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Some representations of SL*(2,A)

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SOME REPRESENTATIONS OF $\mathrm{SL}_*(2, A)$

by

Syvillia Ann Averett

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

July 2012

Thesis Supervisor: Professor Philip Kutzko

ABSTRACT

The study of groups has been of interest to mathematicians since the 19th century. Although much is known about the structure of groups, many group theoretic problems remain unsolved. Representation theory allows us to employ linear algebra to solve such problems. The representation theory of linear groups over finite fields has been a particularly interesting topic. Studying these representations is of interest to mathematicians and other scientists as it relates to physics and modern number theory.

In the 1960s Andre Weil introduced a method for finding a special unitary representation for symplectic groups over locally compact fields. This unitary representation is now referred to as the Weil representation. In 2010 Luis Gutiérrez, José Pantoja and Jorge Soto-Andrade were able to generalize Weil's method to a larger class of linear groups namely the $*$-analogue of Sl_2.

Originally, Weil constructed this unitary representation, decomposed it into irreducibles and, in this way, produced the irreducible complex representations of $Sp(2n, k)$. Later, Shalika went in the other direction, first finding the irreducible representations and then computing their multiplicities in the Weil representation. We intend to follow Shalika's method.

In this thesis we look to explore the representation theory of $Sl_*(2, \mathcal{A})$ where $\mathcal{A}$ is the direct sum of the upper and lower $n \times n$ block matrices in $M(2n, k)$, k a finite field. We use Wigner and Mackey's Method of Little Groups to construct these representations.

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the July 2012 graduation.

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To my grandmother, Virginia L. Smith, for the lessons you taught me years ago that have helped mold me into the woman I am today.

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The study of groups has been of interest to mathematicians since the 19th century. Although much is known about the structure of groups, many group theoretic problems remain unsolved. Representation theory allows us to employ linear algebra to solve such problems. The representation theory of linear groups over finite fields has been a particularly interesting topic. Studying these representations is of interest to mathematicians and other scientists as it relates to physics and modern number theory.

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In this thesis we look to explore the representation theory of $Sl_*(2, \mathcal{A})$ where $\mathcal{A}$ is the direct sum of the upper and lower $n \times n$ block matrices in $M(2n, k)$, k a finite field. We use Wigner and Mackey's Method of Little Groups to construct these representations.

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INTRODUCTION

1.1 Motivation

Group representations have been studied since the late 19th century. They allow us to view groups as groups of matrices with the group product being given by matrix multiplication. The representations of reductive groups over p-adic fields is of particular interest to those who study number theory.

In 1964 Andre Weil produced a special unitary representation, called the Weil representation, for symplectic groups over a locally compact field. Several mathematicians have since looked to expand this theory to other classical groups. In 2010 Gutiérrez, Pantoja and Soto-Andrade developed a method for constructing a generalized Weil representation for $*$-analogues of $Sl(2, A)$ where A is a unitary ring with involution $*$. They showed that if A was a simple artinian ring with involution, $*$, then one could construct a generalized Weil representation of $Sl_*(2, A)$.

In this thesis we look to study $Sl_*(2, A)$ for a particular ring A. Our goal is to construct the irreducible representations of this group using the Method of Little Groups developed by Wigner and Mackey.

1.2 Overview

In Chapter 2, we discuss the background material necessary for our work. We will begin with the basic theory for group representations, some important results related to induced representations, and Wigner and Mackey's Method of Little Groups.

We will also discuss the construction of the $*$-analogue of Sl_2.

We will introduce the structure of our ring $\mathcal{A}$ in the beginning Chapter 3. We will do this in the most general setting, but very quickly restrict ourselves to the case where $n = 1$. The remainder of the chapter will be dedicated to constructing the irreducibles representations of $Sl_*(2, \mathcal{A})$ in the case where $n = 1$.

In Chapter 4 we will construct the irreducible representations in the case where $n = 2$. This will provide us with significant insight into what issues arise in the general case.

Finally, in Chapter 5, we will discuss future research involving the $*$-analogue of $Sl(2)$, Bruhat presentations, Weil representations, and p-adic fields.

CHAPTER 2
DEFINITIONS AND BACKGROUND

2.1 Preliminaries

We begin here with some basic definitions. Let G be a finite group and F an algebraically closed field. We also assume that the characteristic of the field F does not divide the order of the group G.

Definition 2.1. A **linear representation of** G **over** F is a pair (π, V) where V is an F-vector space and $\pi : G \to \mathrm{Aut}_F(V)$ is a homomorphism between G and the group of invertible linear transformations of V onto itself. The **degree** of the representation is the F-dimension of V.

If we let $V = F$, then (π, V) is called the **trivial representation** if π maps every element of G to the identity element in $F^\times$.

Definition 2.2. Let G be a group and (π, V) a representation of G over F. We say that a subspace V' of V is **stable** under G if $\pi(g)v$ lies in V' for all g in G and v in V'. The action of G on V restricts to an action on V'. The representation (π, V') is called a **subrepresentation** of (π, V).

Definition 2.3. Let (π, V) be a representation of G. Then (π, V) is said to be **irreducible** if the only stable subspaces of V are 0 and V. Otherwise, we say that (π, V) is **reducible**. We denote the set of irreducible representations of G by $\mathrm{Irr}(G)$.

Maschke proved the following important result.

Theorem 2.4. *(Maschke's Theorem) Let (π, V) be a representation of a finite group G over F. Assume that the characteristic of F does not divide the order of G. Then π is completely reducible, i.e. V is the direct sum of irreducible subspaces.*

Proof. See [3] Theorem 10.8. $\square$

Definition 2.5. Let G_1 and G_2 be groups. Let (π_1, V_1) and (π_2, V_2) be representations of G_1 and G_2 respectively. We define a representation $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$ of $G_1 \times G_2$ by $\pi_1 \otimes \pi_2(g_1, g_2)(v_1 \otimes v_2) = \pi_1(g_1)(v_1) \otimes \pi_2(g_2)(v_2)$, where $v_1 \in V_1$ and $v_2 \in V_2$. We call this representation the **tensor product** of π_1 and π_2.

Theorem 2.6. *Let G_1, G_2 be groups and let (π_1, V_1) and (π_2, V_2) be irreducible representations of G_1 and G_2 respectively. Then:*

1. *$\pi_1 \otimes \pi_2$ is irreducible.*

2. *All irreducible representations of $G_1 \times G_2$ have this form.*

Proof. See [12] § 3.2 Theorem 10 $\square$

In linear algebra we say that two matrices are similar if there exists an invertible matrix that conjugates one to the other. We define isomorphic representations in an analogous manner.

Definition 2.7. Let $(\pi,\ V)$ and $(\pi',\ V')$ be representations of G. We say that a homomorphism $T : V \to V'$ is a G-**map** if $T \circ \pi(g)v = \pi'(g) \circ T(v)$ for all $g \in G$, $v \in V$. We say that $(\pi,\ V)$ is **isomorphic** to $(\pi',\ V')$ if there exists a G-map T that is an isomorphism. We denote the set of all G-maps by $\mathrm{Hom}_G(\pi, \pi')$.

We denote the category of all F-representations of G by $\mathcal{R}_F(G)$. The objects of $\mathcal{R}_F(G)$ are the pairs (π, V) and morphisms are the intertwining maps.

The following result, due to Schur, describes the behavior of intertwining maps of irreducible representations.

Lemma 2.8. *(Schur's Lemma) Let (ρ_1, V_1) and (ρ_2, V_2) be two irreducible representations of a finite group G, and let f be a linear mapping of V_1 into V_2 such that $\rho_2(g) \circ f = f \circ \rho_1(g)$ for all $g \in G$. Then:*

1. *If ρ_1 and ρ_2 are not isomorphic, $f = 0$.*

2. *If $V_1 = V_2$ and $\rho_1 = \rho_2$, f is a scalar multiple of the identity.*

Proof. See [12] §2.2 Proposition 4. □

A concept that will arise repeatedly throughout this thesis is that of the **conjugate** representation.

Definition 2.9. Let G be a group and $(\pi,\ V)$ a linear representation of G over F. Let g be an element of G. We define $\pi^{(g)} : G \to \mathrm{Aut}_F(V)$ by $\pi^{(g)}(h) = \pi(g^{-1}hg)$. We call the pair $(\pi^{(g)},\ V)$ a **conjugate** of (π, V).

Definition 2.10. Let G be a finite group and let (π, V) be a representation of G over F of dimension n. For each $g \in G$, we set $\chi_\pi(g) = \mathrm{tr}(\pi(g))$, where $\mathrm{tr}(\pi(g))$ is the trace of the matrix $\pi(g)$ with respect to some basis of V. The F valued function χ_π is called the **character** of (π, V).

Remark 1. *Note that χ_π is independent of the choice of basis for V. This is easy to see since* $\mathrm{tr}(ABA^{-1}) = \mathrm{tr}(B)$ *for all A, $B \in Gl_n(F)$.*

Proposition 2.1. *Let G be a finite group and let (π, V) and (ρ, W) be representations of G over a field F,* $\mathrm{char}(F) \nmid |G|$. *Then (π, V) is isomorphic to (ρ, W) if and only if they have the same character.*

Proof. See [12] § 2.3 Corollary 2. □

2.2 Induction and Restriction

Suppose (π, V) is a representation of G over F. One may wonder what occurs when we restrict π to a subgroup H of G. Do we obtain a representation of H? Is there a "reverse" process? More specifically, if given a representation of H, are we then able to construct a representation of G? These questions are answered by induction and restriction.

Definition 2.11. Let H be a subgroup of the group G. Let $(\pi,\, V)$ be a representation of G. Let $\mathrm{res}_H^G\,\pi : H \to \mathrm{Aut}_F(V)$ be defined by $\mathrm{res}_H^G\,\pi(h)v = \pi(h)v$. Then $(\mathrm{res}_H^G\,\pi, V)$ is a representation of H called the **restriction** of $(\pi,\, V)$.

Definition 2.12. Let H be a subgroup G and let $(\tau,\, W)$ be a representation of H. The representation $(\mathrm{ind}_H^G\,\tau,\, \mathrm{ind}_H^G\,W)$ is called the **induced representation** of $(\tau,\, W)$, where $\mathrm{ind}_H^G\,W = \{f : G \to W \mid f(hx) = \tau(h)f(x), \quad x \in G, h \in H\}$ and the action of G on $\mathrm{ind}_H^G\,W$ is defined as follows:

$$\mathrm{ind}_H^G\,\tau(x)f(y) = f(yx)$$

for all $x, y \in G$.

Lemma 2.13. *Let H be a subgroup G and let (τ, W) be a representation of H. The representation $(\operatorname{ind}_H^G \tau, \operatorname{ind}_H^G W)$ has degree $[G : H] \dim_F(W)$.*

Proof. See [3] § 12D. □

2.3 Frobenius Reciprocity and Mackey's Irreducibility Criterion

We have seen in Proposition 2.1 that representations are isomorphic if they have the same character. The following theorem allows us to compare representations using induction and restriction.

Theorem 2.14. *(Frobenius Reciprocity) Let G be a group and let H be a subgroup of G. Let (π, V) be a representation of G and (τ, W) a representation of H over F. Then.*

$$\operatorname{Hom}_G(V, \operatorname{ind}_H^G(W)) \cong \operatorname{Hom}_H(\operatorname{Res}_H^G(V), W).$$

Proof. See [3]. □

The following theorem allows us to determine the irreducibility of an induced representation.

Theorem 2.15. *(Mackey's Irreducibility Criterion) Let G be a finite group. Let F be a field so that $\operatorname{char}(F) \nmid |G|$. Let H be a subgroup of G and (τ, W) a representation of H over F. Then $(\operatorname{ind}_H^G \tau, \operatorname{ind}_H^G W)$ is irreducible if and only if:*

1. (τ, W) *is irreducible.*

2. For each $s \in G/H$ *the representations* $\tau^{(s)}$ *and* $\mathrm{Res}^H_{H_s}\tau$ *are disjoint, where* $H_s = sHs^{-1} \cap H$. *(We call two representations disjoint if they share no irreducible components.)*

Proof. See [12] § 7.4 Proposition 23. □

2.4 Method of Little Groups

Let G be a finite group. Let F be an algebraically closed field such that $\mathrm{char}(F) \nmid |G|$. Suppose G contains two subgroups A and H, where A is abelian. Assume also that $G \cong A \rtimes H$. Wigner and Mackey developed a process, called the **Method of Little Groups**, by which they construct the irreducible representations of G using representations of particular subgroups of H.

Since A is abelian and F is algebraically closed, all of the irreducible representations of of A are 1-dimensional. These representations form a group which we denote by $\widehat{A}$. Let G act on $\widehat{A}$ by conjugation and let $\{\chi_i\}$ be a set of representatives for the orbits of H under this action. For each i, let $H_i = \{h \in H |\ \chi_i^{(h)} = \chi_i\}$. Let $G_i = A \cdot H_i$. We may extend χ_i to G_i by $\chi_i(ah) = \chi(a)$. Using this extension we may now view χ_i as a character of G_i.

Let ρ be an irreducible representation of H_i. Then we may inflate ρ to G_i by setting $\tilde{\rho}(ah) = \rho(h)$. Then $\tilde{\rho}$ is an irreducible representation of G_i. Now take the tensor product of χ_i and $\tilde{\rho}$. As stated above, $\chi_i \otimes \tilde{\rho}$ is an irreducible representation of G_i.

Proposition 2.2. *Let $\theta_{i,\rho}$ be the representation of G induced by $\chi_i \otimes \tilde{\rho}$.*

1. *$\theta_{i,\rho}$ is irreducible.*

2. *If $\theta_{i,\rho}$ and $\theta_{i',\rho'}$ are isomorphic, then $i = i'$ and ρ is isomorphic to ρ'.*

3. *Every irreducible representation of G is isomorphic to one of the $\theta_{i,\rho}$.*

Proof. See [12] Proposition 25. □

2.5 The Construction of $Sl_*(2, A)$

Let A be any ring with identity and let $a \mapsto a^*$ be an involution on A, i.e $* : A \to A$ is an isomorphism with the properties $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all $a \in A$. Consider the ring $M(2, A)$. For T, an element of $M(2, A)$, we define T^* by $(T^*)_{ij} = (T_{ji})^*$.

Proposition 2.3. *The map $T \mapsto T^*$ defines an involution on $M(2, A)$.*

Let $J = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Let $M_*(2, A)$ be the collection of matrices g in $M(2, A)$ such that $g^*JgJ^{-1} \in Z(A)I_2$, where $Z(A)$ is the center of A and I_2 is the identity matrix in $M(2, A)$. For each g in $M_*(2, A)$ let $\delta(g)$ be the element in $Z(A)$ satisfying the equation

$$g^*JgJ^{-1} = \delta(g)I_2.$$

.

Let $Gl_*(2, A)$ be the set of invertible elements in $M_*(2, A)$.

Definition 2.16. Let $\det_* : M_*(2, A) \to A$ be the map defined by

$$det_*(g) = ad^* - bc^*$$

for $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. We call this function the **$*$-determinant**. We define $Sl_*(2, A)$ to be the subset of $M_*(2, A)$ containing all elements with $*$-determinant 1.

Proposition 2.4. *Let A be a unitary ring with involution $*$. Let $Sl_*(2, A)$ be defined as above. Then $Sl_*(2, A)$ is a group.*

Proof. See [11] $\square$

Definition 2.17. Let A be a ring with identity and let $*$ be an involution on A. Let $h_t = \begin{bmatrix} t & 0 \\ 0 & {t^*}^{-1} \end{bmatrix}$, $u_r = \left[\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right]$, and $w = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, where $t \in A^\times$ and $r \in A_s$, the symmetric elements in A. The set of all h_t, u_r, w, $t \in A^\times$, $r \in A_s$ is called the set of **Bruhat generators** for $Sl_*(2, A)$.

The set of Bruhat generators satisfy the following relations:

$$h_t h_{t'} = h_{tt'}, \qquad u_r u_s = u_{r+s}, \qquad h_t u_r = u_{trt^*} h_t,$$

$$w^2 = h_{-1}, \qquad w h_t = h_{{t^*}^{-1}} w, \qquad u_t w u_{t^{-1}} w u_t = w h_{-t^{-1}}.$$

These are called the Bruhat relations.

Proposition 2.5. *Let $B = \{g = \left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \mid g \in \mathrm{Sl}_*(2, A)\}$, $D = \{h_t \in B \mid t \in A^\times\}$, and $N = \{u_r \in B \mid r \in A_s\}$. Then $B = DN$.*

Proof. If $g = \left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]$ then $g = h_a u_{a^{-1}b}$. $\square$

Definition 2.18. Let B and w be as described above. Let $SSl_*(2, A)$ be the subgroup of $Sl_*(2, A)$ given by

$$SSl_*(2, A) = \bigcup_{j=0}^{\infty} (BwB)^j, \quad \text{where} \quad (BwB)^0 = B \tag{2.1}$$

Definition 2.19. We say that $Sl_*(2, A)$ has a Bruhat decomposition if $Sl_*(2, A) = SSl_*(2, A)$. If the union is finite, then the minimal n so that $Sl_*(2, A) = \bigcup_{j=0}^{n}(BwB)^j$ is called the length of the Bruhat decomposition.

If we let $A = M(n, k)$ and let $*$ be transposition we get that $Sl_*(2, A) = Sp(2n, k)$. Weil studied the representations of $Sp(2n, k)$, k a locally compact field. Letting V be a two dimensional k-vector space and Q a quadratic form on V, Weil produced a naturally occurring unitary representation, $(\omega, L^2(V))$ of $Sl_2(k)$ on the space $L^2(V)$ using the Heisenberg group. We call this representation the Weil representation. If one varies V and Q and decomposes ω into irreducibles, then one recovers all irreducible complex representations of $Sp(2n, k)$.

In the case of $Sl(2, k)$ Cartier observed that if one knew only the images of the Bruhat generators under ω and that these images satisfied the Bruhat relations, then one could extend these images to a representation of $Sl(2, k)$ without knowing a priori that the Weil representation exists. (The key point here is that the Bruhat generators along with the Bruhat relations give a *presentation* of $Sl(2, k)$.) Cartier hypothesized that the Weil representation could be generalized to $Sl_*(2, A)$ as long as there was a Bruhat presentation for the group. Gutiérrez, Pantoja, and Soto-Andrade produced an explicit construction for this representation in the case that A is finite[6]. Our ultimate goal is to decompose the generalized Weil representation constructed by Gutiérrez, Pantoja, and Soto-Andrade into irreducible representations. In this thesis we begin this process by constructing the irreducible representations of $Sl_*(2, \mathcal{A})$.

CHAPTER 3
REPRESENTATIONS OF $SL_*(2, \mathcal{A})$: CASE N=1

For the remainder of this thesis we let $F = \mathbb{C}$. Let k be a finite field of $q = p^f$ elements. Let n be a positive integer. Let $\mathcal{U}$ and $\mathcal{L}$ be defined as follows,

$$\mathcal{U} = \left\{ \begin{bmatrix} X & Y \\ & Z \end{bmatrix} \mid X, Y, Z \in M(n,k) \right\}$$
$$\mathcal{L} = \left\{ \begin{bmatrix} X & \\ W & Z \end{bmatrix} \mid X, W, Z \in M(n,k) \right\}$$

where X, Y, W, and Z are matrices in $M(n,k)$. Let $\mathcal{A} = \mathcal{U} \oplus \mathcal{L}$. $\mathcal{A}$ is a unitary ring. Define $\Phi : \mathcal{U} \to \mathcal{L}$ by $\Phi(A) = A^t$. We define an involution, $*$ on $\mathcal{A}$ by $(A, B)^* = (\Phi^{-1}(B), \Phi(A))$, for all $A \in \mathcal{U}$ and $B \in \mathcal{L}$.

Pantoja and Soto-Andrade proved in [10] Proposition 5.1 that $Sl_*(2, \mathcal{A})$ is isomorphic to $Gl(2, \mathcal{U})$. We may therefore study the representation theory of this general linear group. We have the following Lemma

Lemma 3.1. *$Gl(2, \mathcal{U})$ is isomorphic to $P = P_{(2n,2n)}(4n, k)$, the $(2n, 2n)$ parabolic subgroup of $Gl_{4n}(k)$.*

Proof. Let E be the following matrix.

$$E = \begin{bmatrix} I_n & & & \\ & & I_n & \\ & I_n & & \\ & & & I_n \end{bmatrix}$$

We claim that the map $\Gamma : Gl(2, \mathcal{U}) \to P$ defined by $\Gamma(g) = E^{-1} g E$ is an isomorphism. We first note that for $g \in Gl(2, \mathcal{U})$, conjugating by E yields a block upper triangular matrix. Secondly, $\det(E^{-1}gE) = \det(g)$ and therefore $E^{-1}gE$ lies in P.

For $g,\, h \in Gl(2,\mathcal{U})$

$$\Gamma(gh) = E^{-1}ghE = (E^{-1}gE)(E^{-1}hE) = \Gamma(g)\Gamma(h)$$

Therefore, Γ is a homomorphism.

An element $g \in Gl(2,\mathcal{U})$ lies in $\ker(\Gamma)$ if and only if $\Gamma(g) = E^{-1}gE = I_{4n}$. This implies that $g = I_{4n}$. So Γ is injective.

Lastly, if A lies in P, then a simple calculation shows that $g = EAE^{-1}$ lies in $Gl(2,\mathcal{U})$. Hence, Γ is an isomorphism. $\square$

Let $\mathcal{N}$ and $\mathcal{T}$ be the following subgroups of P.

$$\mathcal{N} = \left\{ n_Y = \begin{bmatrix} I_{2n} & Y \\ & I_{2n} \end{bmatrix} \mid Y \in M(2n,k) \right\}$$

$$\mathcal{T} = \left\{ \begin{bmatrix} X & \\ & Z \end{bmatrix} \mid X, Z \in Gl(2n,k) \right\}$$

Lemma 3.2. *$P \cong \mathcal{N} \rtimes \mathcal{T}$.*

Remark 2. *The following proof will be used several times throughout this thesis. We will go through the proof here. The later claims follow by the same argument.*

Proof. We need the following:

1. $P = \mathcal{N}\mathcal{T}$

2. $\mathcal{N} \cap \mathcal{T} = \{I_{4n}\}$

3. $\mathcal{N}$ is a normal subgroup of P

Property 2 is clear. Let $g = \left[\begin{smallmatrix} X & Y \\ & Z \end{smallmatrix}\right]$ be an element of P. Then $g = \left[\begin{smallmatrix} I_{2n} & YZ^{-1} \\ & I_{2n} \end{smallmatrix}\right] \left[\begin{smallmatrix} X & \\ & Z \end{smallmatrix}\right]$. Since the intersection of $\mathcal{N}$ and $\mathcal{T}$ is the identity we have that ths product is unique, thus $P = \mathcal{N}\mathcal{T}$.

N is isomorphic to $(M(n,k),+)$ and is therefore an abelian subgroup of P. To see that $\mathcal{N}$ is normal, we need only show that $g^{-1}n_Y g$ lies in $\mathcal{N}$ for all g in $\mathcal{T}$.

$$\begin{aligned} g^{-1}n_Y g &= \begin{bmatrix} X^{-1} & 0 \\ 0 & Z^{-1} \end{bmatrix} \begin{bmatrix} I_{2n} & Y \\ & I_{2n} \end{bmatrix} \begin{bmatrix} X & \\ & Z \end{bmatrix} \\ &= \begin{bmatrix} I_{2n} & X^{-1}YZ \\ & I_{2n} \end{bmatrix} \end{aligned}$$

This shows that $\mathcal{N}$ is normal in P. Our result follows. □

Let ψ be a nontrivial additive character of k. For each n_A in $\mathcal{N}$ we define a character ψ_A of $\mathcal{N}$ by $\psi_A(n_B) = \psi(\mathrm{tr}(AB))$. Let $\widehat{\mathcal{N}}$ be the collection of characters of $\mathcal{N}$. We want to explore $\widehat{\mathcal{N}}$. In particular, we want to understand the action of $\mathcal{L}$ on $\widehat{\mathcal{N}}$ so that we may use the Method of Little Groups to determine the irreducible representations of P. To understand this action we are going to construct a $\mathcal{T}$-equivariant map of $M(2n,k)$ with $\widehat{\mathcal{N}}$.

Proposition 3.1. *Let P, $\mathcal{N}$, and $\mathcal{T}$ be as before. Define an action of $\mathcal{T}$ on $M(2n,k)$ by $\left[\begin{smallmatrix} X & \\ & Z \end{smallmatrix}\right] \circ A = ZAX^{-1}$. Then the map $(M(2n,k),+) \to \widehat{\mathcal{N}}$ defined by $A \mapsto \psi_A$ is a $\mathcal{T}$-equivariant isomorphism of abelian groups.*

Proof. We need to show that for every $t \in \mathcal{T}$ $\psi_A^{(t)} = \psi_{t\circ A}$, where the action on the left is given by conjugation. If we let $t = \left[\begin{smallmatrix} X & \\ & Z \end{smallmatrix}\right]$ then it is clear that $\psi_{t\circ A} = \psi_{ZAX^{-1}}$. We calculate $\psi_A^{(t)}$.

$$\begin{aligned}
\psi_A^{(t)}(n_B) &= \psi_A(t^{-1}n_B g) \\
&= \psi_A\left(\left[\begin{smallmatrix} I_{2n} & X^{-1}BZ \\ 0 & I_{2n} \end{smallmatrix}\right]\right) \\
&= \psi(\mathrm{tr}(AX^{-1}BZ)) \\
&= \psi(\mathrm{tr}(ZAX^{-1}B)) \\
&= \psi_{ZAX^{-1}}(n_B) \\
&= \psi_{t\circ A}
\end{aligned}$$

The map $A \mapsto \psi_A$ is an homomorphism since $\psi_{A+B} = \psi_A\psi_B$. The kernel of the homomorphism is the group of A such that ψ_A is the trivial character. If $A \neq 0$ then we may find a matrix B so that $AB \neq 0$ and $\mathrm{tr}(AB)$ does not lie in the kernel of ψ. Therefore, $A = 0$ is the only element in the kernel of the map $A \mapsto \psi_A$. Our result follows. $\square$

Proposition 3.2. *There exists $t \in \mathcal{T}$ so that $\psi_A^{(t)} = \psi_B$ if and only if* $\mathrm{rk}(A) = \mathrm{rk}(B)$.

Proof. Suppose $\psi_A^{(t)} = \psi_B$, then there exist X and Z in $\mathrm{Gl}_4(k)$ so that $ZAX^{-1} = B$. Since X and Z are invertible we have that the $\mathrm{rk}(ZAX^{-1}) = \mathrm{rk}(A)$.

Conversely, assume that A and B have the same rank. Then there are invertible matrices X_0, X_1, Y_0 and Y_1 so that $X_0AY_0 = \left[\begin{smallmatrix} I_r & \\ & 0_{4-r} \end{smallmatrix}\right] = X_1BY_1$. By letting $X = X_1^{-1}X_0$ and $Y = Y_1Y_0^{-1}$ we get that $B = XAY^{-1}$ and therfore ψ_A is conjugate to ψ_B. $\square$

Thus, we get a $\mathcal{T}$-orbit for each possible rank of A. There are $2n+1$ of these. We choose our representatives to be ψ_{A_i}, where $A_i = \left[\begin{smallmatrix} I_i & \\ & 0_{2n-i} \end{smallmatrix}\right]$, $0 \le i \le 2n$.

At this point we will restrict ourselves to the $n = 1$ case.

3.1 Representations of $Sl_*(2, \mathcal{A})$, $n = 1$

Using the results of Proposition 3.2 we know that we have three conjugacy classes for $\{\psi_A\}_{A \in M(2,k)}$. Let $\{\psi_{A_i}\}_{i=0,1,2}$ be a set of representatives, where $A_0 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $A_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, and $A_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Set $\chi_i = \psi_{A_i}$. In order to determine the stabilizer $\mathcal{T}_i = \{g \in \mathcal{T} |\ \chi_i^{(g)} = \chi_i\}$ we must solve the equation $ZA_iX^{-1} = A_i$.

<u>Case 1: i=0:</u> $\mathcal{T}_0 = \{g \in \mathcal{T} |\ \chi_0^{(g)} = \chi_0\}$. Since A_0 is the zero matrix any X and Z will suffice and thus $\mathcal{T}_0 = \mathcal{T}$.

<u>Case 2: i=1:</u> $\mathcal{T}_1 = \{g \in \mathcal{T} |\ \chi_1^{(g)} = \chi_1\}$. If $ZA_1X^{-1} = A_1$. A simple calculation shows that $x_{11} = z_{11}$, and $x_{12} = 0 = z_{21}$. This shows that g lies in $\mathcal{T}_1$ if and only if it is of the form $\left[\begin{smallmatrix} x_{11} & & & \\ x_{21} & x_{22} & & \\ & & x_{11} & z_{12} \\ & & & z_{22} \end{smallmatrix}\right]$ where $x_{11}, x_{22}, z_{22} \in k^\times$ and $x_{21}, z_{12} \in k$.

<u>Case 3: i=2:</u> $\mathcal{T}_2 = \{g \in \mathcal{T} |\ \chi_2^{(g)} = \chi_2\}$. Since A_2 is the identity matrix we must have $X = Z$ and g is of the form $\left[\begin{smallmatrix} X & \\ & X \end{smallmatrix}\right]$ where $X \in Gl_2(k)$.

For each i let $P_i = N \cdot T_i$. Extend χ_i to P_i by setting $\chi_i(n_A t) = \chi(n_A)$. This defines a character of P_i.

Remark 3. *It is clear at this point that in order to find all irreducible representations of P we will need to know the irreducible representations of $Gl_2(k)$ and $Gl_3(k)$. Steinberg covers the representations theory of both of these groups in [14].*

3.1.1 The Irreducible Representations of $\mathcal{T}_i$

We first note that the representation theory of $\mathcal{T}_2$ reduces to the representation theory of $Gl(2,k)$ which can be found in [8]. For $\mathcal{T}_0$, we see that $\mathcal{T}_0 \cong Gl(2,k) \times Gl(2,k)$, therefore we obtain all irreducible representations of $\mathcal{T}_0$ by taking the tensor product of irreducible representations of $Gl(2,k)$. We focus on the representation theory of $\mathcal{T}_1$.

Let N and T be the following subgroups of $\mathcal{T}_1$.

$$N = \left\{ \begin{bmatrix} 1 & & & \\ a & 1 & & \\ & & 1 & b \\ & & & 1 \end{bmatrix} | a, b \in k \right\}$$

$$T = \left\{ \begin{bmatrix} x & & & \\ & y & & \\ & & x & \\ & & & z \end{bmatrix} | x, y, z \in k^\times \right\}$$

The group $\mathcal{T}_1$ is isomorphic to $N \rtimes T$ by the same argument used to show that $P = \mathcal{N} \rtimes \mathcal{T}$. N is an abelian subgroup of $\mathcal{T}_1$ as it is isomorphic to $k \oplus k$.

Let ψ be a nontrivial additive character of k. For $(a,b) \in k \oplus k$ define $\psi_{a,b} : k \oplus k \mapsto \mathbb{C}^\times$ by $\psi_{a,b}(c,d) = \psi_a(c)\psi_b(d)$. By the same reasoning as before we have that $(a,b) \mapsto \psi_{a,b}$ is a T equivariant isomorphism between $k \oplus k$ and $\widehat{N}$, the characters of N. The action of T on $k \oplus k$ is given by $t \circ (a,b) = (xay^{-1}, zbx^{-1})$. Therefore, the T orbits in $\widehat{N}$ are determined by the equations $a' = xay^{-1}$ and $b' = zbx^{-1}$. These equations are determined by the invertibility of a and b respectively. We therefore choose our representatives in $\widehat{N}$ to be $\psi_{0,0}$, $\psi_{1,0}$, $\psi_{0,1}$, and $\psi_{1,1}$.

Using the equations $a = xay^{-1}$ and $b = zbx^{-1}$ we determine the stabilizers in

T for each of these representatives as follows:

$$T_{0,0} = \{t \in T | \psi_{0,0}^{(t)} = \psi_{0,0}\} = \left\{ \begin{bmatrix} x & & & \\ & y & & \\ & & x & \\ & & & z \end{bmatrix} | x, y, z \in k^\times \right\} = T$$

$$T_{1,0} = \{t \in T | \psi_{1,0}^{(t)} = \psi_{1,0}\} = \left\{ \begin{bmatrix} x & & & \\ & x & & \\ & & x & \\ & & & z \end{bmatrix} | x, z \in k^\times \right\}$$

$$T_{0,1} = \{t \in T | \psi_{0,1}^{(t)} = \psi_{0,1}\} = \left\{ \begin{bmatrix} x & & & \\ & y & & \\ & & x & \\ & & & x \end{bmatrix} | x, y \in k^\times \right\}$$

$$T_{1,1} = \{t \in T | \psi_{1,1}^{(t)} = \psi_{1,1}\} = \left\{ \begin{bmatrix} x & & & \\ & x & & \\ & & x & \\ & & & x \end{bmatrix} | x \in k^\times \right\}$$

As all of these groups are isomorphic to the direct product of copies of $k^\times$ we know that their irreducible representations are given by way the tensor product. We proceed to find the irreducible representations of $\mathcal{T}_1$. Let $\mathcal{T}_1^{0,0} = NT_{0,0}$, $\mathcal{T}_1^{1,0} = NT_{1,0}$, $\mathcal{T}_1^{0,1} = NT_{0,1}$, and $\mathcal{T}_1^{1.1} = NT_{1,1}$.

The irreducible representations of $\mathcal{T}_1$ have one of the following forms:

1. $\psi_{0,0} \otimes (\rho_1 \otimes \widetilde{\rho_2} \otimes \rho_3)$, where $\rho_j \in \widehat{k^\times}$.

2. $\mathrm{ind}_{\mathcal{T}_1^{1,0}}^{\mathcal{T}_1}(\psi_{1,0} \otimes (\widetilde{\rho_1 \otimes \rho_2}))$, where $\rho_j \in \widehat{k^\times}$.

3. $\mathrm{ind}_{\mathcal{T}_1^{0,1}}^{\mathcal{T}_1}(\psi_{0,1} \otimes (\widetilde{\rho_1 \otimes \rho_2}))$, where $\rho_j \in \widehat{k^\times}$.

4. $\mathrm{ind}_{\mathcal{T}_1^{1,1}}^{\mathcal{T}_1}(\psi_{1,1} \otimes \widetilde{\rho})$, where $\rho \in \widehat{k^\times}$.

Now that we have the irreducible representations of $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$ we may construct the irreducible representations of P. More precisely, if π is an irreducible representation of P, then π is of one of the following forms:

1. $\chi_0 \otimes \widetilde{\mu_1 \otimes \mu_2}$, where $\mu_j \in \mathrm{Irr}(Gl_2(k))$.

2. $\mathrm{ind}_{\mathcal{N}\mathcal{T}_1}^{P}(\chi_1 \otimes \widetilde{\mu})$, where $\mu \in \mathrm{Irr}(\mathcal{T}_1)$.

3. $\mathrm{ind}_{\mathcal{N}\mathcal{T}_2}^{P}(\chi_2 \otimes \widetilde{\mu})$, where $\mu \in \mathrm{Irr}(Gl_2(k))$.

CHAPTER 4
REPRESENTATIONS OF $SL_*(2,\mathcal{A})$: CASE N=2

We now study the representations of $Sl_*(2,\mathcal{A})$ when $n=2$. In this case, unlike the case $n=1$, we get more complicated stabilizers for our subgroup $\mathcal{T}$, which make for much more complex representations of P.

From Proposition 3.1 we know that we have 5 T- orbits in $\widehat{\mathcal{N}}$. For each $0 \le i \le 4$ let $A_i = \left[\begin{smallmatrix} I_i & \\ & 0_{4-i}\end{smallmatrix}\right]$ be the representative for the $\mathcal{T}$-orbits in $M(4,k)$. Let $\chi_i = \psi_{A_i}$. We look now to find a explicit description of $\mathcal{T}_i = \{g \in \mathcal{T} | \chi_i^{(g)} = \chi_i\}$. We use the following lemma to give us the structure of $\mathcal{T}_i$.

Lemma 4.1. *Let A_i be as before. Let X and Z be elements of $Gl(4,k)$. Then $ZA_i = A_iX$ if and only if*

$$X = \begin{bmatrix} X_{11} & \\ X_{21} & X_{22} \end{bmatrix}$$

and

$$Z = \begin{bmatrix} X_{11} & Z_{12} \\ & Z_{22} \end{bmatrix}$$

where $X_{11} \in Gl_i(k)$, $X_{22}, Z_{22} \in Gl(4-i,k)$, and X_{21} (respectively Z_{12}^t) is a $4-i \times i$ (resp. $i \times 4-i$) matrix.

Proof. We first note that A_i is an idempotent matrix, i.e. $A_i^2 = A_i$. Let X and Z be elements of $Gl(2n,k)$ so that $ZA_i = A_iX$. Then $A_iZA_i = A_i^2X = A_iX$. Thus, the first i rows of X have the form

$$\begin{matrix} z_{11} & \dots & z_{1i} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_{i1} & \dots & z_{ii} & 0 & \dots & 0 \end{matrix}$$

Similarly, $ZA_i = ZA_i^2 = A_iXA_i$, therefore the first i columns of Z have the form

$$\begin{matrix} x_{11} & \dots & x_{1i} \\ \vdots & \ddots & \vdots \\ x_{i1} & \dots & x_{ii} \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix}$$

Going back to our original equation $ZA_i = A_iX$ we see that $x_{jk} = z_{jk}$ for all j, k in $\{1, 2, ..., i\}$. Thus, X has the form

$$X = \begin{bmatrix} X_{11} & \\ X_{21} & X_{22} \end{bmatrix}$$

and Z has the form

$$Z = \begin{bmatrix} X_{11} & Z_{12} \\ & Z_{22} \end{bmatrix}$$

$\square$

We have therefore shown that for each i between 0 and 4, elements of $\mathcal{T}_i$ have the form $\left[\begin{smallmatrix} X & 0 \\ 0 & Z \end{smallmatrix}\right]$ where X and Z have the form shown in Lemma 4.1.

Now that we have determined the structure of $\mathcal{T}_i$ we must now determine the irreducible representations for each of these groups. As $\mathcal{T}_0$ is isomorphic to $Gl_4(k) \times Gl_4(k)$ the irreducible representations of $\mathcal{T}_0$ have the form $\sigma_1 \otimes \sigma_2$ where σ_1 and σ_2 are elements of $\mathrm{Irr}(Gl_4(k))$. In the case of $\mathcal{T}_4$ the irreducible representations correspond to those of $Gl_4(k)$. For $1 \leq i \leq 3$ we consider the following subgroups of $\mathcal{T}_i$,

$$\mathcal{N}_i = \left\{ \{n_{l,u} = \begin{bmatrix} \begin{bmatrix} I_i & \\ l & I_{4-i} \end{bmatrix} & \\ & \begin{bmatrix} I_i & u \\ & I_{4-i} \end{bmatrix} \end{bmatrix} \mid l \text{ and } u^t \text{ are } 4-i \times i \text{ matrices} \right\}$$

$$\mathcal{D}_i = \left\{ d_{X,Y,Z} = \begin{bmatrix} \begin{bmatrix} X & \\ & Y \end{bmatrix} & \\ & \begin{bmatrix} X & \\ & Z \end{bmatrix} \end{bmatrix} \mid X \in Gl(i,k) \text{ and } Y, Z \in Gl(4-i,k) \right\}$$

Notice that an element of $\mathcal{T}_i$ may be written in the form

$$\begin{bmatrix} \begin{bmatrix} I_i & \\ lX^{-1} & I_{4-i} \end{bmatrix} & \\ & \begin{bmatrix} I_i & uZ^{-1} \\ & I_{4-i} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} X & \\ & Y \end{bmatrix} & \\ & \begin{bmatrix} X & \\ & Z \end{bmatrix} \end{bmatrix}$$

This product is unique because an element, $t \in \mathcal{T}_i$, lies in both $\mathcal{N}_i$ and $\mathcal{D}_i$ if and only if $l = u^t = 0$, $X = I_i$, and $Y = Z = I_{4-i}$, i.e. the intersection of these subgroups is trivial. If $n_{l,u}$ and $n_{l',u'}$ are elements of $\mathcal{N}_i$, then the product $n_{l,u}n_{l',u'} = n_{l+l',u+u'}$. Using this, it is easy to see that $\mathcal{N}_i$ is isomorphic to the direct sum of the $4-i \times i$ matrices over k and the $i \times 4-i$ matrices over k. This group is abelian, therefore $\mathcal{N}_i$ is as well.

If we conjugate an element of $\mathcal{N}_i$ by an element of $\mathcal{D}_i$ we get,

$$\begin{bmatrix} \begin{bmatrix} X^{-1} & \\ & Y^{-1} \end{bmatrix} & \\ & \begin{bmatrix} X^{-1} & \\ & Z^{-1} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} I_i & \\ l & I_{4-i} \end{bmatrix} & \\ & \begin{bmatrix} I_i & u \\ & I_{4-i} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} X & \\ & Y \end{bmatrix} & \\ & \begin{bmatrix} X & \\ & Z \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I_i & \\ Y^{-1}lX & I_{4-i} \end{bmatrix} & \\ & \begin{bmatrix} I_i & X^{-1}uZ \\ & I_{4-i} \end{bmatrix} \end{bmatrix}$$

Hence, $\mathcal{N}_i$ is a normal subgroup of $\mathcal{T}_i$. We also get that $\mathcal{T}_i \cong \mathcal{N}_i \rtimes \mathcal{D}_i$. Lastly, since $\mathcal{N}_i$ is a finite abelian group we know that all of its irreducible representations are 1-dimensional. We denote the group of characters of $\mathcal{N}_i$ by $\widehat{\mathcal{N}_i}$.

Remark 4. *We note here that since $\mathcal{N}_i$ is isomorphic to a direct sum of groups, we may view the irreducible representations of $\mathcal{N}_i$ as the tensor product of irreducible representations of the $(4-i)\times i$ matrices over the field k, denoted $M_{4-i,i}(k)$, and the irreducible representations of the $i\times(4-i)$ matrices over k, $M_{i,4-i}(k)$. The irreducible representations of $M_{4-i,i}(k)$ may be obtained as follows:*

1. *Choose an additive character $\psi : k \to \mathbb{C}^\times$.*

2. *For each l in $M_{4-i,i}(k)$ define $\psi_l : M_{4-i,i}(k) \to \mathbb{C}^\times$ by $\psi_l(l') = \psi(\mathrm{tr}(l^t l'))$.*

Each of these ψ_l defines a unique irreducible representation of $M_{4-i,i}(k)$. The irreducible representations of $M_{i,4-i}$ may be realized in an analogous manner.

Consider, now, the subgroup $\mathcal{D}_i$ and define the map $d_{X,Y,Z} \mapsto (X,Y,Z)$ from $\mathcal{D}_i$ to $Gl_(k) \times Gl_{4-i})(k) \times Gl_{4-i}(k)$. This map is a group homomorphism. The kernel of the map is trivial and, as a result, the map is an isomorphism.

By an argument similar to that used in Chapter 2, we have a $\mathcal{D}_i$ equivariant isomorphism $M_{4-i,i}(k) \oplus M_{i,4-i}(k) \to \widehat{\mathcal{N}_i}$ defined by $(l,u) \mapsto \psi_{l,u}$. The action of $\mathcal{D}_\rangle$ on $M_{4-i,i}(k) \oplus M_{i,4-i}(k)$ is given by

$$\begin{bmatrix} \begin{bmatrix} X & \\ & Y \end{bmatrix} & \\ & \begin{bmatrix} X & \\ & Z \end{bmatrix} \end{bmatrix} \circ (l,u) = ((Y^{-1})^t l X^t, (X^{-1})^t u Z^t).$$

This is the correct action since conjugation in $\widehat{\mathcal{N}_i}$ gives the equation:

$$\begin{aligned} \psi^{(t)}_{l,u}(n_{l',u'}) &= \psi_{l,u}(t^{-1} n_{l',u'} t) \\ &= \psi_{l,u}\left(\begin{bmatrix} \begin{bmatrix} X^{-1} & \\ & Y^{-1} \end{bmatrix} & \\ & \begin{bmatrix} X^{-1} & \\ & Z^{-1} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} I_i & \\ l' & I_{4-i} \end{bmatrix} & \\ & \begin{bmatrix} I_i & u' \\ & I_{4-i} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} X & \\ & Y \end{bmatrix} & \\ & \begin{bmatrix} X & \\ & Z \end{bmatrix} \end{bmatrix} \right) \end{aligned}$$

$$
\begin{aligned}
&= \psi_{l,u}\left(\begin{bmatrix} \begin{bmatrix} I_i & \\ Y^{-1}l'X & I_{4-i} \end{bmatrix} & \\ & \begin{bmatrix} I_i & X^{-1}u'Z \\ & I_{4-i} \end{bmatrix} \end{bmatrix}\right) \\
&= \psi_l(\begin{bmatrix} I_i & \\ Y^{-1}l'X & I_{4-i} \end{bmatrix}) \otimes \psi_u(\begin{bmatrix} I_i & X^{-1}u'Z \\ & I_{4-i} \end{bmatrix}) \\
&= \psi(\mathrm{tr}(l^tY^{-1}l'X)) \otimes \psi(\mathrm{tr}(u^tX^{-1}u'Z)) \\
&= \psi(\mathrm{tr}(Xl^tY^{-1}l')) \otimes \psi(\mathrm{tr}(Zu^tX^{-1}u')) \\
&= \psi_{(Y^{-1})^tlX^t,\ (X^{-1})^tuZ^t}(n_{l',u'})
\end{aligned}
$$

Making calculations similar to the ones made in Proposition 3.1 show that the map is an isomorphism. Therefore, $\psi_{l,u}$ is conjugate to $\psi_{l',u'}$ if and only if there exists X an element of $\mathrm{Gl}_i(k)$ and matrices Y, Z in $\mathrm{Gl}_{4-i}(k)$ so that the following equations are satisfied.

$$
\begin{aligned}
u' &= (X^{-1})^tuZ^t \\
l' &= (Y^{-1})^tlX^t
\end{aligned}
$$

From this point we will handle each T_i separately.

4.1 The Irreducible Representations of $\mathcal{T}_1$

First, we may assume that u is in rank form due to the equation $u' = (X^{-1})^tuZ^t$. We denote the representatives for the rank matrices of u by u_0 and u_1. We then proceed by determining what happens to l' as x is varied.

<u>Case 1:</u> $u_0 = [\,0\ 0\ 0\,]$: Since any x and Z will fix u_0 we are reduced to the equation $l' = (Y^{-1})^tlx$ which is determined by the rank of l.

<u>Case 2:</u> $u_1 = [\,1\ 0\ 0\,]$: In this case we must determine what effect multiplication by

x on the right has on the possible reduced row echelon forms for l. We are able to do this since Y is an arbitrary matrix in $Gl_3(k)$. Since l is a 3×1 matrix we may choose the two representatives for the reduced row echelon form to be $l_0 = \left[\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right]$ and $l_1 = \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]$. Since no element $x \in k^\times$ can change the rank of our l representatives we get that l_0 and l_1 determine distinct classes. We label the representatives of $\psi_{l,u}$ as ψ_{l_j,u_k}, $0 \leq j, k \leq 1$. We must now determine the stabilizers, denoted $D_1^{i,j}$, for each $\psi_{i,j}$. $D_1^{0,0} = D_1$ is seen immediately as any elements $x \in k^\times$, $Y, Z \in Gl_3(k)$ will satisfy the equations $u_0 = (X^{-1})^t u_0 Z^t$ and $l_0 = (Y^{-1})^t l_0 X^t$.

For $D_1^{0,1}$ we see that $l_0 = (Y^{-1})^t l_0 X^t$ is satisfied by any x and Y, therefore we need only determine the structure of Z using the equation $u_1 = (X^{-1})^t u_1 Z^t$. We get the following:

$$\begin{aligned} [x]\left[\begin{smallmatrix} 1 & 0 & 0 \end{smallmatrix}\right] &= \left[\begin{smallmatrix} 1 & 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} x & 0 & 0 \end{smallmatrix}\right] &= \left[\begin{smallmatrix} z_{11} & z_{12} & z_{13} \end{smallmatrix}\right] \end{aligned}$$

Thus, $z_{11} = x$ and $z_{12} = z_{13} = 0$. Replacing Z with Z^t we get that the elements of $D_1^{0,1}$ have the following form

$$\begin{bmatrix} \left[\begin{smallmatrix} x & \\ & Y \end{smallmatrix}\right] & \\ & \left[\begin{smallmatrix} x & \\ & \left[\begin{smallmatrix} x & r & s \\ & z'_{11} & z'_{12} \\ & z'_{21} & z'_{22} \end{smallmatrix}\right] \end{smallmatrix}\right] \end{bmatrix}$$

where $r, s \in k$ and $Z' = (z'_{ij}) \in Gl_2(k)$.

For $D_1^{1,0}$ we have that 3.1 is satisfied by arbitrary x and Z so we have a result similar to that for $D_1^{0,1}$. More specifically, elements of $D_1^{1,0}$ have the form

$$\begin{bmatrix} \left[\begin{smallmatrix} x & \\ & \left[\begin{smallmatrix} x & & \\ v & y'_{11} & y'_{12} \\ w & y'_{21} & y'_{22} \end{smallmatrix}\right] \end{smallmatrix}\right] & \\ & \left[\begin{smallmatrix} x & \\ & z \end{smallmatrix}\right] \end{bmatrix}$$

where $v, w \in k$ and $Y' = (y'_{ij}) \in \mathrm{Gl}_2(k)$.

Our final stabilizer, $D_1^{1,1}$ consists of elements of the form

$$\begin{bmatrix} \begin{bmatrix} x & \\ & \begin{bmatrix} x & & \\ v & y'_{11} & y'_{12} \\ w & y'_{21} & y'_{22} \end{bmatrix} \end{bmatrix} & \\ & \begin{bmatrix} x & \\ & \begin{bmatrix} x & r & s \\ & z'_{11} & z'_{12} \\ & z'_{21} & z'_{22} \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

where $u, v, r, s \in k$ and $Y', Z' \in Gl_2(k)$.

We must now study the representations of each of these stabilizer groups. Notice that the irreducible representations of D_1 are given by the irreducible representations of $k^\times \times Gl_3(k) \times Gl_3(k)$. Also note that the irreducible representations of $D_1^{0,1}$ may be obtained in analogous to those of $D_1^{1,0}$ as the two groups are isomorphic. Therefore, we focus our attention on determining the irreducible representations of $D_1^{1,0}$.

4.1.1 The Irreducible Representations of $D_1^{1,0}$

Let $A_{1,0}$ be the subgroup of $D_1^{1,0}$ containing elements of the form

$$\begin{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & v & \\ & w & I_2 \end{bmatrix} & \\ & \begin{bmatrix} 1 & \\ & I_3 \end{bmatrix} \end{bmatrix}$$

$v, w \in k$ and let $T_{1,0}$ be the subgroup consisting of elements of the form

$$\begin{bmatrix} \begin{bmatrix} x & & \\ & x & \\ & & Y \end{bmatrix} & \\ & \begin{bmatrix} x & \\ & Z \end{bmatrix} \end{bmatrix}$$

where $Y \in Gl_2(k)$ and $Z \in Gl_3(k)$. The following are easily checked:

1. $A_{1,0}$ is an abelian subgroup of $D_1^{1,0}$
2. $A_{1,0}$ is isomorphic to the group $(M_{2,1}, +)$

3. $A_{1,0}T_{1,0} = D_1^{1,0}$

4. $A_{1,0} \cap T_{1,0} = \{I_8\}$

From items 1, 3, and 4 we get that $D_1^{1,0} \cong A_{1,0} \rtimes T_{1,0}$. We define the irreducible representations of $A_{1,0}$ using a nontrivial additive character ψ of k. More specifically, for each $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \in M_{2,1}(k)$ we define $\psi_{\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]} : A_{1,0} \to \mathbb{C}^\times$ by $\psi_{\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]}(a_{v',w'}) = \psi(\mathrm{tr}(\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]^t \left[\begin{smallmatrix} v' \\ w' \end{smallmatrix}\right]))$. We denote by $\widehat{A_{1,0}}$ the collection of all characters of $A_{1,0}$.

By an argument similar to that for Proposition 3.1 we get that the map $M_{2,1}(k) \to \widehat{A_{1,0}}$ defined by $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \mapsto \psi_{\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]}$ is a $T_{1,0}$ equivariant isomorphism. The action of $T_{1,0}$ on $M_{2,1}(k)$ is given by $t \circ \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] = (Y^{-1})^t \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] x$. We know that this is the appropriate action since conjugation of $\psi_{\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]}$ in $\widehat{A_{1,0}}$ is as below.

$$
\begin{aligned}
\psi^{(t)}_{\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]}(a_{v',w'}) &= \psi_{\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]}\left(\begin{bmatrix} \left[\begin{smallmatrix} x^{-1} & & \\ & x^{-1} & \\ & & Y^{-1} \end{smallmatrix}\right] & \\ & \left[\begin{smallmatrix} x^{-1} & \\ & Z^{-1} \end{smallmatrix}\right] \end{bmatrix} \begin{bmatrix} \left[\begin{smallmatrix} 1 & & \\ & 1 & \\ & \begin{smallmatrix} v' \\ w' \end{smallmatrix} & I_2 \end{smallmatrix}\right] & \\ & \left[\begin{smallmatrix} 1 & \\ & I_3 \end{smallmatrix}\right] \end{bmatrix} \begin{bmatrix} \left[\begin{smallmatrix} x & & \\ & x & \\ & & Y \end{smallmatrix}\right] & \\ & \left[\begin{smallmatrix} x & \\ & Z \end{smallmatrix}\right] \end{bmatrix} \right) \\
&= \psi_{\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]}\left(\begin{bmatrix} \left[\begin{smallmatrix} 1 & & \\ & 1 & \\ & Y^{-1}\left[\begin{smallmatrix} v' \\ w' \end{smallmatrix}\right] x & I_2 \end{smallmatrix}\right] & \\ & \left[\begin{smallmatrix} 1 & \\ & I_3 \end{smallmatrix}\right] \end{bmatrix} \right) \\
&= \psi(\mathrm{tr}(\left[\begin{smallmatrix} v & w \end{smallmatrix}\right] Y^{-1} \left[\begin{smallmatrix} v' \\ w' \end{smallmatrix}\right] x)) \\
&= \psi(\mathrm{tr}(x \left[\begin{smallmatrix} v & w \end{smallmatrix}\right] Y^{-1} \left[\begin{smallmatrix} v' \\ w' \end{smallmatrix}\right])) \\
&= \psi_{(Y^{-1})^t \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] x}(a_{v',w'})
\end{aligned}
$$

Thus $\psi_{\left[\begin{smallmatrix} v' \\ w' \end{smallmatrix}\right]}$ is in the orbit of $\psi_{\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]}$ if and only if there exists $Y \in Gl_2(k)$ and $x \in k^\times$ so that $\left[\begin{smallmatrix} v' \\ w' \end{smallmatrix}\right] = (Y^{-1})^t \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] x$. Thus, there are two $T_{1,0}$ orbits in $\widehat{A_{1,0}}$ which are determined by the rank of $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$. We choose the representatives $\psi_{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]}$ and $\psi_{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]}$.

The stabilizer of $\psi_{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]}$ is $T_{1,0}$ since for any x and Y, $(Y^{-1})^t \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] x = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. In the case of$\psi_{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]}$ we use the equation $(Y^{-1})^t \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] x = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ to show that the stabilizer is the group of matrices of the following form:

$$\begin{bmatrix} \begin{bmatrix} x & & \\ & x & \\ & & \begin{bmatrix} x & 0 \\ \alpha & \beta \end{bmatrix} \end{bmatrix} & \\ & \begin{bmatrix} x & \\ & Z \end{bmatrix} \end{bmatrix}$$

where $\alpha \in k$, x, $\beta \in k^{\times}$, and $Z \in Gl_3(k)$. We denote this group by $T_{1,0}^{1}$. Our next step is to again determine the irreducible representations of these stabilizer groups. In the case of $T_{1,0}^{0} = T_{1,0}$ we obtain the irreducible representations using the tensor product of irreducible representations of $k^{\times}$, $Gl_2(k)$, and $Gl_3(k)$. We shift our focus to the representations of $T_{1,0}^{1}$.

4.1.1.1 Representations of $T_{1,0}^{1}$

For $T_{1,0}^{1}$ we employ the Method of Little Groups again with the following subgroups:

$$\begin{aligned} M &= \left\{ m_{\alpha} = \begin{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & \alpha & 1 \end{bmatrix} & \\ & \begin{bmatrix} 1 & \\ & I_3 \end{bmatrix} \end{bmatrix} \mid \ \alpha \in k \right\} \\ L &= \left\{ l_{x,\beta,Z} = \begin{bmatrix} \begin{bmatrix} x & & & \\ & x & & \\ & & x & \\ & & & \beta \end{bmatrix} & \\ & \begin{bmatrix} x & \\ & Z \end{bmatrix} \end{bmatrix} \mid \ x, \beta \in k^{\times} \text{and} Z \in \mathrm{Gl}_3(k) \right\} \end{aligned}$$

By arguments similar to before we have the following:

1. M is an abelian subgroup of $T_{1,0}^{1}$

2. M is isomorphic to the additive group of k

3. $ML = T_{1,0}^{1}$

4. $M \cap L = I_8$

and, as before, $T^1_{1,0} \cong M \rtimes L$. For each α in k we define ψ_α by $\psi_\alpha(m_\beta) = \psi(\alpha\beta)$. ψ_α defines an irreducible representation of M. We denote the group of characters of M by $\widehat{M}$.

The isomorphism $\alpha \mapsto \psi_\alpha$ is a L equivariant map where the action of L on k is given by $l_{x,\beta,Z} \circ \alpha = x\alpha\beta^{-1}$. To verify this is the appropriate action of L we simply check conjugation of ψ_α by an element of L.

$$\begin{aligned} \psi_\alpha^{(l_{x,\beta,Z})}(m_\gamma) &= \psi_\alpha(l_{x^{-1},\beta^{-1},Z^{-1}} m_\gamma l_{x,\beta,Z}) \\ &= \psi_\alpha(m_{\beta^{-1}\gamma x}) \\ &= \psi(\alpha\beta^{-1}\gamma x) \\ &= \psi(x\alpha\beta^{-1}\gamma) = \psi_{x\alpha\beta^{-1}}(m_\gamma) \end{aligned}$$

Therefore, we have that ψ_α is conjugate to ψ_β if and only if both α and β are invertible or both are 0. This gives us two classes, ψ_0 and ψ_1. The stabilizer of ψ_0 is L and the stabilizer for ψ_1 is the group L_1 which contains matrices of the following form:

$$\begin{bmatrix} \begin{bmatrix} x & & & \\ & x & & \\ & & x & \\ & & & x \end{bmatrix} & \\ & \begin{bmatrix} x & \\ & Z \end{bmatrix} \end{bmatrix}$$

where $x \in k^\times$ and $Z \in Gl_3(k)$. The irreducible representations of both $L_0(= L)$ and $L_1(\cong k^\times \times Gl_3(k))$ may be obtained using the tensor product.

Let ${}_0T^1_{1,0} = ML_0$. Since $M \cong k$ we know that we may view elements of $\widehat{k}$ as characters of M. We view these characters as characters of ${}_0T^1_{1,0}$ by acting trivially on L. On the other hand, the irreducible representations of $L \cong k^\times \times k^\times \times Gl_3(k)$ have the form $\rho_1 \otimes \rho_2 \otimes \tau$, where ρ_1, $\rho_2 \in \widehat{k^\times}$ and $\tau \in \mathrm{Irr}(Gl_3(k))$. Since $ML = T^1_{1,0}$ we get that the irreducible representations of $T^1_{1,0}$ coming from this group are of the form $\psi_0 \otimes (\widetilde{\rho_1 \otimes \rho_2 \otimes \tau})$, where $(\widetilde{\rho_1 \otimes \rho_2 \otimes \tau})$ indicates inflation to ${}_0T^1_{1,0}$.

Now let ${}_1T^1_{1,0} = ML_1$. As L_1 is isomorphic to $k^\times \times Gl_3(k)$ we may view all irreducible representations as $\rho \otimes \tau$, where $\rho \in \widehat{k^\times}$ and $\tau \in \mathrm{Irr}(Gl_3(k))$. We let $\widetilde{\rho \otimes \tau}$ denote the inflation of $\rho \otimes \tau$ to ${}_1T^1_{1,0}$. By Proposition 2.2 we get that $\mathrm{ind}^{T^1_{1,0}}_{{}_1T^1_{1,0}}\, \rho \otimes \widetilde{\rho \otimes \tau}$ is an irreducible representation of $T^1_{1,0}$.

Now that we have a complete set of irreducible representations $T^0_{1,0}$ and $T^1_{1,0}$ we may construct the irreducible representations of $D^{1,0}_1$. First, note that ${}_0D^{1,0}_1 = A_{1,0}T^0_{1,0} = D^{1,0}_1$ since $T^0_{1,0} = T_{1,0}$. Therefore we get irreducible representations of $D^{1,0}_1$ by taking the tensor product $\psi_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]} \otimes (\widetilde{\rho \otimes \nu \otimes \tau})$, where $\psi_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}$ is viewed as a character of $D^{1,0}_1$ and $(\widetilde{\rho \otimes \nu \otimes \tau})$ is the inflation of an irreducible representation of $T^0_{1,0}$ to $D^{1,0}_1$ ($\rho \in \widehat{k^\times}$, $\nu \in \mathrm{Irr}(Gl_2(k))$, $\tau \in \mathrm{Irr}(Gl_3(k))$). If we let ${}_1D^{1,0}_1 = A_{1,0}T^1_{1,0}$ then $\mathrm{ind}^{D^{1,0}_1}_{{}_1D^{1,0}_1}\, \psi_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \otimes \widetilde{\mu}$ is an irreducible representation of $D^{1,0}_1$, where $\psi_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}$ is viewed as a character of $D^{1,0}_1$ and $\widetilde{\mu}$ is the inflation of the representation $\mu \in \mathrm{Irr}(T^1_{1,0})$. This completes the representation theory of $D^{1,0}_1$.

4.1.2 The Irreducible Representations of $D^{1,1}_1$

We now focus on the representations of $D^{1,1}_1$. Consider the following subgroups of $D^{1,1}_1$:

$$A_{1,1} = \left\{ \{a_{v,w,r,s} = \begin{bmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ & v & 1 &&&&& \\ & w && 1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 & r & s \\ &&&&&& 1 & \\ &&&&&&& 1 \end{bmatrix} \mid v, w, r, s \in k \right\}$$

and

$$T_{1,1} = \left\{ t_{x,Y,Z} = \begin{bmatrix} x &&&&& \\ & x &&&& \\ && Y &&& \\ &&& x && \\ &&&& x & \\ &&&&& Z \end{bmatrix} \mid x \in k^\times, Y \in Gl_2(k), Z \in Gl_2(k) \right\}$$

$A_{1,1}$ is an abelian normal subgroup of $D_1^{1,1}$, $T_{1,1}$ is a subgroup, and $D_1^{1,1} \cong A_{1,1} \rtimes T_{1,1}$. $A_{1,1}$ is isomorphic to the direct sum of $M_{2,1}(k)$ and $M_{1,2}(k)$.

Let ψ be a nontrivial additive character on k. For each $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \in M_{2,1}(k)$ and $\left[\begin{smallmatrix} r & s \end{smallmatrix}\right] \in M_{1,2}(k)$ we may define a character of $A_{1,1}$ as follows:

$$\psi_{\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right],\left[\begin{smallmatrix} r & s \end{smallmatrix}\right]}(a_{v',w',r',s'}) = \psi_{\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]}(\left[\begin{smallmatrix} v' \\ w' \end{smallmatrix}\right])\psi_{\left[\begin{smallmatrix} r & s \end{smallmatrix}\right]}(\left[\begin{smallmatrix} r' & s' \end{smallmatrix}\right])$$

where $\psi_{\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]}(\left[\begin{smallmatrix} v' \\ w' \end{smallmatrix}\right]) = \psi(\mathrm{tr}(\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]^t \left[\begin{smallmatrix} v' \\ w' \end{smallmatrix}\right]))$ and $\psi_{\left[\begin{smallmatrix} r & s \end{smallmatrix}\right]}$ is defined similarly. Let $\widehat{A_{1,1}}$ denote the group of characters of $A_{1,1}$. Notice that $\psi_{\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right],\left[\begin{smallmatrix} r & s \end{smallmatrix}\right]}$ is trivial if and only if both $\psi_{\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]}$ and $\psi_{\left[\begin{smallmatrix} r & s \end{smallmatrix}\right]}$ are trivial. This occurs only if both $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} r & s \end{smallmatrix}\right]$ are zero. A simple calculation shows that the map $M_{2,1}(k) \oplus M_{1,2}(k) \to \widehat{A_{1,1}}$ is a homomorphism. The kernel of this map is $\{(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right])\}$, therefore we have an isomorphism.

We let $D_1^{1,1}$ act on $\widehat{A_{1,1}}$ by conjugation. The $T_{1,1}$ orbits under this action are determined by the equations

$$\begin{aligned} \left[\begin{smallmatrix} v' \\ w' \end{smallmatrix}\right] &= (Y^{-1})^t \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] x \\ \left[\begin{smallmatrix} r' & s' \end{smallmatrix}\right] &= x^{-1} \left[\begin{smallmatrix} r & s \end{smallmatrix}\right] Z^t \end{aligned}$$

If we take $\left[\begin{smallmatrix} r & s \end{smallmatrix}\right]$ to be in rank form we may determine the possible forms for $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$ by using the possible reduced row echelon forms for this matrix. As these forms correspond to the rank of the matrix $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$ we may choose $\psi_{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right],\left[\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right]}$, $\psi_{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right],\left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]}$, $\psi_{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right],\left[\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right]}$, and $\psi_{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right],\left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]}$ as representatives for the $T_{1,1}$ orbits of $\widehat{A_{1,1}}$. We proceed to find the irreducible representations for the stabilizers in $T_{1,1}$ of each of these representatives.

We note that the stabilizer of $\psi_{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right],\left[\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right]}$ is $T_{1,1}$ as there are no restrictions on x, Y, and Z. Since $T_{1,1}$ is isomorphic to $k^\times \times Gl_2(k) \times Gl_3(k)$ we may find all irreducible representations of $T_{1,1}$ by way of the tensor product. The elements in the stabilizer of $\psi_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}1&0\end{smallmatrix}\right]}$ must satisfy the equation $x^{-1}\left[\begin{smallmatrix}1&0\end{smallmatrix}\right]Z^t=\left[\begin{smallmatrix}1&0\end{smallmatrix}\right]$. A simple calculation shows that Z is of the form

$$\begin{bmatrix}\left[\begin{smallmatrix}x&&\\&x&\\&&Y\end{smallmatrix}\right]&\\&\left[\begin{smallmatrix}x&&\\&x&\\&&\left[\begin{smallmatrix}x&z_{12}\\0&z_{22}\end{smallmatrix}\right]\end{smallmatrix}\right]\end{bmatrix}.$$

A similar calculation shows that elements of the stabilizer of $\psi_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}0&0\end{smallmatrix}\right]}$ have the form

$$\begin{bmatrix}\left[\begin{smallmatrix}x&&\\&x&\\&&\left[\begin{smallmatrix}x&\\y_{21}&y_{22}\end{smallmatrix}\right]\end{smallmatrix}\right]&\\&\left[\begin{smallmatrix}x&&\\&x&\\&&Z\end{smallmatrix}\right]\end{bmatrix}.$$

Lastly, the elements of the stabilizer of $\psi_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}1&0\end{smallmatrix}\right]}$ are of the form

$$\begin{bmatrix}\left[\begin{smallmatrix}x&&\\&x&\\&&\left[\begin{smallmatrix}x&\\y_{21}&y_{22}\end{smallmatrix}\right]\end{smallmatrix}\right]&\\&\left[\begin{smallmatrix}x&&\\&x&\\&&\left[\begin{smallmatrix}x&z_{12}\\0&z_{22}\end{smallmatrix}\right]\end{smallmatrix}\right]\end{bmatrix}.$$

We denote the stabilizers of $\psi_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}1&0\end{smallmatrix}\right]}$, $\psi_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}0&0\end{smallmatrix}\right]}$, and $\psi_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}1&0\end{smallmatrix}\right]}$ by $T_{1,1}^{0,1}$, $T_{1,1}^{1,0}$ and $T_{1,1}^{1,1}$ respectively. It is clear by inspection that $T_{1,1}^{0,1}$ and $T_{1,1}^{1,0}$ are isomorphic. We now determine the irreducible representations of the groups $T_{1,1}^{0,1}$ and $T_{1,1}^{1,1}$.

4.1.2.1 The Irreducible Representations of $T_{1,1}^{0,1}$

For $T_{1,1}^{0,1}$ we consider the following subgroups,

$$\begin{aligned}
M &= \left\{m_\alpha=\begin{bmatrix}\left[\begin{smallmatrix}1&&\\&1&\\&&I_2\end{smallmatrix}\right]&\\&\left[\begin{smallmatrix}1&&\\&1&\\&&\left[\begin{smallmatrix}1&\alpha\\&1\end{smallmatrix}\right]\end{smallmatrix}\right]\end{bmatrix} \mid \alpha\in k\right\}\\
L &= \left\{l_{x,\beta,Y}=\begin{bmatrix}\left[\begin{smallmatrix}x&&\\&x&\\&&Y\end{smallmatrix}\right]&\\&\left[\begin{smallmatrix}x&&\\&x&\\&&\left[\begin{smallmatrix}x&\\&\beta\end{smallmatrix}\right]\end{smallmatrix}\right]\end{bmatrix} \mid x,b\in k^\times\right\}
\end{aligned}$$

We have that M is isomorphic to the additive group k. By arguments analogous to those used earlier in this thesis we get that $T_{1,1}^{0,1}\cong M\rtimes L$. We denote by $\widehat{M}$ the group of characters of M. Since the map $k\to\widehat{M}$ defined by $a\mapsto\psi_a$ is an L-equivariant isomorphism we may view the characters of M as left translations of a nontrivial additive character ψ of k.

Let $T_{1,1}^{0,1}$ act on the set $\widehat{M}$ by conjugation. By restricting this action to the subgroup L we see that ψ_α is conjugate to ψ_β if and only if both α and β are invertible, or both are zero. We let ψ_0 and ψ_1 be representatives for the L orbits under this action. The stabilizer in L of ψ_0 is L and the stabilizer of ψ_1 is the group of matrices in L of the form

$$\begin{bmatrix} \begin{bmatrix} x & & \\ & x & \\ & & Y \end{bmatrix} & \\ & \begin{bmatrix} x & & \\ & x & \\ & & \begin{bmatrix} x & \\ & x \end{bmatrix} \end{bmatrix} \end{bmatrix}.$$

We denote these stabilizers by L_0 and L_1 respectively.

The irreducible representations of L are given by the irreducible representations of $k^\times \times k^\times \times Gl_2(k)$, which may be obtained by way of the tensor product. The irreducible representations of the stabilizer of ψ_1 are found by taking the tensor product of irreducible representations of $k^\times$ and $Gl_2(k)$.

Let ${}_0T_{1,1}^{0,1} = ML_0 = ML$. If we view ψ_0 as a character of $T_{1,1}^{0,1}$ then there are irreducible representations of $T_{1,1}^{0,1}$ having the form $\psi_0 \otimes \rho_1 \widetilde{\otimes \rho_2} \otimes \tau$, where $\rho_i \in \widehat{k^\times}$, $\tau \in \mathrm{Irr}(Gl_2(k))$, and $\rho_1 \widetilde{\otimes \rho_2} \otimes \tau$ indicates the inflation of $\rho_1 \otimes \rho_2 \otimes \tau$ to $T_{1,1}^{0,1}$. Now let ${}_1T_{1,1}^{0,1} = ML_1$. Then we get irreducible representations of $T_{1,1}^{0,1}$ by taking $\mathrm{ind}_{{}_1T_{1,1}^{0,1}}^{T_{1,1}^{0,1}}(\psi_1 \otimes \widetilde{\rho \otimes \tau})$, where $\rho \in \widehat{k^\times}$ and $\tau \in Gl_2(k)$.

4.1.2.2 The Irreducible Representations of $T_{1,1}^{1,1}$

In the case of $T_{1,1}^{1,1}$ we consider the following subgroups:

$$M = \left\{ m_{\alpha,\beta} = \begin{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \alpha \\ & & & 1 \end{bmatrix} & \\ & \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \beta \\ & & & 1 \end{bmatrix} \end{bmatrix} \mid \alpha, \beta \in k \right\}$$

$$L = \left\{ l_{x,\gamma,\delta} = \begin{bmatrix} \begin{bmatrix} x & & & \\ & x & & \\ & & x & \\ & & & \gamma \end{bmatrix} & \\ & \begin{bmatrix} x & & & \\ & x & & \\ & & x & \\ & & & \delta \end{bmatrix} \end{bmatrix} \mid x, \gamma, \delta \in k^{\times} \right\}$$

For the same reasoning as throughout this thesis we have that $T_{1,1}^{1,1} \cong M \rtimes L$. As M is isomorphic to the direct sum of two copies of k we know that we may identify elements in the group of irreducible representations of M, denoted $\mathrm{Irr}(M)$, with the tensor product of elements in $\widehat{k}$. More specifically, if we let ψ be a nontrivial additive character of k and (α, β) be an element of $k \otimes k$, then we define an irreducible representation $\psi_{\alpha,\beta}$ on M by $\psi_{\alpha,\beta}(m_{\alpha',\beta'}) = \psi_\alpha(\alpha') \otimes \psi_\beta(\beta')$, where ψ_α and ψ_β are left translations of ψ.

By arguments similar to those used before, we have that $(\alpha, \beta) \mapsto \psi_{\alpha,\beta}$ is an L equivariant isomorphism between $k \oplus k$ and $\widehat{M}$. The orbits under the action of L on $\widehat{M}$ are determined by the following equations:

$$\begin{aligned} \alpha' &= (x^{-1}\gamma)\alpha \\ \beta' &= (x^{-1}\delta)\beta \end{aligned}$$

where x, γ, $\delta \in k^{\times}$. This depends on the invertibility of α and β respectively. We choose $\psi_{0,0}$, $\psi_{1,0}$, $\psi_{0,1}$, and $\psi_{1,1}$ as our representatives. Using the equations above we see that the stabilizer in L for $\psi_{0,0}$ is L. For $\psi_{1,0}$, $\psi_{1,0}$ and $\psi_{1,1}$ we get matrices of the form

$$L_{1,0} = \left\{ \begin{bmatrix} \begin{bmatrix} x & & \\ & x & \\ & & x \end{bmatrix} & \\ & \begin{bmatrix} x & & \\ & x & \\ & & \delta \end{bmatrix} \end{bmatrix} \mid x, \delta \in k^\times \right\}$$

$$L_{0,1} = \left\{ \begin{bmatrix} \begin{bmatrix} x & & \\ & x & \\ & & \gamma \end{bmatrix} & \\ & \begin{bmatrix} x & & \\ & x & \\ & & x \end{bmatrix} \end{bmatrix} \mid x, \gamma \in k^\times \right\}$$

$$L_{1,1} = \left\{ \begin{bmatrix} \begin{bmatrix} x & & \\ & x & \\ & & x \end{bmatrix} & \\ & \begin{bmatrix} x & & \\ & x & \\ & & x \end{bmatrix} \end{bmatrix} \mid x \in k^\times \right\}.$$

As each of the stabilizer groups is isomorphic to the direct product of copies of $k^\times$, the irreducible representations for each of them may be obtained by way of the tensor product.

Let ${}_0T_{1,1}^{1,1} = ML_{0,0}$. Since $L_{0,0} = L$ we get irreducible representations of $T_{1,1}^{1,1}$ by taking the tensor products of the form $\psi_{0,0} \otimes (\widetilde{\rho_1 \otimes \rho_2 \otimes \rho_3})$ where $\psi_{0,0}$ is viewed as a character of $T_{1,1}^{1,1}$ and $(\widetilde{\rho_1 \otimes \rho_2 \otimes \rho_3})$ is the inflation of $\rho_1 \otimes \rho_2 \otimes \rho_3$ to $T_{1,1}^{1,1}$.

Now let ${}_1T_{1,1}^{1,1} = ML_{1,0}$. If $\rho_1 \otimes \rho_2$ is an element of $\mathrm{Irr}(L_{1,0})$, then we may inflate this representation to an irreducible representation, $\widetilde{\rho_1 \otimes \rho_2}$ of ${}_1T_{1,1}^{1,1}$ and $\mathrm{ind}_{{}_1T_{1,1}^{1,1}}^{T_{1,1}^{1,1}}(\psi_{1,0} \otimes \widetilde{\rho_1 \otimes \rho_2})$ is an irreducible representation of $T_{1,1}^{1,1}$.

Remark 5. $\mathrm{Irr}(L_{0,1}) \cong \mathrm{Irr}(L_{1,0})$ *therefore the irreducible representations of* $T_{1,1}^{1,1}$ *induced by irreducible representations of* $ML_{0,1}$ *are of the form* $\mathrm{ind}_{{}_1T_{1,1}^{1,1}}^{T_{1,1}^{1,1}}(\psi_{0,1} \otimes \widetilde{\rho_1 \otimes \rho_2})$.

Finally, if we let ${}_2T_{1,1}^{1,1} = ML_{1,1}$, then for each $a \in k^\times$ we get an irreducible representation of $T_{1,1}^{1,1}$ of the form $\mathrm{ind}_{{}_2T_{1,1}^{1,1}}^{T_{1,1}^{1,1}}(\psi_{1,1} \otimes \widetilde{\rho})$.

Now that we have found all irreducible representations for $T_{1,1}^{0,0}$, $T_{1,1}^{1,0}$, $T_{1,1}^{0,1}$, and $T_{1,1}^{1,1}$ we may now construct the irreducible representations of $D_1^{1,1}$. First, let

${}_0D_1^{1,1} = A_{1,1}T_{1,1}^{0,0}$. Since $T_{1,1}^{0,0} = T_{1,1}$ we get that ${}_0D_1^{1,1} = D_1^{1,1}$. The irreducible representations of $D_1^{1,1}$ coming from this subgroup are of the form $\psi_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right],[0\ 0]} \otimes \tilde{\mu}$, where $\psi_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right],[0\ 0]}$ is viewed as a character of $D_1^{1,1}$ and $\tilde{\mu}$ is an irreducible representation of $T_{1,1}$.

Now let ${}_1D_1^{1,1} = A_{1,1}T_{1,1}^{1,0}$. We view the character $\psi_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],[0\ 0]}$ as a character of ${}_1D_1^{1,1}$ and we obtain an irreducible representations of $D_1^{1,1}$ by taking $\mathrm{ind}_{{}_1D_1^{1,1}}^{D_1^{1,1}}(\psi_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],[0\ 0]} \otimes \widetilde{\mu})$, where $\mu \in \mathrm{Irr}(T_{1,1}^{0,1})$. We find the irreducible representations of $D_1^{1,1}$ coming from ${}_2D_1^{1,1} = A_{1,1}T_{1,1}^{0,1}$ in the same way by replacing $\psi_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],[0\ 0]}$ with $\psi_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right],[1\ 0]}$.

Lastly, we let ${}_3D_1^{1,1} = A_{1,1}T_{1,1}^{1,1}$. We obtain irreducible representations of $D_1^{1,1}$ from this group by taking $\mathrm{ind}_{{}_3D_1^{1,1}}^{D_1^{1,1}}(\psi_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],[1\ 0]} \otimes \widetilde{\mu})$, where $\mu \in \mathrm{Irr}(T_{1,1}^{1,1})$.

Finally, for the representations of $\mathcal{T}_1$ we consider the subgroups ${}_0\mathcal{T}_1 = \mathcal{N}_1 D_1^{0,0}$, ${}_1\mathcal{T}_1 = \mathcal{N}_1 D_1^{1,0}$, ${}_2\mathcal{T}_1 = \mathcal{N}_1 D_1^{0,1}$, and ${}_3\mathcal{T}_1 = \mathcal{N}_1 D_1^{1,1}$. We get irreducible representations of $\mathcal{T}_1$ from these groups as follows:

$\underline{{}_0\mathcal{T}_1}$: The irreducible representations are of the form $\psi_{l_0,u_0} \otimes \widetilde{\phi}$, $\phi \in \mathrm{Irr}(D_1)$.

$\underline{{}_1\mathcal{T}_1}$: The irreducible representations are of the form $\mathrm{ind}_{{}_1\mathcal{T}_1}^{\mathcal{T}_1}(\psi_{l_1,u_0} \otimes \widetilde{\phi})$, $\phi \in \mathrm{Irr}(D_1^{1,0})$.

$\underline{{}_2\mathcal{T}_1}$: The irreducible representations are of the form $\mathrm{ind}_{{}_2\mathcal{T}_1}^{\mathcal{T}_1}(\psi_{l_0,u_1} \otimes \widetilde{\phi})$, $\phi \in \mathrm{Irr}(D_1^{0,1})$.

$\underline{{}_3\mathcal{T}_1}$: The irreducible representations are of the form $\mathrm{ind}_{{}_3\mathcal{T}_1}^{\mathcal{T}_1}(\psi_{l_1,u_1} \otimes \widetilde{\phi})$, $\phi \in \mathrm{Irr}(D_1^{1,1})$.

4.2 The Irreducible Representations of $\mathcal{T}_2$

As in the case of $\mathcal{T}_1$ we begin by assuming that u is in rank form. We denote the representatives as u_0, u_1, and u_2. We have the following cases:

<u>**Case 1**</u>: u is rank 0: In this case both X and Z are independent and we are reduced to the rank of the matrix l. We obtain the classes ψ_{l_0,u_0}, ψ_{l_1,u_0}, and ψ_{l_2,u_0}.

Case 2: u is rank 1: We note here that in order for $u = (X^{-1})^t u Z^t$ we must have that X is a lower triangular matrix. We must determine what multiplication by X on the right does to the reduced row echelon forms for 2×2 matrices over k. We know that the possible reduced row echelon forms of l are $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 & \alpha \\ 0 & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, where $\alpha \in k$. We note that we cannot multiply the zero matrix by an invertible matrix and obtain a matrix of positive rank. Similarly, multiplying the identity matrix by an invertible matrix will only give another rank 2 matrix. Therefore, we need only determine if we may obtain a matrix of the form $\left[\begin{smallmatrix} 1 & \alpha \\ 0 & 0 \end{smallmatrix}\right]$ by multiplying the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ on the right by an invertible upper triangular matrix X^t, but this would imply that $1 = 0$, a contradiction. We choose ψ_{l_0,u_1}, ψ_{l_1,u_1}, $\psi_{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right],u_1}$, and ψ_{l_2,u_1} to be our class representatives.

Case 3: u is rank 2: Notice that, in this case, $Z^t = X^t$, but there are no restrictions on X, therefore the class of (l, u_2) is determined by the rank of l. We let our last three class representatives be ψ_{l_0,u_2}, ψ_{l_1,u_2} and ψ_{l_2,u_2}.

For each $\psi_{l,u}$ in our set of representatives we denote these stabilizer in $\mathcal{D}_2$ by $D_2^{l,u}$. In each case we calculate $u = (X^{-1})^t u Z^t$ and $l = (Y^{-1})^t l X^t$ and list the stabilizers in Table 4.1.

First note that $\mathcal{D}_2$ is isomorphic to $Gl_2(k) \times Gl_2(k) \times Gl_2(k)$ and therefore the irreducible representations of $\mathcal{D}_2$ may be obtained via the tensor product. Also, $D_2^{l_2,u_0}$ and $D_2^{l_0,u_2}$ are isomorphic to $Gl_2(k) \times Gl_2(k)$ and, again, we are able to use the tensor product to determine the irreducible representations of these groups. Lastly, $D_2^{l_2,u_2}$ is isomorphic to $Gl_2(k)$ and the irreducible representations here are determined as well. We turn our attention to the other stabilizer groups.

Notice that $D_2^{l_1,u_0} \cong D_2^{l_0,u_1}$ and $D_2^{l_2,u_1} \cong D_2^{l_1,u_2}$ therefore we need only study the irreducible representations for $D_2^{l_1,u_0}$, $D_2^{l_1,u_1}$, $D_2^{\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right],u_1}$, and $D_2^{l_2,u_1}$.

4.2.1 The Irreducible Representations of $D_2^{l_1,u_0}$

Let N_{l_1,u_0} and T_{l_1,u_0} be the following subgroups of $D_2^{l_1,u_0}$:

$$\begin{aligned} N_{l_1,u_0} &= \left\{ n_{\alpha,\beta} = \begin{bmatrix} \left[\begin{smallmatrix}1&\alpha\\&1\end{smallmatrix}\right] & & & \\ & \left[\begin{smallmatrix}1&\\\beta&1\end{smallmatrix}\right] & & \\ & & \left[\begin{smallmatrix}1&\alpha\\&1\end{smallmatrix}\right] & \\ & & & I_2 \end{bmatrix} \mid \alpha,\beta \in k \right\} \\ T_{l_1,u_0} &= \left\{ \begin{bmatrix} \left[\begin{smallmatrix}x&\\&w\end{smallmatrix}\right] & & & \\ & \left[\begin{smallmatrix}x&\\&y\end{smallmatrix}\right] & & \\ & & \left[\begin{smallmatrix}x&\\&w\end{smallmatrix}\right] & \\ & & & Z \end{bmatrix} \mid x,y,w \in k^\times \text{and } Z \in Gl_2(k) \right\}. \end{aligned}$$

N_{l_1,u_0} is an abelian normal subgroup of $D_2^{l_1,u_0}$. Every element of $D_2^{l_1,u_0}$ may be written as a product of elements in N_{l_1,u_0} and T_{l_1,u_0} and $N_{l_1,u_0} \cap T_{l_1,u_0} = I_8$, therefore $D_2^{l_1,u_0} \cong N_{l_1,u_0} \rtimes T_{l_1,u_0}$. Let $\widehat{N_{l_1,u_0}}$ be the group of characters of N_{l_1,u_0}. Let ψ be an nontrivial additive character of k, then for each (α,β) in $k \oplus k$ we can define an character, $\psi_{\alpha,\beta}$, of N_{l_1,u_0} by $\psi_{\alpha,\beta}(n_{\alpha',\beta'}) = \psi_\alpha(\alpha')\psi_\beta(\beta')$. Here ψ_a is a left translation of the character ψ. By an argument similar to the one used in Proposition 3.1 we have that the map $(\alpha,\beta) \mapsto \psi_{\alpha,\beta}$ is a T_{l_1,u_0} equivariant isomorphism. The action of T_{l_1,u_0} is given by $t \circ (\alpha,\beta) = (\alpha w x^{-1}, \beta x y^{-1})$. The T_{l_1,u_0} orbits in $\widehat{N_{l_1,u_0}}$ are determined by the equations:

$$\begin{aligned} \alpha' &= w\alpha x^{-1} = \alpha(wx^{-1}) \\ \beta' &= x\beta y^{-1} = \beta(xy^{-1}) \end{aligned}$$

The classes are therefore determined by whether α and β are invertible respectively.We take the representatives to be $\psi_{0,0}$, $\psi_{1,0}$, $\psi_{0,1}$, and $\psi_{1,1}$. The stabilizer in T_{l_1,u_0} of

$\psi_{l,u}$ representative	$D_2^{l,u}$
ψ_{l_0,u_0}	D_2
ψ_{l_1,u_0}	$\left\{ \begin{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ & x_{22} \end{bmatrix} & & & \\ & \begin{bmatrix} x_{11} & \\ y_{21} & y_{22} \end{bmatrix} & & \\ & & \begin{bmatrix} x_{11} & x_{12} \\ & x_{22} \end{bmatrix} & \\ & & & Z \end{bmatrix} \mid \; Z \in Gl_2(k) \right\}$
ψ_{l_2,u_0}	$\left\{ \begin{bmatrix} X & & & \\ & X & & \\ & & X & \\ & & & Z \end{bmatrix} \mid \; X, Z \in Gl_2(k) \right\}$
ψ_{l_0,u_1}	$\left\{ \begin{bmatrix} \begin{bmatrix} x_{11} & \\ x_{21} & x_{22} \end{bmatrix} & & & \\ & Y & & \\ & & \begin{bmatrix} x_{11} & \\ x_{21} & x_{22} \end{bmatrix} & \\ & & & \begin{bmatrix} x_{11} & z_{12} \\ & z_{22} \end{bmatrix} \end{bmatrix} \mid \; Y \in Gl_2(k) \right\}$
ψ_{l_1,u_1}	$\left\{ \begin{bmatrix} \begin{bmatrix} x_{11} & \\ & x_{22} \end{bmatrix} & & & \\ & \begin{bmatrix} x_{11} & \\ y_{21} & y_{22} \end{bmatrix} & & \\ & & \begin{bmatrix} x_{11} & \\ & x_{22} \end{bmatrix} & \\ & & & \begin{bmatrix} x_{11} & z_{12} \\ & z_{22} \end{bmatrix} \end{bmatrix} \right\}$
$\psi_{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right],u_1}$	$\left\{ \begin{bmatrix} \begin{bmatrix} x_{11} & \\ x_{21} & x_{22} \end{bmatrix} & & & \\ & \begin{bmatrix} x_{22} & \\ y_{21} & y_{22} \end{bmatrix} & & \\ & & \begin{bmatrix} x_{11} & \\ x_{21} & x_{22} \end{bmatrix} & \\ & & & \begin{bmatrix} x_{11} & z_{12} \\ & z_{22} \end{bmatrix} \end{bmatrix} \right\}$
ψ_{l_2,u_1}	$\left\{ \begin{bmatrix} \begin{bmatrix} x_{11} & \\ x_{21} & x_{22} \end{bmatrix} & & & \\ & \begin{bmatrix} x_{11} & \\ x_{21} & x_{22} \end{bmatrix} & & \\ & & \begin{bmatrix} x_{11} & \\ x_{21} & x_{22} \end{bmatrix} & \\ & & & \begin{bmatrix} x_{11} & z_{12} \\ & z_{22} \end{bmatrix} \end{bmatrix} \right\}$
ψ_{l_0,u_2}	$\left\{ \begin{bmatrix} X & & & \\ & Y & & \\ & & X & \\ & & & X \end{bmatrix} \mid \; X, Y \in Gl_2(k) \right\}$
ψ_{l_1,u_2}	$\left\{ \begin{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ & x_{22} \end{bmatrix} & & & \\ & \begin{bmatrix} x_{11} & \\ y_{21} & y_{22} \end{bmatrix} & & \\ & & \begin{bmatrix} x_{11} & x_{12} \\ & x_{22} \end{bmatrix} & \\ & & & \begin{bmatrix} x_{11} & x_{12} \\ & x_{22} \end{bmatrix} \end{bmatrix} \right\}$
ψ_{l_2,u_2}	$\left\{ \begin{bmatrix} X & & & \\ & X & & \\ & & X & \\ & & & X \end{bmatrix} x \in Gl_2(k) \right\}$

Table 4.1: Stabilizers in D_2 of $\psi_{l,u}$ representatives

$\psi_{\alpha,\beta}$ representative	$T_{l_1,u_0}^{\alpha,\beta} = \{t \in T_{l_1,u_0} \mid \psi_{\alpha,\beta}^{(t)} = \psi_{\alpha,\beta}\}$
$\psi_{0,0}$	T_{l_1,u_0}
$\psi_{1,0}$	$\left\{ \begin{bmatrix} \begin{bmatrix} x & \\ & x \end{bmatrix} & & & \\ & \begin{bmatrix} x & \\ & y \end{bmatrix} & & \\ & & \begin{bmatrix} x & \\ & x \end{bmatrix} & \\ & & & Z \end{bmatrix} \mid x, y \in k^\times \text{and } Z \in Gl_2(k) \right\}$
$\psi_{0,1}$	$\left\{ \begin{bmatrix} \begin{bmatrix} x & \\ & w \end{bmatrix} & & & \\ & \begin{bmatrix} x & \\ & x \end{bmatrix} & & \\ & & \begin{bmatrix} x & \\ & w \end{bmatrix} & \\ & & & Z \end{bmatrix} \mid x, w \in k^\times \text{and } Z \in Gl_2(k) \right\}$
$\psi_{1,1}$	$\left\{ \begin{bmatrix} \begin{bmatrix} x & \\ & x \end{bmatrix} & & & \\ & \begin{bmatrix} x & \\ & x \end{bmatrix} & & \\ & & \begin{bmatrix} x & \\ & x \end{bmatrix} & \\ & & & Z \end{bmatrix} \mid x \in k^\times \text{and } Z \in Gl_2(k) \right\}$

Table 4.2: Stabilizers in T_{l_1,u_0} of $\psi_{\alpha,\beta}$ representatives

each of the representatives is as in Table 4.2. We have that T_{l_1,u_0} is isomorphic to $k^\times \times k^\times \times k^\times \times Gl_2(k)$, thus all irreducible representations of T_{l_1,u_0} may be obtained using the tensor product. Likewise, the irreducible representations of $T_{l_1,u_0}^{1,0}$ and $T_{l_1,u_0}^{0,1}$, as they are both isomorphic to $k^\times \times k^\times \times Gl_2(k)$, may be constructed by taking products of irreducible representations of $k^\times$ and $Gl_2(k)$. The irreducible representations of $T_{l_1,u_0}^{1,1}$ may be obtained via the tensor product as well since it is isomorphic to $k^\times \times Gl_2(k)$.

Let ${}_0D_2^{l_1,u_0} = N_{l_1,u_0} T_{l_1,u_0}^{0,0}$, then we get representations of $D_2^{l_1,u_0}$ by taking the tensor product of $\psi_{0,0}$ with $(\rho_1 \otimes \widetilde{\rho_2 \otimes \rho_3} \otimes \tau)$ where $\rho_j \in \widehat{k^\times}$ and $\tau \in \mathrm{Irr}(Gl_2(k))$. Representations of $D_2^{l_1,u_0}$ coming from the subgroup ${}_1D_2^{l_1,u_0} = N_{l_1,u_0} T_{l_1,u_0}^{1,0}$ are of the form $\mathrm{ind}_{{}_1D_2^{l_1,u_0}}^{D_2^{l_1,u_0}} (\psi_{1,0} \otimes (\rho_1 \widetilde{\otimes \rho_2} \otimes \tau))$. Similarly, the representations coming from ${}_2D_2^{l_1,u_0} = N_{l_1,u_0} T_{l_1,u_0}^{0,1}$ are of the form $\mathrm{ind}_{{}_2D_2^{l_1,u_0}}^{D_2^{l_1,u_0}} (\psi_{0,1} \otimes (\rho_1 \widetilde{\otimes \rho_2} \otimes \tau))$. Lastly, we have representations of $D_2^{l_1,u_0}$ of the form $\mathrm{ind}_{{}_3D_2^{l_1,u_0}}^{D_2^{l_1,u_0}}(\psi_{1,1} \otimes \widetilde{(\rho \otimes \tau)})$, where ${}_3D_2^{l_1,u_0} = N_{l_1,u_0} T_{l_1,u_0}^{1,1}$.

4.2.2 The Irreducible Representations of $D_2^{l_1,u_1}$

We now consider the following subgroups of $D_2^{l_1,u_1}$,

$$N_{l_1,u_1} = \left\{ \begin{bmatrix} I_2 & \begin{bmatrix} 1 & \\ \alpha & 1 \end{bmatrix} & & \\ & & I_2 & \\ & & & \begin{bmatrix} 1 & \beta \\ & 1 \end{bmatrix} \end{bmatrix} \mid \alpha, \beta \in k \right\}$$

$$T_{l_1,u_1} = \left\{ \begin{bmatrix} \begin{bmatrix} x & \\ & w \end{bmatrix} & & & \\ & \begin{bmatrix} x & \\ & y \end{bmatrix} & & \\ & & \begin{bmatrix} x & \\ & w \end{bmatrix} & \\ & & & \begin{bmatrix} x & \\ & z \end{bmatrix} \end{bmatrix} \mid x, y, w, z \in k^\times \right\}.$$

Then, by arguments similar to those made before, we have $D_2^{l_1,u_1} = N_{l_1,u_1} \rtimes T_{l_1,u_1}$. N_{l_1,u_1} is abelian. We use a nontrivial additive character of k, ψ, to define a character of N_{l_1,u_1} as we did for N_{l_1,u_0}. Again we label these representations $\psi_{\alpha,\beta}$. As in the previous case the map $(\alpha, \beta) \mapsto \psi_{\alpha,\beta}$ is a T_{l_1,u_1} equivariant map. Here the action of T_{l_1,u_1} on $k \oplus k$ is given by $t \circ (\alpha, \beta) = (\alpha x y^{-1}, \beta x^{-1} z)$. We see that $\psi_{\alpha,\beta}$ is conjugate to $\psi_{\alpha',\beta'}$ if and only if the following hold:

$$\begin{aligned} \alpha' &= \alpha(y^{-1}x) \\ \beta' &= \beta(x^{-1}z) \end{aligned}$$

As in the previous case we get that the equivalence classes depend on whether α and β are invertible respectively. Again, we have take the orbit representatives to be $\psi_{0,0}$, $\psi_{0,1}$, $\psi_{1,0}$, and $\psi_{1,1}$. The stabilizers of each of these representations are seen in Table 4.3. As all of these stabilizers are isomorphic to direct products of copies of $k^\times$ we

$\psi_{\alpha,\beta}$ representative	$T^{\alpha,\beta}_{l_1,u_1} = \{t \in T_{l_1,u_1} \vert \psi^{(t)}_{\alpha,\beta} = \psi_{\alpha,\beta}\}$
$\psi_{0,0}$	T_{l_1,u_1}
$\psi_{0,1}$	$\left\{ \begin{bmatrix} \begin{bmatrix} x & \\ & w \end{bmatrix} & & & \\ & \begin{bmatrix} x & \\ & y \end{bmatrix} & & \\ & & \begin{bmatrix} x & \\ & w \end{bmatrix} & \\ & & & \begin{bmatrix} x & \\ & x \end{bmatrix} \end{bmatrix} \vert\ x, y, w \in k^\times \right\}$
$\psi_{1,0}$	$\left\{ \begin{bmatrix} \begin{bmatrix} x & \\ & w \end{bmatrix} & & & \\ & \begin{bmatrix} x & \\ & x \end{bmatrix} & & \\ & & \begin{bmatrix} x & \\ & w \end{bmatrix} & \\ & & & \begin{bmatrix} x & \\ & z \end{bmatrix} \end{bmatrix} \vert\ x, w, z \in k^\times \right\}$
$\psi_{1,1}$	$\left\{ \begin{bmatrix} \begin{bmatrix} x & \\ & w \end{bmatrix} & & & \\ & \begin{bmatrix} x & \\ & x \end{bmatrix} & & \\ & & \begin{bmatrix} x & \\ & w \end{bmatrix} & \\ & & & \begin{bmatrix} x & \\ & x \end{bmatrix} \end{bmatrix} \vert\ x, w \in k^\times \right\}$

Table 4.3: Stabilizers in T_{l_1,u_1} of $\psi_{\alpha,\beta}$ representatives

may determine their irreducible representations by way of the tensor product.

Now consider the subgroup ${}_0D_2^{l_1,u_1} = N_{l_1,u_1}T^{0,0}_{l_1,u_1}$ of $D_2^{l_1,u_1}$. Then there are irreducible representations of $D_2^{l_1,u_1}$ of the form $\psi_{0,0} \otimes (\rho_1 \otimes \widetilde{\rho_2 \otimes \rho_3} \otimes \rho_4)$, where $\rho_j \in \widehat{k^\times}$.

The representations of $D_2^{l_1,u_1}$ produced from the subgroup ${}_1D_2^{l_1,u_1} = N_{l_1,u_1}T^{1,0}_{l_1,u_1}$ are of the form $\mathrm{ind}^{D_2^{l_1,u_1}}_{{}_1D_2^{l_1,u_1}}(\psi_{1,0} \otimes (\rho_1 \widetilde{\otimes \rho_2 \otimes} \rho_3))$. The representations coming from the subgroup ${}_2D_2^{l_1,u_1} = N_{l_1,u_1}T^{0,1}_{l_1,u_1}$ have the form replacing ${}_1D_2^{l_1,u_1}$ with ${}_2D_2^{l_1,u_1}$ and $\psi_{1,0}$ with $\psi_{0,1}$.

Lastly, the representations induced from the subgroup ${}_3D_2^{l_1,u_1} = N_{l_1,u_1}T^{1,1}_{l_1,u_1}$ are of the form $\mathrm{ind}^{D_2^{l_1,u_1}}_{{}_3D_2^{l_1,u_1}}(\psi_{1,1} \otimes (\widetilde{\rho_1 \otimes \rho_2}))$.

4.2.3 The Irreducible Representations of $D_2^{l_1',u_1}$

We denote the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ by l_1'. Let N_{l_1',u_1} and T_{l_1',u_1} be the following subgroups of $D_2^{l_1',u_1}$:

$$N_{l_1',u_1} = \left\{ n_{\alpha,\beta.\gamma} = \begin{bmatrix} \left[\begin{smallmatrix} 1 & \\ \alpha & 1 \end{smallmatrix}\right] & & & \\ & \left[\begin{smallmatrix} 1 & \\ \beta & 1 \end{smallmatrix}\right] & & \\ & & \left[\begin{smallmatrix} 1 & \\ \alpha & 1 \end{smallmatrix}\right] & \\ & & & \left[\begin{smallmatrix} 1 & \gamma \\ & 1 \end{smallmatrix}\right] \end{bmatrix} \mid \alpha, \beta, \gamma \in k \right\}$$

$$T_{l_1',u_1} = \left\{ \begin{bmatrix} \left[\begin{smallmatrix} x & \\ & w \end{smallmatrix}\right] & & & \\ & \left[\begin{smallmatrix} w & \\ & y \end{smallmatrix}\right] & & \\ & & \left[\begin{smallmatrix} x & \\ & w \end{smallmatrix}\right] & \\ & & & \left[\begin{smallmatrix} x & \\ & z \end{smallmatrix}\right] \end{bmatrix} \mid x, y, w, z \in k^\times \right\}.$$

Then $D_2^{l_1',u_1} \cong N_{l_1',u_1} \rtimes T_{l_1',u_1}$. As it is N_{l_1',u_1} is abelian and isomorphic to $k \oplus k \oplus k$, we may view the irreducible representations here as tensor products of characters of k. More specifically, for each $(\alpha, \beta, \gamma) \in k \oplus k \oplus k$ define $\psi_{\alpha,\beta,\gamma}$ on N_{l_1',u_1} by $\psi_{\alpha,\beta,\gamma}(n_{\alpha',\beta',\gamma'}) = \psi_\alpha(\alpha')\psi_\beta(\beta')\psi_\gamma(\gamma')$. If we denote the group of characters of N_{l_1',u_1} by $\widehat{N_{l_1',u_1}}$ then the map $(\alpha, \beta, \gamma) \mapsto \psi_{\alpha,\beta,\gamma}$ is a T_{l_1',u_1} equivariant isomorphism. Therefore the T_{l_1',u_1} orbits under the action of T_{l_1',u_1} on $\widehat{N_{l_1',u_1}}$ defined by conjugation are determined by the equations:

$$\begin{aligned} \alpha' &= \alpha(w^{-1}x) \\ \beta' &= \beta(y^{-1}w) \\ \gamma' &= \gamma(x^{-1}z) \end{aligned}$$

Since x, y, z, and w are arbitrary elements in $k^\times$ we have that our T_{l_1',u_1} orbit representatives are determined by the invertability of α, β, and γ. We choose our representatives to be $\psi_{0,0,0}$, $\psi_{1,0,0}$, $\psi_{0,1,0}$, $\psi_{0,0,1}$, $\psi_{1,1,0}$, $\psi_{1,0,1}$, $\psi_{0,1,1}$, and $\psi_{1,1,1}$. Using the equations above we find that the stabilizers in T_{l_1',u_1} of these representatives are as in Table 4.4. Since each of the stabilizers is isomorphic to the direct product of copies of $k^\times$ we may determine the irreducible representations of each $T^{\alpha,\beta,\gamma}_{l_1',u_1}$ using the tensor product of characters of $k^\times$.

We present the forms for the irreducible representations of $D_2^{l_1',u_1}$ obtained from the each of the ${}_jD_2^{l_1',u_1}$ in Table 4.5. In each case ρ_j is an element of $\widehat{k^\times}$ and $\widetilde{\rho}$ indicates the inflation of the representation to ${}_jD_2^{l_1',u_1}$.

4.2.4 The Irreducible representations of $D_2^{l_2,u_1}$

Let N_{l_2,u_1} and T_{l_2,u_1} be the following subgroups of $D_2^{l_2,u_1}$:

$$N_{l_2,u_1} = \left\{ n_{\alpha,\beta} = \begin{bmatrix} \left[\begin{smallmatrix} 1 & \\ \alpha & 1 \end{smallmatrix}\right] & & & \\ & \left[\begin{smallmatrix} 1 & \\ \alpha & 1 \end{smallmatrix}\right] & & \\ & & \left[\begin{smallmatrix} 1 & \\ \alpha & 1 \end{smallmatrix}\right] & \\ & & & \left[\begin{smallmatrix} 1 & \beta \\ & 1 \end{smallmatrix}\right] \end{bmatrix} \mid \alpha, \beta \in k \right\}$$

$$T_{l_2,u_1} = \left\{ \begin{bmatrix} \left[\begin{smallmatrix} x & \\ & w \end{smallmatrix}\right] & & & \\ & \left[\begin{smallmatrix} x & \\ & w \end{smallmatrix}\right] & & \\ & & \left[\begin{smallmatrix} x & \\ & w \end{smallmatrix}\right] & \\ & & & \left[\begin{smallmatrix} x & \\ & z \end{smallmatrix}\right] \end{bmatrix} \mid x, w, z \in k^\times \right\}.$$

N_{l_2,u_1} is abelian and normal in $D_2^{l_2,u_1}$. By the same argument used in previous cases we have that $D_2^{l_2,u_1} \cong N_{l_2,u_1} \rtimes T_{l_2,u_1}$. We identify $\widehat{N_{l_2,u_1}}$, the group of characters of N_{l_2,u_1} with $k \oplus k$ via the T_{l_2,u_1} equivariant isomorphism $(\alpha,\beta) \mapsto \psi_{\alpha,\beta}$, where $\psi_{\alpha,\beta}$ is defined as in the previous section. The T_{l_2,u_1} orbits under the conjugation action defined on $\widehat{N_{l_2,u_1}}$ are determined by the equations:

$$\begin{aligned} \alpha' &= \alpha(xw^{-1}) \\ \beta' &= \beta(x^{-1}z) \end{aligned}$$

$\psi_{\alpha,\beta,\gamma}$ representative	$T^{\alpha,\beta,\gamma}_{l'_1,u_1} = \{t \in T_{l'_1,u_1} \mid \psi^{(t)}_{\alpha,\beta,\gamma} = \psi_{\alpha,\beta,\gamma}\}$
$\psi_{0,0,0}$	$T_{l'_1,u_1}$
$\psi_{1,0,0}$	$\left\{ \begin{bmatrix} \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] & & & \\ & \left[\begin{smallmatrix} x & \\ & y \end{smallmatrix}\right] & & \\ & & \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] & \\ & & & \left[\begin{smallmatrix} x & \\ & z \end{smallmatrix}\right] \end{bmatrix} \mid x, y \in k^\times \right\}$
$\psi_{0,1,0}$	$\left\{ \begin{bmatrix} \left[\begin{smallmatrix} x & \\ & w \end{smallmatrix}\right] & & & \\ & \left[\begin{smallmatrix} w & \\ & w \end{smallmatrix}\right] & & \\ & & \left[\begin{smallmatrix} x & \\ & w \end{smallmatrix}\right] & \\ & & & \left[\begin{smallmatrix} x & \\ & z \end{smallmatrix}\right] \end{bmatrix} \mid x, w, z \in k^\times \right\}$
$\psi_{0,0,1}$	$\left\{ \begin{bmatrix} \left[\begin{smallmatrix} x & \\ & w \end{smallmatrix}\right] & & & \\ & \left[\begin{smallmatrix} w & \\ & y \end{smallmatrix}\right] & & \\ & & \left[\begin{smallmatrix} x & \\ & w \end{smallmatrix}\right] & \\ & & & \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] \end{bmatrix} \mid x, y, w \in k^\times \right\}$
$\psi_{1,1,0}$	$\left\{ \begin{bmatrix} \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] & & & \\ & \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] & & \\ & & \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] & \\ & & & \left[\begin{smallmatrix} x & \\ & z \end{smallmatrix}\right] \end{bmatrix} \mid x, z \in k^\times \right\}$
$\psi_{1,0,1}$	$\left\{ \begin{bmatrix} \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] & & & \\ & \left[\begin{smallmatrix} x & \\ & y \end{smallmatrix}\right] & & \\ & & \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] & \\ & & & \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] \end{bmatrix} \mid x, y \in k^\times \right\}$
$\psi_{0,1,1}$	$\left\{ \begin{bmatrix} \left[\begin{smallmatrix} x & \\ & w \end{smallmatrix}\right] & & & \\ & \left[\begin{smallmatrix} w & \\ & w \end{smallmatrix}\right] & & \\ & & \left[\begin{smallmatrix} x & \\ & w \end{smallmatrix}\right] & \\ & & & \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] \end{bmatrix} \mid x \in k^\times \right\}$
$\psi_{1,1,1}$	$\left\{ \begin{bmatrix} \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] & & & \\ & \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] & & \\ & & \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] & \\ & & & \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] \end{bmatrix} \mid x \in k^\times \right\}$

Table 4.4: Stabilizers in $T_{l'_1,u_1}$ of $\psi_{\alpha,\beta,\gamma}$ representatives

${}_jD_2^{l_1',u_1}$	Irreducible Rep'ns of $D_2^{l_1',u_1}$
${}_0D_2^{l_1',u_1} = N_{l_1',u_1} T_{l_1'u_1}^{0,0,0}$	$\psi_{0,0,0} \otimes (\rho_1 \otimes \widetilde{\rho_2 \otimes \rho_3} \otimes \rho_4)$
${}_1D_2^{l_1',u_1} = N_{l_1',u_1} T_{l_1'u_1}^{1,0,0}$	$\mathrm{ind}_{{}_1D_2^{l_1'u_1}}^{D_2^{l_1',u_1}} (\psi_{1,0,0} \otimes (\rho_1 \widetilde{\otimes \rho_2 \otimes} \rho_3))$
${}_2D_2^{l_1',u_1} = N_{l_1',u_1} T_{l_1'u_1}^{0,1,0}$	$\mathrm{ind}_{{}_2D_2^{l_1'u_1}}^{D_2^{l_1',u_1}} (\psi_{0,1,0} \otimes (\rho_1 \widetilde{\otimes \rho_2 \otimes} \rho_3))$
${}_3D_2^{l_1',u_1} = N_{l_1',u_1} T_{l_1'u_1}^{0,0,1}$	$\mathrm{ind}_{{}_3D_2^{l_1'u_1}}^{D_2^{l_1',u_1}} (\psi_{0,0,1} \otimes (\rho_1 \widetilde{\otimes \rho_2 \otimes} \rho_3))$
${}_4D_2^{l_1',u_1} = N_{l_1',u_1} T_{l_1'u_1}^{1,1,0}$	$\mathrm{ind}_{{}_4D_2^{l_1'u_1}}^{D_2^{l_1',u_1}} (\psi_{1,1,0} \otimes (\widetilde{\rho_1 \otimes \rho_2}))$
${}_5D_2^{l_1',u_1} = N_{l_1',u_1} T_{l_1'u_1}^{1,0,1}$	$\mathrm{ind}_{{}_5D_2^{l_1'u_1}}^{D_2^{l_1',u_1}} (\psi_{1,0,1}(\widetilde{\rho_1 \otimes \rho_2}))$
${}_6D_2^{l_1',u_1} = N_{l_1',u_1} T_{l_1'u_1}^{0,1,1}$	$\mathrm{ind}_{{}_6D_2^{l_1'u_1}}^{D_2^{l_1',u_1}} (\psi_{0,1,1} \otimes (\widetilde{\rho_1 \otimes \rho_2}))$
${}_7D_2^{l_1',u_1} = N_{l_1',u_1} T_{l_1'u_1}^{1,1,1}$	$\mathrm{ind}_{{}_7D_2^{l_1'u_1}}^{D_2^{l_1',u_1}} (\psi_{1,1,1} \otimes \widetilde{\rho})$

Table 4.5: Irreducible Representations of $D_2^{l_1',u_1}$

$\psi_{\alpha,\beta}$ representative	$T^{\alpha,\beta}_{l_2,u_1} = \{t \in T_{l_2,u_1} \mid \psi^{(t)}_{\alpha,\beta,\gamma} = \psi_{\alpha,\beta,\gamma}\}$
$\psi_{0,0}$	T_{l_2,u_1}
$\psi_{1,0}$	$\left\{ \begin{bmatrix} \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] & & & \\ & \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] & & \\ & & \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] & \\ & & & \left[\begin{smallmatrix} x & \\ & z \end{smallmatrix}\right] \end{bmatrix} \mid x, z \in k^\times \right\}$
$\psi_{0,1}$	$\left\{ \begin{bmatrix} \left[\begin{smallmatrix} x & \\ & w \end{smallmatrix}\right] & & & \\ & \left[\begin{smallmatrix} x & \\ & w \end{smallmatrix}\right] & & \\ & & \left[\begin{smallmatrix} x & \\ & w \end{smallmatrix}\right] & \\ & & & \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] \end{bmatrix} \mid x, w, z \in k^\times \right\}$
$\psi_{1,1}$	$\left\{ \begin{bmatrix} \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] & & & \\ & \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] & & \\ & & \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] & \\ & & & \left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] \end{bmatrix} \mid x \in k^\times \right\}$

Table 4.6: Stabilizers in T_{l_2,u_1} of $\psi_{\alpha,\beta,\gamma}$ representatives

We, again, choose the representatives $\psi_{0,0}$, $\psi_{1,0}$, $\psi_{0,1}$, and $\psi_{1,1}$. The stabilizers of these representatives are shown in Table 4.6. All of these stabilizer groups are the direct product of copies of $k^\times$ so we may determine all irreducible representations by taking products in $\widehat{k^\times}$.

Let ${}_0D_2^{l_2,u_1} = N_{l_2,u_1}T^{0,0}_{l_2,u_1}$. The irreducible representations of $D_2^{l_2,u_1}$ induced from this subgroup are of the form $\psi_{0,0} \otimes (\rho_1 \widetilde{\otimes \rho_2 \otimes} \rho_3)$ where ρ_j are elements of $\widehat{k^\times}$. As usual, $\widetilde{\rho}$ indicates inflation.

The irreducible representations of $D_2^{l_2,u_1}$ induced from the subgroup ${}_1D_2^{l_2,u_1} = N_{l_2,u_1}T^{1,0}_{l_2,u_1}$ are of the form $\mathrm{ind}^{D_2^{l_2,u_1}}_{{}_1D_2^{l_2,u_1}}(\psi_{1,0}\otimes(\widetilde{\rho_1 \otimes \rho_2}))$, where ρ_j are in $\widehat{k^\times}$. The elements of $\mathrm{Irr}(D_2^{l_2,u_1})$ determined by ${}_2D_2^{l_2,u_1} = N_{l_2,u_1}T^{0,1}_{l_2,u_1}$ are of the form $\mathrm{ind}^{D_2^{l_2,u_1}}_{{}_2D_2^{l_2,u_1}}(\psi_{0,1} \otimes \widetilde{\rho_1 \otimes \rho_2})$, where ρ_j are in $\widehat{k^\times}$.

${}_j\mathcal{T}_2$	Irreducible rep'ns of $\mathcal{T}_2$ from ${}_j\mathcal{T}_2$
${}_0\mathcal{T}_2 = \mathcal{N}_2 D_2^{l_0,u_0}$	$\psi_{l_0,u_0} \otimes (\mu_1 \otimes \widetilde{\mu_2} \otimes \mu_3)$, $\quad \mu_1,\ \mu_2,\ \mu_3 \in \mathrm{Irr}(D_2^{l_0,u_0})$
${}_1\mathcal{T}_2 = \mathcal{N}_2 D_2^{l_1,u_0}$	$\mathrm{ind}_{{}_1\mathcal{T}_2}^{\mathcal{T}_2}(\psi_{l_1,u_0} \otimes \widetilde{\mu})$, $\quad \mu \in \mathrm{Irr}(D_2^{l_1,u_0})$
${}_2\mathcal{T}_2 = \mathcal{N}_2 D_2^{l_2,u_0}$	$\mathrm{ind}_{{}_2\mathcal{T}_2}^{\mathcal{T}_2}(\psi_{l_2,u_0} \otimes \widetilde{\mu_1 \otimes \mu_2}$, $\quad \mu_1),\ \mu_2 \in \mathrm{Irr}(\mathrm{Gl}_2(k))$
${}_3\mathcal{T}_2 = \mathcal{N}_2 D_2^{l_0,u_1}$	$\mathrm{ind}_{{}_3\mathcal{T}_2}^{\mathcal{T}_2}(\psi_{l_0,u_1} \otimes \widetilde{\mu})$, $\quad \mu \in \mathrm{Irr}(D_2^{l_0,u_1})$
${}_4\mathcal{T}_2 = \mathcal{N}_2 D_2^{l_1,u_1}$	$\mathrm{ind}_{{}_4\mathcal{T}_2}^{\mathcal{T}_2}(\psi_{l_1,u_1} \otimes \widetilde{\mu})$, $\quad \mu \in \mathrm{Irr}(D_2^{l_1,u_1})$
${}_5\mathcal{T}_2 = \mathcal{N}_2 D_2^{l_1',u_1}$	$\mathrm{ind}_{{}_5\mathcal{T}_2}^{\mathcal{T}_2}(\psi_{l_0,u_0} \otimes \widetilde{\mu})$, $\quad \mu \in \mathrm{Irr}(D_2^{l_1',u_1})$
${}_6\mathcal{T}_2 = \mathcal{N}_2 D_2^{l_2,u_1}$	$\mathrm{ind}_{{}_6\mathcal{T}_2}^{\mathcal{T}_2}(\psi_{l_2,u_1} \otimes \widetilde{\mu})$, $\quad \mu \in \mathrm{Irr}(D_2^{l_2,u_1})$
${}_7\mathcal{T}_2 = \mathcal{N}_2 D_2^{l_0,u_2}$	$\mathrm{ind}_{{}_7\mathcal{T}_2}^{\mathcal{T}_2}(\psi_{l_0,u_2} \otimes \widetilde{\mu_1 \otimes \mu_2})$, $\quad \mu_1,\ \mu_2 \in \mathrm{Irr}(\mathrm{Gl}_2(k))$
${}_8\mathcal{T}_2 = \mathcal{N}_2 D_2^{l_1,u_2}$	$\mathrm{ind}_{{}_8\mathcal{T}_2}^{\mathcal{T}_2}(\psi_{l_1,u_2} \otimes \widetilde{\mu})$, $\quad \mu \in \mathrm{Irr}(D_2^{l_1,u_2})$
${}_9\mathcal{T}_2 = \mathcal{N}_2 D_2^{l_2,u_2}$	$\mathrm{ind}_{{}_9\mathcal{T}_2}^{\mathcal{T}_2}(\psi_{l_2,u_2} \otimes \widetilde{\mu_1 \otimes \mu_2})$, $\quad \mu_1,\ \mu_2 \in \mathrm{Irr}(\mathrm{Gl}_2(k))$

Table 4.7: Irreducible Representations of $\mathcal{T}_2$

Lastly, the irreducible representations induced from ${}_3D_2^{l_2,u_1} = N_{l_2,u_1} T_{l_2,u_1}^{1,1}$ are of the form $\mathrm{ind}_{{}_3D_2^{l_2,u_1}}^{D_2^{l_2,u_1}}(\psi_{1,1} \otimes \widetilde{\rho})$.

Now that we have determined the irreducible representations of $D_2^{l,u}$ we may construct the irreducible representations of $\mathcal{D}_2$. These are shown in Table 4.7.

4.3 The Irreducible Representations of $\mathcal{T}_3$

As in the previous two cases we assume that u is in rank form.

Case 1: u has rank 0: If this is the case then X and Z may be chosen arbitrarily and we are reduced to the rank of l. We choose the representatives ψ_{u_0,l_0} and ψ_{u_0,l_1}.

Case 2: u has rank 1: In this case we may assume that l is in reduced row echelon form. It is clear that multiplication by an invertible matrix on the right does not change the rank of l. We therefore need only determine which rank 1 reduced row echelon forms for l may be obtained from another by multiplication on the right by an invertible matrix.

The rank 1 reduced row echelon forms for l are $\left[\begin{smallmatrix} 1 & \alpha & \beta \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 & 1 & \alpha \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} 0 & 0 & 1 \end{smallmatrix}\right]$, where α and β are in k. In order that $u_1 = (X^{-1})^t u_1 Z^t$ we get that X must be of the form

$$X = \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

If we take the product $\left[\begin{smallmatrix} 1 & \alpha & \beta \end{smallmatrix}\right] X^t$, then we see that we cannot get either of the other reduced row echelon forms for l (the first entry is 0 in both forms). If we take $X = \left[\begin{smallmatrix} 1 & 0 & 0 \\ -\alpha-\beta & 1 & 1 \\ -\beta & 0 & 1 \end{smallmatrix}\right]$, then we see that all $\left[\begin{smallmatrix} 1 & \alpha & \beta \end{smallmatrix}\right]$ are in the same orbit. We let ψ_{l_1,u_0} be the representative for this orbit.

For the reduced row echelon forms $\left[\begin{smallmatrix} 0 & 1 & \alpha \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} 0 & 0 & 1 \end{smallmatrix}\right]$ we consider the following invertible matrices

$$X_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \beta-\alpha & 1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\alpha & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$\psi_{l,u}$ representative	$D_3^{l,u} = \{d \in \mathcal{D}_3 \mid \psi_{l,u}^{(d)} = \psi_{l,u}\}$
ψ_{l_0,u_0}	$\mathcal{D}_3$
ψ_{l_1,u_0}	$\left\{ \begin{bmatrix} \begin{bmatrix} x & \alpha & \beta \\ & X & \end{bmatrix} & & & \\ & x & & \\ & & \begin{bmatrix} x & \alpha & \beta \\ & X & \end{bmatrix} & \\ & & & z \end{bmatrix} \mid x, z \in k^\times, \alpha, \beta \in k, \text{and } X \in Gl_2(k) \right\}$
ψ_{l_0,u_1}	$\left\{ \begin{bmatrix} \begin{bmatrix} x & \\ \gamma & X \\ \delta & \end{bmatrix} & & & \\ & y & & \\ & & \begin{bmatrix} x & \\ \gamma & X \\ \delta & \end{bmatrix} & \\ & & & x \end{bmatrix} \mid x, y \in k^\times, \gamma, \delta \in k, \text{and } X \in Gl_2(k) \right\}$
ψ_{l_1,u_1}	$\left\{ \begin{bmatrix} \begin{bmatrix} x & \\ & X \end{bmatrix} & & & \\ & x & & \\ & & \begin{bmatrix} x & \\ & X \end{bmatrix} & \\ & & & x \end{bmatrix} \mid x \in k^\times, \text{and } X \in Gl_2(k) \right\}$
ψ_{l_1',u_1}	$\left\{ \begin{bmatrix} \begin{bmatrix} x_{11} & & \\ x_{21} & x_{22} & \\ x_{31} & x_{32} & x_{33} \end{bmatrix} & & & \\ & x_{33} & & \\ & & \begin{bmatrix} x_{11} & & \\ x_{21} & x_{22} & \\ x_{31} & x_{32} & x_{33} \end{bmatrix} & \\ & & & x_{11} \end{bmatrix} \mid x_{11}, x_{22}, x_{33} \in k^\times \right\}$

Table 4.8: Stabilizers in $\mathcal{D}_3$ of $\psi_{l,u}$ representatives

If we take the form $[\, 0\ 1\ \beta \,]$, then multiplication of $[\, 0\ 1\ \alpha \,]$ on the right by X_1^t gives us $[\, 0\ 1\ \beta \,]$. If we multiply $[\, 0\ 1\ \alpha \,]$ on the right by X_2 we get $[\, 0\ 0\ 1 \,]$, so these reduced row echelon forms are in the same orbit. We let $\psi_{l_1',u_1} = \psi_{[0\ 0\ 1],u_1}$ be our representative. Lastly we let ψ_{l_0,u_1} be the representative for the rank 0 reduced row echelon form for l.

The stabilizers in $\mathcal{D}_3$ of each of the orbit representatives are given in Table 4.8. The irreducible representations of $\mathcal{D}_3$ are obtained by taking the tensor product of two characters of $k^\times$ and an irreducible representation of $Gl_3(k)$. The irreducible representations of $D_3^{l_1,u_1}$ are given by the tensor product of a character of $k^\times$ and an irreducible representation of $Gl_2(k)$. As $D_3^{l_0,u_1}$ and $D_3^{l_1,u_0}$ are isomorphic we need only find the irreducible representations of one of them. We choose $D_3^{l_1,u_0}$.

4.3.1 The Irreducible Representations of $D_3^{l_1,u_0}$

Let M and L be the following subgroups of $D_3^{l_1,u_0}$:

$$M = \left\{ m_{\alpha,\beta} = \begin{bmatrix} \begin{bmatrix} 1 & \alpha\ \beta \\ & I_2 \end{bmatrix} & & \\ & 1 & \\ & & \begin{bmatrix} 1 & \alpha\ \beta \\ & I_2 \end{bmatrix} & \\ & & & 1 \end{bmatrix} \mid \alpha, \beta \in k \right\}$$

$$L = \left\{ \begin{bmatrix} \begin{bmatrix} x & \\ & X \end{bmatrix} & & & \\ & x & & \\ & & \begin{bmatrix} x & \\ & X \end{bmatrix} & \\ & & & z \end{bmatrix} \mid x, z \in k^\times \text{and } X \in Gl_2(k) \right\}$$

M is an abelian subgroup that is isomorphic to $M_{1,2}(k)$. Each element in $D_3^{l_1,u_0}$ may be written as

$$\begin{bmatrix} \begin{bmatrix} 1 & [\alpha\ \beta]X^{-1} \\ & I_2 \end{bmatrix} & & & \\ & 1 & & \\ & & \begin{bmatrix} 1 & [\alpha\ \beta]X^{-1} \\ & I_2 \end{bmatrix} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x & \\ & X \end{bmatrix} & & & \\ & x & & \\ & & \begin{bmatrix} x & \\ & X \end{bmatrix} & \\ & & & z \end{bmatrix}.$$

Since $M \cap L$ is trivial we get that this product is unique and $D_3^{l_1,u_0} = ML$. If we conjugate an element in M by an element in L we get $m_{x^{-1}[\alpha\ \beta]X}$, thus M is a normal subgroup of $D_3^{l_1,u_0}$ and we have that $D_3^{l_1,u_0} \cong M \rtimes L$. We let $\widehat{M}$ denote the group of characters of M.

Let ψ be an nontrivial character of k. For each $[\,\alpha\ \beta\,] \in M_{1,2}(k)$ we define $\psi_{[\alpha\ \beta]}$ as follows:

$$\psi_{[\alpha\ \beta]}(m_{\alpha',\beta'}) = \psi(\mathrm{tr}([\,\alpha'\ \beta'\,]\,[\,\alpha\ \beta\,]^t))$$

the map $[\,\alpha\ \beta\,] \mapsto \psi_{[\alpha\ \beta]}$ is an L equivariant isomorphism by the same argument as before. The L orbits under conjugation in $\widehat{M}$ are determined by the equation

$$[\,\alpha'\ \beta'\,] = x^{-1}\,[\,\alpha\ \beta\,]\,X^t$$

therefore, the orbits in L are determined by the rank of $[\,\alpha\ \beta\,]$. We choose $\psi_{[0\ 0]}$ and $\psi_{[1\ 0]}$ as the representatives for these orbits.

The stabilizer of $\psi_{[0\ 0]}$ is L. The stabilizer in L of $\psi_{[1\ 0]}$, denoted L_1 contains elements of the form

$$\begin{bmatrix} \begin{bmatrix} x & & \\ & x & x_{12} \\ & & x_{22} \end{bmatrix} & & & \\ & x & & \\ & & \begin{bmatrix} x & & \\ & x & x_{12} \\ & & x_{22} \end{bmatrix} & \\ & & & z \end{bmatrix}.$$

We may construct the irreducible representations of $L \cong k^\times \times k^\times \times Gl_2(k)$ by way of the tensor product. It is left to determine the irreducible representations of L_1.

L_1 is isomorphic to the semidirect product of k with $k^\times \times k^\times \times k^\times$ therefore we may use the Method of Little Groups to determine the irreducible representations.

The orbits obtained by letting $k^\times \times k^\times \times k^\times$ act on k by $(x, y, z) \circ \alpha = \alpha y x^{-1}$ are determined by the invertibility of α. We therefore choose ψ_0 and ψ_1 as our representatives in $\widehat{k}$. The stabilizer of ψ_0 is $k^\times \times k^\times \times k^\times$ and the stabilizer of ψ_1 is $k^\times \times k^\times$. The irreducible representations of both of these stabilizers are determined by taking tensor products of characters of $k^\times$. As a result all irreducible representations of L_1 have the form $\psi_0 \otimes (\rho_1 \widetilde{\otimes \rho_2 \otimes} \rho_3)$ or $\operatorname{ind}_{L_1^1}^{L_1}(\psi_1 \otimes \widetilde{\rho_1 \otimes \rho_2})$ where $\rho_j \in \widehat{k^\times}$. L_1^1 is the product of the groups N and T shown below:

$$N = \left\{ \begin{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & \alpha & \\ & & 1 & \\ & & & 1 \end{bmatrix} & \\ & \begin{bmatrix} 1 & & & \\ & 1 & \alpha & \\ & & 1 & \\ & & & 1 \end{bmatrix} \end{bmatrix} \mid \alpha \in k \right\}$$

$$T = \left\{ \begin{bmatrix} \begin{bmatrix} x & & & \\ & x & & \\ & & x & \\ & & & x \end{bmatrix} & \\ & \begin{bmatrix} x & & & \\ & x & & \\ & & x & \\ & & & z \end{bmatrix} \end{bmatrix} \mid x, z \in k^\times \right\}.$$

The irreducible representations of $D_3^{l_1,u_0}$ obtained from the subgroup ${}_0D_3^{l_1,u_0} =$

ML_0 are of the form $\psi_{[0\ 0]} \otimes \widetilde{(\rho \otimes \tau)}$. The irreducible representations induced from ${}_0D_3^{l_1,u_0} = ML_1$ are of the form $\operatorname{ind}_{{}_1D_3^{l_1,u_0}}^{D_3^{l_1,u_0}}(\psi_{[1\ 0]} \otimes \tilde{\sigma})$, where σ is an irreducible representation of L_1.

4.3.2 The Irreducible Representations of $D_3^{l_1',u_1}$

Let M and L be the following subgroups of $D_3^{[0\ 0\ 1],u_1}$

$$M = \left\{ \begin{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ x_{31} & x_{32} & 1 & \\ & & & 1 \end{bmatrix} & \\ & \begin{bmatrix} 1 & & & \\ & 1 & & \\ x_{31} & x_{32} & 1 & \\ & & & 1 \end{bmatrix} \end{bmatrix} \right\}$$

$$L = \left\{ \begin{bmatrix} \begin{bmatrix} x_{11} & & & \\ x_{21} & x_{22} & & \\ & & x_{33} & \\ & & & x_{33} \end{bmatrix} & \\ & \begin{bmatrix} x_{11} & & & \\ x_{21} & x_{22} & & \\ & & x_{33} & \\ & & & x_{11} \end{bmatrix} \end{bmatrix} \right\}.$$

One observes that $M \cong (M_{1,2}(k), +)$. By arguments similar to those given before we have that $D_3^{[0\ 0\ 1],u_1} \cong M \rtimes L$. We also have an L-equivariant isomorphism from $M_{1,2}(k)$ onto $\widehat{M}$, the group of characters of M.

The orbits in $\widehat{M}$ given by conjugation by elements of L are determined by the equation

$$\begin{bmatrix} \alpha' & \beta' \end{bmatrix} = x_{33}^{-1} \begin{bmatrix} \alpha & \beta \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} \\ & x_{22} \end{bmatrix}.$$

We choose our representatives to be $\psi_{[0\ 0]}$, $\psi_{[1\ 0]}$, and $\psi_{[0\ 1]}$ (we define these characters as we did in previous cases).

The stabilizer in L of $\psi_{[0\ 0]}$ is L. The stabilizer in L of $\psi_{[1\ 0]}$ consists of matrices of the form

$$\begin{bmatrix} \begin{bmatrix} x_{11} & & & \\ & x_{22} & & \\ & & x_{11} & \\ & & & x_{11} \end{bmatrix} & \\ & \begin{bmatrix} x_{11} & & & \\ & x_{22} & & \\ & & x_{11} & \\ & & & x_{11} \end{bmatrix} \end{bmatrix}$$

and the stabilizer in L of $\psi_{[0\ 1]}$ consists of matrices of the form

$$\begin{bmatrix} \begin{bmatrix} x_{11} & & & \\ x_{21} & x_{22} & & \\ & & x_{22} & \\ & & & x_{22} \end{bmatrix} & \\ & \begin{bmatrix} x_{11} & & & \\ x_{21} & x_{22} & & \\ & & x_{22} & \\ & & & x_{11} \end{bmatrix} \end{bmatrix}.$$

The irreducible representations of $L_{[1\ 0]}$ are obtained using the tensor product. We note that $L_{[0\ 0]} \cong (k \rtimes (k^\times \times k^\times) \times k^\times)$ and $L' = L_{[0\ 1]} \cong k \rtimes (k^\times \times k^\times)$, therefore, if we determine the irreducible representations of $L_{[0\ 1]}$, then we will be able to determine the irreducible representations for each of our stabilizer groups via the tensor product.

4.3.2.1 The Irreducible Representations of L'

We use the Method of Little Groups to determine the representations in this case. We let N and T be the following subgroups of $L' = L_{[0\ 1]}$.

$$N = \left\{ \begin{bmatrix} \begin{bmatrix} 1 & & & \\ x_{21} & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} & \\ & \begin{bmatrix} 1 & & & \\ x_{21} & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \end{bmatrix} \right\}$$

$$T = \left\{ \begin{bmatrix} \begin{bmatrix} x_{11} & & & \\ & x_{22} & & \\ & & x_{22} & \\ & & & x_{22} \end{bmatrix} & \\ & \begin{bmatrix} x_{11} & & & \\ & x_{22} & & \\ & & x_{22} & \\ & & & x_{11} \end{bmatrix} \end{bmatrix} \right\}$$

Then $L' \cong N \rtimes T$. $N \cong k$ and we define a T-equivariant isomorphism between k and $\widehat{N}$ by $\alpha \mapsto \psi_\alpha$. The T orbits in $\widehat{N}$ are determined by the equation

$$\alpha' = x_{11}\alpha x_{22}^{-1}.$$

The orbits are therefore determined by the invertibility of α. We choose ψ_0 and ψ_1 as our representatives. The stabilizer in T of ψ_0 is T and the stabilizer of ψ_1 consists of matrices of the form

$$\begin{bmatrix} \begin{bmatrix} x_{11} & & & \\ & x_{11} & & \\ & & x_{11} & \\ & & & x_{11} \end{bmatrix} & \\ & \begin{bmatrix} x_{11} & & & \\ & x_{11} & & \\ & & x_{11} & \\ & & & x_{11} \end{bmatrix} \end{bmatrix}.$$

The irreducible representations of $L_{[0\ 1]}$ have one of the following forms

1. $\psi_0 \otimes (\widetilde{\rho_1 \otimes \rho_2})$, where $\rho_j \in \widehat{k^\times}$.

2. $\mathrm{ind}_{NT_1}^{L_{[0\ 1]}}(\psi_1 \otimes \widetilde{\rho})$, where $\rho \in \widehat{k^\times}$.

We obtain representations of $L_{[0\ 0]}$ by taking the tensor product of these representations with an element of $\widehat{k^\times}$.

Now that we have determined the irreducible representations of $L_{[0\ 0]}$, $L_{[0\ 1]}$, and $L_{[1\ 0]}$ we know that the irreducible representations of $D_3^{l_1',u_1}$ have one of the following forms:

1. $\psi_{[0\ 0]} \otimes \widetilde{\mu}$, where μ is an irreducible representation of $L_{[0\ 0]}$.

2. $\mathrm{ind}_{ML_{[1\ 0]}}^{D_3^{l_1',u_1}}(\psi_{[1\ 0]} \otimes \widetilde{\rho_1 \otimes \rho_2})$, where $\rho_j \in \widehat{k^\times}$.

3. $\mathrm{ind}_{ML_{[0\ 1]}}^{D_3^{l_1',u_1}}(\psi_{[0\ 1]} \otimes \widetilde{\mu})$, where $\mu \in \mathrm{Irr}(L_{[0\ 1]})$.

Now let ${}_0\mathcal{T}_3 = \mathcal{N}_3 D_3^{l_0,u_0}$, ${}_1\mathcal{T}_3 = \mathcal{N}_3 D_3^{l_1,u_0}$, ${}_2\mathcal{T}_3 = \mathcal{N}_3 D_3^{l_0,u_1}$, ${}_3\mathcal{T}_3 = \mathcal{N}_3 D_3^{l_1,u_1}$, and ${}_4\mathcal{T}_3 = \mathcal{N}_3 D_3^{l_1',u_1}$. The irreducible representations of $\mathcal{T}_3$ are obtained as follows:

$\underline{{}_0\mathcal{T}_3}$: The irreducible representations of $\mathcal{T}_3$ are of the form $\psi_{l_0,u_0} \otimes (\rho_1 \widetilde{\otimes \rho_2} \otimes \tau)$ where $\rho_j \in \widehat{k^\times}$ and $\tau \in \mathrm{Irr}(Gl_3(k))$.

$\underline{{}_1\mathcal{T}_3}$: The irreducible representations of $\mathcal{T}_3$ are of the form $\mathrm{ind}_{{}_1\mathcal{T}_3}^{\mathcal{T}_3}(\psi_{l_1,u_0} \otimes \widetilde{\mu})$ where $\mu \in \mathrm{Irr}(D_3^{l_1,u_0})$.

$\underline{{}_2\mathcal{T}_3}$: The irreducible representations of $\mathcal{T}_3$ are of the form $\mathrm{ind}_{{}_2\mathcal{T}_3}^{\mathcal{T}_3}(\psi_{l_0,u_1} \otimes \widetilde{\mu})$ where

$\mu \in \mathrm{Irr}(D_3^{l_0,u_1})$.

$\underline{{}_3\mathcal{T}_3}$: The irreducible representations of $\mathcal{T}_3$ are of the form $\mathrm{ind}_{{}_3\mathcal{T}_3}^{\mathcal{T}_3}(\psi_{l_1,u_1} \otimes \widetilde{\rho \otimes \tau})$ where $\rho \in \widehat{k^\times}$ and $\tau \in \mathrm{Irr}(\mathrm{Gl}_2(k))$.

$\underline{{}_4\mathcal{T}_3}$: The irreducible representations of $\mathcal{T}_3$ are of the form $\mathrm{ind}_{{}_4\mathcal{T}_3}^{\mathcal{T}_3}(\psi_{l_1',u_1} \otimes \widetilde{\mu})$ where $\mu \in \mathrm{Irr}(D_3^{l_1',u_1})$.

The only task left for us in this case is to give the irreducible representations of P. Since we have determined all of the irreducible representations of $\mathcal{T}_0$, $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, and $\mathcal{T}_4$, this is done using Proposition 2.2. More precisely, if π is an irreducible representation of P, then π has one of the following forms:

1. $\chi_0 \otimes (\widetilde{\tau_1 \otimes \tau_2})$ where τ_1, $\tau_2 \in \mathrm{Irr}(Gl_4(k))$.

2. $\mathrm{ind}_{\mathcal{N}T_1}^{P}(\chi_1 \otimes \widetilde{\mu})$, where μ is an irreducible representation of $\mathcal{T}_1$.

3. $\mathrm{ind}_{\mathcal{N}T_2}^{P}(\chi_2 \otimes \widetilde{\mu})$, where μ is an irreducible representation of $\mathcal{T}_2$.

4. $\mathrm{ind}_{\mathcal{N}T_3}^{P}(\chi_3 \otimes \widetilde{\mu})$, where μ is an irreducible representation of $\mathcal{T}_3$.

5. $\mathrm{ind}_{\mathcal{N}T_4}^{P}(\chi_4 \otimes \widetilde{\tau})$, where τ is an irreducible representation of $Gl_4(k)$.

where χ_i is as defined at the beginning of this chapter.

CHAPTER 5
FUTURE RESEARCH

In this thesis we have explored the representation theory for our group $Sl_*(2,\mathcal{A})$ in the cases n=1 and n=2. Ultimately, we would like to find a Weil-like representation for our group when we take k to be a p-adic field. We believe that answering following questions can lead to solving this problem.

5.1 The n-dimensional case, $n \geq 3$, k a finite field

In this thesis we have determined the representations for the cases $n = 1$ and $n = 2$. We would like to determine the representations for an arbitrary $n \geq 3$. The process that we have used for cases $n = 1, 2$ seems promising in this case as well.

The reduced row echelon forms of matrices played a particularly important role in the case where $n = 2$. I believe that these will also be instrumental to solving the general case. We will also need the representation theory for $Gl_n(k)$ for an arbitrary n. This is covered in [4].

5.2 A Bruhat Presentation for $Sl_*(2, A)$

In [11] Pantoja and Soto-Andrade determine that if A is a simple artinian ring with involution, then there is a Bruhat presentation for $SSl_*(2, A)$ (See definition 2.18). They also show that the index of $SSl_*(2, A)$ in $Sl_*(2, A)$ is at most 2. In our case it follows from [10] that this index is in fact 1. We would like to find out whether we have a Bruhat presentation in our case. If this is the case then we may construct a generalized Weil representation as in [6].

5.3 Representations $Sl_*(2, A)$, k a p-adic field

As mentioned before, the representation theory of $Sl_*(2, M(n, k))$, k a locally compact topological field, gives that of $Sp(2n, k)$. This group is of particular importance in number theory. We would like to determine if $Sl_*(2, \mathcal{A})$, where k is a p-adic field, plays a similar role.

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