extremal, we will establish conditions for the existence of a representing measure by checking consistency. In other words, for any polynomial p of degree at most 6 vanishing on $\mathcal{Z}(q_7)$ we must verify that the Riesz functional Λ vanishes on p; in symbols, $\Lambda(p) = 0$. Another aspect we have to note is that the variety condition gives arise to another column relation. Since $M(2)$ is invertible, the columns $\mathit{1}, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2$, and $\bar{Z}Z^2$ must be linearly independent. This means the new column relation simultaneously contains $\bar{Z}Z^2$ and $\bar{Z}^2Z$. Using the functional calculus, we can show that, in the presence of a representing measure, the new column relation must be

$$\bar{Z}^2Z + i\bar{Z}Z^2 - iuZ - u\bar{Z} = 0,$$

which will be shown in the Proof of Theorem 5.8.65.

In the sequel, $\mathbb{C}_6[z, \bar{z}]$ will denote the space of complex polynomials in z and $\bar{z}$ of degree at most 6, and let

$$\begin{aligned} q_{LC}(z, \bar{z}) &:= \bar{z}^2z + i\bar{z}z^2 - iuz - u\bar{z} \\ &= (\bar{z} + iz)(\bar{z}z - u). \end{aligned}$$

Observe that q_{LC} is the product of the equations of a line and a circle, and that $\mathcal{Z}(q_7)$ is contained in $\mathcal{Z}(q_{LC})$.

5.6 Ideal-Like Property of $\ker\,\Lambda$

We now present an ideal-like property of $\ker\,\Lambda$. Given $\gamma \equiv \gamma^{(2n)}$, let $\mathcal{V} \equiv \mathcal{V}_\gamma$. Let us define an ideal

$$\mathcal{I}(\mathcal{V}) := \{p \in \mathcal{P} : p|_{\mathcal{V}} \equiv 0\}.$$

Since $\mathcal{V}$ is a set of complex numbers, $\mathcal{I}(\mathcal{V})$ is a complex ideal, which will be called *complex ideal* of γ. Let

$$\mathcal{N}_n := \{p \in \mathcal{P}_n : M(n)\hat{p} = 0\}.$$

If $p \in \mathcal{P}_n$ and $M(n)\hat{p} = 0$, then $p|_{\mathcal{V}} \equiv 0$ by the definition of $\mathcal{V}$. Therefore, $p \in \mathcal{I}(\mathcal{V})$, which implies $\mathcal{N}_n \subseteq \mathcal{I}(\mathcal{V})$. For general γ, the consistency condition can be rephrased in terms of $\mathcal{I}(\mathcal{V})$ as

$$p \in \mathcal{I}(\mathcal{V}) \cap \mathcal{P}_{2n} \Longrightarrow \Lambda(p) = 0.$$

Given $\gamma \equiv \gamma^{(2n)}$, let $\{p_1, \cdots, p_s\}$ denote a basis for $\mathcal{N}_n$. Denote by $\mathcal{J} \equiv \mathcal{J}_\gamma$ the smallest ideal containing the polynomials $p_1, \cdots, p_s$. Since $\mathcal{V}$ is the set of all common zeros of $p_1, \cdots, p_s$, it follows that $\mathcal{J} \subseteq \mathcal{I}(\mathcal{V})$, and hence that

$$\dim\,(\mathcal{J} \cap \mathcal{P}_k) \leq \dim\,(\mathcal{I}(\mathcal{V}) \cap \mathcal{P}_k) \quad (k \geq 0).$$

Note that if $f \in \mathcal{N}_n$ and $g \in \mathcal{P}_n$, then $p := fg \in \mathcal{J} \cap \mathcal{P}_{2n}$ and $\Lambda(p) = \left\langle M(n)\hat{f}, \hat{g} \right\rangle = 0$.

5.7 Representation of Polynomials Vanishing on a 7-Point Set

Let $\mathbb{C}_n[z, \bar{z}]$ be the set of all complex polynomials of the form:

$$a_{00} + a_{01}z + a_{10}\bar{z} + a_{02}z^2 + a_{11}\bar{z}z + a_{20}\bar{z}^2 + \cdots + a_{0n}z^n + \cdots + a_{n0}\bar{z}^n,$$

where all a_{ij}'s are complex numbers. The set of monomials $\{1, z, \bar{z}, \cdots, z^n, \cdots, \bar{z}^n\}$ is a basis for $\mathbb{C}_n[z,\bar{z}]$ and hence $\mathbb{C}_n[z,\bar{z}]$ is a vector space of dimension $(n+1)(n+2)/2$.

Checking consistency requires us to find a representation theorem of polynomials that vanish on the algebraic variety. This can be done through dimension analysis between vector spaces.

Lemma 5.7.64. Let $\mathcal{Q}_6 := \{p \in \mathbb{C}_6[z,\bar{z}] : p|_{\mathcal{Z}(q_7)} \equiv 0\}$ and let $\mathcal{I} := \{p \in \mathbb{C}_6[z,\bar{z}] : p = f\ q_7 + g\ \bar{q}_7 + h\ q_{LC}$ for some $f, g, h \in \mathbb{C}_3[z,\bar{z}]\}$. Then $\mathcal{Q}_6 = \mathcal{I}$.

Proof. Clearly, $\mathcal{I}$ is contained in $\mathcal{Q}_6$. We shall show that dim $\mathcal{I}$ = dim $\mathcal{Q}_6$. Let $T : \mathbb{C}^{30} \to \mathbb{C}_6[z,\bar{z}]$ be given by

$$\begin{aligned}(a_{00}, a_{01}, a_{10}, \cdots, a_{30}, b_{00}, b_{01}, b_{10}, \cdots, b_{30}, c_{00}, c_{01}, c_{10}, \cdots, c_{30}) \mapsto \\ (a_{00} + a_{01}z + a_{10}\bar{z} + \cdots + a_{30}\bar{z}^3)q_7 \\ +(b_{00} + b_{01}z + b_{10}\bar{z} + \cdots + b_{30}\bar{z}^3)\bar{q}_7 \\ +(c_{00} + c_{01}z + c_{10}\bar{z} + \cdots + c_{30}\bar{z}^3)q_{LC}.\end{aligned}$$

To determine the rank of T, we reduce the matrix form of the linear mapping using *Mathematica* and see that rank $T = 21$ if $tu \neq 0$. Since Ran $T = \mathcal{I}$, it follows that dimension of $\mathcal{I}$ is 21.

Now, consider an evaluation mapping $S : \mathbb{C}_6[z,\bar{z}] \to \mathbb{C}^7$ defined by

$$\begin{aligned}S(p(z,\bar{z})) := (&p(w_0,\bar{w}_0), p(w_1,\bar{w}_1), p(w_2,\bar{w}_2), \\ &p(w_3,\bar{w}_3), p(w_4,\bar{w}_4), p(w_5,\bar{w}_5), p(w_6,\bar{w}_6))\,,\end{aligned}$$

where $w_i \in \mathcal{V}$ for $i = 0, \cdots, 6$. Using Lagrange's interpolation, it is easy to verify that S is onto. In detail, if we define

$$\ell_j(z,\bar{z}) := \prod_{\substack{i=0\\ i\neq j}}^{6} \frac{z-w_i}{w_j-w_i},$$

then $S(\ell_j(z,\bar{z})) = e_{j+1}$, where e_j is the Euclidean basis element in $\mathbb{C}^7$ for $j = 0, \cdots, 6$.

Furthermore, note that the ker $S = \mathcal{Q}_6$. Because S is onto, we know that $\mathbb{C}_6[z,\bar{z}]/(\ker S) \cong \mathbb{C}^7$. Therefore, the dimension ker S is 21, which means $\mathcal{Q}_6$ has dimension 21. This completes the proof. $\square$

5.8 The Main Theorem

Theorem 5.8.65. Let $M(3) \geq 0$, with $M(2) > 0$ and $q_7(Z,\bar{Z}) = 0$. The following are equivalent:

(i) There exists a representing measure for $M(3)$;

(ii) $\Lambda(q_{LC}) = 0$ and $\Lambda(zq_{LC}) = 0$;

(iii) $\operatorname{Re}\gamma_{12} - \operatorname{Im}\gamma_{12} = u(\operatorname{Re}\gamma_{01} - \operatorname{Im}\gamma_{01})$, $\gamma_{22} = (t+u)\gamma_{11} - 2u \operatorname{Im}\gamma_{02}$;

(iv) $q_{LC}(Z,\bar{Z}) = 0$.

Proof. It is straight forward to see that (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv). Thus, it is enough to show (i)$\Longrightarrow$(iv) and (ii)$\Longrightarrow$(i).

(i) $\Longrightarrow$ (iv): First, recall that in the presence of a representing measure μ for γ, for $p \in \mathcal{P}_n$, card $\mu \subseteq \mathcal{Z}(p)$ if and only if $p(Z,\bar{Z}) = 0$. Let $\mathcal{V}$ be the algebraic variety of $M(3)$. Since there is the column relation $q_7(Z,\bar{Z}) = 0$ in $M(3)$, it follows that card $\mu \subseteq \mathcal{Z}(q_7) = \mathcal{V}$. Furthermore, since $\mathcal{V} \subseteq \mathcal{Z}(q_{LC})$, by applying the above argument again we see that there is another column relation $q_{LC}(Z,\bar{Z}) = 0$ in $M(3)$.

(ii) $\Longrightarrow$ (i): Observe that (ii) gives rise to the new moment matrix column relation $q_{LC}(Z, \bar{Z}) = 0$. Now, for the existence of a representing measure, we must have $7 = \text{rank } M(3) \leq \text{card } \mathcal{V}(\gamma) \leq |\mathcal{Z}(q_7)| = 7$, which means the cardinality of $\mathcal{V}(\gamma)$ must be seven. Thus, in this case the moment problem becomes extremal. To complete the proof, it suffices to show that this moment problem is consistent. That is, we have to show:

$$p \in \mathcal{Q}_6 \Longrightarrow \Lambda_\gamma(p) = 0.$$

By Lemma 5.7.64, $\mathcal{Q}_6$ is generated by $q_7, \bar{q}_7$, and q_{LC}, and hence if we can check

$$\Lambda_\gamma(\bar{z}^i z^j p(\bar{z}, z)) = 0 \qquad \text{for } 0 \leq i + j \leq 3,$$

for $p = q_7, \bar{q}_7, q_{LC}$, then we can show consistency of $M(3)$. For q_7 and $\bar{q}_7$, it is immediate from the column relation $Z^3 = itZ + u\bar{Z}$. For q_{LC}, it suffices to have $\Lambda_\gamma(q_{LC}) = 0$ and $\Lambda_\gamma(zq_{LC}) = 0$ due to the following observation:

First, we can reduce each polynomial $\bar{z}^i z^j q_{LC}$ for $2 \leq i+j \leq 3$ by $z^3 = itz + u\bar{z}$. For example,

$$\begin{aligned}
z^2 q_{LC} &= z^2(\bar{z}^2 z + i\bar{z}z^2 - iuz - u\bar{z}) \\
&= \bar{z}^2 z^3 + i\bar{z}z^4 - u\bar{z}z^2 - iuz^3 \\
&= \bar{z}^2(itz + u\bar{z}) + i\bar{z}z(itz + u\bar{z}) - u\bar{z}z^2 - iu(itz + u\bar{z}) \\
&= i(t+u)q_{LC}.
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
\bar{z}zq_{LC} &= -(t+u)q_{LC};\\
z^3q_{LC} &= (t+u)\bar{z}q_7;\\
\bar{z}z^2q_{LC} &= i(t+u)\bar{z}q_7.
\end{aligned}
$$

Second, note that the Riesz functional is invariant under the conjugate and $q_{LC} = \bar{q}_{LC}$. Now, we compute the Riesz functional for each polynomial. Suppose that $\Lambda_\gamma(q_{LC}) = 0$ and $\Lambda_\gamma(zq_{LC}) = 0$. Then,

$$
\begin{aligned}
\Lambda_\gamma(\bar{z}q_{LC}) &= \Lambda_\gamma(\bar{z}\bar{q}_{LC}) = \overline{\Lambda_\gamma(zq_{LC})} = 0;\\
\Lambda_\gamma(z^2q_{LC}) &= i(t+u)\Lambda_\gamma(q_{LC}) = 0;\\
\Lambda_\gamma(\bar{z}zq_{LC}) &= -i(t+u)\Lambda_\gamma(q_{LC}) = 0;\\
\Lambda_\gamma(\bar{z}^2q_{LC}) &= \Lambda_\gamma(\bar{z}^2\bar{q}_{LC}) = \overline{\Lambda_\gamma(z^2q_{LC})} = 0;\\
\Lambda_\gamma(z^3q_{LC}) &= (t+u)\Lambda_\gamma(\bar{z}q_{LC}) = 0;\\
\Lambda_\gamma(\bar{z}z^2q_{LC}) &= i(t+u)\Lambda_\gamma(\bar{z}q_{LC}) = 0;\\
\Lambda_\gamma(\bar{z}^2zq_{LC}) &= \overline{\Lambda_\gamma(\bar{z}z^2q_{LC})} = 0;\\
\Lambda_\gamma(\bar{z}^3q_{LC}) &= \overline{\Lambda_\gamma(z^3q_{LC})} = 0.
\end{aligned}
$$

□

5.8.1 A Concrete Example

Example 5.8.66. Consider $M(3)(\gamma)$

$$\begin{pmatrix}
1 & 0 & 0 & \frac{11i}{14} & \frac{13}{14} & -\frac{11i}{14} & 0 & 0 & 0 & 0 \\
0 & \frac{13}{14} & -\frac{11i}{14} & 0 & 0 & 0 & \frac{7i}{8} & \frac{59}{56} & -\frac{7i}{8} & -\frac{23}{56} \\
0 & \frac{11i}{14} & \frac{13}{14} & 0 & 0 & 0 & -\frac{23}{56} & \frac{7i}{8} & \frac{59}{56} & -\frac{7i}{8} \\
-\frac{11i}{14} & 0 & 0 & \frac{59}{56} & -\frac{7i}{8} & -\frac{23}{56} & 0 & 0 & 0 & 0 \\
\frac{13}{14} & 0 & 0 & \frac{7i}{8} & \frac{59}{56} & -\frac{7i}{8} & 0 & 0 & 0 & 0 \\
\frac{11i}{14} & 0 & 0 & -\frac{23}{56} & \frac{7i}{8} & \frac{59}{56} & 0 & 0 & 0 & 0 \\
0 & -\frac{7i}{8} & -\frac{23}{56} & 0 & 0 & 0 & \frac{277}{224} & -\frac{227i}{224} & -\frac{97}{224} & -\frac{61i}{224} \\
0 & \frac{59}{56} & -\frac{7i}{8} & 0 & 0 & 0 & \frac{227i}{224} & \frac{277}{224} & -\frac{227i}{224} & -\frac{97}{224} \\
0 & \frac{7i}{8} & \frac{59}{56} & 0 & 0 & 0 & -\frac{97}{224} & \frac{227i}{224} & \frac{277}{224} & -\frac{227i}{224} \\
0 & -\frac{23}{56} & \frac{7i}{8} & 0 & 0 & 0 & \frac{61i}{224} & -\frac{97}{224} & \frac{227i}{224} & \frac{277}{224}
\end{pmatrix}.$$

Using nested determinants and Smul'jan's theorem, it can be verified that $M(3)$ is positive semidefinite and is of rank 7 with three column relations $Z^3 = 2iZ + \frac{5}{4}\bar{Z}$, $\bar{Z}^2 Z = i\frac{5}{4}Z + \frac{5}{4}\bar{Z} - i\bar{Z}Z^2$, and $\bar{Z}^3 = -2i\bar{Z} + \frac{5}{4}Z$. As in Example 5.5.62, the algebraic variety consists of exactly seven points $z_1 = 0$, $z_2 = 1 + \frac{1}{2}i$, $z_3 = \frac{1}{2} + i$, $z_4 = -1 - \frac{1}{2}i$, $z_5 = -\frac{1}{2} - i$, $z_6 = \frac{\sqrt{6}}{4} + \frac{\sqrt{6}}{4}i$, and $z_7 = -\frac{\sqrt{6}}{4} - \frac{\sqrt{6}}{4}i$. Thus γ is extremal and the main theorem implies that $M(3)$ has a 7-atomic representing measure μ. Solving a Vandermonde equation, we compute the densities and may write $\mu = \sum_{i=1}^{7} \rho_i \delta_{z_i}$, where the densities $\rho_i = \frac{1}{7}$ for $1 \le i \le 7$.

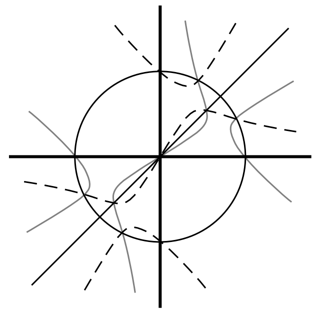


Figure 5.3: Re q_7 (in gray), Im q_7 (dashed line), and q_{LC} (in black)

The next example shows a column relation allowing 7 points does not guarantee the existence of a representing measure even though the moment matrix is positive semidefinite.

Example 5.8.67. Consider $M(3)(\gamma)$

$$\begin{pmatrix}
1 & 0 & 0 & \frac{11i}{14} & \frac{13}{14} & -\frac{11i}{14} & 0 & 0 & 0 & 0 \\
0 & \frac{13}{14} & -\frac{11i}{14} & 0 & 0 & 0 & \frac{7i}{8} & \frac{21}{20} & -\frac{7i}{8} & -\frac{23}{56} \\
0 & \frac{11i}{14} & \frac{13}{14} & 0 & 0 & 0 & -\frac{23}{56} & \frac{7i}{8} & \frac{21}{20} & -\frac{7i}{8} \\
-\frac{11i}{14} & 0 & 0 & \frac{21}{20} & -\frac{7i}{8} & -\frac{23}{56} & 0 & 0 & 0 & 0 \\
\frac{13}{14} & 0 & 0 & \frac{7i}{8} & \frac{21}{20} & -\frac{7i}{8} & 0 & 0 & 0 & 0 \\
\frac{11i}{14} & 0 & 0 & -\frac{23}{56} & \frac{7i}{8} & \frac{21}{20} & 0 & 0 & 0 & 0 \\
0 & -\frac{7i}{8} & -\frac{23}{56} & 0 & 0 & 0 & \frac{277}{224} & -\frac{161i}{160} & -\frac{7}{16} & -\frac{61i}{224} \\
0 & \frac{21}{20} & -\frac{7i}{8} & 0 & 0 & 0 & \frac{161i}{160} & \frac{277}{224} & -\frac{161i}{160} & -\frac{7}{16} \\
0 & \frac{7i}{8} & \frac{21}{20} & 0 & 0 & 0 & -\frac{7}{16} & \frac{161i}{160} & \frac{277}{224} & -\frac{161i}{160} \\
0 & -\frac{23}{56} & \frac{7i}{8} & 0 & 0 & 0 & \frac{61i}{224} & -\frac{7}{16} & \frac{161i}{160} & \frac{277}{224}
\end{pmatrix}$$

Applying Smul'jan' theorem, we know $M(3)$ is positive semidefinite if and only if

$M(3)_{\mathcal{B}}$, the compression of $M(3)$ to rows and columns indexed by the basis $\mathcal{B}$ for $\mathcal{C}_{M(3)}$, is positive semidefinite. Since all nested determinants of $M(3)_{\mathcal{B}}$ are positive, it follows that $M(3)$ is positive semidefinite. Row reduction via *Mathematica* shows $M(3)$ has two column relations ($Z^3 = 2iZ + \frac{5}{4}\bar{Z}$ and its conjugate) and hence, $M(3)$ has rank 8. As seen in Example 5.8.66, the zero set of the polynomial, and therefore the algebraic variety, has at most 7 points. Clearly, this case violates the variety condition and there is no representing measure.

We end this section with introducing another class of cubic harmonic polynomial with only real coefficients. This class was discovered independently from the previous class using symmetry. Later, we learned that the two classes were equivalent under a degree-one transformation.

Corollary 5.8.68. A cubic complex polynomial $w^3 = 2\alpha w - \beta\bar{w}$ $(\alpha, \beta \in \mathbb{C})$ has seven zeros if and only if $\alpha < \beta < \frac{2\sqrt{3}}{3}\alpha$. In other words, the zero set of a cubic polynomial $w^3 = 2\left(m^2 - n^2\right) w - \left(m^2 + n^2\right) \bar{w}$ has seven points if and only if $\left[m - (2 + \sqrt{3})n\right] \left[m + (2 + \sqrt{3})n\right] < 0$.

This corollary to Proposition 5.4.61 can be easily verified by the degree-one transformation $\varphi(z) = (1 + i)\bar{z}$ used in proof of the next lemma.

Lemma 5.8.69. Suppose $M(3)(\widehat{\gamma})$, the associated moment matrix of a moment sequence $\widehat{\gamma}$, is positive semidenifinite and satisfies the column relation $W^3 = 2\alpha W - \beta\overline{W}$ for $\alpha < \beta < \frac{2\sqrt{3}}{3}\alpha$ and $\widehat{M}(2) > 0$. Then $\widehat{\gamma}$ has a representing measure if and only if

$$\Lambda(\hat{q}_{LC}) = 0 \text{ and } \Lambda(w\hat{q}_{LC}) = 0,$$

where $\hat{q}_{LC}(w,\bar{w}) = \bar{w}^2 w - \bar{w}w^2 + \beta w - \beta \bar{w}$.

Proof. We will prove this using the equivalence of truncated moment problems under degree-one transformations. Consider the degree-one transformation

$$\varphi(z,\bar{z}) = (1+i)\bar{z}$$

and let $M(3)$ be a new moment matrix under φ. Now, we can transform a column relation in $M(3)$ using an identity of column relations in two moment matrices, $p(W,\overline{W}) = J^*(p \circ \Phi)(Z,\bar{Z})$ as in Subsection 2.6.3:

If $p(z,\bar{z}) = z$, then $Z = (1+i)J^*\overline{W}$, and so $\overline{W} = \frac{1-i}{2}(J^*)^{-1}Z$;

If $p(z,\bar{z}) = \bar{z}$, then $\bar{Z} = \overline{(1+i)}J^*W$, and so $W = \frac{1+i}{2}(J^*)^{-1}\bar{Z}$;

If $p(z,\bar{z}) = \bar{z}^3$, then $\bar{Z}^3 = \overline{(1+i)}^3 J^*W^3$, and so $W^3 = \frac{-1+i}{4}(J^*)^{-1}\bar{Z}^3$.

Thus, we rewrite $W^3 = 2\alpha W - \beta\overline{W}$ as

$$\frac{-1+i}{4}(J^*)^{-1}\bar{Z}^3 = 2\alpha \cdot \frac{1+i}{2}(J^*)^{-1}\bar{Z} - \beta \cdot \frac{1-i}{2}(J^*)^{-1}Z$$
$$\Longrightarrow \bar{Z}^3 = -4\alpha i\bar{Z} + 2\beta Z \Longrightarrow Z^3 = 4\alpha i Z + 2\beta\bar{Z}.$$

Therefore, the given moment problem is equivalent to that solved by Theorem 5.8.65, which says there is a representing measure of $M(3)$ if and only if

$$\Lambda(q_{LC}) = 0 \qquad \text{and} \qquad \Lambda(zq_{LC}) = 0,$$

where $q_{LC}(z,\bar{z}) = \bar{z}^2 z + i\bar{z}z - iuz - u\bar{z}$.

Again, to find the transformed column relation we proceed as follows. If $p(z,\bar{z}) = \bar{z}^2 z$, then $\bar{Z}^2 Z = (1+i)(1-i)^2 J^*\overline{W}W^2$ and so $\bar{Z}Z^2 = (1-i)(1+i)^2 J^*\overline{W}^2 W$.

Thus, we can define $\hat{q}_{LC}(w, \bar{w}) := \overline{w}^2 w - \bar{w}w^2 + \beta w - \beta\bar{w}$ and show that:

$$
\begin{aligned}
\Lambda(q_{LC}(z,\bar{z})) = 0 &\iff \Lambda(\hat{q}_{LC}(w,\bar{w})) = 0; \\
\Lambda(zq_{LC}(z,\bar{z})) = 0 &\iff \Lambda(\bar{w}\hat{q}_{LC}(w,\bar{w})) = 0 \\
&\iff \Lambda(\bar{w}\overline{\hat{q}_{LC}(w,\bar{w})}) = 0 \text{ (since } \hat{q}_{LC} \text{ is real)} \\
&\iff \overline{\Lambda(w\hat{q}_{LC}(w,\bar{w}))} = 0 \\
&\iff \Lambda(w\hat{q}_{LC}(w,\bar{w})) = 0.
\end{aligned}
$$

$\square$

A special case of this lemma when $\alpha = 3$ and $\beta = -2$ appeared in [28]. In our first case of a cubic harmonic column relation we saw that q_7 can be decomposed as two irreducible cubics and q_{LC} is a product of a line and a circle. On the other hand, in the second case all polynomials from column relations turn out to be reducible cubics which are products of a line and a conic.

Example 5.8.70. Consider $M(3)(\gamma)$

$$\begin{pmatrix}
1 & 0 & 0 & \frac{11}{7} & \frac{13}{7} & \frac{11}{7} & 0 & 0 & 0 & 0 \\
0 & \frac{13}{7} & \frac{11}{7} & 0 & 0 & 0 & \frac{7}{2} & \frac{59}{14} & \frac{7}{2} & \frac{23}{14} \\
0 & \frac{11}{7} & \frac{13}{7} & 0 & 0 & 0 & \frac{23}{14} & \frac{7}{2} & \frac{59}{14} & \frac{7}{2} \\
\frac{11}{7} & 0 & 0 & \frac{59}{14} & \frac{7}{2} & \frac{23}{14} & 0 & 0 & 0 & 0 \\
\frac{13}{7} & 0 & 0 & \frac{7}{2} & \frac{59}{14} & \frac{7}{2} & 0 & 0 & 0 & 0 \\
\frac{11}{7} & 0 & 0 & \frac{23}{14} & \frac{7}{2} & \frac{59}{14} & 0 & 0 & 0 & 0 \\
0 & \frac{7}{2} & \frac{23}{14} & 0 & 0 & 0 & \frac{277}{28} & \frac{227}{28} & \frac{97}{28} & -\frac{61}{28} \\
0 & \frac{59}{14} & \frac{7}{2} & 0 & 0 & 0 & \frac{227}{28} & \frac{277}{28} & \frac{227}{28} & \frac{97}{28} \\
0 & \frac{7}{2} & \frac{59}{14} & 0 & 0 & 0 & \frac{97}{28} & \frac{227}{28} & \frac{277}{28} & \frac{227}{28} \\
0 & \frac{23}{14} & \frac{7}{2} & 0 & 0 & 0 & -\frac{61}{28} & \frac{97}{28} & \frac{227}{28} & \frac{277}{28}
\end{pmatrix}$$

In a way entirely similar to previous examples, we see $M(3)$ is positive. Row reduction shows that $M(3)$ has three column relations $f(Z,\bar{Z}) := Z^3 - 4Z + \frac{5}{2}\bar{Z} = 0$, $\bar{f}(Z,\bar{Z}) = 0$, and $g(Z,\bar{Z}) := \bar{Z}^2 Z - \bar{Z}Z^2 + \frac{5}{2}Z - \frac{5}{2}\bar{Z} = 0$. Note that f and g are exactly in the form of $\hat{q}_7$ and $\hat{q}_{LC}$ in Lemma 5.8.69. Indeed, with $z := x + iy$ and $\bar{z} := x - iy$ we write $f = \frac{1}{2}x\left(-3 + 2x^2 - 6y^2\right) - \frac{1}{2}y\left(13 - 6x^2 + 2y^2\right)i$ and $g = -y\left(-5 + 2x^2 + 2y^2\right)i$. Using the resultant, we find the algebraic variety: $\mathcal{V} := \left\{0, \pm\frac{\sqrt{6}}{2}, \frac{3}{2} \pm \frac{1}{2}i, -\frac{3}{2} \pm \frac{1}{2}i\right\}$. Thus, this is an extremal problem. To apply Lemma 5.8.69, compute

$$\begin{aligned}
\Lambda(g) &= \gamma_{21} - \gamma_{12} + \frac{5}{2}\gamma_{01} - \frac{5}{2}\gamma_{10} = 0, \\
\Lambda(zg) &= \gamma_{22} - \gamma_{13} + \frac{5}{2}\gamma_{02} - \frac{5}{2}\gamma_{11} \\
&= \frac{59}{14} - \frac{7}{2} + \frac{5}{2} \cdot \frac{11}{7} - \frac{5}{2} \cdot \frac{13}{7} = 0
\end{aligned}$$

and it follows that there is a representing measure. By solving a Vandermonde equation, we find the densities and the measure is written as $\mu = \sum_{i=1}^{7} \rho_i \delta_{z_i}$, where the densities $\rho_i = \frac{1}{7}$ for $1 \leq i \leq 7$ and $z_i \in \mathcal{V}$. Lastly, we note that this problem is equivalent to the problem in Example 5.5.62 under the degree-one transformation $\varphi(z) = (1+i)\bar{z}$ as in Lemma 5.8.69.

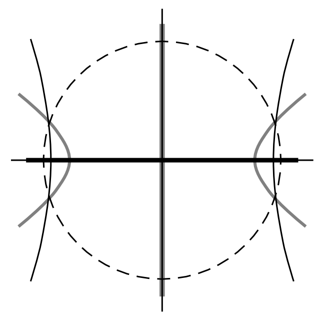


Figure 5.4: Re f (in gray), Im f (in black) and g (dotted line)

5.9 $\mathcal{M}(3)$ satisfying $Y = X^3$

In this dissertation we focus almost entirely on truncated moment problems with finite algebraic variety; however, there are solutions of truncated moment problems with infinite algebraic variety as well. In [21], a solution of the general $\mathcal{M}(n)(\beta)$ satisfying $Y = X^3$ was announced, regardless of cardinality of the algebraic variety. Together with positivity of $\mathcal{M}(n)$, a numerical condition $\beta_{1,2n-1} > \psi(\beta)$ (where $\psi(\beta)$ is some quantity dependent on the moments) is part of the solution. This is a totally different type of condition from those used in the solution of quadratic and quartic moment problems. While similar condition is not necessary for planar curves of degree 1 or 2, this new numerical condition is essentially related to the cubic $y = x^3$. Thus, this solution prompted us to find such conditions in the study of $\mathcal{M}(3)$. In our solutions presented in Chapter 6, we will confirm that the suggested conditions turn out to help solve the problem. Our solutions will need information on new cubics whose zero sets contain the original algebraic variety. Once this is done, the verification of Consistency will depend on a parameter, just as in L. Fialkow's approach.

CHAPTER 6
THE DIVISION ALGORITHM IN TMP

In this chapter, we will outline a solution of the truncated singular sextic moment problem. The main contribution comes from an application of the Division Algorithm from real algebraic geometry. The algorithm involves polynomials in several indeterminates.

6.1 The Division Algorithm for Multivariable Polynomials

Theorem 6.1.71 (Division Algorithm)**.** [7] Fix a monomial order $>$ on $\mathbb{Z}^n_{\geq 0}$, and let $F = (f_1, \cdots, f_s)$ be an ordered s-tuple of polynomials in $\mathbb{R}[x_1, \cdots, x_n]$. Then every $f \in \mathbb{R}[x_1, \cdots, x_n]$ can be written as

$$f = a_1 f_1 + \cdots + a_s f_s + r,$$

where $a_i, r \in \mathbb{R}[x_1, \cdots, x_n]$, and either $r = 0$ or r is a linear combination of monomials with coefficients in $\mathbb{R}$, none of which is divisible by any leading terms of $f_1, \cdots, f_s$. We call r a *remainder* of f on division by F. Furthermore, if $a_i f_i \neq 0$, then we have

$$\text{multideg}(f) \geq \text{multideg}(a_i f_i).$$

Using the Division Algorithm we will be able to construct a structure theorem for polynomials that vanish on the algebraic variety of $\mathcal{M}(3)$.

While not necessary to solve the moment problem, we note that one big difference between the one-dimensional and multi-dimensional versions of the division algorithm is the fact that a remainder might not be unique in the latter case.

Example 6.1.72. [7] Let $f_1 = xy + 1$, $f_2 = y^2 - 1 \in \mathbb{R}[x, y]$. We will use the lexicographic order with $x > y$. Dividing $f = xy^2 - x$ by $F = (f_1, f_2)$, the result is

$$f = y \cdot f_1 + 0 \cdot f_2 + (-x - y).$$

With $F = (f_2, f_1)$, however, we have

$$f = x \cdot f_2 + 0 \cdot f_1 + 0.$$

Thus, one might think that the Division Algorithm is an imperfect generalization of the traditional (one-variable) version. However, there is a remedy, the so-called *Gröebner bases*. For further study on this topic, the reader is referred to [7].

6.2 Extremal Sextic Moment Problems

6.2.1 Importance of Extremal Moment Problems

The crux of the truncated moment problem is to find a moment extension sequence ending with a flat extension. In other words, we need to predict the existence and estimate the minimal value of an integer k satisfying rank $M(n + k) =$ rank $M(n + k + 1)$. However, the process is troublesome; even building $M(4)$ from $M(3)$ is not straightforward as seen in Example 5.1.59 in Chapter 5. According to the existence criterion in [11, Theorem 1.5], we might need to check all the extension matrices up to $k = 2n^2 + 6n + 6$; for example, if $n = 3$, then $k \leq 42$. The following theorem significantly reduces the length of an extension sequence.

Theorem 6.2.73. [23] Suppose $v < \infty$. Then β admits a representing measure if and only if $\mathcal{M}(n)(\beta)$ has a positive extension $\mathcal{M}(n+v-r+1)$ satisfying rank $\mathcal{M}(n+v-r+1) \leq \text{card}\, \mathcal{V}_{\mathcal{M}(n+v-r+1)}$.

For the existence of a representing measure, $\mathcal{M}(n)$ has a convergent extension (rank-increasing) sequence with length $k = v - r$. For instance, if $\mathcal{M}(3)$ is singular with the invertible $\mathcal{M}(2)$ and $v < \infty$, then k is less than or equal to $v - r = 9 - 7 = 2$. Thus, we might check an extension up to $\mathcal{M}(3+2+1) = \mathcal{M}(6)$. Using this argument, we classify all kinds of sextic truncated moment problems. Given a convergent sequence of moment matrix extensions:

$$\mathcal{M}(n) \to \mathcal{M}(n+1) \to \mathcal{M}(n+2) \to \cdots,$$

denote $r_n :=$ rank $\mathcal{M}(n)$ and $v_n := \text{card}\, \mathcal{V}(\mathcal{M}(n))$. Since each extension must satisfy the variety condition, the following inequality is true:

$$r_n \leq r_{n+1} \leq r_{n+2} \leq \cdots \leq v_{n+2} \leq v_{n+1} \leq v_n.$$

We are interested in $\mathcal{M}(3)$ with the invertible $\mathcal{M}(2)$ block. Thus, we will assume $\mathcal{M}(2) > 0$. Applying Theorem 6.2.73, we see what the maximal extension is in each case, in terms of the rank and the cardinality of the algebraic variety. We now summarize what is known for moment sequences with or without a representing measure.

Table 6.1: Sextic moment problems according to r and v

r_3	v_3	$v_3 - r_3$	Max Extension		Eg w/ meas.	Eg wo/ meas.
7	7	0	$\mathcal{M}(4)$	extremal	known	unknown
7	8	1	$\mathcal{M}(5)$		unknown	unknown
7	9	2	$\mathcal{M}(6)$		unknown	unknown
7	∞	N/A	N/A		known	known
8	8	0	$\mathcal{M}(4)$	extremal	known	known
8	9	1	$\mathcal{M}(5)$		known	known
8	∞	N/A	N/A		known	known
9	9	0	N/A		impossible	impossible
9	∞	N/A	N/A		known	known
10	∞	N/A	N/A		known	known

First, observe that the case of $r_3 = v_3 = 9$ cannot happen. For, if the rank is 9, there is only one column relation, and hence the algebraic variety is an entire algebraic curve which means v_3 is infinity. Our focus from now on is on the extremal cases and then we will make a modest application of extremal cases to non-extremal cases. We need one more results from [18] together with Theorem 5.5.63.

Theorem 6.2.74. [18, Theorem 2.4] Assume that $\mathcal{M}(n) \geq 0$ admits a flat extension $\mathcal{M}(n+1)$. Then $\mathcal{V}(\mathcal{M}(n+2)) = \mathcal{V}(\mathcal{M}(n+1))$.

Combining two Theorems, it follows that

$$r_n = r_{n+1} \Longrightarrow \left\{ \begin{array}{rcl} r_n & = & v_{n+1} \\ \\ v_{n+1} & = & v_{n+2} \end{array} \right\} \Longrightarrow r_n = v_{n+1} = v_{n+2}. \tag{6.2.1}$$

Consider $\mathcal{M}(3) \geq 0$ with $r_3 = 8$ and $v_3 = 9$ and apply (6.2.1). We then have only two feasible cases: Notice that in both cases an extension $\mathcal{M}(4)$ is extremal.

Table 6.2: The cases when $r_3 = 8$ and $v_3 = 9$

r_3	$\leq$	r_4	$\leq$	r_5	$\leq$	v_5	$\leq$	v_4	$\leq$	v_3
8		**8**		8		8		**8**		9
8		**9**		9		9		**9**		9

Therefore, after building $\mathcal{M}(4)$ with proper higher order moments, it might possible to solve the problem by checking Consistency. This approach would be a part of our future research.

6.3 The Case rank $M(3) = \text{card } \mathcal{V} = 7$

For computational convenience, we will deal with this problem in the real version of TMP, that is, $\{\mathit{1}, X, Y, X^2, XY, Y^2, X^3, X^2Y, XY^2, Y^3\}$ denotes the set of columns in $\mathcal{M}(3)$. In the sequel, assume rank $\mathcal{M}(3) := r$ and card $\mathcal{V}(\beta) := v$; let $W_{\mathcal{B}}$ denote a compression of the generalize Vandermonde matrix to a basis $\mathcal{B}$ for $\mathcal{C}_{\mathcal{M}(3)}$. In this section we study the extremal moment problem for a moment matrix $\mathcal{M}(3)$

satisfying

$$\mathcal{M}(3) \geq 0, \mathcal{M}(2) > 0, \text{ and } r = v = 7. \tag{6.3.1}$$

Set $\mathcal{V} \equiv \mathcal{V}(\beta) := \{(x_1, y_1), \cdots, (x_7, y_7)\}$. Because of the rank condition and the invertibility of $\mathcal{M}(2)$, there is only one linearly independent column amongst X^3, X^2Y, XY^2, and Y^3. Thus, the basis of $\mathcal{C}_{\mathcal{M}(3)}$ is one of the following:

Case 1. $\mathcal{B}_1 := \{1, X, Y, X^2, XY, Y^2, X^3\}$

Case 2. $\mathcal{B}_2 := \{1, X, Y, X^2, XY, Y^2, X^2Y\}$

Case 3. $\mathcal{B}_3 := \{1, X, Y, X^2, XY, Y^2, XY^2\}$

Case 4. $\mathcal{B}_4 := \{1, X, Y, X^2, XY, Y^2, Y^3\}$

Recall that consistency of β is a necessary condition for the existence of a representing measure, and it is also sufficient in the extremal case. Thus, the key to the solution of TMP is checking Consistency of the moment sequence. For Case 1, Weak Consistency and a condition about moments solves the problem.

Theorem 6.3.75. Suppose $\mathcal{M}(3)(\beta)$ satisfies (6.3.1). Let $\mathcal{B}_1$ be a basis for $\mathcal{C}_{\mathcal{M}(3)}$. Then β has a representing measure if and only if $\mathcal{M}(3)$ is weakly consistent and for $0 \leq i + j \leq 2$,

$$\Lambda_\beta(x^i y^j (x^4 - a_{00} - a_{10}x - a_{01}y - a_{20}x^2 - a_{11}xy - a_{02}y^2 - a_{30}x^3)) = 0,$$

where $(a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, a_{30})^T = W_{\mathcal{B}_1}^{-1}(x_1^4, \cdots, x_7^4)^T$.

Proof. Let $q_k(X,Y) = 0$ is the column relation in ith column for $k = 8, 9$, and 10. Since the following compression of the generalized Vandermonde matrix $W_{\mathcal{B}_1}$,

$$W_{\mathcal{B}_1} = \begin{pmatrix} 1 & x_1 & y_1 & {x_1}^2 & x_1y_1 & {y_1}^2 & {x_1}^3 \\ 1 & x_2 & y_2 & {x_2}^2 & x_2y_2 & {y_2}^2 & {x_2}^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_7 & y_7 & {x_7}^2 & x_7y_7 & {y_7}^2 & {x_7}^3 \end{pmatrix} \tag{6.3.2}$$

is invertible, there exist a unique polynomial with the leading monomial x^4 that vanishes on the variety $\mathcal{V}$, say,

$$r_1(x,y) := x^4 - (a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3),$$

where $(a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, a_{30})^T = W_{\mathcal{B}_1}^{-1}(x_1^4, \cdots, x_7^4)^T$.

Set $\mathcal{I} := \{p \in \mathcal{P}_6 : p|_{\mathcal{V}} \equiv 0\}$. Applying the Division Algorithm, any $p \in \mathcal{I}$ can be written as,

$$p = Aq_8 + Bq_9 + Cq_{10} + Dr_1 + r,$$

where $A, B, C \in \mathcal{P}_3$, $D \in \mathcal{P}_2$ and $r(x,y) = c_{00}+c_{10}x+c_{01}y+c_{20}x^2+c_{11}xy+c_{02}y^2+c_{30}x^3$ for some $c_{00}, \cdots, c_{02}, c_{30} \in \mathbb{R}$.

We now claim that $\mathcal{I} = \{fq_8 + gq_9 + hq_{10} + qr_1 : f, g, h \in \mathcal{P}_3, q \in \mathcal{P}_2\}$ by showing $r(x,y) \equiv 0$. Note that since p vanishes on $\mathcal{V}$, so does r, which leads to the matrix form of a linear system:

$$W_{\mathcal{B}_1} \begin{pmatrix} c_{00} & c_{10} & c_{01} & c_{20} & c_{11} & c_{02} & c_{30} \end{pmatrix}^T = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}^T.$$

Since the matrix in the left hand side is invertible, we know $c_{00} = c_{10} = c_{01} = c_{20} = c_{11} = c_{02} = c_{30} = 0$, which means $r(x,y) = 0$.

Consequently, β is consistent if and only if

$$\begin{cases} \Lambda_\beta(x^i y^j q_k(x,y)) = 0 \quad (0 \le i+j \le 3;\ k = 8, 9, 10); \\ \Lambda_\beta(x^t y^u r_1(x,y)) = 0 \quad (0 \le t+u \le 2). \end{cases}$$

But both conditions are immediate from the column relations in $\mathcal{M}(3)$ and from the hypothesis. $\square$

For Case 2, again only Weak Consistency is enough to solve the problem and the proof is almost identical as that of Case 1.

Theorem 6.3.76. Suppose $\mathcal{M}(3)(\beta)$ satisfies (6.3.1). Let be a basis for $\mathcal{C}_{\mathcal{M}(3)}$. Then β admits a representing measure if and only if $\mathcal{M}(3)$ is weakly consistent.

Proof. Let $q_k(X,Y) = 0$ is the column relation in ith column for $k = 7, 9$, and 10. Set $\mathcal{I} := \{p \in \mathcal{P}_6 : p|_{\mathcal{V}} \equiv 0\}$. Due to the Division Algorithm, we may write any $p \in \mathcal{I}$ as,

$$p = Aq_7 + Bq_9 + Cq_{10} + r,$$

where $A, B, C \in \mathcal{P}_3$ and $r(x,y) = c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{21}x^2y$ for some $c_{00}, \cdots, c_{02}, c_{21} \in \mathbb{R}$.

Once we establish that $r(x,y) \equiv 0$, we will need to verify that $\mathcal{I} = \{fq_7 + gq_9 + hq_{10} + r : f, g, h \in \mathcal{P}_3\}$. Note that $p|_{\mathcal{V}} \equiv 0 \Longrightarrow r|_{\mathcal{V}} \equiv 0$. This argument brings up the matrix form of a linear system:

$$W_{\mathcal{B}_2} \begin{pmatrix} c_{00} & c_{10} & c_{01} & c_{20} & c_{11} & c_{02} & c_{21} \end{pmatrix}^T = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}^T.$$

Invertibility of $W_{\mathcal{B}_2}$ implies $c_{00} = c_{10} = c_{01} = c_{20} = c_{11} = c_{02} = c_{21} = 0$, i.e. $r(x,y) = 0$. Therefore, we need only three polynomials attained from column relations in $\mathcal{M}(3)$ to check consistency of β. The test is straightfoward as follows: β is consistent if and only if

$$\Lambda_\beta(x^i y^j q_k(x,y)) = 0 \quad (0 \le i+j \le 3;\, k = 7, 9, 10),$$

which is inherent in $\mathcal{M}(3)$. This completes the proof. $\square$

In order to solve Case 3, we need to go through the same approach we used for Case 2, and therefore the proof will be omitted.

Theorem 6.3.77. Suppose $\mathcal{M}(3)(\beta)$ satisfies (6.3.1). Let $\mathcal{B}_3$ be a basis for $\mathcal{C}_{\mathcal{M}(3)}$. Then β admits a representing measure if and only if $\mathcal{M}(3)$ is weakly consistent.

As for Case 4, we can embed this case in the study of Case 1 via a degree-one transformation. Indeed, observe first that $X^3, YX^2, Y^2X \in \text{Ran } \mathcal{M}(2)$ and $Y^3 \notin \text{Ran } \mathcal{M}(2)$. Consider the degree-one transformation that interchanges X and Y, and set $X = \tilde{Y}$ and $Y = \tilde{X}$. Let $\tilde{\mathcal{M}}(3)$ denote the moment matrix obtained under the transformation. The basis for $\tilde{\mathcal{M}}(3)$ is $\{\tilde{\mathit{1}}, \tilde{X}, \tilde{Y}, \tilde{X}^2, \tilde{X}\tilde{Y}, \tilde{Y}^2, \tilde{X}^3\}$ and we see that $\tilde{X}^3, \tilde{X}^2\tilde{Y}, \tilde{X}\tilde{Y}^2 \in \text{Ran } \tilde{\mathcal{M}}(2)$ and $\tilde{Y}^3 \notin \text{Ran } \tilde{\mathcal{M}}(2)$. Thus, this case is a subcase of Case 1.

6.4 The Case rank $\mathcal{M}(3) = \text{card } \mathcal{V} = 8$

In this section we discuss the other extremal moment problem for a moment matrix $\mathcal{M}(3)$ satisfying

$$\mathcal{M}(3) \ge 0, \mathcal{M}(2) > 0, \text{ and } r = v = 8. \tag{6.4.1}$$

Write $\mathcal{V} \equiv \mathcal{V}(\beta) := \{(x_1, y_1), \cdots, (x_8, y_8)\}$. Since we assumed the invertibility of the minor block $\mathcal{M}(2)$ and rank $\mathcal{M}(3) = 8$, there are two linearly independent columns among X^3, X^2Y, XY^2, and Y^3. Thus, the basis of $\mathcal{C}_{\mathcal{M}(3)}$ is one of the following:

Case 1. $\mathfrak{B}_1 := \{1, X, Y, X^2, XY, Y^2, X^3, X^2Y\}$

Case 2. $\mathfrak{B}_2 := \{1, X, Y, X^2, XY, Y^2, X^3, XY^2\}$

Case 3. $\mathfrak{B}_3 := \{1, X, Y, X^2, XY, Y^2, X^3, Y^3\}$

Case 4. $\mathfrak{B}_4 := \{1, X, Y, X^2, XY, Y^2, X^2Y, XY^2\}$

Case 5. $\mathfrak{B}_5 := \{1, X, Y, X^2, XY, Y^2, X^2Y, Y^3\}$

Case 6. $\mathfrak{B}_6 := \{1, X, Y, X^2, XY, Y^2, XY^2, Y^3\}$

Again, the problem will be solved once we show that the moment sequence is Consistent. The proof is very similar to the ones in the previous section but we need to find anther polynomial vanishing on the algebraic variety. Also, we can reduce some cases to a subcase of another through the use of degree-one transformations, that is, interchanging X and Y. In the sequel, set the transformation as $X = \tilde{Y}$ and $Y = \tilde{X}$ and let $\tilde{\mathcal{M}}(3)$ denote the moment matrix obtained under the transformation.

Claim 1. Case 6 is a subcase of Case 1.

Proof. Since $X^3, X^2Y \in \text{Ran } \mathcal{M}(2)$, it follows that $\tilde{X}\tilde{Y}^2, \tilde{Y}^3 \in \text{Ran } \tilde{\mathcal{M}}(2)$. In addition, the transformation preserves the rank of matrices and so rank $\tilde{\mathcal{M}}(3) = 8$, which completes the proof. $\square$

Claim 2. Case 5 is a subcase of Case 2.

Proof. Write two column relations in $\mathcal{M}(3)$ as $XY^2 = p(X,Y)$ and $X^3 = \alpha X^2Y + q(X,Y)$ for some $p, q \in \mathcal{P}_2$ and for some $\alpha \in \mathbb{C}$. After applying the degree-one transformation, we get $\tilde{X}^2\tilde{Y} = p(\tilde{Y}, \tilde{X})$ and $\tilde{Y}^3 = \alpha\tilde{X}\tilde{Y}^2 + q(\tilde{Y}, \tilde{X})$. Clearly, the 8th and 10th columns are dependent, which proves that this case is a subcase of Case 2. $\square$

Consequently, we focus on the first 4 cases and present their solutions. The detailed proofs are omitted but instead we present here a sketch. Obviously, the "only if" parts of the proofs are trivial, so we focus on the converses. We have to show that the Riesz functional is zero for any polynomial vanishing on the algebraic variety, with the degree up to 6. Thus, it is essential to construct a representation of such polynomials, which is done by the Division Algorithm. The representing sets contain at most 4 polynomials, two of which come from column relations in $\mathcal{M}(3)$ and the other polynomials (they are quartic) are found by invertibility of the compression of the generalized Vandermonde matrix (equivalently, by the Weak Consistency of β). Multiplying polynomials in the representing set by monomials, we can check that the Riesz functional is zero for higher order polynomials.

Theorem 6.4.78. Suppose $\mathcal{M}(3)(\beta)$ satisfies (6.4.1). Let $\mathfrak{B}_1$ be a basis for $\mathcal{C}_{\mathcal{M}(3)}$. Then β has a representing measure if and only if $\mathcal{M}(3)$ is weakly consistent and for $0 \le i + j \le 2$,

$$\Lambda_\beta(x^i y^j (x^4 - a_0 - a_1 x - a_2 y - a_3 x^2 - a_4 xy - a_5 y^2 - a_6 x^3 - a_7 x^2 y)) = 0$$

and

$$\Lambda_\beta(x^i y^j(x^3y - b_0 - b_1x - b_2y - b_3x^2 - b_4xy - b_5y^2 - b_6x^3 - a_7x^2y)) = 0,$$

where $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)^T = W_{\mathfrak{B}_1}^{-1}(x_1^4, \cdots, x_8^4)^T$

and $(b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7)^T = W_{\mathfrak{B}_1}^{-1}(x_1^2y_1, \cdots, x_8^2y_8)^T$.

Theorem 6.4.79. Suppose $\mathcal{M}(3)(\beta)$ satisfies (6.4.1). Let $\mathfrak{B}_2$ be a basis for $\mathcal{C}_{\mathcal{M}(3)}$. Then β has a representing measure if and only if $\mathcal{M}(3)$ is weakly consistent and for $0 \le i + j \le 2$,

$$\Lambda_\beta(x^i y^j(x^4 - a_0 - a_1x - a_2y - a_3x^2 - a_4xy - a_5y^2 - a_6x^3 - a_7x^2y)) = 0,$$

where $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)^T = W_{\mathfrak{B}_2}^{-1}(x_1^4, \cdots, x_8^4)^T$.

Theorem 6.4.80. Suppose $\mathcal{M}(3)(\beta)$ satisfies (6.4.1). Let $\mathfrak{B}_3$ be a basis for $\mathcal{C}_{\mathcal{M}(3)}$. Then β has a representing measure if and only if $\mathcal{M}(3)$ is weakly consistent and for $0 \le i + j \le 2$,

$$\Lambda_\beta(x^i y^j(x^4 - a_0 - a_1x - a_2y - a_3x^2 - a_4xy - a_5y^2 - a_6x^3 - a_7x^2y)) = 0$$

and

$$\Lambda_\beta(x^i y^j(y^4 - b_0 - b_1x - b_2y - b_3x^2 - b_4xy - b_5y^2 - b_6x^3 - a_7x^2y)) = 0,$$

where $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)^T = W_{\mathfrak{B}_3}^{-1}(x_1^4, \cdots, x_8^4)^T$

and $(b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7)^T = W_{\mathfrak{B}_3}^{-1}(y_1^4, \cdots, y_8^4)^T$.

Theorem 6.4.81. Suppose $\mathcal{M}(3)(\beta)$ satisfies (6.4.1). Let $\mathfrak{B}_4$ be a basis for $\mathcal{C}_{\mathcal{M}(3)}$. Then β has a representing measure if and only if $\mathcal{M}(3)$ is weakly consistent and for $0 \le i+j \le 2$,

$$\Lambda_\beta(x^i y^j(x^2y^2 - a_0 - a_1x - a_2y - a_3x^2 - a_4xy - a_5y^2 - a_6x^3 - a_7x^2y)) = 0,$$

where $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)^T = W_{\mathfrak{B}_4}^{-1}(x_1^2y_1^2, \cdots, x_8^2y_8^2)^T$.

In [20], R. Curto, L. Fialkow, and M. Möller provided an example of $\mathcal{M}(3)$ satisfying $Y = X^3$ in $\mathcal{C}_{\mathcal{M}(3)}$ and $r = v = 8$, which is Weakly Consistent but not Consistent. Consequently, $\beta^{(6)}$ has no representing measure. They presented a class of examples but we use a specific case to show that the numerical conditions about moments in the main theorems are essential.

Example 6.4.82. [20, Theorem 5.2] Consider $\beta^{(6)}$ with following moments;

$\beta_{00} := 14,\quad \beta_{10} := \frac{7}{2},\quad \beta_{01} := -\frac{67}{8},$

$\beta_{20} := \frac{79}{4},\quad \beta_{11} := \frac{1055}{16},\quad \beta_{02} := \frac{18195}{64},$

$\beta_{30} := -\frac{67}{8},\quad \beta_{21} := -\frac{1935}{32},\quad \beta_{12} := -\frac{43115}{128},\quad \beta_{03} := -\frac{926695}{512},$

$\beta_{40} := \frac{1055}{16},\quad \beta_{31} := \frac{18195}{64},\quad \beta_{22} := \frac{336151}{256},$

$\beta_{13} := \frac{6407195}{1024},\quad \beta_{04} := \frac{124731423}{4096},$

$\beta_{50} := -\frac{1935}{32},\quad \beta_{41} := -\frac{43115}{128},\quad \beta_{32} := -\frac{926695}{512},$

$\beta_{23} := -\frac{19736547}{2048},\quad \beta_{14} := -\frac{419176415}{8192},\quad \beta_{05} := -\frac{8894873563}{32768},$

$\beta_{60} := \frac{18195}{64},\quad \beta_{51} := \frac{336151}{256},\quad \beta_{42} := \frac{6407195}{1024},\quad \beta_{33} := \frac{124731423}{4096},$

$\beta_{24} := \frac{2469281827}{16384},\quad \beta_{15} := \frac{49568350247}{65536},\quad \beta_{15} := \frac{1006568996907}{262144}.$

After building $\mathcal{M}(3)(\beta)$, we see that there are two column relations $f(X,Y) := X^3 - Y = 0$ and $g(X,Y) := Y^3 - 3X + \frac{3}{4}Y + 13X^2 - \frac{65}{4}XY + \frac{13}{4}Y^2$

$-12X^3 + 22X^2Y - \frac{35}{4}XY^2 = 0$ in $\mathcal{C}_{\mathcal{M}(3)}$. This moment matrix satisfies (6.4.1) with the basis $\mathfrak{B}_4$ as in Case 4. A calculation shows that the algebraic variety is $\mathcal{V}(\beta) := \left\{(0,0), (-2,-8), (2,8), (1,1), \left(-\frac{1}{2} - \frac{\sqrt{13}}{2}, -5 - 2\sqrt{13}\right), (-1,-1), \left(\frac{1}{2}, \frac{1}{8}\right)\right\}$. In order to apply Theorem 6.4.81, we need to find the new polynomial, denoted as $h(x,y)$, vanishing on $\mathcal{V}$ by using the compression of the generalized Vandermonde matrix:

$$h(x,y) = x^4 + 6x - \frac{11}{2}y - 14x^2 + \frac{43}{2}xy - \frac{17}{2}y^2 - x^2y + \frac{1}{2}xy^2.$$

Now evaluate the Riesz functional:

$$\begin{aligned}\Lambda_\beta(h) &= \beta_{40} + 6\beta_{10} - \frac{11}{2}\beta_{01} - 14\beta_{20} + \frac{43}{2}\beta_{11} - \frac{17}{2}\beta_{02} - \beta_{21} + \frac{1}{2}\beta_{12} \\ &= -\frac{320081}{256},\end{aligned}$$

which is different from zero. To summarize, even though h vanishes on $\mathcal{V}$, its Riesz functional is not zero. We just verified that β admits no representing measure via the new result.

CHAPTER 7
DIRECTIONS FOR FUTURE RESEARCH

As seen in Chapter 2, there are 78 species of cubics. In [4], we can find a nice classification of irreducible cubics according to singularity and the existence of a flex (i.e., a generalized inflection point). This categorization shows that any irreducible cubic can be transformed into one of the form

$$y^2 = x^3 + fx^2 + gx + h \tag{7.0.1}$$

for some $f, g, h \in \mathbb{R}$. This is accomplished through a series of affine transformations.

Theorem 7.0.83. [4] A nonsingular, irreducible cubic has a flex if and only if it can be transformed into

$$y^2 = x(x-1)(x-w) \tag{7.0.2}$$

or

$$y^2 = x(x^2 + kx + 1) \tag{7.0.3}$$

for $w > 1$ and $-2 < k < 2$.

The extremal truncated moment problem we have studied in detail in this dissertation, and for which we have found a complete solution in Theorem 5.8.65, is associated with an algebraic curve of the form (7.0.2). As we know, all other truncated moment problems related to this one via a degree-one transformation are equivalent, and we can therefore draw the same conclusion, that they admit a representing measure if and only if a pair of conditions, similar to those listed in Theorem

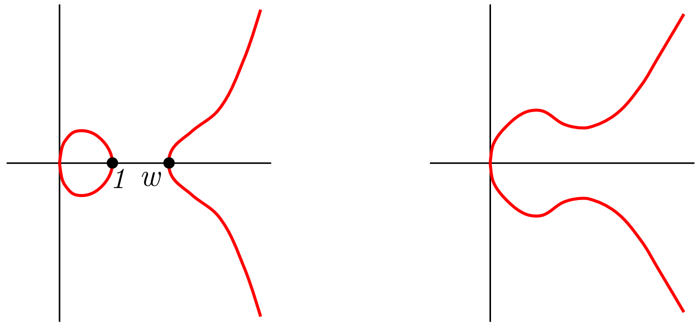


Figure 7.1: The two cubics described in Theorem 7.0.83.

5.8.65 (iii), hold. Of course every degree-one transformation is affine, but the converse is not true. That is, while we only have two equivalence classes of nonsingular, irreducible cubics with a flex, it is perfectly possible to have many more equivalence classes of such cubics under degree-one transformations.

From the perspective of projective geometry, we only have two inequivalent cubics, given by (7.0.2) and (7.0.3). This means that we have at least two distinct kinds of associated sextic truncated moment problems. On the other hand, from the perspective of truncated moment problems it is anticipated that each cubic will give rise to more than one equivalence class of TMP. To illustrate this point, we return to the discussion of quartic moment problems in Chapter 4. There we observed that, while all nondegenerate conics are affine equivalent to a circle, there are three inequivalent classes of TMP: the trigonometric TMP, the hyperbolic TMP and the parabolic TMP. It is then natural to expect that each of the cubics in Theorem 7.0.83 will be associated to more than one equivalence classes of TMP.

In this dissertation, we first exhibited Theorem 5.8.65 as providing the complete solution of sextic TMP with a column relation given by an harmonic cubic polynomial. In view of the stability of TMP under degree-one transformations, along with Theorem 5.8.65 we have a whole collection of soluble sextic TMP associated with (7.0.2). We are aware that there might be other sextic TMP, inequivalent to those covered by Theorem 5.8.65, but still associated to (7.0.2). Our first proposed problem attempts to unravel this mystery.

Problem 7.0.84. Let $\beta^{(6)}$ be a sextic moment sequence, and assume that $\mathcal{M}(3)$ is positive semidefinite, with $\mathcal{M}(2)$ invertible, and with algebraic variety of the form given in (7.0.2). Must $\beta^{(6)}$ admit a representing measure?

Similarly, we consider sextic TMP with associated algebraic variety given by (7.0.3). Clearly, this TMP is not equivalent to the one discussed in Theorem 5.8.65. It is straightforward to attempt to build a TMP with this algebraic variety; that is, one would use each point in (7.0.3) as atom, and assign a density of $\frac{1}{v}$ to each atom, where v is the cardinality of $\mathcal{V}(\beta)$.

Problem 7.0.85. Under the above mentioned conditions, is rank $\mathcal{M}(3) = v$?

In case the answer to Problem 7.0.85 is affirmative, we would be in the presence of an extremal TMP, and once again we would seek conditions on $\beta^{(6)}$ to guarantee Consistency.

Finally, within the class of TMP with associated cubic (7.0.3), we would like to determine how many inequivalent TMP can be found.

Problem 7.0.86. Let $\beta \equiv \beta^{(6)}$ be a sextic moment sequence with associated algebraic variety given by (7.0.3). How many inequivalent TMP can β give rise to?

Having understood a key sextic TMP in Theorem 5.8.65, and with the additional tools provided by the Division Algorithm, as demonstrated in Chapter 6, our hope is to be able to classify, up to degree-one transformations, all sextic TMP with finite algebraic variety. We have developed a reasonably good understanding of the issues involved when the TMP is extremal, and we know very little in other cases. Thus, in the future we plan to devote considerable effort to dealing with non-extremal cases, especially the case $r = 7$ and $v = 8$.

APPENDIX

In the following example, we study the sextic truncated moment problem generated initially by an 8-atomic measure, five of whose atoms are carefully chosen to lie on the horizontal line $y = 1$. However, we externally modify one of the moments, β_{60}, and leave it as a parameter. The resulting moment matrix $\mathcal{M}(3)$ is positive semidefinite, it depends on β_{60}, and it has rank 6 or 7 according to the value of the β_{60}. Needless to say, we are interested in the case rank $\mathcal{M}(3) = 7$. We use Gaussian elimination to find the necessary and sufficient condition for rank $\mathcal{M}(3) = 7$, while maintaining positive semidefiniteness. We then identify the three polynomials arising from the column relations in $\mathcal{M}(3)$. We prove that the intersection of the zero sets of these polynomials consists of the line $y = 1$ together with three additional points in the plane. Thus, the algebraic variety is infinite, while the rank is 7. We proceed to obtain a flat extension of $\mathcal{M}(3)$, called $\mathcal{M}(4)$, which therefore means that the TMP associated with $\mathcal{M}(3)$ is soluble, with a representing measure supported on the 7-point algebraic variety. To our surprise, the value of β_{60}, which plays a role in determining both the rank of $\mathcal{M}(3)$ and the positive semidefiniteness of $\mathcal{M}(3)$, plays no role in the calculation of the flat extension. Thus, the example automatically generates a family of examples, indexed by the β_{60}.

Clear[x,y,a,b,c1,c2,d1, d2,e1,e2,f1,f2,g1,g2, h1,h2,i1,i2,j2,j1,k,p1, p2,p3,s,g,NMR]

x[1]=c1; x[2]=d1; x[3]=e1; x[4]=f1; x[5]=g1; x[6]=h1; x[7]=i1; x[8]=j1;

y[1]=c2; y[2]=d2; y[3]=e2;

y[4]=1; y[5]=1; y[6]=1; y[7]= 1; y[8]=1;

h[i_]:=$\{1, x[i], y[i], x[i]^2, y[i]x[i], y[i]^2, x[i]^3, y[i]x[i]^2, y[i]^2x[i], y[i]^3, x[i]^4,$

$y[i]x[i]^3, y[i]^2x[i]^2, y[i]^3\ x[i], y[i]^4\}$

GV = Table[$h[i], \{i, 8\}$];

GV//MatrixForm

GV//RowReduce//MatrixForm

$$\begin{pmatrix}
1 & 3 & 3 & 9 & 9 & 9 & 27 & 27 & 27 & 27 & 81 & 81 & 81 & 81 & 81 \\
1 & 2 & 2 & 4 & 4 & 4 & 8 & 8 & 8 & 8 & 16 & 16 & 16 & 16 & 16 \\
1 & -1 & -7 & 1 & 7 & 49 & -1 & -7 & -49 & -343 & 1 & 7 & 49 & 343 & 2401 \\
1 & -3 & 1 & 9 & -3 & 1 & -27 & 9 & -3 & 1 & 81 & -27 & 9 & -3 & 1 \\
1 & -5 & 1 & 25 & -5 & 1 & -125 & 25 & -5 & 1 & 625 & -125 & 25 & -5 & 1 \\
1 & 6 & 1 & 36 & 6 & 1 & 216 & 36 & 6 & 1 & 1296 & 216 & 36 & 6 & 1 \\
1 & 7 & 1 & 49 & 7 & 1 & 343 & 49 & 7 & 1 & 2401 & 343 & 49 & 7 & 1 \\
1 & 10 & 1 & 100 & 10 & 1 & 1000 & 100 & 10 & 1 & 10000 & 1000 & 100 & 10 & 1
\end{pmatrix}$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 6 & 6 & 0 & 30 & 36 & 36 & 36 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -7 & -8 & -15 & 0 & -27 & -33 & -27 & 15 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -4 & -3 & 4 & 0 & -22 & -27 & -33 & -75 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 7 & 9 & 15 & 0 & 27 & 33 & 28 & -15 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & -3 & -9 & 0 & -8 & -9 & -3 & 40 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Clear[s]

s[i_, j_]:=Sum[($x[k]^i$ $y[k]^j$), {k, 1, 8}]

$g[0,0] = s[0,0]$//Together;

$g[0,1] = s[0,1]$//Together;

$g[1,0] = s[1,0]$//Together;

$g[0,2] = s[0,2]$//Together;

$g[1,1] = s[1,1]$//Together;

$g[2,0] = s[2,0]$//Together;

$g[0,3] = s[0,3]$//Together;

$g[1,2] = s[1,2]$//Together;

$g[2,1] = s[2,1]$//Together;

$g[3,0] = s[3,0]$//Together;

$g[0,4] = s[0,4]$//Together;

$g[1,3] = s[1,3]$//Together;

$g[2,2] = s[2,2]$//Together;

$g[3,1] = s[3,1]$//Together;

$g[4,0] = s[4,0]$//Together;

$g[0,5] = s[0,5]$//Together;

$g[1,4] = s[1,4]$//Together;

$g[2,3] = s[2,3]$//Together;

$g[3,2] = s[3,2]$//Together;

$g[4,1] = s[4,1]$//Together;

$g[5,0] = s[5,0]$//Together;

$g[0,6] = s[0,6]$//Together;

$g[1,5] = s[1,5]$//Together;

$g[2,4] = s[2,4]$//Together;

$g[3,3] = s[3,3]$//Together;

$g[4,2] = s[4,2]$//Together;

$g[5,1] = s[5,1]$//Together;

$g[6,0] = \mathrm{g}_{60}$//Together;

M3:=

$$\begin{pmatrix}
g[0,0] & g[1,0] & g[0,1] & g[2,0] & g[1,1] & g[0,2] & g[3,0] & g[2,1] & g[1,2] & g[0,3] \\
g[1,0] & g[2,0] & g[1,1] & g[3,0] & g[2,1] & g[1,2] & g[4,0] & g[3,1] & g[2,2] & g[1,3] \\
g[0,1] & g[1,1] & g[0,2] & g[2,1] & g[1,2] & g[0,3] & g[3,1] & g[2,2] & g[1,3] & g[0,4] \\
g[2,0] & g[3,0] & g[2,1] & g[4,0] & g[3,1] & g[2,2] & g[5,0] & g[4,1] & g[3,2] & g[2,3] \\
g[1,1] & g[2,1] & g[1,2] & g[3,1] & g[2,2] & g[1,3] & g[4,1] & g[3,2] & g[2,3] & g[1,4] \\
g[0,2] & g[1,2] & g[0,3] & g[2,2] & g[1,3] & g[0,4] & g[3,2] & g[2,3] & g[1,4] & g[0,5] \\
g[3,0] & g[4,0] & g[3,1] & g[5,0] & g[4,1] & g[3,2] & g[6,0] & g[5,1] & g[4,2] & g[3,3] \\
g[2,1] & g[3,1] & g[2,2] & g[4,1] & g[3,2] & g[2,3] & g[5,1] & g[4,2] & g[3,3] & g[2,4] \\
g[1,2] & g[2,2] & g[1,3] & g[3,2] & g[2,3] & g[1,4] & g[4,2] & g[3,3] & g[2,4] & g[1,5] \\
g[0,3] & g[1,3] & g[0,4] & g[2,3] & g[1,4] & g[0,5] & g[3,3] & g[2,4] & g[1,5] & g[0,6]
\end{pmatrix}$$

M3//MatrixForm

$$\begin{pmatrix} 8 & 19 & 3 & 233 & 35 & 67 & 1441 & 247 & 1 & -303 \\ 19 & 233 & 35 & 1441 & 247 & 1 & 14501 & 1511 & 365 & 455 \\ 3 & 35 & 67 & 247 & 1 & -303 & 1511 & 365 & 455 & 2503 \\ 233 & 1441 & 247 & 14501 & 1511 & 365 & 121489 & 14671 & 1633 & 151 \\ 35 & 247 & 1 & 1511 & 365 & 455 & 14671 & 1633 & 151 & -2111 \\ 67 & 1 & -303 & 365 & 455 & 2503 & 1633 & 151 & -2111 & -16527 \\ 1441 & 14501 & 1511 & 121489 & 14671 & 1633 & g_{60} & 122015 & 15245 & 2543 \\ 247 & 1511 & 365 & 14671 & 1633 & 151 & 122015 & 15245 & 2543 & 3413 \\ 1 & 365 & 455 & 1633 & 151 & -2111 & 15245 & 2543 & 3413 & 17615 \\ -303 & 455 & 2503 & 151 & -2111 & -16527 & 2543 & 3413 & 17615 & 118447 \end{pmatrix}$$

NMR:=M3//RowReduce

NMR//MatrixForm

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 6 & 6 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -7 & -8 & -15 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -4 & -3 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 7 & 9 & 15 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & -3 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\mathrm{Clear}[g_{60}]$

$g_{60} = g_{60};$

$\mathrm{GaussJordan}[\mathrm{A0\_}, \mathrm{n\_}] :=$

$\mathrm{Module}[\{A = \mathrm{A0}, i, p\},$

$\mathrm{Print}[\mathrm{MatrixForm}[A]];$

$\mathrm{For}[p = 1, p \leq n, p{+}{+},$

$A_{[[p]]} = \frac{A_{[[p]]}}{A_{[[p,p]]}};$

$\mathrm{For}[i = 1, i \leq n, i{+}{+},$

$\mathrm{If}[i \neq p,$

$A_{[[i]]} = A_{[[i]]} - A_{[[i,p]]} A_{[[p]]}\,];\,];$

$\mathrm{Print}[A//\mathrm{Together}//\mathrm{MatrixForm}];\ ];\ ]$

$\mathrm{GaussJordan}[\mathrm{M3}, 10]$

$$\begin{pmatrix}
8 & 19 & 3 & 233 & 35 & 67 & 1441 & 247 & 1 & -303 \\
19 & 233 & 35 & 1441 & 247 & 1 & 14501 & 1511 & 365 & 455 \\
3 & 35 & 67 & 247 & 1 & -303 & 1511 & 365 & 455 & 2503 \\
233 & 1441 & 247 & 14501 & 1511 & 365 & 121489 & 14671 & 1633 & 151 \\
35 & 247 & 1 & 1511 & 365 & 455 & 14671 & 1633 & 151 & -2111 \\
67 & 1 & -303 & 365 & 455 & 2503 & 1633 & 151 & -2111 & -16527 \\
1441 & 14501 & 1511 & 121489 & 14671 & 1633 & g_{60} & 122015 & 15245 & 2543 \\
247 & 1511 & 365 & 14671 & 1633 & 151 & 122015 & 15245 & 2543 & 3413 \\
1 & 365 & 455 & 1633 & 151 & -2111 & 15245 & 2543 & 3413 & 17615 \\
-303 & 455 & 2503 & 151 & -2111 & -16527 & 2543 & 3413 & 17615 & 118447
\end{pmatrix}$$

$$\begin{pmatrix}
1 & \frac{19}{8} & \frac{3}{8} & \frac{233}{8} & \frac{35}{8} & \frac{67}{8} & \frac{1441}{8} & \frac{247}{8} & \frac{1}{8} & \frac{-303}{8} \\
0 & \frac{1503}{8} & \frac{223}{8} & \frac{7101}{8} & \frac{1311}{8} & \frac{-1265}{8} & \frac{88629}{8} & \frac{7395}{8} & \frac{2901}{8} & \frac{9397}{8} \\
0 & \frac{223}{8} & \frac{527}{8} & \frac{1277}{8} & \frac{-97}{8} & \frac{-2625}{8} & \frac{7765}{8} & \frac{2179}{8} & \frac{3637}{8} & \frac{20933}{8} \\
0 & \frac{7101}{8} & \frac{1277}{8} & \frac{61719}{8} & \frac{3933}{8} & \frac{-12691}{8} & \frac{636159}{8} & \frac{59817}{8} & \frac{12831}{8} & \frac{71807}{8} \\
0 & \frac{1311}{8} & \frac{-97}{8} & \frac{3933}{8} & \frac{1695}{8} & \frac{1295}{8} & \frac{66933}{8} & \frac{4419}{8} & \frac{1173}{8} & \frac{-6283}{8} \\
0 & \frac{-1265}{8} & \frac{-2625}{8} & \frac{-12691}{8} & \frac{1295}{8} & \frac{15535}{8} & \frac{-83483}{8} & \frac{-15341}{8} & \frac{-16955}{8} & \frac{-111915}{8} \\
0 & \frac{88629}{8} & \frac{7765}{8} & \frac{636159}{8} & \frac{66933}{8} & \frac{-83483}{8} & \frac{-2076481+8g_{60}}{8} & \frac{620193}{8} & \frac{120519}{8} & \frac{456967}{8} \\
0 & \frac{7395}{8} & \frac{2179}{8} & \frac{59817}{8} & \frac{4419}{8} & \frac{-15341}{8} & \frac{620193}{8} & \frac{60951}{8} & \frac{20097}{8} & \frac{102145}{8} \\
0 & \frac{2901}{8} & \frac{3637}{8} & \frac{12831}{8} & \frac{1173}{8} & \frac{-16955}{8} & \frac{120519}{8} & \frac{20097}{8} & \frac{27303}{8} & \frac{141223}{8} \\
0 & \frac{9397}{8} & \frac{20933}{8} & \frac{71807}{8} & \frac{-6283}{8} & \frac{-111915}{8} & \frac{456967}{8} & \frac{102145}{8} & \frac{141223}{8} & \frac{855767}{8}
\end{pmatrix}$$

$$\begin{pmatrix}
1 & 0 & \frac{34}{1503} & \frac{2990}{167} & \frac{1154}{501} & \frac{15592}{1503} & \frac{20078}{501} & \frac{9614}{501} & \frac{-2234}{501} & \frac{-79244}{1503} \\
0 & 1 & \frac{223}{1503} & \frac{789}{167} & \frac{437}{501} & \frac{-1265}{1503} & \frac{29543}{501} & \frac{2465}{501} & \frac{967}{501} & \frac{9397}{1503} \\
0 & 0 & \frac{92794}{1503} & \frac{4664}{167} & \frac{-18256}{501} & \frac{-457910}{1503} & \frac{-337228}{501} & \frac{67748}{501} & \frac{200812}{501} & \frac{3670846}{1503} \\
0 & 0 & \frac{4664}{167} & \frac{588048}{167} & \frac{-47196}{167} & \frac{-140164}{167} & \frac{4538784}{167} & \frac{519348}{167} & \frac{-18264}{167} & \frac{572192}{167} \\
0 & 0 & \frac{-18256}{501} & \frac{-47196}{167} & \frac{11512}{167} & \frac{150200}{501} & \frac{-216560}{167} & \frac{-42404}{167} & \frac{-28336}{167} & \frac{-906784}{501} \\
0 & 0 & \frac{-457910}{1503} & \frac{-140164}{167} & \frac{150200}{501} & \frac{2718610}{1503} & \frac{-556636}{501} & \frac{-570952}{501} & \frac{-908900}{501} & \frac{-19540130}{1503} \\
0 & 0 & \frac{-337228}{501} & \frac{4538784}{167} & \frac{-216560}{167} & \frac{-556636}{501} & \frac{-152445147+167g_{60}}{167} & \frac{3843592}{167} & \frac{-1055176}{167} & \frac{-6084388}{501} \\
0 & 0 & \frac{67748}{501} & \frac{519348}{167} & \frac{-42404}{167} & \frac{-570952}{501} & \frac{3843592}{167} & \frac{512824}{167} & \frac{121568}{167} & \frac{3501380}{501} \\
0 & 0 & \frac{200812}{501} & \frac{-18264}{167} & \frac{-28336}{167} & \frac{-908900}{501} & \frac{-1055176}{167} & \frac{121568}{167} & \frac{453064}{167} & \frac{7708228}{501} \\
0 & 0 & \frac{3670846}{1503} & \frac{572192}{167} & \frac{-906784}{501} & \frac{-19540130}{1503} & \frac{-6084388}{501} & \frac{3501380}{501} & \frac{7708228}{501} & \frac{149739274}{1503}
\end{pmatrix}$$

$$\begin{pmatrix}
1 & 0 & 0 & \frac{830226}{46397} & \frac{107490}{46397} & \frac{486498}{46397} & \frac{1870842}{46397} & \frac{888042}{46397} & \frac{-213702}{46397} & \frac{-2487750}{46397} \\
0 & 1 & 0 & \frac{216091}{46397} & \frac{44533}{46397} & \frac{-5080}{46397} & \frac{2810993}{46397} & \frac{213203}{46397} & \frac{44861}{46397} & \frac{17760}{46397} \\
0 & 0 & 1 & \frac{20988}{46397} & \frac{-27384}{46397} & \frac{-228955}{46397} & \frac{-505842}{46397} & \frac{101622}{46397} & \frac{301218}{46397} & \frac{1835423}{46397} \\
0 & 0 & 0 & \frac{162789072}{46397} & \frac{-12347508}{46397} & \frac{-32546964}{46397} & \frac{1275121008}{46397} & \frac{141450444}{46397} & \frac{-13486680}{46397} & \frac{107710056}{46397} \\
0 & 0 & 0 & \frac{-12347508}{46397} & \frac{2200488}{46397} & \frac{5566920}{46397} & \frac{-78598512}{46397} & \frac{-8077932}{46397} & \frac{3103632}{46397} & \frac{-17094960}{46397} \\
0 & 0 & 0 & \frac{-32546964}{46397} & \frac{5566920}{46397} & \frac{14168040}{46397} & \frac{-205661232}{46397} & \frac{-21914604}{46397} & \frac{7598160}{46397} & \frac{-44008560}{46397} \\
0 & 0 & 0 & \frac{1275121008}{46397} & \frac{-78598512}{46397} & \frac{-205661232}{46397} & \frac{-42693765553}{46397}+g_{60} & \frac{1136253888}{46397} & \frac{-90402912}{46397} & \frac{671973408}{46397} \\
0 & 0 & 0 & \frac{141450444}{46397} & \frac{-8077932}{46397} & \frac{-21914604}{46397} & \frac{1136253888}{46397} & \frac{128734128}{46397} & \frac{-6957576}{46397} & \frac{76062456}{46397} \\
0 & 0 & 0 & \frac{-13486680}{46397} & \frac{3103632}{46397} & \frac{7598160}{46397} & \frac{-90402912}{46397} & \frac{-6957576}{46397} & \frac{5138208}{46397} & \frac{-21828960}{46397} \\
0 & 0 & 0 & \frac{107710056}{46397} & \frac{-17094960}{46397} & \frac{-44008560}{46397} & \frac{671973408}{46397} & \frac{76062456}{46397} & \frac{-21828960}{46397} & \frac{139652640}{46397}
\end{pmatrix}$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & \frac{24920271}{6782878} & \frac{95388615}{6782878} & \frac{-338602164}{3391439} & \frac{24361935}{6782878} & \frac{-10593069}{3391439} & \frac{-221998101}{3391439} \\
0 & 1 & 0 & 0 & \frac{17813061}{13565756} & \frac{11146801}{13565756} & \frac{81747728}{3391439} & \frac{7437533}{13565756} & \frac{9175549}{6782878} & \frac{-18305817}{6782878} \\
0 & 0 & 1 & 0 & \frac{-1885299}{3391439} & \frac{-16428988}{3391439} & \frac{-48991938}{3391439} & \frac{6095127}{3391439} & \frac{22144956}{3391439} & \frac{133147163}{3391439} \\
0 & 0 & 0 & 1 & \frac{-1028959}{13565756} & \frac{-2712247}{13565756} & \frac{26565021}{3391439} & \frac{11787537}{13565756} & \frac{-561945}{6782878} & \frac{4487919}{6782878} \\
0 & 0 & 0 & 0 & \frac{92388537}{3391439} & \frac{226469313}{3391439} & \frac{1324433700}{3391439} & \frac{193781133}{3391439} & \frac{152088894}{3391439} & \frac{-652395762}{3391439} \\
0 & 0 & 0 & 0 & \frac{226469313}{3391439} & \frac{559975689}{3391439} & \frac{3602027268}{3391439} & \frac{465333813}{3391439} & \frac{358296750}{3391439} & \frac{-1642741506}{3391439} \\
0 & 0 & 0 & 0 & \frac{1324433700}{3391439} & \frac{3602027268}{3391439} & \frac{-3850829103355}{3391439}+\mathrm{g}_{60} & \frac{2066981364}{3391439} & \frac{1113821496}{3391439} & \frac{-12551739912}{3391439} \\
0 & 0 & 0 & 0 & \frac{193781133}{3391439} & \frac{465333813}{3391439} & \frac{2066981364}{3391439} & \frac{425800305}{3391439} & \frac{348028758}{3391439} & \frac{-1281287322}{3391439} \\
0 & 0 & 0 & 0 & \frac{152088894}{3391439} & \frac{358296750}{3391439} & \frac{1113821496}{3391439} & \frac{348028758}{3391439} & \frac{293909796}{3391439} & \frac{-943337340}{3391439} \\
0 & 0 & 0 & 0 & \frac{-652395762}{3391439} & \frac{-1642741506}{3391439} & \frac{-12551739912}{3391439} & \frac{-1281287322}{3391439} & \frac{-943337340}{3391439} & \frac{4998737124}{3391439}
\end{pmatrix}$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \frac{51914016}{10265393} & \frac{-1565562318}{10265393} & \frac{-42235674}{10265393} & \frac{-94149690}{10265393} & \frac{-405633786}{10265393} \\
0 & 1 & 0 & 0 & 0 & \frac{-24606689}{10265393} & \frac{54205061}{10265393} & \frac{-22644373}{10265393} & \frac{-8303077}{10265393} & \frac{67479306}{10265393} \\
0 & 0 & 1 & 0 & 0 & \frac{-35739919}{10265393} & \frac{-66485706}{10265393} & \frac{30418266}{10265393} & \frac{76423578}{10265393} & \frac{362720843}{10265393} \\
0 & 0 & 0 & 1 & 0 & \frac{-143768}{10265393} & \frac{91570452}{10265393} & \frac{10552929}{10265393} & \frac{431304}{10265393} & \frac{1293912}{10265393} \\
0 & 0 & 0 & 0 & 1 & \frac{25163257}{10265393} & \frac{147159300}{10265393} & \frac{21531237}{10265393} & \frac{16898766}{10265393} & \frac{-72488418}{10265393} \\
0 & 0 & 0 & 0 & 0 & \frac{14644224}{10265393} & \frac{1075991616}{10265393} & \frac{-29288448}{10265393} & \frac{-43932672}{10265393} & \frac{-131798016}{10265393} \\
0 & 0 & 0 & 0 & 0 & \frac{1075991616}{10265393} & \frac{-11713369120885+10265393\mathrm{g}_{60}}{10265393} & \frac{-2151983232}{10265393} & \frac{-3227974848}{10265393} & \frac{-9683924544}{10265393} \\
0 & 0 & 0 & 0 & 0 & \frac{-29288448}{10265393} & \frac{-2151983232}{10265393} & \frac{58576896}{10265393} & \frac{87865344}{10265393} & \frac{263596032}{10265393} \\
0 & 0 & 0 & 0 & 0 & \frac{-43932672}{10265393} & \frac{-3227974848}{10265393} & \frac{87865344}{10265393} & \frac{131798016}{10265393} & \frac{395394048}{10265393} \\
0 & 0 & 0 & 0 & 0 & \frac{-131798016}{10265393} & \frac{-9683924544}{10265393} & \frac{263596032}{10265393} & \frac{395394048}{10265393} & \frac{1186182144}{10265393}
\end{pmatrix}$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \frac{-118968}{227} & 6 & 6 & 6 \\
0 & 1 & 0 & 0 & 0 & 0 & \frac{658863}{3632} & -7 & -8 & -15 \\
0 & 0 & 1 & 0 & 0 & 0 & \frac{905585}{3632} & -4 & -3 & 4 \\
0 & 0 & 0 & 1 & 0 & 0 & \frac{4517}{454} & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & \frac{-602087}{3632} & 7 & 9 & 15 \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{266863}{3632} & -2 & -3 & -9 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{227}(-260767531+227\mathrm{g}_{60}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 6 & 6 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -7 & -8 & -15 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -4 & -3 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 7 & 9 & 15 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & -3 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ⟩⟩

∞::indet : Indeterminate expression 0ComplexInfinity encountered. ⟩⟩

∞::indet : Indeterminate expression 0ComplexInfinity encountered. ⟩⟩

∞::indet : Indeterminate expression 0ComplexInfinity encountered. ⟩⟩

General::stop : Further output of ∞::indet will be suppressed during this calculation. ⟩⟩

M3[[{1,2}, {1,2}]]//Det

M3[[{1,2,3}, {1,2,3}]]//Det

M3[[{1,2,3,4}, {1,2,3,4}]]//Det

M3[[{1,2,3,4,5}, {1,2,3,4,5}]]//Det

M3[[{1,2,3,4,5,6}, {1,2,3,4,5,6}]]//Det

M3[[{1,2,3,4,5,6,7}, {1,2,3,4,5,6,7}]]//Det

1503

92794

325578144

8869299552

12652609536

$-14534756605329408 + 12652609536\mathrm{g}_{60}$

Reduce[$-14534756605329408 + 12652609536\mathrm{g}_{60} > 0, \mathrm{g}_{60}$]

$\mathrm{g}_{60} > \frac{260767531}{227}$

p1[x_, y_]:=

x^2y − (NMR[[1, 8]] + NMR[[2, 8]]x + NMR[[3, 8]]y + NMR[[4, 8]]x^2 + NMR[[5, 8]]yx

+NMR[[6, 8]]y^2 + NMR[[7, 8]]x^3)

p2[x_, y_]:=

xy^2 − (NMR[[1, 9]] + NMR[[2, 9]]x + NMR[[3, 9]]y + NMR[[4, 9]]x^2 + NMR[[5, 9]]yx

+NMR[[6, 9]]y^2 + NMR[[7, 9]]x^3)

p3[x_, y_]:=

y^3 − (NMR[[1, 10]] + NMR[[2, 10]]x + NMR[[3, 10]]y + NMR[[4, 10]]x^2

+NMR[[5, 10]]yx + NMR[[6, 10]]y^2 + NMR[[7, 10]]x^3)

p1[x, y]

p2[x, y]

p3[x, y]

p1[x, y]//Factor

$-6+7x-x^2+4y-7xy+x^2y+2y^2$

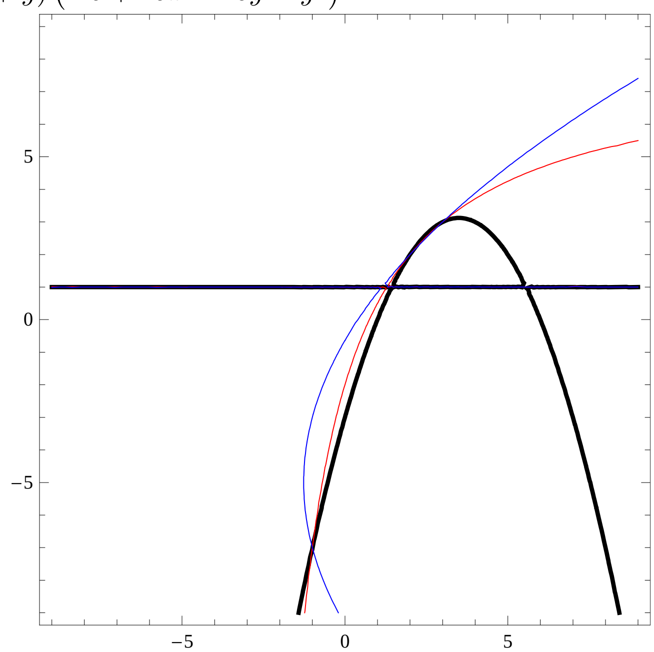
$-6+8x+3y-9xy+3y^2+xy^2$

$-6+15x-4y-15xy+9y^2+y^3$

$(-1+y)\left(6-7x+x^2+2y\right)$

$(-1+y)(6-8x+3y+xy)$

$-(-1+y)\left(-6+15x-10y-y^2\right)$



Solve::svars : Equations may not give solutions for all "solve" variables. ⟩⟩

$\{\{x\to -1, y\to -7\}, \{x\to 2, y\to 2\}, \{x\to 3, y\to 3\}, \{y\to 1\}\}$

Clear[M4]

M4:={{8, 19, 3, 233, 35, 67, 1441, 247, 1, −303, 14501, 1511, 365, 455, 2503},

{19, 233, 35, 1441, 247, 1, 14501, 1511, 365, 455, 121489, 14671, 1633, 151, −2111},

{3, 35, 67, 247, 1, −303, 1511, 365, 455, 2503, 14671, 1633, 151, −2111, −16527},

{233, 1441, 247, 14501, 1511, 365, 121489, 14671, 1633, 151,

$g_{60}, 122015, 15245, 2543, 3413\},$

$\{35, 247, 1, 1511, 365, 455, 14671, 1633, 151,$

$-2111, 122015, 15245, 2543, 3413, 17615\},$

$\{67, 1, -303, 365, 455, 2503, 1633, 151,$

$-2111, -16527, 15245, 2543, 3413, 17615, 118447\},$

$\{1441, 14501, 1511, 121489, 14671, 1633, g_{60},$

$122015, 15245, 2543, g[7, 0], g[6, 1], g[5, 2], g[4, 3], g[3, 4]\},$

$\{247, 1511, 365, 14671, 1633, 151, 122015, 15245, 2543, 3413,$

$g[6, 1], g[5, 2], g[4, 3], g[3, 4], g[2, 5]\},$

$\{1, 365, 455, 1633, 151, -2111, 15245, 2543, 3413, 17615,$

$g[5, 2], g[4, 3], g[3, 4], g[2, 5], g[1, 6]\},$

$\{-303, 455, 2503, 151, -2111, -16527, 2543, 3413, 17615, 118447,$

$g[4, 3], g[3, 4], g[2, 5], g[1, 6], g[0, 7]\},$

$\{14501, 121489, 14671, g_{60}, 122015, 15245,$

$g[7, 0], g[6, 1], g[5, 2], g[4, 3], g[8, 0], g[7, 1], g[6, 2], g[5, 3], g[4, 4]\},$

$\{1511, 14671, 1633, 122015, 15245, 2543,$

$g[6, 1], g[5, 2], g[4, 3], g[3, 4], g[7, 1], g[6, 2], g[5, 3], g[4, 4], g[3, 5]\},$

$\{365, 1633, 151, 15245, 2543, 3413,$

$g[5, 2], g[4, 3], g[3, 4], g[2, 5], g[6, 2], g[5, 3], g[4, 4], g[3, 5], g[2, 6]\},$

$\{455, 151, -2111, 2543, 3413, 17615,$

$g[4, 3], g[3, 4], g[2, 5], g[1, 6], g[5, 3], g[4, 4], g[3, 5], g[2, 6], g[1, 7]\},$

$\{2503, -2111, -16527, 3413, 17615, 118447,$

$g[3, 4], g[2, 5], g[1, 6], g[0, 7], g[4, 4], g[3, 5], g[2, 6], g[1, 7], g[0, 8]\}\}$

```
B3:=M4[[Range[1, 10], {11, 12, 13, 14, 15}]]

M3B3:=Join[Transpose[M3[[Range[1,10],Range[1,7]]]],Transpose[B3]]//Transpose

M3B3//MatrixForm;

M3 − M4[[Range[1, 10], Range[1, 10]]];

M4 − Transpose[M4];


g[7,0]=g70;

g[6,1]=g61;

g[5,2]=g52;

g[4,3]=g43;

g[3,4]=g34;

g[2,5]=g25;

g[1,6]=g16;

g[0,7]=g07;


Clear[g60]

g60 = g60;

M3B3//MatrixForm

GaussJordan[M3B3, 10]
```

$\{\{1, 0, 0, 0, 0, 0, 0, 6, \frac{71361680121719-125581106g_{60}+59g_{60}{}^{2}-475872g70}{1043070124-908g_{60}},$

$\frac{8003143482+112158g_{60}-118968g61}{260767531-227g_{60}}, 36+\frac{118968(-123481+g52)}{-260767531+227g_{60}}, 36+\frac{118968(-16375+g43)}{-260767531+227g_{60}},$

$36+\frac{118968(-1321+g34)}{-260767531+227g_{60}}\},$

$\{0,1,0,0,0,0,0,-7,\frac{405757234285168277-71315488870g_{60}+33955g_{60}{}^{2}-332066952g70}{8064(-260767531+227g_{60})},$

$\frac{9(12627676886+62311g_{60}-73207g61)}{-4172280496+3632g_{60}},-33+\frac{658863(-123481+g52)}{4172280496-3632g_{60}},-27+\frac{658863(-16375+g43)}{4172280496-3632g_{60}},$

$15+\frac{658863(-1321+g34)}{4172280496-3632g_{60}}\},$

$\{0,0,1,0,0,0,0,-4,\frac{79386074135010197-139824960314g_{60}+65165g_{60}{}^{2}-456414840g70}{8064(-260767531+227g_{60})},$

$\frac{93161226602+825681g_{60}-905585g61}{-4172280496+3632g_{60}},-27+\frac{905585(-123481+g52)}{4172280496-3632g_{60}},-33+\frac{905585(-16375+g43)}{4172280496-3632g_{60}},$

$-75+\frac{905585(-1321+g34)}{4172280496-3632g_{60}}\},$

$\{0,0,0,1,0,0,0,1,\frac{158920318941865-284261554g_{60}+145g_{60}{}^{2}-2276568g70}{1008(-260767531+227g_{60})},$

$\frac{4517(1514+g_{60}-g61)}{-521535062+454g_{60}},1+\frac{4517(-123481+g52)}{521535062-454g_{60}},\frac{4517(-16375+g43)}{521535062-454g_{60}},\frac{4517(-1321+g34)}{521535062-454g_{60}}\},$

$\{0,0,0,0,1,0,0,7,\frac{2243172129258775-3952571614g_{60}+1855g_{60}{}^{2}-14450088g70}{100134731904-87168g_{60}},$

$\frac{113563133110+504023g_{60}-602087g61}{4172280496-3632g_{60}},$

$33+\frac{602087(-123481+g52)}{-4172280496+3632g_{60}},28+\frac{602087(-16375+g43)}{-4172280496+3632g_{60}},-15+\frac{602087(-1321+g34)}{-4172280496+3632g_{60}}\},$

$\{0,0,0,0,0,1,0,-2,\frac{21808532152355755-38408817094g_{60}+17971g_{60}{}^{2}-134498952g70}{8064(-260767531+227g_{60})},$

$\frac{33782274550+237807g_{60}-266863g61}{-4172280496+3632g_{60}},-9$

$+\frac{266863(-123481+g52)}{4172280496-3632g_{60}},-3+\frac{266863(-16375+g43)}{4172280496-3632g_{60}},40+\frac{266863(-1321+g34)}{4172280496-3632g_{60}}\},$

$\{0,0,0,0,0,0,1,0,\frac{458419227-4517g_{60}+454g70}{-521535062+454g_{60}},$

$\frac{261111209-227g61}{260767531-227g_{60}},\frac{-123481+g52}{-\frac{260767531}{227}+g_{60}},\frac{-16375+g43}{-\frac{260767531}{227}+g_{60}},\frac{-1321+g34}{-\frac{260767531}{227}+g_{60}}\},$

$\{0,0,0,0,0,0,0,0,-1514-g_{60}+g61,-123481+g52,-16375+g43,-1321+g34,14273+g25\},$

$\{0,0,0,0,0,0,0,0,-123481+g52,-16375+g43,-1321+g34,14273+g25,115319+g16\},$

$\{0,0,0,0,0,0,0,0,-16375+g43,-1321+g34,14273+g25,115319+g16,821223+g07\}\}$

TT:=%

Reduce[TT[[Range[8, 10], Range[11, 15]]]

== {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}];

$\mathbf{g52 = 123481; g43 = 16375; g34 = 1321; g25 = -14273; g16 = -115319;}$

$\mathbf{g07 = -821223; g61 = g_{60} + 1514;}$

$WWW = M3B3//RowReduce;$

$\mathbf{WW = WWW[[Range[1, 10], Range[11, 15]]];}$

$\mathbf{WWs = WW//Transpose//Conjugate//ComplexExpand;}$

$\mathbf{M3.WW - M4[[Range[1, 10], Range[11, 15]]]//Together;}$

$\mathbf{CCb = WWs.M3.WW//Together;}$

$\mathbf{CCb}$

$\{\{\frac{-1759067328678042971 33+471171789008859\mathrm{g}_{60}-424603887\mathrm{g}_{60}{}^2}{1008(-260767531+227\mathrm{g}_{60})}$

$+\frac{+145\mathrm{g}_{60}{}^3+462086580816\mathrm{g}70-4553136\mathrm{g}_{60}\mathrm{g}70+228816\mathrm{g}70^2}{1008(-260767531+227\mathrm{g}_{60})},$

$4510 + \mathrm{g}70, 6072 + \mathrm{g}_{60}, 128375, 23621\},$

$\{4510 + \mathrm{g}70, 6072 + \mathrm{g}_{60}, 128375, 23621, 25031\},$

$\{6072 + \mathrm{g}_{60}, 128375, 23621, 25031, 124685\},$

$\{128375, 23621, 25031, 124685, 830375\},$

$\{23621, 25031, 124685, 830375, 5771623\}\}$

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