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Superposition of zeros of automorphic L-functions and functoriality

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SUPERPOSITION OF ZEROS OF AUTOMORPHIC L-FUNCTIONS AND FUNCTORIALITY

by

Timothy Lee Gillespie

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

July 2011

Thesis Supervisor: Professor Yangbo Ye

ABSTRACT

In this paper we deduce a prime number theorem for the L-function $L(s, AI_{E/\mathbb{Q}}(\pi) \times AI_{F/\mathbb{Q}}(\pi'))$ where π and π' are automorphic cuspidal representations of GL_n/E and GL_m/F, respectively, with E and F solvable algebraic number fields with a Galois invariance assumption on the representations. Here $AI_{E/\mathbb{Q}}$ denotes the automorphic induction functor. We then use the proof of the prime number theorem to compute the n-level correlation function of a product of L-functions defined over cyclic algebraic number fields of prime degree.

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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ABSTRACT

In this paper we deduce a prime number theorem for the L-function $L(s, AI_{E/\mathbb{Q}}(\pi) \times AI_{F/\mathbb{Q}}(\pi'))$ where π and π' are automorphic cuspidal representations of GL_n/E and GL_m/F, respectively, with E and F solvable algebraic number fields with a Galois invariance assumption on the representations. Here $AI_{E/\mathbb{Q}}$ denotes the automorphic induction functor. We then use the proof of the prime number theorem to compute the n-level correlation function of a product of L-functions defined over cyclic algebraic number fields of prime degree.

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CHAPTER 1
INTRODUCTION

1.1 n-Level Correlation

The n-level correlation of nontrivial zeroes of automorphic L-functions was first computed by Rudnick and Sarnak [34] for an automorphic L-function attached to a cuspidal representation π of $GL_m(\mathbb{A}_{\mathbb{Q}})$, based on early work of Montgomery [15], Odlyzko [26] [27], and Hejhal [10]. In [22], [23] Liu and Ye computed the n-level correlation of nontrivial zeroes of the product $L(s,\pi_1)...L(s,\pi_k)$ of L-functions, where $\{\pi_j\}_{j=1,...,k}$ are cuspidal representations of $GL_{m_j}(\mathbb{A}_{\mathbb{Q}})$ and proved that this distribution follows the superposition of Gaussian Unitary Ensemble (GUE) models of individual L-functions and products of lower rank GUE's. These results are unconditional when $m_1,...,m_k$ are small and are under Hypothesis H in other cases. In [20] Liu and Ye also computed the n-level correlation of zeros of $L(s,\pi)$ for π being a cuspidal representation of $GL_m(\mathbb{A}_E)$ where E is a Galois extension of $\mathbb{Q}$.

Aside from giving insight into the spectral nature of the zeros of such a product, in the sense that the zeroes correspond in some way to the eigenvalues of a symmetric operator, we may also use the n-level correlation to shed light on the factorization of (primitive) L-functions defined over number fields into products of L-functions (primitive) over intermediate fields, as has already been demonstrated in [20] in the special case when $F_j = \mathbb{Q}$ for $j = 1,...,k$. All of our computations are done for test functions $f(x_1,...,x_n)$ whose Fourier transform $\Phi(\xi_1,...,\xi_n)$ has a restricted support, this is important for the possible application of the n-level correlation of zeros to the distribution of primes.

First off, let π be an automorphic cuspidal representation of $GL_m(\mathbb{A}_{\mathbb{Q}})$ with unitary central character. Let $\rho^{(\pi)} = 1/2 + i\gamma^{(\pi)}$ be a nontrivial zero of $L(s,\pi)$, here nontrivial means $\rho^{(\pi)}$ does not coincide with a pole coming from a gamma factor attached to $L(s,\pi)$ (in other words $\rho^{(\pi)}$ is a zero of the complete L-function $\Lambda(s,\pi)$ attached to π-see the next section for more details). Without assuming the Riemann Hypothesis for $L(s,\pi)$, $\gamma^{(\pi)}$ is a complex number; moreover, if we let $\gamma^{(\pi)} = \sigma + it$ then $\rho^{(\pi)} = \frac{1}{2} - t + i\sigma$, so in particular we have $Im(\rho^{(\pi)}) = Re(\gamma^{(\pi)})$ and the Riemann Hypothesis claims that in any case $t = 0$. Using the functional equation of the complete L-function

$$\Lambda(s,\pi) = \epsilon(s,\pi)\Lambda(1-s,\tilde{\pi})$$

where $\tilde{\pi}$ denotes the representation contragredient to π, we may order the nontrivial zeros $\rho^{(\pi)} = 1/2 + i\gamma^{(\pi)}$ with multiplicities by

$$.. \le Re(\gamma_{-2}^{(\pi)}) \le Re(\gamma_{-1}^{(\pi)}) < 0 \le Re(\gamma_1^{(\pi)}) \le Re(\gamma_2^{(\pi)}) \le ...$$

Also, the number of $\gamma^{(\pi)}$ with $T < Re(\gamma^{(\pi)}) \le T+1$ is asymptotically $m\log(T)/2\pi$, so the normalization $\tilde{\gamma}_j^{(\pi)} = m\gamma_j^{(\pi)}(\log|\gamma_j^{(\pi)}|)/2\pi$ will have unit mean spacing with respect to the real part. More precisely

$$Re(\tilde{\gamma}_j^{(\pi)}) \sim j \text{ as } j \to \infty$$

The n-level correlation function measures the correlation between differences of n zeroes from a collection $B_N = \{\tilde{\gamma}_1, ..., \tilde{\gamma}_N\}$. In other words, if we let $Q = \{\mathbf{x} \in$

$\mathbb{R}^{n-1} | a_i \leq x_i \leq b_i$ for $1 \leq i \leq n-1\}$, then set

$$R_n(B_N, Q) = \frac{1}{N} |\{j_1, ..., j_n \leq N \text{ distinct } : (\tilde{\gamma}_{j_1} - \tilde{\gamma}_{j_2}, ..., \tilde{\gamma}_{j_{n-1}} - \tilde{\gamma}_{j_n}) \in Q\}|$$

We would like to determine the asymptotic behavior of $R_n(B_N, Q)$ as $N \to \infty$. A more tractable problem is to consider the sum

$$R_n(B_N, f) = \frac{n!}{N} \sum_{\substack{S \subset B_N \\ |S|=n}} f(S)$$

for a suitable test function $f(x_1, ..., x_n)$ defined on $\mathbb{C}^n$ and smooth on $\mathbb{R}^n$. Here $f(S) = f(a_1, ..., a_n)$ if $S = \{a_1, ..., a_n\}$ and we also assume the test function $f(x)$ satisfies

TF1.$f(x_1, ..., x_n)$ is symmetric

TF2.$f(x_1 + t, ..., x_n + t) = f(x_1, ..., x_n)$ for any $t \in \mathbb{R}$

TF3.$f(x) \to 0$ rapidly as $|x| \to \infty$ in the hyperplane $\sum_j x_j = 0$

TF2 gives $f(x)$ is a function of the successive differences and **TF1** justifies the notation $f(S)$ above. Condition **TF3** localizes the sum outside of the hyperplane $\sum_j x_j = 0$ so we may think of $R_n(B_N, f)$ as a measurement of the number of clusters of size n in B_N. Under a technical Hypothesis **H**, the Riemann Hypothesis, and for test functions f whose Fourier transform $\hat{f}(\xi)$ is supported in $\{\xi \in \mathbb{R}^n | \sum_{i=1}^n |\xi_i| < 2/m\}$, Rudnick and Sarnak proved the universal asymptotic formula for the high nontrivial zeroes of $L(s, \pi)$ [34]

$$R_n(B_N, f) \to \int_{\mathbb{R}^n} f(x) W_n(x) \delta\left(\frac{x_1 + ... + x_n}{n}\right) dx_1 ... dx_n$$

as $N \to \infty$. Here the density function $W_n(x) = W_n(x_1, ..., x_n)$ is given by

$$W_n(x) = \det(K(x_i - x_j))$$

with $K(x) = \frac{\sin \pi x}{\pi x}$ for $x \neq 0$, $K(x) = 1$ for $x = 0$, and $\delta(x)$ denotes the dirac mass at zero. This is the same density function as discovered by Dyson [6] when he determined the limiting behavior of the n-level correlation for the Gaussian Unitary Ensemble (GUE) from random matrix theory (see [28]). If $L = m \log T$ and h is an entire rapidly decreasing function on $\mathbb{C}$, in [34] Rudnick and Sarnak first compute an asymptotic formula for the smoothly weighted sums without assuming the Riemann Hypothesis

$$R_n(T, f, h) = \sum_{j_1, \ldots, j_n}' h(\frac{\gamma_{j_1}}{T}) ... h(\frac{\gamma_{j_n}}{T}) f\left(\frac{L}{2\pi}\gamma_{j_1}, ..., \frac{L}{2\pi}\gamma_{j_n}\right)$$

here the prime denotes the sum over distinct indices. They then assume RH to remove the smoothness restriction on h and choose it to be the characteristic function of an interval, thereby proving the formula for $R_n(B_N, f)$. Note that the smooth cutoff function h forces the restriction $|\gamma_{j_i}| \ll T$, and this also gives that the two different normalizations $\frac{L}{2\pi}\gamma$ and $\tilde{\gamma} = \frac{m}{2\pi}\gamma \log |\gamma|$ will produce the same main term in the expansion of $R_n(T, f, h)$. In [18] Liu and Ye computed the n-level correlation of nontrivial zeroes of a product $L(s, \pi) = L(s, \pi_1) ... L(s, \pi_k)$ with $\{\pi_i\}_{i=1,\ldots,k}$ distinct cuspidal automorphic representations of $GL_m(\mathbb{A}_{\mathbb{Q}})$. In this case, as predicted by random matrix theory, they were able to show that the zeroes of distinct primitive automorphic L-functions are uncorrelated. In particular they proved the density function attached to the correlation function is given by a superposition of GUE ensembles. More precisely, recall that a set partition $\underline{\mathrm{H}}$ of $\underline{\mathrm{N}} = (1, 2, ..., n)$ is a decomposition of $\underline{\mathrm{N}}$ into disjoint subsets $\underline{\mathrm{H}} = [H_1, ..., H_\nu]$ where $\nu = \nu(\underline{\mathrm{H}})$ is the number of subsets in $\underline{\mathrm{H}}$. Given $\underline{\mathrm{H}}$ define

$$W_n^{\underline{\mathrm{H}}}(x_1, ..., x_n) = \prod_{1 \le \ell \le \nu(\underline{\mathrm{H}})} \det(K(x_i - x_j)_{i,j \in H_\ell})$$

Again under Hypothesis H and for test functions f whose Fourier transform $\hat{f}$ is supported in $\{\xi \in \mathbb{R}^n | \sum_{i=1}^n |\xi_i| < 2/m\}$

$$\begin{aligned} &R_n(T, f, h_1, ..., h_n) \sim \\ &\frac{\kappa(\mathbf{h})}{2\pi} TL \sum_{\substack{\underline{\mathrm{H}} \\ \nu(\underline{\mathrm{H}}) \le k}} \begin{pmatrix} & k \\ \\ \nu(\underline{\mathrm{H}}) & \end{pmatrix} \int_{\mathbb{R}^n} f(x) W_n^{\underline{\mathrm{H}}}(x) \delta \left(\frac{x_1 + ... + x_n}{n} \right) dx_1 ... dx_n \end{aligned}$$

as $T \to \infty$. Where $R_n(T, f, h_1, ..., h_n)$ denotes the sum

$$\sum_{j_1, ..., j_n} h_1\left(\frac{\gamma_{j_1}}{T}\right) ... h_n\left(\frac{\gamma_{j_n}}{T}\right) f\left(\frac{L\gamma_{j_1}}{2\pi}, ..., \frac{L\gamma_{j_n}}{2\pi}\right)$$

which runs over all indices $j_1, ..., j_n$ of nontrivial zeros of $L(s, \pi) = \prod_{i=1}^k L(s, \pi_i)$ such that if γ_{j_r} and γ_{j_s} are from zeros of the same L-function $L(s, \pi_\mu)$ then the indices j_r and j_s are distinct. In this case the smooth cutoff functions are given as Fourier transforms

$$h_j(r) = \int_{\mathbb{R}} g_j(u) e^{iru} \tag{1.1}$$

with g_j a compactly supported smooth function on $\mathbb{R}$ for $j = 1, ..., n$, and finally for $\mathbf{h} = (h_1, ..., h_n)$

$$\kappa(\mathbf{h}) = \int_{\mathbb{R}} h_1(r) h_2(r) ... h_n(r) dr$$

Given a compactly supported C^1 function Φ on $\mathbb{R}^n$, we set

$$f(x) = \int_{\mathbb{R}^n} \Phi(\xi)\delta(\xi_1 + ... + \xi_n)e(-x \cdot \xi)d\xi \tag{1.2}$$

where $e(x) = e^{2\pi i x}$, $x = (x_1, ..., x_n)$, $\xi = (\xi_1, ..., \xi_n)$, and $\delta(t)$ is the Dirac mass at zero. Note that f is a smooth function on $\mathbb{R}^n$ and satisfies the identity

$$f(x_1 + t, ..., x_n + t) = f(x_1, ..., x_n)$$

for any $t \in \mathbb{R}$; moreover,

$$f(x) \to 0 \text{ rapidly as } |x| \to \infty \text{ on } \sum_j x_j = 0$$

We generalize the above results for a product of L-functions

$$L(s, \pi_1)L(s, \pi_2)...L(s, \pi_k), \tag{1.3}$$

with each π_i for $i = 1, ..., k$ an automorphic cuspidal representation on $GL_{m_i}(\mathbb{A}_{F_i})$ such that F_i is a Galois extension of $\mathbb{Q}$ that is cyclic of prime degree. In all cases we make a Galois invariance assumption on the representations with respect to the action

$$\pi_i^{\sigma_i}(g) = \pi_i(\sigma_i^{-1}(g)),$$

for $\sigma_i \in Gal(F_i/\mathbb{Q})$. Denote by $C_n(f, \mathbf{h}, T, \pi_1, ..., \pi_k)$ the sum

$$\sum_{\gamma_1, ..., \gamma_n} h_1(\frac{\hat{\gamma}_1}{T})...h_n(\frac{\hat{\gamma}_n}{T})f(\frac{L}{2\pi}\hat{\gamma}_1, ..., \frac{L}{2\pi}\hat{\gamma}_n) \tag{1.4}$$

where $L = \log(T)$, and if γ_i comes from a zero of the L-function $L(s, \pi_j)$ then $\hat{\gamma}_i = m_j\gamma_i$. Here the sum runs over the nontrivial zeros of the product in (1.3). We again compute an asymptotic formula for the n-level correlation function above as $T \to \infty$, but first we prove the following prime number theorems which give a combinatorial argument we need to compute (1.4). In particular, for π and π' coming as base change lifts of the cuspidal representations over $\mathbb{Q}$

$$\{\pi_{\mathbb{Q}} \otimes \eta^{i}_{E/\mathbb{Q}}\}_{i=0}^{\ell-1},$$

$$\{\pi'_{\mathbb{Q}} \otimes \psi^{j}_{F/\mathbb{Q}}\}_{j=0}^{\ell'-1},$$

respectively, where $\eta_{E/\mathbb{Q}}$ and $\psi_{F/\mathbb{Q}}$ are the class characters associated to the cyclic prime degree Galois extensions E and F. For future reference we let

$$T = \{(\sigma_{\mathbb{Q}}, \sigma'_{\mathbb{Q}}) | \sigma_{\mathbb{Q}} \in BC^{-1}_{E/\mathbb{Q}}(\pi), \sigma'_{\mathbb{Q}} \in BC^{-1}_{F/\mathbb{Q}}(\pi'), \sigma_{\mathbb{Q}} \cong \sigma'_{\mathbb{Q}} \otimes |\det|^{i\tau} \exists \tau \in \mathbb{R}\},$$

where $BC_{K/\mathbb{Q}}$ denotes the base change map for any number field K. First we obtain a theorem in the cyclic case

Theorem 1.1.1. *Let π and π' be unitary automorphic cuspidal representations of $GL_n(\mathbb{A}_E)$ and $GL_m(\mathbb{A}_F)$, respectively, with $E/\mathbb{Q}$ and $F/\mathbb{Q}$ of prime degrees ℓ and ℓ' respectively with $E \neq F$. Suppose that π and π' are invariant under the action of $Gal(E/\mathbb{Q})$ and $Gal(F/\mathbb{Q})$, respectively, and with notation as above suppose that at least one of $\pi_{\mathbb{Q}}$ or $\pi'_{\mathbb{Q}}$ is self-contragredient. For $T \neq \phi$, we have*

$$\sum_{n \le x} \Lambda(n) a_{\pi \times_{E,F} \tilde{\pi}'}(n) =$$

$$= \begin{cases} \frac{x^{1+i\tau(\pi,\pi')}}{1+i\tau(\pi,\pi')} + O\{x \exp(-c\sqrt{\log x})\} \textit{if } BC_{EF/E}(\pi) \textit{ is cuspidal} \\ \frac{\ell x^{1+i\tau(\pi,\pi')}}{1+i\tau(\pi,\pi')} + O\{x \exp(-c\sqrt{\log x})\} \textit{ if } \ell = \ell' \textit{ and } BC_{EF/E}(\pi) \textit{ is not cuspidal.} \end{cases}$$

Now let E and F be solvable Galois extensions of $\mathbb{Q}$ with filtrations coming from the cyclic prime degree composition factors of the Galois groups

$$E = E_0 \supset E_1 \supset ... \supset E_k \supset E_{k+1} = \mathbb{Q},$$

$$F = F_0 \supset F_1 \supset ... \supset F_r \supset F_{t+1} = \mathbb{Q}.$$

Since the Galois group $Gal(EF/(E \cap F))$ is a solvable group, we have a filtration

$$EF = (EF)_0 \supset (EF)_1 \supset ... \supset (EF)_c \supset (EF)_{c+1} = E \cap F.$$

Using the Galois correspondence we get that for every $0 \leq k \leq c+1$ there exists a unique $0 \leq j \leq t+1$ so that $(EF)_k = EF_j$. Also consider the filtrations

$$E \cap F \supset E_{d+1} \cap F \supset ... \supset E_{d+e} \cap F \supset \mathbb{Q}$$

$$E \cap F \supset F_{d'+1} \cap E \supset ... \supset F_{d'+e} \cap E \supset \mathbb{Q}$$

Let α_K be equal to the module for any number field K. We obtain one final prime number theorem in the solvable Galois invariant case.

Theorem 1.1.2. *Let π and π' be as above, and let E and F be solvable Galois extensions. Suppose that for every $i = 0, ..., k-1$ and $j = 0, ..., t-1$ the elements of the fibers $BC^{-1}_{E/E_i}(\pi)$ and $BC^{-1}_{F/F_j}(\pi')$ are invariant under the action of $Gal(E_i/E_{i+1})$ and $Gal(F_j/F_{j+1})$ respectively, then for $BC_{EF/E}(\pi)$ cuspidal and $T \neq \phi$ we have*

$$\sum_{n \leq x} \frac{\Lambda(n) a_{\pi \times_{E,F} \pi'}(n) \log(n)}{n} = \frac{[E \cap F : \mathbb{Q}]}{2} \log^2 x + O(\log x).$$

The theorem above gives a cuspidality criterion for the base change of π up to the composite extension EF, and measures how many cuspidal components the two representations $AI_{E/\mathbb{Q}}(\pi)$ and $AI_{F/\mathbb{Q}}(\pi')$ have in common when decomposed into isobaric sums of cuspidal representations. Alternatively, on the Galois side the above formula gives a criterion for the induced representation to be irreducible.

Denoting by $L(s,\pi)$ the product in (1.3), we can write

$$L(s,\pi) = L(s,\pi_{\iota_1})^{r_1} L(s,\pi_{\iota_2})^{r_2} ... L(s,\pi_{\iota_\gamma})^{r_\gamma},$$

where $BC_{F_{\iota_i}F_{\iota_j}/F_{\iota_i}}(\pi_{\iota_i}) \not\cong BC_{F_{\iota_i}F_{\iota_j}/F_{\iota_j}}(\pi_{\iota_j})$ for $i \neq j$. Let $\underline{\mathrm{K}} = [K_1, ..., K_\gamma]$ be the set partition corresponding to this factorization, and write $\ell \in K_\alpha$ if K_α contains an element π_{ι_δ} such that $[F_{\iota_\delta} : \mathbb{Q}] = \ell$. Also, for any prime ℓ let

$$S_{\ell,\alpha} = \{\pi_{\iota_\delta} \in K_\alpha | \ell_{\iota_\delta} = \ell\}$$

$$a_{\ell,\alpha} = |\{\pi_{\iota_\delta} \in S_{\ell,\alpha} | BC_{F_{\iota_\alpha}F_{\iota_\delta}/F_{\iota_\alpha}}(\pi_{\iota_\alpha}) \text{ is not cuspidal }\}|$$

$$b_{\ell,\alpha} = |\{\pi_{\iota_\delta} \in S_{\ell,\alpha} | BC_{F_{\iota_\alpha}F_{\iota_\delta}/F_{\iota_\alpha}}(\pi_{\iota_\alpha}) \text{ is cuspidal }\}|$$

We also denote by $\nu_\alpha = \nu(\prod_{\pi_{\iota_\delta} \in K_\alpha} \ell_{\iota_\delta})$ the number of prime factors occurring in the partition class K_α which have multiplicity one. Using the above theorems we obtain the formula for the correlation function in (1.4).

Theorem 1.1.3. *For each $i = 1, ..., k$ let π_i be a unitary automorphic cuspidal representation of $GL_{m_i}(\mathbb{A}_{F_i})$ with F_i a cyclic algebraic number field of prime degree ℓ_i. Suppose that $F_i \neq F_j$ for $i \neq j$ and suppose each π_i is invariant under the action of $Gal(F_i/\mathbb{Q})$. Let $g_1, ..., g_n \in C_c^\infty(\mathbb{R})$ and let $\Phi \in C^1(\mathbb{R}^n)$ be a symmetric function supported in $|\xi_1| + ... + |\xi_n| < 2/[m_1, ..., m_k]$. Define $h_1, ..., h_n$ and f as in (1.1) and (1.2), respectively. Then under Hypothesis H, or if all $m_j \leq 4$ we have*

$$
\begin{aligned}
&C_n(f,\mathbf{h},T,\pi_1,...,\pi_k) \sim \\
&\frac{\kappa(\mathbf{h})}{2\pi} TL\Big\{ a^n \Phi(0,...,0) + \sum_{1\le r\le n/2} \frac{n!a^{n-2r}}{r!(n-2r)!2^r} b^r \\
&\times \int_{\mathbb{R}^r} |v_1|...|v_r| \Phi(v_1,...,v_r,-v_1,....,-v_r,0,...,0) dv \Big\} + O(T)
\end{aligned}
$$

where

$$
a = \sum_{i=1}^{\gamma} r_i + \sum_{\alpha=1}^{\gamma} \Big(\big(\sum_{\substack{\ell_{\iota_\alpha} \in K_\alpha \\ |S_{\ell_{\iota_\alpha},\alpha}|=1}} \ell_{\iota_\alpha} \big) - \nu_\alpha + \sum_{\substack{\ell_{\iota_\alpha} \in K_\alpha \\ |S_{\ell_{\iota_\alpha},\alpha}|\ge 2}} a_{\ell_{\iota_\alpha},\alpha}(\ell_{\iota_\alpha}-1) + b_{\ell_{\iota_\alpha},\alpha}(\ell_{\iota_\alpha}-1) \Big),
$$

$$
b = \sum_{i=1}^{\gamma} r_i^2 + \sum_{\alpha=1}^{\gamma} \Big(\big(\sum_{\substack{\ell_{\iota_\alpha} \in K_\alpha \\ |S_{\ell_{\iota_\alpha},\alpha}|=1}} \ell_{\iota_\alpha} \big) - \nu_\alpha + \sum_{\substack{\ell_{\iota_\alpha} \in K_\alpha \\ |S_{\ell_{\iota_\alpha},\alpha}|\ge 2}} a^2_{\ell_{\iota_\alpha},\alpha}(\ell_{\iota_\alpha}-1) + b_{\ell_{\iota_\alpha},\alpha}(\ell_{\iota_\alpha}-1) \Big).
$$

Using Theorem 1.2 from [23], we can put necessary conditions on the factorization of an L-function into a product of L-functions defined over cyclic algebraic number fields of prime degree.

Corollary 1.1.4. *Let E be a Galois extension of $\mathbb{Q}$ of degree ℓ. Let π and π_j, $j = 1,...,k$ be automorphic cuspidal representations of $GL_m(\mathbb{A}_E)$ and $GL_{m_j}(\mathbb{A}_{F_j})$ respectively, with $F_j/\mathbb{Q}$ cyclic of prime degree ℓ_j. Let c be the size of the subgroup G_π in $Gal(E/\mathbb{Q})$ such that $\pi^\sigma \cong \pi$ for $\sigma \in G_\pi$. Suppose $m, m_1, ..., m_k \le 4$ or otherwise assume Hypothesis H. Also assume Conjecture 2.1.1 when $m \ge 3$ and there is a prime factor p of ℓ which is $\le (m^2+1)/2$. Then $L(s,\pi)$ and $L(s,\pi_1)^{r_1}...L(s,\pi_\gamma)^{r_\gamma}$ have the same n-level correlation of normalized nontrivial zeros if and only if $b = c$ and*

$$
m_1 + ... + m_k = m\ell
$$

1.2 Overview of Proof

In chapter 3 we prove a prime number theorem in the case $E = F$ for the classical Rankin-Selberg L-function $L(s, \pi \times \tilde{\pi}')$ and show that, under Hypothesis H and an upper bound on the unramified local parameters, the main term comes from those primes which split completely in the extension. In chapter 4 we consider the more general case, and in the end the proof, broken into two forms, requires putting a group structure on the fibers of the base change map; in particular, the group in question is the direct product of the prime degree cyclic composition factors coming from the Galois group. When the degrees of the extensions are relatively prime this method suffices to give the full result. Otherwise we employ the isomorphism of Galois groups $Gal(EF/E \cap F) \cong Gal(E/E \cap F) \times Gal(F/E \cap F)$ and knowledge of the fibers of the base change map due to Rajan [32]. In chapter 5 we use the previous results to compute the zero statistics of a product of L-functions attached to cyclic algebraic number fields, giving necessary conditions for the factorization of an L-function into such a product. In chapter 6 we return to the general problem, and solve it in its entirety in some special cases using the elementary theory of Artin-L functions.

CHAPTER 2
L-FUNCTIONS, BASE CHANGE, AND AUTOMORPHIC INDUCTION

2.1 L-Functions

We refer to [11] for the necessary background on L-functions attached to GL_n. Let π be an automorphic cuspidal representation of $GL_m(\mathbb{A}_E)$ with unitary central character, where E is a Galois extension of $\mathbb{Q}$. The symbol $\mathbb{A}_E$ will always denote the ring of adeles of E, so that $\mathbb{A}_E$ is the weak direct product of E_ν over all places ν of E. We view E as a subset of $\mathbb{A}_E$ by means of the diagonal embedding in which it has discrete image, and if S is a finite set of places of E, we let $\mathbb{A}_E^S{=}\prod'_{\nu\notin S} E_\nu$. We have the decomposition $\pi = \otimes'_\nu \pi_\nu$ (see [7]), with π_ν an irreducible admissible unitary generic representation of $GL_m(E_\nu)$, and $\otimes'_\nu$ denotes the restricted tensor product over all places ν of E. For all but finitely many ν, π_ν is unramified in the sense that the representation space for π_ν contains a nonzero fixed vector under the maximal compact subgroup $GL_m(\mathcal{O}_\nu)$, where $\mathcal{O}_\nu$ denotes the maximal compact subring of E_ν. Let G be a locally compact group and H a closed subgroup. If (τ, W) is a representation of H, let

$$Ind_H^G(W) = \{f : G \longrightarrow W | f(hg) = \delta_{G/H}^{-1/2}(h)\tau(h)f(g)\forall h \in H, g \in G\}$$

where $\delta_{G/H} = \delta_G/\delta_H$ denotes the modulus. For $G = GL_m$ let B denote the subgroup of upper triangular matrices, T the diagonal matrices, and N the standard unipotent subgroup. Then $B = TN$ and It follows that for all but finitely many ν, π_ν is an irreducible subquotient of the induced representation

$$Ind_B^G(\mu_{\nu,1} \otimes ... \otimes \mu_{\nu,n})$$

with $\mu_{\nu,i}$ a character on $E_\nu^\times$ trivial on the units $\mathcal{O}_\nu^\times$ for $i = 1, ..., n$ (see [4] Prop 3.9.1). Here $\mu_{\nu,1} \otimes ... \otimes \mu_{\nu,n}$ is a representation on T made trivial on N to obtain a representation of B and Ind_B^G is the unitarily induced representation from B as defined before. For non-archimedean ν, if we let ω_ν denote a generator of the unique nonzero prime ideal $\mathfrak{P}$ of $\mathcal{O}_\nu$ and $\mu_{\nu,i}(\omega_\nu) = \alpha_\pi(i,\nu)$, then we define the local L-factor for π at the unramified place ν by

$$L(s,\pi_\nu) = \prod_{i=1}^{n}(1 - \alpha_\pi(i,\nu)N(\mathfrak{P})^{-s})^{-1}$$

where $N(\mathfrak{P}) = |\mathcal{O}_\nu/\mathfrak{P}|$. Note that if $\mathfrak{P}$ lies over the prime $p \in \mathbb{Z}_p$, then $\mathcal{O}_\nu/\mathfrak{P}$ is a vector space over $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$. Since $E/\mathbb{Q}$ is Galois, any two primes lying over a prime p in $\mathbb{Z}$ are conjugate under the Galois group $Gal(E/\mathbb{Q})$; consequently, any two completions of E lying over $\mathbb{Z}_p$ are isomorphic, so the index $f_p = [\mathcal{O}_\nu/\mathfrak{P} : \mathbb{Z}_p/p\mathbb{Z}_p]$ only depends on p, and $N(\mathfrak{P}) = p^{f_p}$. We can write the finite-part local L-factor more explicitly as

$$L(s,\pi_\nu) = \prod_{i=1}^{m}(1 - \alpha_\pi(i,\nu)p^{-sf_p})^{-1} \text{ for } \nu|p$$

For ramified ν the definition of $L(s,\pi_\nu)$ requires much more work and we refer to the appendix in [34] for a discussion. In the ramified case we define $L(s,\pi_\nu)$ as before with the convention that some of the parameters $\{\alpha_\pi(i,\nu)\}_{i=1}^n$ may be zero. Denote by $L(s,\pi)$ and $\Lambda(s,\pi)$ the finite-part and complete L-functions attached to π, respectively. Then $L(s,\pi) = \prod_{\nu<\infty} L(s,\pi_\nu)$ and $\Lambda(s,\pi) = L(s,\pi_\infty)L(s,\pi)$ for

$Re(s) > 1$. Here $L(s, \pi_\infty) = \prod_{\nu|\infty} \prod_{k=1}^{m} \Gamma_\nu(s + \mu_\pi(k, \nu))$, where $\Gamma_\nu(s) = \pi^{-s/2}\Gamma(s/2)$ for ν real and $\Gamma_\nu(s) = (2\pi)^{-s}\Gamma(s)$ for ν complex, and $\mu_\pi(k, \nu)$ for $k = 1, ..., m$ are a set complex numbers associated to π by the Langland's correspondence. Recall that $\Lambda(s, \pi)$ extends to an entire function with the exception of π coming as an unramified character of $\mathbb{A}_E^\times / E^\times$. $\Lambda(s, \pi)$ also satisfies a functional equation

$$\Lambda(s, \pi) = \epsilon(s, \pi)\Lambda(1 - s, \tilde{\pi})$$

where $\tilde{\pi}$ denotes the contragredient representation of π and $\epsilon(s, \pi) = \tau(\pi)Q_\pi^{-s}$. Here $Q_\pi > 0$ is the conductor of π, $\tau(\pi) \in \mathbb{C}^\times$, $Q_\pi = Q_{\tilde{\pi}}$, and $\tau(\pi)\tau(\tilde{\pi}) = Q_\pi$. Denote

$$a_\pi(p^{kf_p}) = f_p \sum_{\nu|p} \sum_{1 \le j \le m} \alpha_\pi(j, \nu)^k$$

where $a_\pi(p^k) = 0$ if $f_p \nmid k$, $c_\pi(n) = \Lambda(n)a_\pi(n)$, and $\Lambda(n)$ denotes the Von Mangoldt function. Then $\overline{a_\pi(n)} = a_{\tilde{\pi}}(n)$, and for $Re(s) > 1$ we have that

$$\frac{L'}{L}(s, \pi) = -\sum_{n \ge 1} \frac{\Lambda(n)a_\pi(n)}{n^s}$$

We will need the following bound for $\alpha_\pi(i, \nu)$:

$$|\alpha_\pi(i, \nu)| \le p^{f_p(1/2 - 1/(\ell m^2 + 1))} \tag{1.1}$$

where $\ell = [E : \mathbb{Q}]$. This bound holds for any π_ν either ramified or unramified. It was first observed by Serre [36] and appeared in published form in [25]. A complete proof is given in [34] for the case $E = \mathbb{Q}$ using an argument of Landau [16]. Note that the generalized Ramanujan conjecture states that for π_ν unramified

$$|\alpha_\pi(i, \nu)| = 1$$

The best known bounds towards this over an arbitrary number field are

$$|\alpha_\pi(i,\nu)| \leq p^{f_p/9} \text{ for } \nu|p$$

when $m = 2$ [14], and

$$|\alpha_\pi(i,\nu)| \leq p^{f_p(1/2-1/(m^2+1))} \tag{1.2}$$

for general m [25] for $\nu|p$. We will not assume the generalized Ramanujan conjecture, but assume a bound θ_p toward it for any p which is unramified and does not split completely in E.

Conjecture 2.1.1. *For any p which is unramified and does not split completely in E, we have for any $\nu|p$ that*

$$|\alpha_\pi(i,\nu)| \leq p^{f_p\theta_p} \tag{1.3}$$

where $\theta_p = 1/2 - 1/(2f_p) - \epsilon$ for a small $\epsilon > 0$.

Note that if $\ell_p = [E_\nu : \mathbb{Q}_p] = e_p f_p$ where e_p denotes the ramification index, then for ν unramified we have that $e_p = 1$ so that $\ell_p = f_p$. Since p does not split completely in E, we know that $f_p \geq 2$. Consequently Conjecture 2.1 is known for $m = 2$, is trivial for $m = 1$, and is known when all prime factors of ℓ are $> (m^2+1)/2$. For $m = 3$ this means that any $p|\ell$ is ≥ 7, while for $m = 4$, it is true when any $p|\ell$ is ≥ 11. We will also need Hypothesis H generalized to E

Hypothesis H 2.1.2. *Let π be an automorphic cuspidal representation of $GL_m(\mathbb{A}_E)$ with unitary central character. Then for any fixed $k \geq 2$*

$$\sum_p \frac{\log(p)^2}{p^{kf_p}} \sum_{\nu|p} \left| \sum_{1\leq j\leq m} \alpha_\pi(j,\nu)^k \right|^2 < \infty$$

Note that Hypothesis H follows easily from the generalized Ramanujan conjecture. Since there are only finitely many p which ramify in E, the sum may be taken over all unramified p. As we have assumed Conjecture 2.1, if we let S_π be the set of all primes p such that p is unramified and does not spit completely in E then we know that

$$\sum_{p\in S_\pi} \frac{\log(p)^2}{p^{kf_p}} \sum_{\nu|p} \left| \sum_{1\leq j\leq m} \alpha_\pi(j,\nu)^k \right|^2 \ll \sum_{p\in S_\pi} \frac{\log(p)^2}{p^{kf_p}} p^{2kf_p\theta_p} < \infty$$

so that under Conjecture 2.1, Hypothesis H claims that for any fixed $k \geq 2$

$$\sum_{\text{p splits completely}} \frac{\log(p)^2}{p^k} \sum_{\nu|p} \left| \sum_{1\leq j\leq m} \alpha_\pi(j,\nu)^k \right|^2 < \infty$$

Finally, we will also need the basic properties of the Rankin-Selberg L-function. We will use the Rankin-Selberg L-functions $L(s, \pi \times \widetilde{\pi}')$ as developed by Jacquet, Piatetski-Shapiro, and Shalika [9], Shahidi [37], and Moeglin and Waldspurger [29], for π' an automorphic cuspidal representation of $GL_{m'}(\mathbb{A}_E)$. It is again defined as a product of local factors $L(s, \pi \times \tilde{\pi}') = \prod_{\nu<\infty} L(s, \pi_\nu \times \tilde{\pi}_\nu{}')$, where the local factor at the finite place ν is given by

$$L(s, \pi_\nu \times \pi'_\nu) = \prod_{i=1}^{m} \prod_{j=1}^{m'} (1 - \alpha_\pi(i,\nu)\overline{\alpha_{\pi'}(j,\nu)} p^{-f_p s})^{-1} \text{ for } \nu|p$$

We will need the following properties of $L(s, \pi \times \widetilde{\pi}')$

RS1. The Euler product defining $L(s, \pi \times \tilde{\pi}')$ converges absolutely for $\sigma > 1$ (Jacquet and Shalika [12]).

RS2. The complete L-function $\Lambda(s, \pi \times \tilde{\pi}')$ has an analytic continuation to the entire complex plane and satisfies a functional equation

$$\Lambda(s, \pi \times \tilde{\pi}') = \varepsilon(s, \pi \times \tilde{\pi}')\Lambda(1 - s, \tilde{\pi} \times \pi')$$

with

$$\varepsilon(s, \pi \times \tilde{\pi}') = \tau(\pi \times \tilde{\pi}')Q_{\pi\times\tilde{\pi}'}^{-s},$$

where $Q_{\pi\times\tilde{\pi}'} > 0$ and $\tau(\pi \times \tilde{\pi}') = \pm Q_{\pi\times\tilde{\pi}'}^{1/2}$ (Shahidi [37], [38], [39], and [40]).

RS3. Denote $\alpha(g) = |\det(g)|$. When $\pi' \not\cong \pi \otimes \alpha^{it}$ for any $t \in \mathbb{R}$, $\Lambda(s, \pi \times \tilde{\pi}')$ is holomorphic. When $m = m'$ and $\pi' \cong \pi \otimes \alpha^{i\tau_0}$ for some $\tau_0 \in \mathbb{R}$, the only poles of $\Lambda(s, \pi \times \tilde{\pi}')$ are simple poles at $s = i\tau_0$ and $1 + i\tau_0$ coming from $L(s, \pi \times \tilde{\pi}')$ (Jacquet and Shalika [12] and [13], Moeglin and Waldspurger [29]).

RS4. $\Lambda(s, \pi \times \tilde{\pi}')$ is meromorphic of order one away from its poles, and bounded in vertical strips (Gelbart and Shahidi [8]).

RS5. $L(s, \pi \times \tilde{\pi}')$ is non-zero in $\sigma \geq 1$ (Shahidi [37]). Furthermore, if at least one of π or π' is self-contragredient, it is zero-free in the region

$$\sigma \geq 1 - \frac{c}{\log(Q_\pi Q_{\pi'}(|t| + 2))} \quad |t| \geq 1$$

where c is an explicit constant depending only on m and n. Moreover $L(s, \pi \times \tilde{\pi})$ has at most one exceptional zero in the region

$$\sigma \geq 1 - \frac{c}{\log(Q_\pi Q_{\pi'}(|t| + 2))} \quad |t| \leq 1$$

(see Sarnak [35] and Moreno [30],[31]).

2.2 Orthogonality

For π and π' cuspidal representations of $GL_m(\mathbb{A}_{\mathbb{Q}})$ and $GL_{m'}(\mathbb{A}_{\mathbb{Q}})$ respectively Liu, Wang and Ye [24] proved the following Selberg orthogonality for $\pi \ncong \pi'$

$$\sum_{n\leq x} \frac{(\log n)\Lambda(n)a_\pi(n)\overline{a_{\pi'}(n)}}{n} \ll \log x$$

This followed from a stronger weighted prime number theorem for a Rankin-Selberg L-function, assuming that at least one of π or π' is self-contragredient. The contragredient assumption was necessary for the application of a zero free region of the classical type [30], [31]. On the other hand in [19] the self-contragredient assumption was removed by first proving a weighted version of the above orthogonality and then applying a method of Landau [16] to remove the weight. More precisely in [20] they prove the following theorem:

Lemma 2.2.1. *Let π and π' be automorphic cuspidal representations of $GL_m(\mathbb{A}_E)$ and $GL_{m'}(\mathbb{A}_E)$ with unitary central characters, respectively, then*

$$\begin{aligned}\sum_{n\leq x} \frac{(\log n)\Lambda(n)a_{\pi\times\tilde{\pi}'}(n)}{n} &= \frac{1}{2}\log^2 x + O(\log x) \ \textit{if}\ \pi' \cong \pi \\ &= O(\log x) \ \textit{if}\ \pi' \ncong \pi\end{aligned}$$

In order to control sums over prime powers in the computation of the n-level correlation function we will need a version of orthogonality where π and π' come from representations of $GL_m(\mathbb{A}_E)$ and $GL_{m'}(\mathbb{A}_F)$, respectively, when $E \neq F$. This is the main obstacle in the computation, and to use the same methods as in [20] and [19], ostensibly this would require the definition of a Rankin-Selberg L-function over different fields, with analytic continuation, functional equation, and zero-free region. We will need the Rankin-Selberg L-function defined as follows: write

$$L(s,\pi)=\prod_p\prod_{\nu|p}\prod_{j=1}^{m}\prod_{a=0}^{f_p-1}(1-\alpha_\pi(j,\nu)^{1/f_p}\omega_{f_p}^a p^{-s})^{-1}$$

where ω_{f_p} is a primitive f_p-th root of unity. Similarly

$$L(s,\pi')=\prod_p\prod_{\omega|p}\prod_{i=1}^{m'}\prod_{b=0}^{f'_p-1}(1-\alpha_{\pi'}(i,\omega)^{1/f'_p}\omega_{f'_p}^b p^{-s})^{-1}$$

then define

$$L(s,\pi\times_{E,F}\pi')=\prod_p\prod_{\nu|p,\omega|p}\prod_{i,j=1}^{m,m'}\prod_{a,b=0}^{f_p-1,f'_p-1}(1-\alpha_\pi(i,\nu)^{1/f_p}\alpha_{\pi'}(j,\omega)^{1/f'_p}\omega_{f_p}^a\omega_{f'_p}^b p^{-s})^{-1}$$

We will show the absolute convergence of the above product for $Re(s)>1$ in the appendix. Thus, for $Re(s)>1$ we have

$$\frac{L'}{L}(s,\pi\times_{E,F}\pi')=-\sum_{n\geq 1}\frac{\Lambda(n)a_{\pi\times_{E,F}\pi'}(n)}{n^s}$$

where for $n=p^k$

$$a_{\pi\times_{E,F}\pi'}(n)=\sum_{\nu,\omega|p}\sum_{i,j}^{m,m'}\sum_{a,b=0}^{f_p-1,f'_p-1}\alpha_\pi(i,\nu)^{k/f_p}\alpha_{\pi'}(j,\omega)^{k/f'_p}\omega_{f_p}^{ka}\omega_{f'_p}^{kb}$$

the above sum is zero unless $f_p|k$ and $f'_p|k$ if and only if $[f_p,f'_p]|k$, where $[f_p,f'_p]$ denotes the least common multiple of f_p and f'_p. In this case we have

$$a_{\pi\times_{E,F}\pi'}(n)=f_pf'_p\sum_{\nu,\omega|p}\sum_{i,j=1}^{m,m'}\alpha_\pi(i,\nu)^{k'[f_p,f'_p]/f_p}\alpha_{\pi'}(j,\omega)^{k'[f_p,f'_p]/f'_p}$$

$$=f_p\Big(\sum_{i=1}^{m}\sum_{\nu|p}\alpha_\pi(i,\nu)^{k'[f_p,f'_p]/f_p}\Big)f'_p\Big(\sum_{j=1}^{m'}\sum_{\omega|p}\alpha_{\pi'}(j,\omega)^{k'[f_p,f'_p]/f'_p}\Big)$$

for $k'[f_p,f'_p]=k$. Given two automorphic representations π and Π of $GL_n(\mathbb{A}_L)$ and $GL_{nq}(\mathbb{A}_K)$, respectively, where $L\supset K$ is a Galois extension of number fields such that $[L:K]=q$, we say that Π is *automorphically induced* from π if for any place α of K we have

$$L(s,\Pi_\alpha)=\prod_{\beta|\alpha} L(s,\pi_\beta)$$

There is at most one Π satisfying these conditions by strong multiplicity one [13], and we denote $\Pi = AI_{K/L}(\pi)$. Under the functoriality principle, Π should correspond to a representation of $Gal(\overline{K}/L)$ induced from a representation of $Gal(\overline{K}/K)$ corresponding to π (see [33]). Also, corresponding to restriction of representations on the Galois side there is the base change functor on the automorphic side. An automorphic representation π of $GL_n(\mathbb{A}_L)$ is said to be the *base change lift* of a representation σ of $GL_n(\mathbb{A}_K)$ if for all but finitely many places γ lying over β we have

$$\{\alpha_\sigma(i,\beta)\}_{i=1,\dots,n}^{f_{\gamma|\beta}}=\{\alpha_\pi(j,\gamma)\}_{j=1,\dots,n}$$

here $f_{\gamma|\beta}$ denotes the index $[\mathcal{O}_\gamma/\mathfrak{P}_\gamma : \mathcal{O}_\beta/\mathfrak{P}_\beta]$. There is again at most one representation π satisfying these conditions by strong multiplicity one, so that we can write $BC_{L/K}(\sigma)=\pi$

In order to obtain an orthogonality relation for the sum

$$\sum_{n\le x} a_{\pi\times_{E,F}\pi'}(n)\Lambda(n),$$

for large $x>0$, we will assume throughout that the automorphic induction $AI_{E/\mathbb{Q}}(\pi)$ of π (resp. $AI_{F/\mathbb{Q}}(\pi')$ of π') exists. Using this we can deduce the analytic continuation, functional equation, and zero-free region of the L-function $L(s,\pi\times_{E,F}\pi')$ since in this case we can write $L(s,\pi\times_{E,F}\pi')$ as a product of the classical Rankin-Selberg L-functions. In fact, knowing the above properties of $L(s,\pi\times_{E,F}\pi')$ would give the existence of the automorphic induction $AI_{E/\mathbb{Q}}(\pi)$ of π by setting $F = \mathbb{Q}$ and using a standard converse theorem of Cogdell [5]. Indeed, it follows from this that knowing the analytic properties of $L(s, \pi \times_{E,F} \pi')$ is equivalent to knowing the existence of $AI_{E/\mathbb{Q}}(\pi)$ and $AI_{F/\mathbb{Q}}(\pi')$, since if the latter exists we can simply define

$$L(s, \pi \times_{E,F} \pi') = L(s, AI_{E/\mathbb{Q}}(\pi) \times AI_{F/\mathbb{Q}}(\pi'))$$

Now let (π_i, V_{π_i}) be an automorphic cuspidal representation of $GL_{m_i}(\mathbb{A}_{F_i})$ with F_i a finite Galois extension for $1 \leq i \leq k$. The main difficulty in using the automorphic induction functor to find an orthogonality relation for $L(s, \pi \times_{E,F} \pi')$ is the presence of multiple poles for the L-function. In other words, if we write

$$\mathcal{A}_{E/\mathbb{Q}}(\pi) = \boxplus_{b=1}^{k} \pi_b$$

$$\mathcal{A}_{F/\mathbb{Q}}(\pi') = \boxplus_{a=1}^{t} \pi'_a$$

with π_b and π'_a cuspidal automorphic representations on $GL_{n_b}(\mathbb{A}_{\mathbb{Q}})$ and $GL_{n_a}(\mathbb{A}_{\mathbb{Q}})$ respectively, then

$$L(s, \pi \times_{E,F} \pi') = \prod_{b=1}^{k} \prod_{a=1}^{t} L(s, \pi_b \times \pi'_a)$$

where the right hand side is a product of the classical Rankin-Selberg L-functions. Thus we may have multiple twisted equivalent pairs: $\pi_b \cong \tilde{\pi_a}' \otimes |\det|^{i\tau}$, which will contribute to multiple main terms in computing the asymptotic behavior of the sum

$$\sum_{n \leq x} \frac{\log(n)\Lambda(n) a_{\pi \times_{E,F} \pi'}(n)}{n}$$

With this in mind denote by $N(\pi, \pi')$ the number of twisted equivalent pairs (π_b, π'_a) for $b = 1, ..., k,\ a = 1, ..., t$. Then from the previous orthogonality relations we can write the above sum as

$$\begin{cases} \frac{N(\pi,\pi')}{x} \log x^2 + O(\log x) \text{ if } \pi_b \cong \pi'_a \text{ for some } 1 \le b \le k \text{ },1 \le a \le t \\ O(\log x) \text{ if } \pi_b \ncong \pi'_a \text{ for any} 1 \le b \le k \text{ },1 \le a \le t \end{cases}$$

It is possible to compute $N(\pi, \pi')$ in many special cases where the automorphic induction and base change functors exist. In particular in the cases when the extensions are cyclic or more generally solvable with a Galois invariance assumption.

Corollary 2.2.2. *Let the notation and assumptions be as in Theorem 1.1.1, then* $N(\pi, \pi')$ *is nonzero only when* $\tau_0 = 0$*. Moreover, when* $\tau_0 = 0$ *we have* $N(\pi, \pi') = 1$ *for* $BC_{EF/E}(\pi)$ *cuspidal or* $(\ell, \ell') = 1$ *and* $N(\pi, \pi') = \ell$ *otherwise.*

2.3 L-Groups and Functoriality

We follow [2] in the discussion of L-groups and base change that follows. Let G denote an arbitrary reductive algebraic matrix group defined over a number field F. Recall that a connected affine algebraic group is called *reductive* if it contains no nontrivial normal unipotent subgroups. Associated to G there is a certain Lie algebra called the root data which uniquely determines G up to isogeny (up to a morphism with finite kernel). If E is an extension of F over which G splits, in that G contains a maximal torus T which is isomorphic to a direct product of copies of $E^{\times}$, then we define the L-group as the semi-direct product

$$^{L}G = \hat{G} \rtimes Gal(E/F)$$

where $\hat{G}$ is the complex Lie group whose root system is dual to that of G. Denote by $\Pi(G(F_\nu))$ the set of equivalence classes of irreducible admissible representations of $G(F_\nu)$. For almost every ν a place of F, $G(\mathcal{O}_\nu)$ is a maximal compact subgroup of $G(F_\nu)$ and $G(F_\nu)$ contains a maximal connected solvable subgroup defined over F_ν; moreover, $G(F_\nu)$ contains a torus which is defined over an unramified extension of F_ν. In this case the representations $\pi_\nu \in \Pi(G(F_\nu))$ having a nonzero $G(\mathcal{O}_\nu)$-fixed vector are in one-to-one correspondence with the semisimple conjugacy classes $\sigma(\pi_\nu)$ in ${}^LG$ whose projection onto the factor $Gal(E/F)$ equals the Frobenius class at ν. So given any irreducible admissible representation π of the group $G(\mathbb{A}_F)$, by [7] we can write

$$\pi = \otimes'_\nu \pi_\nu$$

where π_ν contains a $G(\mathcal{O}_\nu)$ fixed vector for all ν outside a finite set S. This gives rise to a family

$$\sigma(\pi) = \{\sigma(\pi_\nu) | \nu \notin S\}$$

of semisimple conjugacy classes in ${}^LG$. To get started we take a finite dimensional representation of the L-group

$$r :^L G \longrightarrow GL_n(\mathbb{C})$$

as well as an automorphic representation π of G which gives rise to a family

$$\{r(\sigma(\pi_\nu)) | \nu \notin S\}$$

of semisimple conjugacy classes in $GL_n(\mathbb{C})$. The automorphic L-function attached to this family is defined as the product

$$L_S(s, \pi, r) = \prod_{\nu \notin S} \det(I - r(\sigma(\pi_\nu)) q_\nu^{-s})^{-1}$$

for $s \in \mathbb{C}$ and q_ν denotes the cardinality of the residue field of F_ν. And it can be shown that this product converges absolutely in some right half plane. Here we allow ourselves to inflate the extension E to the algebraic closure of F, and we can also take $\hat{G}$ to be trivial combined with a continuous n-dimensional representation of $Gal(\bar{F}/F)$ to obtain a representation of the L-group. Moreover, since $\sigma(\pi_\nu)$ projects onto the Frobenius class at ν the local factor will correspond to the local factor of the Artin L-function at the place ν, so that the Artin L-functions are a special case of the general automorphic L-functions.

Suppose for example that $E/\mathbb{Q}$ is a cyclic extension of prime degree ℓ. We can take the algebraic group G defined by restriction of scalars of the general linear group defined over E denoted $Res_{E/\mathbb{Q}}(GL_{n,E})$ (for a nice introduction see Weil [41]). The L-group in this case is

$$^LG = \prod_{i=1}^{\ell} GL_n(\mathbb{C}) \rtimes Gal(E/\mathbb{Q})$$

where the cyclic Galois group acts by permuting the factors of the direct product. Here $\hat{G}$ is embedded diagonally as a Levi factor and $Gal(E/\mathbb{Q})$ is mapped to the set of permutation matrices which permute the blocks of the Levi factor. Since $G(\mathbb{A}_\mathbb{Q}) \cong GL_n(\mathbb{A}_E)$, an automorphic representation π of G can be identified with an automorphic representation Π of $GL_{n,E}$, and moreover

$$L_S(s, \pi, r) = L_S(s, \Pi)$$

Underlying this example is Langlands' functoriality principle (see [3]), which pertains to maps $\rho :^L G \longrightarrow^L G'$ between L-groups. We call such a map an *L-homomorphism* if E' is a subfield of E, and if the composition of ρ with the projection of ${}^LG'$ onto $Gal(E'/F)$ equals the restriction map of $Gal(E/F)$ onto $Gal(E'/F)$. One part of the conjecture states that given a homomorphism of L-groups, and any automorphic representation π of G, there is an automorphic representation π' of G' such that $\rho(\sigma(\pi_\nu)) = \sigma(\pi'_\nu)$ for all ν outside a finite set S. In particular

$$L_S(s, \pi, r \circ \rho) = L_S(s, \pi', r)$$

for any finite dimensional representation r of ${}^LG'$. Returning to our previous example, there is an L-homorphism that gives rise to the automorphic induction functor; indeed, let

$$\rho :{}^L G \longrightarrow GL_{n\ell}(\mathbb{C})$$

where ρ is defined by

$$\rho(g_1, g_2, ..., g_\ell, \sigma) = \begin{pmatrix} g_{\sigma(1)} & \cdots & & \\ & g_{\sigma(2)} & \cdots & \\ & & \ddots & \\ & & & g_{\sigma(\ell)} \end{pmatrix}$$

where we have identified the Galois group with a subgroup of the permutation group S_ℓ. Then functoriality is known in this case [1] and the corresponding representation π' is the automorphic induction of π. For general $E/\mathbb{Q}$ to find the desired automorphic representation corresponding to the generalized Rankin-Selberg L-function $L(s, \pi \times_{E,F} \pi')$ this would require verifying the functoriality conjecture for the Lhomomorphisms

$$\rho_1 : \prod_{i=1}^{\ell} GL_n(\mathbb{C}) \rtimes Gal(E/\mathbb{Q}) \longrightarrow GL_{n\ell}(\mathbb{C})$$
$$\rho_2 : \prod_{j=1}^{\ell'} GL_m(\mathbb{C}) \rtimes Gal(F/\mathbb{Q}) \longrightarrow GL_{m\ell'}(\mathbb{C})$$
$$\rho_3 : GL_{n\ell}(\mathbb{C}) \times GL_{m\ell'}(\mathbb{C}) \longrightarrow GL_{mn\ell\ell'}(\mathbb{C})$$

where ρ_3 is given by the tensor product. In fact in our case we are most interested in the map $\rho_3(\rho_1, \rho_2)$ which (should) correspond to the generalized Rankin-Selberg product. Of course this daunting task is beyond the scope of this discussion, but nevertheless the functoriality conjecture gives us a guiding post in terms of the corresponding representation theory underlying the manipulation of the L-series.

CHAPTER 3
A PRIME NUMBER THEOREM FOR RANKIN-SELBERG L-FUNCTIONS OVER NUMBER FIELDS

3.1 The Classical Case

We now proceed to deduce a prime number theorem in these cases and then the orthogonality relations will follow, but first we prove a prime number theorem in the classical context.

Theorem 3.1.1. *Let E be Galois extension of $\mathbb{Q}$ of degree ℓ. Let π and π' be irreducible cuspidal representations of $GL_m(\mathbb{A}_E)$ and $GL_{m'}(\mathbb{A}_E)$, respectively. Assume that at least one of π or π' is self-contragredient. Then*

$$\sum_{n \le x} \Lambda(n) a_{\pi \times \widetilde{\pi}'}(n)$$
$$= \begin{cases} \dfrac{x^{1+i\tau_0}}{1+i\tau_0} + O\{x \exp(-c\sqrt{\log x})\} \\ \qquad\qquad \text{if } \pi' \cong \pi \otimes |\det|^{i\tau_0} \text{ for some } \tau_0 \in \mathbb{R}; \\ O\{x \exp(-c\sqrt{\log x})\} \\ \qquad\qquad \text{if } \pi' \not\cong \pi \otimes |\det|^{it} \text{ for any } t \in \mathbb{R}. \end{cases}$$

Let π and π' be as in the above theorem, we will first need a modified version of Lemma 4.1 of [21]. It is a weighted prime number theorem in the diagonal case.

Lemma 3.1.2. *Let π be a self-contragredient automorphic irreducible cuspidal representation of GL_m over E. Then*

$$\sum_{n \le x} \left(1 - \frac{n}{x}\right) \Lambda(n) a_{\pi \times \widetilde{\pi}}(n) = \frac{x}{2} + O\{x \exp(-c\sqrt{\log x})\}$$

Proof. By **RS1**, we have for $\sigma > 1$,

$$J(s) := -\frac{d}{ds}\log(L(s, \pi \times \tilde{\pi})) = \sum_{n=1}^{\infty} \frac{\Lambda(n) a_{\pi\times\tilde{\pi}}(n)}{n^s}$$

Now note that

$$\frac{1}{2\pi i}\int_{(b)} \frac{y^s}{s(s+1)} ds = \left\{ \begin{array}{r} 1 - 1/y \quad \text{if } y \geqq 1 \\ \\ 0 \quad \text{if } 0 < y < 1 \end{array} \right\}.$$

where (b) means the line $\sigma = b > 0$. Taking $b = 1 + 1/\log x$, we have

$$\sum_{n\le x}\Big(1 - \frac{n}{x}\Big)\Lambda(n) a_{\pi\times\tilde{\pi}}(n) = \frac{1}{2\pi i}\int_{(b)} J(s)\frac{x^s}{s(s+1)} ds$$

$$= \frac{1}{2\pi i}\Big(\int_{b-iT}^{b+iT} + \int_{b-i\infty}^{b-iT} + \int_{b+iT}^{b+i\infty}\Big)$$

The last two integrals are bounded by

$$\ll \int_T^{\infty} \frac{x}{t^2} dt \ll \frac{x}{T}$$

Thus,

$$\sum_{n\le x}\Big(1 - \frac{n}{x}\Big)\Lambda(n) a_{\pi\times\tilde{\pi}}(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} J(s)\frac{x^s}{s(s+1)} ds + O\Big(\frac{x}{T}\Big)$$

For completeness we now recall an argument made in Liu and Ye [20] Lemma 6.1. Let $\lambda(s) = \min_{n\le 0}|s - n|$, and let $\mathbb{C}(m)$ be the region in the complex plane with the following discs removed:

$$|s - 2n + \mu_{\pi\times\tilde{\pi}}(i, j, \nu)| < \frac{1}{8m^2\ell},\ n \le 0,\ 1 \le i \le m,\ 1 \le j \le m,\ \nu|\infty,$$

if ν is real, and

$$|s - n + \mu(i, j, \nu)| < \frac{1}{8m^2\ell},\ n \le 0,\ 1 \le i \le m,\ 1 \le j \le m,\ \nu|\infty,$$

if ν is complex. Then for $s \in \mathbb{C}(m)$ and all i, j, and $\nu|\infty$,

$$\lambda(\frac{s+\mu_{\pi\times\tilde{\pi}}(i,j,\nu)}{2}) \geq \frac{1}{16m^2\ell},$$

for ν real and

$$\lambda(s+\mu_{\pi\times\tilde{\pi}}(i,j,\nu)) \geq \frac{1}{16m^2\ell},$$

if ν is complex. Let $\beta(i,j,\nu)$ be the fractional part of $Re(\mu_{\pi\times\tilde{\pi}}(i,j,\nu))$. In addition let $\beta(0,0,\nu)=0$ and $\beta(m+1,m+1,\nu)=1$. Then all $\beta(i,j,\nu)\in[0,1]$, so there exist $\beta(i_1,j_1,\nu_1)$ and $\beta(i_2,j_2,\nu_2)$ so that $\beta(i_2,j_2,\nu_2)-\beta(i_1,j_1,\nu_1) \geqq 1/(3m^2\ell)$and there is no $\beta(i,j,\nu)$ such that $\beta(i_1,j_1,\nu_1)<\beta(i,j,\nu)<\beta(i_2,j_2,\nu_2)$. It follows that the strip

$$S_0=\{s|\beta(i_1,j_1,\nu_1)+1/(8m^2\ell) \leqq Re(s) \leqq \beta(i_2,j_2,\nu_2)-1/(8m^2\ell)\}$$

is contained inside $\mathbb{C}(m)$. Consequently, for all $n=0,-1,-2,...$, the strips

$$S_n=\{s|n+\beta(i_1,j_1,\nu_1)+1/(8m^2\ell) \leqq Re(s) \leqq n+\beta(i_2,j_2,\nu_2)-1/(8m^2\ell)\}$$

are all contained inside $\mathbb{C}(m)$. Thus we can choose a real number a with $-2<a<-1$ such that the line Re(s)=a is contained in the strip $S_{-2}\subset\mathbb{C}(m)$, and for large $T>0$ consider the contour

$$C_1: b\geq\sigma\geq a, t=-T;$$

$$C_2: \sigma=a, -T\leq t\leq T;$$

$$C_3: a\leq\sigma\leq b, t=T.$$

Note that the three poles $s=1,0,-1$, some trivial zeros, and certain nontrivial zeros $\rho=\beta+i\gamma$ of $L(s,\pi\times\tilde{\pi})$ are passed by the shifting of the contour. Also note that $s=0$ is a double pole. The trivial zeros can be determined by **RS2** and by the fact that $Re(\mu_{\pi\times\tilde{\pi}}(i,j,\nu)) > -1$: $s=-\mu_{\pi\times\tilde{\pi}}(i,j,\nu)$ with $a < -Re(\mu_{\pi\times\tilde{\pi}}) < 1$ and $s = -2-\mu_{\pi\times\tilde{\pi}}(i,j,\nu)$ with $a+2 < -Re(\mu_{\pi\times\tilde{\pi}}(i,j,\nu) < 1$, here we have used $-2 < a < -1$. Thus we get

$$\begin{aligned}
& \frac{1}{2\pi i}\int_{b-iT}^{b+iT} J(s)\frac{x^s}{s(s+1)}ds \\
= \ & \frac{1}{2\pi i}\left(\int_{C_1}+\int_{C_2}+\int_{C_3}\right) + Res_{s=1,0,-1}J(s)\frac{x^s}{s(s+1)} \\
& + \sum_{a<-Re(\mu_{\pi\times\tilde{\pi}}(i,j,\nu))<1} Res_{s=-\mu_{\pi\times\tilde{\pi}}(i,j,\nu)}J(s)\frac{x^s}{s(s+1)} \\
& + \sum_{a+2<-Re(\mu_{\pi\times\tilde{\pi}}(i,j,\nu))<1} Res_{s=-2-\mu_{\pi\times\tilde{\pi}}(i,j,\nu)}J(s)\frac{x^s}{s(s+1)} \\
& + \sum_{|\gamma|\le T} Res_{s=\rho}J(s)\frac{x^s}{s(s+1)}
\end{aligned}$$

By the same proof as in Liu Ye [19] Lemma 4.1 (d), using the complete L-function, functional equation, and the bound (1.1) in 2.1, for any large $\tau > 0$ we can choose T in $\tau < T < \tau+1$ such that, when $-1 \le \sigma \le 2$,

$$J(\sigma \pm iT) \ll \log^2(Q_{\pi\times\tilde{\pi}}T)$$

so that

$$\int_{C_1} \ll \int_a^b \log^2(Q_{\pi\times\tilde{\pi}}T)\frac{x^\sigma}{T^2}d\sigma \ll \frac{x\log^2(Q_{\pi\times\tilde{\pi}}T)}{T^2}$$

The same bound holds for the integral C_3. By Lemma 4.2 in [19] we can choose a so that, when $|t| \le T$,

$$J(a+it) \ll 1$$

which gives

$$\int_{C_2} \ll \int_{-T}^{T} \frac{x^a}{(|t|+1)^2}dt \ll \frac{1}{x}$$

So if we take $T \gg \exp\{\sqrt{\log(x)}\}$, we can bound the three integrals over C_1, C_2, and C_3 by

$$\ll x\exp\{-c\sqrt{\log x}\}$$

The function

$$J(s)\frac{x^s}{s(s+1)}$$

has only simple poles at $s = 1, -1$, and a double pole at $s = 0$; the residues are $\frac{x}{2}$, $O(x^{-1})$, and $O(\log x)$, respectively. Thus

$$Res_{s=1,0,-1}J(s)\frac{x^s}{s(s+1)} = \frac{x}{2} + O(\log x)$$

Near a trivial zero $s = -\mu_{\pi\times\tilde{\pi}}(i,j,\nu)$ of order ℓ, we can express $J(s)$ as $\frac{-\ell}{(s+\mu_{\pi\times\tilde{\pi}}(i,j,\nu))}$ plus an analytic function. By the trivial bound for the archimedean parameters we know that $Re(\mu_{\pi\times\tilde{\pi}}(i,j,\nu)) \geq -1+\delta$ for some $\delta > 0$. Thus,

$$\sum_{a<-Re(\mu_{\pi\times\tilde{\pi}}(i,j,\nu))<1} Res_{s=-\mu_{\pi\times\tilde{\pi}}(i,j,\nu)}J(s)\frac{x^s}{s(s+1)} \ll x^{1-\delta}$$

$$\sum_{a+2<-Re(\mu_{\pi\times\tilde{\pi}}(i,j,\nu))<1} Res_{s=-2-\mu_{\pi\times\tilde{\pi}}(i,j,\nu)}J(s)\frac{x^s}{s(s+1)} \ll x^{-1-\delta}$$

To compute the residues corresponding to nontrivial zeros, first use **RS4** and **RS5** to get

$$\sum_{\rho}\frac{1}{|\rho(\rho+1)|} < \infty$$

so that,

$$\sum_{|\gamma|\leq T} Res_{s=\rho}J(s)\frac{x^s}{s(s+2)} = -\sum_{|\gamma|\leq T} Res_{s=\rho}\frac{1}{s-\rho}\frac{x^s}{s(s+1)} \ll \sum_{|\gamma|\leq T}\left|\frac{x^\rho}{\rho(\rho+1)}\right|$$

$$= \Big(\sum_{|\gamma|\leq T,\rho\in\mathcal{E}} + \sum_{|\gamma|\leq T,\rho\notin\mathcal{E}}\Big)\frac{x^\beta}{|\rho(\rho+1)|}$$

where $\mathcal{E}$ is the set of exceptional zeroes in **RS5**. We have $|\mathcal{E}| \leq 1$, and hence the sum over $\rho \in \mathcal{E}$ is clearly $\ll x^{1-\delta}$ for some $\delta > 0$. By the zero free region in **RS5**, the sum over $\rho \notin \mathcal{E}$ is

$$\ll x \exp\{-c\frac{\log x}{2\log(Q_{\pi\times\tilde{\pi}}T)}\} \sum_{\rho} \frac{1}{|\rho(\rho+1)|} \ll x \exp\{-c\sqrt{\log x}\}$$

by taking $T = \exp\{\sqrt{\log x}\} + d$ for some d with $0 < d < 1$. So that the above is bounded by $x \exp\{-c\sqrt{\log x}\}$, combining all the above estimates, the lemma follows.

□

The next lemma again closely follows [21], and allows the removal of the weight $\left(1 - \frac{n}{x}\right)$ from the previous lemma. The proof involves a standard argument due to de la Vallee Poussin.

Lemma 3.1.3. *Let π be a self-contragredient automorphic irreducible cuspidal representation of GL_m over E. Then*

$$\sum_{n \leq x} \Lambda(n) a_{\pi\times\tilde{\pi}}(n) = x + O\{x \exp(-c\sqrt{\log x})\}.$$

Proof. Since the coefficients of the left hand side are non-negative, the proof follows as in Lemma 5.1 of [21] with no modification. □

The next lemma also follows exactly as in Lemma 5.2 of [21], and doesn't require π to be self-contragredient. The proof is an application of a Tauberian theorem of Ikehara.

Lemma 3.1.4. *For any automorphic irreducible cuspidal unitary representation π of*

GL_m over the number field E, we have

$$\sum_{n\leq x}\Lambda(n)a_{\pi\times\tilde{\pi}}(n)\sim x.$$

With the above lemmas we may now prove the theorem.

Proof of Theorem 3.0.1

We suppose throughout that π is self-contragredient. When $\pi'\cong\pi$, the theorem reduces to Lemma 3.0.4. We will first consider the case when π and π' are twisted equivalent, so suppose that $\pi'\cong\pi\otimes|det|^{i\tau_0}$ for some $\tau_0\in\mathbb{R}$. By Lemma 3.0.5, we obtain a bound for the short sum

$$\sum_{x<n\leq x+y}\Lambda(n)a_{\pi\times\tilde{\pi'}}(n)\ll y$$

for $y\ll x\exp\{-c\sqrt{\log x}\}$. Note that π' is not necessarily self-contragredient; nevertheless, by Lemma 3.0.4 we get for $0<y\leq x$ that

$$\sum_{x<\leq x+y}\Lambda(n)a_{\pi\times\tilde{\pi}'}(n)\ll\sum_{x<n\leq 2x}\Lambda(n)a_{\pi\times\tilde{\pi}'}(n)\ll x$$

By definition of the coefficients $a_{\pi\times\tilde{\pi}'}(n)$, we have that for $n=p^k$

$$|\Lambda(n)a_{\pi\times\tilde{\pi}'}(n)|\leq\log(p)\sum_{\nu|p}f_p\left|(\sum_{j=1}^{m}\alpha_\pi(j,\nu)^k)\right|\left|(\sum_{i=1}^{n}\overline{\alpha_{\pi'}(i,\nu)}^k)\right|$$

$$\begin{aligned}&\leq\log(p)\left(\sum_{\nu|p}f_p|\sum_{j=1}^{m}\alpha_\pi(j,\nu)^k)|^2\right)^{1/2}\left(\sum_{\nu|p}f_p|\sum_{i=1}^{n}\alpha_{\pi'}(i,\nu)^k)|^2\right)^{1/2}\\ =\ &\log(p)(a_{\pi\times\tilde{\pi}}(n)^{1/2}a_{\pi'\times\tilde{\pi}'}(n)^{1/2})\end{aligned}$$

Thus we have,

$$\leq\sum_{x<n\leq x+y}|\Lambda(n)a_{\pi\times\tilde{\pi}'}(n)|\leq\sum_{x<n\leq x+y}\Lambda(n)a_{\pi\times\tilde{\pi}}(n)^{1/2}a_{\pi'\times\tilde{\pi}'}(n)^{1/2}$$

$$\leq \left(\sum_{x<n\leq x+y} \Lambda(n) a_{\pi\times\tilde{\pi}}(n) \right)^{1/2} \left(\sum_{x<n\leq x+y} \Lambda(n) a_{\pi'\times\tilde{\pi}'}(n) \right)^{1/2} \ll \sqrt{yx}$$

Now let $T \gg \exp\{\sqrt{\log x}\}$. Then

$$\sum_{x-x/\sqrt{T}<n\leq x+x/\sqrt{T}} |\Lambda(n) a_{\pi\times\tilde{\pi}'}(n)| \ll \sqrt{\left(\frac{x}{\sqrt{T}}\right) x} = \frac{x}{T^{1/4}}$$

We still need an upper bound for

$$B(\sigma) = \sum_{n=1}^{\infty} \frac{\Lambda(n) a_{\pi\times\tilde{\pi}'}(n)}{n^{\sigma}} \ll \left\{ \sum_{n=1}^{\infty} \frac{\Lambda(n) a_{\pi\times\tilde{\pi}}(n)}{n^{\sigma}} \right\}^{1/2} \left\{ \sum_{n=1}^{\infty} \frac{\Lambda(n) a_{\pi'\times\tilde{\pi}'}(n)}{n^{\sigma}} \right\}^{1/2}$$

But by Lemma 3.0.4, for $1 < \sigma \leq 2$

$$\frac{1}{u^{\sigma}} \sum_{n\leq u} \Lambda(n) a_{\pi\times\tilde{\pi}}(n) \ll u^{1-\sigma}$$

and tends to 0 when $u \to \infty$. Thus we get,

$$\sum_{n=1}^{\infty} \frac{\Lambda(n) a_{\pi\times\tilde{\pi}}(n)}{n^{\sigma}} = \int_{1}^{\infty} \frac{1}{u^{\sigma}} d \left\{ \sum_{n\leq u} \Lambda(n) a_{\pi\times\tilde{\pi}}(n) \right\}$$
$$\ll 1 + \sigma \int_{1}^{\infty} \frac{du}{u} \ll \frac{1}{\sigma - 1}$$

Note this same bound also holds for π'. Applying this to both sums on the right side of the bound for $B(\sigma)$ we get for $1 < \sigma \leq 2$ that

$$B(\sigma) \ll \frac{1}{\sigma - 1}$$

Now we apply Corollary 2.2 of [21] with $b = 1 + 1/\log x$ and $T \gg \exp\{\sqrt{\log x}\}$ to $a_n = \Lambda(n) a_{\pi(n)\times\tilde{\pi}'}(n)$ and the rest of the proof follows as in Liu Ye [21] $\square$

We can also rewrite the previous Theorem as sums over primes using Conjecture 2.1.1 and Hypothesis H to show that the main term comes from those primes which split completely in the field extension.

Theorem 3.1.5. *Let the notations be as in the above Theorem 3.0.1. Assume Hypothesis H and Conjecture 2.1.1, then*

$$\sum_{\substack{p\le x \\ p \text{ splits completely}}} (\log p) a_{\pi\times\tilde{\pi}'}(p) = \frac{x^{1+i\tau_0}}{1+i\tau_0} + O\{x\exp(-c\sqrt{\log x})\} \text{ for } \pi \cong \pi'\otimes|\det|^{i\tau_0}$$

Proof. Note that Theorem 3.0.1 says

$$\sum_{\substack{p,k \\ p^{kf_p}\le x}} \log(p) \sum_{\nu|p} f_p \sum_{i=1}^{m}\sum_{j=1}^{m'} \alpha_\pi(i,\nu)^k \overline{\alpha_{\pi'}(j,\nu)}^k$$

$$= \frac{x^{1+i\tau_0}}{1+i\tau_0} + O\{x\exp(-c\sqrt{\log x})\} \text{ for } \pi \cong \pi'\otimes|\det|^{i\tau_0}$$

We can apply the bound (1.1) in 2.1 to the sum

$$\sum_{\substack{k>(m^2\ell+1)/2 \\ p^{f_p k}\le x}} \log(p) \sum_{\nu|p} f_p \sum_{i=1}^{m}\sum_{j=1}^{m'} \alpha_\pi(i,\nu)^k \overline{\alpha_{\pi'}(j,\nu)}^k$$

$$\ll \sum_{\substack{k>(m^2\ell+1)/2 \\ p^{kf_p}\le x}} (\log p) p^{2kf_p(1/2-1/(m^2\ell+1))} \ll x\Big(\sum_{n\le x}\frac{\Lambda(n)}{n^{1+\epsilon}}\Big)$$

for small $\epsilon>0$. By partial summation we get

$$\sum_{n\le x}\frac{\Lambda(n)}{n^{1+\epsilon}} = \Big(x+O\{x\exp(-c\sqrt{\log x})\}\Big)\frac{1}{x^{1+\epsilon}} - \int_1^x \Big(t+O\{t\exp(-c\sqrt{\log t})\}\Big)\frac{1}{t^{2+\epsilon}}(-1-\epsilon)dt$$

$$= O\{x\exp(-c\sqrt{\log x})\}$$

Note that Hypothesis H gives that for fixed $k\ge 2$

$$\sum_p \frac{|a_{\pi\times\tilde{\pi}'}(p^{kf_p})|(\log(p^{kf_p}))^2}{p^{kf_p}} < \infty$$

using this and partial summation for fixed $k\ge 2$ we can write for $y=\exp(\exp(c\sqrt{\log x}))$ and x sufficiently large (note that $m=m'$)

$$\sum_{p^{kf_p}\le x} \log p \sum_{\nu|p} f_p \sum_{i=1}^{m}\sum_{j=1}^{m'} \alpha_\pi(i,\nu)^k \overline{\alpha_{\pi'}(j,\nu)}^k \ll x \sum_{p^{kf_p}\le y} \frac{\log^2 p}{p^{kf_p}} \sum_{\nu|p} \Big|\sum_{i=1}^{m}\alpha_\pi(i,\nu)^k\Big|^2 \frac{1}{\log p}$$

$$= x\Big(O(1)\,\frac{1}{\log t}\Big|_2^y - \int_2^y O(1)\frac{1}{t(\log t)^2}dt\Big) \ll x\exp\{-c\sqrt{\log x}\}$$

Finally, using Conjecture 2.1.1 we get

$$\sum_{\substack{p^{f_p}\le x\\ \text{p not split}}} \log(p)a_{\pi\times\tilde{\pi}'}(p^{f_p}) \ll \sum_{\substack{p^{f_p}\le x\\ \text{p not split}}} \log(p)f_p\sum_{\nu|p}\Big|\sum_{i=1}^{m}\alpha_\pi(i,\nu)\Big|^2$$

$$\ll \sum_{\substack{p^{f_p}\le x\\ \text{p not split}}} (\log p)p^{2f_p(1/2-1/(2f_p)-\epsilon)} \ll x\Big(\sum_{n\le x}\frac{\Lambda(n)}{n^{1+\epsilon}}\Big) \ll x\exp\{-c\sqrt{\log x}\}$$

and this completes the proof of the Theorem. $\square$

CHAPTER 4
A PRIME NUMBER THEOREM FOR RANKIN-SELBERG L-FUNCTIONS OVER DIFFERENT FIELDS

4.1 The Cyclic Prime Degree Case

Let E be a cyclic Galois extension of $\mathbb{Q}$ of degree ℓ. Let π be an automorphic cuspidal representation of $GL_m(\mathbb{A}_E)$ with unitary central character. Suppose that π is stable under the action of $\mathrm{Gal}(E/\mathbb{Q})$. Thanks to Arthur and Clozel [1], π is the base change lift of exactly ℓ nonequivalent cuspidal representations

$$\pi_{\mathbb{Q}}, \pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}, ..., \pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{\ell-1}$$

of $GL_m(\mathbb{A}_{\mathbb{Q}})$, where $\eta_{E/\mathbb{Q}}$ is a nontrivial character of $\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}$ attached to the field extension E according to class field theory. Consequently, we have $L(s,\pi) = L(s,\pi_{\mathbb{Q}})L(s,\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}) \cdots L(s,\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{\ell-1})$ where the L-functions on the right side are distinct.

Similarly, let F be a cyclic Galois extension of $\mathbb{Q}$ of degree q. Let π' be an automorphic cuspidal representation of $GL_{m'}(\mathbb{A}_F)$, and suppose that π' is stable under the action of $\mathrm{Gal}(F/\mathbb{Q})$. Then we can write

$$L(s,\pi') = \prod_{j=0}^{q-1} L(s,\pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j})$$

where $\pi'_{\mathbb{Q}}$ is an irreducible cuspidal representation of $GL_{m'}(\mathbb{A}_{\mathbb{Q}})$ and $\psi_{F/\mathbb{Q}}$ is a nontrivial character of $\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}$ attached to the field extension F. Then we define the Rankin-Selberg L-function over the different number fields E and F by

$$L(s,\pi \times_{E,F} \widetilde{\pi}') = \prod_{\substack{0\le i\le \ell-1 \\ 0\le j\le q-1}} L(s,\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i} \times \widetilde{\pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j}})$$

where $L(s,\pi\otimes\eta^i\times\widetilde{\pi'\otimes\psi^j})$, $0\le i\le \ell-1,\ 0\le j\le q-1$ are the usual Rankin-Selberg L-functions over $\mathbb{Q}$ with unitary central characters. Then for $\sigma>1$, we have

$$-\frac{d}{ds}\log L(s,\pi\times_{E,F}\widetilde{\pi}')=\sum_{n=1}^{\infty}\frac{\Lambda(n)a_{\pi\times_{E,F}\widetilde{\pi}'}(n)}{n^s},$$

where

$$a_{\pi\times_{E,F}\widetilde{\pi}'}(n)=\sum_{0\le i\le\ell-1}\sum_{0\le j\le q-1}a_{\pi_{\mathbb{Q}}\otimes\eta^i_{E/\mathbb{Q}}}(n)a_{\widetilde{\pi'_{\mathbb{Q}}\otimes\psi^j_{F/\mathbb{Q}}}}(n).$$

By a prime number theorem for Rankin-Selberg L-functions $L(s,\pi\times_{E,F}\pi')$ over number fields E and F, we mean the asymptotic behavior of the sum

$$\sum_{n\le x}\Lambda(n)a_{\pi\times_{E,F}\widetilde{\pi}'}(n)=\sum_{n\le x}\sum_{0\le i\le\ell-1}\sum_{0\le j\le q-1}\Lambda(n)a_{\pi_{\mathbb{Q}}\otimes\eta^i_{E/\mathbb{Q}}}(n)a_{\widetilde{\pi'_{\mathbb{Q}}\otimes\psi^j_{F/\mathbb{Q}}}}(n)$$

Using the main theorem in Liu and Ye [21], we obtain a prime number theorem over different number fields E and F.

Theorem 4.1.1. *Let E and F be two cyclic Galois extensions of $\mathbb{Q}$ of prime degrees ℓ and q, respectively, with $(\ell,q)=1$. Let π and π' be unitary automorphic cuspidal representations of $GL_m(\mathbb{A}_E)$ and $GL_{m'}(\mathbb{A}_F)$, respectively. Assume both π and π' are invariant are the action of the respective Galois groups, and suppose that $\pi_{\mathbb{Q}}$ is self contragredient, then*

$$\sum_{n\le x}\Lambda(n)a_{\pi\times_{E,F}\widetilde{\pi}'}(n)
=\begin{cases}\dfrac{x^{1+i\tau_0}}{1+i\tau_0}+O\{x\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'_{\mathbb{Q}}\otimes\psi^{j_0}_{F/\mathbb{Q}}\cong\pi_{\mathbb{Q}}\otimes\eta^{i_0}_{E/\mathbb{Q}}\otimes|\det|^{i\tau_0} \text{ for some } \tau_0\in\mathbb{R} \text{ and some } i_0,j_0; \\ O\{x\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'_{\mathbb{Q}}\otimes\psi^{j}_{F/\mathbb{Q}}\ncong\pi_{\mathbb{Q}}\otimes\eta^{i}_{E/\mathbb{Q}}\otimes|\det|^{i\tau} \text{ for any } i,j \text{ and } \tau\in\mathbb{R}.\end{cases}$$

Proof. We begin with a lemma which shows that the τ which gives a twisted equivalent pair is unique.

Lemma 4.1.2. *Suppose that* $\pi_{\mathbb{Q}}\otimes\eta_{E/\mathbb{Q}}^{i_0} \cong \pi'_{\mathbb{Q}}\otimes\psi_{F/\mathbb{Q}}^{j_0}\otimes|\det|^{i\tau_0}$, *for some* $0 \le i_0 \le \ell-1$, $0 \le j_0 \le q-1$ *and* $\tau_0 \in \mathbb{R}$. *Then*

$$\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i_0} \cong \pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j} \otimes |\det|^{i\tau}$$

implies that $\tau = \tau_0$ *and* $j = j_0$. *Moreover, if*

$$\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i} \cong \pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j} \otimes |\det|^{i\tau}$$

for some i *and* j, *and* $\tau \in \mathbb{R}$, *then* $\tau = \tau_0$.

Proof. . By class field theory, $\eta_{E/\mathbb{Q}}, \psi_{F/\mathbb{Q}}$ are finite order idele class characters, so they are actually primitive Dirichlet characters. Assume that

$$\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i_0} \cong \pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j} \otimes |\det|^{i\tau},$$

for some $0 \le i \le \ell - 1$, $0 \le j \le q - 1$ and $\tau \in \mathbb{R}$. Then we have

$$\pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j_0} \otimes |\det|^{i\tau_0} \cong \pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j} \otimes |\det|^{i\tau}.$$

For any unramified p, we get

$$\{\alpha_{\pi'_{\mathbb{Q}}}(p,j)\psi_{F/\mathbb{Q}}^{j_0}|p|_p^{i\tau_0}\}_{j=1}^m = \{\alpha_{\pi'_{\mathbb{Q}}}(p,j)\psi_{F/\mathbb{Q}}^{j}(p)|p|_p^{i\tau}\}_{j=1}^m.$$

Hence,

$$(\psi_{F/\mathbb{Q}}^{j_0}p^{-i\tau_0})^m = (\psi_{F/\mathbb{Q}}^{j}(p)p^{-i\tau})^m.$$

Since $\psi_{F/\mathbb{Q}}$ is of finite order, we get by multiplicity one for characters $\tau = \tau_0$, so that $j = j_0$. The last conclusion of the lemma follows from the same argument just given. $\square$

Lemma 4.1.3. *Suppose that $\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i_0} \cong \pi_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j_0} \otimes |\det|^{i\tau_0}$ for some $0 \le i_0 \le \ell - 1$, $0 \le j_0 \le q-1$ and $\tau_0 \in \mathbb{R}$. Then the number of twisted equivalent pairs $(\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i}, \pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j})$ with $0 \le i \le \ell - 1$, $0 \le j \le q - 1$ divides the greatest common divisor of ℓ and q.*

Proof. By relabeling the collection $\{\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i}\}_{0 \le i \le \ell-1}$ if necessary we may assume that $\pi_{\mathbb{Q}} \cong \pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j_0} \otimes |\det|^{i\tau_0}$. Now let

$$G = (\{\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i}, 0 \le i \le \ell - 1\}, *)$$

where we define

$$\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i_1} * \pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i_2} = \pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i_1+i_2}.$$

Since the character $\eta_{E/\mathbb{Q}}$ has order ℓ we have $G \cong \mathbb{Z}/\ell\mathbb{Z}$. Now let

$$H = \{\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i} : \exists\, 0 \le j \le q-1, \tau \in \mathbb{R}, \text{ such that } \pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i} \cong \pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j} \otimes |\det|^{i\tau}\}.$$

By hypothesis, we have $\pi_{\mathbb{Q}} \in H$. Assume that

$$\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i_1} \cong \pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j_1} \otimes |\det|^{i\tau_1}$$

and

$$\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i_2} \cong \pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j_2} \otimes |\det|^{i\tau_2},$$

then

$$\begin{aligned} \pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i_1-i_2} &\cong \pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j_1} \otimes \eta_{E/\mathbb{Q}}^{-i_2} \otimes |\det|^{i\tau_1} \\ &\cong \pi_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j_1-j_2} \otimes |\det|^{i(\tau_1-\tau_2)} \\ &\cong \pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j_0+j_1-j_2} \otimes |\det|^{i(\tau_1-\tau_2+\tau_0)}. \end{aligned}$$

Hence H is a subgroup of G. By Lemma 4.1.2, each $\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i}$ is twisted equivalent to at most one $\pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j}$ so by Lagrange's theorem the number of twisted equivalent pairs divides ℓ, and by symmetry of the above argument we also have that it divides q, so the lemma follows. □

The above lemmas are simple but give the following: the second conclusion says that we can have at most one twisted equivalent pair when ℓ and q are relatively prime, and the first conclusion says that if the L-function $L(s, \pi \times_{E,F} \pi')$ has poles at $1 + i\tau_0$ and $i\tau_0$, then these are the only poles, with orders possibly bigger than one. If one considers the diagonal case

$$L(s, \pi \times_{E,F} \widetilde{\pi}) = \prod_{i=0}^{\ell-1} \prod_{j=0}^{q-1} L(s, \pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i} \times \widetilde{\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{j}})$$

then we get by **RS2** a simple pole of order ℓ at $s = 1$ for each factor in the left hand side, since the other factors on the right hand side are nonzero at $s = 1$ by **RS3**. Now assuming $\pi_{\mathbb{Q}}$ to be self-contragredient, and applying Lemma 4.1.3 to the L-function $L(s, \pi \times_{E,F} \widetilde{\pi}')$, we can use the zero-free region in **RS3** to obtain the same error term as before since

$$L(s, \pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i-1} \times \widetilde{\pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j-1}}) = L(s, \pi_{\mathbb{Q}} \times \widetilde{\pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j-1}} \otimes \eta_{E/\mathbb{Q}}^{-(i-1)}).$$

We can apply the zero-free region to all the factors in the definition of $L(s, \pi \times_{E,F} \widetilde{\pi}')$, to get the same error term as before and the Theorem follows. $\square$

In a similar fashion we can show the main term comes from those primes which split completely in the composite extension EF

Theorem 4.1.4. *Let the notations be as above and suppose for some i_0, j_0 and $\tau_0 \in \mathbb{R}$ that $\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i_0} \cong \pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j_0} \otimes |\det|^{i\tau_0}$. Assume Hypothesis H and Conjecture 2.1.1, also suppose that for any prime p for which both π_ν and π'_ω are unramified for any $\nu|p$, $\omega|p$ that we have the following: suppose that there exist primes $\mathfrak{p}$ and $\mathfrak{q}$ in the ring of integers of E and F ,respectively, lying above p which also lie below primes $\mathfrak{P}$ and $\mathfrak{Q}$ inside the ring of integers of EF with the restriction $f_{\mathfrak{Q}/\mathfrak{q}} \leq f_p$ and $f_{\mathfrak{P}/\mathfrak{p}} \leq f_p$. Then*

$$\sum_{\substack{p \leq x \\ p \text{ splits completely in } EF}} (\log p) a_{\pi \times_{BC} \widetilde{\pi}'}(p) = \frac{x^{1+i\tau_0}}{1+i\tau_0} + O\{x \exp(-c\sqrt{\log x})\}$$

Remark: Note the above theorem says that to obtain the main term we need only consider those summands for which $f_p = f'_p = 1$. Such conditions are useful in controlling sums over primes in the computation of the n-level correlation function attached to a cuspidal representation of $GL_n(\mathbb{A}_E)$ over a number field E (see [20]).

Proof. We can do a similar calculation as before by first noting that

$$\sum_{\nu|p} f_p \sum_{i=1}^{m} \sum_{j=1}^{m'} \alpha_\pi(i,\nu)^k = \sum_{a=0}^{\ell-1} \sum_{i=1}^{m} \alpha_{\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^a}(i,p)^{f_p k}$$

for $n = p^{kf_p}$, and similarly

$$\sum_{\omega|p} f'_p \sum_{j=1}^{m'} \alpha_{\pi'}(j,\nu)^k = \sum_{b=0}^{q-1} \sum_{j=1}^{m'} \alpha_{\pi_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^b}(j,\nu)^{f'_p k}$$

Thus we get

$$\sum_{n\le x}\Lambda(n)a_{\pi\times_{BC}\widetilde{\pi}'}(n)=\sum_{p^{k_1f_p}=p^{k_2f'_p}\le x}(\log p)\sum_{\nu|p}\sum_{\omega|p}f_pf'_p\sum_{i=1}^{m}\sum_{j=1}^{m'}\alpha_\pi(i,\nu)^{k_1}\overline{\alpha_{\pi'}(j,\omega)}^{k_2}$$

again by the bound for the local parameters (1.1) in 2.1

$$\begin{aligned}&\sum_{\substack{p^{k_1f_p}=p^{k_2f'_p}\le x\\ k_1>\min\{2/(m^2\ell+1),2/(m'^2q+1)\}}}(\log p)\sum_{\nu|p}\sum_{\omega|p}f_pf'_p\sum_{i=1}^{m}\sum_{j=1}^{m'}\alpha_\pi(i,\nu)^{k_1}\overline{\alpha_{\pi'}(j,\omega)}^{k_2}\\ &\ll\sum_{\substack{p^{k_1f_p}=p^{k_2f'_p}\le x\\ k_1>\min\{2/(m^2\ell+1),2/(m'^2q+1)\}}}(\log p)p^{k_1f_p(1/2-1/(m^2\ell+1))+k_2f'_p(1/2-1/(m'^2q+1))}\\ &\ll x\Big(\sum_{n\le x}\frac{\Lambda(n)}{n^{1+\epsilon}}\Big)\ll x\exp\{-c\sqrt{\log x}\}\end{aligned}$$

Now consider the sum

$$\sum_{\substack{p^{k_1f_p}=p^{k_2f'_p}\le x\\ \text{p not split in EF}}}(\log p)\sum_{\nu|p}\sum_{\omega|p}f_pf'_p\sum_{i=1}^{m}\sum_{j=1}^{m'}\alpha_\pi(i,\nu)^{k_1}\overline{\alpha_{\pi'}(j,\omega)}^{k_2}$$

since p doesn't split in EF we have $f_{\mathfrak{P}/p},f_{\mathfrak{Q}/p}\ge 2$ and since $f_{\mathfrak{P}/p}=f_{\mathfrak{P}/\mathfrak{p}}f_p\le f_p^2$ and$f_{\mathfrak{Q}/p}=f'_pf_{\mathfrak{Q}/\mathfrak{q}}\le f'^2_p$ we know that $f_p,f'_p\ge 2$ so that p doesn't split in E or F. Hence under Conjecture 2.1.1 we have the above sum is bounded by

$$\sum_{p^{k_1f_p}=p^{k_2f'_p}\le x}(\log p)p^{k_1f_p\theta_p+k_2f'_p\theta'_p}\ll x\Big(\sum_{n\le x}\frac{\Lambda(n)}{n^{1+\epsilon}}\Big)\ll x\exp\{-c\sqrt{\log x}\}$$

So if the collection of twisted equivalent pairs is nonempty we have the estimate

$$\sum_{\substack{k_1\le\min\{2/(m^2\ell+1),2/(m'^2q+1)\}\\ p^{k_1}=p^{k_2}\le x\text{ p splits in }EF}}(\log p)a_{\pi\times_{E,F}\pi'}(p^{k_1f_p})=\frac{x^{1+i\tau_0}}{1+i\tau_0}+O\{x\exp(-c\sqrt{\log x})\}$$

Using Hypothesis H as before and the above estimates we can restrict the sum to $k_1=k_2=1$ to get

$$\sum_{\substack{p\le x\\ \text{p splits completely in }EF}}(\log p)a_{\pi\times_{BC}\pi'}(p)=\frac{x^{1+i\tau_0}}{1+i\tau_0}+O\{x\exp(-c\sqrt{\log x})\}$$

as desired. $\square$

We now complete the cyclic case when the degrees of the extensions are not relatively prime.

Proof of Theorem 1.1.1. First note that since E and F are of prime degree, if $E \neq F$ then we have an isomorphism of Galois groups $Gal(EF/\mathbb{Q}) \cong Gal(E/\mathbb{Q}) \times Gal(F/\mathbb{Q})$ given by restriction. We also have that $EF/\mathbb{Q}$ is a solvable extension of degree ℓ^2, so that the base change map $BC_{EF/\mathbb{Q}}$ exists, and we denote $\pi_{EF} = BC_{EF/E}(\pi)$. Suppose first that π_{EF} is cuspidal, then from [32] we get $BC^{-1}_{EF/\mathbb{Q}}(\pi_{EF}) = \{\pi_{\mathbb{Q}} \otimes \chi_i\}_{i \in I}$ for some idele class characters of the quotient

$$\chi_i : \mathbb{A}_{\mathbb{Q}}^{\times}/(\mathbb{Q}^{\times} N_{EF/\mathbb{Q}}(\mathbb{A}_{EF}^{\times})) \longrightarrow \mathbb{C}^{\times}$$

Note that there are ℓ^2 distinct representations which lift to π_{EF} if π_{EF} is cuspidal. To see this take

$$BC^{-1}_{EF/E}(\pi_{EF}) = \{\pi \otimes \eta^{j}_{EF/E}\}_{j=0}^{\ell-1}$$

and these representations are distinct. Now let $C_K = \mathbb{A}_K^{\times}/K^{\times}$ for any number field K, and consider the character $\psi_{F/\mathbb{Q}} \circ N_{E/\mathbb{Q}}$ which is a character on C_E trivial on $N_{EF/E}(C_{EF})$. Thus we have that for some $0 \leq i \leq \ell - 1$, $\eta^{i}_{EF/E} = \psi_{F/\mathbb{Q}} \circ N_{E/\mathbb{Q}}$ and if the character on the right hand side is trivial we get that $N_{E/\mathbb{Q}}(C_E) \subseteq ker(\psi_{F/\mathbb{Q}}) = N_{F/\mathbb{Q}}(C_F)$ so by class field theory $F \subseteq E$ which is a contradiction, so that $\psi_{F/\mathbb{Q}} \circ N_{E/\mathbb{Q}}$ is nontrivial and so has order ℓ. Hence $\eta_{EF/E} = \eta_{EF/\mathbb{Q}} \circ N_{E/\mathbb{Q}}$ for some class field theory character $\eta_{EF/\mathbb{Q}}$, in other words $\eta_{EF/E}$ lies in the image of the base change map so that we can take

$$BC_{E/\mathbb{Q}}^{-1}(\pi \otimes \eta_{EF/E}^{j}) = \{\pi^j \otimes \eta_{E/\mathbb{Q}}^{i}\}_{i=0}^{\ell-1}$$

and again these are distinct for each fixed j. Consider the collection $\{\pi^j \otimes \eta_{E/\mathbb{Q}}^{i}\}_{0\le i,j\le \ell-1}$, which all lift to π_{EF} by the transitivity of the base change map; moreover they are distinct since given $\pi^{j_1} \otimes \eta_{E/\mathbb{Q}}^{i_1} \cong \pi^{j_2} \otimes \eta_{E/\mathbb{Q}}^{i_2}$ then applying $BC_{E/\mathbb{Q}}$ gives

$$\pi \otimes \eta_{EF/E}^{j_1} \cong \pi \otimes \eta_{EF/E}^{j_2}$$

which implies $j_1 = j_2 \bmod(\ell)$ so that $i_1 = i_2 \bmod(\ell)$ and this proves the claim. Finally using the isomorphism of Galois groups we get that any character on $\mathbb{A}_{\mathbb{Q}}^{\times}/(\mathbb{Q}^{\times} N_{EF/\mathbb{Q}}(\mathbb{A}_{EF}^{\times}))$ may be written as $\eta_{E/\mathbb{Q}}^{i} \otimes \psi_{F/\mathbb{Q}}^{j}$ for some $0 \le i,j \le \ell-1$. From this and the preceding remarks it follows that the representations

$$\{\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i} \otimes \psi_{F/\mathbb{Q}}^{j}\}_{0\le i,j\le \ell-1}$$

are distinct. Now suppose the set T of twisted equivalent pairs is nonempty, so that for some $0 \le i_0, j_0 \le \ell - 1$ and $\tau_0 \in \mathbb{R}$ we have

$$\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i_0} \cong \pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j_0} \otimes |\det|^{i\tau_0}$$

If we have another twisted equivalent pair, say

$$\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i_1} \cong \pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j_1} \otimes |\det|^{i\tau_1}$$

Then by Lemma 4.1.2 we may suppose that $\tau_0 = \tau_1$, thus we get

$$\begin{aligned} \pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i_1} \otimes \psi_{F/\mathbb{Q}}^{j_0} &\cong \pi'_{\mathbb{Q}} \otimes \psi_{F/\mathbb{Q}}^{j_1} \otimes \psi_{F/\mathbb{Q}}^{j_0} \otimes |\det|^{i\tau_0} \\ &\cong \pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{i_0} \otimes \psi_{F/\mathbb{Q}}^{j_1} \end{aligned}$$

so that $i_1 = i_0 \bmod(\ell)$ and $j_1 = j_0 \bmod(\ell)$ as desired. Finally suppose that π_{EF} is not cuspidal, then from [1] we get that $\ell | n$ and

$$\pi \otimes \eta_{EF/E} \cong \pi$$

for some nontrivial character of $\mathbb{A}_E^\times/(E^\times N_{EF/E}(\mathbb{A}_{EF}^\times))$. As before we get that $\eta_{EF/E} = \psi^k_{F/\mathbb{Q}} \circ N_{E/\mathbb{Q}}$ for some $1 \leq r \leq \ell - 1$, hence $\pi_{\mathbb{Q}} \cong \pi_{\mathbb{Q}} \otimes \eta^s_{E/\mathbb{Q}} \otimes \psi^r_{F/\mathbb{Q}}$ for some $0 \leq s \leq \ell - 1$ again from [1]. As before suppose that

$$\pi_{\mathbb{Q}} \otimes \eta^{i_0}_{E/\mathbb{Q}} \cong \pi'_{\mathbb{Q}} \otimes \psi^{j_0}_{F/\mathbb{Q}} \otimes |\det|^{i\tau_0}$$

Then by a simple calculation we get another twisted equivalent pair

$$\pi_{\mathbb{Q}} \otimes \eta^{s+i_0}_{E/\mathbb{Q}} \cong \pi'_{\mathbb{Q}} \otimes \psi^{j_0-r}_{F/\mathbb{Q}} \otimes |\det|^{i\tau_0}$$

and these two pairs are distinct, so by Lemma 4.1.3 we get $|T| = \ell$, and the result follows.

□

4.2 The Solvable Galois Invariant Case

Let $E/\mathbb{Q}$ and $F/\mathbb{Q}$ be any solvable Galois extensions of degrees ℓ and ℓ', and let π and π' denote automorphic cuspidal representations of $GL_n(\mathbb{A}_E)$ and $GL_m(\mathbb{A}_F)$, respectively. Moreover suppose that both π and π' admit base change from $\mathbb{Q}$; in other words assume that the sets $BC^{-1}_{E/\mathbb{Q}}(\pi)$ and $BC^{-1}_{F/\mathbb{Q}}(\pi')$ are nonempty. Then by Theorem 2 of [32] we can write

$$BC^{-1}_{E/\mathbb{Q}}(\pi) = \{\pi_{\mathbb{Q}} \otimes \chi_i\}_{i \in I}$$

for some idele class characters χ_i trivial on $\mathbb{Q}^\times N_{E_{ab}/\mathbb{Q}}(\mathbb{A}^\times_{E_{ab}})$, where E_{ab} denotes the fixed field of the commutator subgroup $[Gal(E/\mathbb{Q}), Gal(E/\mathbb{Q})]$. Similarly we can write

$$BC^{-1}_{F/\mathbb{Q}}(\pi') = \{\pi'_{\mathbb{Q}} \otimes \psi_j\}_{j \in J}$$

for some idele class characters of $\mathbb{A}^\times_{\mathbb{Q}}$ trivial on $\mathbb{Q}^\times N_{F_{ab}/\mathbb{Q}}(\mathbb{A}^\times_{F_{ab}})$. Consider the towers of extensions coming from the cyclic composition factors of $Gal(E/\mathbb{Q})$ and $Gal(F/\mathbb{Q})$ using the Galois correspondence.

$$E = E_0 \supset E_1 \supset ... \supset E_k \supset E_{k+1} = \mathbb{Q} \tag{2.1}$$

$$F = F_0 \supset F_1 \supset ... \supset F_r \supset F_{t+1} = \mathbb{Q} \tag{2.2}$$

with $[E_i, E_{i+1}] = \ell_{i+1}$ of prime degree for $0 \leq i \leq k$ and $[F_j : F_{j+1}] = q_{j+1}$ of prime degree for $0 \leq j \leq t$. We will actually need stronger assumptions on the Galois invariance of the representations in the fibers $BC^{-1}_{E/E_i}(\pi)$ for all i. More precisely suppose that π is invariant under the action of $Gal(E/E_1)$ and that the representations in the fiber $BC^{-1}_{E/E_i}(\pi)$ are invariant under the action of $Gal(E_i/E_{i+1})$ for any $2 \leq i \leq k$, then we define the Rankin-Selberg L-function over the fields E and F by

$$\begin{aligned} L(s, \pi \times_{BC} \pi') &= \prod_{i \in I} \prod_{j \in J} L(s, \pi_{\mathbb{Q}} \otimes \chi_i \times \pi'_{\mathbb{Q}} \otimes \psi_j) \\ &= L(s, AI_{E/\mathbb{Q}}(\pi) \times AI_{F/\mathbb{Q}}(\pi)) \end{aligned}$$

To simplify notation first consider the two step case

$$E = E_0 \supset E_1 \supset E_2 = \mathbb{Q} \tag{2.3}$$

Then by assumption the ℓ_1 distinct representations

$$BC^{-1}_{E/E_1}(\pi) = \{\pi_{E_1} \otimes \eta^i_{E/E_1}\}_{i=0}^{\ell_1-1}$$

are all invariant under the action of $Gal(E_1/E_2)$ and from the proof of Theorem 2 in [32] this forces $\eta^\sigma_{E/E_1} \cong \eta_{E/E_1}$ for all $\sigma \in Gal(E_1/E_2)$ so that $\eta_{E/E_1} = \eta_{E/\mathbb{Q}} \circ N_{E_1/\mathbb{Q}}$ for some idele character on $\mathbb{A}^\times_{\mathbb{Q}}$ trivial on $\mathbb{Q}^\times N_{E_{ab}/\mathbb{Q}}(\mathbb{A}^\times_{E_{ab}})$. Thus we can write

$$BC^{-1}_{E/\mathbb{Q}} = \{\pi_{\mathbb{Q}} \otimes \eta^i_{E/\mathbb{Q}} \otimes \eta^j_{E_1/\mathbb{Q}}\}_{\substack{0\leq j\leq \ell_2-1\\ 0\leq i\leq \ell_1-1}}$$

for some cuspidal automorphic $\pi_{\mathbb{Q}}$ on $GL_n(\mathbb{A}_{\mathbb{Q}})$. Note the above representations are distinct, which can be seen using the fact that $BC_{E/E_1}(\pi_{\mathbb{Q}} \otimes \eta^i_{E/\mathbb{Q}}) = BC_{E/E_1}(\pi_{\mathbb{Q}}) \otimes \eta^i_{E/E_1}$. In other words if we have $\pi_{\mathbb{Q}} \otimes \eta^{i_1}_{E/\mathbb{Q}} \otimes \eta^{j_1}_{E_1/\mathbb{Q}} \cong \pi_{\mathbb{Q}} \otimes \eta^{i_2}_{E/\mathbb{Q}} \otimes \eta^{j_2}_{E_1/\mathbb{Q}}$ then the preceding remark implies that $i_1 = i_2 \bmod(\ell_1)$ so that $j_1 = j_2 \bmod(\ell_2)$. Thus inductively we get that in the general case we have the ℓ distinct representations which lift to π from $\mathbb{Q}$

$$BC^{-1}_{E/\mathbb{Q}}(\pi) = \{\pi_{\mathbb{Q}} \otimes (\otimes^k_{a=0}\eta^{i_a}_{E_a/\mathbb{Q}})\}_{i_a=0}^{\ell_{a+1}-1}$$

If we make similar Galois invariance assumptions for π' then we can write

$$BC^{-1}_{F/\mathbb{Q}}(\pi') = \{\pi'_{\mathbb{Q}} \otimes (\otimes^t_{b=0}\psi^{j_b}_{F_b/\mathbb{Q}})\}_{j_b=0}^{q_{b+1}-1}$$

and these are also distinct. By a similar calculation as before we put a group structure on the above representations and show that the set of twisted equivalent pairs divides the order of this group.

Theorem 4.2.1. *Let notation and assumptions be as in Theorem 1.1.2, and suppose* *further that* $(\ell, \ell') = 1$, *then*

$$\sum_{n \le x} a_{\pi \times_{E,F} \tilde{\pi}'}(n)\Lambda(n) = \frac{x^{1+i\tau(\pi,\pi')}}{1+i\tau(\pi,\pi')} + O\{x \exp(-c\sqrt{\log x})\}$$

We rely heavily on the description of the fibers of the base change map as proved in [32], and using Theorem 2 from [32] combined with Lemma 4.1.2 we get that there is at most one distinct pole of $L(s, \pi \times_{E,F} \pi')$ with multiplicity equal to one.

Proof. For completeness we first state a Lemma which is the analogue of Lemma 4.1.2, we omit the proof as it is almost identical to the one previously given.

Lemma 4.2.2. *Suppose that* $\pi_{\mathbb{Q}} \otimes (\otimes_{a=0}^{k} \eta_{E_a/\mathbb{Q}}^{i_{a,0}}) \cong \pi'_{\mathbb{Q}} \otimes (\otimes_{b=0}^{t} \psi_{F_b/\mathbb{Q}}^{j_{b,0}}) \otimes |\det|^{i\tau_0}$ *for some* $0 \le i_{a,0} \le \ell_{a+1}$ *and* $0 \le j_{b,0} \le q_{b+1}$ *then*

$$\pi_{\mathbb{Q}} \otimes (\otimes_{a=0}^{k} \eta_{E_a/\mathbb{Q}}^{i_{a,0}}) \cong \pi'_{\mathbb{Q}} \otimes (\otimes_{b=0}^{t} \psi_{F_b/\mathbb{Q}}^{j_b}) \otimes |\det|^{i\tau}$$

implies that $j_b = j_{b,0}$ *for* $b = 0, ..., t$ *and* $\tau = \tau_0$. *Moreover, if for some* i_a *and* j_b *with* $a = 0, ..., k,\ b = 0, ..., t$

$$\pi_{\mathbb{Q}} \otimes (\otimes_{a=0}^{k} \eta_{E_a/\mathbb{Q}}^{i_a}) \cong \pi'_{\mathbb{Q}} \otimes (\otimes_{b=0}^{t} \psi_{F_b/\mathbb{Q}}^{j_b}) \otimes |\det|^{i\tau}$$

then $\tau = \tau_0$

By Lemma 4.2.2 if the set T is nonempty the exponent τ is uniquely determined and we denote this by $\tau(\pi, \pi')$. We will use the notation as in the introduction. Since the representations

$$BC_{E/\mathbb{Q}}^{-1}(\pi) = \{\pi_{\mathbb{Q}} \otimes (\otimes_{a=0}^{k} \eta_{E_a/\mathbb{Q}}^{i_a})\}_{i_a=0}^{\ell_{a+1}-1}$$

are distinct, we get a well-defined group operation by setting

$$(\pi_{\mathbb{Q}} \otimes (\otimes_{a=0}^{k} \eta_{E_a/\mathbb{Q}}^{i_a})) * (\pi_{\mathbb{Q}} \otimes (\otimes_{a=0}^{k} \eta_{E_a/\mathbb{Q}}^{i'_a}))$$

$$= \pi_{\mathbb{Q}} \otimes (\otimes_{a=0}^{k} \eta_{E_a/\mathbb{Q}}^{i_a+i'_a})$$

and we denote this group of order ℓ by $(\mathcal{G}, *)$. Now suppose the set T is nonempty, then

$$\sigma_{\mathbb{Q}} \cong \sigma'_{\mathbb{Q}} \otimes |\det|^{i\tau(\pi,\pi')}$$

for any $(\sigma_{\mathbb{Q}}, \sigma'_{\mathbb{Q}}) \in T$, and by relabeling if necessary we may assume that $(\pi_{\mathbb{Q}}, \pi'_{\mathbb{Q}} \otimes (\otimes_{b=0}^{t} \psi_{F_b/\mathbb{Q}}^{j_{b,0}})) \in T$ for some $\pi'_{\mathbb{Q}} \otimes (\otimes_{b=0}^{t} \psi_{F_b/\mathbb{Q}}^{j_{b,0}})) \in BC_{F/\mathbb{Q}}^{-1}(\pi')$. Moreover given two twisted equivalent pairs

$$\pi_{\mathbb{Q}} \otimes (\otimes_{a=0}^{k} \eta_{E_a/\mathbb{Q}}^{i_a}) \cong \pi'_{\mathbb{Q}} \otimes (\otimes_{b=0}^{t} \psi_{F_b/\mathbb{Q}}^{j_b}) \otimes |\det|^{i\tau(\pi,\pi')}$$

$$\pi_{\mathbb{Q}} \otimes (\otimes_{a=0}^{k} \eta_{E_a/\mathbb{Q}}^{i'_a}) \cong \pi'_{\mathbb{Q}} \otimes (\otimes_{b=0}^{t} \psi_{F_b/\mathbb{Q}}^{j'_b}) \otimes |\det|^{i\tau(\pi,\pi')}$$

we obtain

$$\pi_{\mathbb{Q}} \otimes (\otimes_{a=0}^{k} \eta_{E_a/\mathbb{Q}}^{i_a-i'_a}) \cong \pi'_{\mathbb{Q}} \otimes (\otimes_{b=0}^{t} \psi_{F_b/\mathbb{Q}}^{j_b}) \otimes (\otimes_{a=0}^{k} \eta_{E_a/\mathbb{Q}}^{-i'_a}) \otimes |\det|^{i\tau(\pi,\pi')}$$

$$\cong \pi_{\mathbb{Q}} \otimes (\otimes_{b=0}^{t} \psi_{F_b/\mathbb{Q}}^{j_b-j'_b}) \cong \pi'_{\mathbb{Q}} \otimes (\otimes_{b=0}^{t} \psi_{F_b/\mathbb{Q}}^{j_{b,0}+j_b-j'_b}) \otimes |\det|^{i\tau(\pi,\pi')}$$

so it follows that the subset $\mathcal{H} \subset \mathcal{G}$ defined by

$$\mathcal{H} = \{\sigma_{\mathbb{Q}} \in \mathcal{G} | (\sigma_{\mathbb{Q}}, \sigma'_{\mathbb{Q}}) \in T, \exists \sigma'_{\mathbb{Q}} \in BC_{F/\mathbb{Q}}^{-1}(\pi')\}$$

forms a subgroup of $\mathcal{G}$ so by LaGrange's theorem we get $|\mathcal{H}|$ divides ℓ. Moreover Lemma 4.2.2 gives that for $(\sigma_{\mathbb{Q}}, \sigma'_{\mathbb{Q}}) \in T$ then $\sigma_{\mathbb{Q}}$ is twisted equivalent to at most one

$\sigma'_{\mathbb{Q}} \in BC^{-1}_{F/\mathbb{Q}}(\pi')$, hence $|T| = |\mathcal{H}|$. Thus applying the same argument above to the collection

$$\{\pi'_{\mathbb{Q}} \otimes (\otimes_{b=0}^{t} \psi^{j_b}_{F_b/\mathbb{Q}})\}_{j_b=0}^{q_{b+1}-1}$$

we get that the cardinality of T also divides ℓ', so that $|T| = 1$ since $(\ell, \ell') = 1$. Finally, assuming $\pi_{\mathbb{Q}}$ to be self-contragredient and using the equality

$$L(s, \pi_{\mathbb{Q}} \times \chi \times \widetilde{\pi'_{\mathbb{Q}} \otimes \xi}) = L(s, \pi_{\mathbb{Q}} \times \widetilde{\pi'_{\mathbb{Q}} \otimes \chi^{-1}\xi})$$

valid for any finite order idele class characters we can apply the zero-free region to obtain the desired error term, and the theorem follows. □

Proof of Theorem 1.1.2

Again write

$$BC^{-1}_{E/\mathbb{Q}}(\pi) = \{\pi_{\mathbb{Q}} \otimes \eta_i\}_{i \in I}$$

$$BC^{-1}_{F/\mathbb{Q}}(\pi') = \{\pi'_{\mathbb{Q}} \otimes \xi_j\}_{j \in J}$$

for some idele class characters η_i and ξ_j assigned to the field extensions E and F by class field theory. So that

$$L(s, AI_{E/\mathbb{Q}}(\pi) \times \widetilde{AI_{F/\mathbb{Q}}(\pi')}) = \prod_{\substack{i \in I \\ j \in J}} L(s, \pi_{\mathbb{Q}} \otimes \eta_i \times \widetilde{\pi'_{\mathbb{Q}} \otimes \xi_j})$$

and the right side has a pole if and only if $\pi_{\mathbb{Q}} \otimes \eta_{i_0} \cong \pi'_{\mathbb{Q}} \otimes \xi_{j_0} \otimes |\det|^{i\tau_0}$ for some $i_0 \in I$, $j_0 \in J$ and $\tau_0 \in \mathbb{R}$. First suppose that $E \cap F = \mathbb{Q}$, then we have an isomorphism of Galois groups $Gal(EF/\mathbb{Q}) \cong Gal(E/\mathbb{Q}) \times Gal(F/\mathbb{Q})$, so that any class character $\eta_{EF/\mathbb{Q}}$ associated to EF may be written as $\eta_i \otimes \xi_j$ for some $i \in I$ and $j \in J$, and by the same argument as before it suffices to count the number of distinct characters of this form. If $BC_{EF/\mathbb{Q}}(\pi) = \pi_{EF}$ is cuspidal we consider the fiber

$$BC^{-1}_{EF/EF_1}(\pi_{EF}) = \{\pi_{EF_1} \otimes \eta^i_{EF/EF_1}\}_{i=0}^{q_1-1}$$

these representations are also invariant under the action of $Gal(EF/EF_1)$, since if we consider the isomorphism $\phi_1 : Gal(EF/EF_1) \longrightarrow Gal(F/F_1)$ given by restriction, then composing with the Artin maps

$$A_{EF/EF_1} : \mathbb{A}^\times_{EF_1}/(EF_1)^\times N_{EF/EF_1}(\mathbb{A}^\times_{EF}) \longrightarrow Gal(EF/EF_1)$$

$$A_{F/F_1} : \mathbb{A}^\times_{F_1}/F_1^\times N_{F/F_1}(\mathbb{A}^\times_F) \longrightarrow Gal(F/F_1)$$

we get that the corresponding function $\psi = A^{-1}_{F/F_1} \circ \phi_1 \circ A_{EF/EF_1}$ is given by the norm N_{EF_1/F_1}, since for $a \in \mathbb{A}^\times_{EF_1}$

$$\phi_1 \circ A_{EF/EF_1}(a) = res_F(a, EF/EF_1) = (N_{EF_1/F_1}(a), F/F_1) = A_{F/F_1}(N_{EF_1/F_1}(a))$$

by Theorem 4 Ch X section 3 of [17]. It follows that ψ induces an isomorphism of character groups

$$\psi^* : \left(\mathbb{A}^\times_{F_1}/F_1^\times N_{F/F_1}(\mathbb{A}^\times_F)\right)^* \longrightarrow \left(\mathbb{A}^\times_{EF_1}/(EF_1)^\times N_{EF/EF_1}(\mathbb{A}^\times_{EF})\right)^*$$

given by $\psi^*(\chi) = \chi \circ \psi$ for $\chi \in \left(\mathbb{A}^\times_{F_1}/F_1^\times N_{F/F_1}(\mathbb{A}^\times_F)\right)^*$. Moreover, for $\sigma \in Gal(EF_1/EF_2)$ we have $\psi^*(\chi^{\phi_1(\sigma)}) = (\psi^*(\chi))^\sigma$, this follows since if P_K denotes the places of any number field $K/\mathbb{Q}$, σ induces an action on any completion

$$\sigma : (EF_1)_\alpha \longrightarrow (EF_1)_{\sigma(\alpha)}$$

with the absolute value on $(EF_1)_{\sigma(\alpha)}$ given by $|x|_{\sigma(\alpha)} = |\sigma^{-1}(x)|_\alpha$ for $x \in (EF_1)$ followed by completion. Thus for any $\beta \in P_{EF_2}$

$$\begin{aligned} \sigma : \prod_{\alpha|\beta}(EF_1)_\alpha &\longrightarrow \prod_{\alpha|\beta}(EF_1)_\alpha \\ (a_\alpha)_{\alpha|\beta} &\longmapsto (\sigma(a_\alpha))_{\sigma(\alpha)|\beta} \end{aligned}$$

So that for any $a \in \mathbb{A}^\times_{EF_1}$

$$\psi^*(\chi)^\sigma(a) = \chi(N_{EF_1/F_1}(\sigma^{-1}(a))) = \prod_{\nu \in P_{F_1}} \chi_\nu(\prod_{\substack{\omega|\nu \\ \omega \in P_{EF_1}}} N_{(EF_1)_\omega/(F_1)_\nu}(\sigma^{-1}(a_\omega)))$$

where $N_{(EF_1)_\omega/(F_1)_\nu}(a_\omega) = \prod_{\tau_\omega \in \Gamma_{(EF_1)_\omega/(F_1)_\omega}} \tau_\omega(\sigma^{-1}(a_\omega))$.

Now consider the automorphism $\sigma\tau_\omega\sigma^{-1} : (EF_1)_{\sigma(\omega)} \longrightarrow (EF_1)_{\sigma(\omega)}$ and note that

$$\begin{aligned} N_{(EF_1)_\omega/(F_1)_\nu}(\sigma^{-1}(a_\omega)) &= \sigma^{-1}\Big(\prod_{\tau_\omega \in \Gamma_{(EF_1)_\omega/(F_1)_\omega}} \sigma\tau_\omega\sigma^{-1}(a_{\sigma(\omega)})\Big) \\ &= \sigma^{-1}\Big(\prod_{\tau_{\sigma(\omega)} \in \Gamma_{(EF_1)_{\sigma(\omega)}/\sigma((F_1)_\nu)}} \tau_{\sigma(\omega)}(a_{\sigma(\omega)})\Big) \\ &= res_{\sigma((F_1)_\nu)}(\sigma^{-1})(N_{(EF_1)_{\sigma(\omega)}/\sigma((F_1)_\nu)}(a_{\sigma(\omega)})) \end{aligned}$$

Hence

$$\begin{aligned} &\prod_{\nu \in P_{F_1}} \chi_\nu(\prod_{\substack{\omega|\nu \\ \omega \in P_{EF_1}}} N_{(EF_1)_\omega/(F_1)_\nu}(\sigma^{-1}(a_\omega))) \\ = &\prod_{\nu \in P_{F_1}} \chi_\nu\Big(res_{\sigma((F_1)_\nu)}(\sigma^{-1})\Big(\prod_{\omega|\sigma(\nu)} N_{(EF_1)_\omega/\sigma((F_1)_\nu)}(a_\omega)\Big)\Big) \\ = &\ \chi(res_{F_1}(\sigma^{-1})(N_{EF_1/F_1}(a)) = \chi^{\phi_1(\sigma)}(N_{EF_1/F_1}(a)) \\ = &\ \psi^*(\chi^{\phi_1(\sigma)})(a) \end{aligned}$$

as desired. Thus any character inside $\Big(\mathbb{A}^\times_{EF_1}/(EF_1)^\times N_{EF/EF_1}(\mathbb{A}^\times_{EF})\Big)^*$ is invariant under the action of $Gal(EF_1/EF_2)$, and hence any representation in the fiber

$$BC^{-1}_{EF/EF_1}(\pi_{EF}) = \{\pi_{EF_1} \otimes \eta^i_{EF/EF_1}\}_{i=0}^{q_1-1}$$

is invariant under $Gal(EF_1/EF_2)$. In a similar fashion any representation in the fiber

$$BC^{-1}_{EF_i/EF_{i+1}}(\pi_{EF_i})$$

will be invariant under $Gal(EF_{i+1}/EF_{i+2})$ for any $0 \leq i \leq c-1$, so we get $\ell\ell'$ distinct representations on $GL_n(\mathbb{A}_{\mathbb{Q}})$ which lift to π_{EF}

$$\{\pi_{\mathbb{Q}} \otimes \eta_k\}_{k \in K}$$

and hence we get $\ell\ell'$ distinct characters η_k, $k \in K$. Moreover, for any α we can write $\eta_\alpha = \left(\otimes_{a=1}^k \eta^{i_{a,\alpha}}_{E_a/\mathbb{Q}} \right) \otimes \left(\otimes_{b=1}^t \psi^{j_{b,\alpha}}_{F_b/\mathbb{Q}} \right)$, so that given two twisted equivalent pairs

$$\begin{aligned} \pi_{\mathbb{Q}} \otimes \left(\otimes_{a=1}^k \eta^{i_{a,0}}_{E_a/\mathbb{Q}} \right) &\cong \pi'_{\mathbb{Q}} \otimes \left(\otimes_{b=1}^t \psi^{j_{b,0}}_{F_b/\mathbb{Q}} \right) \otimes \alpha_{\mathbb{Q}}^{i\tau(\pi,\pi')} \\ \pi_{\mathbb{Q}} \otimes \left(\otimes_{a=1}^k \eta^{i_{a,1}}_{E_a/\mathbb{Q}} \right) &\cong \pi'_{\mathbb{Q}} \otimes \left(\otimes_{b=1}^t \psi^{j_{b,1}}_{F_b/\mathbb{Q}} \right) \otimes \alpha_{\mathbb{Q}}^{i\tau(\pi,\pi')} \end{aligned} \tag{2.4}$$

we get

$$\pi_{\mathbb{Q}} \otimes \left(\otimes_{a=1}^k \eta^{i_{a,0}}_{E_a/\mathbb{Q}} \right) \otimes \left(\otimes_{b=1}^t \psi^{j_{b,1}}_{F_b/\mathbb{Q}} \right) \cong \pi_{\mathbb{Q}} \otimes \left(\otimes_{a=1}^k \eta^{i_{a,1}}_{E_a/\mathbb{Q}} \right) \otimes \left(\otimes_{b=1}^t \psi^{j_{b,0}}_{F_b/\mathbb{Q}} \right)$$

hence $i_{a,0} = i_{a,1} \bmod(\ell_a)$ and $j_{b,0} = j_{b,1} \bmod(q_b)$ for any a and b so that the number of twisted equivalent pairs is equal to one in this case. If $E \cap F \neq \mathbb{Q}$ we use the transitivity of the base change map by first considering $BC^{-1}_{EF/E\cap F}(\pi_{EF})$ which contains $[EF : E \cap F]$ distinct representations which lift to π_{EF}. If we also consider the towers coming from the filtrations for E and F respectively

$$E \cap F \supset E_{d+1} \cap F \supset ... \supset E_{d+e} \cap F$$

$$E \cap F \supset F_{d'+1} \cap E \supset ... \supset F_{d'+e} \cap E$$

We can obtain these as fixed fields of composition series for the groups $Gal(E \cap F/\mathbb{Q})$ as follows: consider the composition series

$$Gal(E/\mathbb{Q}) \lhd Gal(E/E_k) \lhd ... \lhd Gal(E/E_1) \lhd < e >$$

$$Gal(F/\mathbb{Q}) \lhd Gal(F/F_t) \lhd ... \lhd Gal(F/F_1) \lhd < e >$$

Now with the notation as in the proof of Theorem 4.2.1, if we let

$$\pi_{\mathbb{Q}} \otimes \left(\otimes_{a=1}^{k} \eta_{E_a/\mathbb{Q}}^{i_a} \right) = \pi_{\mathbb{Q}}(i_a)_{a=1,\dots,k}$$

$$\pi'_{\mathbb{Q}} \otimes \left(\otimes_{b=1}^{t} \xi_{F_b/\mathbb{Q}}^{j_b} \right) = \pi'_{\mathbb{Q}}(j_b)_{b=1,\dots,t}$$

if we denote $G_K = Gal(E/K)$ and $G'_K = Gal(F/K)$ for any K then since $E \cap F$ is Galois over $\mathbb{Q}$ we get composition series for the quotients $G_{\mathbb{Q}}/G_{E\cap F} \cong Gal(E\cap F/\mathbb{Q}) \cong G'_{\mathbb{Q}}/G'_{E\cap F}$

$$G_{\mathbb{Q}}/G_{E\cap F} \lhd G_{E_k}G_{E\cap F}/G_{E\cap F} \lhd ... \lhd G_{E_1}G_{E\cap F}/G_{E\cap F} \lhd < e >$$

$$G'_{\mathbb{Q}}/G'_{E\cap F} \lhd G'_{F_t}G'_{E\cap F}/G'_{E\cap F} \lhd ... \lhd G'_{F_1}G'_{E\cap F}/G'_{E\cap F} \lhd < e >$$

with composition factors $\Big(G_{E_i}G_{E\cap F}/G_{E\cap F}\Big)\Big/\Big(G_{E_{i-1}}G_{E\cap F}/G_{E\cap F}\Big) \cong G_{E_i}/G_{E_{i-1}}$ for any i which gives a nontrivial factor (similarly the composition factors for F will be isomorphic to $G_{F_j}/G_{F_{j-1}}$). Computing fixed fields we get

$$Fix_E(G_{E_i}G_{E\cap F}) = E_i \cap F \;\; i = d+1, ..., d+e = k$$

$$Fix_F(G'_{F_j}G'_{E\cap F}) = F'_j \cap E \;\; j = d'+1, ..., d'+e = t$$

so that $[E_i \cap F : E_{i+1} \cap F] = [E_i : E_{i+1}]$ and $[F_j \cap E : F_{j+1} \cap E] = [F_j : F_{j+1}]$ are of prime degree. Consequently the map

$$res_{E_i \cap F} : Gal(E_i/E_{i+1}) \longrightarrow Gal(E_i \cap F/E_{i+1} \cap F)$$

given by restriction is an isomorphism. Thus by the same argument as before get that any class character $\eta_{E_i\cap F/E_{i+1}\cap F}$ is invariant under the action of $Gal(E_{i+1}\cap F/E_{i+2}\cap F)$, so that $BC^{-1}_{E\cap F/\mathbb{Q}}(\sigma)$ contains $[E\cap F:\mathbb{Q}]$ distinct representations for any σ which occurs as a cuspidal component of $AI_{E/E\cap F}(\pi)$. Now suppose we have a twisted equivalent pair

$$\pi_{\mathbb{Q}}(i_a)_{a=1,\dots,k} \cong \pi'_{\mathbb{Q}}(j_b)_{b=1,\dots,t} \otimes \alpha_{\mathbb{Q}}^{i\tau(\pi,\pi')}$$

then applying the base change map to both sides we get

$$\pi_{E\cap F}(i_a)_{a=1,\dots,d} \cong \pi'_{E\cap F}(j_b)_{b=1,\dots,d'} \otimes \alpha_{E\cap F}^{i\tau(\pi,\pi')} \tag{2.5}$$

where

$$\pi_{E\cap F}(i_a)_{a=1,\dots,d} = \pi_{E\cap F} \otimes \Big(\otimes_{a=1}^{d} \eta_{E_a/\mathbb{Q}}^{i_a} \circ N_{E\cap F/\mathbb{Q}}\Big)$$

$$\pi'_{E\cap F}(j_b)_{b=1,\dots,d'} = \pi'_{E\cap F} \otimes \Big(\otimes_{b=1}^{d'} \xi_{F_b/\mathbb{Q}}^{j_b} \circ N_{E\cap F/\mathbb{Q}}\Big)$$

Then by the same proof when $E\cap F = \mathbb{Q}$ we get that i_a and j_b are unique for $a = 1, ..., d$ and $b = 1, ..., d'$ and

$$\begin{aligned} AI_{E\cap F/\mathbb{Q}}(\pi_{E\cap F}(i_a)_{a=1,\dots,d}) &= AI_{E\cap F/\mathbb{Q}}(\pi'_{E\cap F}(j_b)_{b=1,\dots,d'}) \otimes \alpha_{\mathbb{Q}}^{i\tau(\pi,\pi')} \\ \boxplus_{\gamma=1}^{\ell_{d+1}\dots\ell_{d+e}} \pi_\gamma &= \boxplus_{\delta=1}^{q_1\dots q_{d'+e}} \pi'_\delta \otimes \alpha_{\mathbb{Q}}^{i\tau(\pi,\pi')} \end{aligned} \tag{2.6}$$

here γ (resp. δ) runs over the $\ell_{d+1}...\ell_{d+e}$ distinct representations which lift to $\pi_{E\cap F}(i_a)_{a=1,\dots,d}$ (resp. $\pi'_{E\cap F}(j_b)_{b=1,\dots,d'}$), so that $\ell_{d+1}...\ell_{d+e} = q_{d'+1}...q_{d'+e} = [E\cap F:\mathbb{Q}]$ and after re-ordering $\pi_\gamma \cong \pi'_\delta \otimes \alpha_{\mathbb{Q}}^{i\tau(\pi,\pi')}$ for any $1 \le \gamma \le [E\cap F:\mathbb{Q}]$, $1 \le \delta \le [E\cap F:\mathbb{Q}]$. Moreover, we exactly $[E\cap F:\mathbb{Q}]$ twisted equivalent pairs in this case, since given any other pair

$$\pi_{\mathbb{Q}}(i'_a)_{a=1,\ldots,k} \cong \pi'_{\mathbb{Q}}(j'_b)_{b=1,\ldots,t} \otimes \alpha_{\mathbb{Q}}^{i\tau(\pi,\pi')}$$

the base change of this pair up to $E \cap F$ must be equivalent to (2.8) hence $\pi_{\mathbb{Q}}(i'_a)_{a=1,\ldots,k}$ must occur as a cuspidal component of (2.9), and this completes the proof.

$\square$

CHAPTER 5
FACTORIZATION OVER DIFFERENT FIELDS THROUGH N-LEVEL CORRELATION

5.1 Proof of the Main Theorem

Proof of Theorems 1.1.3-1.1.4

Suppose we have the product $L(s,\pi_1)...L(s,\pi_k)$ with π_i an automorphic cuspidal representation of $GL_{n_i}(\mathbb{A}_{F_i})$ and F_i a finite Galois extension of $\mathbb{Q}$ for $i = 1,...,k$. Suppose that for any $i = 1,...,k$ we have the factorization

$$L(s,\pi_i) = \prod_{j_i=1}^{a_i} L(s,\pi_{j_i})$$

with π_{j_i} an automorphic cuspidal representation of $GL_{n_{j_i}}(\mathbb{A}_{\mathbb{Q}})$. From [23] we immediately get a formula for the n-level correlation function.

Proposition 5.1.1. *Denote by $\underline{L}$ a set partition of $\underline{K} = (1,2,...,a_1+...+a_k)$ with $\underline{L} = [L_1,...,L_\beta]$, where $\beta = \beta(\underline{L})$ is the number of subsets in $\underline{L}$. Assume that $\pi_{j_i} \cong \pi_{j_i'}$ for j_i, j_i' in the same subset L_α, and $\pi_{j_i} \not\cong \pi_{j_i'}$ otherwise. Denote by k_α the number of elements in L_α. Let $g_1,...,g_n \in C_c^\infty(\mathbb{R}^n)$ and $\Phi \in C^1(\mathbb{R}^n)$ with Φ supported in $|\xi_1| + ... + |\xi_n| \le 2/m$ where m is the least common multiple of the collection $\{n_{j_i}\}_{\substack{1\le j_i\le a_i\\ 1\le i\le k}}$. Define $h_1,...,h_n$ and f as before, and assume Hypothesis H or that all the n_{j_i}'s are ≤ 4, then*

$$\begin{aligned}
&\sum_{\gamma_1,...,\gamma_n} h_1(\frac{\hat{\gamma}_1}{T})...h_n(\frac{\hat{\gamma}_n}{T}) f(\frac{L}{2\pi}\hat{\gamma}_1,...,\frac{L}{2\pi}\hat{\gamma}_n) \\
= &\ \frac{\kappa(\mathbf{h})}{2\pi} TL\Big\{(a_1+...+a_k)^n\Phi(0) + \sum_{1\le r\le n/2} \frac{n!(a_1+...+a_r)^{n-2r}}{r!(n-2r)!2^r}(k_1^2+...+k_\beta^2)^r \\
&\times \int_{\mathbb{R}^r} |v_1|...|v_r|\Phi(v_1,...,v_r,-v_1,...,-v_r,0,...,0)dv\Big\} + O(T)
\end{aligned}$$

Using the above Proposition and the proofs of the prime number theorems above we obtain the correlation function for a product of L-functions over different fields. More precisely, given a product

$$L(s,\pi) = L(s,\pi_1)...L(s,\pi_k)$$

with π_i an automorphic cuspidal representation of $GL_{n_i}(\mathbb{A}_{F_i})$ with F_i a cyclic Galois extension of degree ℓ_i for $i = 1, ..., k$ such that $F_i \neq F_j$ for $i \neq j$. Suppose that each representation π_i is invariant under the action of $Gal(F_i/\mathbb{Q})$. As in the above Lemma we have $a_i = \ell_i$ for $i = 1, ..., k$ and if

$$BC^{-1}_{F_i/\mathbb{Q}}(\pi_i) = \{\pi_{\mathbb{Q}}(i) \otimes \eta^{j_i}_{F_i/\mathbb{Q}}\}^{\ell_i - 1}_{j_i = 0}$$

then

$$\pi_{\mathbb{Q}}(i_1) \otimes \eta^{j_{i_1}}_{F_{i_1}/\mathbb{Q}} \cong \pi_{\mathbb{Q}}(i_2) \otimes \eta^{j_{i_2}}_{F_{i_2}/\mathbb{Q}}$$

for some i_1 and i_2 if and only if

$$BC_{F_{i_1}F_{i_2}/F_{i_1}}(\pi_{i_1}) \cong BC_{F_{i_1}F_{i_2}/F_{i_2}}(\pi_{i_2})$$

Moreover from the proof of the prime number theorem above we know that if

$$L_\alpha = \{\pi_{\mathbb{Q}}(i) \otimes \eta^{j_i}_{F_i/\mathbb{Q}}\}_{i \in I_\alpha}$$

then $BC_{F_iF_j/F_i}(\pi_i) \ncong BC_{F_iF_j/F_j}(\pi_j)$ for $i \in I_\alpha$ and $j \in I_\beta$ with $\alpha \neq \beta$. Further we have that

$$|I_\alpha| = |\{\pi_i | BC_{F_{i_0}F_i/F_{i_0}}(\pi_{i_0}) \cong BC_{F_{i_0}F_i/F_i}(\pi_i)\}|$$

Thus, if we write

$$L(s,\pi_1)...L(s,\pi_k) = L(s,\pi_{\iota_1})^{r_1}...L(s,\pi_{\iota_t})^{r_t}$$

with $BC_{F_{\iota_i}F_{\iota_j}/F_{\iota_i}}(\pi_{\iota_i}) \ncong BC_{F_{\iota_i}F_{\iota_j}/F_{\iota_j}}(\pi_{\iota_j})$ for $i \neq j$. For fixed $1 \leq \alpha \leq \gamma$ any two representations σ and σ' in K_α will have at least one common cuspidal component when automorphically induced to $\mathbb{Q}$. If we collect all the primes $\ell \in K_\alpha$ such that $|S_{\ell,\alpha}| = 1$ then we can count the number of distinct L-functions occurring in the automorphic induction of any $\sigma \in S_{\ell,\alpha}$ such that ℓ has multiplicity one. In this case we get $\sum_{\substack{\pi_{\iota_\delta} \in K_\alpha \\ \ell_{\iota_\delta} \text{ has multiplicity one}}} \ell_{\iota_\delta} - \nu_\alpha$ singleton classes for the multiplicity one primes. If $|S_{\alpha,\ell}| \geq 2$ we get $\ell - 1$ distinct classes each of cardinality $a_{\ell,\alpha}$ and $(\ell-1)b_{\ell,\alpha}$ singleton classes, and finally we have one more class of cardinality r_α. It follows that

$$\begin{aligned}
& k_1^2 + \ldots + k_\beta^2 \\
& = r_1^2 + \ldots + r_\gamma^2 \\
& + \sum_{\alpha=1}^{\gamma} \Big((\sum_{\substack{\ell_{\iota_\alpha} \in K_\alpha \\ |S_{\ell_{\iota_\alpha},\alpha}|=1}} \ell_{\iota_\alpha}) - \nu_\alpha + \sum_{\substack{\ell_{\iota_\alpha} \in K_\alpha \\ |S_{\ell_{\iota_\alpha},\alpha}| \geq 2}} a^2_{\ell_{\iota_\alpha},\alpha}(\ell_{\iota_\alpha} - 1) + b_{\ell_{\iota_\alpha},\alpha}(\ell_{\iota_\alpha} - 1) \Big)
\end{aligned}$$

which gives the formula for the correlation function in the cyclic case. Moreover this gives that if an L-function $L(s,\pi)$ with π an automorphic cuspidal representation of $GL_n(\mathbb{A}_E)$ with E a finite Galois extension of $\mathbb{Q}$, factors as before then

$$\begin{aligned}
a &= r_1^2 + \ldots + r_\gamma^2 \\
& + \sum_{\alpha=1}^{\gamma} \Big((\sum_{\substack{\ell_{\iota_\alpha} \in K_\alpha \\ |S_{\ell_{\iota_\alpha},\alpha}|=1}} \ell_{\iota_\alpha}) - \nu_\alpha + \sum_{\substack{\ell_{\iota_\alpha} \in K_\alpha \\ |S_{\ell_{\iota_\alpha},\alpha}| \geq 2}} a^2_{\ell_{\iota_\alpha},\alpha}(\ell_{\iota_\alpha} - 1) + b_{\ell_{\iota_\alpha},\alpha}(\ell_{\iota_\alpha} - 1) \Big)
\end{aligned}$$

where $G_\pi = \{\sigma \in Gal(E/\mathbb{Q}) | \pi^\sigma \cong \pi\}$ and $a = |G_\pi|$.

CHAPTER 6
CONVERGENCE OF THE EULER PRODUCT

6.1 Some Results for a General Galois Extension

We first prove the absolute convergence of $L(s, \pi \times_{E,F} \pi')$. Consider the Euler product

$$L(s, \pi \times_{E,F} \sigma) = \prod_p \prod_{\nu,\omega|p} \prod_{i=1,j=1}^{n,m} \prod_{\mu=0,\vartheta=0}^{f_p-1,f_p'-1} (1 - \alpha_\pi(j,\nu)^{1/f_p} \alpha_\sigma(i,\omega)^{1/f_p{}'} \omega_{f_p}^{\mu} \omega_{f_p{}'}^{\vartheta} p^{-s})^{-1}$$

Expanding into a geometric series, we get

$$= \prod_p \prod_{\nu,\omega|p} \prod_{i=1,j=1}^{n,m} \prod_{\mu=0,\vartheta=0}^{f_p-1,f_p'-1} \Big(\sum_{k=0}^{\infty} \frac{\alpha_\pi(i,\nu)^{k/f_p} \alpha_\sigma(j,\omega)^{k/f_p'} \omega_{f_p}^{k\mu} \omega_{f_p'}^{k\vartheta}}{p^{ks}} \Big)$$

Now taking absolute values and applying the triangle inequality

$$|L(s, \sigma \times \pi)| \leq \prod_p \prod_{\nu,\omega|p} \prod_{i=1,j=1}^{n,m} \prod_{\mu=0,\vartheta=0}^{f_p-1,f_p'-1} \Big(\sum_{k=0}^{\infty} \frac{|\alpha_\pi(i,\nu)^{k/f_p} \omega_{f_p}^{k\mu}||\alpha_\sigma(j,\omega)^{k/f_p'} \omega_{f_p'}^{k\vartheta}|}{p^{kRe(s)}} \Big)$$

$$\leq \prod_p \prod_{\nu,\omega|p} \prod_{i=1,j=1}^{n,m} \prod_{\mu=0,\vartheta=0}^{f_p-1,f_p'-1} \Big(\sum_{k=0}^{\infty} \frac{|\alpha_\pi(i,\nu)|^{2k/f_p} |\omega_{f_p}^{2k\mu}|}{p^{kRe(s)}} \Big)^{1/2} \Big(\sum_{a=0}^{\infty} \frac{|\alpha_\sigma(j,\omega)|^{2a/f_p'} |\omega_{f_p'}^{2a\vartheta}|}{p^{aRe(s)}} \Big)^{1/2}$$

$$= \prod_p \prod_{\nu,\omega|p} \prod_{i=1,j=1}^{n,m} \prod_{\mu=0,\vartheta=0}^{f_p-1,f_p'-1} \Big(\sum_{k=0}^{\infty} \frac{\alpha_\pi(i,\nu)^{k/f_p} \omega_{f_p}^{\mu k} \alpha_\pi(\bar{i},\nu)^{k/f_p} \bar{\omega_{f_p}}^{\mu k}}{p^{kRe(s)}} \Big)^{1/2}$$

$$\times \Big(\sum_{a=0}^{\infty} \frac{\alpha_\sigma(j,\omega)^{a/f_p'} \omega_{f_p'}^{a\vartheta} \alpha_\sigma(\bar{j},\omega)^{a/f_p'} \bar{\omega_{f_p'}}^{a\vartheta}}{p^{aRe(s)}} \Big)^{1/2}$$

$$= (\prod_p \prod_{\nu|p} \prod_{i=1}^{n} \prod_{\mu=0}^{f_p-1} (1 - |\alpha_\pi(i,\nu)|^{2/f_p} \omega_{f_p}^{\mu} \bar{\omega_{f_p}}^{\mu} p^{-Re(s)})^{-1})$$

$$\times\left(\prod_{p}\prod_{\omega|p}\prod_{j=1}^{m}\prod_{\vartheta=0}^{f_p'-1}(1-|\alpha_\sigma(j,\omega)|^{2/f_p'}\omega_{f_p'}^{\vartheta}\bar{\omega_{f_p'}}^{\vartheta}p^{-Re(s)})^{-1}\right)$$

This last product is a subproduct of $L(Re(s),\pi\times\tilde{\pi})^{1/2}L(Re(s),\sigma\times\tilde{\sigma})^{1/2}$, which is known to be absolutely convergent for Re(s) > 1, hence the result follows.□

For any two Galois extensions E and F of $\mathbb{Q}$ consider the Dedekind zeta functions

$$\zeta_E(s)=\prod_{p}\prod_{\nu|p}(1-p^{-f_ps})^{-1}=\prod_{p}\prod_{a=0}^{f_p-1}(1-\omega_{f_p}^{a}p^{-s})^{-t_p}$$

$$\zeta_F(s)=\prod_{p}\prod_{\omega|p}(1-p^{-f_p's})^{-1}=\prod_{p}\prod_{b=0}^{f_p'-1}(1-\omega_{f_p'}^{b}p^{-s})^{-t_p'}$$

Taking the convolution and expanding we get the Euler product

$$\prod_{p}\prod_{\nu,\omega|p}\prod_{a=0}^{f_p-1}\prod_{b=0}^{f_p'-1}(1-\omega_{f_p}^{a}\omega_{f_p'}^{b}p^{-s})^{-1}=\prod_{p}(1-p^{-f_pf_p's})^{-t_pt_p'}$$

As in the proof of the prime number theorems the most tractable case is when the degrees of the extensions are relatively prime. So assume that if E has degree ℓ over $\mathbb{Q}$ and F has degree ℓ' over $\mathbb{Q}$ that $(\ell,\ell')=1$. Then since for any prime p we have the identities $\ell=e_pf_pt_p$ and $\ell'=e_p'f_p't_p'$, and since the degree of the composite extension $EF/\mathbb{Q}$ is $\ell\ell'$, we get that the modular degree of any prime lying in the ring of integers of $EF/\mathbb{Q}$ over p is given by f_pf_p'. Moreover the number of prime ideals in the integral closure of EF lying over p is t_pt_p'. Thus we get that the above Euler product is the Dedekind zeta function of the composite

$$\zeta_{EF}(s)=\prod_{p}(1-p^{-f_pf_p's})^{-t_pt_p'}$$

Hence we obtain analytic continuation and a functional equation for free in this case. Since $\zeta_E(s)$ is the Artin L-function attached to the trivial representation 1_E of the absolute Galois group $G(\overline{\mathbb{Q}}/E) = \Gamma_E$, the induced representation $Ind_{\Gamma_E}^{\Gamma_{\mathbb{Q}}}(1_E)$ can be canonically identified with the group algebra $\mathbb{C}[\Gamma_{E/\mathbb{Q}}]$ of the finite group $\Gamma_{E/\mathbb{Q}} = \Gamma_{\mathbb{Q}}/\Gamma_E = Gal(E/\mathbb{Q})$. Moreover, the factorization of $\zeta_E(s)$ corresponds to the decomposition of $\mathbb{C}[\Gamma_{E/\mathbb{Q}}]$ into irreducibles under the right regular representation. In particular we can write

$$\zeta_E(s) = \prod_{\sigma \in \hat{\Gamma_{E/\mathbb{Q}}}} L(s, \sigma)$$

where $L(s, \sigma)$ denotes the Artin L-function attached to the irreducible representation $\sigma \in \hat{\Gamma_{E/\mathbb{Q}}}$. More precisely given an irreducible representation $\bar{\sigma}$ of the finite group $\Gamma_{E/\mathbb{Q}}$ we obtain σ by composing with the projection

$$\pi_E : \Gamma_{\mathbb{Q}} \longrightarrow \Gamma_{E/\mathbb{Q}}$$

so that $\sigma = \bar{\sigma} \circ \pi_E$. Similarly, write

$$\zeta_F(s) = \prod_{\tau \in \hat{\Gamma_{F/\mathbb{Q}}}} L(s, \tau)$$

then the convolution may be rewritten as

$$L(s, 1_E \times_{E,F} 1_F) = \prod_{\sigma \in \hat{\Gamma_{E/\mathbb{Q}}}} \prod_{\tau \in \hat{\Gamma_{F/\mathbb{Q}}}} L(s, \sigma \otimes \tau)$$

The tensor product $\sigma \otimes \tau$ factors through the subgroup $\Gamma_E \cap \Gamma_F = \Gamma_{EF}$ which gives a representation $\bar{\alpha}$ of $\Gamma_{EF/\mathbb{Q}}$. Again letting $\alpha = \bar{\alpha} \circ \pi_{EF}$ we get an identity of L-factors

$$L(s, \sigma \otimes \tau) = L(s, \alpha)$$

On the other hand if $E \cap F = \mathbb{Q}$ we have an isomorphism

$$\psi : \Gamma_{EF/\mathbb{Q}} \longrightarrow \Gamma_{E/\mathbb{Q}} \times \Gamma_{F/\mathbb{Q}}$$

given by $\psi(g\Gamma_{EF}) = (g\Gamma_E, g\Gamma_F)$. Suppose that we are given an irreducible representation $\bar{\alpha}$ of $\Gamma_{EF/\mathbb{Q}}$ with corresponding space V. Then we obtain an irreducible representation $\bar{\alpha} \circ \psi^{-1}$ of the direct product $\Gamma_{E/\mathbb{Q}} \times \Gamma_{F/\mathbb{Q}}$. Thus there exist irreducible representations $(\bar{\sigma}, V_1)$ and $(\bar{\tau}, V_2)$ of $\Gamma_{E/\mathbb{Q}}$ and $\Gamma_{F/\mathbb{Q}}$, respectively, and an intertwining map

$$\phi : V \longrightarrow V_1 \otimes V_2$$

such that for any $(g_1, g_2) \in \Gamma_{E/\mathbb{Q}} \times \Gamma_{F/\mathbb{Q}}$

$$\phi^{-1} \circ (\bar{\sigma} \otimes \bar{\tau})(g_1, g_2) \circ \phi = (\bar{\alpha} \circ \psi^{-1})(g_1, g_2)$$

For any finite prime p we can choose an embedding $\iota_p : \mathbb{Q} \hookrightarrow \mathbb{Q}_p$ which induces a homomorphism of Galois groups

$$\iota_p^* : \Gamma_{\mathbb{Q}_p} \longrightarrow \Gamma_{\mathbb{Q}}$$

Moreover, we have a short exact sequence

$$1 \longrightarrow I_p \longrightarrow \Gamma_{\mathbb{Q}_p} \longrightarrow Gal(\overline{\mathbb{F}_p}/\mathbb{F}_p) \longrightarrow 1$$

and $Gal(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ is topologically generated by the Frobenius automorphism $\phi_p : x \mapsto x^p$. In other words $Gal(\overline{\mathbb{F}_p}/\mathbb{F}_p) = \overline{< \phi_p >}$ where the closure takes place in the Krull topology. We choose $Fr_p \in \Gamma_{\mathbb{Q}_p}$ which maps to ϕ_p and the local Artin L-factor at p is by definition

$$L_p(s, \alpha) \quad = \quad \det(I - p^{-s}\alpha(\iota_p^*(Fr_p)) \mid V^{I_p})^{-1}$$

where $V^{I_p} = \{v \in V | \alpha(g)(v) = v \ \forall g \in I_p\}$. Unraveling this gives

$$\begin{aligned} L_p(s,\alpha) &= \det(I - p^{-s}(\bar{\alpha}(\iota_p^*(Fr_p)\Gamma_{EF})) \mid V^{I_p})^{-1} \\ &= \det(I - p^{-s}(\bar{\alpha} \circ \psi^{-1})(\iota_p^*(Fr_p)\Gamma_E, \iota_p^*(Fr_p)\Gamma_F) \mid V^{I_p})^{-1} \\ &= \det(I - p^{-s}[\phi^{-1} \circ \beta(\iota_p^*(Fr_p)\Gamma_E, \iota_p^*(Fr_p)\Gamma_F) \circ \phi] \mid V^{I_p})^{-1} \end{aligned}$$

where we have written $\beta(g_1, g_2) = (\bar{\sigma} \otimes \bar{\tau})(g_1, g_2)$ $g_1 \in \Gamma_E,\ g_2 \in \Gamma_F$. Note that we have the identity

$$(\phi^{-1} \circ \beta(g_1, g_2) \circ \phi) \mid V^{I_p} = \phi^{-1} \circ \beta(g_1, g_2) \mid (V_1 \otimes V_2)^{I_p} \circ \phi$$

which follows since $v \in V^{I_p} \iff \forall g \in I_p$

$$\begin{aligned} (\phi^{-1} \circ \beta(g\Gamma_E, g\Gamma_F)\phi)(v) &= (\bar{\alpha} \circ \psi^{-1})(g\Gamma_E, g\Gamma_F)(v) \\ &= \bar{\alpha}(g\Gamma_{EF})(v) = v \end{aligned}$$

if and only if

$$\beta(g\Gamma_E, g\Gamma_F)(\phi(v)) = \phi(v) \ \forall g \in I_p$$

if and only if $\phi(v) \in (V_1 \otimes V_2)^{I_p}$ which proves the claim. Thus we can write $L_p(s,\alpha)$ as

$$\begin{aligned} &\det(I - p^{-s}[\beta(\iota_p^*(Fr_p)\Gamma_E, \iota_p^*(Fr_p)\Gamma_F) \mid (V_1 \otimes V_2)^{I_p})^{-1} \\ &= \det(I - p^{-s}[\sigma(\iota_p^*(Fr_p)) \otimes \tau(\iota_p^*(Fr_p))] \mid (V_1 \otimes V_2)^{I_p})^{-1} \end{aligned}$$

but this is precisely $L_p(s, \sigma \otimes \tau)$, and it follows that

$$\prod_{\sigma \in \hat{\Gamma_{E/\mathbb{Q}}}} \prod_{\tau \in \hat{\Gamma_{F/\mathbb{Q}}}} L(s, \sigma \otimes \tau) = \prod_{\alpha \in \hat{\Gamma_{EF/\mathbb{Q}}}} L(s,\alpha) = \zeta_{EF}(s)$$

so we again obtain all the analytic properties for free.

REFERENCES

[1] J. Arthur and L. Clozel. *Simple algebras, base change, and the advanced theory of the trace formula.* Princeton University Press, 1989.

[2] J. Arthur and S. Gelbart. *Lectures on Automorphic L-Functions.*

[3] A. Borel. Automorphic l-functions. *Borel and Casselman part 2*, pages 27–61, 1979.

[4] Daniel Bump. *Automorphic Forms and Representations.* Cambridge University Press, 1998.

[5] J. Cogdell and Piateski-Shapiro. Converse theorems for gl(n). *Publ. Math. IHES*, 79:157–214, 1994.

[6] F. Dyson. Statistical theory of the energy levels of complex systems, iii. *Math. Physics*, 3:166–175, 1962.

[7] D. Flath. Decomposition of representations into tensor products. *Borel and Casselman part 1*, pages 179–184, 1979.

[8] S. Gelbart and F. Shahidi. Boundedness of automorphic l-functions in vertical strips. *Amer. Math. Soc.*, 14:79–107, 2001.

[9] I.I. H. J̃acquet, P̃iatetski-Shapiro, and J. Shalika. Rankin-selberg convolutions. *Amer. J. Math.*, 105:367–464, 1983.

[10] A. Hejhal. On the triple correlation of zeros of the zeta function. *Intern Math. Res. Notice*, pages 293–302, 1994.

[11] H. Jacquet. Principal l-functions of the linear group. *Proc. Sympos. Pure Math*, 33:63–86, 1979.

[12] H. Jacquet and J. Shalika. Rankin-selberg convolutions: Archimedean theory israel mathematical conference proceedings in honor of i.i. piateski-shapiro on the occasin of his sixtieth birthday. part i: Papers in representation theory. *Weizmann Science press of Israel.*

[13] H. Jacquet and J. A. Shalika. On euler products and the classification of automorphic representations i, ii. *Amer. J. Math.*, 103:499–558, 777–815, 1981.

[14] H. Kim and F. Shahidi. Cuspidality of symmetric powers with applications. *Duke Math. J.*, 112:177–197, 2002.

[15] Montgomery H L. The pair correlation of zeros of the zeta function. *Proc Symp Pure Math*, 24:181–193, 1973.

[16] E. Landau. Uber die anzahl der gitterpunkte in gewissen bereichen. *Gott. Nachr.*, pages 209–243, 1915.

[17] S. Lang. *Algebraic Number Theory*, volume 110. Graduate Texts in Mathematics.

[18] Ye Y B. Liu J Y. Superposition of zeros of distinct l-functions. *Forum Math.*, 14:419–455, 2002.

[19] Ye Y B. Liu J Y. Weighted selberg orthogonality and uniqueness of factorization of automorphic l-functions. *Forum Math*, 17:493–512, 2005.

[20] Ye Y B. Liu J Y. Zeroes of automorphic l-functions and noncyclic base change. *Number Theory: Tradition and Modernization*, pages 119–152, 2005.

[21] Ye Y B. Liu J Y. Perron's formula and the prime number theorem for automorphic l-functions. *Pure and Applied Mathematics Quarterly*, 3:481–497, 2007.

[22] Ye Y B. Liu J Y. Correlation of zeroes of automorphic l-functions. *Sci China Ser A-Math*, 51:1147–1166, 2008.

[23] Ye Y B. Liu J Y. Functoriality of automorphic l-functions through their zeros. *Science in China Series A*, 51:1–16, 2008.

[24] Ye Y.B. Liu J.Y., Wang Y. A proof of selberg's orthogonality conjecture for automorphic l-functions.

[25] W. Luo, Z. Rudnick, and P. Sarnak. On the generalized ramanujan conjecture for $gl(n)$. *Proc. Symp. Pure Math*, 66:301–310, 1999.

[26] Odlyzko A M. On the distribution of spacings between zeros of the zeta function. *Math Comp.*, 48:273–308, 1987.

[27] Odlyzko A M. The 10^{20} zero of the riemann zeta function and 70 million of its neighbors. 1989.

[28] M. L. Mehta. *Random Matrices.* Academic Press, Boston, 1991.

[29] C. Moeglin and J.-L. Waldspurger. Le spectre résiduel de $gl(n)$. *Ann. Sci. École Norm. Sup*, 22:605–674, 1989.

[30] C. J. Moreno. Explicit formulas in the theory of automorphic forms. *Lecture Notes Math.*, 626:73–216, 1977.

[31] C. J. Moreno. Analytic proof of the strong multiplicity one theorem. *Amer. J. Math*, 107:163–206, 1985.

[32] C.S. Rajan. On the image and fibres of solvable base change. *Mathematical Research Letters*, 9:499–508, 2002.

[33] E. Rogalwski. Functoriality and the artin conjecture. *Proceedings of Symposia in Pure Mathematics*, 61:331–353, 1997.

[34] Sarnak P. Rudnick Z. Zeroes of principal l-functions and random matrix theory. *Duke Math J*, 88:269–322, 1996.

[35] P. Sarnak. Non-vanishing of l-functions on r(s)=1. *Contributions to automorphic forms, geometry, and number theory*, pages 719–732, 2004.

[36] J.-P. Serre. Letter to j.-m deshoouillers. 1981.

[37] F. Shahidi. On certain l-functions. *Amer. J. Math.*, 103:297–355, 1981.

[38] F. Shahidi. Fourier transforms of intertwining operators and plancherel measures for $gl(n)$. *Amer. J. Math.*, 106:67–111, 1984.

[39] F. Shahidi. Local coefficients as artin factors for real groups. *Duke Math. J.*, 52:973–1007, 1985.

[40] F. Shahidi. A proof of langlands' conjecture on plancherel measures; complementary series for p-adic groups. *Ann. Math.*, 132:273–330, 1990.

[41] A. Weil. *Adeles and Algebraic Groups*. Institute for Advanced Study, 1961.