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# N-parameter Fibonacci AF C*-Algebras

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# $N$-PARAMETER FIBONACCI AF $C^*$-ALGEBRAS

by

Cecil Buford Flournoy Jr.

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy degree
in Mathematics in
the Graduate College of
The University of Iowa

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Thesis Supervisor: Associate Professor Richard Baker

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An $n$-parameter Fibonacci AF-algebra is determined by a constant incidence matrix $K$ of a special form. The form of the matrix K is defined by a given $n$-parameter Fibonacci sequence. We compute the K-theory of certain Fibonacci AF-algebra, and relate their K-theory to the K-theory of an AF-algebra defined by incidence matrices that are the transpose of K.

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Thesis Supervisor

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Date

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CERTIFICATE OF APPROVAL

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PH.D. THESIS

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This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the July 2011 graduation.

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## ABSTRACT

An $n$-parameter Fibonacci AF-algebra is determined by a constant incidence matrix $K$ of a special form. The form of the matrix K is defined by a given $n$-parameter Fibonacci sequence. We compute the K-theory of certain Fibonacci AF-algebra, and relate their K-theory to the K-theory of an AF-algebra defined by incidence matrices that are the transpose of K.

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# CHAPTER 1
# INTRODUCTION

## 1.1 Overview

It is the goal of this thesis to link the theory of AF $C^*$-algebras and their K-theory with the generalization of the Fibonacci sequence. We will focus on a well known result and present it in a new light that should lead to some interesting results.

Much of the work done in this thesis was inspired by papers written by Palle Jorgensen and Ola Bratteli [1]. Another inspiration was information on the classical Fibonacci algebra given in Davidson's book [4]. In [4], Davidson gives a proof of the K-theory of the Fibonacci Algebra. It turns out that one can generalize this work to arrive at many other interesting results. In [1], Bratteli and Jorgensen examine a group of matrices that turn out to be the transpose of the matrix generated by the Fibonacci sequence that makes the generalized Fibonacci algebra, which we will later define. The work in this thesis will largely be defining the generalized n-parameter Fibonacci sequence, proving its K-theory, and checking the conditions on which the algebras generated by the Fibonacci sequence are order isomorphic as AF $C^*$-algebras to the AF $C^*$-algebras generated by their transpose, that were studied in [1].

**Definition 1.1.** *An* ***Inductive Sequence*** *is a sequence* $\{A_n\}_{n=1}^{\infty}$ *of* $C^*$*-algebras and a sequence of* $\varphi_n : A_n \to A_{n+1}$ *of* $*$*-homomorphisms written*

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

*For* $m > n$ *consider the composed* $*$*-homomorphism* $\varphi_{m,n} = \varphi_{m-1} \circ \ldots \circ \varphi_n : A_n \to A_m$ *which together with* $\varphi_n$ *are called the connecting maps. We will also define* $\varphi_{n,n}$ *to be the identity map on* $A_n$ *and* $\varphi_{m,n} = 0$ *if* $m < n$

**Definition 1.2.** *An **inductive limit** of the inductive sequence*

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

*is a system $(A, \{\mu_n\})$ where $A$ is a $C^*$-algebra, and where $\mu_n : A_n \to A$ is a $*$-homomorphism for each $n \in \mathbb{N}$ satisfiying*

*i) $\mu_n = \mu_{n+1} \circ \varphi_n$ for all $n$*

*ii) Satisfies the universal property. If $(B, \{\lambda_n\})$ is a system that satisfies i), for all $n$, there exist a unique $*$-homomorphism $\lambda : A \to B$ such that $\lambda_n = \lambda \circ \mu_n$.*

Note that an Inductive limit is denoted $A = \lim\limits_{\to}(A_n, \varphi_n)$

**Definition 1.3.** *An **AF-Algebra**, Approximately finite dimensional algebra, is a $C^*$-algebra which is the inductive limit of a sequence of finite dimensional $C^*$-algebras.*

**Remark**: We will use $\mathfrak{A}$ to represent an AF algebra.

From [8], we see that if $\mathfrak{A} = \lim\limits_{\to}(A_n, \varphi_n)$ is an AF algebra, then $\mathfrak{A} = \bigcup\limits_{n=1}^{\infty} \varphi_n(A_n)$.

AF algebras are a special type of $C^*$-algebra because they are completely classified. This classification is described in terms of a group called $K_0$. Elliott proved in [5] that AF algebras are classified by their ordered $K_0$ groups.

For more info on AF algebras see the appendix.

## 1.2 K-theory

K-theory is a pair of functors called $K_0$ and $K_1$ that to each $C^*$-algebra $A$, we associate two abelian groups $K_0(A)$ and $K_1(A)$. With K-theory, one can learn about the structure of a given $C^*$-algebra. The $K_0$ group of a $C^*$-algebra is built up from the projections in the infinite dimensional matrices over the $C^*$-algebra. $K_1$ is built up from unitaries in the infinite matrices. For AF algebras, the $K_1$ group is always zero, which can be seen in [8]. For AF algebras, the $K_0$ group has an ordering which makes $K_0(A)$ an ordered abelian group. All info about an AF algebras structure is contained in its ordered $K_0$ group.

### 1.3 Fibonacci algebra

Now we will define the Fibonacci algebra. We will denote the $n \times n$ full matrices over $\mathbb{C}$ as $M_n$. These $M_n$ are $C^*$-algebras.

Let $f_0, f_1, f_2, \ldots$ be the Fibonacci sequence given by $f_0 = 0$, $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$. Let $\mathfrak{A}_n = M_{f_n} \bigoplus M_{f_{n-1}}$ and let $\varphi : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ be a unital *-homomorphism sending $X \bigoplus Y \to \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \bigoplus X$. Observe that these imbeddings are the standard imbeddings given by the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. The resulting sequence

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

defines an AF algebra known as the Fibonacci algebra. Therefore $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \varphi_n)$ is the Fibonacci algebra.

From this point there are two ways of generalizing the Fibonacci sequence. One way is to make the recursive formula dependent on more than just the previous two numbers. We say that the standard Fibonacci sequence is of order two since it is dependent on the previous two terms. We will define the Fibonacci sequence of order $n$, or the generalized Fibonacci sequence.

Let $\{f_n\}_{n=0}^{\infty}$ be the general Fibonacci sequence given by $f_0 = f_1 = \cdots = f_{n-2} = 0$ and $f_{n-1} = 1$ and $f_{n+1} = \sum_{i=0}^{k-1} f_{n-i}$. Let $\mathfrak{A}_{n+1} = M_{f_n} \bigoplus M_{f_{n-1}} \cdots \bigoplus M_{f_{n-k}}$ and let $\varphi : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ be a unital *-homomorphism sending $X_1 \bigoplus X_2 \bigoplus \cdots \bigoplus X_n \to \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & X_n \end{pmatrix} \bigoplus X_1 \bigoplus \cdots \bigoplus X_{n-1}$. Observe that these imbeddings are the standard imbeddings given by the matrix

$A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$. The resulting sequence

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

defines an AF algebra known as the general Fibonacci algebra. Therefore $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \varphi_n)$ is the general Fibonacci algebra.

The second way to generalize the Fibonacci sequence is to put parameters on the previous terms. The next term will be the previous two terms multiplied by constants and added together. So for the standard Fibonacci sequence we can have a 2 parameter Fibonacci sequence.

Let $f_0, f_1, f_2, \ldots$ be the 2-parameter Fibonacci sequence given by $f_0 = 0$ and $f_1 = 1$ and $f_{n+1} = m_1 f_n + m_2 f_{n-1}$ for $n \geq 2$ and $m_1,\ m_2 \in \mathbb{Z}_+$. Let $\mathfrak{A}_n = M_{f_n} \bigoplus M_{f_{n-1}}$ and let $\varphi : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ be a unital *-homomorphism sending $X \bigoplus Y \to \begin{pmatrix} X & 0 & \cdots & & \cdots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & X & 0 & & \\ & & 0 & Y & \ddots & \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & & \cdots & 0 & Y \end{pmatrix} \bigoplus X$ where $X$ is repeated down the diagonal $m_1$ times and $Y$ is repeated $m_2$ times down the diagonal. Observe that these imbeddings are the standard imbeddings given by the matrix $A = \begin{pmatrix} m_1 & m_2 \\ 1 & 0 \end{pmatrix}$. The resulting sequence

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

defines an AF algebra known as the 2-parameter Fibonacci algebra. Therefore $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \varphi_n)$ is the Fibonacci algebra.

For the generalized Fibonacci sequence of order $n$, we can have an $n$-parameter Fibonacci sequence of order $n$. Combining the two generalizations creates the $n$-parameter Fibonacci algebra which will be looked at later.

# CHAPTER 2
# THE FIBONACCI ALGEBRA

## 2.1 Machinery for new proof of Fibonacci Algebra

In the literature, we see that the Fibonacci Algebra has been used in the tiling of the plane [3]. In [4], Davidson computes $K_0$ of the classical Fibonacci AF $C^*$-algebra. We decided to come up with new machinery to recompute the K-theory of the Fibonacci Algebra so that we could generalize this result and make it easier to understand. So first we will go through the machinery to compute the K-theory. Our first proof relates the Fibonacci sequence to the $n^{th}$ power of the incidence matrix.

**Lemma 2.1.** *Let* $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ *and let* $f_{n+1} = f_n + f_{n-1}$ *be the Fibonacci sequence where* $f_0 = 0$ *and* $f_1 = 1$. *Then*

$$A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}.$$

*Proof.* $A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} f_3 & f_2 \\ f_2 & f_1 \end{pmatrix}$. So its true for $n = 2$.

So assume its true for $k$, that is $A^k = \begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix}$.

Then $A{\cdot}A^k = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix} = \begin{pmatrix} f_{k+1} + f_k & f_k + f_{k-1} \\ f_{k+1} & f_k \end{pmatrix} = \begin{pmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{pmatrix}$.

By induction, it is true for all $n$. □

From this we see that powers of this matrix give the Fibonacci sequence as its entries. This result will be helpful in establishing other results.

**Theorem 2.2.** *(Perron-Frobenius Theorem) Let* $A$ *be an* $n \times n$ *positive matrix, i.e.* $A > 0$. *Then the following is true.*

*1 There is a positive real number* $\tau$, *called the Perron-Frobenius eigenvalue, such that*

*$\tau$ is an eigenvalue of $A$ and any other eigenvalue $\lambda$ is strictly less than $\tau$ in absolute value.*

*2 The Perron-Frobenius eigenvalue is simple, i.e. $\tau$ is a simple root of the characteristic polynomial of $A$. $\tau$ has multiplicity one.*

*Proof.* This can be found [7] □

**Definition 2.3.** *A matrix is* ***primitive*** *if it is non-negative and its $m^{th}$ power is positive for some natural number $m$.*

**Remark** Everything that is true about positive matrices in the Perron-Frobenius theorem is also true for primitive matrices. As one can check, our matrices are primitive and therefore have a Perron-Frobenius eigenvalue.

Just as we were able to link the powers of our matrix to the Fibonacci sequence, we will also link the Perron-Frobenius eigenvalue, $\tau$ to the Fibonacci sequence. This allows us, for sufficiently large $n$, to look at them interchangeably.

**Lemma 2.4.** *Let $\tau$ be the Perron-Frobenius eigenvalue of*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

*and let $f_{n+1} = f_n + f_{n-1}$ be the Fibonacci sequence where $f_0 = 0$ and $f_1 = 1$. Then $\tau = \lim_{n\to\infty} \frac{f_{n+1}}{f_n}$.*

*Proof.* $A$ is a diagonalizable matrix since the characteristic polynomial of $A$, $x^2 - x - 1$, has 2 distinct eigenvalues. One of the roots is $\tau$ and the other root is $\gamma$. Since $\tau$ is the Perron-Frobenius eigenvalue we know that $\tau > |\gamma|$. Since $A$ is diagonalizable, there exists an invertible matrix $P$ such that $A = PDP^{-1}$ where $D$ is a diagonal matrix with the eigenvalues of $A$ as its entries.

So since $A = PDP^{-1}$ then $A^n = (PDP^{-1})^n = PDP^{-1} \cdot PDP^{-1} \cdot \ldots \cdot PDP^{-1}$ $= PD^nP^{-1}$

but $A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau^n & 0 \\ 0 & \gamma^n \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$$= \begin{pmatrix} a_{11}b_{11}\tau^n + a_{12}b_{21}\gamma^n & * \\ * & * \end{pmatrix}$$

so we see that $f_{n+1} = a_{11}b_{11}\tau^n + a_{12}b_{21}\gamma^n$.

Repeating this process for $A^{n-1}$ we get that

$$A^{n-1} = \begin{pmatrix} f_n & f_{n-1} \\ f_{n-1} & f_{n-2} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11}\tau^{n-1} + a_{12}b_{21}\gamma^{n-1} & * \\ * & * \end{pmatrix}$$

so $f_n = a_{11}b_{11}\tau^{n-1} + a_{12}b_{21}\gamma^{n-1}$.

we now have that $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \lim_{n\to\infty} \frac{a_{11}b_{11}\tau^n + a_{12}b_{21}\gamma^n}{a_{11}b_{11}\tau^{n-1} + a_{12}b_{21}\gamma^{n-1}}$

$$= \lim_{n\to\infty} \frac{a_{11}b_{11}\frac{\tau^n}{\tau^{n-1}} + a_{12}b_{21}\frac{\gamma^n}{\tau^{n-1}}}{a_{11}b_{11}\frac{\tau^{n-1}}{\tau^{n-1}} + a_{12}b_{21}\frac{\gamma^{n-1}}{\tau^{n-1}}} = \lim_{n\to\infty} \frac{a_{11}b_{11}\tau + a_{12}b_{21}\gamma\left(\frac{\gamma}{\tau}\right)^{n-1}}{a_{11}b_{11} + a_{12}b_{21}\left(\frac{\gamma}{\tau}\right)^{n-1}} = \frac{a_{11}b_{11}\tau}{a_{11}b_{11}} = \tau$$

Therefore $\tau = \lim_{n\to\infty} \frac{f_{n+1}}{f_n}$. □

This result is not best for generalizing to the general 2-parameter Fibonacci sequence or higher dimension incidence matrices because of messy calculations. We will prove this again using a method using difference equations that will be more general. For basic information on difference equations see the Appendix.

**Lemma 2.5.** *Let $\tau$ be the Perron-Frobenius eigenvalue of*

$$A = \begin{pmatrix} m_1 & m_2 \\ 1 & 0 \end{pmatrix}$$

*and let $f_{k+1} = m_1 f_k + m_2 f_{k-1}$ be the 2-parameter Fibonacci sequence where $f_0 = 0$ and $f_1 = 1$. Then $\tau = \lim_{k\to\infty} \frac{f_{k+1}}{f_k}$.*

*Proof.* From [6], we see that the homogeneous equation for the 2-parameter Fibonacci algebra is $f_{k+2} - m_1 f_{k+1} - m_2 f_k = 0$. From this we see that the auxiliary equation is $x^2 - m_1 x - m_2 = 0$, which is an algebraic equation of degree 2 with exactly 2 roots. The auxiliary equation is the characteristic polynomial for $\begin{pmatrix} m_1 & m_2 \\ 1 & 0 \end{pmatrix}$.

Since $\begin{pmatrix} m_1 & m_2 \\ 1 & 0 \end{pmatrix}$ is a primitive matrix, by the Perron-Frobenius Theorem, there exist $\tau$, the Perron-Frobenius eigenvalue which is unique and greater than the other root in absolute value. So the other root also has to be a real unrepeated root since $\tau$ is unique and complex roots come in pairs.

The general solution of the homogeneous equation is $f_k = C_1\tau^k + C_2\gamma^k$, where $\gamma$ is the other root of the auxiliary equation and $C_1, C_2$ are arbitrary constants. We have $f_{k+1} = C_1\tau^{k+1} + C_2\gamma^{k+1}$.

Hence, $\lim\limits_{k\to\infty} \frac{f_{k+1}}{f_k} = \lim\limits_{k\to\infty} \frac{C_1\tau^{k+1}+C_2\gamma^{k+1}}{C_1\tau^k+C_2\gamma^k} = \lim\limits_{k\to\infty} \frac{C_1\frac{\tau^{k+1}}{\tau^k}+C_2\frac{\gamma^{k+1}}{\tau^k}}{C_1\frac{\tau^k}{\tau^k}+C_2\frac{\gamma^k}{\tau^k}} = \frac{C_1\tau}{C_1} = \tau.$ $\square$

This method gives us a nice result that can be further generalized to higher dimension incidence matrices. Now with the new machinery in place we shall take a look at the positive cone of the Fibonacci Algebra. Before we get to the positive cone we state some useful Lemmas and theorems.

**Lemma 2.6** (Lemma A)**.** *If $a \in K$ is algebraic over $F$, then $a$ is a zero of a unique monic irreducible polynomial $m_a(x)$ which divides any polynomial $g(x)$ of which $a$ is a root.*

*Proof.* See [9][page 267]. $\square$

**Lemma 2.7** (Lemma B)**.** *Let $\theta \in F$ be algebraic of degree $n$ over $K$ and let $g$ be the minimal polynomial of $\theta$ over $K$. Then:*

*(i) $K(\theta)$ is isomorphic to $K[x]/(g)$.*

*(ii) $[K(\theta) : K] = n$ and $\{1, \theta, \ldots, \theta^{n-1}\}$ is a basis of $K(\theta)$ over $K$.*

*(iii) Every $\alpha \in K(\theta)$ is algebraic over $K$ and its degree over $K$ is a divisor of $n$.*

*Proof.* See [7, Theorem 1.86]. $\square$

**Theorem 2.8** (Theorem C)**.** *Let $F \subseteq K$ be fields. Suppose $p(x) \in F[x]$ is monic irreducible of degree $n$ over $F$. If $\theta \in K$ satisfies $p(\theta) = 0$ then $\{1, \theta, \ldots, \theta^{n-1}\}$ are linearly independent over $F$.*

*Proof.* By Lemma A, $p(x)$ is the minimal polynomial of $\theta$ over $F$. Then by (ii) of Lemma B, $\{1, \theta, \ldots, \theta^{n-1}\}$ are linearly independent over $F$. $\square$

We will now look to see that the polynomials that we will be discussing are all irreducible. A proof can be found [?] but we will include for completeness.

**Theorem 2.9.** *Show that* $n \geq 2$ *the polynomial* $x^n - x^{n-1} - x^{n-2} - \cdots - x - 1$ *is irreducible over the* $\mathbb{Q}$.

*Proof.* Let $P_n(x) = x^n - x^{n-1} - x^{n-2} - \cdots - x - 1$ for each $n \geq 2$ and consider $Q_n(x) = (x-1)P_n(x) = x^{n+1} - 2x^n + 1$. Now $Q_n(1) = 0$ and $Q_n'(r) < 0$ for $1 < r < \frac{2n}{n+1}$, so that $Q_n(r) < 0$ for every $r$ satisfying $1 < r < \frac{2n}{n+1}$. If $|z| = r$, $1 < r < \frac{2n}{n+1}$, we have $|1 - 2z| \geq 2|z|^n - 1 = 2r^n - 1 > r^{n+1} = |z|^{n+1}$, which shows that the function $1 - 2z$ strictly dominates the function $z^{n+1}$ on the circle $|z| = r$. Thus *Rouché′s* theorem implies that $Q_n(z)$ and $1 - 2z^n$ have exactly the same number of zeroes inside $|z| = r$ for every $r$ satisfying $1 < r < \frac{2n}{n+1}$, and hence the same number of zeros inside or on $|z| = 1$. The function $1 - 2z^n$ obviously has $n$ zeros inside $|z| = 1$, while $Q_(z) = 0$ with $|z| = 1$ forces $z = 1$. Thus $Q_n(z)$ has exactly $(n-1)$ zeros satisfying $|z| < 1$. Since $P_n(1) < 0 < P_n(2)$, we see that $P_n(z)$ has $(n-1)$ zeros satisfying $|z| < 1$ and one zero satisfying $|z| > 1$. In $P_n(x)$ were reducible over the rationals, Gauss's Lemma would tell us that $P_n(x) = G(x)H(x)$ for suitable monic polynomials $G(x)$, $H(x)$ of positive degrees with coefficients which are rational integers. One of these polynomials, say $G(x)$, must have all its zeros of modulus strictly less than one. However this implies that $|G(0)| < 1$, contradicting the fact that constant term of the polynomial $G(x)$ must be a nonzero integer. □

**Theorem 2.10.** *If* $m_1 \geq m_2 \geq \ldots \geq m_n > 0$, *then the polynomial*

$$x^n - m_1 x^{n-1} - \cdots - m_{n-1}x - m_n$$

*is irreducible over* $\mathbb{Q}$.

*Proof.* This can be found in [2]. □

## 2.2 New proof of Fibonacci Algebra

**Remark** Our definition of the positive cone with $K$ being the incidence matrix is $K_0^+(\mathfrak{A}_K) = \bigcup_{n=0}^{\infty} K^{-n}\mathbb{Z}_+^2$. So if $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0^+(\mathfrak{A}_K)$ then there exist $n \geq 0$ and $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \mathbb{Z}_+^2$ such that $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = K^{-n} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$.

Hence, $K^n \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \geq 0$. So we can rewrite the definition of the positive cone as $K_0^+(\mathfrak{A}_K) = \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \mid \exists n \geq 0 \mid K^n \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in \mathbb{Z}_+^2 \right\}$.

**Lemma 2.11.** *Show that* $K_0^+ (\mathfrak{A}_K) = \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in \mathbb{Z}^2 \mid n_1 + \tau^{-1} n_2 \geq 0 \right\}$

*Proof.* By our remark of $K_0^+ (\mathfrak{A}_K)$, we have that

$$K_0^+ (\mathfrak{A}_K) = \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in \mathbb{Z}^2 \mid \exists n \geq 0 \mid A^n \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in \mathbb{Z}_+^2 \right\}.$$

First we will show $K_0^+ (\mathfrak{A}_K) \subseteq \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in \mathbb{Z}^2 \mid n_1 + \tau^{-1} n_2 \geq 0 \right\}$.

Suppose that $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0^+ (\mathfrak{A}_K)$. Then there exist an $m \geq 0$ such that $K^m \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \geq 0$. Take $n$ arbitrary such that $n \geq m$. If $n = m$, we get $K^n \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \geq 0$. Now suppose $n > m$ and that $k = n - m$. Then $K^n \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = K^{(n-m)} \left( K^m \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \right) = K^{(n-m)} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$, where $K^m \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \geq 0$.

So $K^k \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} m_1 f_{k+1} + m_2 f_k \\ m_1 f_k + m_2 f_{k-1} \end{pmatrix} \geq 0$, since $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \geq 0$

and $\{f_k\}_{k=1} \geq 0$ for all $k \geq 0$.

So $K^n \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \geq 0$ for all $n \geq m$.

With this we see that $K^n \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} n_1 f_{n+1} + n_2 f_n \\ n_1 f_n + n_2 f_{n-1} \end{pmatrix} \geq 0$,

hence $\begin{pmatrix} n_1 \frac{f_{n+1}}{f_n} + n_2 \\ n_1 + n_2 \frac{f_{n-1}}{f_n} \end{pmatrix} \geq 0$, for $n \geq 1$. So we have $n_1 \frac{f_{n+1}}{f_n} + n_2 \geq 0$ and $n_1 + n_2 \frac{f_{n-1}}{f_n} \geq 0$. Taking limits as $n \to \infty$ and using the fact that $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \tau$, we get that $\tau n_1 + n_2 \geq 0$ and $n_1 + \tau^{-1} n_2 \geq 0$ which are the same. So $K_0^+(\mathfrak{A}_K) \subseteq \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in \mathbb{Z}^2 \mid n_1 + \tau^{-1} n_2 \geq 0 \right\}$.

Now we will show $K_0^+(\mathfrak{A}_K) \supseteq \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in \mathbb{Z}^2 \mid n_1 + \tau^{-1} n_2 \geq 0 \right\}$.

Let $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \mathbb{Z}^2$ such that $m_1 + \tau^{-1} m_2 \geq 0$. If $m_1 = m_2 = 0$, then $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in K_0^+(\mathfrak{A}_K)$. If $m_1$ and $m_2$ are not both zero, then by Theorem C, $\{1, \tau\}$ are linearly independent over $\mathbb{Q}$ since the characteristic polynomial $x^2 - x - 1 = 0$ is irreducible over $\mathbb{Q}$ by previous Theorem. This implies that $m_1 + \tau^{-1} m_2 > 0$. Hence, for some $n_0 \geq 1$ if $n \geq n_0$, we have that $m_1 + \left(\frac{f_n}{f_{n+1}}\right) m_2 > 0$ and $m_1 + \frac{f_{n-1}}{f_n} m_2 > 0$. Hence $m_1 f_{n+1} + m_2 f_n > 0$ and $m_1 f_n + m_2 f_{n-1} > 0$,

so $\begin{pmatrix} m_1 f_{n+1} + m_2 f_n \\ m_1 f_n + m_2 f_{n-1} \end{pmatrix} > 0$,

so $K^n \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} > 0$. So $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in K_0^+(\mathfrak{A}_K)$.

Therefore, $K_0^+(\mathfrak{A}_K) = \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in \mathbb{Z}^2 \mid n_1 + \tau^{-1} n_2 \geq 0 \right\}$. $\square$

We now compute $K_0(\mathfrak{A}_K)$.

**Proposition 2.12.** *Let* $K = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\tau$ *be the Perron-Frobenius eigenvalue, and let* $\mathbb{F} = \mathbb{Z} + \tau^{-1}\mathbb{Z}$. *Then*

$$\left( K_0\left(\mathfrak{A}_K\right), K_0^+\left(\mathfrak{A}_K\right), \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \simeq (\mathbb{F}, \mathbb{F} \cap \mathbb{R}_+, 1).$$

*Proof.* Let $f : K_0(\mathfrak{A}_K) \longrightarrow \mathbb{F}$ with $f\left(\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}\right) = n_1 + \tau^{-1}n_2$. So we check that its a homomorphism. $f\left(\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}\right) + f\left(\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}\right) = n_1 + \tau^{-1}n_2 + m_1 + \tau^{-1}m_2 = n_1m_1 + \tau^{-1}n_2 + \tau^{-1}m_2 = n_1 + m_1 + \tau^{-1}(n_2 + m_2) = f\left(\begin{pmatrix} n_1 + m_1 \\ n_2 + m_2 \end{pmatrix}\right)$. So $f$ is a homomorphism.

Next check that $f$ is a 1-1.
$ker\ f = \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \mid f\left(\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}\right) = 0 \right\} = \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \mid n_1 + \tau^{-1}n_2 = 0 \right\}$. Since $\{1, \tau\}$ is linearly independent over $\mathbb{Q}$, $n_1 = n_2 = 0$. So the only thing in the kernel is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. So $f$ is 1-1.

Next check that $f$ is onto.
Let $m \in \mathbb{F}$. Then there exists an $a$, $b \in \mathbb{Z}$ such that $m = a + \tau^{-1}b$. So $f\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = a + \tau^{-1}b = m$. So $f$ is onto. Therefore $f$ is a bijection.

Lets examine the positive cone, $K_0^+(\mathfrak{A}_K)$. Let $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0^+(\mathfrak{A}_K)$. Then $f\left(\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}\right) = \tau^{-1}n_1 + n_2 > 0$. But this is exactly the set $\mathbb{F} \cap \mathbb{R}_+$.
So $f(K_0^+(\mathfrak{A}_K)) \cong \mathbb{F} \cap \mathbb{R}_+$.

Next we need to check the order unit goes to the order unit. So $f\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) =$

$1+\tau^{-1}(0)=1$.

Therefore $\left(K_0(\mathfrak{A}_K), K_0^+(\mathfrak{A}_K), \begin{pmatrix}1\\ \\0\end{pmatrix}\right) \cong (\mathbb{F}, \mathbb{F}\cap\mathbb{R}_+, 1)$. $\square$

# CHAPTER 3
# 2-PARAMETER FIBONACCI ALGEBRAS

## 3.1 2-paramteter Fibonacci algebra

Doing a search of the literature we see that in [1], Bratteli is examining matrices that happen to be the transpose of our matrices to create a family of AF algebras. A natural question that arises is if an incidence matrix and its transpose yield the same AF-algebras. We will explore that here for the general 2-parameter Fibonacci algebra that has a $2 \times 2$ incidence matrix. First we shall examine the $n^{th}$ power of $K = \begin{pmatrix} m_1 & m_2 \\ 1 & 0 \end{pmatrix}$.

**Lemma 3.1.** *Let*

$$K = \begin{pmatrix} m_1 & m_2 \\ 1 & 0 \end{pmatrix}$$

*where* $m_1, m_2 \in \mathbb{Z}_+$ *and let* $f_{n+1} = m_1 f_n + m_2 f_{n-1}$ *be the 2-parameter Fibonacci sequence where* $f_0 = 0$ *and* $f_1 = 1$*. Then*

$$K^n = \begin{pmatrix} f_{n+1} & m_2 f_n \\ f_n & m_2 f_{n-1} \end{pmatrix}$$

*Proof.* $K^2 = \begin{pmatrix} m_1 & m_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m_1 & m_2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} m_1^2 + m_2 & m_1 m_2 \\ m_1 & m_2 \end{pmatrix} = \begin{pmatrix} f_3 & f_2 \\ f_2 & f_1 \end{pmatrix}$. So its true for $n = 2$.

So assume its true for $k$, that is $K^k = \begin{pmatrix} f_{k+1} & m_2 f_k \\ f_k & m_2 f_{k-1} \end{pmatrix}$.

Then $K \cdot K^k = \begin{pmatrix} m_1 & m_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{k+1} & m_2 f_k \\ f_k & m_2 f_{k-1} \end{pmatrix}$

$= \begin{pmatrix} m_1 f_{k+1} + m_2 f_k & m_1 m_2 f_k + m_2^2 f_{k-1} \\ f_{k+1} & m_2 f_k \end{pmatrix} = \begin{pmatrix} f_{k+2} & m_2 f_{k+1} \\ f_{k+1} & m_2 f_k \end{pmatrix}$

By induction it is true for all $n$. □

Recall from the previous chapter that $\tau = \lim_{n\to\infty} \frac{f_{n+1}}{f_n}$, where $\tau$ is the Perron-Frobenius eigenvalue of $K$. Next we will look at the positive cone of our matrix $K$ and its transpose $J$.

**Lemma 3.2.** *Let* $K = \begin{pmatrix} m_1 & m_2 \\ 1 & 0 \end{pmatrix}$ *and* $m_1, m_2 \in \mathbb{Z}_+^2$, *with* $x^2 - m_1x - m_2$ *irreducible over* $\mathbb{Q}$. *Then we have that*

$$K_0^+(\mathfrak{A}_K) = \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0(\mathfrak{A}_K) \mid n_1 + m_2\tau^{-1}n_2 \geq 0 \right\}$$

*where* $\tau$ *is the Perron-Frobenius eigenvalue of* $K$.

*Proof.* We first note from the previous Lemma that $K^n = \begin{pmatrix} f_{n+1} & m_2 f_n \\ f_n & m_2 f_{n-1} \end{pmatrix}$ with $n \geq 1$.

By our definition of $K_0^+(\mathfrak{A}_K)$, we have that

$$K_0^+(\mathfrak{A}_K) = \bigcup_{n=0}^{\infty} K^{-n}(\mathbb{Z}_+^2)$$

In the case where $m_1 = m_2 = 1$, $K^{-n}(\mathbb{Z}^2) = \mathbb{Z}^2$. When the constants are arbitrary, $K^{-n}(\mathbb{Z}^2) \subseteq \mathbb{Q}^2$ since $K^{-n}$ does not have entries in $\mathbb{Z}$.

First we will show $K_0^+(\mathfrak{A}_K) \subseteq \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0(\mathfrak{A}_K) \mid n_1 + m_2\tau^{-1}n_2 \geq 0 \right\}$. From a similar argument for from the previous chapter, we see that for $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0^+(\mathfrak{A}_K)$ there exist $n_0$ such that for all $n \geq n_0$ $K^n \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \geq 0$.

So $K^n \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} n_1 f_{n+1} + m_2 n_2 f_n \\ n_1 f_n + m_2 n_2 f_{n-1} \end{pmatrix} \geq 0$

$\Rightarrow \begin{pmatrix} n_1 \frac{f_{n+1}}{f_n} + m_2 n_2 \\ n_1 + m_2 n_2 \frac{f_{n-1}}{f_n} \end{pmatrix} \geq 0$. So we have $n_1 \frac{f_{n+1}}{f_n} + m_2 n_2 \geq 0$ and $n_1 + m_2 n_2 \frac{f_{n-1}}{f_n} \geq 0$.

Taking limits we get that $\tau n_1 + m_2 n_2 \geq 0$ and $n_1 + m_2\tau^{-1}n_2 \geq 0$ which are the same.

Therefore $K_0^+(\mathfrak{A}_K) \subseteq \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0(\mathfrak{A}_K) \mid n_1 + m_2\tau^{-1}n_2 \geq 0 \right\}$.

Now we will show $K_0^+(\mathfrak{A}_K) \supseteq \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0(\mathfrak{A}_K) \mid n_1 + m_2\tau^{-1}n_2 \geq 0 \right\}$.

Let $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0(\mathfrak{A}_K)$ such that $n_1 + m_2\tau^{-1}n_2 \geq 0$. If $n_1 = n_2 = 0$, then $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in K_0^+(\mathfrak{A}_K)$. If $n_1$ and $n_2$ are not both zero, then by Theorem C, the set $\{1, \tau\}$ is linearly independent since the characteristic polynomial $x^2 - m_1x - m_2 = 0$ is irreducible over $\mathbb{Q}$. This implies that $n_1 + m_2\tau^{-1}n_2 > 0$. Hence, there exist an $n_0 \geq 1$ such that for all $n \geq n_0$, we have that $n_1 + \left(\frac{f_n}{f_{n+1}}\right) m_2n_2 > 0$ and $n_1 + \left(\frac{f_{n-1}}{f_n}\right) m_2n_2 > 0$.

Hence, $n_1f_{n+1} + m_2n_2f_n > 0$ and $n_1f_n + m_2n_2f_{n-1} > 0$

So $\begin{pmatrix} n_1f_{n+1} + m_2n_2f_n \\ n_1f_n + m_2n_2f_{n-1} \end{pmatrix} > 0$

Which implies $K^n \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} > 0$. So $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0^+(\mathfrak{A}_K)$.

Therefore, $K_0^+(\mathfrak{A}_K) = \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0(\mathfrak{A}_K) \mid n_1 + m_2\tau^{-1}n_2 \geq 0 \right\}$ □

### 3.2 The transpose algebra of the 2-parameter Fibonacci algebra

In [1], the authors show by other methods that for $J = \begin{pmatrix} m_1 & 1 \\ m_2 & 0 \end{pmatrix}$ and $\mathbb{F} = \mathbb{Z} + \tau^{-1}\mathbb{Z}$ we have

$$\left( K_0(\mathfrak{A}_J), K_0^+(\mathfrak{A}_J), \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \cong (\mathbb{F}, \mathbb{F} \cap \mathbb{R}_+, 1)$$

if $x^2 - m_1x - m_2$ is irreducible over $\mathbb{Q}$. We will examine $K_0^+$ using our method.

**Lemma 3.3.** *Let* $J = K^T = \begin{pmatrix} m_1 & 1 \\ m_2 & 0 \end{pmatrix}$ *and* $m_1, m_2 \in \mathbb{Z}_+^2$ *with* $x^2 - m_1x - m_2$

*irreducible over $\mathbb{Q}$. Then we have that*

$$K_0^+(\mathfrak{A}_J) = \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0(\mathfrak{A}_J) \mid n_1 + \tau^{-1} n_2 \geq 0 \right\}$$

*Where $\tau$ is the Perron-Frobenius eigenvalue of $J$.*

*Proof.* We first note from the previous Lemma that $J^n = (K^T)^n = (K^n)^T = \begin{pmatrix} f_{n+1} & f_n \\ m_2 f_n & m_2 f_{n-1} \end{pmatrix}$ with $n \geq 1$.

By our definition of $K_0^+(\mathfrak{A}_J)$, we have that

$$K_0^+(\mathfrak{A}_J) = \bigcup_{n=0}^{\infty} J^{-n}(\mathbb{Z}_+^2)$$

In the case where $m_1 = m_2 = 1$, $J^{-n}(\mathbb{Z}^2) = \mathbb{Z}^2$. When the constants are arbitrary, $J^{-n}(\mathbb{Z}^2) \subseteq \mathbb{Q}^2$ since $J^{-n}$ does not have entries in $\mathbb{Z}$.

First we will show $K_0^+(\mathfrak{A}_J) \subseteq \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0(\mathfrak{A}_J) \mid n_1 + \tau^{-1} n_2 \geq 0 \right\}$. From a similar argument from the previous chapter, we see that for $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0^+(\mathfrak{A}_J)$ there exist $n_0$ such that for all $n \geq n_0$ $J^n \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \geq 0$.

So $J^n \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} n_1 f_{n+1} + n_2 f_n \\ m_2 n_1 f_n + m_2 n_2 f_{n-1} \end{pmatrix} \geq 0$

$\Rightarrow \begin{pmatrix} n_1 \frac{f_{n+1}}{f_n} + n_2 \\ m_2 n_1 + m_2 n_2 \frac{f_{n-1}}{f_n} \end{pmatrix} \geq 0$. So we have $n_1 \frac{f_{n+1}}{f_n} + n_2 \geq 0$ and $m_2 n_1 + m_2 n_2 \frac{f_{n-1}}{f_n} \geq 0$. Taking limits we get that $\tau n_1 + n_2 \geq 0$ and $m_2 n_1 + m_2 \tau^{-1} n_2 \geq 0$ which are the same.

Therefore $K_0^+(\mathfrak{A}_J) \subseteq \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0(\mathfrak{A}_J) \mid n_1 + \tau^{-1} n_2 \geq 0 \right\}$.

Now we will show $K_0^+(\mathfrak{A}_J) \supseteq \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0(\mathfrak{A}_J) \mid n_1 + \tau^{-1} n_2 \geq 0 \right\}$.

Let $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0(\mathfrak{A}_J)$ such that $n_1 + \tau^{-1} n_2 \geq 0$. If $n_1 = n_2 = 0$, then $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in K_0^+(\mathfrak{A}_J)$. If $n_1$ and $n_2$ are not both zero, then by Theorem C, the set $\{1, \tau\}$ is linearly independent since the characteristic polynomial $x^2 - m_1 x - m_2 = 0$ is irreducible over $\mathbb{Q}$. This implies that $n_1 + \tau^{-1} n_2 > 0$. Hence, there exist an $n_0 \geq 1$ such that for all $n \geq n_0$, we have that $n_1 + \left(\frac{f_n}{f_{n+1}}\right) n_2 > 0$ and $m_2 n_1 + \left(\frac{f_{n-1}}{f_n}\right) m_2 n_2 > 0$.

Hence, $n_1 f_{n+1} + n_2 f_n > 0$ and $m_2 n_1 f_n + m_2 n_2 f_{n-1} > 0$

So $\begin{pmatrix} n_1 f_{n+1} + n_2 f_n \\ m_2 n_1 f_n + m_2 n_2 f_{n-1} \end{pmatrix} > 0$

Which implies $J^n \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} > 0$. So $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0^+(\mathfrak{A}_J)$.

Therefore, $K_0^+(\mathfrak{A}_J) = \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in K_0(\mathfrak{A}_J) \mid n_1 + \tau^{-1} n_2 \geq 0 \right\}$ □

Now that we have established the positive cone for $K_0(\mathfrak{A}_J)$ and $K_0(\mathfrak{A}_K)$ we can now focus our attention on seeing if there is an order isomorphisms from $K_0(\mathfrak{A}_J)$ to $K_0(\mathfrak{A}_K)$. In our next proof we will show that if there exist an order isomorphism between $K_0(\mathfrak{A}_J)$ and $K_0(\mathfrak{A}_K)$ then it is unique.

**Proposition 3.4.** *If $\theta : K_0(\mathfrak{A}_J) \to K_0(\mathfrak{A}_K)$ is an order isomorphism, then $\theta$ must have the form $\theta = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix}$.*

*Proof.* For an order isomorphism $\theta$ we know that $\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ since $\theta$ sends the order unit to the order unit. Once we know where $\theta$ sends $\begin{pmatrix}0\\1\end{pmatrix}$ we can express where $\theta$ sends any element of $K_0(\mathfrak{A}_J)$. Set $\theta\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix} \in K_0\left(\mathfrak{A}_K\right) = \bigcup_{n=0}^{\infty} K^{-n}\left(\mathbb{Z}^2\right)$, and where $a,\, b \in \mathbb{Q}$ Then for $\begin{pmatrix}k_1\\k_2\end{pmatrix} \in K_0(\mathfrak{A}_J)$ we have $\theta\begin{pmatrix}k_1\\k_2\end{pmatrix} = \theta\left(k_1\begin{pmatrix}1\\0\end{pmatrix} + k_2\begin{pmatrix}0\\1\end{pmatrix}\right) = k_1\theta\begin{pmatrix}1\\0\end{pmatrix} + k_2\theta\begin{pmatrix}0\\1\end{pmatrix} = k_1\begin{pmatrix}1\\0\end{pmatrix} + k_2\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}k_1\\0\end{pmatrix} + \begin{pmatrix}ak_2\\bk_2\end{pmatrix} = \begin{pmatrix}k_1+ak_2\\bk_2\end{pmatrix}$.

That is $\theta\begin{pmatrix}k_1\\k_2\end{pmatrix} = \begin{pmatrix}k_1+ak_2\\bk_2\end{pmatrix}$. Let $\begin{pmatrix}k_1\\k_2\end{pmatrix} \in K_0(\mathfrak{A}_J)$ which means that $k_1 + \tau^{-1}k_2 \geq 0$. Then $\begin{pmatrix}k_1\\k_2\end{pmatrix} = J^{-0}\begin{pmatrix}k_1\\k_2\end{pmatrix}$, $k_1+\tau^{-1}k_2 \geq 0$, hence $\begin{pmatrix}k_1\\k_2\end{pmatrix} \in K_0^+\left(\mathfrak{A}_J\right)$. So $\theta\begin{pmatrix}k_1\\k_2\end{pmatrix} = \begin{pmatrix}k_1+ak_2\\bk_2\end{pmatrix} \in K_0^+\left(\mathfrak{A}_K\right) = \bigcup_{n=0}^{\infty} K^{-n}\left(\mathbb{Z}_+^2\right)$. So we get that for some $m \geq 0$, $\begin{pmatrix}k_1+ak_2\\bk_2\end{pmatrix} = K^{-m}\begin{pmatrix}n_1\\n_2\end{pmatrix}$, where $\begin{pmatrix}n_1\\n_2\end{pmatrix} \in \mathbb{Z}_+^2$ i.e., $K^m\begin{pmatrix}k_1+ak_2\\bk_2\end{pmatrix} = \begin{pmatrix}n_1\\n_2\end{pmatrix} \in \mathbb{Z}_+^2$. Hence for $n \geq m$ we get $K^n\begin{pmatrix}k_1+ak_2\\bk_2\end{pmatrix} \geq 0$ but $K^n = \begin{pmatrix}f_{n+1} & m_2f_n\\ f_n & m_2f_{n-1}\end{pmatrix}$ for $n \geq 1$. So $0 \leq K^n\begin{pmatrix}k_1+ak_2\\bk_2\end{pmatrix} = \begin{pmatrix}f_{n+1} & m_2f_n\\ f_n & m_2f_{n-1}\end{pmatrix}\begin{pmatrix}k_1+ak_2\\bk_2\end{pmatrix}$

$= \begin{pmatrix}f_{n+1}\left(k_1+ak_2\right)+m_2f_nbk_2\\ f_n\left(k_1+ak_2\right)+m_2f_{n-1}bk_2\end{pmatrix} = \begin{pmatrix}f_{n+1}k_1+af_{n+1}k_2+m_2bf_nk_2\\ f_nk_1+af_nk_2+m_2f_{n-1}bk_2\end{pmatrix}$

$= \begin{pmatrix}f_{n+1}k_1+\left(af_{n+1}+m_2bf_n\right)k_2\\ f_nk_1+\left(af_n+m_2bf_{n-1}\right)k_2\end{pmatrix}$. Therefore, for all $n > m$ we get $f_{n+1}k_1 +$

$(af_{n+1} + m_2 b f_n)\, k_2 \geq 0$ i.e., $k_1 + \left(a + m_2 b \left(\frac{f_n}{f_{n+1}}\right)\right) k_2 \geq 0$

Now take the limit as $n \to \infty$ to get $k_1 + (a + m_2 b\tau^{-1})\, k_2 \geq 0$. This shows that for all $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in K_0^+(\mathfrak{A}_J)$ which has $k_1 + \tau^{-1} k_2 \geq 0$ then $k_1 + (a + m_2 b\tau^{-1})\, k_2 \geq 0$. Taking $k_1 = 0$ and $k_2 = 1$ we get $0 \leq k_1 + (a + m_2 b\tau^{-1})\, k_2 = a + m_2 b\tau^{-1}$, so $a + m_2 b\tau^{-1} \geq 0$. Because $\begin{pmatrix} a \\ b \end{pmatrix} \in \bigcup_{n=0}^{\infty} K^{-n}(\mathbb{Z}^2)$ we get that $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Q}^2$. We have that $a + m_2 b\tau^{-1} > 0$ otherwise $a = 0$ and $b = 0$, assuming $m_2 \geq 1$. But then $\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which is a contradiction, contradicting the fact that $\theta$ is an isomorphism. Set $\gamma^{-1} = a + m_2 b\tau^{-1} > 0$. Then for all $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in K_0^+(\mathfrak{A}_J)$ which has $k_1 + \tau^{-1} k_2 \geq 0 \Rightarrow k_1 + \gamma^{-1} k_2 \geq 0$. So let's look at the relation between $\tau$ and $\gamma$. We can view $k_1 + \tau^{-1} k_2$ and $k_1 + \gamma^{-1} k_2$ as linear equations. The only way two lines with the same points can have the same positive area is if their slopes are the same. Therefore $\tau = \gamma$. This means that $a = 0$ and $bm_2 = 1$. So $b = \frac{1}{m_2}$. Hence
$\theta \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} k_1 + ak_2 \\ bk_2 \end{pmatrix} = \begin{pmatrix} k_1 \\ \frac{k_2}{m_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$. □

**Lemma 3.5.** *Let $\theta$, $J$, and $K$ be from the previous Proposition. Then*
$\theta \left( J^{-n} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \right) = K^{-n} \begin{pmatrix} k_1 \\ \frac{k_2}{m_2} \end{pmatrix}$ *for $n \geq 0$*

*Proof.* For $n = 0$, we have $\theta J^{-0} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \theta \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} k_1 \\ \frac{k_2}{m_2} \end{pmatrix} = K^{-0} \begin{pmatrix} k_1 \\ \frac{k_2}{m_2} \end{pmatrix}$.
So its true for $n = 0$.
Assume its true $k$, that is $\theta \left( J^{-k} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \right) = K^{-k} \begin{pmatrix} k_1 \\ \frac{k_2}{m_2} \end{pmatrix}$
Hence, $J^{-k} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \theta^{-1} K^{-k} \begin{pmatrix} k_1 \\ \frac{k_2}{m_2} \end{pmatrix}$.

Then $\theta\left(J^{-(k+1)}\begin{pmatrix}k_1\\k_2\end{pmatrix}\right)=\theta J^{-1}\left(J^{-k}\begin{pmatrix}k_1\\k_2\end{pmatrix}\right)=\theta J^{-1}\left(\theta^{-1}K^{-k}\begin{pmatrix}k_1\\\frac{k_2}{m_2}\end{pmatrix}\right)=$
$\begin{pmatrix}1&0\\0&\frac{1}{m_2}\end{pmatrix}\begin{pmatrix}0&1\\m_2&m_1\end{pmatrix}\begin{pmatrix}1&0\\0&m_2\end{pmatrix}K^{-k}\begin{pmatrix}k_1\\\frac{k_1}{m_2}\end{pmatrix}$
$\begin{pmatrix}0&m_2\\1&-m_1\end{pmatrix}K^{-k}\begin{pmatrix}k_1\\\frac{k_2}{m_2}\end{pmatrix}=K^{-1}K^{-k}\begin{pmatrix}k_1\\\frac{k_2}{m_2}\end{pmatrix}=K^{-(m+1)}\begin{pmatrix}k_1\\\frac{k_2}{m_2}\end{pmatrix}$. So its true for all $n$. □

**Corollary 3.6.** *Let $\theta$, $J$, and $K$ be as in previous Proposition.*
*Then* $\theta\left(J^{-n}\begin{pmatrix}1\\0\end{pmatrix}\right)=K^{-n}\begin{pmatrix}1\\0\end{pmatrix}$ *and* $\theta\left(J^{-n}\begin{pmatrix}0\\m_2\end{pmatrix}\right)=K^{-n}\begin{pmatrix}0\\1\end{pmatrix}$

*Proof.* This follows from previous Lemma. □

**Lemma 3.7.** *Let* $K=\begin{pmatrix}m_1&m_2\\1&0\end{pmatrix}$. *Then* $\begin{pmatrix}0\\\frac{1}{m_2}\end{pmatrix}K_0^+(\mathfrak{A}_K)$

*Proof.* Since $K^n=\begin{pmatrix}f_{n+1}&m_2f_n\\f_n&m_2f_{n-1}\end{pmatrix}$ we have that

$K^{-n}=\frac{1}{m_2f_{n+1}f_{n-1}-m_2f_n^2}\begin{pmatrix}m_2f_{n-1}&-m_2f_n\\-f_n&f_{n+1}\end{pmatrix}$. So for $n=1$,

$K^{-1}\begin{pmatrix}1\\0\end{pmatrix}=\frac{1}{m_2f_2f_0-m_2f_1^2}\begin{pmatrix}m_2f_0&-m_2f_1\\-f_1&f_2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$

$=-\frac{1}{m_2}\begin{pmatrix}0&-m_1\\-1&m_1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\\frac{1}{m_2}\end{pmatrix}\in K_0^+(\mathfrak{A}_K)$ □

From this we see that $\theta$ is an order isomorphism from $K_0(\mathfrak{A}_K)$ and $K_0(\mathfrak{A}_J)$. This completes the $2\times 2$ case.

# CHAPTER 4
# GENERAL N-PARAMETER FIBONACCI AF $C^*$-ALGEBRA

## 4.1 Tribonacci Algebra

The Fibonacci sequence of order 3, otherwise known as the Tribonacci sequence, will be the focus of this first section. We will call this the standard Tribonacci sequence. The standard Tribonacci sequence is defined similarly to the standard Fibonacci sequence except that the next term depends on the previous three terms. So we have $f_{n+1} = f_n + f_{n-1} + f_{n-2}$ where $f_0 = f_1 = 0$ and $f_2 = 1$. The incidence matrix that goes along with the Tribonacci sequence is $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ with characteristic polynomial of $x^3 - x^2 - x - 1$. Looking at this case will help us to generalize to the $n \times n$ case since the $2 \times 2$ case does not give enough information. We will start with similar machinery used in the $2 \times 2$ case.

**Lemma 4.1.** *Let*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

*and let* $f_{n+1} = f_n + f_{n-1} + f_{n-2}$ *be the Fibonacci sequence where* $f_0 = f_1 = 0$ *and* $f_2 = 1$. *Then*

$$A^n = \begin{pmatrix} f_{n+2} & f_{n+3} - f_{n+2} & f_{n+4} - f_{n+3} - f_{n+2} \\ f_{n+1} & f_{n+2} - f_{n+1} & f_{n+3} - f_{n+2} - f_{n+1} \\ f_n & f_{n+1} - f_n & f_{n+2} - f_{n+1} - f_n \end{pmatrix}$$

*Proof.* $A^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 4-2 & 7-4-2 \\ 1 & 2-1 & 4-2-1 \\ 1 & 1-1 & 2-1-1 \end{pmatrix} =$

$\begin{pmatrix} f_4 & f_5 - f_4 & f_6 - f_5 - f_4 \\ f_3 & f_4 - f_3 & f_5 - f_4 - f_3 \\ f_2 & f_3 - f_2 & f_4 - f_3 - f_2 \end{pmatrix}$. So its true for $n = 2$.

So assume its true for $k$, that is $A^k = \begin{pmatrix} f_{k+2} & f_{k+3} - f_{k+2} & f_{k+4} - f_{k+3} - f_{k+2} \\ f_{k+1} & f_{k+2} - f_{k+1} & f_{k+3} - f_{k+2} - f_{k+1} \\ f_k & f_{k+1} - f_k & f_{k+2} - f_{k+1} - f_k \end{pmatrix}$

We will represent the matrix as a column vector where the $j$ represents the $j^{th}$ column of the matrix. So $A^k = \begin{pmatrix} f_{k+2} & f_{k+3} - f_{k+2} & f_{k+4} - f_{k+3} - f_{k+2} \\ f_{k+1} & f_{k+2} - f_{k+1} & f_{k+3} - f_{k+2} - f_{k+1} \\ f_k & f_{k+1} - f_k & f_{k+2} - f_{k+1} - f_k \end{pmatrix}$

$$= \begin{pmatrix} f_{k+(j+1)} - \sum_{i=0}^{j-2} f_{k+j-i} \\ f_{k+j} - \sum_{i=0}^{j-2} f_{k+j-1-i} \\ f_{k+(j-1)} - \sum_{i=0}^{j-2} f_{k+j-2-i} \end{pmatrix}$$

Then $A \cdot A^k = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f_{k+(j+1)} - \sum_{i=0}^{j-2} f_{k+j-i} \\ f_{k+j} - \sum_{i=0}^{j-2} f_{k+j-1-i} \\ f_{k+(j-1)} - \sum_{i=0}^{j-2} f_{k+j-2-i} \end{pmatrix}$

$$= \begin{pmatrix} f_{k+j+1} - \sum_{i=0}^{j-2} f_{k+j-i} + f_{k+j} - \sum_{i=0}^{j-2} f_{k+j-1-i} + f_{k+j-1} \sum_{i=0}^{j-2} f_{k+j-2-i} \\ f_{k+j+1} - \sum_{i=0}^{j-2} f_{k+j-i} \\ f_{k+j} - \sum_{i=0}^{j-2} f_{k+j-1-i} \end{pmatrix}$$

$$= \begin{pmatrix} f_{k+j+2} - \sum_{i=0}^{j-2} f_{k+j-i} + f_{k+j-1-i} + f_{k+j-2-i} \\ f_{k+j+1} - \sum_{i=0}^{j-2} f_{k+j-i} \\ f_{k+j} - \sum_{i=0}^{j-2} f_{k+j-1-i} \end{pmatrix} = \begin{pmatrix} f_{k+j+2} - \sum_{i=0}^{j-2} f_{k+j+1-i} \\ f_{k+j+1} - \sum_{i=0}^{j-2} f_{k+j-i} \\ f_{k+j} - \sum_{i=0}^{j-2} f_{k+j-1-i} \end{pmatrix}$$

$$= \begin{pmatrix} f_{k+3} & f_{k+4} - f_{k+3} & f_{k+5} - f_{k+4} - f_{k+3} \\ f_{k+2} & f_{k+3} - f_{k+2} & f_{k+4} - f_{k+3} - f_{k+2} \\ f_{k+1} & f_{k+2} - f_{k+1} & f_{k+3} - f_{k+2} - f_{k+1} \end{pmatrix}$$

By induction it is true for all $n$. □

**Remark** Another form of this matrix is $\begin{pmatrix} f_{n+3} & f_{n+2} + f_{n+1} & f_{n+2} \\ f_{n+2} & f_{n+1} + f_n & f_{n+1} \\ f_{n+1} & f_n + f_{n-1} & f_n \end{pmatrix}$. We get this by the Tribonacci sequence. $f_{n+3} = f_{n+2} + f_{n+1} + f_n \to f_{n+3} - f_{n+2} = f_{n+1} + f_n$ and other relations.

As we can see from the previous Lemma, the $n^{th}$ power of the matrix generated by the Tribonacci sequence could not have been gotten from the $2 \times 2$ case. This formulation of the $n^{th}$ power of our matrix can be easily generalized which we will do in the next section. We can also do a similar calculation of the Perron-Frobenius eigenvalue, $\tau$ as for the $2 \times 2$ case.

**Lemma 4.2.** *Let $\tau$ be the Perron-Frobenius eigenvalue of*

$$A = \begin{pmatrix} m_1 & m_2 & m_2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

*and let $f_{n+1} = m_1 f_n + b m_2 f_{n-1} + m_3 f_{n-2}$ be the 3-parameter Fibonacci sequence where $f_0 = f_1 = 0$, and $f_2 = 1$. Then $\tau = \lim_{n\to\infty} \frac{f_{n+1}}{f_n}$.*

*Proof.* Similar to the 2-parameter Fibonacci sequence we see that the homogeneous equation is $f_{k+3} - m_1 f_{k+2} - m_2 f_{k+1} - m_3 f_k = 0$ with an auxiliary equation of $x^3 - m_1 x^2 - m_2 x - m_3 = 0$. This is an algebraic equation of degree 3 with exactly 3 roots. The auxiliary equation is the characteristic polynomial of $\begin{pmatrix} m_1 & m_2 & m_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Since $\begin{pmatrix} m_1 & m_2 & m_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is a primitive matrix there exist $\tau$, the Perron-Frobenius eigenvalue, that has multiplicity one and is greater than the other roots in absolute value. The other roots are either distinct real roots, repeated real roots, or a pair of complex conjugates. The general solution of the homogeneous equation is $f_k = C_1\tau^k + C_2\gamma_1^k + C_3\gamma_2^k + (B_1 + B_2k)\lambda^k + Ar^k\cos(k\theta + D)$, where $C_1, C_2, C_3, B_1, B_2, D$ and $A$ are arbitrary constants. Now we will look at the different cases for the possible different roots showing that dividing by $\tau$ and taking limits the roots go to zero.

Case 1. There are distinct real roots. Suppose $\gamma$ is a distinct real root other than $\tau$. Then $\tau > |\gamma|$ since $\tau$ is the Perron-Frobenius eigenvalue. $\Rightarrow \frac{|\gamma|}{\tau} < 1$.
$\Rightarrow \lim\limits_{k\to\infty} |\frac{\gamma}{\tau}|^k = 0 \Rightarrow \lim\limits_{k\to\infty} \left(\frac{\gamma}{\tau}\right)^k = 0$.

Case 2. There are repeated real roots. For repeated roots, we need to examine $\lim\limits_{k\to\infty} \frac{(B_1+B_2k)\lambda^k}{\tau^k}$, where $\lambda$ is a repeated root and $B_1$ and $B_2$ are arbitrary constants. Since $\lambda$ is a root other than $\tau$ we have that $\tau > |\lambda| \Rightarrow \lim\limits_{k\to\infty} \left(\frac{\lambda}{\tau}\right)^k = 0$. Let $h = \frac{\lambda}{\tau}$.
So $\lim\limits_{k\to\infty} (B_1 + B_2k)\, h^k = \lim\limits_{k\to\infty} \frac{B_1+B_2k}{h^{-k}} = \lim\limits_{k\to\infty} \frac{B_2}{-h^{-k}\ln h} = 0$.
Therefore $\lim\limits_{k\to\infty} \frac{(B_1+B_2k)\lambda^k}{\tau^k} = 0$.

Case 3. There are a pair of complex conjugates. For complex conjugates we need to examine $\lim\limits_{k\to\infty} \frac{Ar^k\cos(k\theta+D)}{\tau^k}$, where $A$ and $D$ are arbitrary constants and $r$ is a complex root. Since $r$ is a root, we have that $\tau > |r| \Rightarrow \lim\limits_{k\to\infty} \left(\frac{r}{\tau}\right)^k = 0$.
So $\lim\limits_{k\to\infty} |\frac{Ar^k\cos(k\theta+D)}{\tau^k}| \leq \lim\limits_{k\to\infty} |\frac{Ar^k}{\tau^k}| = 0$

Therefore with $f_k = C_1\tau^k + C_2\gamma_1^k + C_3\gamma_2^k + (B_1 + B_2k)\lambda^k + Ar^k\cos(k\theta + D)$, we see that

$$\lim_{k\to\infty} \frac{f_{k+1}}{f_k} = \lim_{k\to\infty} \frac{C_1\tau^{k+1}+C_2\gamma_1^{k+1}+C_3\gamma_2^{k+1}+(B_1+B_2(k+1))\lambda^{k+1}+Ar^{k+1}\cos((k+1)\theta+D)}{C_1\tau^k+C_2\gamma_1^k+C_3\gamma_2^k+(B_1+B_2k)\lambda^k+Ar^k\cos(k\theta+D)} = \cdots = \tau$$

□

With the machinery taken care of we move on to proving the $K$-theory of the

AF-algebra generated by the standard Tribonacci sequence.

**Lemma 4.3.** *Let* $K = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. *Then*

$$K_0^+(\mathfrak{A}_K) = \left\{ \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in \mathbb{Z}^3 \mid n_1 + (\tau - 1)n_2 + (\tau^2 - \tau - 1)n_3 \geq 0 \right\}$$

*Proof.* Similarily as shown in the remark of the previous chapter, we have that

$$K_0^+(\mathfrak{A}) = \left\{ \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in \mathbb{Z}^3 \mid \exists n \geq 1 \mid K^n \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in \mathbb{Z}^3_+ \right\}$$

This is true only because $K^{-1}$ has entries in $\mathbb{Z}$. First we will show

$$K_0^+(\mathfrak{A}_K) \subseteq \left\{ \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in \mathbb{Z}^3 \mid n_1 + (\tau - 1)n_2 + (\tau^2 - \tau - 1)n_3 \geq 0 \right\}$$

.

Suppose that $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in K_0^+(\mathfrak{A}_K)$. Then there exist an $m \geq 1$ such that $K^m = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \geq 0$. Take $n$ arbitrary such that $n \geq m$. If $n = m$, we get $K^n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \geq 0$. Now suppose $n > m$ and that $k = n - m$. Then $K^n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} =$ $K^{(n-m)} \left( K^m \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \right) = K^m \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$, where $K^m = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \geq 0$.

So $K^k \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} f_k+2 & f_{k+3}-f_{k+2} & `f_{k+4}-f_{k+3}-f_{k+2} \\ f_{k+1} & f_{k+2}-f_{k+1} & f_{k+3}-f_{k+2}-f_{k+1} \\ f_k & f_{k+1}-f_k & f_{k+2}-f_{k+1}-f_k \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$

$= \begin{pmatrix} m_1 f_{k+2} + m_2(f_{k+3}-f_{k+2}) + m_3(f_{k+4}-f_{k+3}-f_{k+2}) \\ m_1 f_{k+1} + m_2(f_{k+2}-f_{k+1}) + m_3(f_{k+3}-f_{k+2}-f_{k+1}) \\ m_1 f_k + m_2(f_{k+1}-f_k) + m_3(f_{k+2}-f_{k+1}-f_k) \end{pmatrix}$

$= \begin{pmatrix} m_1 f_{k+2} + m_2(f_{k+1}+f_k) + m_3 f_{k+1} \\ m_1 f_{k+1} + m_2(f_k+f_{k-1}) + m_3 f_k \\ m_1 f_k + m_2(f_{k-1}+f_{k-2}) + m_3 f_{k-1} \end{pmatrix} \geq 0$, since $\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \geq 0$ and $\{f_k\}_{k=1} \geq 0$

for all $k \geq 0$.

So $K^n \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \geq 0$ for all $n \geq m$.

With this we see that $K^n \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$

$= \begin{pmatrix} f_{n+2} & f_{n+3}-f_{n+2} & f_{n+4}-f_{n+3}-f_{n+2} \\ f_{n+1} & f_{n+2}-f_{n+1} & f_{n+3}-f_{n+2}-f_{n+1} \\ f_n & f_{n+1}-f_n & f_{n+2}-f_{n+1}-f_n \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$

$= \begin{pmatrix} n_1 f_{n+2} + n_2(f_{n+3}-f_{n+2}) + n_3(f_{n+4}-f_{n+3}-f_{n+2}) \\ n_1 f_{n+1} + n_2(f_{n+2}-f_{n+1}) + n_3(f_{n+3}-f_{n+2}-f_{n+1}) \\ n_1 f_n + n_2(f_{n+1}-f_n) + n_3(f_{n+2}-f_{n+1}-f_n) \end{pmatrix} \geq 0.$

Hence $f_n n_1 + (f_{n+1}-f_n)n_2 + (f_{n+2}-f_{n+1}-f_n)n_3 \geq 0$.

So $n_1 + \left(\frac{f_{n+1}}{f_n} - 1\right) n_2 + \left(\frac{f_{n+2}}{f_n} - \frac{f_{n+1}}{f_n} - 1\right) n_3 \geq 0$.

Taking limits we get that $n_1 + (\tau - 1)n_2 + (\tau^2 - \tau - 1)n_3 \geq 0$.

Therefore $K_0^+(\mathfrak{A}_K) \subseteq \left\{ \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in \mathbb{Z}^3 \mid n_1 + (\tau-1)n_2 + (\tau^2-\tau-1)n_3 \geq 0 \right\}$.

Now we will show

$$K_0^+(\mathfrak{A}_K) \supseteq \left\{ \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in \mathbb{Z}^3 \mid n_1\tau^{-2} + n_2(\tau^{-1} - \tau^{-2}) + n_3(1 - \tau^{-1} - \tau^{-2}) \geq 0 \right\}.$$

Suppose $\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \in \mathbb{Z}^3$ such that $m_1 + (\tau - 1)m_2 + (\tau^2 - \tau - 1)m_3 \geq 0$. If $\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \in K_0^+(\mathfrak{A}_K)$. If $m_1$, $m_2$, and $m_3$ are not all zero, then by Theorem C, the set $\{1, \tau, \tau^2\}$ is linearly independent over $\mathbb{Q}$ since $x^3 - x^2 - x - 1$ is irreducible over $\mathbb{Q}$. This implies that $m_1 + (\tau - 1)m_2 + (\tau^2 - \tau - 1)m_3 > 0$. Hence, there exist an $n_0 > 0$ such that for all $n \geq n_0 > 0$, we have that $m_1 + \left(\frac{f_{n+1}}{f_n} - 1\right) m_2 + \left(\frac{f_{n+2}}{f_n} - \frac{f_{n+1}}{f_n} - 1\right) m_3 > 0$.

$\Rightarrow f_n m_1 + (f_{n+1} - f_n)m_2 + (f_{n+2} - f_{n+1} - f_n)m_3 > 0$

$$\Rightarrow \begin{pmatrix} m_1 f_{n+2} + m_2(f_{n+3} - f_{n+2}) + m_3(f_{n+4} - f_{n+3} - f_{n+2}) \\ m_1 f_{n+1} + m_2(f_{n+2} - f_{n+1}) + m_3(f_{n+3} - f_{n+2} - f_{n+1}) \\ m_1 f_n + m_2(f_{n+1} - f_n) + m_3(f_{n+2} - f_{n+1} - f_n) \end{pmatrix} > 0$$

$\Rightarrow K^n \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} > 0$. So $\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \in K_0^+(\mathfrak{A}_K)$.

Therefore, $K_0^+(\mathfrak{A}_K) = \left\{ \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in \mathbb{Z}^3 \mid n_1 + (\tau - 1)n_2 + (\tau^2 - \tau - 1)n_3 \geq 0 \right\}$ $\square$

**Lemma 4.4.** *Let* $K = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, *$\tau$ be the Perron-Frobenius eigenvalue of $K$ and* $\mathbb{F} = \mathbb{Z} + (\tau - 1)\mathbb{Z} + (\tau^2 - \tau - 1)\mathbb{Z}$. *Then*

$$\left(K_0\left(\mathfrak{A}_K\right), K_0^+\left(\mathfrak{A}_K\right), \begin{pmatrix}1\\0\\0\end{pmatrix}\right) \simeq (\mathbb{F}, \mathbb{F}\cap\mathbb{R}_+, 1)$$

*Proof.* Let $\theta : K_0(\mathfrak{A}_K) \longrightarrow \mathbb{Z} + (\tau-1)\mathbb{Z} + (\tau^2-\tau-1)\mathbb{Z}$ with $f\left(\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}\right) =$ $n_1+(\tau-1)n_2+(\tau^2-\tau-1)n_3$. So we check that its a homomorphism. $f\left(\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}\right) +$ $f\left(\begin{pmatrix}m_1\\m_2\\m_3\end{pmatrix}\right) = n_1+(\tau-1)n_2+(\tau^2-\tau-1)n_3+m_1+(\tau-1)m_2+(\tau^2-\tau-1)m_3 =$ $(n_1+m_1)+(\tau-1)(n_2+m_2)+(\tau^2-\tau-1)(n_3+m_3) = \theta\left(n_1+m_1n_2+m_2n_3+m_3\right)$ So $f$ is a homomorphism.

Next we check that $f$ is 1-1.

$ker\ f = \left\{\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix} \mid f\left(\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}\right) = 0\right\} =$ $\left\{\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix} \mid n_1+(\tau-1)n_2+(\tau^2-\tau-1)n_3 = 0\right\}$. By Theorem C, we have that the set $\{1,\tau,\tau^2\}$ are linearly independent so the only way $n_1+(\tau-1)n_2+(\tau^2-\tau-1)n_3 =$ $0$ is if $n_1 = n_2 = n_3 = 0$. Hence the kernel of $f$ is $f = \begin{pmatrix}0\\0\\0\end{pmatrix}$. So $f$ is 1-1.

Next check that $f$ is onto.

Let $m \in \mathbb{F}$. Then there exists an $a$, $b$, and $c \in \mathbb{Z}$ such that $m = a+(\tau-1)b+(\tau^2-$

$\tau - 1)c$. So $f\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = a + (\tau-1)b + (\tau^2-\tau-1)c = m$. So $f$ is onto. Therefore $f$ is a bijection.

Lets examine the positive cone, $K_0^+(\mathfrak{A}_K)$. Let $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in K_0^+(\mathfrak{A}_K)$. Then from our previous Lemma, $\theta\left(\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}\right) = n_1 + (\tau-1)n_2 + (\tau^2-\tau-1)n_3 \geq 0$. But this is exactly the set $\mathbb{F} \cap \mathbb{R}_+$.

So $f(K_0^+(\mathfrak{A}_K)) = \mathbb{F} \cap \mathbb{R}_+$.

Next we need to check the order unit goes to the order unit. So $f\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) =$ $1 + (\tau-1)(0) + (\tau^2-\tau-1)(0) = 1$. So we have that the order unit goes to the order unit.

Therefore $\left(K_0(\mathfrak{A}_K), K_0^+(\mathfrak{A}_K), \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) \cong$ $(\mathbb{F}, \mathbb{F} \cap \mathbb{R}_+, 1)$. $\square$

We will now look at the relation between the standard Tribonacci algebra and the AF $C^*$-algebra whose incidence matrix is $J = K^T$. These are a part of the family of algebras that Bratteli and Jorgensen studied in [1]. First, we will start with looking at the $K$-theory of the algebra generated by the transpose.

**Lemma 4.5.** *Let* $J = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. *Then*

$$K_0^+(\mathfrak{A}_J) = \left\{ \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in \mathbb{Z}^3 \mid n_1 + \tau^{-1} n_2 + \tau^{-2} n_3 \geq 0 \right\}$$

*Proof.* By our definition of $K_0^+(\mathfrak{A}_J)$, we have that

$$K_0^+(\mathfrak{A}_J) = \left\{ \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in \mathbb{Z}^3 \mid \exists n \geq 0 \mid J^n \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in \mathbb{Z}_+^3 \right\}$$

First we will show $K_0^+(\mathfrak{A}_J) \subseteq \left\{ \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in \mathbb{Z}^3 \mid n_1 + \tau^{-1} n_2 + \tau^{-2} n_3 0 \right\}$.

By similar resoning as for $K_0^+(\mathfrak{A}_K)$ if $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in K_0^+(\mathfrak{A}_J)$ there exist $m$ such that

$J^m \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \geq 0 \Rightarrow J^n \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \geq 0$ for all $n \geq m$.

With this we see that

$$J^n \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} f_{n+2} & f_{n+1} & f_n \\ f_{n+3} - f_{n+2} & f_{n+2} - f_{n+1} & f_{n+1} - f_n \\ f_{n+4} - f_{n+3} - f_{n+2} & f_{n+3} - f_{n+2} - f_{n+1} & f_{n+2} - f_{n+1} - f_n \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$= \begin{pmatrix} f_{n+2} n_1 + f_{n+1} n_2 + f_n n_3 \\ (f_{n+3} - f_{n+2}) n_1 + (f_{n+2} - f_{n+1}) n_2 + (f_{n+1} - f_n) n_3 \\ (f_{n+4} - f_{n+3} - f_{n+2}) n_1 + (f_{n+3} - f_{n+2} - f_{n+1}) n_2 + (f_{n+2} - f_{n+1} - f_n) n_3 \end{pmatrix} \geq 0$$

$\Rightarrow f_{n+2} n_1 + f_{n+1} n_2 + f_n n_3 \geq 0$

$\Rightarrow n_1 + \left(\frac{f_{n+1}}{f_{n+2}}\right) n_2 + \left(\frac{f_n}{f_{n+2}}\right) n_3 \geq 0$

Taking limits we get that $n_1 + \tau^{-1}n_2 + \tau^{-2}n_3 \geq 0$.

Therefore $K_0^+(\mathfrak{A}_J) \subseteq \left\{ \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in \mathbb{Z}^3 \mid n_1 + \tau^{-1}n_2 + \tau^{-2}n_3 \geq 0 \right\}$.

Now we will show

$K_0^+(\mathfrak{A}_J) \supseteq \left\{ \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in \mathbb{Z}^3 \mid n_1 + \tau^{-1}n_2 + \tau^{-2}n_3 \geq 0 \right\}$.

Suppose $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in \mathbb{Z}^3$ such that $n_1 + \tau^{-1}n_2 + \tau^{-2}n_3 \geq 0$. If $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in K_0^+(\mathfrak{A}_J)$. If $n_1$, $n_2$, and $n_3$ are not all zero, then by Theorem C, the set $\{1, \tau, \tau^2\}$ is linearly independent over $\mathbb{Q}$ since $x^3 - x^2 - x - 1$ is irreducible over $\mathbb{Q}$. This implies that $n_1 + \tau^{-1}n_2 + \tau^{-2}n_3 > 0$. Hence, there exist an $n_0 > 2$ such that for all $n \geq n_0 > 2$, we have that $n_1 + \left(\frac{f_{n-1}}{f_n}\right) n_2 + \left(\frac{f_{n-2}}{f_n}\right) n_3 > 0$.

$\Rightarrow f_n n_1 + f_{n-1}n_2 + f_{n-2}n_3 > 0$.

$\Rightarrow f_{n+1}n_1 + f_n n_2 + f_{n-1}n_3 > 0$ and $f_{n+2}n_1 + f_{n+1}n_2 + f_n n_3 > 0$

$\Rightarrow f_{n+1}n_1 + f_n n_2 + f_{n-1}n_3 + f_n n_1 + f_{n-1}n_2 + f_{n-2}n_3 > 0$

$$\Rightarrow \begin{pmatrix} f_{n+2}n_1 + f_{n+1}n_2 + f_n n_3 \\ (f_{n+1} + f_n)n_1 + (f_n + f_{n-1})n_2 + (f_{n-1} + f_{n-2})n_3 \\ f_{n+1}n_1 + f_n n_2 + f_{n-1}n_3 \end{pmatrix} > 0$$

$$\Rightarrow \begin{pmatrix} f_{n+2}n_1 + f_{n+1}n_2 + f_n n_3 \\ (f_{n+3} - f_{n+2})n_1 + (f_{n+2} - f_{n+1})n_2 + (f_{n+1} - f_n)n_3 \\ (f_{n+4} - f_{n+3} - f_{n+2})n_1 + (f_{n+3} - f_{n+2} - f_{n+1})n_2 + (f_{n+2} - f_{n+1} - f_n) \end{pmatrix} > 0$$

$\Rightarrow J^n \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} > 0$. So $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in K_0^+(\mathfrak{A}_J)$.

Therefore, $K_0^+(\mathfrak{A}_J) = \left\{ \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in \mathbb{Z}^3 \mid n_1 + \tau^{-1}n_2 + \tau^{-2}n_3 \geq 0 \right\}$ $\square$

**Lemma 4.6.** *Let* $J = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, *$\tau$ be the Perron-Frobenius eigenvalue of $J$, and* $\mathbb{F} = \mathbb{Z} + \tau^{-1}\mathbb{Z} + \tau{-}2\mathbb{Z}$. *Then*
$\left( K_0(\mathfrak{A}_J), K_0^+(\mathfrak{A}_J), \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right)$
$\simeq (\mathbb{F}, )\mathbb{F} \cap \mathbb{R}_+, 1)$

*Proof.* This proof is similar to the proof of $K$, which is the transpose except the map $g$ is $g\left(\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}\right) = n_1 + \tau^{-1}n_2 + \tau^{-2}n_2$. $\square$

**Lemma 4.7.** *Let* $K = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ *and* $J = K^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

*Then* $\left( K_0(\mathfrak{A}_K), K_0^+(\mathfrak{A}_K), \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \cong \left( K_0(\mathfrak{A}_J), K_0^+(\mathfrak{A}_J), \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right)$

*Proof.* From previous Propositions we have seen that

$K_0^+(\mathfrak{A}_J)(\mathbb{Z} + \tau^{-1}\mathbb{Z} + \tau^{-2}\mathbb{Z}) \cap \mathbb{R}_+$ and

$K_0^+(\mathfrak{A}_K) \cong (\mathbb{Z} + (\tau - 1)\mathbb{Z} + (\tau^2 - \tau - 1)\mathbb{Z}) \cap \mathbb{R}_+$. Now we will show that $\mathbb{Z} + \tau^{-1}\mathbb{Z} +$

$\tau^{-2}\mathbb{Z} = \mathbb{Z} + (\tau - 1)\mathbb{Z} + (\tau^2 - \tau - 1)\mathbb{Z})$.

$\tau$ satisfies the equation $\tau^3 - \tau^2 - \tau - 1 = 0$. With a little manipulation, we see that $\tau - 1 = \tau^{-1} + \tau^{-2}$ and $\tau^2 - \tau - 1 = \tau^{-1}$. This tells us that $\mathbb{Z} + (\tau - 1)\mathbb{Z} + (\tau^2 - \tau - 1)\mathbb{Z}$ $= \mathbb{Z} + (\tau^{-1} + \tau^{-2})\mathbb{Z} + \tau^1\mathbb{Z}$.

Next we will show that $\mathbb{Z} + (\tau^{-1} + \tau^{-2})\mathbb{Z} + \tau^1\mathbb{Z} = \mathbb{Z} + \tau^{-1}\mathbb{Z} + \tau^{-2}\mathbb{Z}$

Let $m \in \mathbb{Z} + (\tau^{-1} + \tau^{-2})\mathbb{Z} + \tau^1\mathbb{Z}$. Then there exist $m_1, m_2, m_3 \in \mathbb{Z}$ such that $m = m_1 + (\tau^{-1} + \tau^{-2})m_2 + \tau^1 m_2 = m_1 + \tau^{-1}(m_2 + m_3) + \tau^{-2}m_2 \in \mathbb{Z} + \tau^{-1}\mathbb{Z} + \tau^{-2}$. Let $n \in \mathbb{Z} + \tau^{-1}\mathbb{Z} + \tau^{-2}\mathbb{Z}$. Then there exist $n_1, n_2, n_3 \in \mathbb{Z}$ such that $n = n_1 + \tau^{-1}n_2 + \tau^{-2}n_3$. Select $m_1 = n_1, m_2 = n_3$, and $m_3 = n_2 - m_2$. So $n = n_1 + \tau^{-1}n_2 + \tau^{-2}n_3 = m_1 + \tau^{-1}(m_2 + m_3) + \tau^{-2}m_2 = m_1 + (\tau^{-1} + \tau^{-2})m_2 + \tau^{-1}m_3$.

So $\mathbb{Z} + (\tau^{-1} + \tau^{-2})\mathbb{Z} + \tau^1\mathbb{Z} = \mathbb{Z} + \tau^{-1}\mathbb{Z} + \tau^{-2}\mathbb{Z}$. We have the same groups with the same positive cone, and the order units are inherited from $\mathbb{R}$ since they are subgroups of $\mathbb{R}$.

Therefore $\left(K_0\left(\mathfrak{A}_J\right), K_0^+\left(\mathfrak{A}_J\right), \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) \simeq \left(K_0\left(\mathfrak{A}_K\right), K_0^+\left(\mathfrak{A}_K\right), \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$. $\square$

## 4.2 $n$-nacci algebra

The Fibonacci sequence of order $n$, known as the $n$-nacci sequence will be covered next. The standard $n$-nacci sequence is defined similarly to the standard Tribonacci sequence except that the next term depends on the previous n terms. So we have $f_{n+1} = \sum_{i=0}^{k-1} f_{n-i}$ where $f_0 = f_1 = \cdots = f_{n-2} = 0$ and $f_{n-1} = 1$. The incidence matrix that goes along with the standard $n$-nacci sequence is $A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$ with a characteristic polynomial of $x^n - x^{n-1} - \ldots - x - 1$. We will start with similar machinery used in the $3 \times 3$ case.

**Lemma 4.8.** *Let*

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

*a $k \times k$ matrix and let $f_{n+1} = \sum_{i=0}^{k-1} f_{n-i}$ be the Fibonacci sequence where $f_0 = f_1 = \ldots = f_{k-2} = 0$ and $f_{k-1} = 1$. Then*

$$A^n = \begin{pmatrix} f_{n+k-1} & f_{n+k} - f_{n+k-1} & \cdots & f_{n+2k-3} - \sum_{i=k-1}^{2k-4} f_{n+i} & f_{n+2k-2} - \sum_{i=k-1}^{2k-3} f_{n+i} \\ f_{n+k-2} & f_{n+k-1} - f_{n+k-2} & \cdots & f_{n+2k-4} - \sum_{i=k-2}^{2k-5} f_{n+i} & f_{n+2k-3} - \sum_{i=k-2}^{2k-4} f_{n+i} \\ f_{n+k-3} & f_{n+k-2} - f_{n+k-3} & \cdots & f_{n+2k-5} - \sum_{i=k-3}^{2k-6} f_{n+i} & f_{n+2k-4} - \sum_{i=k-3}^{2k-5} f_{n+i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_n & f_{n+1} - f_n & \cdots & f_{n+k-2} - \sum_{i=0}^{k-3} f_{n+i} & f_{n+k-1} - \sum_{i=0}^{k-2} f_{n+i} \end{pmatrix}$$

*Proof.* $A^1 = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} f_k & f_{k+1}-f_k & \cdots & f_{2k-2}-\sum\limits_{i=k-1}^{2k-4} f_{i+1} & f_{2k-1}-\sum\limits_{i=k-1}^{2k-3} f_{i+1} \\ f_{k-1} & f_k-f_{k-1} & \cdots & f_{2k-3}-\sum\limits_{i=k-2}^{2k-5} f_{i+1} & f_{2k-2}-\sum\limits_{i=k-2}^{2k-4} f_{i+1} \\ f_{k-2} & f_{k-3}-f_{k-2} & \cdots & f_{2k-4}-\sum\limits_{i=k-3}^{2k-6} f_{i+1} & f_{2k-3}-\sum\limits_{i=k-3}^{2k-5} f_{i+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_1 & f_2-f_1 & \cdots & f_{k-1}-\sum\limits_{i=0}^{k-3} f_{i+1} & f_k-\sum\limits_{i=0}^{k-2} f_{i+1} \end{pmatrix}$. So its true for $n = 1$.

So assume its true for $m$, that is

$$A^m = \begin{pmatrix} f_{m+k-1} & f_{m+k}-f_{m+k-1} & \cdots & f_{m+2k-2}-\sum\limits_{i=k-1}^{2k-3} f_{m+i} \\ f_{m+k-2} & f_{m+k-1}-f_{m+k-2} & \cdots & f_{m+2k-3}-\sum\limits_{i=k-2}^{2k-4} f_{m+i} \\ f_{m+k-3} & f_{m+k-2}-f_{m+k-3} & \cdots & f_{m+2k-4}-\sum\limits_{i=k-3}^{2k-5} f_{m+i} \\ \vdots & \vdots & \ddots & \vdots \\ f_m & f_{m+1}-f_m & \cdots & f_{m+k-1}-\sum\limits_{i=0}^{k-2} f_{m+i} \end{pmatrix}$$

As in the $3 \times 3$ case we will represent the matrix as a column vector where the $j$ represents the $j^{th}$ column of the matrix. So

$$A^m = \begin{pmatrix} f_{m+k-1} & f_{m+k}-f_{m+k-1} & \cdots & f_{m+2k-2}-\sum\limits_{i=k-1}^{2k-3} f_{m+i} \\ f_{m+k-2} & f_{m+k-1}-f_{m+k-2} & \cdots & f_{m+2k-3}-\sum\limits_{i=k-2}^{2k-4} f_{m+i} \\ f_{m+k-3} & f_{m+k-2}-f_{m+k-3} & \cdots & f_{m+2k-4}-\sum\limits_{i=k-3}^{2k-5} f_{m+i} \\ \vdots & \vdots & \ddots & \vdots \\ f_m & f_{m+1}-f_m & \cdots & f_{m+k-1}-\sum\limits_{i=0}^{k-2} f_{m+i} \end{pmatrix}$$

$$= \begin{pmatrix} f_{m+(j+k-2)} - \sum_{i=0}^{j-2} f_{n+(j+k-3-i)} \\ f_{m+(j+k-3)} - \sum_{i=0}^{j-2} f_{n+(j+k-4-i)} \\ \vdots \\ f_{m+(j-1)} - \sum_{i=0}^{j-2} f_{n+(j-i-2)} \end{pmatrix}$$

Then $A \cdot A^m = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} f_{m+(j+k-2)} - \sum_{i=0}^{j-2} f_{n+(j+k-3-i)} \\ f_{m+(j+k-3)} - \sum_{i=0}^{j-2} f_{n+(j+k-4-i)} \\ \vdots \\ f_{m+(j-1)} - \sum_{i=0}^{j-2} f_{n+(j-i-2)} \end{pmatrix}$

$$= \begin{pmatrix} f_{m+(j+k-2)} - \sum_{i=0}^{j-2} \left(f_{m+(j+k-3-i)}\right) + \ldots + f_{m+j-1} - \sum_{i=0}^{j-2} \left(f_{n+(j-i-2)}\right) \\ f_{m+(j+k-2)} - \sum_{i=0}^{j-2} \left(f_{m+(j+k-3-i)}\right) \\ \vdots \\ f_{m+j} - \sum_{i=0}^{j-2} \left(f_{n+(j-i-1)}\right) \end{pmatrix}$$

$$= \begin{pmatrix} f_{m+(j+k-1)} - \sum_{i=0}^{j-2} \left(f_{m+(j+k-3-i)} + f_{m+(j+k-4-i)} + \ldots + f_{m+(j-i-2)}\right) \\ f_{m+(j+k-2)} - \sum_{i=0}^{j-2} \left(f_{m+(j+k-3-i)}\right) \\ \vdots \\ f_{m+j} - \sum_{i=0}^{j-2} \left(f_{n+(j-i-1)}\right) \end{pmatrix}$$

$$= \begin{pmatrix} f_{m+(j+k-1)} - \sum_{i=0}^{j-2} \left(f_{m+(j+k-2-i)}\right) \\ f_{m+(j+k-2)} - \sum_{i=0}^{j-2} \left(f_{m+(j+k-3-i)}\right) \\ \vdots \\ f_{m+j} - \sum_{i=0}^{j-2} \left(f_{n+(j-i-1)}\right) \end{pmatrix} = A^{m+1}$$

By induction it is true for all $n$. $\square$

As we can see from the previous Lemma, the $m^{th}$ power of the matrix could not have been gotten from the $2 \times 2$ case. The $3 \times 3$ was sufficient to get the $m^{th}$

power. We can also do a similar calculation of the Perron-Frobenius eigenvalue, $\tau$ as for the $3 \times 3$ case.

**Lemma 4.9.** *Let $\tau$ be the Perron-Frobenius eigenvalue of*

$$A = \begin{pmatrix} m_1 & m_2 & \cdots & m_{n-1} & m_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

*and let $f_{k+1} = \sum_{i=0}^{k-1} f_{n-i}$ be the n-parameter n-nacci sequence where $f_0 = f_1 = \cdots = f_{n-2} = 0$ and $f_{n-1} = 1$. Then $\tau = \lim_{n\to\infty} \frac{f_{n+1}}{f_n}$*

*Proof.* This follows from the $3 \times 3$ case of the same problem. $\square$

With the machinery taken care of we move on to proving the $K$-theory of the AF-algebra generated by the standard n-nacci sequence.

**Lemma 4.10.** *Let $K = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$ and $\tau$ be the Perron-Frobenius eigenvalue of $K$, where $K$ is an $k \times k$ matrix. Then*

$$K_0^+(\mathfrak{A}_K) = \left\{ \begin{pmatrix} m_1 \\ \vdots \\ m_k \end{pmatrix} \in \mathbb{Z}^k \mid m_1 + (\tau - 1)m_2 + \ldots + (\tau^{k-1} - \sum_{i=2}^{k} \tau^{k-i})m_k \geq 0 \right\}$$

*Proof.* This follows from the $3 \times 3$ case of the same problem.

$\square$

**Lemma 4.11.** *Let* $K = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$, *$\tau$ be the Perron-Frobenius eigenvalue, and* $\mathbb{F} = \mathbb{Z} + (\tau - 1)\mathbb{Z} + \ldots + (\tau^{k-1} - \sum_{i=2}^{k} \tau^{k-i})\mathbb{Z}$. *Then*

$$\left( K_0 \left( \mathfrak{A}_K \right), K_0^+ \left( \mathfrak{A}_K \right), \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \simeq \left( \mathbb{F}, \mathbb{F} \cap \mathbb{R}_+, 1 \right)$$

*Proof.* This follows from the $3 \times 3$ case of the same problem. $\square$

We will now look at the relation between the standard n-nacci algebra and its transpose, which is part of the family of algebras that Bratteli and Jorgensen studied in [1]. First, we will start with looking at the $K$-theory of the algebra generated by the transpose.

**Lemma 4.12.** *Let* $J = K^T = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}^T$. *Then*

$$K_0^+ \left( \mathfrak{A}_J \right) = \left\{ \begin{pmatrix} n_1 \\ \vdots \\ n_k \end{pmatrix} \in \mathbb{Z}^k \mid \sum_{i=1}^{k-1} \tau^{-i+1} n_i \geq 0 \right\}$$

*Proof.* This follows from the $3 \times 3$ case of the same problem. $\square$

**Lemma 4.13.** *Let* $J = K^T = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}^T$ *,* $\tau$ *be the Perron-Frobenius eigenvalue of* $J$ *and* $\mathbb{F} = \sum_{i=0}^{k-1} \tau^{-i}\mathbb{Z}$*. Then*

$$\left( K_0\left(\mathfrak{A}_J\right), K_0^+\left(\mathfrak{A}_J\right), \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \simeq \left(\mathbb{F}, \mathbb{F} \cap \mathbb{R}_+, 1\right)$$

*Proof.* This proof is similar to the proof of $K$, which is the transpose except the map $g$ is $g = \left( \begin{pmatrix} n_1 \\ \vdots \\ n_k \end{pmatrix} \right) = \sum_{i=1}^{k-1} \tau^{-i+1} n_i$. □

**Lemma 4.14.** *Let* $K = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$ *and* $J = K^T$*.*

*Then* $\left( K_0\left(\mathfrak{A}_K\right), K_0^+(\mathfrak{A}_K), \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) \cong \left( K_0(\mathfrak{A}_J), K_0^+(\mathfrak{A}_J), \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right)$

*Proof.* This follows from the $3 \times 3$ case of the same problem. □

# APPENDIX
# BASICS

In this appendix we offer relevant definitions and review some classical $C^*$-algebra theory results. We do this so that the casual reader may reference any terminology with which he or she may be unfamiliar. It also serves the purpose of making this document more self-contained. Some of the following appears in the main body of this thesis; we will not worry about repeating anything in favor of continuity for the reader.

## A.1 $C^*$-algebra

We begin with some very basic definitions and theorems working toward .

**Theorem A.1.** *Every finite dimensional $C^*$-algebra $\mathfrak{A}$ is $*$-isomorphic to the direct sum of full matrix algebras*

$$\mathfrak{A} \cong M_{n_1} \bigoplus \cdots \bigoplus M_{n_k}.$$

*In particular, every finite dimensional $C^*$-algebra is unital.*

**Theorem A.2.** *Let $\mathfrak{A} = \lim\limits_{\rightarrow}(A_n, \varphi_n)$ be an AF algebra. We then have that $\mathfrak{A}$ is $*$-isomorphic to an AF-algebra $\mathfrak{B} = \lim\limits_{\rightarrow}(B_k, \sigma_k)$ where $B_k$ is a direct sum of full matrix algebras over $\mathbb{C}$, i.e. $B_k = M_{n_1} \bigoplus \cdots \bigoplus M_{n_k}$*

**Theorem A.3** (Elliott's Classification Theorem)**.** *Two unital AF algebras $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic if and only if the triples $(K_0(\mathfrak{A}), K_0^+(\mathfrak{A}), [1_{\mathfrak{A}}])$ and $(K_0(\mathfrak{B}), K_0^+(\mathfrak{B}), [1_{\mathfrak{B}}])$ are isomorphic, in other words, if and only if there is a group isomorphism $\alpha : K_0(\mathfrak{A}) \rightarrow K_0(\mathfrak{B})$ that satisfies $\alpha(K_0^+(\mathfrak{A})) = K_0^+(\mathfrak{B})$ and $\alpha([1_{\mathfrak{A}}]) = [1_{\mathfrak{B}}]$. Moreover, for any such group isomorphism $\alpha$ there is a $*$-isomorphism $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ with $K_0(\varphi) = \alpha$.*

### A.2 K-theory

Here is the K-theory of some $C^*$-algebras of interest. $K_0(\mathbb{C}) = \mathbb{Z}$
$K_0(M_n(\mathbb{C})) = \mathbb{Z}$

### A.3 Difference Equation

Let $y_{k+n} + a_1 y_{k+n-1} + \cdots + a_{n-1} y_{k+1} + a_n y_k = 0$ be the homogeneous difference equation of order $n$. The auxiliary equation or this homogeneous system is $m^n + a_1 m^{n-1} + \ldots + a_{n-1} m + a_n = 0$, which is an algebraic equation of degree $n$ with exactly $n$ roots. These roots may be real or complex numbers, and any root may be distinct from the others or may be repeated. The fundamental set of solutions is obtained as follows.

(i) For each real unrepeated root $m$, we write the solution

$$C_1 m^k$$

where $C_1$ is an arbitrary constant.

(ii) If a real root $m$ is repeated $p$ times, we write the solution

$$\left(C_1 + C_2 k + C_3 k^2 + \cdots + C_p k^{p-1}\right) m^k$$

where $C_i$ is an arbitrary constant for $1 \leq i \leq p$.

(iii) For each pair of unrepeated complex conjugate roots with modulus $r$ and amplitude $\theta$, we write the solution

$$A r^k \cos\left(k\theta + B\right)$$

where $A, B$ are arbitrary constants.

(iv) If a pair of complex conjugate roots is repeated $p$ times, write the solution as

$$r^k \left[A_1 \cos(k\theta + B_1) + A_2 k \cos(k\theta + B_2) + \cdots + A_p k^{p-1} \cos(k\theta + B_p)\right]$$

where $A_i$ and $B_i$ are arbitrary constants for $1 \leq i \leq p$.

Now taking sums of solutions found in (i)-(iv) gives the general solution of the homogeneous solution. $(y_k)$

Here is an example of how this works for further illustration.

The third order difference equation

$$y_{k+3} - 9y_{k+2} + 26y_{k+1} - 24y_k = 0$$

has the auxiliary equation

$$m^3 + -9m^2 + 26m - 24 = 0$$

This equation can be factored into

$$(m-2)(m-3)(m-4) = 0$$

So the roots are 2, 3, and 4.

The general solution of the homogeneous equation is given by

$$y_k = C_1 2^k + C_2 3^k + C_3 4^k$$

This completes the basic background material for this thesis.

## REFERENCES

[1] O. Bratteli and P.E.T. Jorgensen. Representation theory and numerical af-invariants. *Mem. Amer. Math. Soc.*, 168(797), 2004.

[2] A. Brauer. On algebraic equations with all but one root in the interior of the unit circle. *Math. Nachrichten 4*, (pp. 250-257), 1950/51.

[3] A. Connes. *Noncommutative Geometry.* Academic Press, San Diego, CA, 1994.

[4] K.R. Davidson. *$C^*$-Algebras by example.* Amer. Math. Soc., Providence, R.I., 1996. Fields Institute Monographs.

[5] G.A. Elliott. On the classifcation of inductive limits of sequences of semisimple finite-dimensional algebras. *J. Algebra*, (pp. 29-44), 1976.

[6] S. Goldberg. *Introduction to Difference Equations.* John Wiley & Sons Inc., Mineola, NY, 1986.

[7] R. Lidl. *Finite Fields.* Cambridge University Press, Cambridge, U.K., 1997.

[8] M. Rørdam and N.J. Laustsen F. Larsen. *An Introduction to $K$-theory for $C^*$-algebras.* Cambridge University Press, New York, NY, 2000. London Mathematical Society.

[9] O.F.G. Schilling and W.S. Piper. *Basic Abstract Algebra.* Allyn and Bacon Inc., Boston, MA, 1975.