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The Milnor fiber associated to an arrangement of hyperplanes

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THE MILNOR FIBER ASSOCIATED TO AN ARRANGEMENT OF HYPERPLANES

by

Kristopher John Williams

An Abstract

Of a thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Mathematics in the Graduate College of The University of Iowa

July 2011

Thesis Supervisor: Professor Richard Randell

ABSTRACT

Let f be a non-constant, homogeneous, complex polynomial in n variables. We may associate to f a fibration with typical fiber F known as the Milnor fiber. For regular and isolated singular points of f at the origin, the topology of the Milnor fiber is well-understood. However, much less is known about the topology in the case of non-isolated singular points. In this thesis we analyze the Milnor fiber associated to a hyperplane arrangement, ie, f is a reduced, homogeneous polynomial with degree one irreducible components in n variables. If $n > 2$ then the origin will be a non-isolated singular point.

In particular, we use the fundamental group of the complement of the arrangement in order to construct a regular CW-complex that is homotopy equivalent to the Milnor fiber. Combining this construction with some local combinatorics of the arrangement, we generalize some known results on the upper bounds for the first betti number of the Milnor fiber. For several classes of arrangements we show that the first homology group of the Milnor fiber is torsion free. In the final section, we use methods that depend on the embedding of the arrangement in the complex projective plane (ie not necessarily combinatorial data) in order to analyze arrangements to which the known results on arrangements do not apply.

Abstract Approved: ____________________________

Thesis Supervisor

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Title and Department

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Date

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Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Kristopher John Williams

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the July 2011 graduation.

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ABSTRACT

Let f be a non-constant, homogeneous, complex polynomial in n variables. We may associate to f a fibration with typical fiber F known as the Milnor fiber. For regular and isolated singular points of f at the origin, the topology of the Milnor fiber is well-understood. However, much less is known about the topology in the case of non-isolated singular points. In this thesis we analyze the Milnor fiber associated to a hyperplane arrangement, ie, f is a reduced, homogeneous polynomial with degree one irreducible components in n variables. If $n > 2$ then the origin will be a non-isolated singular point.

In particular, we use the fundamental group of the complement of the arrangement in order to construct a regular CW-complex that is homotopy equivalent to the Milnor fiber. Combining this construction with some local combinatorics of the arrangement, we generalize some known results on the upper bounds for the first betti number of the Milnor fiber. For several classes of arrangements we show that the first homology group of the Milnor fiber is torsion free. In the final section, we use methods that depend on the embedding of the arrangement in the complex projective plane (ie not necessarily combinatorial data) in order to analyze arrangements to which the known results on arrangements do not apply.

TABLE OF CONTENTS

LIST OF FIGURES

Figure

CHAPTER 1
INTRODUCTION

In the study of hyperplane arrangements, a question of interest is to what extent the combinatorics of an arrangement affects the topology of the complement of the arrangement. One of the early results in this area was by Orlik and Solomon in [18] where they showed that the cohomology algebra of the complement was determined by the combinatorics. In 1993, Rybnikov presented examples of arrangements that had the same combinatorics, but whose fundamental groups were not isomorphic.

In this thesis, we study the Milnor fiber, a topological space associated to the complement of the arrangement. While many have studied the Milnor fiber [15] [7] [17] [4] [3] [12] [14], most of the work concerns computing the homology with respect to complex coefficients. Some classes of arrangements have been studied in regards to general coefficients, but the general case has been difficult to examine.

Cohen, Denham, and Suciu [3] studied a class of "multi"-arrangements such that the corresponding Milnor fiber possess torsion in its first homology group. This dissertation focuses on the question of existence of torsion in the first homology group of the Milnor fiber associated to arrangements, which so far appears to be torsion free. While we have not arrived at a definitive answer, we present several theorems and results which show further the evidence that torsion elements would be quite rare.

The remainder of this chapter covers the basic theory of arrangements and the Milnor fiber needed for the rest of the thesis. Those familiar with arrangements may feel free to skip this chapter and refer back as needed.

In Chapter 2, we use combinatorial group theory to construct a cell complex homotopy equivalent to the Milnor fiber associated to an arrangement. A large family of complements of arrangements with fundamental groups isomorphic to a finitely generated non-trivial direct product of groups will be used to motivate and understand the theorems proven in this chapter.

In Chapter 3, we state and prove a combinatorial theorem that establishes an upper bound on the first betti number of the Milnor fiber with respect to any field. As a corollary, we establish conditions that guarantee minimality of the first betti number, thereby ensuring that the first homology group is torsion free.

Chapter 4 contains several geometric theorems that may be used to show when the first homology group of the Milnor fiber is torsion free. As motivation, we show explicit calculations for the $(9_3)_1$ and $(9_3)_2$ arrangements, as well as prove a classification theorem for arrangements with points of multiplicity one, two and three only.

1.1 Arrangements

In this section, we review the ideas of arrangement theory we will need throughout the thesis. For a complete reference see the work of Orlik and Terao [18].

Let $\mathbb{K}$ be a field and let $V_{\mathbb{K}}$ be a vector space over $\mathbb{K}$ of dimension l. A hyperplane is an affine subspace of $V_{\mathbb{K}}$ with $\mathbb{K}$-codimension one.

Definition 1.1. An arrangement is a finite set of hyperplanes in $\mathbb{K}^l$.

Unless otherwise specified, we will work in a complex vector space denoted by

$\mathbb{C}^l$. Let $\mathcal{A} = \{H_1, H_2, \cdots, H_n\}$ be an arrangement in $\mathbb{C}^l$. For each H_i we may choose a degree one polynomial $\alpha_{H_i} \in \mathbb{C}[z_1, \ldots, z_l]$ such that $H_i = \text{Ker } \alpha_{H_i}$. It will often be convenient to denote an arrangement by a polynomial $Q(\mathcal{A}) = \alpha_1 \cdots \alpha_n$, where the arrangement is given by $\mathcal{A} = \{\text{Ker } \alpha_{H_1}, \ldots, \text{Ker } \alpha_{H_n}\}$. One can see that the polynomial defining a hyperplane is only defined up to multiplication by a non-zero constant. We express this idea by writing $p \doteq q$ if $p = cq$ for some $c \in \mathbb{C}^*$.

Definition 1.2. Any polynomial of the form

$$Q(\mathcal{A}) \doteq \alpha_1 \cdots \alpha_n$$

is a defining polynomial of $\mathcal{A}$.

Definition 1.3. An arrangement $\mathcal{A}$ will be called centerless if the intersection of all of the hyperplanes in the arrangement, $T = \bigcap_{H \in \mathcal{A}}$, is the empty set. If $T \neq \emptyset$, then one may choose coordinates such that T contains the origin. As a result, each hyperplane contains the origin and we call $\mathcal{A}$ a central arrangement. If $\mathcal{A}$ is central, we may choose coordinates such that each α_H is a linear form and $Q(\mathcal{A})$ is a homogeneous.

In order to discuss the topological spaces induced by an arrangement, we introduce the following standard definitions.

Definition 1.4. The variety of an arrangement is the zero locus of the polynomial associated to the arrangement. For an arrangement $\mathcal{A}$ given by a polynomial Q, we denote the variety by $N(\mathcal{A}) := \{\bar{z} \in \mathbb{C}^l : Q(\bar{z}) = 0\}$.

Definition 1.5. The complement of an arrangement is the complement of the set defined by the variety in the vector space. For an arrangement $\mathcal{A}$ we denote the complement by $M(\mathcal{A}) := \mathbb{C}^l \setminus N(\mathcal{A}) = \mathbb{C}^l \setminus \bigcup_{H\in\mathcal{A}} H$.

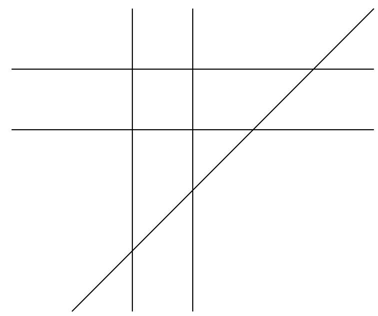

Figure 1.1: The real part of the arrangement defined by $Q(\mathcal{A}) = xy(x-1)(y-1)(y-x+2)$.

Example 1.1. In Figure 1.1 we have depicted the real part of the arrangement defined by $Q(\mathcal{A}) = xy(x-1)(y-1)(y-x+2)$, ie, the real valued solutions to the equation $Q(\mathcal{A}) = 0$.

From this diagram, we see our motivation for studying arrangements in complex vector spaces rather than in real vector spaces. Considered as an arrangement in $\mathbb{R}^2$, the complement of the arrangement is divided into a number of chambers. Each chamber is homeomorphic to $\mathbb{R}^2$. As an arrangement in $\mathbb{C}^2$, the complement is path-connected. In fact, the complement of a complex arrangement in any dimension is path-connected. $\boxplus$

For any hyperplane H in $\mathbb{C}^n$, choose coordinates so that $H = \mathrm{Ker}(z_1)$, and no other hyperplane intersects H in a ball of radius ϵ centered at the origin with

$0 < \epsilon << 1$. Then we may construct a small loop around the hyperplane by

$$p : [0, 1] \to \mathbb{C}^n$$
$$p(t) = \left(\frac{\epsilon}{2}e^{2\pi i t}, 0, \cdots, 0\right).$$

Using this path, we may connect the chambers that were separated in the case of real arrangements. It is easy to see that these loops are not contractible. Therefore, they indicate that the fundamental group of $M(\mathcal{A})$ will not be trivial.

Figure 1.1 also shows the intersection pattern of the hyperplanes. We formalize the pattern of intersections by associating to every arrangement a partially ordered set (poset).

Definition 1.6. Let $L(\mathcal{A})$ be the set of non-empty intersections of hyperplanes in $\mathcal{A}$. Define a partial order $\leq$ on the elements of $L(\mathcal{A})$ given by reverse inclusion:

$$X \leq Y \quad \iff \quad X \supseteq Y.$$

$(L(\mathcal{A}), \leq)$ is the intersection poset of $\mathcal{A}$.

We will often refer to the intersection poset by just the set $L(\mathcal{A})$ when no confusion will arise.

Example 1.2. Let $H_{i,j} := H_i \cap H_j$. Then Figure 1.2 depicts the Hasse diagram of the intersection poset of the arrangement $\mathcal{A}$ in Figure 1.1. $\boxplus$

The intersection poset of a central arrangement forms a geometric lattice: the intersection poset has a unique maximal and minimal element, any two elements have a unique supremum and unique infimum, any element of the poset may be written as

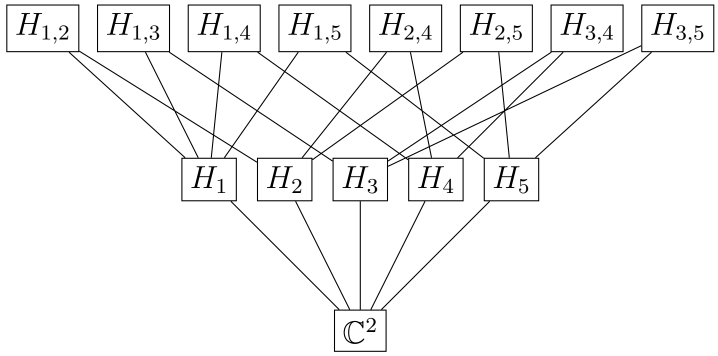

Figure 1.2: The intersection poset of the arrangement defined by $Q(\mathcal{A}) = xy(x-1)(y-1)(y-x+2)$.

the intersection of hyperplanes. Later, we will want to make use of this extra structure imposed by central arrangements. Therefore, we give the following standard coning construction that associates a central arrangement to every affine arrangement.

Definition 1.7. Let $\mathcal{A}$ be an affine arrangement in $\mathbb{C}^l$ defined by a polynomial $Q(\mathcal{A}) \in \mathbb{C}[z_1, \ldots, z_l]$. Let $Q' \in \mathbb{C}[z_0, z_1, \ldots, z_l]$ be the polynomial $Q(\mathcal{A})$ homogenized with respect to z_0. The <u>cone over $\mathcal{A}$</u> is the central arrangement in $\mathbb{C}^{l+1}$ denoted by $\mathbf{c}\mathcal{A}$ and defined by the polynomial $Q(\mathbf{c}\mathcal{A}) = z_0 Q'$.

We note that $|\mathbf{c}\mathcal{A}| = |\mathcal{A}| + 1$ and the arrangement $\mathcal{A}$ is embedded in $\mathbf{c}\mathcal{A}$ by identifying the spaces associated with $\mathcal{A}$ with the affine subspace $\mathrm{Ker}(z_0 - 1)$ in $\mathbf{c}\mathcal{A}$. From this description, we see that the coning operation has an inverse. Let $\mathcal{A}$ now denote any central arrangement in $\mathbb{C}^{l+1}$ and choose coordinates such that some hyperplane $H \in \mathcal{A}$ is defined by $H = \mathrm{Ker}(z_0)$. If $Q(\mathcal{A}) \in \mathbb{C}[z_0, \ldots, z_l]$ is a defining polynomial for $\mathcal{A}$, then we obtain a polynomial for an arrangement $\mathbf{d}\mathcal{A}$ by setting $z_0 = 1$ in $Q(\mathcal{A})$. This construction is called <u>deconing</u> and can be seen as the inverse operation to coning.

One may also see the operations of coning and deconing from a more geometric perspective. Recall the Hopf map $h : \mathbb{C}^{l+1} \setminus \{\vec{0}\} \to \mathbb{CP}^l$ which identifies any $z \in \mathbb{C}^{l+1} \setminus \{\vec{0}\}$ with γz for any $\gamma \in \mathbb{C}^*$. For a central arrangement $\mathcal{A}$, the defining polynomial $Q(\mathcal{A})$ is homogeneous and therefore defines an arrangement in $\mathbb{C}^{l+1}$ as well as in $\mathbb{CP}^l$. In the projective image of the complement of the arrangement $h(M(\mathcal{A}))$, we may identify any projective image of a hyperplane with a copy of $\mathbb{CP}^{l-1}$ and thus identify $h(M(\mathcal{A}))$ with the complement of an arrangement in affine space. This arrangement is denoted by $\mathbf{d}\mathcal{A}$. Reversing the procedure corresponds to coning the affine arrangement.

The construction has a topological implication.

Theorem 1.8 ([18], Prop 5.1)**.** *Let $\mathcal{A}$ be a central arrangement. The restriction of the Hopf map $h|_{M(\mathcal{A})} : M(\mathcal{A}) \to M(\mathbf{d}\mathcal{A})$ is a trivial bundle. Therefore,*

$$M(\mathcal{A}) \cong M(\mathbf{d}\mathcal{A}) \times \mathbb{C}^*.$$

Proof. From the discussion preceding the theorem, we see that $h(M(\mathcal{A})) = M(\mathbf{d}\mathcal{A})$. For any $H \in \mathcal{A}$, $h(M(\{H\})) = \mathbb{CP}^l \setminus \mathbb{CP}^{l-1} \cong \mathbb{C}^l$. As the Hopf map is a fiber bundle map with fiber $\mathbb{C}^*$, the restriction of the Hopf map to $M(\{H\})$ is also a bundle map by restriction. As the bundle induced by $h|_{M(\{H\})}$ has base space $\mathbb{C}^l$, it is a trivial bundle. Finally, as $h|_{M(\mathcal{A})}$ is a restriction of the trivial bundle $h|_{M(\{H\})}$, it is also trivial. Therefore we have $M(\mathcal{A}) \cong M(\mathbf{d}\mathcal{A}) \times \mathbb{C}^*$. $\square$

Example 1.3. In examples 1.1 and 1.2 we defined an arrangement $\mathcal{A}$ by $Q(\mathcal{A}) =$

$xy(x-1)(y-1)(y-x+2)$. The cone over the arrangement $\mathbf{c}\mathcal{A}$ is defined by the polynomial $Q(\mathbf{c}\mathcal{A}) = zxy(x-z)(y-z)(y-x+2z)$. The projective image is often given as in Figure 1.3 where the "line at infinity" is the line associated to the hyperplane defined by $z = 0$. Note that the lines in the affine portion of the diagram meet at the line at infinity. In Figure 1.4 we depict the intersection poset of the arrangement considered as an arrangement in $\mathbb{C}^3$. ⊞

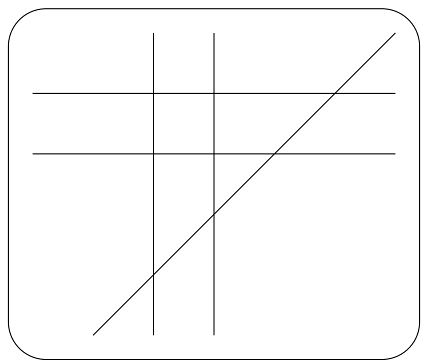

Figure 1.3: The real part of the arrangement defined by $Q(\mathbf{c}\mathcal{A}) = zxy(x-z)(y-z)(y-x+2z)$.

1.1.1 Fundamental Group

We begin this section with a quote from Hirzebruch [11]: "The topology of the complement of an arrangement of lines in the projective plane is very interesting, the investigation of the fundamental group of the complement very difficult."

The fundamental group of the complement of an arrangement possesses an intriguing structure. While the groups have been studied for many years, basic ques-

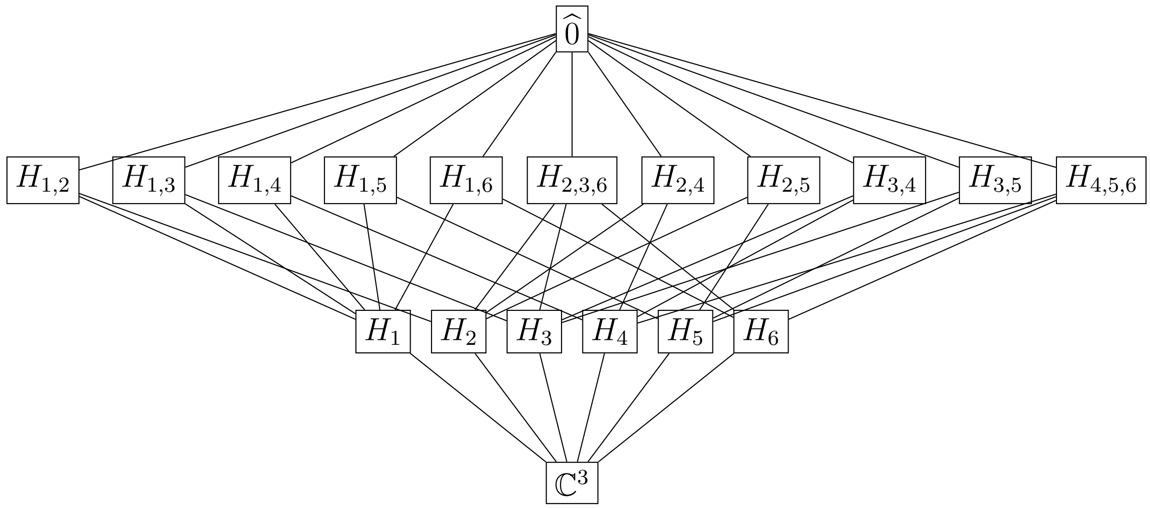

Figure 1.4: The intersection poset of the arrangement defined by $Q(\mathbf{c}\mathcal{A}) = zxy(x - z)(y - z)(y - x + 2z)$.

tions remain unanswered. The most famous question asks whether or not any group arising as the fundamental group of complement of an arrangement has torsion elements. While we do not treat this question in this thesis, we will work closely with the group, and therefore give the necessary background here.

Finding a presentation for the fundamental group of the complement of an arbitrary arrangement is a difficult task. However, we are aided by Zariski's theorem:

Theorem 1.9 (Zariski's theorem on fundamental groups, [21])**.** *Let Z be a closed m-dimensional subvariety of $\mathbb{CP}^n$. Then the inclusion $j_H : (\mathbb{CP}^n \setminus Z) \cap H \to \mathbb{CP}^n \setminus Z$ induces an epimorphism*

$$\pi_{m-1}(j_H) : \pi_{m-1}((\mathbb{CP}^n \setminus Z) \cap H) \to \pi_{m-1}(\mathbb{CP}^n \setminus Z)$$

and isomorphism for all $i < m - 1$

$$\pi_i(j_H) : \pi_i((\mathbb{CP}^n \setminus Z) \cap H) \to \pi_i(\mathbb{CP}^n \setminus Z)$$

for any "generic" hyperplane H in $\mathbb{CP}^n$.

In the case of an arrangement $\mathcal{A}$, the closed subvariety Z can be taken to be the variety associated to the arrangement $N(\mathcal{A})$. A "generic" hyperplane will be any hyperplane that is transversal to every element of the intersection poset. That is $\operatorname{codim}_{\mathbb{C}}(H \cap P) = \operatorname{codim}_{\mathbb{C}}(P)$ for all $P \in L(\mathcal{A})$ such that $\operatorname{codim}_{\mathbb{C}}(P) < n$. By repeatedly applying Theorem 1.9, we see that we may determine the fundamental group of the complement of an arrangement by examining affine arrangements in $\mathbb{C}^2$.

Several approaches to the problem of finding presentations for the fundamental group have been explored [19], [1], [5]. In this thesis, we will only make use of the presentation developed by Randell in [19], and we briefly describe it here for convenience. We follow the algorithm presented by Falk in [8] without proof.

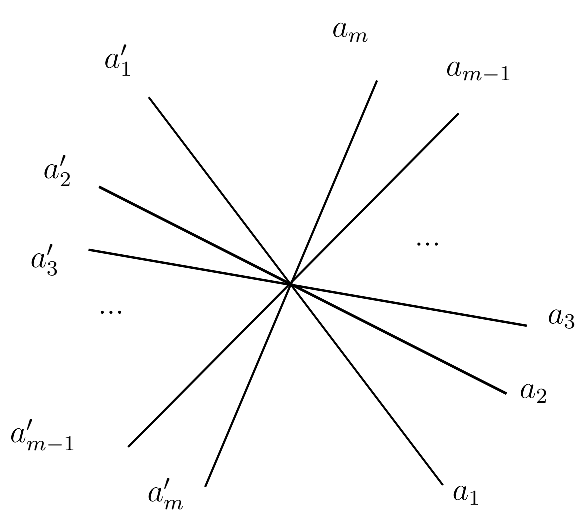

Figure 1.5: Local depiction of an intersection of lines in $\mathbb{C}^2$

As stated earlier, we assume that $\mathcal{A}$ is a complexified-real arrangement in $\mathbb{C}^2$. As such, each line has a real defining form, and we may consider the variety $N(\mathcal{A})$ as a graph $G(\mathcal{A})$ in the real plane. The graph consists of a vertex for each point where at least two hyperplanes intersect, and edges that lie on the hyperplanes. We note that edges may be rays. Give each edge a unique label, and let these be the generators in the presentation $\mathcal{P}(\mathcal{A})$ of the fundamental group. Each intersection of lines in the arrangement locally has the form shown in Figure 1.5.

At each vertex of $G(\mathcal{A})$, we introduce the following conjugation relators,

$$\begin{aligned} a_1' a_1^{-1} \\ a_2' (a_2^{a_1})^{-1} \\ a_3' (a_3^{a_2 a_1})^{-1} \\ \vdots \\ a_{m-1}' (a_{m-1}^{a_{m-2} \cdots a_1})^{-1} \\ a_m' a_m^{-1} \end{aligned}$$

where $a^b = b^{-1}ab$. We also introduce the commutation relations,

$$\begin{aligned} [a_m, a_{m-1} a_{m-2} \cdots a_2 a_1] \\ [a_m a_{m-1}, a_{m-2} \cdots a_2 a_1] \\ \vdots \\ [a_m a_{m-1} a_{m-2} \cdots, a_2 a_1] \\ [a_m a_{m-1} a_{m-2} \cdots a_2, a_1] \end{aligned}$$

which may be economically written as

$$[a_m, a_{m-1}, \cdots, a_1].$$

The resulting presentation $\mathcal{P}(\mathcal{A})$ is called the Arvola-Randell Presentation and is a presentation for $\pi_1(M(\mathcal{A}))$. (The method presented above was originally developed by Randell [19] for complexified-real arrangements and then extended by Arvola [1] for all complex arrangements. We will note when results depend on complexified-real arrangements and when they may be extended to arbitrary arrangements). One may see from the conjugation relators that the number of generators in $\mathcal{P}(\mathcal{A})$ can be reduced. One may reduce the number of generators by using Tietze transformations after the presentation has been given. However, it is easier in practice to reduce the number of generators while constructing the presentation as follows.

Note that each hyperplane in the arrangement is associated to (at most) two distinct rays in the graph $G(\mathcal{A})$. By choosing an arbitrary system of coordinates for $\mathbb{R}^2$, we may pick a distinguished ray as follows. For each hyperplane $H \in \mathcal{A}$, let ρ_1 and ρ_2 be the rays associated to the hyperplane. Let ρ be the ray with the greatest x-coordinates, or if these are all equal, then the ray with the largest y-coordinate, and let γ_H be the corresponding label. By proceeding away from the ray along the line H, we may use the conjugation relators that involve γ_H to express the generators coming from the other edges in terms of $\{\gamma_H\}_{H \in \mathcal{A}}$.

Continuing this procedure for each hyperplane in the arrangement, we arrive at a presentation with one generator for each hyperplane and $m-1$ relators for each point of multiplicity m in the arrangement.

Each generator γ_H may be identified with a meridional loop chosen compatibly with respect to the other generators around each hyperplane in the arrangement. These generators appear in both the Arvola-Randell presentation [8] and the braid monodromy presentation [5]. We will call any of these equivalent presentations geometric as the generators correspond to meridional loops about each line.

Example 1.4. Let $\mathcal{A}_n$ be the arrangement in $\mathbb{C}^n$ defined by

$$Q(\mathcal{A}_n) = \prod_{1 \le i < j \le n} (x_i - x_j).$$

These are the braid n-arrangements. They are so named because the fundamental group of the complement of the braid n-arrangement is isomorphic to the pure braid group on n strands. $\boxplus$

1.2 Milnor Fiber

Let $f \in \mathbb{C}[x_0, \ldots, x_l]$ such that $f(\vec{0}) = 0$ and f has a hypersurface singularity at the origin. Let $V(f)$ denote the variety associated to f. Let S_ϵ denote the $(2l+1)$-sphere of radius $\epsilon > 0$ centered at the origin in $\mathbb{C}^{l+1}$, and let $K = V(f) \cap S_\epsilon$.

Theorem 1.10 (Milnor [15]). *There exists $\epsilon > 0$ sufficiently small so that*

$$\phi : S_\epsilon \setminus K \to S^1 \quad \phi(\vec{z}) = \frac{f(\vec{z})}{|f(\vec{z})|}$$

is a smooth fiber bundle over S^1.

The fiber is called the Milnor fiber and will denoted by F.

If the origin is a regular point of f, then the Milnor fiber is diffeomorphic to $\mathbb{R}^{2l}$ [15]. In the case of an isolated singular point, Milnor proves the following theorem.

Theorem 1.11 ([15], pp. 57, Theorem 6.5). *If the origin is an isolated critical point of f, then the Milnor fiber has the homotopy type of a wedge of spheres. In fact,*

$$F \sim \vee_\mu S^l$$

where

$$\mu = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_0, \dots, x_l\}}{(Jf)}$$

and (Jf) is the ideal generated by partial derivatives of f, ie, $\left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_l}\right)$.

In general, the following facts are also known.

Theorem 1.12 ([15]). *The Milnor fiber is parallelizable and has the homotopy type of a finite CW-complex of dimension n.*

If we assume that f is homogeneous, then one may show that F is homotopy equivalent to $f^{-1}(1)$. Let $E = \mathbb{C}^{l+1} - V(f)$, we have the following fiber bundle.

$$F \xhookrightarrow{i} E \xrightarrow{f|_E} \mathbb{C}^* \tag{1.1}$$

From this fiber bundle, the following facts follow immediately.

Proposition 1.13.

1. *The following sequence is exact*

$$0 \longrightarrow \pi_1(F) \xrightarrow{i_\#} \pi_1(E) \xrightarrow{(f|_E)_\#} \pi_1(\mathbb{C}^*) = \mathbb{Z}$$
$$\longrightarrow \pi_0(F) \longrightarrow \pi_0(E) \longrightarrow \pi_0(S^1)$$

2. *For $i \geq 2$, $\pi_i(F)$ is isomorphic to $\pi_i(E)$.*

1.3 Arrangements and the Milnor Fiber

Let $\mathcal{A}$ be an central arrangement of d hyperplanes in $\mathbb{C}^{n+1}$ with complement $M(\mathcal{A})$. Combining the Hopf map from Theorem 1.8 with the Milnor fiber construction in (1.1), one may construct the commutative diagram in Figure 1.6.

As f is homogeneous we may define the geometric monodromy $g : F \to F$ by $g(x_0, \ldots, x_n) = (\lambda x_0, \ldots, \lambda x_n)$ where $\lambda = e^{\frac{2\pi i}{d}}$. We note that g generates a cyclic group with order d acting freely on F. By restricting the Hopf bundle $h_F : F \to M(\mathbf{d}\mathcal{A})$, we obtain the orbit map of the free action of the geometric monodromy g. That is, $(h_f, M(\mathbf{d}\mathcal{A}))$ forms a $\mathbb{Z}_d$-bundle over F, hence $F/\mathbb{Z}_d$ is homeomorphic to $M(\mathbf{d}\mathcal{A})$.

$$\begin{tikzcd}
 & F \arrow[r, "p"] \arrow[d, hook] & F/\mathbb{Z}_d \arrow[d, "\cong"] \\
\mathbb{C}^* \arrow[r, hook] & M(\mathcal{A}) \arrow[r, two heads, "h|_{M(\mathcal{A})}"'] \arrow[d, two heads, "Q(\mathcal{A})|_{M(\mathcal{A})}"'] & M(d\mathcal{A}) \\
 & \mathbb{C}^* &
\end{tikzcd}$$

Figure 1.6: Commutative diagram associated to the Milnor fiber.

The induced map $p : F \to M(\mathbf{d}\mathcal{A})$ is a covering map of order d induced by the map $\phi : \pi_1(M(\mathbf{d}\mathcal{A})) \to \mathbb{Z}_d$ which sends a meridional generator to $1 \in \mathbb{Z}_d$. The homotopy sequences of the fibrations fit together into the commutative diagram of

Figure 1.7.

$$\begin{tikzcd}
 & \pi_1(F) \arrow[r, equal] \arrow[d] & \pi_1(F) \arrow[d, "p_\#"] \\
\pi_1(\mathbb{C}^*) \arrow[r] & \pi_1(M(\mathcal{A})) \arrow[r, "(h|_{M(\mathcal{A})})_\#"'] \arrow[d, "(Q(\mathcal{A})|_{M(\mathcal{A})})_\#"'] & \pi_1(M(d\mathcal{A})) \arrow[d, "\phi"] \\
 & \mathbb{Z} & \mathbb{Z}_d
\end{tikzcd}$$

Figure 1.7: Homotopy groups associated to the Milnor fiber.

Cohen and Suciu [4] use the above construction and the Leray-Serre spectral sequence to decompose the homology of the Milnor fiber with constant coefficients in the complex numbers into a direct sum of local system homology groups induced by rank one representations.

Theorem 1.14 ([4]). *The homology of the Milnor fiber associated to a central hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^n$ may be computed as*

$$H_*(F;\mathbb{C}) \cong \bigoplus_{k=0}^{d-1} H_*(M(\mathbf{d}\mathcal{A});\mathcal{V}_k) \tag{1.2}$$

where the rank one local system $\mathcal{V}_k$ is induced by the representation $t_k : \pi_1(M(\mathbf{d}\mathcal{A})) \to \mathbb{C}^*$ where t_k sends a meridional generator to $e^{\frac{2\pi ik}{d}}$.

In particular, as $H_*(M(\mathbf{d}\mathcal{A});\mathcal{V}_0) \cong H_*(M(\mathbf{d}\mathcal{A});\mathbb{C})$, we have that

$$b_m(F) \geq b_m(M(\mathbf{d}\mathcal{A})) \tag{1.3}$$

for $0 \leq m \leq n$.

Cohen and Suciu then use Fox calculus to determine an algorithm for computing these rank one local systems in the case of line arrangements. While the method is efficient, there are some drawbacks.

First, one may generalize the method to compute the homology of F with coefficients in any algebraically closed field as along as the characteristic of the field does not divide d, the number of lines in the arrangement $\mathcal{A}$. This leaves open the question of whether or not torsion is possible in the first homology group of the Milnor fiber.

Second, the method of computation is algebraic and dependent on the presentation of the fundamental group of $M(\mathbf{d}\mathcal{A})$. Therefore, very few conjectures are possible on what numbers to expect in terms of local data on the arrangement. More clearly, one must examine the entire arrangement in order to compute the first betti number of the Milnor fiber.

CHAPTER 2
COMPLEXES HOMOTOPY EQUIVALENT TO THE MILNOR FIBER

In this chapter we develop the theory which will be used in Chapters 3 and 4. We begin with a review of combinatorial group theory and use it to construct a CW-complex that is homotopy equivalent to the Milnor fiber. Section 2.2 gives a large class of examples for which we may easily apply the theory developed in this chapter. Unless otherwise specified, all homology modules are computed with integer coefficients. That is, we compute homology as a group.

2.1 Construction for group presentations

The material in Section 2.1.1 and 2.1.2 is standard combinatorial group theory. For more information consult the work of Magnus, Karrass and Solitar [13].

2.1.1 CW-complex for a group

Let $\mathcal{P} = \langle g_\alpha, \alpha \in A : r_\beta, \beta \in B \rangle$ be a group presentation and $G(\mathcal{P})$ the group associated to the presentation. We will construct a two dimensional CW-complex with fundamental group isomorphic to G. Start with a single 0-cell denoted by v. By abuse of notation, we will attach an oriented 1-cell g_α for every generator in the group presentation by attaching both ends to the vertex v. Finally, attach a 2-cell r_β, for each relator in the presentation by attaching along the 1-cells corresponding to the letters in the word associated to the relator r_β. This is made clear in the next example. We will refer to the resulting complex as $C(\mathcal{P})$ or simply C when no confusion will arise.

Example 2.1. Let $\mathcal{P} = \langle g_1, g_2 : 1 = g_1 g_2 g_1^{-1} g_2^{-1} \rangle$. The 1-skeleton is homeomorphic to $S^1 \vee S^1$. The 2-cell is attached along g_1 with positive orientation, g_2 with positive orientation, g_1 with negative orientation, and finally g_2 with negative orientation. The result is a CW-complex homeomorphic to a torus. $\boxplus$

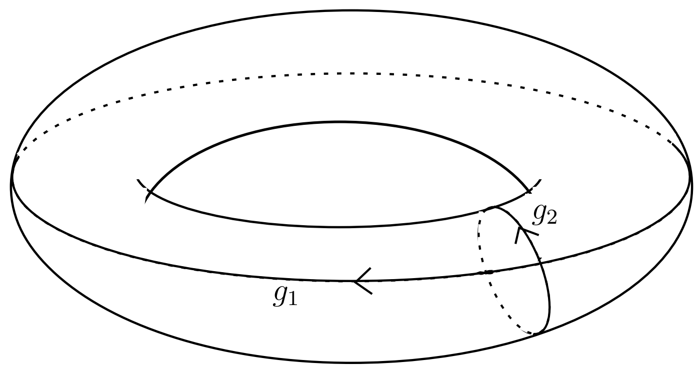

Figure 2.1: The CW-complex coming from Example 2.1.

Remark 2.1. Given any two finite presentations of a group, it is possible to transform one presentation into the other via a sequence of Tietze transformations. However, one must exercise care when performing these transformations as they can alter the homotopy type of the associated CW-complex. In [8], Falk lists the following transformations as not affecting the homotopy type:

(i) Replace any relator r by $w^{-1} r^{\pm 1} w$ where w is any word in the generators.

(ii) Delete a generator g and a relator $g w^{-1}$ where w is a word in the generators that does not contain g, and in each relator replace g with w.

(iii) For any distinct relators r and s, replace r with rs.

Also listed is a transformation that changes the homotopy type by wedging the complex with a copy of S^2:

(iv) Insert a relator that is a consequence of other relators.

Any two presentations of a group may be attained by a sequence of these transformations and their inverses. $\boxplus$

2.1.2 Covering of a CW complex

We now review the construction of a covering space associated to an epimorphism $\phi\colon \pi_1(C) \twoheadrightarrow \mathbb{Z}_m$ when C is a two dimensional CW-complex.

Let $\mathcal{P} = \langle g_1, g_2, \dots, g_n : r_\beta,\ \beta \in B\rangle$ be a group presentation for $\pi_1(C)$ and let $\langle x : x^m\rangle$ be a presentation for $\mathbb{Z}_m$. We may define a homomorphism $\phi\colon \pi_1(C) \to \mathbb{Z}_m$ by $\phi(g_i) = x^{d_i}$ for all $g_i \in G$ and $0 \le d_i < m$ as long as the order of g_i is divisible by d_i or is infinite. Call the tuple $(d_1, d_2, \dots, d_i, \dots, d_n)$ the weights associated to the cover. If ϕ is surjective, we may assume that for some i we have $d_i = 1$. We will use the structure of the CW-complex C in order to construct a cylic covering space.

Begin by taking a set of zero-cells denoted by $\{v, xv, x^2v, \cdots, x^{m-1}\}$, corresponding to lifting v from C via the homomorphism. We then (by abuse of notation) attach m oriented one-cells for each generator of the group presentation. Denote the lifts by $x^j g_i$ for $0 \le j \le m$ and $1 \le i \le n$. These are attached such that the 1-cell is positively oriented from $x^j v$ to $x^{j+d_i}v$.

Finally, we attach a 2-cell at each vertex by following the associated lift of the relator to each vertex. We denote this complex by $\tilde{C}(\mathcal{P}, \phi)$ or just $\tilde{C}$ when no confusion will arise. The following example should make this clear.

Example 2.2. Let $\mathcal{P} = \langle g_1, g_2 : 1 = g_1 g_2 g_1^{-1} g_2^{-1} \rangle$, $C = C(\mathcal{P})$, and let $\phi\colon \pi_1(C) \twoheadrightarrow \mathbb{Z}_4$ be a homomorphism defined by $\phi(g_1) = 1, \phi(g_2) = 2$. Attach the cell coming from $1 = g_1 g_2 g_1^{-1} g_2^{-1}$ by attaching a cell, $\tilde{D}$, with ordered boundary $g_1 + x^1 g_2 - x^{1+2-1} g_1 - x^{1+2-1-2} g_2$. This starts and ends at the vertex v. We then attach a 2-cell at xv with ordered boundary $xg_1 + x^{1+1} g_2 - x^{1+1+2-1} g_1 - x^{1+1+2-1-2} g_2$. Continuing on in this manner, we have the following collection of 2-cells $\{x^j \tilde{D} : 0 \leq j < 4 \quad , \quad \partial(x^j \tilde{D}) = x^j (g_1 + x^1 g_2 - x^{1+2-1} g_1 - x^{1+2-1-2} g_2)\}$. (Note: $x^4 = x^0 = 1$). ⊞

Remark 2.2. In the case that $\phi(g_i) = 1$ for all generators of the group, we will employ the following more compact notation. Let $\mathcal{P}$ and C be as in the last example. In this case the relator is attached with boundary $x^0 g_1 + x^1 g_2 - x^1 g_1 - x^0 g_2$. We will denote this boundary by $(g_1 - g_2, g_2 - g_1, 0, 0)$, where the position in the tuple indicates the power of x associated to the 1-cell. In this way we define an action of $\langle x : x^m = 1 \rangle$ on a tuple by $x.(a_1, a_2, \ldots, a_m) = (a_m, a_1, a_2, \ldots, a_{m-1})$, ie, the action is to shift each entry in the tuple one space to the right. ⊞

2.1.3 Arrangements

In Chapter 1, Section 1.1.1 we recalled the Arvola-Randell presentation for the fundamental group of the complement of an arrangement. We know by work of Falk [8] and Cohen and Suciu [5] that the two dimensional CW-complex modeled on either the Arvola-Randell or braid monodromy presentation for $\pi_1(M(\mathcal{A}))$ is homotopy equivalent to $M(\mathcal{A})$.

Example 2.3. Let $\mathcal{A}$ be the arrangement in $\mathbb{C}^2$ with defining polynomial $Q(\mathcal{A}) = xy$. The Arvola-Randell algorithm yields

$$\langle a, b : [a, b] \rangle$$

as a presentation for $\pi_1(M(\mathcal{A}))$. The associated CW-complex is the same as given in Example 2.1. ⊞

2.1.4 Milnor Fiber

A central arrangement $\mathcal{A}$ of n hyperplanes in $\mathbb{C}^{l+1}$ may be given by a defining polynomial Q such that $Q(0) = 0$ and Q is homogeneous and factors into degree one terms. Therefore, we will refer to $F = Q^{-1}(1)$ as the Milnor fiber of $\mathcal{A}$.

In Section 1.2 we showed that restricting the Hopf fibration to the Milnor fiber induces a n-fold covering map $p\colon F \to M(\mathbf{d}\mathcal{A})$. This covering map is classified by an epimorphim $\phi\colon \pi_1(M(\mathcal{A})) \to \mathbb{Z}_n$. This map is defined by sending all meridional generators of the fundamental group γ_H to the same generator of $\mathbb{Z}_n$. Without loss of generality, we may assume $\phi(\gamma_H) = 1$ [4].

Combining this with the results of section 2.1.2, the following theorem follows.

Theorem 2.1. *Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^2$ and let $\mathcal{P}$ be a standard presentation for $\pi_1(M(\mathcal{A}))$. Then $\tilde{C}(\mathcal{P}, \phi)$ is homotopy equivalent to the Milnor fiber associated to $\mathbf{c}\mathcal{A}$.*

Remark 2.3. It should be noted that different presentations of a group do lead to different cell complexes. However, the complexes differ (up to homotopy type) only by wedges with 2-spheres. For complexes K and $K \vee S^2$, the coverings of these two have the same homotopy type up to wedging with n 2-spheres (n is cardinality of covering). Therefore, as long as we are only concerned with calculating the first homology groups of the covers, we will get the correct answer. Also note that the characteristic map specifies precisely where to send a meridional generator. If we have a different presentation for the group, we need to send it through the map ϕ. That is, suppose $\beta \in Aut(G)$ and $\phi\colon G \twoheadrightarrow \mathbb{Z}_n$ is our characteristic map. Then we use $\phi \circ \beta$ to build our complex from K associated with this new presentation of G. ⊞

2.2 Direct Products

As the methods described thus far depend on the presentation of the fundamental group, it is natural to begin our investigation with the groups and arrangements admitting simple presentations. These elementary examples also serve to model the general behavior we will encounter later. Unless otherwise stated, all homology groups will have integer coefficients.

2.2.1 Parallel Lines

Let $\mathcal{A} = \{H_i\}_{i=1}^{n+1}$ be a pencil of lines in $\mathbb{CP}^2$, ie, the intersection of all of the lines is exactly one point. Letting H_{n+1} be the line at infinity, we may assume that $\mathbf{d}\mathcal{A}$ has defining polynomial $Q(\mathbf{d}\mathcal{A}) = (x-1)(x-2)\cdots(x-n)$. Therefore $M(\mathbf{d}\mathcal{A}) \cong \mathbb{C} \setminus \{1, 2, \ldots, n\} \times \mathbb{C}$, and we clearly have that $\pi_1(M(\mathcal{A})) \cong \mathbb{F}_n$, a free group of rank n. In an unpublished note [9], Fan proves the converse of the statement, establishing the following lemma:

Lemma 2.2 ([9])**.** *Let* $\mathbf{d}\mathcal{A}$ *be an arrangement of* n *lines in* $\mathbb{C}^2$*. Then* $\pi_1(M(\mathbf{d}\mathcal{A})) \cong \mathbb{F}_n$ *if and only if* $\mathbf{d}\mathcal{A}$ *consists of* n *parallel lines.*

We may give an Arvola-Randell presentation of $\pi_1(M(\mathbf{d}\mathcal{A}))$ by

$$\langle g_1, \ldots, g_n \ : \ - \ \rangle$$

where g_i is the homotopy class coming from a meridional loop oriented in the positive direction about the hyperplane H_i.

Lemma 2.3. *Let* $\mathcal{A}$ *be a pencil of* $n+1$ *lines in* $\mathbb{CP}^2$*, and let* F *be the Milnor fiber associated to* $\mathcal{A}$*. Then* $H_1(F;\mathbb{Z}) \cong \mathbb{Z}^{n^2}$*.*

Proof. From the remark, we have a presentation for our fundamental group by

$$\pi_1(M(\mathcal{A})) \cong \langle\ g_1, \ldots, g_n \ : \ - \ \rangle.$$

The complex C coming from this presentation is homotopy equivalent to a wedge of 1-spheres $\vee_n S^1$. Therefore, the $(n+1)-$fold cyclic cover coming from the homomorphism $\phi\colon \pi_1(M(\mathcal{A})) \twoheadrightarrow \mathbb{Z}_{n+1}$ sending each g_i to $1 \in \mathbb{Z}_{n+1}$ has 0-cells $\{v, xv, x^2v, \ldots, x^nv\}$ and 1-cells $\{x^j g_i\}$ for $0 \le j \le n$, $1 \le i \le n$. See Figure 2.2. Thus we may collapse along the maximal tree given by $\{x^j g_1\}$ for $0 \le j \le n-1$. We are left with a wedge of n^2 1-spheres. Thus $H_1(F) \cong \mathbb{Z}^{n^2}$. $\square$

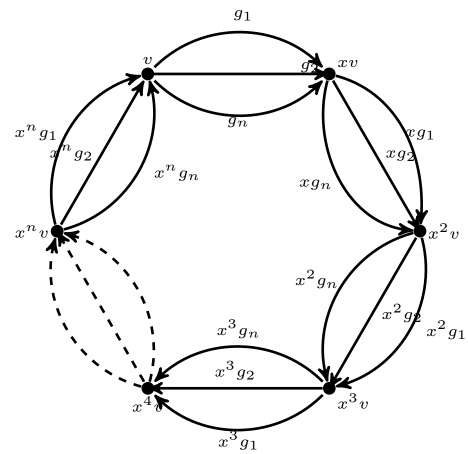

Figure 2.2: The 1-skeleton associated to a cover of an arrangement complement.

2.2.2 One point of intersection

Let $\mathbf{d}\mathcal{A} = \{H_1, \ldots, H_n\}$ be a pencil of lines in $\mathbb{C}^2$. Note that the intersection of the hyperplanes is no longer on the line at infinity.

Without loss of generality, we may assume that the arrangement has defining polynomial $Q = (y - x)(y - 2x) \cdots (y - nx)$. Letting g_k be homotopy classes corresponding to a meridional loops about $(y - kx)$, we have a group presentation given by $\mathcal{P} \cong \langle g_1, \ldots, g_n : [g_1, \ldots, g_n] \rangle$, where $[g_1, \ldots, g_n]$ represents the relations:

$$g_n g_{n-1} \cdots g_2 g_1 = g_{n-1} \cdots g_2 g_1 g_n = \cdots = g_1 g_n g_{n-1} \cdots g_2$$

Let us instead consider the more general situation of an intersection of lines in an arbitrary arrangement. Let $\mathcal{P} = \langle g_1, \ldots, g_p : S \rangle$ be a presentation of a group G with the subset of relations given by $[g_1, \ldots, g_n]$ denoted by R $(n \leq p)$. Let $\phi_m \colon G \twoheadrightarrow \mathbb{Z}_m$ be a homomorphism such that $\phi(g_i) = 1$ and $n \leq m$. Let $C(\mathcal{P})$ be the CW-complex associated to $\mathcal{P}$ and let $\tilde{C}(\mathcal{P}, \phi)$ be the cover of $C(\mathcal{P})$ associated to the map ϕ.

The relations given by $[g_1, \ldots, g_n]$ are

$$g_n g_{n-1} \cdots g_2 g_1 = g_{n-1} g_{n-2} \cdots g_2 g_1 g_n = \cdots = g_1 g_n \cdots g_2$$

We are able to rewrite these relations as the following relators:

$$R_1 := g_n g_{n-1} \cdots g_2 g_1 g_n^{-1} g_1^{-1} g_2^{-1} \ldots g_{n-1}^{-1}$$

$$R_2 := g_{n-1} g_{n-2} \cdots g_2 g_1 g_n g_{n-1}^{-1} g_n^{-1} g_1^{-1} \ldots g_{n-2}^{-1}$$

$$\ldots\ldots$$

$$R_{n-1} := g_2 g_1 g_n \cdots g_3 g_2^{-1} g_3^{-1} \ldots g_n^{-1} g_1^{-1}$$

These are the boundaries of 2-cells in $C(\mathcal{P})$, thus we lift them to boundaries of 2-cells in $\tilde{C}(\mathcal{P}, \phi)$ as

$$x^i.\tilde{R}_1 := x^i.(g_n - g_{n-1}, g_{n-1} - g_{n-2}, \ldots, g_1 - g_n, 0, \ldots, 0)$$

$$x^i.\tilde{R}_2 := x^i.(g_{n-1} - g_{n-2}, g_{n-2} - g_{n-3}, \ldots, g_1 - g_n, g_n - g_{n-1}, 0, \ldots, 0)$$

$$\ldots\ldots$$

$$x^i.\tilde{R}_{n-1} := x^i.(g_2 - g_1, g_1 - g_n, \ldots, g_3 - g_2, 0, \ldots, 0)$$

for $0 \leq i \leq m-1$ and denote this set by $\tilde{R}$.

These expressions form part of the set of relators $\tilde{S}$. We use parenthesis to denote abelian presentations and can see that

$$H_1(\tilde{C}) = \left(\begin{array}{l|l} (g_1, \ldots, g_1) & \tilde{S} \\ & \\ x^i(g_{j+1} - g_j, 0, \ldots, 0), 1 \leq j < n, 0 \leq i \leq m-1 & \end{array} \right).$$

We will use Tietze transformations to determine a more useful presentation of this group. . Thus we have $\{x^i.\tilde{R}_j\}_{i,j}$ as the generating set for the relators. We now replace $x^i.\tilde{R}_1$ by $x^i.\tilde{R}_1^* = x^i.\tilde{R}_1 - x^{i+1}.\tilde{R}_2$ for $0 \leq i \leq m-1$. Continue replacing $x^i.\tilde{R}_j$ by $x^i.\tilde{R}_j^* = x^i.\tilde{R}_j - x^{i+1}.\tilde{R}_{j+1}$ for $0 \leq i \leq m-1$ and $1 \leq j \leq n-2$.

Thus we end up with

$$x^i.\tilde{R}_1^* := x^i.(g_n - g_{n-1}, 0, \ldots, 0, -g_n + g_{n-1}, 0 \ldots, 0)$$

$$x^i.\tilde{R}_2^* := x^i.(g_{n-1} - g_{n-2}, 0, \ldots, 0, -g_{n-1} + g_{n-2}, 0 \ldots, 0)$$

$$\ldots\ldots$$

$$x^i.\tilde{R}_{n-2}^* := x^i.(g_3 - g_2, 0, \ldots, 0, -g_3 + g_2, 0 \ldots, 0)$$

$$x^i.\tilde{R}_{n-1} := x^i.(g_2 - g_1, g_1 - g_n, \ldots, g_3 - g_2, 0, \ldots, 0)$$

with second non-zero entry in the $(n+1)$-th place in the tuple.

Let us now inspect $x^i.\tilde{R}_1^*$. Expanding with respect to i, we may see that this set of relators is equivalent to

$$x^i(g_n - g_{n-1}, 0, \ldots, 0, -(g_n - g_{n-1}), 0, \ldots, 0)$$

where the second non-zero entry is in the $(w+1)$-th position of the tuple, where $w = 1$ if $m = 2$, otherwise $w = \gcd(m, n)$. Let us denote this new set of expressions by $x^i\tilde{R}_1^{**}$. We have a similar result for each set of expressions $x^i\tilde{R}_j^*$, so call these new relators $x^i\tilde{R}_j^{**}$.

To summarize, we have the set of relators

$$\left\{ \begin{array}{rl|c} x^i.\tilde{R}_j^{**} & := x^i.(g_{n-(j-1)} - g_{n-j}, \ldots, -(g_{n-(j-1)} - g_{n-j}), \ldots, 0) & 0 \leq i \leq m-1 \\ x^i.\tilde{R}_{n-1} & := x^i.(g_2 - g_1, g_1 - g_n, \ldots, g_3 - g_2, 0, \ldots, 0) & 1 \leq j \leq n-2 \end{array} \right\} \tag{2.1}$$

where the second non-zero entry is in the $(\gcd(m,n)+1)$-th place in the tuple.

Remark 2.4. Let us assume that $\gcd(m,n) = w = 1$, so that we have the relators

$$\left\{ \begin{array}{rl|c} x^i.\tilde{R}_j^{**} & := x^i.(g_j - g_{j-1}, -(g_j - g_{j-1}), 0 \ldots, 0) & 0 \leq i \leq m-1 \\ x^i.\tilde{R}_{n-1} & := x^i.(g_2 - g_1, g_1 - g_n, \ldots, g_3 - g_2, 0) & 1 \leq j \leq n-2 \end{array} \right\} \tag{2.2}$$

Let us now examine $x^i.\tilde{R}_{n-1}$. We rewrite the set as

$$x^i.\left(g_2 - g_1, \sum_{k=2}^{n}(-1)(g_k - g_{k-1}), g_n - g_{n-1} \ldots, g_3 - g_2, 0\right)$$

By using the relators $x^i.\tilde{R}_j^{**}$, we have cancellation of most terms and have the new relator $x^i.\tilde{R}_{n-1}^{**}$:

$$x^i.(g_2 - g_1, g_1 - g_2, 0, \ldots, 0)$$

Therefore we conclude that $\tilde{R}$ is equivalent to the set of relators below.

$$\left\{ \begin{array}{l|c} x^i.(g_j - g_{j-1}, -(g_j - g_{j-1}), 0, \ldots, 0) & 0 \leq i \leq n \\ & 2 \leq j \leq n. \end{array} \right\} \tag{2.3}$$

⊞

We may now prove the following lemma.

Lemma 2.4. *Let $\mathcal{A}$ be a central arrangement of n lines in $\mathbb{C}^2$, and let F be the Milnor fiber associated to $\mathbf{c}\mathcal{A}$. Then $H_1(F;\mathbb{Z}) \cong \mathbb{Z}^n$.*

Proof. An Arvola-Randell presentation for $\pi = \pi_1(M(\mathcal{A}))$ is given by

$$\mathcal{P} = \langle g_1, \dots, g_n \ : \ R = [g_1, g_2, \dots, g_n] \rangle$$

Thus we let C denote the CW-complex associated with the group presentation $\mathcal{P}$, and let $\tilde{C} = \tilde{C}(\mathcal{P}, \phi)$ denote the CW-complex associated to the group homomorphism $\phi\colon \pi \to \mathbb{Z}_{n+1}$, $\phi(g_i) = 1$ and C.

From Theorem 2.1 we may conclude that

$$\begin{aligned} H_1(F) &\cong H_1(\tilde{C}) \\ &= (x^i.(g_j - g_{j-1}, 0, \dots, 0), (g_1, g_1, \dots, g_1), 0 \le i \le m-1,\ 2 \le j \le n : \tilde{R}) \end{aligned}$$

where $\tilde{R}$ denotes the relators induced by R.

As $\gcd(n, n+1) = 1$, we may use the results of Remark 2.4 and conclude that $\tilde{R}$ is given by

$$\left\{ x^i.(g_j - g_{j-1}, -(g_j - g_{j-1}), 0, \dots, 0) \ \middle| \ \begin{array}{l} 0 \le i \le n \\ \\ 2 \le j \le n \end{array} \right\}$$

Therefore,

$$\begin{aligned} H_1(C) &\cong \sum_{j=2}^{n} \mathbb{Z}\langle (g_j - g_{j-1}, 0, \dots, 0) \rangle \oplus \mathbb{Z}\langle (g_1, \dots, g_1) \rangle \\ &\cong \mathbb{Z}^n. \end{aligned}$$

Thus we may conclude that $H_1(F) \cong \mathbb{Z}^n$. $\square$

2.2.3 Direct Products

We now explore the case when $\pi_1(M(\mathcal{A}))$ decomposes as a direct product of groups. The following theorem proven by Oka and Sakamoto provides us with many examples of arrangements for which we may easily determine the homology groups of the corresponding Milnor fiber.

Theorem 2.5 ([16]). *Let C be an algebraic plane curve in $\mathbb{C}^2$ such that $C = C_1 \cup C_2$, where C_i has degree d_i. If $C_1 \cap C_2$ consists of $d_1 \cdot d_2$ points, then we say that C splits and*

$$\pi_1(\mathbb{C}^2 - (C_1 \cap C_2)) \cong \pi_1(\mathbb{C}^2 - C_1) \oplus \pi_1(\mathbb{C}^2 - C_2).$$

The importance of this theorem comes from its proof. In order to establish the isomorphism, Oka and Sakamoto use a 1-parameter family of curves. More explicitly: let $f(x, y)$ and $g(x, y)$ be defining polynomials of C_1 and C_2 respectively. Letting $C_1(t(s)) = f(t(s)x, y), C_2(\tau(s)) = g(x, \tau(s)y)$, the authors construct a smooth one parameter family of curves $\{C_1(t(s)) \cup C_2(\tau(s)); 0 \leq s \leq 1\}$. This construction is made such that $\mathbb{C}^2 \setminus C_1 \cup C_2$ is homeomorphic to $\mathbb{C}^2 \setminus C_1(t(s)) \cup C_2(\tau(s))$ for all s and $C_1 = C_1(t(0))$, $C_2 = C_2(\tau(0))$.

They then proceed to construct a presentation for $\pi_1(\mathbb{C}^2 \setminus C_1(t(s_0)) \cup C_2(\tau(s_0)))$ given by

$$\mathcal{P} = \langle a_j, b_k : [a_j, b_k], R_a, R_b \rangle \tag{2.4}$$

where R_a (respectively R_b) consists of relations involving only a_i's (respectively b_k's), and $1 \leq j \leq d_1$ ()$1 \leq k \leq d_2$). Additionally, the presentations for $\pi_1(\mathbb{C}^2 \setminus C_1)$ and $\pi_1(\mathbb{C}^2 \setminus C_2)$ are given respectively by

$$\mathcal{P}_1 = \langle a_j : R_a \rangle$$

$$\mathcal{P}_2 = \langle b_k : R_b \rangle$$

In the case when C_1 and C_2 define arrangements, the presentations given above may be taken to be Arvola-Randell presentations.

Using Oka and Sakamoto's presentation of the fundamental group applied to arrangement complements, we may prove the following theorem.

Theorem 2.6. *Let $\mathcal{A}$ be an arrangement of n lines in $\mathbb{C}^2$, which splits via the theorem of Oka and Sakamoto. If F is the Milnor fiber associated to the cone of this arrangement, then $H_1(F) \cong \mathbb{Z}^n$.*

Proof. Assume that the arrangement splits into two components $\mathcal{A}$ and $\mathcal{B}$ of degrees d_1 and d_2 respectively. Then we may conclude by Theorem 2.5 and (2.4) that

$$\mathcal{P} := \langle a_j, b_k : [a_j, b_k], R_a, R_b \rangle$$

where the a_j and b_k are generators corresponding to transverse loops around hyperplanes in $\mathcal{A}$ and $\mathcal{B}$ respectively and $\mathcal{P}$ is an Arvola-Randell presentation for $\pi_1(M(\mathcal{A}))$.

Let $\tilde{C} = \tilde{C}(\mathcal{P}, \phi)$ be the CW-complex constructed from the presentation $\mathcal{P}$ and let $\phi \colon \pi_1(C(\mathcal{P}), k_0) \twoheadrightarrow \mathbb{Z}_{n+1}$ be the homomorphism induced by $a_j \mapsto 1$, $b_k \mapsto 1$. By Theorem 2.1, $\tilde{C}$ has the homotopy type of the Milnor fiber F.

We have the following boundaries for lifts of relators coming from $[a_j, b_k]$ for $0 \leq t \leq n$

$$\left\{ x^t.(a_j - b_k, -a_j + b_k, 0, \ldots, 0) \;\middle|\; \begin{array}{c} 1 \leq j \leq d_1 \\ 1 \leq k \leq d_2. \end{array} \right\} \tag{2.5}$$

Collapse the complex $\tilde{C}$ along the maximal tree given by $\{xa_1, x^2a_1, \ldots, x^na_1\}$.

This allows us to give a presentation

$$
\begin{aligned}
H_1(\tilde{C}) \quad &= \Big((a_1, 0, \ldots, 0), x^t.(a_j, 0, \ldots, 0), x^t.(b_k, 0, \ldots, 0) : \tilde{R}\Big) \\
&\qquad\qquad 0 \le t \le n,\ 2 \le j \le d_1, \text{ and } 1 \le k \le d_2
\end{aligned}
\tag{2.6}
$$

where the relators in $\tilde{R}$ coming from 2.5 are

$$
\left\{ \begin{array}{l|c}
(a_1 - b_k, b_k, 0, \ldots, 0) & 1 \le k \le d_2 \\
x^\tau(-b_k, b_k, 0, \ldots, 0) & 1 \le \tau \le n \\
(b_k - a_1, 0, \ldots, 0, -b_k) &
\end{array} \right\}
\tag{2.7}
$$

and

$$
\left\{ \begin{array}{l|c}
 & 2 \le j \le d_1 \\
x^t.(a_j - b_k, b_k - a_j, 0, \ldots, 0) & 0 \le t \le n \\
 & 1 \le k \le d_2
\end{array} \right\}
\tag{2.8}
$$

Using the relators in 2.7 and standard linear algebra, we may rewrite the relators in 2.8 as

$$
\left\{ \begin{array}{l|c}
(a_1 - a_j, a_j, 0, \ldots, 0) & 2 \le j \le d_1 \\
x^\tau.(-a_j, a_j, 0, \ldots, 0) & 0 \le t \le n \\
(a_j - a_1, 0, \ldots, 0 - a_j) &
\end{array} \right\}
\tag{2.9}
$$

Therefore, by a simple exercise in linear algebra on the basis given for $H_1(\tilde{C})$ in 2.6 we have that the number of generators for $H_1(\tilde{C}) \cong H_1(F)$ is at most $d_1 + d_2$. However by Theorem 1.14, we have that $b_1(F) \ge b_1(M(\mathcal{A})) = d_1 + d_2 = n$. Thus we have $H_1(F) \cong \mathbb{Z}^n$. $\square$

2.3 A Second Proof of Lemma 2.4

Using the results of this section, we have an alternative proof of Lemma 2.4:

Proof. Let L be the line at infinity, H any line in $\mathcal{A}$ and consider the arrangement $\mathcal{A} \cup \{L\}$ in $\mathbb{CP}^2 := \mathbb{C}^2 \cup L$. By deconing $\mathcal{A} \cup L$ with respect to H, we have an arrangement $\mathcal{B}$ in $\mathbb{C}^2$ consisting of $(n-1)$ parallel lines and one line in general position with respect to those lines. As $\mathcal{B}$ and $\mathcal{A}$ are decones of the same arrangement, we may conclude that their respective complements $M(\mathcal{A})$ and $M(\mathcal{B})$ are homeomorphic. Therefore, the corresponding Milnor fibers are homotopy equivalent. The arrangement $\mathcal{B}$ splits via the theorem of Oka and Sakamoto (Theorem 2.5). Finally, by Theorem 2.6 we conclude that $H_1(F) \cong \mathbb{Z}^n$. $\square$

Remark 2.5. These theorems give no new information on the first betti number of product arrangements. However, they do eliminate the possibility of torsion in the first homology group. Cohen and Suciu [6] show that for M_1 and M_2 regular CW-complexes with $b_1(M_1) = n_1$ and $b_2(M_2) = n_2$, that the characteristic variety of $M_1 \times M_2$ decomposes as

$$V_m(M_1 \times M_2) = (V_m(M_1) \times \mathbf{1}) \cup (\mathbf{1} \times V_m(M_2)) \subset (\mathbb{C}^*)^{n_1+n_2}$$

If $\pi_1(M(\mathcal{A}))$ splits in the sense of Oka and Sakamoto, then $\mathcal{A}$ can be realized as a generic slice of a direct product of arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$.

As $H_1(F, \mathbb{C}) = \oplus_{k=0}^{n} H_1(F; \mathcal{L}_k)$ where $\mathcal{L}_k$ is associated to the representation $\rho_k(\gamma_j) = exp(2\pi i/(n+1))^k$. We have that $\rho_k = (exp(2\pi i/(n+1))^k, \ldots, exp(2\pi i/(n+1))^k)$ is not an element of $V_1(\mathcal{A}_1 \oplus \mathcal{A}_2)$ for $k > 0$, therefore the first betti number of the Milnor fiber equals $b_1(M(\mathcal{A}))$ just as in Theorem 2.6. $\boxplus$

CHAPTER 3
COMBINATORIAL CONDITIONS FOR MINIMALITY

In this chapter we make use of the construction from Chapter 2 to prove a combinatorial theorem establishing an upper bound on the first betti number of the Milnor fiber with respect to any field. As a corollary, we establish conditions that guarantee minimality of the first betti number over fields of all characteristic, thereby ensuring that the first homology group is torsion free.

3.1 An upper bound

Our first result is Theorem 3.1 which establishes on upper bound on the first betti number of the Milnor fiber. This theorem extends a result of Cohen, Dimca, and Orlik [2] from the complex numbers to fields of arbitrary characteristic.

Theorem 3.1. *Let $\mathcal{A}$ be a complexified real-arrangement of n hyperplanes in $\mathbb{CP}^2$, and let F be the associated Milnor fiber. For any hyperplane $H \in \mathcal{A}$, let V be the set of multiple points on H, and let m_v denote the multiplicity of the point $v \in V$. Then*

$$b_1(F, \mathbb{K}) \leq (n-1) + \sum_{v \in V} [(m_v - 2)(\gcd(m_v, n) - 1)]$$

As a corollary to the Theorem, we have the following Lemma.

Corollary 3.2. *Let $\mathcal{A}$ be a complexified real-arrangement of n hyperplanes in $\mathbb{CP}^2$, and let F be the associated Milnor fiber. If there exists $H \in \mathcal{A}$ such that for all $v \in V_H$ we have $m(v) = 2$ or $\gcd(m(v), n) = 1$, then $H_1(F; \mathbb{Z}) \cong \mathbb{Z}^{n-1}$.*

Proof. The statement follows immediately from Theorem 3.1 as we have

$$\begin{aligned} b_1(F, \mathbb{K}) &\leq (n-1) + \sum_{v \in V} [(m_v - 2)(\gcd(m_v, n) - 1)] \\ &= (n-1) + 0 \end{aligned}$$

From Theorem 3 we have that $b_1(F, \mathbb{C}) \geq b_1(\mathbf{d}\mathcal{A}) = n-1$. Any p-torsion would be detected by an increase in the betti number with respect to a field of characteristic p. Since the numbers have the same bound for all fields, we may conclude that $H_1(F, \mathbb{Z}) \cong \mathbb{Z}^{n-1}$. □

Corollary 3.3. *If $|\mathcal{A}|$ is prime, then $H_1(F) \cong \mathbb{Z}^{|\mathcal{A}|-1}$.*

Some remarks are needed on the presentation of the fundamental group that will be used in the proof of this theorem. As $\mathcal{A}$ is an arrangement in $\mathbb{CP}^2$, we will let $\mathbf{c}\mathcal{A}$ denote the cone of the arrangement in $\mathbb{C}^3$. By Theorem 1.8, we know that $M(\mathbf{c}\mathcal{A})$ is homeomorphic to $M(\mathcal{A}) \times \mathbb{C}^*$, therefore $\pi_1(M(\mathbf{c}\mathcal{A})) \cong \pi_1(M(\mathcal{A})) \oplus \mathbb{Z}$. In fact, one may gain more precise information by examining fundamental group presentations.

As $\mathbf{c}\mathcal{A}$ is an arrangement in $\mathbb{C}^3$, by Zariski's theorem we may take a generic hyperplane U in $\mathbb{C}^3$ and find that $\pi_1(M(\mathbf{c}\mathcal{A})) \cong \pi_1(U \cap M(\mathbf{c}\mathcal{A}))$. As $U \cap M(\mathbf{c}\mathcal{A})$ is homeomorphic to an arrangement complement in $\mathbb{C}^2$, we may apply the Arvola-Randell algorithm to obtain a presentation $\mathcal{P}$ for $\pi_1(U \cap M(\mathbf{c}\mathcal{A}))$. By careful choice of generators, one may determine the following presentations:

$$\mathcal{P} := \langle \gamma_1, \gamma_2, \ldots, \gamma_n : R \rangle$$

$$\mathcal{P}_2 := \langle \gamma_1, \gamma_2, \ldots, \gamma_n : R = S \cup P, \gamma_1 \cdot \gamma_2 \cdots \gamma_n = 1 \rangle$$

$$\mathcal{P}_3 := \langle \gamma_1, \gamma_2, \ldots, \gamma_{n-1} : S \rangle.$$

$\mathcal{P}_3$ is an Arvola-Randell for $\pi_1(M(\mathcal{A}))$ considered as an arrangement in $\mathbb{C}^2$. $\mathcal{P}_2$ is the projective presentation generated by "pulling" the line at infinity into the arrangement and adding the relators induced by intersections along the line to the presentation. For more information on this procedure see the work of Garber [10]. One should note that the presentations generate isomorphic groups. All relators in R may be derived via the relators in S and $r := \gamma_1 \cdot \gamma_2 \cdots \gamma_n = 1$ via a series of Tietze transformations. While the CW-complex associated to $\mathcal{P}_2$ is not homotopy equivalent to $M(\mathcal{A})$), the construction presented in Chapter 2 will still be useful in computing some information about $H_1(F, \mathbb{Z})$. That is, the difference in homotopy type does not change the first homology groups.

Let $A = C(\mathcal{P}_3, \phi)$, $B = C(\mathcal{P}_2, \phi)$ where $\phi : \pi_1(M(\mathcal{A})) \to \mathbb{Z}_n$ sends γ_j to 1. From the presentations, it is clear that there is an injective map $i : A \to B$. Our aim will be to prove the following lemma.

Lemma 3.4. *The induced map $i_* : H_1(A, \mathbb{Z}) \to H_1(B, \mathbb{Z})$ is injective.*

Proof. Let $[a] \in H_1(A, \mathbb{Z})$ denote the equivalence class of $a \in C_1(A)$ and suppose that $i_*([a]) = 0$. As i is an inclusion of CW-complexes, we have that $C_1(A) \subset C_1(B)$ and $\mathrm{Ker}\partial_{1,A} \subset \mathrm{Ker}\partial_{1,B}$. As $C_2(A) \subset C_2(B)$, we also have that $\mathrm{Im}\ \partial_{2,A} \subset \mathrm{Im}\ \partial_{2,B}$.

As any element in R is a combination of elements in S and r via Tietze transformations, it follows that any element of $C_2(B)$ may be written as a linear combination of lifts of elements of S, ie, $C_2(A)$ and lifts of r. Letting $x^t.r$ denote the possible lifts of r, we may conclude that $\text{Im }\partial_{2,B} = \text{Im }\partial_{2,A} \oplus \partial_{2,B}(x^t.r)$. The sum is direct as the lifts of r contain the γ_n term and elements of $\text{Im }\partial_{2,A}$ contain no such terms. Therefore, as $[a]$ is in the kernel of i_* if and only if $a \in \text{Im }\partial_{2,A} \oplus \partial_{2,B}(x^t.r)$, and $a \in C_1(A)$ implies that a has no γ_n terms, we may conclude that $a \in \text{Im }\partial_{2,A}$. Thus, $[a] = 0$ in $H_1(A, \mathbb{Z})$ and i_* is injective. □

We record the above statements succinctly as the following theorem.

Theorem 3.5. *Let $\mathcal{A}$ be a complexified real-arrangement of n hyperplanes in $\mathbb{CP}^2$, and let F be the Milnor fiber associated to the cone of the arrangement $\mathbf{c}\mathcal{A}$ in $\mathbb{C}^3$. By choosing the γ_j's carefully, we may extend an Arvola-Randell presentation*

$$\mathcal{P}_3 := \langle \gamma_1, \gamma_2, \ldots, \gamma_{n-1} : S \rangle$$

for $\pi_1(M(\mathcal{A}))$ to

$$\mathcal{P}_1 = \langle \gamma_1, \ldots, \gamma_n : R = S \cup P \rangle$$

an Arvola-Randell presentation for $\pi_1(M(\mathbf{c}\mathcal{A}))$. This extension may be accomplished in such a way so that

$$\mathcal{P}_2 = \langle \gamma_1, \ldots, \gamma_n : R = S \cup P, \gamma_1 \cdots \gamma_n = 1 \rangle$$

is the <u>projective presentation</u> for $\pi_1(M(\mathcal{A}))$. From this construction, we may conclude that $H_1(F, \mathbb{Z}) \cong H_1(C(\mathcal{P}_3, \phi), \mathbb{Z})$ is a subgroup of $H_1(C(\mathcal{P}_2, \phi), \mathbb{Z})$.

We will need the following lemmas in order to prove Theorem 3.1. From the previous remarks, we may give the projective presentation for $\pi_1(M(\mathcal{A}))$ as

$$\mathcal{P} = \langle a_1, \ldots, a_n : R_H \cup P \cup R \rangle$$

$$R_H = \left\{ \begin{array}{c} [a_1, a_{1,2}, \ldots, a_{1,m_1}] \\ [a_1, a_{2,2}, \ldots, a_{2,m_2}] \\ [a_1, a_{3,2}, \ldots, a_{3,m_3}] \\ \ldots \\ [a_1, a_{k,2}, \ldots, a_{k,m_k}] \end{array} \right\}$$

$$P = \{a_1 \cdot \prod_{j=2}^{m_1} a_{1,j} \cdots \prod_{j=2}^{m_k} a_{k,j}\}$$

The relators in R_H are the relators coming from multiple points along the hyperplane H, R is used to denote the other relators, and P is the projective relator.

As before, let $\phi : \pi_1(M(\mathcal{A})) \to \mathbb{Z}_n$ be defined by $\phi(a_i) = 1$, and let $\tilde{C} = \tilde{C}(\mathcal{P}, \phi)$.

Lemma 3.6. *$H_1(\tilde{C})$ has a presentation as $(G : \tilde{R} \cup \tilde{P} \cup \tilde{R}_H)$ with*

$$G = \left\{ \begin{array}{l|r} x^j.(a_{i,2} - a_1, 0, \ldots, 0) & 0 \leq j \leq n-1 \\ x^j.(a_{i,p_i+1} - a_{i,p_i}, 0, \ldots, 0) & 1 \leq i \leq k \\ (a_1, a_{1,1}, a_{1,2}, \ldots, a_{k,m_k}) & 2 \leq p_i < m_i \end{array} \right\} \tag{3.1}$$

Proof. Following the construction given in Section 2.1.2, collapse the 1-skeleton of $\tilde{C}$ along edges of the form $x^i a_1$ for $0 \leq i < n-1$. This is a strong deformation retract of the 1-skeleton to a wedge of 1-spheres. Therefore, each remaining edge represents a 1-cycle. By uncollapsing the edges, we may represent every 1-cycle as

$$x^j.(a_{i,k} - a_1, 0, \dots, 0) \text{ or } (a_1, a_1, \dots, a_1)$$

By simple algebraic manipulations we see that the set of generators given by G is equivalent to those given in 3.1 $\square$

Lemma 3.7. *A relator of the form* $[a_1, a_2, \dots, a_m]$ *may be used to reduce the number of distinct homology generators by* $(n-1) + (m-2)(n - \gcd(m,n))$.

Proof. In Subsection 2.2.2 we showed that from a relator of the form given, we have relators in the covering space given by

$$x^j.(a_i - a_{i-1}, 0, \dots, 0, a_i - a_{i-1}, 0, \dots, 0)$$

with the second non-zero entry in the $(w+1)$-th position where $w = \gcd(m,n)$, and

$$x^j.(a_2 - a_1, a_1 - a_m, a_m - a_{m-1}, \dots, a_3 - a_2, 0, \dots, 0)$$

for , $3 \le i \le m$, and $0 \le j < n$.

Using the first set of relators, for each i we may reduce our generating set by $\left(\frac{n}{w} - 1\right) w$ elements by identifying them with a generator of the form $(0, \dots, a_i - a_{i-1}, 0, \dots, 0)$ where the cycle is in the j-th slot for $0 \le j < w$. We repeat this for each of the $(m-2)$ relators.

The second set of relators may be rewritten as

$$\begin{aligned} &x^j.((a_2 - a_1), -(a_m - a_1), a_m - a_{m-1}, \dots, a_3 - a_2, 0, \dots, 0) \\ &= x^j.((a_2 - a_1), -\sum_{p=2}^{m}(a_p - a_{p-1}), a_m - a_{m-1}, \dots, a_3 - a_2, 0, \dots, 0) \end{aligned}$$

Thus we may write any cycle $x^j.(a_2 - a_1, 0, \ldots, 0)$ $(j > 0)$ in terms of other cycles, except for $(a_2 - a_1, 0, \ldots, 0)$. Hence we get another elimination of $n-1$ distinct generators. (This reduction is done via noting that all $a_2 - a_1$ cycles outside the first slot may be written in terms of other cycles. One should note that the cycles are not identified "cyclically" as before.) Combining this with $(m-2)(n - \gcd(m,n))$ generators we reduced earlier, we have our lemma. □

Remark 3.1. From the proof of Lemma 3.7 and the group presentation, it is clear that the relators in $\tilde{R}_H$ coming from each multiple point $v \in V$ may be used to identify a disjoint set of generators. Thus each multiple point reduces the number of distinct homology classes by $(n-1) + (m_v - 2)(n - \gcd(m_v, n)$. ⊞

Lemma 3.8. *The homology class generated by* $(a_1, a_{1,2}, \ldots, a_{k,m_k})$ *is trivial.*

Proof. Recall that the relator P is given by $a_1 \cdot \prod_{j=2}^{m_1} a_{1,j} \cdots \prod_{j=2}^{m_k} a_{k,j}$. In the homology of the covering space, this relator takes the form

$$(a_1, a_{1,2}, \ldots, a_{k,m_k})$$

Therefore $(a_1, a_{1,2}, \ldots, a_{k,m_k})$ is trivial. □

We may now use the preceding lemmas to prove Theorem 3.1.

Proof. By Lemma 3.6, $H_1(F, \mathbb{K})$ is generated by at most $1 + n|V| + n\sum_{v \in V}(m_v - 2)$ homologically distinct cycles. Lemma 3.7 shows that each multiple point reduces a disjoint set of generators. Thus combined with Lemma 3.8, we have

$$
\begin{aligned}
b_1(F,\mathbb{K}) &\leq b_1(\tilde{C},\mathbb{K}) \\
&\leq 1 + n|V| + n\sum_{v\in V}(m_v - 2) - \sum_{v\in V}\left[(m_v-2)(n-\gcd(m_v,n)) + (n-1)\right] - 1 \\
&= n|V| + n\sum_{v\in V}(m_v-2) - n\sum_{v\in V}(m_v-2) \\
&\qquad + \sum_{v\in V}\left[(m_v-2)\gcd(m_v,n)\right] - |V|(n-1) \\
&= |V| + \sum_{v\in V}\left[(m_v-2)\gcd(m_v,n)\right]
\end{aligned}
$$

Note that $(n-1)+|V| = \sum_{v\in V} m_v$, thus we have

$$
\begin{aligned}
&= |V| + \sum_{v\in V}\left[(m_v-2)\gcd(m_v,n)\right] + |V| + (n-1) - \sum_{v\in V} m_v \\
&= (n-1) + \sum_{v\in V}\left[(m_v-2)\gcd(m_v,n)\right] + 2|V| - \sum_{v\in V} m_v \\
&= (n-1) + \sum_{v\in V}\left[(m_v-2)\gcd(m_v,n)\right] + 2\sum_{v\in V} 1 - \sum_{v\in V} m_v \\
&= (n-1) + \sum_{v\in V}\left[(m_v-2)(\gcd(m_v,n)-1)\right]
\end{aligned}
$$

as desired in the statement of the theorem. □

Recall from Theorem 1.14 that

$$H_*(F;\mathbb{C}) \cong \bigoplus_{k=0}^{d-1} H_*(M(\mathbf{d}\mathcal{A});\mathcal{V}_k)$$

Let $b_1(F,\mathbb{C})_k = \dim_{\mathbb{C}} H_1(M(\mathbf{d}\mathcal{A});\mathcal{V}_k)$. We recall the following theorem from the work of Cohen, Dimca, and Orlik [2] below:

Theorem 3.9 (Theorem 13 [2]). *Let $\mathcal{A}$ be a line arrangement in $\mathbb{CP}^2$, with associated Milnor fiber F. Then for any integer $0 < k < n$ and any line H in the arrangement $\mathcal{A}$ we have*

$$b_1(F, \mathbb{C})_k \leq \sum_x (m_x - 2)$$

where the sum is over all points $x \in H$ such that the multiplicity of $\mathcal{A}$ at x is $m_x > 2$ and n divides km_x.

Let $M_k = \{x \in H : n | km_x\}$. From Theorem 1.14 and Theorem 3.9, we have

$$\begin{aligned} b_1(F) &= \sum_{k=0}^{n-1} b_1(F)_k \\ &\leq b_1(F)_0 + \sum_{k=1}^{n-1} \sum_{x \in M_k} (m_x - 2) \\ &= (n-1) + \sum_{k=1}^{n-1} \sum_{x \in M_k} (m_x - 2) \end{aligned}$$

Let $g_x = \gcd(n, m_x)$. As, $g_x = |\{k : 1 \leq k \leq n, n \mid km_x\}|$ (see Proposition 3.10), we conclude that

$$b_1(F, \mathbb{C}) \leq (n-1) + \sum_{x \in V} (m_x - 2)(\gcd(n, m_x) - 1)$$

As a consequence of Theorem 3.9, we see that Theorem 3.1 gives no new information on the complex betti numbers of the Milnor fiber. In fact, Theorem 3.9 is stronger than Theorem 3.1 in terms of computing the betti number. We demonstrate this via the next example.

Example 3.1. Let $\mathcal{A}$ be an arrangement of 12 projective lines in $\mathbb{CP}^2$ such that there is a line $H_3 \in \mathcal{A}$ that contains multiple points of order 2 and 3 only, and contains at least one point of order 3. We also assume that $H_4 \in \mathcal{A}$ is a line that contains points of order 2 and 4 only and contains at least one point of order 4. By Theorem 3.1, $b_1(F, \mathbb{K}) \leq 11 + 2m_3$ where m_3 is the number of multiple points of order 3 contained in H_3.

However, Theorem 3.9 may be applied for $k = 1, 2, 3, 5, 6, 7, 9, 10, 11$ along the line H_3 to conclude that $b_1(F)_k = 0$. For $j = 1, 2, 4, 5, 6, 7, 10, 11$ the Theorem may be applied along H_4 to conclude that $b_1(F)_j = 0$. Therefore, $b_1(F, \mathbb{C}) = 11$. ⊞

Proposition 3.10. *Let n, m, k be positive integers, then*

$$\gcd(n, m) = |\{k : 1 \leq k \leq n, n \mid km\}|$$

Proof. Let $d = \gcd(n, m)$, $K = \{k : 1 \leq k \leq n, n \mid km\}$ and $G = \{\frac{n}{d}, \frac{2n}{d}, \ldots, \frac{dn}{d} = n\}$. Clearly $d = |G|$, therefore we need only show $K = G$.

Let $k \in K$. Then

$$\begin{aligned} n|km \quad &\Rightarrow \quad na = km \quad (a \in \mathbb{Z}) \\ &\Rightarrow \quad \tfrac{n}{d}a = k\tfrac{m}{d} \\ &\Rightarrow \quad \tfrac{m}{d}|a \qquad (\textit{as } \tfrac{m}{d} \textit{ and} \tfrac{m}{d} \textit{ are coprime}) \\ &\Rightarrow \quad \tfrac{np}{d} = k \quad (p \in \mathbb{Z}) \end{aligned}$$

As $k \in K$, we know that $k \leq n$ therefore $p = \frac{k}{n/d} \leq d$. Hence $k = \frac{np}{d}$ where $1 \leq p \leq d$, thus $k \in G$, and $K \subseteq G$.

Let $g \in G$. Then $g = \frac{pn}{d} \in \mathbb{Z}$ and $1 \leq g \leq n$. It is clear that n divides $\frac{pn}{d}m$ as d divides m. Therefore, $g \in K$ and $G \subseteq K$ as desired. □

3.2 Some more conditions on multiple points

In this section we give another condition on the multiple points along a single line in the arrangement. In this section we assume that $\mathcal{A}$ is a complexified real arrangement in $\mathbb{CP}^2$. If $H_0 \in \mathcal{A}$ is such that only one multiple point, $v \in H_0$ satisfies $\gcd(m(v), |\mathcal{A}|) \neq 1$ and $m(v) \geq 3$, then by Theorem 3.1 we may only conclude that

$$b_1(F, \mathbb{K}) \leq |\mathcal{A}| - 1 + (m(v) - 2)(\gcd(m(v), |\mathcal{A}|) - 1)$$

However, we have the following improvement.

Theorem 3.11. *Let $\mathcal{A}$ be an arrangement in $\mathbb{CP}^2$, and F the associated Milnor fiber. If there exists a line $H_0 \in \mathcal{A}$ such that there is only one multiple point $v \in H_0$ with the property that $\gcd(m(v), |\mathcal{A}|) \neq 1$ and $m(v) > 2$, then*

$$H_1(F, \mathbb{Z}) = \mathbb{Z}^{|\mathcal{A}|-1}.$$

Proof. Let H be any line in the arrangement containing the point v except for H_0. Let $\mathbf{d}\mathcal{A}$ be the decone of the arrangement with respect to the line H (ie, the affine arrangement with H the line at infinity). As we assumed that $\mathcal{A}$ was a complexified real arrangement, we may depict the arrangement locally as in Figure 3.1.

We will denote the lines intersecting H_0 by $\{A_{(l,k)}\}_{l=1,k=1}^{m,m_l}$ where the first index is with respect to each point of intersection and the second with respect to the lines in the intersection. The lines parallel to H_0 will be denoted by $\{P_j\}_{j=1}^{p}$. We further assume that the P_j are indexed by increasing distance from H_0. (As they are parallel, the real parts of the lines have a well-defined distance between them. Therefore, we

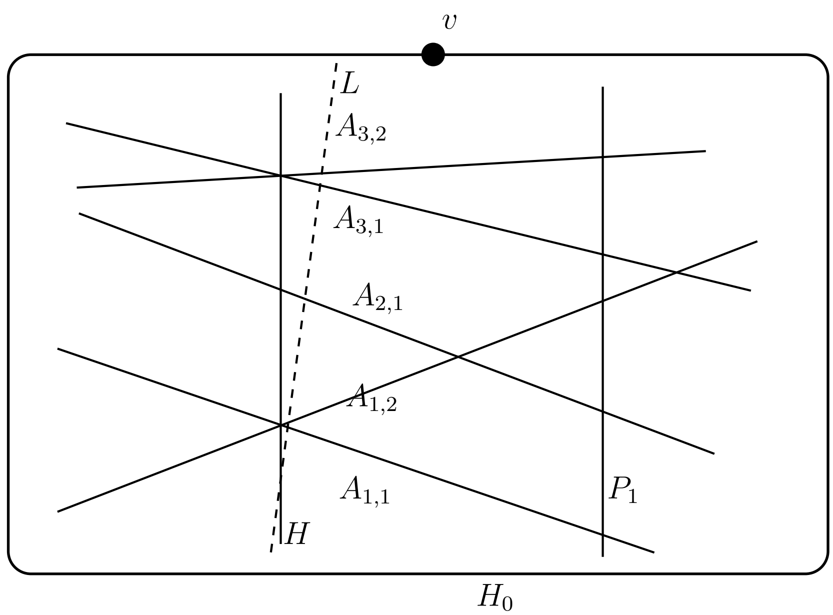

Figure 3.1: An arrangement satisfying the conditions of Theorem 3.11.

assume the distance from H_0 to P_h is less than or equal to the distance from H_0 to P_{h+1}) . By considering a hyperplane slightly askew to H_0 (labeled L in Figure 3.1 and denoted by a dashed line), we associate to each line a generator in the fundamental group coming from a small loop around the line oriented compatibly with all others and denoted by h, $a_{(l,k)}$ or p_j. Thus we construct an Arvola-Randell presentation of the form $\langle h, a_{(l,k)}, p_j : R\rangle$.

Recall from the proof of Lemma 3.6 that we may represent the generators of the first homology group of the Milnor fiber with respect to h by

$$\left\{\begin{array}{l|r} x^i.(a_{(l,k)}-h,0,\ldots,0) & 0\le i\le n-1 \\ x^i.(p_j-h,0,\ldots,0) & 1\le l\le m \\ (h,h,\ldots,h) & 1\le k\le m_l \\ & 1\le j\le p \end{array}\right\} \tag{3.2}$$

for tuples of length $n=|\mathcal{A}|$.

As the order of each multiple point along H_0 is relatively prime to n, the relators in $H_1(F)$ arising from the multiple points along H_0 may be altered by Tietze transformations to have the form

$$\left\{x^i.(a_{(l,k)}-h,-(a_{(l,k)}-h),\ldots,0)\right\}. \tag{3.3}$$

This follows from Remark 2.4.

As the lines P_j are parallel to H_0 and we have at least one line $A_{(l,k)}$, there is a set of relations in R of the form

$$[p_1,\alpha_1^{\Gamma_1},\alpha_2^{\Gamma_2},\ldots,\alpha_{m_1}^{\Gamma_{m_1}}]$$

where each α_w is some $a_{(l,k)}$ and each Γ_w is a word composed of some combination of $a_{(l,k)}$'s.

Consider the set of relators in $H_1(F)$ generated by the relations

$$\alpha_{m_1}^{\Gamma_{m_1}}\cdot p_1\cdot\alpha_1^{\Gamma_1}\cdots\alpha_{m_1-1}^{\Gamma_{m_1-1}}=p_1\cdot\alpha_1^{\Gamma_1}\cdots\alpha_{m_1}^{\Gamma_{m_1}}.$$

One may easily seen that the word $\alpha_{m_1}^{\Gamma_{m_1}}$ has one more positive exponent than negative exponent, thus will move the indexing one space up. As each subsequent α^Γ has the same form, the result will be that all the conjugations do not affect the starting point of each subsequent conjugated term in the relation. Thus, the relators will have the form $x^i.(h-p_1+A_1, p_1-h+A_2, A_3, \ldots, A_{m_1}, 0, \ldots, 0)$, where the A_i are combinations of $\alpha_{(j,k)}$'s. Every $a_{(i,k)}$ occurs an even number of times in the above relations (as they are involved in conjugation or on both sides of the equal sign), thus using the relators from (3.3) we may remove all $(a_{(i,k)} - h)$ terms from the relators. The end result is

$$x^i.(h - p_1, -(h - p_1), 0, \ldots, 0).$$

Any other parallel line P_g will generate relations of the form

$$[p_k, \alpha_1^{\Gamma_1}, \alpha_2^{\Gamma_2}, \ldots, \alpha_{m_g}^{\Gamma_{m_g}}]$$

where the α_i are single letters of the form $a_{(l,k)}$ or p_h where $h < g$. The Γ terms are words in $a_{(l,k)}$ or p_h where $h < g$. By the same arguments as given above, we may conclude that we have relators of the form

$$x^i.(h - p_g, -(h - p_g), 0, \ldots, 0).$$

Therefore, we have a presentation for $H_1(F, \mathbb{Z})$ given by

$$\left(\begin{array}{l|l} x^i.(a_{(l,k)} - h, 0, \ldots, 0) & x^i.(a_{(l,k)} - h, -(a_{(l,k)} - h), \ldots, 0) \\ x^i.(p_j - h, 0, \ldots, 0) & x^i.(h - p_j, -(h - p_j), 0, \ldots, 0) \\ (h, h, \ldots, h) & R* \end{array} \right) \tag{3.4}$$

Therefore, the group has at most $|\mathcal{A}| - 1$ generators, whence must be a free abelian group on $|\mathcal{A}| - 1$ generators as we know that $b_1(F, \mathbb{C}) \geq |\mathcal{A}| - 1$. $\square$

Corollary 3.12. *The conclusion of Theorem 3.11 holds for any m-fold cyclic cover where $m > |\mathcal{A}|$*

Proof. The proof makes no use of the order of the cyclic covering space. $\square$

Corollary 3.13. *Let $\mathcal{A}$ be an arrangement in $\mathbb{CP}^2$. If $b_1(F) > |\mathcal{A}|$ then each line in the arrangement must contain at least two higher order multiple points v such that $\gcd(m(v), |\mathcal{A}|) \neq 1$.*

Let us compare and contrast Theorem 3.9 and Theorem 3.11 by way of the following examples.

Example 3.2. Let $\mathcal{A}$ be an arrangement in $\mathbb{CP}^2$ such that the number of lines in the arrangement is $n = |\mathcal{A}| = 24$. Let H_4 and H_3 be two lines in the arrangement and suppose that the only higher order multiple points on H_j not relatively prime to n are two points of multiplicity j for j equal to 3 or 4.

Applying Theorem 3.9 to the line H_3 for $k \in [n-1] \setminus \{8, 16\}$ we conclude that $b_1(F, \mathbb{C})_k = 0$. Using the theorem again but with line H_4 and $k = 8$ or 16, we conclude that $b_1(F, \mathbb{C})_8 = b_1(F, \mathbb{C})_{16} = 0$. Therefore, $b_1(F) = n - 1$. In fact, the theorem easily extends over any field $\mathbb{K}$ such that the order of the field is not 2 or 3. Thus, the Milnor fiber has minimal betti number and the only torsion possible is of type 2 or 3.

However, Theorem 3.11 does not apply in this instance, as we do not know if some line has only one multiple point relatively prime to 24. $\boxplus$

Example 3.3. Now consider another arrangement $\mathcal{A}$ in $\mathbb{CP}^2$ such that the number of lines in the arrangement is p^d for some prime p greater than two. Also, suppose that all lines in the arrangement have at least one multiple point of order p, and at least one line H has only one multiple point with multiplicity divisible by p. In this case, Theorem 3.9 for any line will at best yield the inequality $b_1(F, \mathbb{C})_k \leq (p-2)$. However, applying Theorem 3.11 to the line H will yield $H_1(F; \mathbb{Z}) \cong \mathbb{Z}^{p^d - 1}$, hence minimal betti number and torsion free. $\boxplus$

CHAPTER 4
GEOMETRIC CONDITIONS FOR MINIMALITY

Cohen and Suciu [4] present examples of two arrangements denoted by $(9_3)_1$ and $(9_3)_2$. These arrangements have "similar" combinatorics in the sense that they both consist of 9 projective lines that each contain points of multiplicity three. However, the intersection posets associated to these arrangements are not isomorphic and the associated Milnor fibers have different first betti numbers. In this section, we present local geometric conditions in the form of Lemmas that show why the Milnor fibers have different betti numbers even though they arise from arrangements with similar combinatorics. As the Lemmas depend on the local geometry of the arrangements, the Lemmas are then used to show how one may determine the first homology group of the Milnor fiber by using only local information of the arrangement. In particular, we give a classification result based on the combinatorics of the arrangement. This method is then compared with other approaches that depend on global information to complete the calculations.

4.1 Definitions and Set-up

We begin by giving several definitions that will be useful in this chapter.

Definition 4.1. Let M be the complement of any plane curve in $\mathbb{C}^2$, and $\mathcal{P}$ a presentation of the fundamental group of M. If M is homotopy equivalent to the standard 2-complex constructed from $\mathcal{P}$, then we will call $\mathcal{P}$ a homotopy-type preserving presentation or HTP presentation for short.

Definition 4.2. Let G be a group such that the abelianization G_{ab} is free abelian with finite rank. A 1-marking is a choice of distinguished ordered basis for G_{ab} as a $\mathbb{Z}$-module. Two 1-marked groups G and G' are 1-marked isomorphic if there exists an isomorphism $\phi : G \to G'$ that preserves the 1-markings.

Definition 4.3. Let $\mathcal{A}$ be an arrangement of hyperplanes. If $G = \pi_1(M(\mathcal{A}))$, then we may choose a geometric (natural) 1-marking by choosing an ordering of the hyperplanes $\{H_i : 1 \leq i \leq n,\ H_i \in \mathcal{A}\}$ and then for each hyperplane H_i we choose the homology class of a small loop γ_{H_i} that goes in the positive direction around the hyperplane H_i as the distinguished basis.

Definition 4.4. Let M be the complement of any arrangement in $\mathbb{C}^2$, and $\mathcal{P}$ a presentation of the fundamental group of M. If the group generated by $\mathcal{P}$ and $\pi_1(M)$ with the geometric 1-marking are 1-marked isomorphic, then $\mathcal{P}$ is a geometric presentation.

Definition 4.5. Let $\mathcal{A} = \{H_i\}_{i=1}^{n}$ be an arrangement of lines in $\mathbb{C}^2$. Let $\Sigma = \{\cap_{i \in I} H_i \neq \emptyset : I \subset \{1, 2, \ldots, n\}, |I| \geq 2\}$ be the set of multiple points of the arrangement.

1. For any $v \in \Sigma$, let $m(v) = |\{H_i : v \in H_i \in \mathcal{A}\}|$ be the multiplicity of the point v.

2. If $m(v) = 2$ we will say that v is a simple multiple point.

3. If $m(v) > 2$, we will say that v is a higher order multiple point.

Let $\mathcal{A}$ be a complexified-real arrangement in $\mathbb{C}^2$ and H any line in the arrangement. By a change of coordinates, we may assume that H is defined by the equation $x = 0$. As such, we may depict the arrangement as in Figure 4.1. Since the arrangement is complexified-real, we may define the positive and negative sides of the line H to be given by $H^+ = \{(x, y) \in \mathbb{R}^2 : x > 0\}$, $H^- = \{(x, y) \in \mathbb{R}^2 : x < 0\}$.

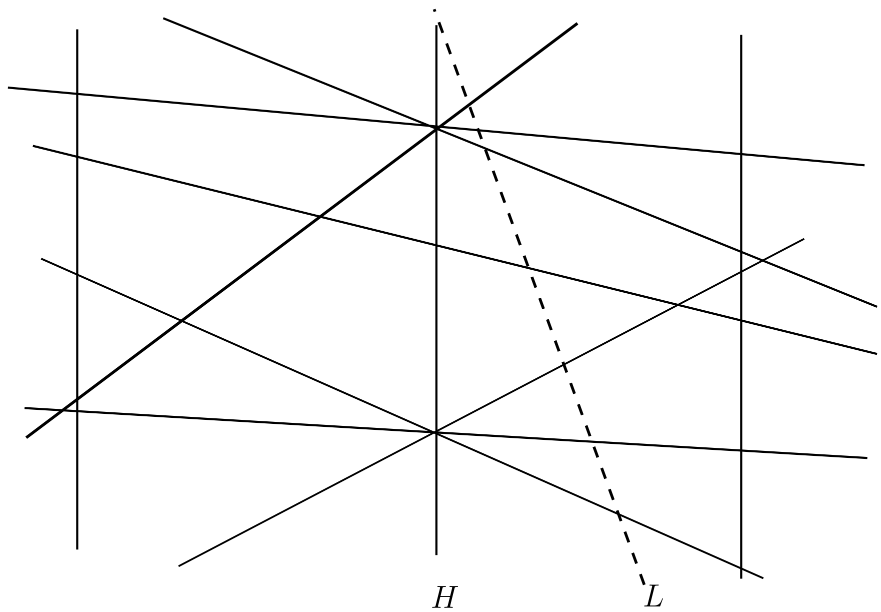

Figure 4.1: The local depiction of an arrangement along H.

Let $\mathcal{A}_\epsilon = (\mathcal{A} \setminus \{H\}) \cup \{H_\epsilon\}$ where H_ϵ is defined by the equation $x = \epsilon$ for some $0 < \epsilon << 1$. As before, we define $H_\epsilon^+ = \{(x, y) \in \mathbb{R}^2 : x > \epsilon\}$, $H_\epsilon^- = \{(x, y) \in \mathbb{R}^2 : x < \epsilon\}$. We may choose ϵ such that $H^+ \cap H_\epsilon^-$ contains no multiple points of the arrangement.

In order to determine a group presentation, we need to choose a line L generic with respect to the arrangement. We may choose this to be the same line with respect to both of the arrangements $\mathcal{A}$ and $\mathcal{A}^+$. For L given by the equation $y = ax+b$, we may

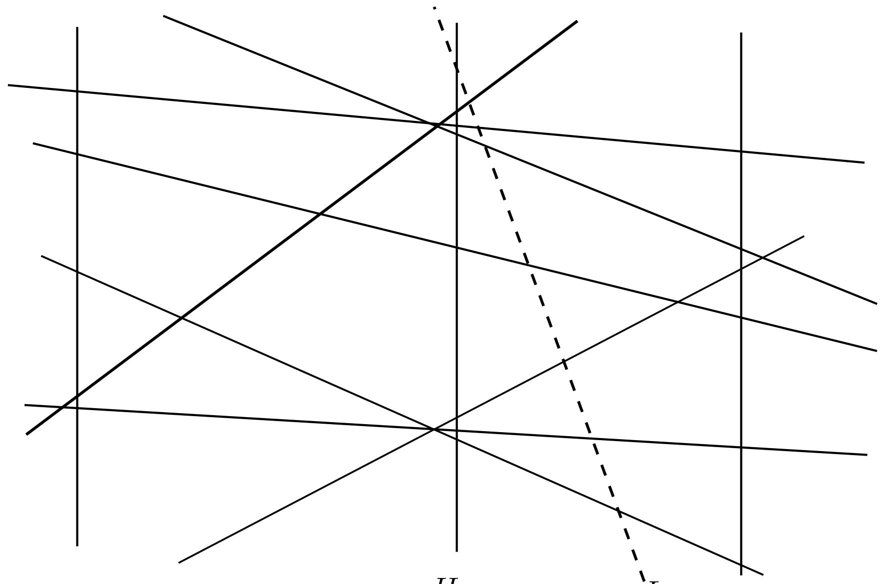

Figure 4.2: The local depiction of an arrangement along H_ϵ

assume that a is negative and b is positive. We let $L^+ = \{(x, y) \in \mathbb{R}^2 : y > ax + b\}$, $L^- = \{(x, y) \in \mathbb{R}^2 : y < ax+b\}$. We may choose a and b so that $L^- \cap H^+$ contains no multiple points, and for all multiple points $v \in H$, we have $v \in L^-$. See the dashed line in Figure 4.1 and Figure 4.2.

Theorem 4.6. *Applying the Arvola-Randell algorithm along the dashed line determines a geometric 1-marked presentations for each arrangement. The presentations are given by the following.*

$$\begin{aligned} \mathcal{P}(\mathcal{A}) &:= \left\langle h, p_1, \ldots, p_k, q_1, \ldots, q_l, a_{(i,j)} : R(h) \right\rangle \\ \mathcal{P}(\mathcal{A}_\epsilon) &:= \left\langle h_\epsilon, p_1, \ldots, p_k, q_1, \ldots, q_l, a_{(i,j)} : \right. \\ &\qquad\qquad \left. [h_\epsilon, a_{(i,j)}], R(h_\epsilon),\ 2 \le j \le m_i - 1 \right\rangle \end{aligned} \tag{4.1}$$

For these presentations

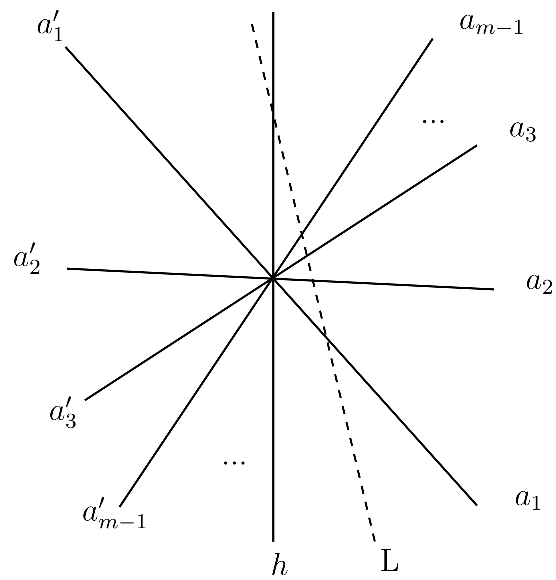

Figure 4.3: Local depiction of a multiple point.

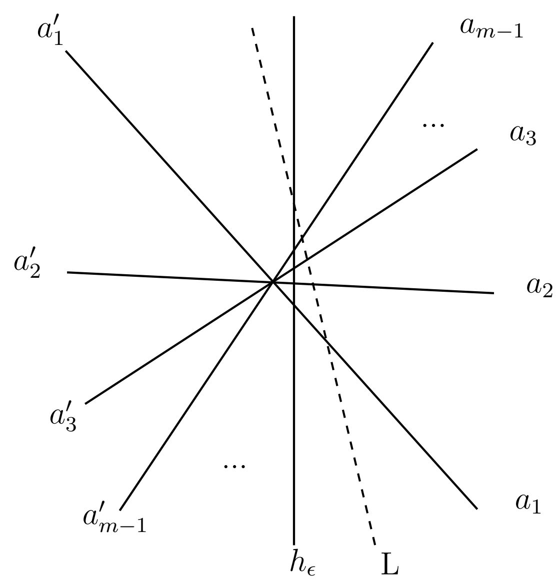

Figure 4.4: Local depiction of a multiple point with respect to h_ϵ.

- *h and h_ϵ corresponds to H and H_ϵ.*

- *The p_i's and q_j's correspond to line(s) parallel to H.*

- *The $a_{(i,j)}$ correspond to the different lines intersecting along H.*

Further, the sets $R(h)$ and $R(h_\epsilon)$ contain identical relators after changing all instances of h_ϵ to h in $R(h_\epsilon)$. We will call these presentations geometrically compatible 1-marked presentations along the distinguished hyperplane H (GCMD presentations along H, for short).

Proof. Consider the local depictions of a multiple point depicted in Figure 4.3. The presentation at a multiple point in $\mathcal{A}$ will include generators

$$G = \left\{h, a_1, \ldots, a_{m-1}, h', a_1', \ldots, a_{m-1}'\right\} \tag{4.2}$$

and relations

$$R = \left\{ \begin{array}{c} ha_{m-1}a_{m-2}\cdots a_1 = a_1 h a_{m-1}\cdots a_2 = \cdots = a_{m-1}\cdots a_1 h \\ a_1' = a_1 \\ a_2' = a_2^{a_1} \\ \cdots = \cdots \\ a_{m-2}' = a_{m-2}^{a_{m-3}\ldots a_1} \\ a_{m-1}' = a_{m-1}^{a_{m-2}a_{m-3}\ldots a_1} \\ h' = h \end{array} \right\} \tag{4.3}$$

In Figure 4.4 we have depicted a multiple point locally with respect to H_ϵ. The generators may be given by

$$G_\epsilon = \left\{h_\epsilon, a_1, \ldots, a_{m-1}, h_\epsilon', a_1', \ldots, a_{m-1}'\right\} \tag{4.4}$$

and relations

$$R_\epsilon = \left\{ \begin{array}{c} a_{m-1}a_{m-2}\cdots a_1 = a_1a_{m-1}\cdots a_2 = \cdots = a_{m-2}\cdots a_1a_{m-1} \\ a_ih_\epsilon = h_\epsilon a_i \\ a_1' = a_1 \\ a_2' = a_2^{a_1} \\ \cdots = \cdots \\ a_{m-2}' = a_{m-2}^{a_{m-3}\ldots a_1} \\ a_{m-1}' = a_{m-1} \\ h_\epsilon' = h_\epsilon \end{array} \right\} \tag{4.5}$$

Using the relation $a_{m-1}a_{m-2}\cdots a_1 = a_{m-2}\cdots a_1a_{m-1}$ we may replace the relation $a_{m-1}' = a_{m-1}$ with $a_{m-1}' = a_{m-1}^{a_{m-2}a_{m-3}\ldots a_1}$ by using a transformation of type (iii) (see Theorem 2.1).

Consider the relations written in 4.5 separately, ie,

$$R_\epsilon = \left\{ \begin{array}{c} a_{m-1}a_{m-2}\cdots a_1 = a_1a_{m-1}\cdots a_2 \\ a_{m-1}a_{m-2}\cdots a_1 = a_2a_1a_{m-1}\cdots a_3 \\ \cdots \\ a_{m-1}a_{m-2}\cdots a_1 = a_{m-2}\cdots a_1a_{m-1}. \end{array} \right\} \tag{4.6}$$

Using the commutators $[a_i, h_\epsilon]$, we may rewrite these relation using transformations of types (i) and (iii) as

$$h_\epsilon a_{m-1}a_{m-2}\cdots a_1 = a_1 h_\epsilon a_{m-1}\cdots a_2$$

$$h_\epsilon a_{m-1}a_{m-2}\cdots a_1 = a_2 a_1 h_\epsilon a_{m-1}\cdots a_3$$

$$\cdots$$

$$h_\epsilon a_{m-1}a_{m-2}\cdots a_1 = a_{m-2}\cdots a_1 h_\epsilon a_{m-1}.$$

This is the set of relations R with (h and h_ϵ identified) except for

$$h a_{m-1}a_{m-2}\cdots a_1 = a_{m-1}\cdots a_1 h.$$

In order to have this relation, we start by writing the commutator $a_{m-1}h_\epsilon = h_\epsilon a_{m-1}$ as a relator

$$a_{m-1}h_\epsilon a_{m-1}^{-1}h_\epsilon^{-1}.$$

We replace the relator with the relator conjugated by a_{m-2}, a move of type (i):

$$a_{m-2}a_{m-1}h_\epsilon a_{m-1}^{-1}h_\epsilon^{-1}a_{m-2}^{-1}.$$

We replace the relator with a the relator right multiplied by the relator

$$a_{m-2}h_\epsilon a_{m-2}^{-1}h_\epsilon^{-1},$$

a move of type (iii):

$$a_{m-2}a_{m-1}h_\epsilon a_{m-1}^{-1}h_\epsilon^{-1}a_{m-2}^{-1}a_{m-2}h_\epsilon a_{m-2}^{-1}h_\epsilon^{-1}$$

We are then able to freely reduce (type (i)):

$$a_{m-2}a_{m-1}h_\epsilon a_{m-1}^{-1}a_{m-2}^{-1}h_\epsilon^{-1}.$$

Continuing this procedure for all a_i from $i = m-3, \ldots, 1$ yields:

$$a_1 \cdots a_{m-1} h_\epsilon a_{m-1}^{-1} a_{m-2}^{-1} \cdots a_1^{-1} h_\epsilon^{-1}.$$

which is simply the relation $ha_{m-1}a_{m-2}\cdots a_1 = a_{m-1}\cdots a_1 h$ (with h and h_ϵ identified).

Therefore, we see that the local relations and line labels are the same in both R and R_ϵ except for the addition of the commutators

$$\{a_i h_\epsilon = h_\epsilon a_i \mid 1 \leq i < m-1\}. \tag{4.7}$$

Repeating this procedure for all multiple points along the lines H and H_ϵ for the arrangements $\mathcal{A}$ and $\mathcal{A}_\epsilon$, we arrive at the presentations listed in 4.1. We note that the relations $R(h)$ and $R(h_\epsilon)$ consist of the same words in the generating sets but do not contain h or h_ϵ. $\square$

Let $C(\mathcal{A})$ and $C(\mathcal{A}_\epsilon)$ be the canonical 2-complexes constructed from the presentations $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{A}_\epsilon)$ respectively. From the generating sets and relations, the next lemma follows easily.

Lemma 4.7. *$C(\mathcal{A})$ is a sub-complex of $C(\mathcal{A}_\epsilon)$.*

In particular,

$$C(\mathcal{A}_\epsilon) = C(\mathcal{A}) \bigcup_{\phi_{i,j} \in \Phi} \sigma_{i,j}^2$$

where $\phi_{i,j}$ is the attaching maps for the 2-cell $\sigma_{i,j}^2$ induced by the commutator relation $a_{i,j}h_\epsilon = h_\epsilon a_{i,j}$ for $1 \leq i \leq d$, $1 \leq j \leq m_i - 2$.

Let $\tilde{C}(\mathcal{A}, \phi)$ and $\tilde{C}(\mathcal{A}_\epsilon, \phi_\epsilon)$ be the covering spaces of $C(\mathcal{A})$ and $C(\mathcal{A}_\epsilon)$ built from the homomorphisms $\phi : \pi_1(M(\mathcal{A})) \to \mathbb{Z}_{|\mathcal{A}|+1}$ and $\phi_\epsilon : \pi_1(M(\mathcal{A}_\epsilon)) \to \mathbb{Z}_{|\mathcal{A}|+1}$ sending a generator to 1. The following lemma follows easily by comparing the complexes.

Lemma 4.8. *$\tilde{C}(\mathcal{A}, \phi)$ is a sub-complex of $\tilde{C}(\mathcal{A}_\epsilon, \phi_\epsilon)$.*

The purpose of constructing $\mathcal{A}_\epsilon$ is in order to have an arrangement such that the homology of the Milnor fiber is computable.

Lemma 4.9. *Let F_ϵ be the Milnor fiber associated to $\mathcal{A}_\epsilon$. Then $H_1(F_\epsilon) = \mathbb{Z}^{|\mathcal{A}_\epsilon|}$.*

Proof. The line H_ϵ has at most one higher order multiple point that is not relatively prime to $|\mathbf{c}\mathcal{A}_\epsilon|$. All multiple points in the affine plane along H_ϵ have multiplicity two, thus the point occurs on the line at infinity. As $H_1(F_\epsilon) \cong H_1(\tilde{C}(\mathcal{A}_\epsilon, \phi_\epsilon))$, from Theorem 3.11 we conclude that $H_1(F_\epsilon) \cong \mathbb{Z}^{|\mathcal{A}|}$. □

Lemma 4.10. *The inclusion map $\psi : \tilde{C}(\mathcal{A}, \phi) \to \tilde{C}(\mathcal{A}_\epsilon, \phi_\epsilon)$ induces an epimorphism of groups $\psi^* : H_1(\tilde{C}(\mathcal{A}, \phi)) \to H_1(\tilde{C}(\mathcal{A}_\epsilon, \phi_\epsilon))$.*

Proof. Consider the chain complexes of $\tilde{C}(\mathcal{A}, \phi)$ and $\tilde{C}(\mathcal{A}_\epsilon, \phi_\epsilon)$:

$$C_\bullet(\tilde{C}(\mathcal{A}, \phi)) = \cdots \to C_2 \to^{d_2} C_1 \to^{d_1} C_0 \to^{d_0} 0$$

$$C_\bullet(\tilde{C}(\mathcal{A}_\epsilon, \phi_\epsilon)) = \cdots \to C_2^\epsilon \to^{d_2^\epsilon} C_1^\epsilon \to^{d_1^\epsilon} C_0^\epsilon \to^{d_0^\epsilon} 0$$

As the 1-skeletons of $\tilde{C}(\mathcal{A})$ and $\tilde{C}(\mathcal{A}_\epsilon, \phi_\epsilon)$ are identical, we have that $C_1 = C_1^\epsilon$. Consider the following long exact sequence of relative homology groups.

$$\longrightarrow H_2(\tilde{C}(\mathcal{A}_\epsilon,\phi_\epsilon),\tilde{C}(\mathcal{A},\phi)) \xrightarrow{\partial} H_1(\tilde{C}(\mathcal{A},\phi)) \xrightarrow{\psi^*}$$

$$H_1(\tilde{C}(\mathcal{A}_\epsilon,\phi_\epsilon)) \longrightarrow H_1(\tilde{C}(\mathcal{A}_\epsilon,\phi_\epsilon),\tilde{C}(\mathcal{A},\phi)) \longrightarrow$$

As $H_1(\tilde{C}(\mathcal{A}_\epsilon,\phi_\epsilon),\tilde{C}(\mathcal{A},\phi))$ is trivial, one may see that ψ^* is surjective. $\square$

4.2 Ker ψ^*

Our goal will be to show that Ker ψ^* is torsion-free. Then we will have the following short exact sequence.

$$0 \longrightarrow \text{Ker } \psi^* \overset{i}{\hookrightarrow} H_1(\tilde{C}(\mathcal{A},\phi)) \overset{\psi^*}{\twoheadrightarrow} H_1(\tilde{C}(\mathcal{A}_\epsilon,\phi_\epsilon)) \longrightarrow 0$$

If Ker ψ^* is torsion-free, then $H_1(\tilde{C}(\mathcal{A},\phi))$ will be torsion-free as we know that $H_1(\tilde{C}(\mathcal{A}_\epsilon,\phi_\epsilon))$ is torsion-free by Lemma 4.9. We will begin by determining a generating set for Ker ψ^*.

Lemma 4.11. *Ker ψ^* is the subgroup of $H_1(\tilde{C}(\mathcal{A},\phi))$ generated by*

$$\left\{x^k.(a_{i,j}-h, h-a_{i,j}, 0, \ldots, 0) \mid 0 \le k \le n,\ 1 \le i \le d,\ 1 \le j \le m_i - 1\right\} \qquad (4.8)$$

Proof. This follows from construction of the chain complexes: the homology groups $H_1(\tilde{C}(\mathcal{A},\phi))$ and $H_1(\tilde{C}(\mathcal{A}_\epsilon,\phi_\epsilon))$ may be given the same generating sets since the 1-skeletons of $\tilde{C}(\mathcal{A},\phi)$ and $\tilde{C}(\mathcal{A}_\epsilon,\phi_\epsilon)$ may be identified. The relations for $H_1(\tilde{C}(\mathcal{A},\phi))$ are a subset of the relations on $H_1(\tilde{C}(\mathcal{A}_\epsilon,\phi_\epsilon))$, with extra relations from the commutators 4.7, leading to the relators in 4.8. As they appear only in the set of relations for $H_1(\tilde{C}(\mathcal{A}_\epsilon,\phi_\epsilon))$, they generate Ker ψ^*. $\square$

As we have generators for a presentation of Ker ψ^*, we now attempt to find some relations.

Lemma 4.12. *If v_i is a multiple point on H, and $\gcd(m(v_i), n+1) = 1$ or $m(v) = 2$, then*

$$x^q.(a_{i,j} - h, h - a_{i,j}, 0, \ldots, 0) = 0$$

is a relation in Ker ψ^* for all $0 \leq q \leq n, 1 \leq j \leq m_i - 1$.

Proof. With these hypothesis, we showed in Subsection 2.2.2 that there are induced relators in $H_1(\tilde{C}(\mathcal{A}, \phi))$ of the form

$$\{x^q.(a_{i,j} - h, h - a_{i,j}, 0, \ldots, 0) \mid 0 \leq q \leq n\}. \tag{4.9}$$

From these relators the lemma follows. □

Lemma 4.13. *If v_i is a multiple point on H, and $k = \gcd(m(v_i), n+1) \neq 1$ and $m(v) \neq 2$, then*

$$x^l(a_{i,j} - h, h - a_{i,j}, 0, \ldots, 0) = x^{l+k}(a_{i,j} - h, h - a_{i,j}, 0, \ldots, 0)$$

is a relation in Ker ψ^* for $1 \leq j \leq m_i - 1$ $l \in \mathbb{Z}$.

Proof. This follows from Subsection 2.2.2 as well. □

At this point, we have attained no more information than one may find by applying Theorem 3.9 or Theorem 3.1. However, from our careful construction, we are now able to make use of multiple points that are not on the distinguished hyperplane H. We begin by making some definitions that will aide our discussion.

Definition 4.14. Let $\{x^q.(a - h, h - a, 0, \ldots, 0) \mid 0 \leq q \leq n\}$ be the <u>generators associated to a</u>, for a any generator in the GCMD presentations along H given in Theorem 4.6.

A generator a will be called <u>trivially collapsing</u> if the generators associated to a are trivial in Ker ψ^*.

For ease of notation, we will write

$$\alpha_q = x^{q-1}.(a-h, h-a, 0, \dots, 0) \tag{4.10}$$

for $1 \leq q \leq n+1$ for a as in the previous definition. With the same hypotheses as Lemma 4.13, we may write

$$\{x^p.(a-h,0,0,\dots,0) = x^{p+k}.(a-h,0,0,\dots,0) \mid 0 \leq p \leq n\}$$

which is equivalent in the generating set $\{\alpha_p\}_{p=1}^{n+1}$ to

$$\left\{ \textstyle\sum_{i=p}^{p+k-1} \alpha_i = 0 \;\middle|\; 1 \leq p \leq n+1 \right\} \tag{4.11}$$

Example 4.1. Consider the relators coming from a triple point in an arrangement with $n = 5$ lines. Let a denote the generator arising from a hyperplane at this triple point. Then

$$(a-h,0,0,0,0,0) = x^3.(a-h,0,0,0,0,0) = (0,0,0,a-h,0,0,0)$$

The generators associated to a are

$$\alpha_1 = (a-h, h-a, 0,0,0,0)$$

$$\alpha_2 = (0, a-h, h-a, 0,0,0)$$

$$\alpha_3 = (0,0,a-h, h-a, 0,0)$$

$$\alpha_4 = (0,0,0,a-h, h-a, 0)$$

$$\alpha_5 = (0,0,0,0,a-h, h-a)$$

$$\alpha_6 = (h-a, 0,0,0,0,a-h)$$

and we can see that we may derive the following:

$$(a-h,0,0,h-a,0,0) = (a-h,h-a+a-h,h-a+a-h,h-a,0,0)$$
$$= \alpha_1 + \alpha_2 + \alpha_3$$

Continuing with the other relators gives the set in (4.11). ⊞

Definition 4.15. A generator a will be called k-cyclically collapsing if the generators associated to a of Ker ψ^* satisfy the relations in (4.11).

Remark 4.1. From this definition, we see that if a is 1-cyclically collapsing, then a is trivially collapsing. ⊞

Definition 4.16. Let v and w be distinct points of at least multiplicity two in the arrangement $\mathcal{A}$ lying on the hyperplane $H \in \mathcal{A}$. If the real part of the line H between v and w contains only points of multiplicity one or two, then we say that v and w are connected multiple points.

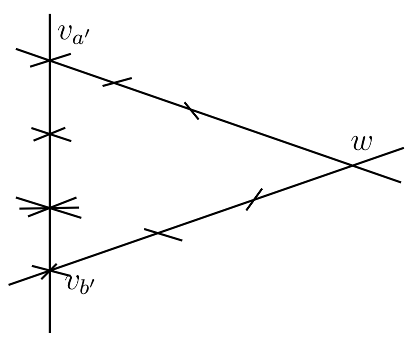

Figure 4.5: Two points $v_{a'}$ and $v_{b'}$ connected to a double point w.

Lemma 4.17. *Let a' and b' be two generators for GCMD presentations along H for $\pi_1(M(\mathcal{A}))$ such that a' is k-cyclically collapsing and b' is j-cyclically collapsing. Let $v_{a'}$ and $v_{b'}$ be the multiple points on H inducing the generators a' and b' respectively. If $v_{a'}$ and $v_{b'}$ are connected to the same point w of multiplicity two (see Figure 4.5), then the generators associated to a' and b' are identified and a' is d-cyclically collapsing for $d = \gcd(k, j)$.*

Proof. Let $\{\alpha_q\}_{q=1}^{n+1}$ and $\{\beta_q\}_{q=1}^{n+1}$ denote the respective generating sets for a' and b' as in 4.10. The double point w induces the relation $a'b' = b'a'$ in the presentation for $\pi_1(M(\mathcal{A}))$ which induces the set of relations

$$\{x^q.(a' - h + h - b', b' - h + h - a', 0, \ldots, 0) \mid 1 \le q \le n+1\}$$

which implies

$$\{\alpha_q = \beta_q \mid 1 \le q \le n+1\} \tag{4.12}$$

in Ker ψ^*.

As a' is k-cyclically collapsing and b' is j-cyclically collapsing we have the following sets of relations.

$$\left\{ \sum_{i=p}^{p+k-1} \alpha_i = 0 \,\middle|\, 1 \le p \le n+1 \right\} \tag{4.13}$$

$$\left\{ \sum_{i=p}^{p+j-1} \beta_i = 0 \,\middle|\, 1 \le p \le n+1 \right\} \tag{4.14}$$

From the relations in (4.12) , we may rewrite the relations in (4.14) as

$$\left\{ \sum_{i=p}^{p+j-1} \alpha_i = 0 \,\middle|\, 1 \le p \le n+1 \right\} \tag{4.15}$$

We may rewrite each set of relations as

$$\left\{ \alpha_l = \alpha_{l+k}, \alpha_{n+1} + \sum_{i=1}^{k-1} \alpha_i = 0 \;\middle|\; 1 \leq l \leq n \right\} \tag{4.16}$$

$$\left\{ \alpha_l = \alpha_{l+j}, \alpha_{n+1} + \sum_{i=1}^{j-1} \alpha_i = 0 \;\middle|\; 1 \leq l \leq n \right\} \tag{4.17}$$

As the relations are being cyclically identified, we may clearly combine these relations into a the following set:

$$\left\{ \begin{array}{l|l} \alpha_l = \alpha_{l+d} & 1 \leq l \leq n \\ \alpha_{n+1} + \displaystyle\sum_{i=1}^{k-1} \alpha_i = 0 & \\ \alpha_{n+1} + \displaystyle\sum_{i=1}^{j-1} \alpha_i = 0 & \end{array} \right\} \tag{4.18}$$

where $d = \gcd(k, j)$.

Using the relations in (4.18), we have that

$$\begin{aligned} 0 &= \alpha_{n+1} + \sum_{i=1}^{k-1} \alpha_i \\ &= \frac{k}{d} \sum_{i=1}^{d} \alpha_i \end{aligned}$$

This follows as α_{n+1} is equivalent to α_k and then we may divide the sum into blocks of length d, each of which is equivalent to $\sum_{i=1}^{d} \alpha_i$. We have a similar result with respect to j.

As $d = \gcd(j, k)$, there exist $s, t \in \mathbb{Z}$ such that $d = sj + tk$. Therefore,

$$\begin{aligned} \sum_{i=1}^{d} \alpha_i &= \left(s\frac{j}{d} + t\frac{k}{d} \right) \sum_{i=1}^{d} \alpha_i \\ &= 0 \end{aligned}$$

Applying the cyclic relations $\{\alpha_i = \alpha_{i+d}\}$ to $\sum_{i=1}^{d} \alpha_i = 0$ we may conclude that we have relations of the form

$$\left\{ \sum_{i=p}^{p+d-1} \alpha_i = 0 \;\middle|\; 1 \leq p \leq n+1 \right\}$$

Therefore, a' is d-cyclically collapsing. $\square$

Lemma 4.18. *Let a' and b' be two generators for GCMD presentations along H for $\pi_1(M(\mathcal{A}))$ such that a' is k-cyclically collapsing and b' is j-cyclically collapsing. Let $v_{a'}$ and $v_{b'}$ be the multiple points on H inducing the generators a' and b' respectively. Let $v_{a'}$ and $v_{b'}$ be connected to the same point w such that they are adjacent at w (see Figure 4.6). If j divides $m(w)$, then a' is d-cyclically collapsing for $d = \gcd(k, m)$ and $m = \gcd(m(w), |\mathcal{A}| + 1)$.*

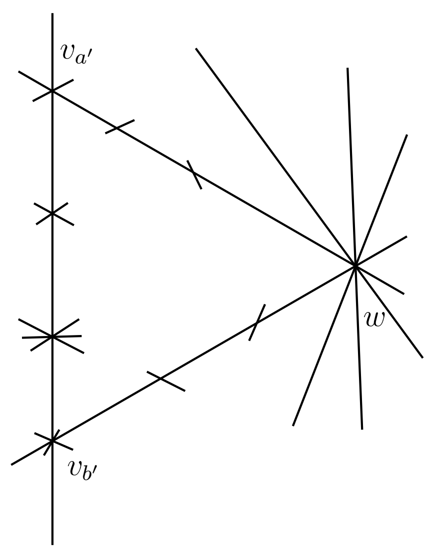

Figure 4.6: Two points $v_{a'}$ and $v_{b'}$ connected and adjacent to a point w.

Proof. Recall that the relators coming from the point w yield the set defined in 2.1 of the form

$$\left\{x^i.(a-b,0,\ldots,0,b-a,\ldots,0)\ :\ 0\le i\le n-1\right\} \tag{4.19}$$

where the second nonzero entry is in the $\gcd(m(w),|\mathcal{A}|+1)$-th slot of the tuple.

One may easily change these relators to be in terms of α and β to yield relators in Ker ψ^* of the form

$$\left\{\sum_{i=p}^{p+m(w)-1}(\alpha_i+\beta_i)\ \middle|\ 1\le p\le|\mathcal{A}|+1\right\}. \tag{4.20}$$

As b' is j-cyclically collapsing, and j divides $m(w)$, these relators become

$$\left\{\sum_{i=p}^{p+m(w)-1}\alpha_i\ \middle|\ 1\le p\le|\mathcal{A}|+1\right\} \tag{4.21}$$

since the sums of β-terms may be divided into blocks of length j that are each zero.

Therefore a' is m-cyclically collapsing. Finally, as we know that a' is k-cyclically collapsing, a proof similar to that of Lemma 4.17 allows us to conclude that a' is $d=\gcd(k,m)$ cyclically collapsing. $\square$

Lemma 4.19. *Let $b'_1,\ldots,b'_k$ be generators for a GCMD presentation along a hyperplane H in an arrangement $\mathcal{A}$ such that $b'_2,\ldots,b'_k$ are trivially collapsing. If the points on H inducing the generators are connected to a point w and then b'_1 is trivially collapsing.*

Proof. For an ordering of these generators, we will have the following relator in the fundamental group presentation.

$$b'_kb'_1\cdots b'_{k-1}=b'_1b'_2\cdots b'_k \tag{4.22}$$

This will induce the set of relators in Ker ψ^* given by

$$\left\{ \sum_{i=1}^{k} (\beta_{k,i+j} - \beta_{i,i+j}) \ : \ 0 \leq j \leq n \right\} \tag{4.23}$$

As $b'_2, \ldots, b'_k$ are trivially collapsing, this set of relators becomes

$$\{-\beta_{1,j} \ : \ 0 \leq j \leq n\} \tag{4.24}$$

whence b'_1 is trivially collapsing. □

One may use Lemma 4.18, Lemma 4.17, and Lemma 4.19 to show that the first homology group of the Milnor fiber is torsion free and of minimal rank. We show this for several examples in the next two sections.

4.3 A Classification Result

Theorem 4.20. *Let $\mathcal{A}$ be a complexified-real arrangement of lines in $\mathbb{CP}^2$ such that the higher order multiple points only have multiplicity three, and there is a line in $\mathcal{A}$ that contains exactly two points of multiplicity three and any number of points with lower multiplicities. Then $\mathcal{A}$ is isomorphic to the arrangement defined by the polynomial $Q(\mathcal{A}_1) = xyz(x-y)(x-z)(y-z)$ or $H_1(F;\mathbb{Z}) \cong \mathbb{Z}^{|\mathcal{A}|-1}$, where F is the Milnor fiber associated to $\mathcal{A}$.*

Proof. As the multiple points are of orders one, two, or three only, we know that the order of $\mathcal{A}$ must be divisible by three otherwise by Theorem 3.1 we have $H_1(F;\mathbb{Z}) \cong \mathbb{Z}^{|\mathcal{A}|-1}$.

If $|\mathcal{A}| \leq 6$ and $\mathcal{A}$ is not isomorphic to $\mathcal{A}_1$, then there must be a line with only one point of multiplicity three. By Theorem 3.11, we have that $H_1(F;\mathbb{Z}) \cong \mathbb{Z}^{|\mathcal{A}|-1}$.

Therefore we assume that $|\mathcal{A}| = 6+3k$ for $k \in \mathbb{Z}_+$. We conclude from Theorem 3.11 that each line contains at least two points of multiplicity three.

Let H denote a line containing exactly two points of multiplicity three. Let $\{H, L, P\}$ denote one set of lines intersecting in one of the triple points. Let $\mathbf{d}\mathcal{A}$ denote the decone of the arrangement with respect to L. Thus P and H are parallel in $\mathbf{d}\mathcal{A}$. Without loss of generality, we may assume that the decone of the arrangement may be depicted as in Figure 4.7.

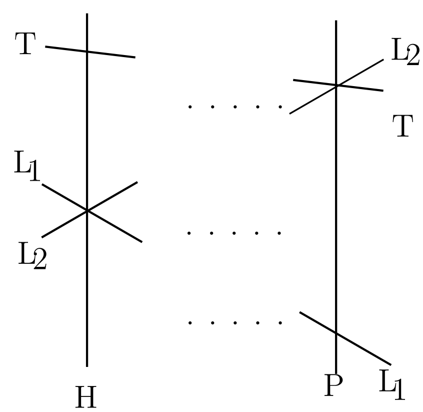

Figure 4.7: Diagram of $\mathbf{d}\mathcal{A}$.

Let $\{H, L_1, L_2\}$ denote the lines intersecting in the other point of multiplicity three on H. We may denote l_1, l_2 as the generators associated to L_1 and L_2 respectively and note that they are both 3-cyclically collapsing. As $|\mathcal{A}| = 6+3k$ and $k \geq 1$, we may conclude that P contains a multiple point that is not contained in L_1 or L_2. (If $L_1 \cap P$ and $L_2 \cap P$ both have multiplicity three, this still leaves $3k-2$ lines that must intersect P non-trivially.)

Consider the following lemma.

Lemma 4.21. *Let* $\{a_1, \ldots, a_n, a'_1, \ldots, a'_n\}$ *be the generators associated to a multiple point in an arrangement* $\mathcal{A}$ *as labeled in Figure 1.5. If each generator in* $\{a_1, \ldots, a_n\}$ *is trivially collapsing, then each* a'_i *is trivially collapsing as well. In fact, we may explicitly identify the generators as*

$$x^q.(a_i - h, 0, \ldots, 0) = x^{q-i+1}(a'_i - h, 0, \ldots, 0).$$

Proof. From the conjugation relations in Chapter 1, we have that

$$a'_i a_1^{-1} a_2^{-1} \cdots a_{i-1}^{-1} a_i a_{i-1} a_{i-2} \cdots a_1 = 1.$$

In $H_1(F; \mathbb{Z})$ this relator becomes the following set of relators:

$$\{x^q.(a_{i-1} - h - (a_i - h), a_{i-2} - h - (a_{i-1} - h), \ldots, a'_i - h - (a_1 - h), 0, \ldots, 0)\}$$

for $0 \leq q \leq |\mathcal{A}| - 1$.

As all of the a_i are trivially collapsing, we have that this is equivalent to the set of relators

$$\{x^q.(-(a_i - h), 0, \cdots, 0, a'_i, 0, \cdots, 0)\}_{q=0}^{|\mathcal{A}|-1}$$

where the first string of zeros has length $i - 2$. Therefore, the a'_i is trivially collapsing and identified as stated. $\square$

Lemma 4.18 and lemma 4.21 allows us to conclude that any of the lines intersecting H in a double point will have associated generators that are trivially collapsing, and will be associated to trivially collapsing generators on the other side of the intersection point as long as they do not intersect the lines L_1 or L_2.

Suppose a line T that is trivially collapsing intersects L_1 in a point between the lines H and P. By Lemma 4.18, we may then conclude that l_1 is trivially collapsing. Similarly if two trivially collapsing lines intersect L_1 between H and P, then l_1 will also be trivially collapsing. In either case, we may conclude that l_2 is trivially collapsing as

$$l_1 l_2 h = l_2 h l_1$$

yields relators of the form

$$\{x^q.(l_1 - h - (l_2 - h), l_2 - h, -(l_1 - h), 0, \ldots, 0)\}.$$

Combining these with the trivial collapse of l_1 yields

$$\{x^q.(-(l_2 - h), l_2 - h, 0, \ldots, 0)\},$$

that is, the trivial collapse of l_2.

One may see a similar behavior in case the roles of L_1 and L_2 are reversed in the previous paragraph.

Therefore we are left with the case that L_1 and L_2 contain no multiple points between H and P, and P has a double point that is not contained in either L_1 or L_2. Let the line inducing the point be denoted by T. Lemma 4.18 and lemma 4.21 allow us to conclude that the line T (technically the generator associated to it at P) is trivially collapsing. Therefore p, the generator associated to P, is also trivially collapsing.

Finally, applying Lemma 4.18 to the points $L_1 \cap P$ and $L_2 \cap P$ allows us to conclude that l_1 and l_2 are trivially collapsing. $\square$

4.4 Examples

Example 4.2. We begin by examining two of the 9_3 arrangements. These arrangements are denoted in such a way since they consist of 9 lines in the projective plane that each contain three points of multiplicity three.

The $(9_3)_1$ arrangement has defining polynomial

$$Q\left((9_3)_1\right) = xyz(x-y)(y-z)(x-y-z)(2x+y+z)(2x+y-z)(-2x+5y-z)$$

In Figure 4.8 we depict the arrangement with the line defined by $z = 0$ as the line at infinity.

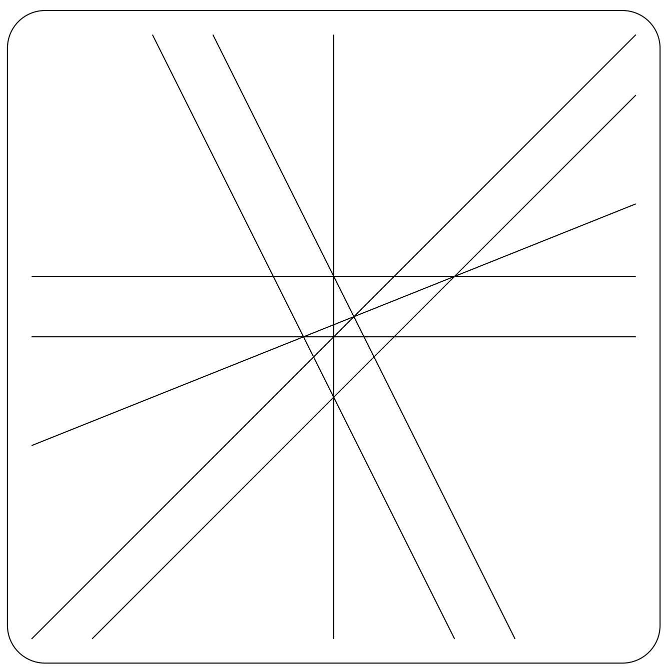

Figure 4.8: The real part of the $(9_3)_1$ arrangement.

Cohen and Suciu [4] show by means of Fox calculus that the first betti number of the Milnor fiber is 10, an excess of 2 over the minimal possible so $H_1(F;\mathbb{Z}) \cong \mathbb{Z}^{10}$.

The $(9_3)_2$ arrangement has defining polynomial

$$Q\left((9_3)_2\right) = xyz(x+y)(y+z)(x+3z)(x+2y+z)(x+2y+3z)(2x+3y+3z)$$

In Figure 4.9 we depict the arrangement with the line defined by $z=0$ as the line at infinity.

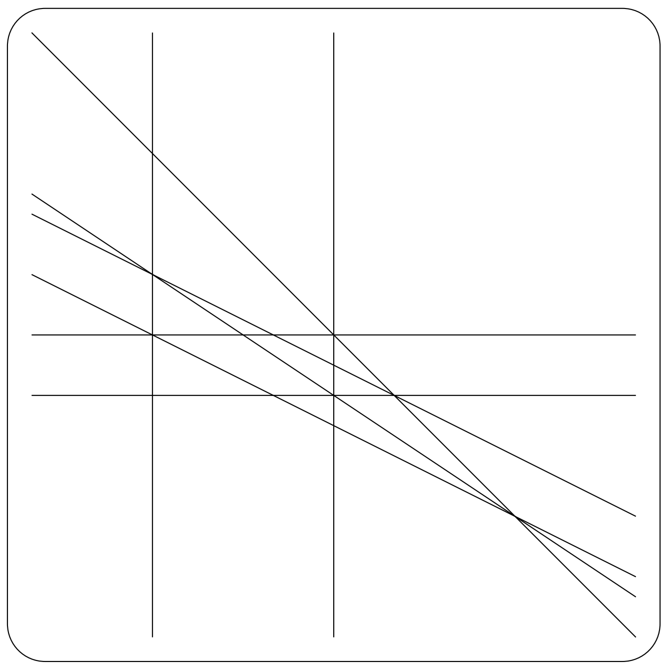

Figure 4.9: The real part of the $(9_3)_2$ arrangement.

Cohen and Suciu compute the first betti number of the associated Milnor fiber to be 8. By using the Lemmas of the previous section, we attain the stronger result that the first homology group of the Milnor fiber is isomorphic to $\mathbb{Z}^8$.

Consider the line defined by $y+z=0$ in the $(9_3)_2$ arrangement and call this line H. We blow up the depiction in Figure 4.10 and label the generators from left to right $a_1, a_2, a_3, a_4, a_5, a_6, a_7$.

We have a generators for Ker ψ^* given by

$$\{\alpha_{i,j} : 2 \leq i \leq 7,\ 1 \leq j \leq 9\} \tag{4.25}$$

where $\alpha_{i,j}$ comes from a_i. As the line inducing the generator a_1 is parallel to H, no generators occur in Ker ψ^*.

We note that a_2 and a_3 are trivially collapsing, and a_4, a_5, a_6, a_7 are all 3-collapsing. By Lemma 4.19, as a_2 and a_3 are trivially collapsing and connected to point 1, we conclude that a_1 is trivially collapsing.

At point 2, we may conclude by Lemma 4.17 that the generators associated to a_1 and a_4 are identified, thus a_4 is trivially collapsing. By applying Lemma 4.17 at point 3 we may identify the generators associated to a_5 and a_6. At point 4 we see that we may identify the generators associated to a_6 and a_1. Finally, at point 5 we apply Lemma 4.19 to the generators associated to a_5 and a_1 to conclude that a_7 is trivially collapsing. As all the generators are trivially collapsing, we conclude that Ker ψ^* is the trivial group, hence the first homology group of the associated Milnor fiber is torsion-free of rank 8. ⊞

Example 4.3. This example shows how one may use the lemmas and theorems of this chapter to show that the first homology group of the Milnor fiber is minimal and torsion free in a case where Theorem 3.9 fails to yield a minimality result. Consider

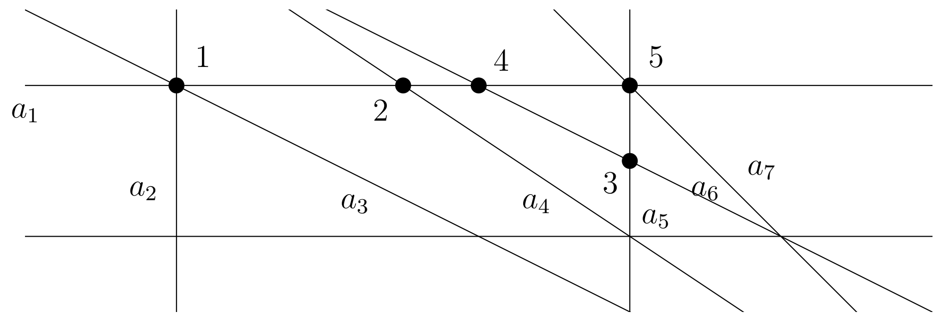

Figure 4.10: A closer look at the $(9_3)_2$ arrangement.

the arrangement $\mathcal{A}$ given by the following defining polynomial:

$$Q(\mathcal{A}) = xyz(x-z)(x+z)(y-z)(y+z)(x-y)$$

$$(x-y+z)(x-y-z)(x+2y-z)(x+2y+z)$$

The real part of this arrangement is depicted in Figure 4.11.

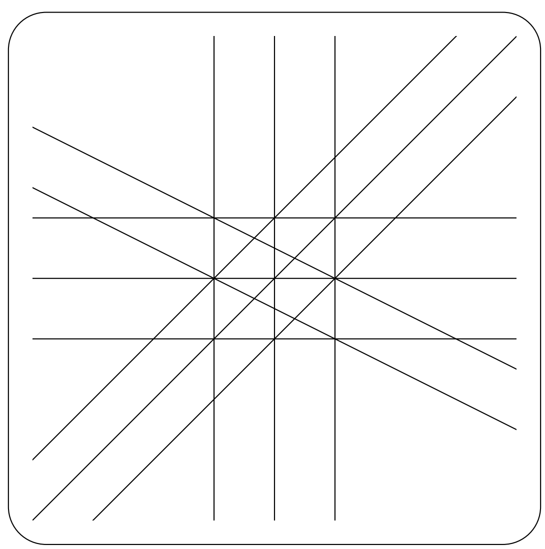

Figure 4.11: The real part of the arrangement $\mathcal{A}$.

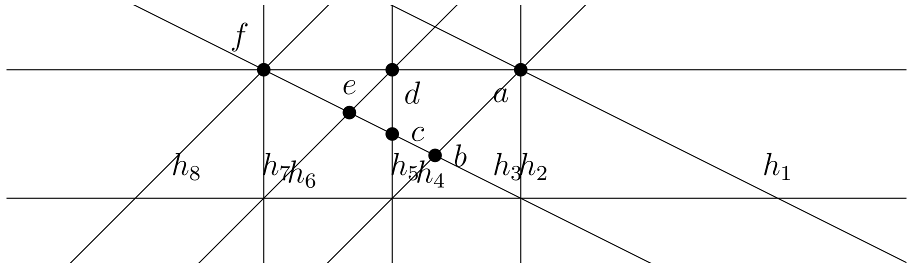

Figure 4.12: A closer look at $\mathcal{A}$ with labels.

Let the line given by $y + 1 = 0$ be the distinguished line H and let the lines intersecting H be denoted by $h_1, h_2, \ldots, h_8$ as in Figure 4.12. Further, label the points of intersections of these lines by a, b, c, d, e, f as in the figure. We know that the generators associated to h_1 and h_8 are already 1- collapsing.

By Lemma 4.18, as h_1 and h_2 meet at point a and h_1 is 1-collapsing, we have that h_2 will be $\gcd(3, 4)$-collapsing. Therefore, it is 1-collapsing.

Similarly, as h_4 and h_2 are adjacent at a, h_4 is also 1-collapsing.

At point b, we may conclude that the generators associated to h_4 and h_3 may be identified, thus h_3 is 1-collapsing by Lemma 4.17.

At point c, we apply Lemma 4.17 to the lines h_5 and h_3 to conclude that h_5 is 1-collapsing. By the same argument at e applied to h_3 and h_6, we have that h_6 is 1-collapsing.

Finally, by Lemma 4.18 applied to the lines h_8 and h_7 at point f, we conclude that h_7 is 1-collapsing. As all generators are therefore trivial, we conclude that the first homology group of the Milnor fiber is isomorphic to $\mathbb{Z}^{11}$. $\boxplus$

CHAPTER 5
FUTURE WORK

5.1 Future Work

As one may plainly see, the work contained in this thesis depends heavily on the Arvola-Randell presentation for the fundamental group of the complement of hyperplane arrangement. Other presentations such as the braid-monodromy presentation [5] and the minimal positive presentation discovered by Yoshinaga [20] exist. In future work, we hope to use these presentations to find other conditions that allow us to conclude that the first homology group of the Milnor fiber is torsion-free and of minimal rank.

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