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Explicit plancherel measure for PGL_2(F)

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EXPLICIT PLANCHEREL MEASURE FOR $\mathrm{PGL}_2(\mathbb{F})$

by

Carlos De la Mora

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
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Thesis Supervisor: Professor Philp Kutzko

ABSTRACT

In this thesis we compute an explicit Plancherel fromula for $\mathrm{PGL}_2(\mathbb{F})$ where $\mathbb{F}$ is a non-archimedean local field by a method developed by Busnell, Henniart and Kutzko. Let G be connected reductive group over a non-archimedean local field $\mathbb{F}$. We show that we can obtain types and covers (as defined by Bushnell and Kutzko in "Smooth representations of reductive p-adic groups: structure theory via types" Pure Appl. Math, 2009.) for G/Z coming from types and covers of G in a very explicit way. We then compute those types and covers for $\mathrm{GL}_2(\mathbb{F})$ which give rise to all types and covers for $\mathrm{PGL}_2(\mathbb{F})$ that are in the principal series. The Bernstein components $\bar{\mathfrak{s}}$ of $\mathrm{PGL}_2(\mathbb{F})$ that correspond to the principal series are of the form $[\bar{\mathbb{T}}, \varphi]_{\bar{G}}$ where $\bar{\mathbb{T}}$ is the diagonal matrices in $\mathrm{GL}_2(\mathbb{F})$ modulo the center and φ can be seen as a smooth character for $\mathbb{F}^{\times}$. Let $(\bar{J}, \bar{\lambda}_{\varphi})$ be a $\bar{\mathfrak{s}}$-type. Then the Hecke algebra $\mathcal{H}(\bar{G}, \bar{\lambda}_{\varphi})$ is a Hilbert algebra and has a measure associated to it called Plancherel measure of $\mathcal{H}(\bar{G}, \bar{\lambda}_{\varphi})$. We show that computing the Plancherel measure for $\mathrm{PGL}_2(\mathbb{F})$ essentially reduces to computing the Plancherel measure for $\mathcal{H}(\bar{G}, \bar{\lambda}_{\varphi})$ for every type $(\bar{J}, \bar{\lambda}_{\varphi})$. We get that the Hilbert algebras $\mathcal{H}(\bar{G}, \bar{\lambda}_{\varphi})$ come in two flavors; they are either $\mathbb{C}[\mathbb{Z}]$ or they are a free algebra in two generators s_1, s_2 subject to the relations $s_1^2 = 1$ and $s_2^2 = (q^{-1/2} - q^{-1/2})s_2 + 1$ and we denote this algebra by $\mathcal{H}(q, 1)$. The Plancherel measure for $\mathbb{C}[\mathbb{Z}]$ as well as the Plancherel measure for $\mathcal{H}(q, 1)$ are known.

Abstract Approved: ______________________________

Thesis Supervisor

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A thesis submitted in partial fulfillment of the
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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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Para mi papá Arturo De la Mora,
de quien heredé el gusto por las matemáticas

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ABSTRACT

In this thesis we compute an explicit Plancherel fromula for $\mathrm{PGL}_2(\mathbb{F})$ where $\mathbb{F}$ is a non-archimedean local field by a method developed by Busnell, Henniart and Kutzko. Let G be connected reductive group over a non-archimedean local field $\mathbb{F}$. We show that we can obtain types and covers (as defined by Bushnell and Kutzko in "Smooth representations of reductive p-adic groups: structure theory via types" Pure Appl. Math, 2009.) for G/Z coming from types and covers of G in a very explicit way. We then compute those types and covers for $\mathrm{GL}_2(\mathbb{F})$ which give rise to all types and covers for $\mathrm{PGL}_2(\mathbb{F})$ that are in the principal series. The Bernstein components $\bar{\mathfrak{s}}$ of $\mathrm{PGL}_2(\mathbb{F})$ that correspond to the principal series are of the form $[\bar{\mathbb{T}}, \varphi]_{\bar{G}}$ where $\bar{\mathbb{T}}$ is the diagonal matrices in $\mathrm{GL}_2(\mathbb{F})$ modulo the center and φ can be seen as a smooth character for $\mathbb{F}^{\times}$. Let $(\bar{J}, \bar{\lambda}_{\varphi})$ be a $\bar{\mathfrak{s}}$-type. Then the Hecke algebra $\mathcal{H}(\bar{G}, \bar{\lambda}_{\varphi})$ is a Hilbert algebra and has a measure associated to it called Plancherel measure of $\mathcal{H}(\bar{G}, \bar{\lambda}_{\varphi})$. We show that computing the Plancherel measure for $\mathrm{PGL}_2(\mathbb{F})$ essentially reduces to computing the Plancherel measure for $\mathcal{H}(\bar{G}, \bar{\lambda}_{\varphi})$ for every type $(\bar{J}, \bar{\lambda}_{\varphi})$. We get that the Hilbert algebras $\mathcal{H}(\bar{G}, \bar{\lambda}_{\varphi})$ come in two flavors; they are either $\mathbb{C}[\mathbb{Z}]$ or they are a free algebra in two generators s_1, s_2 subject to the relations $s_1^2 = 1$ and $s_2^2 = (q^{-1/2} - q^{-1/2})s_2 + 1$ and we denote this algebra by $\mathcal{H}(q, 1)$. The Plancherel measure for $\mathbb{C}[\mathbb{Z}]$ as well as the Plancherel measure for $\mathcal{H}(q, 1)$ are known.

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CHAPTER 1
INTRODUCTION

Let G be a unimodular, locally compact group with Haar measure μ_G. The space $L^1(G)$ becomes an involutive Banach Algebra with involution given by $f^*(s) = \overline{f(s^{-1})}$ for $f \in L^1(G)$, $s \in G$. For a neighbourhood basis of the identity $\mathscr{V}$ construct the positive functions $u_i \in L^1(G)$ for $i \in \mathscr{V}$ such that the support of u_i is in i and the integral equals 1. Then $\{u_i\}_{i\in\mathscr{V}}$ is an approximate identity (see Definition 3.3 for the definition of approximate identity). We can then form the enveloping C^*-algebra of $L^1(G)$ which we denote by $C^*(G)$. (See proposition 3.3 for the construction of the enveloping C^*-algebra).

Let A be a C^*-algebra. We can give to the set of equivalence classes of unitary representations of A a topology; we denote this toplogical space by $\hat{A}$ and refer to it as the spectrum of A. (See Definition 3.8 for the definition of spectrum). In the case where A is a separable, i.e. has a countable dense subset, we can give to the set $\hat{A}$ a Borel structure that we call the Mackey Borel structure (see Definition 3.9 for definition of Borel structure). In the case where A is postliminal we obtain that the Mackey Borel structure on $\hat{A}$ is subordinate to its topology. (see section 3.2 for the definition of Borel structure subordinate to a topology).

If G is separable then $C^*(G)$ is separable. The set of equivalence classes of irreducible unitary representations of G which we denote by $\widehat{G}$ is in one to one correspondence with the set $\widehat{C^*}(G)$ of equivalence classes of unitary representations of $C^*(G)$. We use this correspondence to transfer the topology of $\widehat{C^*}(G)$ to a topology in $\widehat{G}$. Similarly in the case where G is separable we use this correspondence to give $\widehat{G}$ a Mackey Borel structure.

In theorem 3.16 we state that a topological group G that is separable, locally compact, unimodular and such that the enveloping C^*-algebra of G is postliminal has a unique measure ν on the set $\widehat{G}$ (which depends on a Haar measure μ_G) such that for every $u \in L^1(G) \cap L^2(G)$

$$\int_G |u(s)|^2 d\mu_G(s) = \int_{\widehat{G}} tr(\zeta(u)\zeta(u)^*) d\nu(\zeta).$$

We call the measure ν is the *Plancherel measure.*

Let us assume in what follows that G is a connected reductive group defined over a non-archimedean local field $\mathbb{F}$. Then G is totally disconnected and we may form the Hecke algebra $\mathcal{H}(G)$ of all locally constant, compactly supported functions $f : G \to \mathbb{C}$ with the operation of convolution. (See 2.1.2 for the definition of $\mathcal{H}(G)$). According to a theorem of Bernstein [3], G is postliminal. Also G may be embedded in some $\mathrm{GL}_N(\mathbb{F})$ for some positive integer N showing that G is separable since $\mathrm{GL}_N(\mathbb{F})$ is separable [10, p. 121]. It is also well known that a connected reductive group defined over a non-archimedean local field $\mathbb{F}$, is unimodular [10, p. 121]; therefore, satisfies all the conditions for the existence of the Plancherel measure. In this case we can characterize the Plancherel measure in a slightly different and more convenient way. Indeed,

Theorem. *The Plancherel measure ν the unique measure on $\widehat{G}$ such that*

$$f(1) = \int_{\hat{G}} tr(\zeta(f)) d\nu(\zeta), \;\; f \in \mathcal{H}(G).$$

There has been a considerable amount of research on the topics related to the Plancherel measure for connected reductive groups over a non-archimedean local field. One of the most important papers is by Waldspurger following the work of Harish-Chandra [13]. The Plancherel Measure is also used to obtain the celebrated Arthur's Trace formula [1]. The main objective of this thesis is to compute the Plancherel Measure for $\mathrm{PGL}_2(\mathbb{F})$ where $\mathbb{F}$ is a non-archimedean local field, following the paper [5] by Bushnell, Henniart and Kutzko.

1.1 The methodology

Let $\mathfrak{R}(G)$ be the category of smooth representations of G. The Bernstein decomposition (theorem 2.17) gives us that $\mathfrak{R}(G)$ can be decomposed as a product of subcategories $\mathfrak{R}^{\mathfrak{s}}(G)$:

$$\mathfrak{R}(G) = \prod_{\mathfrak{s}\in\mathscr{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G).$$

Here the set $\mathscr{B}(G)$ is given by equivalence classes of pairs (L, σ), where L is a Levi component of a parabolic subgroup of G and σ is a supercuspidal representation of L; the equivalence relation is that of inertial support (see Definition 2.7 for the definition of inertial support). The symbol $\mathfrak{R}^{\mathfrak{s}}(G)$ denotes the full subcategory of $\mathfrak{R}(G)$ where a smooth representation (π, V) is an object of $\mathfrak{R}^{\mathfrak{s}}(G)$ if and only if every irreducible subquotient (τ, E) of (π, V) has inertial support in $\mathfrak{s}$.

Let K be a compact open subgroup of G and let (ρ, W) be an irreducible K-representation. Let $(\check{\rho}, \check{W})$ be the contragradient representation (see subsection 2.1.1 for the definition of contragradient). Let $\mathcal{H}(G, \rho)$ denote the space of compactly supported functions $f : G \longrightarrow End_{\mathbb{C}}(\check{W})$ such that

$$f(k_1gk_2) = \check{\rho}(k_1)f(g)\check{\rho}(k_2), \text{where } k_1, k_2 \in K,\ \ g \in G.$$

The operation of convolution

$$f_1 \star f_2(g) = \int_G f_1(x)f_2(x^{-1}g)d\mu_G(x)\ f_1, f_2 \in \mathcal{H}(G, \rho),$$

gives $\mathcal{H}(G, \rho)$ the structure of an associative complex algebra with identity. We can give to the algebra $\mathcal{H}(G, \rho)$ a structure of a normalized Hilbert algebra (see section 4.4). We can associate to the Hilbert algebra $\mathcal{H}(G, \rho)$ a C^*-algebra that we denote by ${}_rC^*(\mathcal{H}(G, \rho))$ and a Plancherel measure $\nu_{\mathcal{H}(G,\rho)}$ on ${}_r\widehat{C^*}(\mathcal{H}(G, \rho))$ (see theorem 4.6 for the definition of the Plancherel measure for Hilbert algebras).

Let (π, V) be a smooth representation of G and let K and (ρ, W) be as above; the space $\mathrm{Hom}_K(\rho, V)$ has the structure of a $\mathcal{H}(G, \rho)$ module (see lemma 2.21). Given an $\mathfrak{s} \in \mathscr{B}(G)$ there is a functor $\mathbf{M}_\rho : \mathfrak{R}^{\mathfrak{s}}(G) \to \mathcal{H}(G, \rho)$-Mod given by $\mathbf{M}_\rho(\pi, V) = \mathrm{Hom}_K(\rho, V)$. The pair (K, ρ) is an $\mathfrak{s}$-type if an only if the functor $\mathbf{M}_\rho : \mathfrak{R}^{\mathfrak{s}}(G) \to \mathcal{H}(G, \rho)$-Mod gives an equivalence of categories.

Let (π, V) be an irreducible unitary representation of G. Then the representation (π^∞, V^∞) is an irreducible smooth representation of G (see subsection 2.1.1 for the definition of (π^∞, V^∞)). If we denote by ${}_r\widehat{G}$ the subset of $\widehat{G}$ that supports the Plancherel measure then the Bernstein decomposition induces a disjoint union

$${}_r\widehat{G} = \dot{\bigcup}_{\mathfrak{s}\in\mathscr{B}}\ {}_r\widehat{G}(\mathfrak{s})$$

where ${}_r\widehat{G}(\mathfrak{s})$ is the set of all $(\pi, V) \in {}_r\hat{G}$ such that $(\pi^\infty, V^\infty) \in \mathfrak{R}^{\mathfrak{s}}(G)$. The sets ${}_r\widehat{G}(\mathfrak{s})$ are open in ${}_r\widehat{G}$; we conclude that a subset S of ${}_r\widehat{G}$ is Borel if and only if ${}_r\widehat{G}(\mathfrak{s}) \cap S$

is Borel for every $\mathfrak{s}$. The functor $\mathbf{M}_\rho$ induces a homeomorphism $\widehat{\mathbf{M}}_\rho : {}_r\widehat{G}(\mathfrak{s}) \longrightarrow {}_r\widehat{C^*}(\mathcal{H}(G,\rho))$ where $\widehat{\mathbf{M}}_\rho((\pi, V)) = \mathbf{M}_\rho((\pi^\infty, V^\infty))$. Moreover, if S is a Borel subset of ${}_r\hat{G}(\mathfrak{s})$ and ν is the Plancherel measure associated to the Haar measure μ_G, then

$$\nu(S) = \frac{\dim\rho}{\mu_G(K)} \nu_{\mathcal{H}(G,\rho)}(\widehat{\mathbf{M}}_\rho(S)).$$

It follows that in order to obtain an explicit Plancherel measure for G it is enough, for every $\mathfrak{s} \in \mathscr{B}(G)$, to obtain an explicit Plancherel measure $\nu_{\mathcal{H}(G,\rho)}$ where (K, ρ) is an $\mathfrak{s}$-type.

1.2 The New Results on the Thesis

Let G be a connected reductive group defined over a non-archimedean local field $\mathbb{F}$. We denote by $\mathcal{O}$ the ring of integers in $\mathbb{F}$ with maximal prime ideal $\mathfrak{p}$. Let q be the order of the residue field $\mathcal{O}/\mathfrak{p}$, denote by Z the center of G and by $\bar{G}$ the quotient group G/Z. Let J be a compact open subgroup of G and ρ be an irreducible representation of J such that $\rho(z)$ is the identity for all $z \in Z \cap J$. The map from $\mathcal{H}(G,\rho)$ into $\mathcal{H}(\bar{G},\bar{\rho})$ given by $f \to \bar{f}$ where $\bar{f}(\bar{x}) = \int_Z f(zx)\mu_Z(z)$ and μ_Z is the Haar measure on Z is a homomorphism of involutive algebras such that $supp(\bar{f}) \subset \overline{supp(f)}$. It is because of this homomorphism of involutive algebras that we where able to prove the following theorem (theorem 5.6 in this thesis).

Theorem. *Let $L \subset M$ be two Levi components of parabolic subgroups of G. Let $\mathfrak{s}_M \in \mathscr{B}(M)$ and suppose there is a pair $(L, \sigma) \in \mathfrak{s}_M$ such that σ is trivial in Z. If (J_M, τ_M) is a $\mathfrak{s}_M$-type and (J, τ) is a G cover for (J_M, τ_M) then $(\bar{J}, \bar{\tau})$ is a $\bar{G}$ cover for $(\bar{J}_M, \bar{\tau}_M)$ and $(\bar{J}, \bar{\tau})$ is an $[\bar{\mathfrak{s}}]_{\bar{G}}$-type.*

1.2.1 The case for $\mathrm{PGL}_2(\mathbb{F})$

Let us consider the case where $G = \mathrm{GL}_2(\mathbb{F})$. Let φ be a character of $\mathbb{F}^\times$ and let $\mathbb{T}$ denote the diagonal matrices in G. Let $sw(\varphi)$ be the least integer n such that, $n \geq 1$ and $1 + \mathfrak{p}^n \subset \ker \varphi$. Let $J = J_\varphi$ be the compact subgroup of G given by:

$$J = \{[c_{ij}] \in G |\ c_{11}, c_{22} \in \mathcal{O}^\times,\ c_{12} \in \mathcal{O},\ c_{21} \in \mathfrak{p}^{sw(\varphi^2)}\}$$

and define the representation $\lambda = \lambda_\varphi$ on J by $\lambda([c_{ij}]) = \varphi^2(c_{11})\varphi^{-1}(\det[c_{ij}])$. We then get that (J, λ_φ) is a $\mathfrak{s} = [\mathbb{T}, \varphi \otimes \varphi^{-1}]_G$-type (see proposition 5.2 and 5.3 for the proof of the last statement). Therefore $(\bar{J}, \bar{\lambda}_\varphi)$ is a $\bar{\mathfrak{s}} = [\bar{\mathbb{T}}, \varphi]_{\bar{G}}$-type.

Let $q_1 \geq q_2 \geq 1$ be two fixed real numbers and set $\gamma_i = q_i^{1/2}, c_i = \gamma_i - \gamma_i^{-1}$ for $i = 1, 2$. We let $\mathcal{H}(q_1, q_2)$ be the complex algebra with identity $\mathbf{1}$ and two generators $s_i,\ i = 1, 2$ subject to only the relations

$$s_i^2 = c_i s_i + 1,\ i = 1, 2.$$

The algebra $\mathcal{H}(q_1, q_2)$ can be given a structure of Hilbert algebra. Kutzko and Morris [9] compute an explicit Plancherel measure for $\mathcal{H}(q_1, q_2)$. In particular from the results in this paper is easy to deduce the Plancherel measure for $\mathcal{H}(q, 1)$ where q is the number of elements in the residue field of $\mathbb{F}$.

Theorem. *Let $L(1, s) = (1 - q^{-s})^{-1}$. The Plancherel measure $\nu_{\mathcal{H}(q,1)}$ of the Hecke algebra $\mathcal{H}(q, 1)$ can be identified with the interval $[0, \frac{\pi}{\ln q}]$ and a measure $\frac{\ln(q)}{2\pi} P dt$ where dt gives the Lebesgue measure in the interval and*

$$P(t) = q^{-1} \frac{L(1, 1 - 2it) L(1, 1 + 2it)}{L(1, -2it) L(1, 2it)}$$

Union two points ρ_1, ρ_2 *where*

$$\nu_{\mathcal{H}(q,1)}(\rho_1) = \nu_{\mathcal{H}(q,1)}(\rho_2) = \frac{1}{2}\left(\frac{q-1}{q+1}\right)$$

If we let that $\mu_{\bar{G}}(\bar{J}_\varphi) = 1$ we prove at the end of Chapter 5 that for $\varphi^2|_{\mathcal{O}^\times} \neq 1$ the Hecke algebra $\mathcal{H}(\bar{G}, \bar{\lambda}_\varphi)$ is isomorphic to $\mathbb{C}[\mathbb{Z}]$ as Hilbert algebras and that for $\varphi^2|_{\mathcal{O}^\times} = 1$ the Hecke algebra $\mathcal{H}(\bar{G}, \bar{\lambda}_\varphi)$ is isomorphic to $\mathcal{H}(q,1)$. We then translate the Plancherel measure of $\mathbb{C}[\mathbb{Z}]$ and $\mathcal{H}(q,1)$ to the corresponding ${}_r\widehat{G}(\mathfrak{s})$ to obtain an explicit Plancherel measure for $\mathrm{PGL}_2(\mathbb{F})$.

CHAPTER 2
SMOOTH REPRESENTATIONS

2.1 Locally Profinite groups

Definition 2.1. *A* locally profinite group *is a topological group G where the set of compact open subgroups forms a neighbourhood basis around the identity.*

A locally profinite group is locally compact totally disconnected. It can be shown that a locally compact totally disconnected group is locally porfinite.

Example 2.1. Take $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, where $\overline{\mathbb{Q}}$ denotes the algebraic closure of the rational numbers. Give $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ the Krull topology. This group is profinite and thus compact.

Example 2.2. Let $G = G(\mathbb{F})$ be a reductive connected group over a non archimedian local field $\mathbb{F}$.

2.1.1 Smooth representations

Let G be a topological group and let (π, V) be a (complex) representation for G. If K is an open subgroup of G, we define the set $V^K = \{v \in V |\ \pi(k)v = v \ \forall\ k \in K\}$. We also define the *smooth subspace* $V^\infty = \bigcup_K V^K$. Where K ranges over all open subgroups of G. We then get that V^∞ is G stable. We denote restriction of π to V^∞ by (π^∞, V^∞).

Definition 2.2. *Let G be a topological group, a representation (π, V) of G is called a* smooth representation of G *if $(\pi^\infty, V^\infty) = (\pi, V)$*

The category $\mathfrak{R}(G)$ of smooth representations is a full subcategory of the category of representations. It can be shown that direct sums and finite tensor products of smooth representations are smooth. We also have that for a family of smooth representations $\{V_i\}_{i\in I}$ the direct product in the category of smooth representation is $(\Pi_{i\in I}V_i)^{\infty}$.

If (π, V) is a G smooth representation we can form a representation acting on the space of linear functionals $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. For $f \in \mathrm{Hom}(V, \mathbb{C})$ and $v \in V$ let us denote by $< f, v >= f(v)$. We have a G action on the space $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ given by $< g \cdot f, v >=< f, \pi(g)^{-1}v >$. We set $\check{V} = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and denote the resulting smooth representation by $(\check{\pi}, \check{V})$. We refer to it as the *contragradient representation.*

A smooth representation (π, V) of G is said to be *admissible* if $dim(V^K) < \infty$ for any compact open subgroup K of G. For an admissible representation (π, V) we have a canonical isomorphism into its double dual $(\check{\check{\pi}}, \check{\check{V}})$ obtained by the map $v \mapsto \check{\check{v}}$, where $< \check{\check{v}}, \check{w} >=< \check{w}, v >$ for all $v \in V,\ w \in \check{V}$

Let (π, V) be a smooth representation of G, for $\zeta \in V$ and $\check{\zeta} \in \check{V}$. We define the function $f_{\zeta,\check{\zeta}}(g) =< \check{\zeta}, \pi(g^{-1})\zeta >$. We refer to the function $f_{\zeta,\check{\zeta}}$ as a *matrix coefficient.*

2.1.2 Hecke Algebra $\mathcal{H}(G)$

Recall that the category of complex representations of a finite group G may be identified with the category of left modules over the group algebra $\mathbb{C}[G]$. We have an analogous statement for the category $\mathfrak{R}(G)$ for a locally profinite group G.

Let G be a locally profinite group, since G is a locally compact group it has a Haar measure unique up to a constant. Let us fix a Haar measure μ_G for G. Take the set $\mathcal{H}(G)$ to be set of all functions $f : G \longrightarrow \mathbb{C}$ with the properties:

1) there exist a compact open subgroup K of G such that $f(gk) = f(g)$ for all $k \in K$

2) f has compact support.

For $f_1, f_2 \in \mathcal{H}(G)$ we define the function $f_1 \star f_2(y) = \int_G f_1(x) f_2(x^{-1}y) d\mu_G$. We call the function $f_1 \star f_2$ the convolution of f_1 and f_2. With the convolution product the set $\mathcal{H}(G)$ obtains the structure of an associative complex algebra. We refer to $\mathcal{H}(G)$ as a *Hecke algebra*. The Hecke algebra $\mathcal{H}(G)$ has many idempotents, for instance let K be an open compact subgroup. Then define the function

$$e_K(x) = \begin{cases} \frac{1}{\mu_G(K)} & \text{if } x \in K, \\ 0 & \text{otherwise} \end{cases}.$$

The function e_K is in $\mathcal{H}(G)$ and $e_K^2 = e_K$.

Lemma 2.1. *For a finite set $f_1, f_2 \cdots f_n \in \mathcal{H}(G)$ there is an idempotent $e \in \mathcal{H}(G)$ such that $e \star f = f \star e = f$*

In fact if $f \in \mathcal{H}(G)$ note that there is a compact open subgroup K of G, such that $f(kg) = f(gk) = f(g)$ for all $k \in K$, $g \in G$. Then a simple computation shows that $e_K \star f = f \star e_K = f$.

The algebra $\mathcal{H}(G)$ has two natural smooth representation on it, $(\lambda, \mathcal{H}(G))$ and $(\rho, \mathcal{H}(G))$. Where $(\lambda(g)f)(x) = f(g^{-1}x)$ and $(\rho(g)f)(x) = f(xg)$. We refer to $(\lambda, \mathcal{H}(G))$ $[(\rho, \mathcal{H}(G))]$ as the left [right] regular representation.

2.1.2.1 Non-Degenerate $\mathcal{H}(G)$ Modules

Definition 2.3. *A left module M of $\mathcal{H}(G)$ is said to be* non degenerate *if* $\mathcal{H}(G)M = M$.

Let M be a non-degenerate left module for $\mathcal{H}(G)$. Then for $m \in M$ there exist and idempotent $e \in \mathcal{H}(G)$ such that $em = m$. We then define the representation (Γ, M) by $\Gamma(g)m = (\lambda(g)e)m$. The product $\Gamma(g)m$ is well defined and we obtain a smooth representation of G.

Denote by $\mathfrak{R}(G)$ the category of smooth representations. Then for $(\pi, V) \in \mathfrak{R}(G)$ we can construct the non degenerate left module V by saying $fv = \int_G f(x)v d\mu_G(x)$. We thus have a way to go from non degenerate left modules to smooth representations an vice versa. The two functors described above give us an equivalence of categories. We may then talk about objects and morphisms in both categories without any distinction.

2.1.2.2 Modular Function

Since left Haar measure is unique up to a positive constant, we get for $g \in G$ a positive number $\delta_G(g)$ such that

$$\delta_G(g) \int_G f(xg) d\mu_G(x) = \int_G f(x) d\mu_G(x).$$

It is easy to see that $\delta_G : G \longrightarrow \mathbb{R}^+$ is a smooth character, we refer to δ_G as the *modular function* for G . Let H be a closed subgroup of G with modular function δ_H. Let $\delta_{H\backslash G} = \delta_G|_H \delta_H^{-1}$ where $\delta_G|_H$ denotes the restriction of δ_G to H. Consider the space $C_c^\infty(H\backslash G, \delta_{H\backslash G})$ be the space of all functions $f : G \longrightarrow \mathbb{C}$ which satisfy

1) $f(hg) = \delta_{H\backslash G}(h)f(g)\ \forall\ h \in H, g \in G$

2) f is of compact suport modulo H and there is a compact open subgroup K of G such that $f(gk) = f(g)$ for all $k \in K$.

The space $C_c^\infty(H\backslash G, \delta_{H\backslash G})$ carries a smooth G representation ρ given by $\rho(g)f(x) = f(xg)$ for $f \in C_c^\infty(H\backslash G, \delta_{H\backslash G})$. There is a functional $\mu_{H\backslash G} : C_c^\infty(H\backslash G, \delta_{H\backslash G}) \longrightarrow \mathbb{C}$ unique up to a constant such that $< \mu_{H\backslash G}, f >=< \mu_{H\backslash G}, \rho(g)f >$. We denote the complex number $< \mu_{H\backslash G}, f >$ by the integral notation $\int_{H\backslash G} f(x)d\mu_{H\backslash G}$. There is a relation with the Haar measures μ_G and μ_H given by

$$\int_G f(x)d\mu_G = \int_{H\backslash G}\int_H f(hx)d\mu_H d\mu_{H\backslash G}.$$

A case of special interest is when we consider Z the center of a unimodular group G. We have that Z is closed and also unimodular because it is abelian. We thus have an invariant functional $\mu_{Z\backslash G}$ such that

$$\int_{Z\backslash G} f(xg)d\mu_{Z\backslash G} = \int_{Z\backslash G} f(x)d\mu_{Z\backslash G}.$$

This notion allows us to talk about functions that are integrable modulo the center.

Lemma 2.2 (Shur's Lemma). *Let G be a locally profinite group and assume that G has a countable dense set. Let (π, V) be an irreducible smooth G representation. Then* $\mathrm{End}_G(V) \cong \mathbb{C}$

Corollary 2.3. *Let G be a locally profinite group and assume that G has a countable dense set. Let (π, V) be an irreducible smooth G representation. The center Z of G acts on V by a character ω_π, that is $\pi(z)v = \omega_\pi(z)v$ for all $z \in Z$ $v \in V$.*

If a smooth representation (π, V) not necessarily irreducible acts by a character ω_π in the center, we say that ω_π is the *central character* of the representation (π, V).

2.1.3 Compact representations

We assume in this subsection that G is locally profinite group with a countable dense subset. We say that a smooth representation (π, V) is *compact* if all the matrix coefficients have compact support. Compact representations behave nicely, for instance they split the category of smooth representations; that is:

Theorem 2.4. *Let (τ, E) be a compact smooth representation of G. Then for any smooth representation (π, V) of G we have*

$$V = V^\tau \oplus W$$

where V^τ is the direct sum of representations isomorphic to (τ, E), and no irreducible sub-quotient of W is isomorphic to (τ, E).

The proof for GL_n can be found in [2] or see [10] for the general proof.

2.1.4 Res_H^G, Ind_H^G and $c\text{-}\mathrm{Ind}_H^G$ Functors

Let G be a locally profinite group and H be a closed subgroup of G. Then H is a locally profinite group and we have a functor of restriction $\mathrm{Res}_H^G : \mathfrak{R}(G) \longrightarrow \mathfrak{R}(H)$ given by the following. For $(\pi, V) \in \mathfrak{R}(G)$, $\mathrm{Res}_H^G(\pi)(h)(v) = \pi(h)v$.

We can also form the functor of induction $\mathrm{Ind}_H^G : \mathfrak{R}(H) \longrightarrow \mathfrak{R}(G)$ given by the following, if $(\sigma, U) \in \mathfrak{R}(H)$. Consider the space $\mathrm{Ind}_H^G U$ of all functions $f : G \longrightarrow U$ which satisfy

1) $f(hg) = \sigma(h)f(g)\ \forall\ h \in H, g \in G$

2) there is a compact open subgroup K of G such that $f(gk) = f(g)$ for all $k \in K$.

For $f \in \mathrm{Ind}_H^G U$ take $\mathrm{Ind}_H^G(\sigma)(g)(f)(x) = f(xg)$. Then $(\mathrm{Ind}_H^G(\sigma), \mathrm{Ind}_H^G U)$ is a smooth representation of G. As is to be expected Res_H^G is a left adjoint of Ind_H^G;

Theorem 2.5. *(Frobenious Reciprocity) Let H be a closed subgroup of locally profinite group G. Then for a smooth representation (σ, U) of H and a smooth representation (π, V) of G, we have*

$$\mathrm{Hom}_H(\mathrm{Res}_H^G\,\pi, U) \cong \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G\,U)$$

natural in both σ and π.

We have another functor, the functor of compact induction, $c\text{-}\mathrm{Ind}_H^G : \mathfrak{R}(H) \longrightarrow \mathfrak{R}(G)$ given by the following, for $(\sigma, U) \in \mathfrak{R}(H)$. Consider the space $c\text{-}\mathrm{Ind}_H^G\,U$ of all functions $f : G \longrightarrow U$ which satisfy

1) $f(hg) = \sigma(h)f(g)\ \forall\ h \in H, g \in G$

2) f is locally constant and of compact suport modulo H and there is a compact open subgroup K of G such that $f(gk) = f(g)$ for all $k \in K$

and for $f \in c\text{-}\mathrm{Ind}_H^G\,U$ take $c\text{-}\mathrm{Ind}_H^G(\sigma)(g)(f)(x) = f(xg)$. Then $(c\text{-}\mathrm{Ind}_H^G(\sigma), c\text{-}\mathrm{Ind}_H^G\,U)$ is a smooth representation of G. In the case where H is an open subgroup of G we have that $c\text{-}\mathrm{Ind}_H^G$ is a left adjoint for Res_H^G;

Theorem 2.6. *Let H be an open subgroup of locally profinite group G. Then for a smooth representation (σ, U) of H and a smooth representation (π, V) of G, we have*

$$\mathrm{Hom}_G(c\text{-}\mathrm{Ind}_H^G U, \pi) \cong \mathrm{Hom}_H(U, \mathrm{Res}_H^G \pi)$$

natural in both σ and π.

We have a nice relation between compact induction and the contragradient.

Lemma 2.7. *Let H be closed subgroup of locally profinite group G. Then for a smooth representation (σ, U) of H we have*

$$(c\text{-}\mathrm{Ind}_H^G U \otimes \delta_H^{-1})\check{} \cong \mathrm{Ind}_H^G \check{U}.$$

2.2 Representations of Reductive Groups over a p-adic fields

We found the book by D.Renard [10] very useful and all the results in this section (and much more with great explanation) can also be found there.

For the rest of the chapter we take G a connected reductive group over a non archimedean local field $\mathbb{F}$. We denote by $\mathcal{O}$ the ring of integers in $\mathbb{F}$ with maximal prime ideal $\mathfrak{p}$ and by ϖ a prime element in $\mathcal{O}$. Let q be the order of the residue field $\mathcal{O}/\mathfrak{p}$, finally denote by $|\cdot|_{\mathbb{F}}$ the non archimedean valuation such that $|\varpi|_{\mathbb{F}} = q$. All of our representations are assumed to be smooth, we thus drop the prefix smooth and refer to them just as representations. We will write $P = LN$ for a parabolic subgroup P of G with Levi component L and unipotent radical N. We also denote the center of G by Z.

2.2.1 Jacquet Functor and parabolic induction

Let $P = LN$ be a parabolic subgroup of G and (π, V) a G representation. Let $V(N)$ be the span of the set of all vectors of the form $v - \pi(n)v$ for $v \in V$ and $n \in N$. Since L normalizes N the space $V_N := V/V(N)$ can be given a natural L action that we denote π_N given by $\pi_N(x)(v + V(N)) = \pi_N(x)v + V(N)$ for $x \in L$. The representation (π_N, V_N) is smooth. We define the *Jacquet functor* from $\mathfrak{R}(G) \longrightarrow \mathfrak{R}(L)$ by $(\pi, V) \mapsto (\pi_N, V_N)$.

Let us now construct a functor in the other direction. Let $(\sigma, W) \in \mathfrak{R}(L)$; then we can inflate σ to a representation of P by demanding $\sigma(xn) = \sigma(x)$ for all $n \in N$ and all $x \in L$. We then induce it to obtain a representation of G (note that $c\text{-}\operatorname{Ind}_P^G \sigma$ and Ind_H^G are the same since P is parabolic and thus G/P is compact). We call this functor the functor or *parabolic induction* and by abuse of notation we denote it by Ind_P^G. The Jacquet functor and the functor of parabolic induction are exact.

Theorem 2.8. *The Jacquet functor is the right adjoint of the functor of parabolic induction. So for every representation (π, V) of G and (σ, W) of L we have a natural isomorphism*

$$\operatorname{Hom}_L(\pi_N, \sigma) \cong \operatorname{Hom}_G(\pi, \operatorname{Ind}_P^G \sigma)$$

The Jacquet functor and the functor of parabolic induction are exact.

From parabolic induction we can form the functor of *normalized induction* ι_P^G. For (σ, W) as above take $\iota_P^G \sigma := \operatorname{Ind}_P^G \sigma \otimes \delta_P^{-1/2}$; then by 2.7 we get that the functor of normalized induction commutes with the contragradient.

Lemma 2.9. *Let $P = LN$ be a parabolic subgroup of G. Then for a smooth representation (σ, U) of L we have*

$$(\iota_P^G \sigma)\check{} \cong \iota_P^G \check{\sigma}$$

We also form the functor of *normalized parabolic restriction* to be r_P^G. For (π, V) a representation of G we take $r_P^G(\pi, V) = (\pi_N \otimes \delta_P^{-1/2}, V_N)$. Since taking the tensor product with the modular function is exact we have that ι_P^G and r_P^G are exact.

Theorem 2.10. *Let $P = LN$ be a parabolic subgroup of G.For every representation (π, V) of G and (σ, W) of L we have a natural isomorphism*

$$\mathrm{Hom}_L(r_P^G \pi, \sigma) \cong \mathrm{Hom}_G(\pi, \iota_P^G \sigma)$$

Definition 2.4. *A representation (π, V) is called* supercuspidal *if for all proper parabolic subgroups P of G, $r_p^G(V) = 0$.*

Since r_p^G is an exact functor we can see that every sub-quotient representation of a supercuspidal representation is supercuspidal.

Let $P = MN$ and $Q = LU$ be two parabolic subgroups of G with $P \subset Q$. Then $P \cap L = M(N \cap L)$ is a parabolic subgroup of L with levi component M and unipotent radical $N \cap L$.

Lemma 2.11. *Let $P = MN \subset Q = LU$ be two parabolic subgroups of G. Then $\iota_P^G = \iota_Q^G \circ \iota_{P\cap L}^L$ and $r_P^G = r_{P\cap L}^L \circ r_Q^G$.*

We have by the preceding lemma that for an irreducible representation (π, V) of G, there exist a parabolic subgroup $P = MN$ minimal under inclusion such that $r_P^G(V) \neq 0$. We have then by minimality of P that $r_P^G(V)$ is supercuspidal representation of M.

2.2.2 Unramified Characters

Let $X^*(G)$ be the group of rational characters i.e. morphisms of algebraic groups from G into GL_1 defined over $\mathbb{F}$. We define ${}^oG = \bigcap_{\chi \in X^*(G)} \ker |\chi|_{\mathbb{F}}$

Lemma 2.12. *The group ${}^oG$ is a normal open subgroup of G that contains all compact open subgroups of G.*

Definition 2.5. *An* unramified character *of G is a 1-dimensional representation of G that is trivial on ${}^oG$*

2.3 Bernstein Theory

We know that smooth representations are not semisimple; for instance for $G = \mathrm{GL}_2(\mathbb{F})$ we can form the representation $\mathrm{Ind}_B^G 1_B$ where B is the subgroup of upper tirangular matrices. The representation $\mathrm{Ind}_B^G 1_B$ is indecomposable but not irreducible [4]. On the other hand in this section we are going to see how we can obtain a decomposition of the category $\mathfrak{R}(G)$ into subcategories equivalent to categories of modules over rings with identity.

2.3.1 Inertial support

Theorem 2.13. *Let (π, V) be an irreducible representation of G. The following conditions are equivalent:*

i) (π, V) *is supercuspidal.*

ii) The matrix coefficients of (π, V) are compactly supported modulo center Z.

iii) The restriction $\operatorname{Res}^G_{{}^\circ G} \pi$ is compact.

Theorem 2.14. *Let (π, V) be an irreducible representation of G. Then the restriction $\operatorname{Res}^G_{{}^\circ G} \pi$ is semisimple and of finite length. Moreover if (π_1, V_1) and (π_2, V_2) are two irreducible representations then the following are equivalent:*

i) $\operatorname{Res}^G_{{}^\circ G} \pi_1 = \operatorname{Res}^G_{{}^\circ G} \pi_2$

ii) There exist an unramified character χ of G such that $\pi_1 = \pi_2 \otimes \chi$

iii) $\operatorname{Hom}_{{}^\circ G}(\operatorname{Res}^G_{{}^\circ G} \pi_1, \operatorname{Res}^G_{{}^\circ G} \pi_2) \neq 0$

We can form an equvalence relation in the set of supercuspidal representations by saying that two irreducible supercuspidal representations (π_1, V_1) and (π_2, V_2) are related if there exists an unramified character χ of G such that $\pi_1 = \pi_2 \otimes \chi$.

Definition 2.6. *A cuspidal data is a pair $(M, (\sigma, W))$ where M is a levi component of a parabolic subgroup $P = MN$ of G and (σ, W) is an irreducible supercuspidal representation of M.*

We say that two cuspidal data $(M_1, (\sigma_1, W_1))$ and $(M_2, (\sigma_2, W_2))$ are related if there exists an element $g \in G$ such that

$$gM_1g^{-1} = M_2 \text{ and } \sigma_2 = \sigma_1^g.$$

We denote by $((M,(\sigma,W))_G$ the equivalence class set of $(M,(\sigma,W))$.

Theorem 2.15. *Let $P = MN$ be a parabolic subgroup of G and (σ, W) an irreducible supercuspidal representation of M. Let $(\pi,V) = \iota_P^G(\sigma,W)$. Then (π,V) has finite length. Furthermore if $P' = M'N'$ is another parabolic subgroup of G with (σ',W') an irreducible supercuspidal representation of M and $(\pi',V') = \iota_P'^G(\sigma',W')$ then the following are equivalent:*

i) *There exist an element $g \in G$ such that $gMg^{-1} = M'$ and $\sigma' = \sigma^g$.*

ii) $\mathrm{Hom}_G(\pi,\pi') \neq 0$.

iii) *The Jordan-Holder decompositions of π and π' are the same.*

iv) *The Jordan-Holder decompositions of π and π' have at least one element in common.*

Corollary 2.16. *Let $P_i = M_iN_i$, $i = 1,2$ be two parabolic subgroups of G and $(M_i,(\sigma_i,W_i))$ cuspidal data. Take (π,V) an irreducible representation of G such that (σ_i,W_i) is a composition factor of $r_{P_i}^G(\pi,V)$. Then the cuspidal data $(M_1,(\sigma_1,W_1))$ and $(M_2,(\sigma_2,W_2))$ are related.*

Note that the corollary allows us to associate to each irreducible representation of G an equivalence class of cuspidal data in a well defined way. We however have a stronger equivalence relation combining the two previous ones.

Definition 2.7. *We say that two cuspidal data $(M_1,(\sigma_1,W_1))$ and $(M_2,(\sigma_2,W_2))$ define the same* inertial support *if there exist an unramified character χ of M_2 such* *that*

$$gM_1g^{-1} = M_2 \text{ and } \sigma_2 = \sigma_1^g \otimes \chi.$$

We denote by $[(M, (\sigma, W))]_G$ the class of cuspidal data that define the same inertial support as $(M_1, (\sigma_1, W_1))$ and by $\mathscr{B}(G)$ the set of all these equivalence classes. If (π, V) is an irreducible representation, then there exist a parabolic subgroup $P = MN$, a unique element $\mathfrak{s} \in \mathscr{B}(G)$ and a representative $(M, (\sigma, W)) \in \mathfrak{s}$ such that (σ, W) is a composition factor of $r_P^G(\pi, V)$. Saying that (σ, W) is a composition factor of $r_P^G(\pi, V)$ is equivalent to say $\mathrm{Hom}_M(r_P^G(\pi), \sigma) \neq 0$ or by theorem 2.10 is equivalent to say that $\mathrm{Hom}_M(\pi, \iota_P^G(\sigma)) \neq 0$. We say then that (π, V) has *inertial support* in $\mathfrak{s}$ and denote it by $\mathfrak{I}((\pi, V)) \in \mathfrak{s}$.

Let $\mathfrak{s} \in \mathscr{B}(G)$ and define the category $\mathfrak{R}^{\mathfrak{s}}(G)$ to be the full subcategory of $\mathfrak{R}(G)$ given by all the representations (π, V) such that all irreducible sub-quotients of (π, V) have inertial support in $\mathfrak{s}$. We are now ready to state the main theorem in this section.

Theorem 2.17 (Bernstein's Decomposition).

$$\mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathscr{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G).$$

We will see in the next section that the categories $\mathfrak{R}^{\mathfrak{s}}(G)$ are equivalent to a category of modules over a ring with identity.

2.4 Types and Covers

Keep the same notation as in the previous section. For the results in this section we refer to Bushnell and Kutzko [6].

2.4.1 Types

We know that the algebra $\mathcal{H}(G)$ has many idempotents. There are some of special interest to us obtained from irreducible representations of compact open subgroups. If J is compact and open and (ρ, W) is an irreducible representation of J we can form the element $e_\rho \in \mathcal{H}(G)$ by demanding

$$e_\rho(x) = \begin{cases} \frac{dim\rho}{\mu_G(K)} tr(\rho(x)^{-1}) & \text{if } x \in J, \\ 0 & \text{otherwise} \end{cases} .$$

If (π, V) is a G representation then the function e_ρ is the projection onto the ρ-isotypic component, V^ρ of $\mathrm{Res}_J^G V$. We then have that e_ρ is an idempotent and $e_\rho V = V^\rho$. We also have that $e_\rho \star \mathcal{H}(G) \star e_\rho$ is an algebra with identity and has a natural action on $e_\rho V = V^\rho$. Let $\mathfrak{R}_\rho(G)$ be the full subcategory of $\mathfrak{R}(G)$ of representations (π, V) such that $\mathcal{H}(G) \star e_\rho V = V$. We thus have a functor $m_{e_\rho} : \mathfrak{R}_\rho(G) \longrightarrow e_\rho \star \mathcal{H}(G) \star e_\rho - \mathrm{Mod}$ given by sending $e_\rho(V) = V^\rho$. A necessary condition for this to be an equivalence of categories is that $\mathfrak{R}_\rho(G)$ to be closed under sub-quotients. We have that this condition is also sufficient and equivalent to the existence of a finite subset $\mathfrak{S} \subset \mathscr{B}(G)$ such that $\mathfrak{R}_\rho(G) = \Pi_{\mathfrak{s} \in \mathfrak{S}} \mathfrak{R}^{\mathfrak{s}}(G)$. We will summarize this in the following theorem.

Theorem 2.18. *Let (ρ, W) be an irreducible representation of J. For a subcategory $\mathfrak{B}$ of $\mathfrak{R}(G)$ let* $\mathrm{Irr}(\mathfrak{B})$ *denote the irreducible representations in $\mathfrak{B}$. The following are equivalent.*

i) The functor $m_{e_\rho} : \mathfrak{R}_\rho(G) \longrightarrow e_\rho \star \mathcal{H}(G) \star e_\rho - \mathrm{Mod}$ *given by* $e_\rho(V) = V^\rho$ *is an equivalence of Categories*

ii) The category $\mathfrak{R}_\rho(G)$ *is closed under sub-quotients*

iii) There exists a finite subset $\mathfrak{S}$ *of* $\mathscr{B}(G)$ *such that* $\mathfrak{R}_\rho(G) = \Pi_{\mathfrak{s}\in\mathfrak{S}}\mathfrak{R}^{\mathfrak{s}}(G)$

iv) There exists a finite subset $\mathfrak{S}$ *of* $\mathscr{B}(G)$ *such that* $\mathrm{Irr}(\mathfrak{R}_\rho(G)) = \mathrm{Irr}(\Pi_{\mathfrak{s}\in\mathfrak{S}}\mathfrak{R}^{\mathfrak{s}}(G))$

Definition 2.8. *A pair* (J, ρ) *where* (ρ, W) *is an irreducible representation of* J *that satisfies one and thus all of the conditions of theorem 2.18 is said to be an* $\mathfrak{S}$-type*. We will be concerned mainly with the case that* $\mathfrak{S}$ *consists of a single element* $\mathfrak{s}$ *in which case we call it an* $\mathfrak{s}$-type *or just a* type *if there is no risk of confusion.*

Proposition 2.19. *Let* π *be an irreducible supercuspidal representation of* G *of the form*

$$\pi \cong c\text{-}\mathrm{Ind}_{\tilde{J}}^{G}(\tilde{\tau}),$$

for a representation $\tilde{\tau}$ *of some open compact mod the centre subgroup* $\tilde{J}$ *of* G*. Let* $J = \tilde{J} \cap^{o} G$*, and let* τ *be some irreducible component of* $\tilde{\tau}|_J$*. Then* (J, τ) *is a* $[G, \pi]_G$*-type and* J *is the unique maximal compact subgroup of* $\tilde{J}$*.*

Example 2.3. Let $\tilde{\phi} : \mathbb{F}^\times \longrightarrow \mathbb{C}$ be a character. Let $\mathfrak{s} = [\mathbb{F}^\times, \tilde{\phi}]_{\mathbb{F}^\times}$ and $\tilde{\phi}|_{\mathcal{O}^\times} = \phi$. Then $(\phi, \mathcal{O}^\times)$ is an $\mathfrak{s}$-type.

2.4.2 The Hecke Algebra $\mathcal{H}(G, \rho)$

Let (ρ, W) be K representation, where K is an open compact subgroup of G.

Definition 2.9. *Let $\mathcal{H}(G,\rho)$ denote the space of compactly supported functions $f : G \longrightarrow End_{\mathbb{C}}(\check{W})$ such that*

$$f(k_1gk_2) = \check{\rho}(k_1)f(g)\check{\rho}(k_2), \textit{where } k_1, k_2 \in K,\ \ g \in G.$$

The operation of convolution

$$f_1 \star f_2(g) = \int_G f_1(x)f_2(x^{-1}g)d\mu_G(x)\ \ f_1, f_2 \in \mathcal{H}(G,\rho),$$

gives $\mathcal{H}(G,\rho)$ the structure of an associative complex algebra with identity. The identity in $\mathcal{H}(G,\rho)$ is given by the element

$$1_{(G,\rho)}(x) = \begin{cases} \frac{1}{\mu_G(K)}\check{\rho}(x) & \text{if } x \in K, \\ 0 & \text{otherwise} \end{cases}.$$

Let $h \in c\text{-}\operatorname{Ind}_K^G \check{W}$. Then for $f \in \mathcal{H}(G,\rho)$ we get $f \star h(g) = \int_G f(x)h(x^{-1}g)d\mu_G(x) \in c\text{-}\operatorname{Ind}_K^G \check{W}$. We have an isomorphism Φ from the Hecke algebra $\mathcal{H}(G,\rho)$ into the algebra of left endomorphisms $\operatorname{Hom}_G(c\text{-}\operatorname{Ind}_K^G \check{W}, c\text{-}\operatorname{Ind}_K^G \check{W})$ given by $\Phi_f(h) = f \star h$ for $f \in \mathcal{H}(G,\rho)$, $h \in c\text{-}\operatorname{Ind}_K^G \check{W}$. Since ρ an irreducible K representation and K is compact then ρ is finite dimensional and therefore it is canonically isomorphic to $\check{\check{\rho}}$. We use this isomorphism to identify $\mathcal{H}(G,\check{\rho})$ with the space of compactly supported functions $f : G \longrightarrow End_{\mathbb{C}}(W)$ such that

$$f(k_1gk_2) = \rho(k_1)f(g)\rho(k_2), \text{where } k_1, k_2 \in K,\ \ g \in G.$$

We have by the discussion above that $\mathcal{H}(G,\check{\rho})$ is isomorphic to the algebra of left endomorphisms $\operatorname{Hom}_G(c\text{-}\operatorname{Ind}_K^G \check{\check{W}}, c\text{-}\operatorname{Ind}_K^G \check{\check{W}})$. We can hence regard $c\text{-}\operatorname{Ind}_K^G \check{\check{W}}$ as a left module and since is isomorphic to $c\text{-}\operatorname{Ind}_K^G W$ this gives $c\text{-}\operatorname{Ind}_K^G W$ the structure of a left $\mathcal{H}(G,\check{\rho})$ module.

Definition 2.10. *Let (ρ, W) be an irreducible K representation where K is an open compact subgroup of G. Then denote by $\mathcal{I}_G(\rho) = \{x \in G | \operatorname{Hom}_{x^{-1}Kx\cap K}(\rho^x, \rho) \neq 0\}$. Where $\rho^x(y) = \rho(xyx^{-1})$ for $y \in x^{-1}Kx \cap K$.*

Lemma 2.20. *Using the notation of the preceding definition we have the following list of results.*

i) If $x \in \mathcal{I}_G(\rho)$ then $KxK \subset \mathcal{I}_G(\rho)$.

ii) $x \in \mathcal{I}_G(\rho)$ if and only if there exist a function $f \in \mathcal{H}(G, \rho)$ such that the support of f is KxK.

iii) $x \in \mathcal{I}_G(\rho)$ if and only if there exist a function $f \in \mathcal{H}(G, \rho)$ such that the support of f contains KxK.

iv) Let $\mathcal{H}_x(G, \rho) = \{f \in \mathcal{H}(G, \rho) | \operatorname{supp}(f) \subset KxK\}$ then

$$\mathcal{H}(G, \rho) = \bigoplus_{x \in K\backslash \mathcal{I}_G(\rho)/K} \mathcal{H}_x(G, \rho).$$

The anti-isomorphism from ${}^t : End_{\mathbb{C}}(\check{W}) \longrightarrow End_{\mathbb{C}}(W)$ given by $< \check{w}, {}^tAw >=< A\check{w}, w >$ for $\check{w} \in \check{W}$ $A \in End_{\mathbb{C}}(\check{W})$, induces an anti-isomorphism $\check{} : \mathcal{H}(G, \rho) \longrightarrow \mathcal{H}(G, \check{\rho})$ given by $(\check{f})(x) = {}^tf(x^{-1})$ for $f \in \mathcal{H}(G, \rho)$. We can identify the category of left $\mathcal{H}(G, \check{\rho})$ modules with the category of right $\mathcal{H}(G, \rho)$ modules. In particular we have that $c\text{-}\operatorname{Ind}_K^G W$ is a right $\mathcal{H}(G, \rho)$ module. For a representation (π, V) of G we can then give $\operatorname{Hom}_G(c\text{-}\operatorname{Ind}_K^G W, V)$ the structure of a left $\mathcal{H}(G, \rho)$ module. By 2.6 we can give $\operatorname{Hom}_K(W, \operatorname{Res}_K V)$ the structure of a left $\mathcal{H}(G, \rho)$ module. We introduce the notation $V_\rho = \operatorname{Hom}_K(W, \operatorname{Res}_K V)$. If we unravel all the previous definitions we can be explicit about the structure of V_ρ as a left $\mathcal{H}(G, \rho)$ module.

Lemma 2.21. *Let (ρ, W) be an irreducible K representation where K is an open compact subgroup of G and let (π, V) be a representation of G. The left action of $\mathcal{H}(G, \rho)$ in V_ρ is given by $(f \cdot T) : w = \int_G \pi(x) T(\check{f}(x^{-1})w) d\mu_G(x)$ For $f \in \mathcal{H}(G, \rho)$ and $T \in \operatorname{Hom}_K(W, \operatorname{Res}_K V)$*

Definition 2.11. *We define the functor*

$$\mathbf{M}_\rho : \mathfrak{R}(G) \longrightarrow \mathcal{H}(G, \rho) - \mathrm{Mod}$$

$$(\pi, V) \mapsto V_\rho$$

Proposition 2.22. *Let (ρ, W) be an irreducible K representation where K is an open compact subgroup of G. The algebras $\mathcal{H}(G, \rho)$ and $e_\rho \star \mathcal{H}(G) \star e_\rho$ are Morita equivalent.*

Corollary 2.23. *(J, ρ) is an $\mathfrak{s}$-type if and only if the restriction of the functor $\mathbf{M}_\rho$ to $\mathfrak{R}^{\mathfrak{s}}(G)$ gives an equivalence of categories between $\mathfrak{R}^{\mathfrak{s}}(G)$ and $\mathcal{H}(G, \rho) - \mathrm{Mod}$.*

Corollary 2.24. *Let $\mathfrak{S}$ be a finite subset of $\mathscr{B}(G)$. The pair (K, ρ) is then an $\mathfrak{S}$-type in G if and only if, for an irreducible representation $(\pi, V) \in \mathfrak{R}(G)$, we have $\mathfrak{I}(\pi) \in \mathfrak{S}$ if and only if $\operatorname{Hom}_K(\rho, \pi) \neq 0$*

2.4.3 Covers

Let $P_u = MN_u$ be a parabolic subgroup of G. Let us denote by N_ℓ the opposite of N_u relative to M and $P_\ell = MN_\ell$. Consider the pair (J, τ) where J is a compact open subgroup and (τ, W) an irreducible representation of J.

Definition 2.12. *The pair (J, τ) is said to be* decomposed with respect to (M, P_u) *if there following conditions hold:*

i) $J = J \cap N_\ell \cdot J \cap M \cdot J \cap N_u$,

ii) the groups $J \cap N_\ell$ *and* $J \cap N_u$ *are both contained in the kernel of* τ.

Let $\mathscr{Z}(M)$ denote the center of M. Suppose that (J, τ) is decomposed with respect to (M, P_u). We write J_ℓ, J_M and J_u for $J \cap N_\ell$, $J \cap M$ and $J \cap N_u$ respectively.

Definition 2.13. *An element* $z \in M$ *is said to be positive if*

$$zJ_uz^{-1} \subset J_u \text{ and } z^{-1}J_\ell z \subset J_\ell$$

Definition 2.14. *An element* $\zeta \in \mathscr{Z}(M)$ *is said to be strongly* (P_u, J)*-positive if*

i) ζ *is positive,*

ii) for any compact open subgroups H_1,H_2 *of* N_u *there exist an integer* $m \geq 0$ *such that* $\zeta^m H_1 \zeta^{-m} \subset H_2$

iii) for any compact open subgroups K_1,K_2 *of* N_l *there exist an integer* $m \geq 0$ *such that* $\zeta^{-m} K_1 \zeta^m \subset K_2$

Let $\mathfrak{t} = [L, \sigma]_M \in \mathscr{B}(M)$. We have that L is a Levi component of some parabolic subgroup of M and σ a supercuspidal representation of G. Then $[L, \sigma]_M$ determines the element $[L, \sigma]_G \in \mathscr{B}(G)$. We introduce the notation $[\mathfrak{t}]_G = [L, \sigma]_G$. Let $\mathfrak{S}(M)$ be a finite subset of $\mathscr{B}(M)$ and let $\mathfrak{S}(G) = \{[\mathfrak{t}]_G | \ \mathfrak{t} \in \mathfrak{S}(M)\}$. We will show that under certain conditions we can obtain an $\mathfrak{S}(G)$-type from an $\mathfrak{S}(M)$-type.

Definition 2.15. *Let* M *be some Levi subgroup of* G. *Let* J_M *be a compact open subgroup of* M *and* (τ_M, W) *an irreducible representation of* J_M. *The pair* (J, τ) *is a* G-cover *of* (J_M, τ_M) *if the following conditions hold:*

i) *(J,τ) is decomposed with respect to (M,P), for any parabolic subgroup P with Levi component M,*

ii) *$J \cap M = J_M$ and $\tau|_{J_M} \cong \tau_M$,*

iii) *for every parabolic subgroup P with Levi component M there exists an invertible element in $\mathcal{H}(G,\tau)$ supported in the double coset Jz_PJ, where $z_P \in \mathscr{Z}(M)$ is strongly (P,J)-positive (see 2.14 for the definition of strongly (P,J)-positive).*

Proposition 2.25. *Let M be a Levi subgroup of G and let $\mathfrak{S}(M)$ be a finite subset of $\mathscr{B}(M)$. Define $\mathfrak{S}(G)$ as above. Assume that (J_M,τ_M) is an $\mathfrak{S}(M)$-type and that (J,τ) is a G cover of (J_M,τ_M). Then (J,τ) is an $\mathfrak{S}(G)$-type.*

Let R, S be two rings with identity and let $\phi : R \longrightarrow S$ be a ring homomorphism sending the identity in R to the identity in S. We can regard S as a R-S bimodule where R acts on the left by $ra = \phi(r)a$ and S acts on the right by multiplication. We denote by ϕ_* the covariant functor $\phi_* : R\text{-Mod} \longrightarrow S\text{-Mod}$ where $\phi_*(M) = \mathrm{Hom}_R(S,M)$.

Proposition 2.26. *Let (J_M,τ_M) be an $\mathfrak{s}_M$-type and (J,τ) a G cover of (J_M,τ_M) (and hence an $[\mathfrak{s}]_G$-type by 2.25). Let $P = MN$ be a parabolic subgroup of G. Let us choose Haar measures μ_G and μ_M such that $\mu_G(J) = \mu_M(J_M) = 1$ and form the algebras $\mathcal{H}(M,\tau_M)$, $\mathcal{H}(G,\tau)$ with respect to these measures. Then there exists an injective algebra homomorphism $t_P : \mathcal{H}(M,\tau_M) \longrightarrow \mathcal{H}(G,\tau)$ such that if $a \in \mathcal{I}_M(\tau_M)$ is a positive element with respect to (P,J). Then for $h \in \mathcal{H}(M,\tau_M)$ with support $J_M a J_M$, $t_P(h)$ has support in JaJ and $t_P(h)(a) = h(a)$.*

Let us continue with the same notation. Further, let χ be a character of M and $f \in \mathcal{H}(M, \tau_M)$. Set $f \cdot \chi \in \mathcal{H}(M, \tau_M)$ by demanding $f \cdot \chi(m) = f(m)\chi(m)$. We have that the map $f \mapsto f \cdot \chi$ is an automorphism of $\mathcal{H}(M, \tau_M)$. Consider the inverse of the modular function δ_P^{-1}. We can then define an algebra homomorphism $t_P^i : \mathcal{H}(M, \tau_M) \longrightarrow \mathcal{H}(G, \tau)$ by $t_P^i(h) = t_P(h \cdot \delta_P^{-1})$.

Theorem 2.27. *Let (J_M, τ_M) be an $\mathfrak{s}_M$-type and (J, τ) a G cover of (J_M, τ_M). Let $P = MN$ be a parabolic subgroup of G. Let us choose Haar measures μ_G and μ_M such that $\mu_G(J) = \mu_M(J_M) = 1$ and form the algebras $\mathcal{H}(M, \tau_M)$, $\mathcal{H}(G, \tau)$ with respect to these measures. Then the injective algebra homomorphism $t_P^i : \mathcal{H}(M, \tau_M) \longrightarrow \mathcal{H}(G, \tau)$ satisfies:*

$$\begin{tikzcd} \mathfrak{R}^{\mathfrak{s}_M}(M) \arrow[r, "\iota_P^G"] \arrow[d, "\mathbf{M}_{\tau_M}"] & \mathfrak{R}^{[\mathfrak{s}]_G}(G) \arrow[d, "\mathbf{M}_{\tau}"] \\ \mathcal{H}(M, \tau_M)\text{-Mod} \arrow[r, "(t_P^i)_*"] & \mathcal{H}(G, \tau)\text{-Mod} \end{tikzcd}$$

Moreover if $a \in \mathcal{I}_M(\tau_M)$ is a positive element with respect to (P, J). Then for $h \in \mathcal{H}(M, \tau_M)$ with support $J_M a J_M$, $t_P^i(h)$ has support in JaJ and $t_P^i(h)(a) = h(a)\delta_P^{-1}(a)$.

We see that theorem 2.27 gives that the functor of induction Ind_P^G corresponds a functor $(t_P^i)_*$ given by a algebra homomorphism t_P^i. We can obtain a similar result for unitary induction by defining the map $t_P^u(h) = t_P(h \cdot \delta_P^{-1/2})$.

Theorem 2.28. *Let (J_M, τ_M) be an $\mathfrak{s}_M$-type and (J, τ) a G cover of (J_M, τ_M). Let $P = MN$ be a parabolic subgroup of G. Let us choose Haar measures μ_G and μ_M such that $\mu_G(J) = \mu_M(J_M) = 1$ and form the algebras $\mathcal{H}(M, \tau_M)$, $\mathcal{H}(G, \tau)$ with respect* *to these measures. Then the injective algebra homomorphism* $t_P^u : \mathcal{H}(M,\tau_M) \longrightarrow \mathcal{H}(G,\tau)$ *satisfies:*

$$\begin{array}{ccc} \mathfrak{R}^{\mathfrak{s}_M}(M) & \xrightarrow{\iota_P^G} & \mathfrak{R}^{[\mathfrak{s}]_G}(G) \\ \downarrow{\scriptstyle \mathbf{M}_{\tau_M}} & & \downarrow{\scriptstyle \mathbf{M}_{\tau}} \\ \mathcal{H}(M,\tau_M)\text{-Mod} & \xrightarrow{(t_P^u)_*} & \mathcal{H}(G,\tau)\text{-Mod} \end{array}$$

Moreover if $a \in \mathcal{I}_M(\tau_M)$ *is a positive element with respect to* (P,J)*. Then for* $h \in \mathcal{H}(M,\tau_M)$ *with support* $J_M a J_M$*,* $t_P^u(h)$ *has support in* JaJ *and* $t_P^u(h)(a) = h(a)\delta_P^{-1/2}(a)$*.*

2.4.4 Hecke Algebra of a Cover

We end this chapter with a result of Bushnell and Kutzko [7], that is useful to compute the Hecke algebra of a cover. Let $\mathfrak{t} = [L,\sigma]_L \in \mathscr{B}(L)$. The element $\mathfrak{t}$ also defines an element $\mathfrak{s} = [L,\sigma]_G \in \mathscr{B}(G)$.

Assumptions. *i) There is a open compact-mod-center subgroup* $\tilde{J}_L$ *of* L *and and an irreducible smooth representations* $\tilde{\tau_L}$ *of* $\tilde{J}_L$ *such that* $\sigma \cong c\text{-}\mathrm{Ind}_{\tilde{J}_L}^G(\tilde{\tau_L})$*.*

ii) The representation $\tau_L = \tilde{\tau_L}|_{J_L}$ *is irreducible. where* J_L *denotes the unique maximal compact subgroup of* $\tilde{J}_L$*.*

iii) An element $x \in L$ *intertwines* τ_L *if and only if* $x \in \tilde{J}_L$*.*

Let (J,τ) be a G-cover for (J_L,τ_L). We therefore get by 2.25 that (J,τ) is a $\mathfrak{s}$-type. For a parabolic subgroup $P = LN$ of G we get by 2.27 an injective homomorphism $t_P^i : \mathcal{H}(L,\tau_L) \longrightarrow \mathcal{H}(G,\tau)$. We can identify $\mathcal{H}(L,\tau_L)$ with a subalgebra of $\mathcal{H}(G,\tau)$. Let K be a compact subgroup of G that contains J. We can then form the subalgebra of $\mathcal{H}(K,\tau)$ consisting of functions with support contained in K. We can let $N_G(L)$, the G normalizer of L, act on $\mathscr{B}(L)$ by $x \cdot [L,\gamma]_L = [L,\gamma^x]$. We write $\mathbf{W}_{\mathfrak{s}}$ for the (finite) $N_G(L)/L$-stabilizer of $\mathfrak{t}$.

Theorem 2.30. *i) The map*

$$\mathcal{H}(L,\tau_L) \bigotimes \mathcal{H}(K,\tau) \longrightarrow \mathcal{H}(G,\tau)$$

$$(f \otimes \phi) \longmapsto f \star \phi$$

is an injective homomorphism of $\mathcal{H}(L,\tau_L)$*-*$\mathcal{H}(K,\tau)$*-bimodules.*

ii) We have $\dim_{\mathbb{C}}(\mathcal{H}(K,\tau)) \leq |\mathbf{W}_{\mathfrak{s}}|$.

iii) If $\dim_{\mathbb{C}}(\mathcal{H}(K,\tau)) = |\mathbf{W}_{\mathfrak{s}}|$*, then the map of* $i)$ *is an isomorphism.*

CHAPTER 3
POSTLIMINAL C^*-ALGEBRAS

In this Chapter we follow very closely the book by Dixmier [8].

3.1 C^*-Algebras

Let A be an algebra over the complex numbers. An *involution* in A is a map $x \mapsto x^*$ of A into itself such that

i) $(x^*)^*$

ii) $(x + y)^* = x^* + y^*$

iii) $(\lambda x)^* = \bar{\lambda} x^*$

iv) $(xy)^* = y^* x^*$

for any $x, y \in A$ and $\lambda \in \mathbb{C}$. An algebra over $\mathbb{C}$ endowed with an involution is called an *involutive* algebra. A *normed* algebra is an algebra A together with a norm $||\cdot||$ such that $||xy|| \leq ||x||||y||$ for any $x, y \in A$. A *normed involutive algebra* is a normed algebra A with an involution such that $||x|| = ||x^*||$ for each $x \in A$. If in addition, A is complete A is called an *involutive Banach algebra.*

Example 3.1. Let X be a compact topological space, and $C(X)$ the algebra of complex continuous valued functions on X. The map $f \mapsto \bar{f}$, where $\bar{f}(x) = \overline{f(x)}$ makes $C(X)$ an involutive algebra. Let $||f|| = \sup\{|f(x)| : x \in X\}$ makes $C(X)$ a involutive Banach algebra.

Example 3.2. Let H be a Hilbert space and $A = \mathscr{L}(H)$, the algebra of continuous endomorphisms of H. Is an involutive Banach algeba

Example 3.3. Let G be a unimodular, locally compact group and A the convolution algebra $L^1(G)$. For each $f \in L^1(G)$ set $f^*(s) = \overline{f(s^{-1})}(s \in G)$. With this involution A is an involutive algebra. Let $||f|| = \int_G |f(x)| d\mu_G(x)$ where μ_G denotes the Haar measure. Then $|| \cdot ||$ is a norm and makes A into an involutive Banach algebra.

Let A and B be any two normed involutive algebras. Then a morphism of A into B is an algebra homomorphism $f : A \longrightarrow B$ such that $f(x^*) = f(x)^*$. Note that there is no condition on the norms. An ismorphsim on the other hand has to be a norm preserving morphism with an inverse.

Definition 3.1. *A* C^*-algebra *is an involutive Banach algebra A such that $||x||^2 = ||x^*x||$ for every $x \in A$.*

Examples 3.1 and 3.2 are also examples of C^*-algebras. On the other hand the example 3.3 is not a C^*-algebra.

An element x in a C^*-algebra A is said to be *positive* if x is Hermitian (i.e. $x^* = x$) and $x = yy^*$ for a Hermitian element $y \in A$. We denote by A^+ the set of positive elements in A.

Proposition 3.1. *Let A be a C^*-algebra and I a closed two sided ideal of A. Then I is self-adjoint and A/I, endowed with the natural involutive algebra structure $(x+I)^* = x^* + I$ and the quotient norm $||x + I|| = \inf\{||x + i|| : i \in I\}$, is a C^*-algebra.*

Corollary 3.2. *Let A and B be C^*-algebras, ϕ a morphism of A into B and I the kernel of ϕ. Consider the canonical decomposition of ϕ :*

$$A \longrightarrow A/I \xrightarrow{\psi} \phi(A) \longrightarrow B$$

Then I is a closed in A, $\phi(A)$ is closed in B and ψ is an (isometric) isomorphism of the C^*-algebra A/I onto the C^*-algebra $\phi(A)$.

3.1.1 Representations

Definition 3.2. *Let A be an involutive algebra and H a Hilbert space. A* representation *of A in H is a morphism of the involutive algebra A into the involutive algebra $\mathscr{L}(H)$.*

Two representations π and π' of A in H and H' are said to be *equivalent* if there is an unitary operator U from H onto H' such that $U\pi(x) = \pi'(x)U$. A representation π of A in H is *topologically irreducible* if $H \neq 0$ and H has no closed subspaces.invariant under $\pi(A)$. A representation π of A in H is called *non-degenerate* if there is no $\xi \in H$ such that $\pi(x)\xi = 0$ for all $x \in A$.

Definition 3.3. *Let A be a normed algebra. An approximate identity of A is a net $(u_i)_{i \in I}$ of elements of A, possessing the following properties:*

i) $||u_i|| \leq 1$

ii) $||u_i x - x|| \to 0$ *and* $||x - xu_i|| \to 0$ *for every* $x \in A$.

Proposition 3.3. *Let A be an involutive Banach algebra with an approximate identity. Let R be the set of representations of A and R' the set of topologically irreducible* *representations of A. Then for each $x \in A$, we have*

$$\sup_{\pi \in R} ||\pi(x)|| = \sup_{\pi \in R'} ||\pi(x)||$$

If we denote this common value by $||x||'$, we have that the map $x \mapsto ||x||'$ is a seminorm.

In the situation above let I be the set of $x \in A$ such that $||x||' = 0$. Then I is a self-adjoint, closed, two sided ideal of A. The map $x + I \mapsto ||x||'$ is well defined and is a norm on A/I. The completion of A/I with respect to the norm $|| \cdot ||'$ is a C^*-algebra called the *enveloping* C^*-algebra of A. We have that the enveloping C^*-algebra satisfies a universal property.

Proposition 3.4. *Let A be an involutive Banach algebra with an approximate identity, B the enveloping C^*-algebra of A and τ the canonical map of A into B.*

i) If π is a representation of A, there is exactly one representation ρ of B such that $\pi = \rho \circ \tau$, and $\rho(B)$ is the C^*-algebra generated by $\pi(A)$.

ii) The map $\pi \mapsto \rho$ is a bijection of the set of representations of A onto the set of representations of B

iii) ρ is topologically irreducible if and only if π is topologically irreducible.

iv) ρ is non-degenerate if and only if π is non-degenerate.

3.1.1.1 Algebraic Representaitons

Let A be a complex algebra and E a complex vector space. We say that a homomorphism $\pi : A \longrightarrow \mathrm{End}_{\mathbb{C}}(E)$ is an algebraic representation of A in E. An algebraic representation π in E is said to be algebraically irreducible if there is no subspace of E that is invariant under $\pi(A)$. Two representation π_1 and π_2 in the complex vector spaces E_1 and E_2 respectively are said to be *algebraically equivalent* if there is an isomorphism $T : E_1 \to E_2$ such that $T\pi_1(x) = \pi_2(x)T$ for all $x \in A$.

Proposition 3.5. *Let A be a C^*-algebra*

i) Every topologically irreducible representation of the C^*-algebra A is algebraically irreducible.

ii) Every algebraically irreducible representation of A is algebraically equivalent to a topologically irreducible representation if the C^*-algebra A.

iii) Let π and π' be two irreducible representations of the C^*-algebra A in the Hilbert spaces H,H'. If π and π' are algebraically equivalent, then π and π' are equivalent.

We can thus speak of classes of irreducible representations regardless of being topologically irreducible or algebraically irreducible.

An operator $T \in \mathscr{L}(H)$ is said to be *compact*, if the closure image under T of the closed unit ball in H is compact. i.e if $\overline{T(B(0,1))}$ is compact.

Definition 3.4. *A C^*-algebra A is said to be liminal if for every irreducible representation π of A and each $x \in A$, $\pi(x)$ is compact.*

Definition 3.5. *A C^*-algebra A is said to be* postliminal *if every non-zero quotient C^*-algebra of A possesses a non-zero liminal closed two sided ideal.*

3.1.2 Topology of $\hat{A}$

Definition 3.6. *Let A be an involutive algebra. We write $\hat{A}$ for the equivalence class of non-zero topologically irreducible representations.*

Definition 3.7. *A two sided ideal of an algebra A is said to be* primitive*, if it is the kernel of a non-zero algebraically irreducible representation. We denote the set of primitive ideals of A by* $\mathrm{Prim}(A)$

Proposition 3.6. *Let A be a C^*-algebra.*

i) The primitive two-sided ideals of A are just the kernels of the non-zero topologically irreducible representations of A

ii) Every closed two-sided ideal of A is the intersection of the primitive two sided ideals containing it.

iii) The canonical mapping

$$\hat{A} \longrightarrow \mathrm{Prim}(A)$$

sending an irreduicble representations to its kernel is surjective.

The canonical map sending $\pi \in \hat{A}$ to $\ker \pi$ is not necessarily injective. That is, two irreducible representations with the same kernel do not have to be equivalent. Nonetheless this is true if A is a postliminal C^*-algebra. We will make use of the fact that the map given in 3.6 iii) is surjective by first giving a topology to $\mathrm{Prim}(A)$ and then pushing it back to $\hat{A}$.

3.1.2.1 Jacobson Topology

Let A be an algebra over the complex numbers. For each subset T of $\mathrm{Prim}(A)$ we define by $I(T)$ to be the intersection of elements in T, which is a two sided ideal of A. Let $\bar{T}$ denote the set of all primitive ideals of A containing $I(T)$.

Lemma 3.7. *i)* $\overline{\emptyset} = \emptyset$

ii) $T \subset \overline{T}$ *for* $T \subset \mathrm{Prim}(A)$

iii) $\overline{\overline{T}} = \overline{T}$

iv) $\overline{T_1 \cup T_2} = \overline{T_1} \cup \overline{T_2}$ *for* $T_1, T_2 \in \mathrm{Prim}(A)$

It follows that there is a unique topology on $\mathrm{Prim}(A)$ such that for each $T \subset \mathrm{Prim}(A)$, $\overline{T}$ is the closure of T in this topology. This topology is called the *Jacobson topology.*

Definition 3.8. *Let A be a C^*-algebra. The* spectrum *of A is the set $\hat{A}$ endowed with the coarsest topology (fewest open sets) that makes the canonical map $\hat{A} \longrightarrow \mathrm{Prim}(A)$ to be continuous.*

3.2 Borel Spaces

A *Borel* space is a set E together with a family $\mathscr{B}$ of subsets of E with the properties: $E \in \mathscr{B}$, $\emptyset \in \mathscr{B}$ and $\mathscr{B}$ is closed under countable unions, countable intersections and the taking of complements. The elements of $\mathscr{B}$ are called the Borel subsets of E.

Let E be a set, and $\mathscr{A}$ a collection of subsets of E. There is unique smallest family $\mathscr{B}_0$ of subsets of E such that $\mathscr{A} \subset \mathscr{B}_0$ and $(E, \mathscr{B}_0)$ is a Borel space. We say that the Borel structure of $(E, \mathscr{B}_0)$ is *generated* by $\mathscr{A}$.

Let E and F be two Borel spaces. A mapping $f : E \longrightarrow F$ is said to be *Borel* if the inverse image under f of every Borel subset of F is Borel subspace of E.

Let E be a Borel space, and E' a subset of E. The intersections with E' of the Borel subsets of E define a Borel structure on E'. The Borel space E' is said to be a *Borel subspace* of E.

Let E be a Borel space and R an equivalence relation on E. The subsets of E/R whose inverse images in E are Borel define a Borel structure on E/R. The Borel space E/R is said to be the *quotient Borel space* of E by R.

Let (E_α) be a familly of Borel spaces. Let E be the disjoint union of the E_α. The subsets of E whose intersection with E_α are Borel define a Borel structure on E. The Borel space E is said to be the *Borel sum* of the E_α.

Let E be a topological space. The Borel subsets of E for the topology define a Borel structure on E said to be *subordinate to the topology.* If $E' \subset E$, the Borel structure induced on E' by that of E is the structure subordinate to the topology induced on E' by the topology of E.

Let X be a Borel Space. Let $\mathscr{B}$ be the set of Borel subsets of X. A *positive measure* on X (or just a *measure*) is a mapping $\mu : \mathscr{B} \longrightarrow [0, \infty]$ such that:

1) If $X_1, X_2 \cdots$ is a sequence of mutually disjoint elements of $\mathscr{B}$,

$$\mu(X_1 \cup X_2 \cup \cdots) = \mu(X_1) + \mu(X_2) + \cdots$$

2) X is the union of a sequence of Borel subsets $Y_1, Y_2 \ldots$ such that $\mu(Y_i) < \infty$ for

every i.

Let X be a topological space and $\mathscr{B}$ the Borel space subordinate to its topology. Let μ be a positive measure on X. The *support* of μ is the set of all points $x \in X$ with the property that any open neighbourhood U of x $\mu(U) > 0$ [11]. It follows that the support of a measure is a closed set.

Remark. What we call a positive measure is commonly referred in the literature as a σ-finite measure. We can also find a Borel space referred as a measure space [12].

3.3 The Mackey Borel Structure

For every cardinal number n, the Hilbert space of families (ξ_i) of complex numbers indexed by a set of cardinality n such that $\sum |\xi_i|^2 < \infty$ is called the *standard* Hilbert space of dimension n.

Let n be a cardinal, H_n the standard Hilbert space of dimension n, A a C^*-algebra, $\mathrm{Rep}_n(A)$ the set of representations of A in H_n, and $\mathrm{Irr}_n(A)$ the set of non-zero irreducible representations of A in H_n. There is an obvious canonical map $\mathrm{Irr}_n(A) \longrightarrow \hat{A}$, namely that which maps each element of $\mathrm{Irr}_n(A)$ to its class. The canonical image of $\mathrm{Irr}_n(A)$ in $\hat{A}$ is the set $\hat{A}_n$ of classes of non-zero irreducible representations of dimension n.

We endow $\mathrm{Rep}_n(A)$ with the topology of weak pointwise convergence over A. Then $\pi_\lambda \to \pi$, where π_λ, $\pi \in \mathrm{Rep}_n(A)$ means $< \pi_\lambda(a)\xi, \eta > \to < \pi(a)\xi, \eta >$ for any $a \in A,\ \xi, \eta \in H_n$.

Lemma 3.8. *The canonical map of* $\mathrm{Irr}_n(A) \longrightarrow \hat{A}_n$ *is continuous and open.*

For a cardinal number $n \leq \aleph_0$, consider the set $\mathrm{Irr}_n(A)$. The topology on $\mathrm{Rep}_n(A)$ induces a topology on the subset $\mathrm{Irr}_n(A)$. We thus can consider the Borel structure on $\mathrm{Irr}_n(A)$ subordinate to its topology. We now take $\mathrm{Irr}(A)$ to be the Borel sum of all the $\mathrm{Irr}_n(A)$ for $n = 1, 2 \ldots \aleph_0$

Definition 3.9. *For a separable C^*-algebra A, the Mackey Borel structure on $\hat{A}$ is the quotient of the structure of* $\mathrm{Irr}(A)$ *for the canonical map* $\mathrm{Irr}(A) \longrightarrow \hat{A}$

3.4 Integration of Representations

3.4.1 Vector Fields

Let Z be set. A *vector field* on Z is an element of $\Pi_{\zeta \in Z} H(\zeta)$, where $(H(\zeta))_{\zeta \in Z}$ is a family of Hilbert spaces indexed by Z. That is, a vector field on Z is a function f from Z into $\cup_{\zeta \in Z} H(\zeta)$ such that $f(\zeta) \in H(\zeta)$ for every $\zeta \in Z$.

Definition 3.10. *Let Z be a Borel space and μ a measure on Z. A* μ-measurable field of Hilbert spaces over Z *is a pair $\mathscr{E} = ((H(\zeta))_{\zeta \in Z}, \Gamma)$ where $(H(\zeta))_{\zeta \in Z}$ is a family of Hilbert spaces indexed by Z, and where Γ is a set of vector fields satisfying the following conditions:*

i) Γ *is a vector subspace of* $\Pi_{\zeta \in Z} H(\zeta)$;

ii) there exists a sequence $(x_1, x_2, \ldots)$ of elements of Γ such that, for every $\zeta \in Z$ the linear span $\{x_n(\zeta)\}_{n=1}^{\infty}$ is dense in $H(\zeta)$ for all ζ;

iii) For every $x \in \Gamma$ the function $x \rightarrow ||x(\zeta)||$ is μ-measurable;

iv) let x be a vector field; if, for every $y \in \Gamma$, the function $\zeta \rightarrow < x(\zeta), y(\zeta) >$ is μ-measurable, then $x \in \Gamma$.

Under these conditions, the elements of Γ *are called* measurable vector fields *of* $\mathscr{E}$*. If* $x \in \Gamma$ *and* $y \in \Gamma$*, then the function* $\zeta \to < x(\zeta), y(\zeta) >$ *is measurable. Sometimes we forget about* Γ *and just say that* $\zeta \to H(\zeta)$ *is a* μ*-measurable field of Hilbert spaces.*

Definition 3.11. *Let* Z *be a Borel space. A* Borel field $\mathscr{E}$ of Hilbert spaces *over* Z *is a family* $((H(\zeta))_{\zeta \in Z}$ *of Hilbert spaces together with a set,* Γ*, of vector fields satisfying conditions i)-iv) of definition 3.10, where we replace the word "*μ*-measurable" by the word "Borel" throughout.*

We can see that the only difference between the definitions of Borel field of Hilbert spaces and measurable field of of Hilbert spaces is that the latter has a positive measure. Let us then suppose that $\mathscr{E} = ((H(\zeta))_{\zeta \in Z}, \Gamma)$ is Borel field of Hilbert spaces and that Z has a positive measure μ. Then there exist a unique $\mathscr{E} = ((H(\zeta))_{\zeta \in Z}, \Gamma')$ field of μ-measurable Hilbert spaces such that $\Gamma \subset \Gamma'$.

Let Z be Borel space and H_0 a separable Hilbert space. For every $\zeta \in Z$, put $H(\zeta) = H_0$. Let Γ be the set of all mappings from $x : Z \to H_0$ such that $< x(\zeta), a >$ is Borel for every $a \in H_0$. Then $\mathscr{E} = ((H(\zeta))_{\zeta \in Z}, \Gamma)$ is a Borel field of Hilbert spaces over Z called the *constant field* defined by H_0.

Let $\mathscr{E} = ((H(\zeta))_{\zeta \in Z}, \Gamma)$ be a Borel field of Hilbert spaces over Z. Let $Z' \subset Z$ and let Γ' be the set of all $x|_{Z'}$ such that $x \in \Gamma$. Then $\mathscr{E}' = ((H(\zeta))_{\zeta \in Z'}, \Gamma')$ is a Borel field of Hilbert spaces over Z'. We call $\mathscr{E}' = ((H(\zeta))_{\zeta \in Z'}, \Gamma')$ the *restriction* of $\mathscr{E}$ to Z'.

Lemma 3.9. *Let* Z *be a Borel space that is the union of mutually disjoint Borel sets*

$\{Z_1, Z_2 \dots Z_\infty\}$. *Then a field of Hilbert spaces* $\zeta \to H(\zeta)$ *is Borel if and only if the restriction to each* Z_n *for* $n \in \{1, 2 \dots \infty\}$ *is Borel.*

Let A be a separable C^*-algebra and as before let $\hat{A}_n$ be the set of elements of $\hat{A}$ of dimension $1, 2 \dots \infty$. Then we know that with the Mackey Borel structure in $\hat{A}$, a set is Borel, if and only if its pre-image in $\mathrm{Irr}(A)$ is Borel (see definition 3.9). The pre-image of $\hat{A}_n$ is $\mathrm{Irr}_n(A)$ which is Borel in $\mathrm{Irr}(A)$. We conclude that $\hat{A}_n$ is Borel in $\hat{A}$.

Let H_n be the standard field of dimension n. Then for each set $\hat{A}_n$ with the Mackey Borel structure construct the constant field of Hilbert spaces $\mathscr{E}_n = ((H(\zeta))_{\zeta \in \hat{A}_n}, \Gamma_n)$ over $\hat{A}_n$ where $H(\zeta) = H_n$. We have therefore that the Hilbert spaces $\mathscr{E}_n = ((H(\zeta))_{\zeta \in \hat{A}_n}, \Gamma_n)$ are Borel since they are constant and by lemma 3.9 the unique field of Hilbert spaces namely $\mathscr{E} = ((H(\zeta))_{\zeta \in \hat{A}}, \Gamma)$ over $\hat{A}$ that restricts to $\mathscr{E}_n$ in $\hat{A}_n$ is Borel.

Definition 3.12. *With the notation as in the preceding paragraph we say that* $\mathscr{E}$ *is the canonical field.*

3.4.2 Direct Integral of Hilbert Spaces

Let $\mathscr{E} = ((H(\zeta))_{\zeta \in Z}, \Gamma)$ is a μ-measurable field of Hilbert spaces over Z. A vector field $x \in \Gamma$ is said to be square-integrable if $\int_Z ||x(\zeta)||^2 d\mu(\zeta) < \infty$. If $x, y \in \Gamma$ are square-integrable then put

$$\langle x, y \rangle = \int_Z \langle x(\zeta), y(\zeta) \rangle d\mu(\zeta).$$

Then the square integrable vector fields constitute a Hilbert space H called the *direct integral* of the $H(\zeta)$ and denoted by $\int^{\oplus} H(\zeta)d\mu(\zeta)$. (We identify two fields which are equal almost everywhere.) An element x of this space is also written $\int^{\oplus} x(\zeta)d\mu(\zeta)$.

Example 3.4. Let $Z = \{1, 2 \ldots\}$ and our measure μ be counting measure. Let $\{H(n)\}_{n \in Z}$ be a family of Hilbert spaces. With counting measure every function is measurable and therefore $n \to H(n)$ is a μ-measurable field of Hilbert spaces. The space $H = \int^{\oplus} H(n)d\mu(n)$ is the subset of functions $f \in \Pi_{n=1}^{\infty} H(n)$ such that $\sum_{n=1}^{\infty} |f(n)|^2 < \infty$. We immediately recognize this space as $H(1) \bigoplus H(2) \bigoplus \cdots$

Let $\mathscr{E} = ((H(\zeta))_{\zeta \in Z}, \Gamma)$ is a μ-measurable field of Hilbert spaces over Z. For every $\zeta \in Z$, let $T(\zeta) \in \mathscr{L}(H(\zeta))$. We say that $\zeta \to T(\zeta)$ is a *measurable field of operators* if, for every $x \in \Gamma$, the field $T(\zeta)x(\zeta)$ is measurable. If so, the function $\zeta \to ||T(\zeta)||$ is measurable. Suppose further that this function is essentially bounded, in which case the field $\zeta \to T(\zeta)$ is said to be *essentially bounded*. Let $H = \int^{\oplus} H(\zeta)d\mu(\zeta)$. Then for every $x \in H$ the field $Tx : \zeta \to T(\zeta)x(\zeta)$ belongs to H. We have $T \in \mathscr{L}(H)$, and we put $T = \int^{\oplus} T(\zeta)d\mu(\zeta)$. The operators of the form $\int^{\oplus} T(\zeta)d\mu(\zeta)$ on H are called *decomposable*. If $S = \int^{\oplus} S(\zeta)d\mu(\zeta)$ and $T = \int^{\oplus} T(\zeta)d\mu(\zeta)$ are decomposable then we have

$$S + T = \int^{\oplus} (S(\zeta) + T(\zeta))d\mu(\zeta), \quad ST = \int^{\oplus} S(\zeta)T(\zeta)d\mu(\zeta)$$

$$\lambda S = \int^{\oplus} \lambda S(\zeta)d\mu(\zeta), \qquad S^* = \int^{\oplus} S^*(\zeta)d\mu(\zeta).$$

3.4.3 Integration of Representations

Let Z be a Borel space, μ a positive measure on Z, and $\zeta \to H(\zeta)$ a μ-measurable field of Hilbert spaces over Z. For each $\zeta \in Z$, let $\pi(\zeta)$ be a representation of A in $H(\zeta)$: We say that $\zeta \to \pi(\zeta)$ is a *field of representations* of A.

Definition 3.13. *The field of representations $\zeta \to \pi(\zeta)$ is said to be* measurable*, if, for each $x \in A$, the field of operators $\zeta \to \pi(\zeta)(x)$ is measurable.*

We have that $||\pi(\zeta)(x)|| \leq ||x||$ for every $x \in A$ and every $\zeta \in Z$. If the field $\zeta \to \pi(\zeta)$ is measurable we can therefore construct, for each $x \in A$, the continuous operator $\pi(x) = \int_Z^{\oplus} \pi(\zeta)(x) d\mu(\zeta)$ on the Hilbert space $H = \int^{\oplus} H(\zeta) d\mu(\zeta)$.

Definition 3.14. *With the notation above the map $x \to \pi(x)$ is a representation of A and π is said to be the* direct integral *of $\pi(\zeta)$ and we write $\pi = \int_Z^{\oplus} \pi(\zeta) d\mu(\zeta)$*

Proposition 3.10. *Let A be separable, postliminal C^*-algebra and let $\zeta \to H(\zeta)$ be the canonical field (definition 3.12). Then there exists a Borel field of representations $\zeta \to \pi(\zeta)$ where $\pi(\zeta)$ is in the class of ζ. If μ is a positive measure on $\hat{A}$ we get that $\zeta \to \pi(\zeta)$ is μ-measurable.*

3.5 Enveloping C^*-algebra of a Group

Let G be a topological group and H a Hilbert space. A *unitary representation* of G is a homomorphism $\pi : G \to U(H)$ of the group G into the group of unitary operators of $U(H)$, such that $g \mapsto \pi(g)\xi$ is continuous for every $g \in G$ and every $\xi \in H$. We sometimes talk about H_π with the understating that H_π is the Hilbert space where π acts. A non-zero unitary representation π of G is called *irreducible* if there is no proper closed subspaces of H_π invariant under $\pi(G)$.

Example 3.5. Let G be a locally compact group with Haar measure μ_G. We have a unitary representation λ of G in the Hilbert space of square-integrable functions $L^2(G)$ given by $\lambda(s)f(s) = f(s^{-1}t)$ for $f \in L^2(G)$, $s, t \in G$. We call λ the *left regular* representation of G.

Let G be a unimodular locally compact group and fix a Haar measure μ_G throughout. Recall from Example 3.3 that $L^1(G)$ is an involutive Banach algebra. If G is separable then $L^1(G)$ is separable. For a neighbourhood basis of the identity $\mathscr{V}$ construct the positive functions $u_i \in L^1(G)$ for $i \in \mathscr{V}$ such that the support of u_i is in i and the integral equals 1. Then $\{u_i\}_{i\in\mathscr{V}}$ is an approximate identity. Let π be a unitary representation of G in a Hilbert space H. For every $f \in L^1(G)$ construct the operator $\pi(f) \in \mathscr{L}(H)$ by demanding $< \pi(f)\xi, \eta >= \int_G < f(s)\pi(s)\xi, \eta > d\mu_G(s)$ for all $\xi, \eta \in H$. We denote the vector $\pi(f)\xi$ by $\int_G f(s)\pi(s)\xi d\mu_G(s)$. We relate unitary representations of G to representations of the involutive algebra $L^1(G)$ in the following proposition.

Proposition 3.11. *Let π be a unitary representation of G. Then the map $f \mapsto \pi(f)$ is a non degenerate representation of $L^1(G)$.*

We now are going to see that is also possible to associate to a representation π' of the involutive algebra $L^1(G)$ a unitary representation of G. Let $s \in G$ and suppose $\{u_i\}_{i\in\mathscr{V}}$ is an approximate identity as defined above. Consider the functions $\{\lambda_s \cdot u_i\}_{i\in\mathscr{V}}$ where $\lambda_s \cdot u_i(t) = u_i(s^{-1}t)$.

Proposition 3.12. *Let π' be a representation of the involutive algebra $L^1(G)$. Then for every $\xi \in H_\pi$ the net $\pi'(\lambda_s \cdot u_i)\xi$ converges. Moreover, if we denote by $\pi(s)\xi = \lim \pi'(\lambda_s \cdot u_i)\xi$ we get that $\pi(s)$ is a bounded operator and the map $s \mapsto \pi(s)$ is a unitary representation of G*

We can thus relate unitary representations of G with non-degenerate representations of $L^1(G)$. If π is a unitary representation of G and π' is the associated non-degenerate representation of $L^1(G)$ then a subspace is invariant under $\pi(G)$ if and only if it is invariant under $\pi'(L^1(G))$. Hence irreducible representations of G are in one to one correspondence with non-degenerate (non-zero) irreducible representations of $L^1(G)$.

Definition 3.15. *Let G be a unimodular locally compact group. Then we denote by $\widehat{G}$ the set of equivalence classes of unitary representations of G*

Definition 3.16. *Let G be a unimodular locally compact group. Then the involutive algebra $L^1(G)$ satisfies all the hypothesis of proposition 3.4. We denote by $C^*(G)$ the enveloping C^*-algebra of $L^1(G)$ and we call it the* enveloping C^*-algebra *of G.*

We can therefore identify the unitary dual $\widehat{G}$ with the spectrum $\widehat{C^*(G)}$.

Example 3.6. The left regular representation λ of G in $L^2(G)$ can be extended to a representation the representation of $C^*(G)$. We might as well regard λ as a representation of $\widehat{C^*(G)}$. We see that λ when restricted to $L^1(G)$ is a faithful representation but it might have a non-trivial kernel when regarded as a representation of $\widehat{C^*(G)}$. We will comment more on this later.

3.6 Traces

Definition 3.17. *Let A be a C^*-algebra. A trace on A^+ is a function $f : A \to [0, \infty]$ satisfying the following axioms:*

i) If $x, y \in A^+$, we have $f(x+y) = f(x) + f(y)$;

ii) If $x \in A^+$ and λ is a non-negative scalar, then we have $f(\lambda x) = \lambda f(x)$ (with the convention that $0 \cdot \infty = 0$)

iii) If $z \in A$ we have $f(zz^*) = f(z^*z)$.

Proposition 3.13. *Let A be a C^*-algebra, and f a trace on A^+.*

i) Let $\mathfrak{n}$ be the set of all $x \in A$ such that $f(xx^*) < \infty$. Then $\mathfrak{n}$ is a self-adjoint two-sided ideal of A; the two sided ideal $\mathfrak{m} = \mathfrak{n}^2$ is the set of linear combinations of elements $\mathfrak{m}^+ = \mathfrak{m} \cap A^+$, and $\mathfrak{m}^+$ is the set of $x \in A$ such that $f(x) < \infty$.

ii) There exist a unique linear form f' on $\mathfrak{m}$ which coincides with f on $\mathfrak{m}^+$.

The ideal $\mathfrak{m}$ will be called by abuse of language the *ideal of definition* of f.

Lemma 3.14. *Let G be a locally compact unimodular group with Haar measure μ_G. Let ϵ_e be the Dirac measure at the identity. There exists a preferred trace $\tilde{\epsilon_e}$ on $C^*(G)^+$ such that $u * v^*$ is in the ideal of definition of $\tilde{\epsilon_e}$ for every $u, v \in L(G)^1 \cap L^2(G)$ and*

$$\tilde{\epsilon_e}'(u * v^*) = \epsilon_e(u * v^*) = \int_G u(x)\bar{v}(x) d\mu_G(x) < \infty$$

Where $\tilde{\epsilon_e}'$ is the unique linear functional that agrees with $\tilde{\epsilon_e}$ in its ideal of definition.

3.7 The Plancherel measure

We have come to the most important section in this chapter. We are going to state what the Plancherel measure is, and what are some of its properties. We need some preliminary notation and definitions.

Let H_1, H_2 be two Hilbert spaces with norms $<,>_1, <,>_2$ respectively. Consider the complex vector space $H_1 \bigotimes_{\mathbb{C}} H_2$ with the sesquilinear form $< \xi \otimes \eta, \xi' \otimes \eta' >=< \xi, \xi' >_1 < \eta, \eta' >_2$. Then $<,>$ is a positive definite Hermitian form and we can complete $H_1 \bigotimes_{\mathbb{C}} H_2$ to obtain a Hilbert space that we denote by $H_1 \bigotimes H_2$.

Let π and π' be two unitary representations of G. For every $s \in G$, we form, in the Hilbert space tensor product $H_\pi \otimes H_{\pi'}$, the (unitary) operator $\pi(s) \otimes \pi'(s)$. The map $s \to \pi(s) \otimes \pi'(s)$ is a unitary representation of G in $H_\pi \bigotimes H_{\pi'}$, called the *tensor product* of π and π' and denoted by $\pi \otimes \pi'$.

Let H be a Hilbert space with inner-product $\langle , \rangle$. Let $\bar{H}$ denote the vector space with the same underlying additive group as H, but with scalar product $z \cdot \xi = \bar{z}\xi$ for $z \in \mathbb{C}$, $\xi \in \bar{H}$. If we give $\bar{H}$ the inner-product $[,]$ defined by $[\xi, \xi'] = \overline{\langle \xi, \xi' \rangle}$ for $\xi, \xi' \in \bar{H}$, we get that $\bar{H}$ is a Hilbert space. We call $\bar{H}$ the Hilbert space *conjugate* to H.

Let π be a unitary representation of G in H, and let $\bar{H}$ the conjugate Hilbert space conjugate to H. Each $\pi(s)$ can be seen to be a unitary operator in $\bar{H}$ and the map $s \to \pi(s)$ is a unitary representation of G in $\bar{H}$, called the conjugate representation of π, and denoted by $\bar{\pi}$.

Let H be a separable Hilbert space with an orthonormal basis $\{e_n\}$. A positive operator $T \in \mathscr{L}(H)^+$ is said to be *trace-class* if the sum $\sum_{n=1}^{\infty}\langle Te_n, e_n\rangle < \infty$. We then set $\mathrm{tr}(T) = \sum_{n=1}^{\infty} < Te_n, e_n >$. We would like to emphasize that the value of $\mathrm{tr}(T)$ is independent of the choice of the orthonormal basis. An operator S is said to be *Hilbert-Schmidt* if SS^* is trace-class.

Let G be a separable, locally compact, unimodular group with a fixed Haar measure μ_G such that $C^*(G)$ is postliminal. Since G is separable we get that $C^*(G)$ is separable. We know by proposition 3.10 that for the canonical field $\zeta \to K(\zeta)$ we can choose a Borel field of representations $\zeta \to \pi(\zeta)$ such that $\pi(\zeta)$ is in the class of ζ for every $\zeta \in \hat{G}$ and the field is measurable for every positive measure on G. Let $\bar{K}(\zeta)$ be the conjugate Hilbert space of $K(\zeta)$.

Theorem 3.15. *(Plancherel Theorem) With the notation above, let λ be the left regular representation of G (example 3.5). Let $\tilde{\epsilon_e}$ be the preferred trace coming from the Dirac measure at the identity. Then there exist a positive measure ν in $\widehat{G}$ and an isomorphism of W of $L^2(G)$ onto $\int_{\hat{G}}^{\otimes} K(\zeta) \otimes \bar{K}(\zeta)$ such that:*

i) W transforms λ into $\int_{\hat{G}}^{\otimes} \pi(\zeta) d\nu(\zeta)$

ii) The function $\zeta \to tr(\pi(\zeta))$ is measurable and and $\tilde{\epsilon_e}(x) = \int_{\hat{G}} tr(\pi(\zeta)(x)) d\nu(\zeta)$. In particular, if $u \in L^1(G) \cap L^2(G)$, we have

$$\int_G |u(s)|^2 d\mu_G(s) = \int_{\hat{G}} tr((\pi(\zeta)(u))(\pi(\zeta)(u))^*) d\nu(\zeta) \tag{3.1}$$

The theorem 3.15 gives us the existence of a measure in $\widehat{G}$ with many interesting properties. It is pleasant to see that the formula 3.1 is all what we need to obtain uniqueness.

Theorem 3.16. *Let G be a separable, locally compact, unimodular group with a fixed Haar measure μ_G such that $C^*(G)$ is postliminal. Then there exist a unique measure ν such that we have for every $u \in L^1(G) \cap L^2(G)$*

$$\int_G |u(s)|^2 d\mu_G(s) = \int_{\widehat{G}} tr((\pi(\zeta)(u))(\pi(\zeta)(u))^*) d\nu(\zeta).$$

Definition 3.18. *The unique measure of theorem 3.16 is called the* Plancherel Measure *on $\hat{G}$ associated with the Haar measure $\widehat{G}$.*

The Plancherel measure is dependent of the choice of a Haar measure μ_G. We know that two Haar measures differ by a positive scalar $k > 0$. If, say, ν is the Plancherel measure associated with μ_G then $k^{-1}\nu$ is the Plancherel measure associated with $k\mu_G$

Definition 3.19. *Let λ be the left regular representation seen as a representation of $C^*(G)$ and $N = \ker \lambda$. The* reduced dual *is the set of all $\sigma \in \hat{G}$ such that* $\ker \sigma \supset N$. *We denote the reduced dual by ${}_r\widehat{G}$.*

Lemma 3.17. *The set ${}_r\widehat{G}$ is the support of the Plancherel measure.*

Proposition 3.18. *Let G be a separable, locally compact, unimodular group with a fixed Haar measure μ_G such that $C^*(G)$ is postliminal. Let ν be the Plancherel measure corresponding to μ_G. Let $\zeta_0 \in \widehat{G}$. Then ζ_0 is square integrable if and only if $\mu(\{\zeta_0\}) > 0$, and $\mu(\{\zeta_0\})$ is then equal to the formal dimension of $\mu(\zeta_0)$.*

Remark. The Plancherel theorem as stated in Dixmier [8, 18.8.1] also gives a integral decomposition of the right regular representation as well as an integral decomposition of the von Neumann algebras generated by the left and right regular representations. It also gives a decomposition of the trace $\tilde{\epsilon_e}$. On the other hand, the hypotheses of Theorem 3.15 are the same as those assumed in Dixmier [8].

CHAPTER 4
EXPLICIT PLANCHEREL FORMULÆ

The main source for this Chapter is the paper by Bushnell Henniart and Kutzko [5]. We assume that $G = G(\mathbb{F})$ is a connected, reductive group over a non archimedean local field $\mathbb{F}$ with fixed Haar measure μ_G.

4.1 Pre-Unitary Representations

Definition 4.1. *A representation (π, V) is said to be* pre-unitary *if there is a positive definite Hermitian form $\langle\, , \rangle$ on V satisfying*

$$\langle \pi(g)v, \pi(g)w\rangle = \langle v, w\rangle, \qquad g \in G\ v, w \in V.$$

Let (π, V) be a unitary representation of G, recall that (π^∞, V^∞) is a smooth representation obtained form (π, V) (see 2.1.1 for details). We then get that (π^∞, V^∞) is a smooth pre-unitary representation, where the Hermitan form on V^∞ is that inherited from V.

Theorem 4.1. *If (π, V) is a topologically irreducible representation of G, then:*

i) the space V^∞ is non-zero;

ii) the smooth representation (π^∞, V^∞) is irreducible and hence admissible.

We have a way to go from topologically irreducible representation to irreducible pre-unitary representations. We will see that is possible to go on the other direction also. Indeed, if (π, V) is a smooth, pre-unitary, irreducible representation, it has a positive Hermitian form $\langle\, , \rangle$ invariant under the action of G. Since (π, V) is irreducible, Schur's Lemma implies that any Hermitian form is a positive scalar multiple of $\langle\,,\rangle$. We can therefore unambiguously define the completion $\tilde{V}$ of the space V with respect to the norm $v \mapsto \langle v, v\rangle$. We then obtain a unitary representation $\tilde{\pi}$ on the Hilbert space $\tilde{V}$.

Lemma 4.2. *If (π, V) is an irreducible, smooth pre-unitary representation of G, then the unitary representation $(\tilde{\pi}, \tilde{V})$ is topologically irreducible.*

4.2 Plancherel Theorem for $G(\mathbb{F})$

Theorem 4.3. *The C^*-algebra of $C^*(G)$ is liminal. [3]*

We have that G is second countable, hence separable. Moreover, G is unimodular locally compact and the C^*-algebra $C^*(G)$ is liminal. Then we have all the necessary conditions for the Plancherel theorem. Let us consider a function f in the Hecke algebra $\mathcal{H}(G)$. Then there exist a compact open subgroup K of G such that $f * e_K^* = f * e_K = f$. Let ϵ_e be the Dirac measure at the identity (see lemma 3.14 for details), then $\vec{\epsilon}'_e(f * e_K^*) = \epsilon_e(f) = f(1)$. Let ν be the associated Plancherel measure to μ_G. Then $f(1) = \int_{\hat{G}} tr(\pi(\zeta)(f))d\nu(\zeta)$. We state this as a theorem.

Theorem 4.4. *The Plancherel measure ν is the unique measure such that*

$$f(1) = \int_{\hat{G}} tr(\pi(\zeta)(f))d\nu(\zeta),\ f \in \mathcal{H}(G).$$

Uniqueness follows because for $u \in \mathcal{H}(G)$, $\int_G |u(s)|^2 d\mu_G(s) = u * u * (1) = \int_{\hat{G}} tr((\pi(\zeta)(u))(\pi(\zeta)(u))^*)d\nu(\zeta)$. Then use the fact that $\mathcal{H}(G)$ is dense in $C^*(G)$ and mimic the proof of theorem 3.16. The support of the Plancherel measure is denoted by ${}_r\hat{G}$. We therefore obtain.

Theorem 4.5. *The Plancherel measure ν is the unique measure such that*

$$f(1) = \int_{{}_r\hat{G}} tr(\pi(\zeta)(f))d\nu(\zeta).$$

4.3 Hilbert Algebras

Definition 4.2. *A* Hilbert algebra *is a complex algebra A with an involution and carrying a positive Hermitian form $[\,,]$ such that:*

i) $[x, y] = [y^*, x^*], x, y \in A$;

ii) $[xy, z] = [y, x^*z]x, y, z \in A$;

iii) for every $x \in A$ the mapping $y \mapsto xy$ of A into A is continuous with respect the topology induced by $[\,,]$;

iv) the set of elements xy for $x, y \in A$ is dense in A.

A Hilbert algebra is called *normalized* if it has a unit $\mathbf{e}$ and the inner product satisfies $[\mathbf{e}, \mathbf{e}] = \mathbf{1}$. We will assume for the rest of this thesis that our Hilbert algebras are separable.

Let A be a normalized Hilbert algebra, and let $\tilde{A}$ be the Hilbert space obtained by completing A with respect to $[\,,]$. The action of A on itself by left multiplication induces an injection of A into the space $\mathscr{L}(\tilde{A})$. We denote by ${}_rC^*(A)$ the closure of the image of A in $\mathscr{L}(\tilde{A})$.

Let A be a Hilbert algebra and π a topologically irreducible representation into some Hilbert space V. Separability of A implies V is separable, and irreducibility of V gives us that $\mathrm{End}_A(V)$ has countable dimension over $\mathbb{C}$. Then by Shur's lemma $\mathrm{End}_A(V)$ is a division algebra over $\mathbb{C}$ and since it has countable dimension over the complex numbers we get $\mathrm{End}_A(V) \cong \mathbb{C}$. If A has an identity element $\mathbf{e}$, it follows that $\pi(\mathbf{e})$ is a scalar multiple of the identity and an idempotent, thus $\pi(\mathbf{e}) = \mathbf{1_V}$. If we assume that ${}_rC^*(A)$ is liminal then $\pi(\mathbf{e})$ is a compact operator and this implies V is finite dimensional. So ${}_rC^*(A)$ is liminal if and only if all of its irreducible representations are finite dimensional.

Theorem 4.6. *Let A be a normalized Hilbert algebra, with unit element $\mathbf{e}$, such that ${}_rC^*(A)$ is liminal. There is then a unique positive measure ν_A on ${}_r\widehat{C^*(A)}$ such that:*

$$[a, \mathbf{e}] = \int_{{}_r\widehat{C^*(A)}} tr(a)\nu_A(\pi)$$

We refer to ν_A as the *Plancherel measure* for A.

Remark. Recall that the space ${}_r\widehat{C^*(A)}$ is a Borel space with the Mackey Borel structure. The Borel structure in the particular case where ${}_r\widehat{C^*(A)}$ is liminal, is the structure subordinate to its topology. When we speak of a measure on ${}_r\widehat{C^*(A)}$ we want it to be defined in the Borel sets of ${}_r\widehat{C^*(A)}$.

Lemma 4.7. *Let A be a normalized Hilbert algebra such that ${}_rC^*(A)$ is liminal. Let* $\mathrm{Irr}(A)$ *denote the equivalence classes of irreducible algebraic representations of A. The map $(\pi, V) \longrightarrow (\pi|_A, V)$ is an injection from ${}_r\widehat{C^*(A)}$ into* $\mathrm{Irr}(A)$

Theorem 2.17 says that we have a decomposition $\mathfrak{R}(G) = \prod_{\mathfrak{s}\in\mathscr{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G)$. We then obtain a disjoint union ${}_r\hat{G} = \dot{\bigcup}_{\mathfrak{s}\in\mathscr{B}}\ {}_r\hat{G}(\mathfrak{s})$ where ${}_r\hat{G}(\mathfrak{s})$ is the set of all $(\pi, V) \in {}_r\hat{G}$ such that $(\pi^\infty, V^\infty) \in \mathfrak{R}^{\mathfrak{s}}(G)$. Let (K, ρ) be an $\mathfrak{s}$-type then the algebra $e_\rho * \mathcal{H}(G) * e_\rho$

is a Hilbert algebra. Recall that for a function $f \in e_\rho * \mathcal{H}(G) * e_\rho$ we can define an involution by $f^*(x) = \overline{f(x^{-1})}$. The positive definite Hermitian form is given by $[a, b] = a * b^*(1)$ for $a, b \in e_\rho * \mathcal{H}(G) * e_\rho$. If we want this algebra to be normalized we have to choose the Haar measure μ_G such that $\mu_G(K) = 1$.

Theorem 4.8. *With the same notation as above assume further $\nu_{e_\rho * \mathcal{H}(G) * e_\rho}$ is the Plancherel measure of the Hilbert algebra A. Let ν be the Plancherel measure of G associated to the Haar measure μ_G where $\mu_G(K) = 1$. Then*

i) the sets ${}_r\hat{G}(\mathfrak{s})$ are open in ${}_r\hat{G}$;

ii) The map $\hat{m_{e_\rho}} : V \mapsto \pi(\mathbf{e})V$ is a homeomorphism from ${}_r\hat{G}(\mathfrak{s})$ to ${}_r\widehat{C^*}(e_\rho * \mathcal{H}(G) * e_\rho)$

iii) If S is a Borel subset of ${}_r\hat{G}(\mathfrak{s})$ then

$$\nu(S) = e_\rho(1_G)\nu_{e_\rho * \mathcal{H}(G) * e_\rho}(\hat{m_{e_\rho}}(S))$$

Proposition 4.9. *Let (K, ρ) be a type, then the C^*-algebra ${}_r\widehat{C^*}(e_\rho * \mathcal{H}(G) * e_\rho)$ is liminal*

4.4 The Hilbert Algebra $\mathcal{H}(G, \rho)$

Let K be a compact open subgroup of G and let (ρ, W) be an irreducible representation of K. There exists a unique, up to a constant, scalar product $\langle\, ,\rangle$ on W invariant under K. The algebra $\mathrm{End}_{\mathbb{C}}(\hat{W})$ carries an involution $a \to a^*$ given by $\langle a^* w, w'\rangle = \langle w, aw'\rangle$, $a \in \mathrm{End}_{\mathbb{C}}(\hat{W})$ $w, w' \in \hat{W}$. This involution induces an involution $h \to h^*$ on the Hecke algebra $\mathcal{H}(G, \rho)$ given by $h^*(x) = h(x^{-1})^*$ for $h \in$

$\mathcal{H}(G,\rho)$, $g \in G$. We regard $\mathcal{H}(G,\rho)$ as a normalized Hilbert algebra with positive definite Hermitian form $[\,,]$ given by

$$[f,h] = \frac{\mu_G(K)}{\dim\rho} tr(f * h^*(1))$$

Proposition 4.10. *Let (K,ρ) be a type; then the Hilbert algebra $\mathcal{H}(G,\rho)$ is liminal*

Theorem 4.11. *There exists a homeomorphism $\hat{\Phi} : {}_r\widehat{C^*}(e_\rho * \mathcal{H}(G) * e_\rho) \longrightarrow {}_r\widehat{C^*}(\mathcal{H}(G,\rho))$ such that if $S \subset {}_r\widehat{C^*}(\mathcal{H}(G,\rho))$ is Borel set, then $\nu_{e_\rho * \mathcal{H}(G) * e_\rho}(S) = (\dim\rho)^{-1}\nu_{\mathcal{H}(G,\rho)}(\hat{\Phi}(S))$*

We can expect from theorem 4.8 and theorem 4.11 a homeomorphism from ${}_r\hat{G}(\mathfrak{s})$ into ${}_r\widehat{C^*}(\mathcal{H}(G,\rho))$. Recall that we have a map $\mathbf{M}_\rho : \mathfrak{R}^{\mathfrak{s}}(G) \longrightarrow \mathcal{H}(G,\rho)$-Mod given by $\mathbf{M}((\pi,V)) = \mathrm{Hom}_K(\rho,V) = V_\rho$ that is an equivalence of categories. We now state the main theorem of this Chapter.

Theorem 4.12. *Let (K,ρ) be an $\mathfrak{s}$-type.*

i) The C^*-algebra ${}_rC^*(\mathcal{H}(G,\rho))$ is liminal.

ii) Let $(\pi,V) \in {}_r\hat{G}(\mathfrak{s})$. The space V_ρ is non-zero and is a simple $\mathcal{H}(G,\rho)$-module. The natural action of $\mathcal{H}(G,\rho)$ on V_ρ extends uniquely to a representation π_ρ of ${}_rC^*(\mathcal{H}(G,\rho))$.

iii) The map $(\pi,V) \mapsto (\pi_\rho,V_\rho)$ induces a homeomorphism

$$\widehat{\mathbf{M}}_\rho : {}_r\widehat{G}(\mathfrak{s}) \longrightarrow {}_r\widehat{C^*}(\mathcal{H}(G,\rho))$$

iv) If S is a Borel subset of ${}_r\hat{G}(\mathfrak{s})$ and ν is the Plancherel measure associated to the

Haar measure μ_G, then

$$\nu(S) = \frac{\dim\rho}{\mu_G(K)}\nu_{\mathcal{H}(G,\rho)}(\hat{\mathbf{M}}_\rho(S))$$

4.5 The Affine Hecke Algebras $\mathcal{H}(q_1, q_2)$

The results of this section are taken from a paper by P. Kutzko and L. Morris [9]. In the paper the authors compute the Plancherel measure for the affine Hecke algebras $\mathcal{H}(q_1, q_2)$ in two real parameters $q_1 \geq q_2 \geq 1$. This is will be useful for us because we will show later that for certain types (K, ρ) for $\mathrm{PGL}_2(G)$ the Hecke algebra $\mathcal{H}(G, \rho)$ is isomorphic to an affine Hacke $\mathcal{H}(q_1, q_2)$.

Definition 4.3. *Let $q_1 \geq q_2 \geq 1$ be two fixed real numbers and set $\gamma_i = q_i^{1/2}, c_i = \gamma_i - \gamma_i^{-1}$ for $i = 1, 2$. We let $\mathcal{H}(q_1, q_2)$ be the complex algebra with identity $\mathbf{1}$ and two generators $s_i,\ i = 1, 2$ subject to only the relations*

$$s_i^2 = c_i s_i + 1,\ \ i = 1, 2$$

Viewed as a complex vector space the algebra $\mathcal{H}(q_1, q_2)$ has a basis consisting of elements $\Pi_{i=1}^k u_i,\ u_i \in \{s_1, s_2\}$ where $u_i \neq u_{i+1}$ for $1 \leq i \leq k-1$. (We allow for the case $k = 0$ as well; in that case, we set $w = \mathbf{1}$.) We refer to these elements as *words* and denote the set of words by W.

We can give $\mathcal{H}(q_1, q_2)$ the structure of a Hilbert algebra. Let $x \to x^*$ be the characterised by the following properties.

i) $s_i^* = s_i,\ i = 1, 2$

ii) $x \to x^*$ is multiplication reversing and conjugate linear.

We define the functional $\Lambda : \mathcal{H}(q_1, q_2) \to \mathbb{C}$ by setting $\Lambda(\mathbf{1}) = 1$, $\Lambda(w) = 0$, $w \in W$, $w \neq 1$. For $x, y \in \mathcal{H}(q_1, q_2)$ we set $[x, y] = \Lambda(xy^*)$.

Proposition 4.13. *$\mathcal{H}(q_1, q_2)$ is a Hilbert algebra with respect to $[\,,]$.*

Let ρ be a 1-dimensional algebra homomorphism from $\mathcal{H}(q_1, q_2)$ into $\mathbb{C}$. Then $\rho(s_i)^2 = c_i\rho(s_i) + 1$ and this quadratic equation has two solutions i.e $\rho(s_i) = \gamma_i$ or $\rho(s_i) = -\gamma_i^{-1}$. We therefore have 4 possible 1-dimensional algebra homomorphisms.

We set $d = s_1 s_2$ and set $D = \mathbb{C}[d, d^{-1}]$. We then get that $\mathcal{H}(q_1, q_2) = D \bigoplus Ds_1$. We have a functor of induction that we denote by $\mathrm{ind}_D^{\mathcal{H}(q_1,q_2)} : D\text{-Mod} \to \mathcal{H}(q_1, q_2)\text{-Mod}$ given by $\mathrm{ind}_D^{\mathcal{H}(q_1,q_2)} N = \mathrm{Hom}_D(\mathcal{H}(q_1, q_2), N)$ for a left D module N. We consider a 1-dimensional representation $\chi : D \longrightarrow \mathbb{C}$ and denote by $\mathbb{C}_\chi$ the space where χ acts. It is clear that the representation $\mathbb{C}_\chi$ only depends on the value of $\chi(d)$. We set $(\sigma_\chi, M_\chi) = \mathrm{ind}_D^{\mathcal{H}(q_1,q_2)}\mathbb{C}_\chi$ and note that (σ_χ, M_χ) is then a two dimensional representation of $\mathcal{H}(q_1, q_2)$.

Proposition 4.14. *All the irreducible unitary representations of $\mathcal{H}(q_1, q_2)$ have dimension less or equal than 2. Moreover, the two dimensional unitary representations are of the form (σ_χ, M_χ) where $|\chi(d)| = 1$ and $\mathrm{Im}\chi(d) \geq 0$ or $\chi(d) \in (-\frac{\gamma_1}{\gamma_2}, 1) \cup (1, \gamma_1\gamma_2)$.*

Proposition 4.15. *Let $\widehat{\mathcal{H}}(q_1, q_2)$ denote the set of irreducible unitary representations of $\mathcal{H}(q_1, q_2)$. The map $(\pi, V) \mapsto (\pi|_{\mathcal{H}(q_1,q_2)}, V)$ is an injection of ${}_r\widehat{C^*}(\mathcal{H}(q_1, q_2))$ into $\widehat{\mathcal{H}}(q_1, q_2)$.*

We can see that proposition 4.15 implies that all irreducible representations of ${}_rC^*(\mathcal{H}(q_1,q_2))$ are of dimension less or equal than 2. It follows that ${}_rC^*(\mathcal{H}(q_1,q_2))$ is a liminal C^*-algebra. We then have that there exist a Plancherel measure $\nu_{\mathcal{H}(q_1,q_2)}$ for $\mathcal{H}(q_1,q_2)$. We can write $\widehat{{}_rC^*}(\mathcal{H}(q_1,q_2)) = \hat{A}_2 \bigcup \hat{A}_1$ where $\hat{A}_i$ denotes the set of equivalence classes of irreducible representations of dimension $i = 1, 2$

Theorem 4.16. *Let $\hat{A}_2$ be as above. Let $Y = \{s \in \mathbb{C} | \ |s| = 1 \text{ and } \mathrm{Im}(s) \geq 0\}$. Then the map $\theta : Y \to \hat{A}_2$ given by $\theta(s) = (\sigma_\chi, M_\chi)$ where $\chi(d) = s$ is a homeomorphism.*

We thus get a positive measure ν_0 in Y given by $\nu_0(S) = \nu_{\mathcal{H}(q_1,q_2)}(\theta(S))$. We have that s_1 and s_2 are invertible elements in $\mathcal{H}(q_1,q_2)$ therefore $d = s_1 s_2$ is an invertible element. Let $z = d + d^{-1}$ and set $f(z) = z^2 - c_1 c_2 z - (c_1^2 + c_2^2 + 4) = (z - (\gamma_1\gamma_2 + \frac{1}{\gamma_1\gamma_2}))(z + (\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1}))$.

Theorem 4.17. *Let $|dz|$ denote the positive measure on Y given by arclength. Let $\mathtt{h} : Y \to \mathbb{C}^+$ be given by $\mathtt{h}(s) = \frac{\chi(z^2-4)}{\chi(f(z))}$ where $\chi(d) = s$. Then $|dz|$ is absolutely continuous with respect to $d\nu_0$ and $d\nu_0 = \frac{1}{2\pi}\mathtt{h}|dz|$.*

We can see that we have a very explicit description of the Plancherel measure in $\hat{A}_2$. We now take our attention to the $\hat{A}_1$. We have already mentioned that $\hat{A}_1$ has at most 4 points.

Theorem 4.18. *$\hat{A}_1$ has at most 2 points and consists of the representations (non necessarily different) ρ_1, ρ_2 where $\rho_1(d) = -\frac{\gamma_2}{\gamma_1}$ and $\rho_2(d) = \frac{1}{\gamma_2\gamma_1}$. The points have positive measure*

$$\nu_{\mathcal{H}(q_1,q_2)}(\rho_1) = \frac{1}{2}\left(\frac{q_1-1}{q_1+1} + \frac{q_2-1}{q_2+1}\right) \qquad \nu_{\mathcal{H}(q_1,q_2)}(\rho_2) = \frac{1}{2}\left(\frac{q_1-1}{q_1+1} - \frac{q_2-1}{q_2+1}\right).$$

4.5.1 Explicit Plancherel measure for $\mathcal{H}(q,1)$

We have completed the goal of giving a description of the Plancherel measure $\nu_{\mathcal{H}(q_1,q_2)}$. We will make use of this result while computing the Plancherel measure for $\mathrm{PGl}_2(\mathbb{F})$. We will show later that the Hecke algebras obtained from the types for $\mathrm{PGl}_2(\mathbb{F})$ are isomorphic to the algebras $\mathcal{H}(q,1)$ where $q>1$ or the group algebra of the integers $\mathbb{C}[\mathbb{Z}]$.

Let $s \in \mathbb{C}$ and define $L(1,x) = (1-q^{-s})$. The map $t \longrightarrow q^{it}$ from $[0, \frac{\pi}{\ln q}]$ into Y is a Borel isomorphism that takes the measure $\frac{\ln(q)}{2\pi}dt$ to $\frac{1}{2\pi}|dz|$. Let $P(t) = \mathtt{h}(q^{it}) = \frac{\chi(z^2-4)}{\chi(f(z))}$ where $\chi(d) = q^{it}$. We will like to write the function $P(t)$ in terms of functions of the form $L(1,s), s \in \mathbb{C}$. For that end we first compute $\chi(z^2-4) = (q^{it}+q^{it})^2 - 4 = q^{2it} + q^{-2it} - 2 = -(1-q^{2it})(1-q^{2it}) = \frac{-1}{L(1,-2it)L(1,2it)}$. We now compute $\chi(f(z)) = \chi((z-(q^{1/2}+q^{-1/2})) \cdot \chi((z+(q^{1/2}+q^{-1/2}))$. We get the following equations:

$$\chi(z-(q^{1/2}+q^{-1/2})) = (q^{it}+q^{-it}-(q^{1/2}+q^{-1/2})) = -q^{1/2}(1-q^{it-1/2})(1-q^{-it-1/2}) \quad (4.1)$$

$$\chi(z+(q^{1/2}+q^{-1/2})) = (q^{it}+q^{-it}+(q^{1/2}+q^{-1/2})) = q^{1/2}(1+q^{it-1/2})(1+q^{-it-1/2}) \quad (4.2)$$

Multiplying 4.1 and 4.2 we get

$$\chi(f(z)) = -q(1-q^{2it-1})(1-q^{-2it-1}) = \frac{-q}{L(1,1-2it)L(1,1+2it)}$$

We then conclude that

$$P(t) = \mathtt{h}(q^{it}) = \frac{\chi(z^2-4)}{\chi(f(z))} = q^{-1}\frac{L(1,1-2it)L(1,1+2it)}{L(1,-2it)L(1,2it)}$$

We also have that $\mathcal{H}(q,1)$ has two distinct one dimensional representations ρ_1, ρ_2 with positive Plancherel measure where $\rho_1(d) = q^{-1/2}$, $\rho_2(d) = -q^{-1/2}$.

Corollary 4.19. *The Plancherel measure $\nu_{\mathcal{H}(q,1)}$ of the Hecke algebra $\mathcal{H}(q,1)$ can be identified with the interval $[0, \frac{\pi}{\ln q}]$ and a measure $\frac{\ln(q)}{2\pi}Pdt$ where dt gives the Lebesgue measure in the interval and*

$$P(t) = q^{-1}\frac{L(1,1-2it)L(1,1+2it)}{L(1,-2it)L(1,2it)}$$

Union two points ρ_1, ρ_2 where

$$\nu_{\mathcal{H}(q,1)}(\rho_1) = \nu_{\mathcal{H}(q,1)}(\rho_2) = \frac{1}{2}\left(\frac{q-1}{q+1}\right).$$

CHAPTER 5
EXPLICIT PLANCHEREL THEOREM FOR $\mathbf{PGL}_2(\mathbb{F})$

In this Chapter is where we are going to present new results. Namely we are going to compute an explicit Plancherel measure for $\mathrm{PGL}_2(\mathbb{F})$. We first introduce some notation. Unless explicitly stated otherwise, by G we mean $\mathrm{GL}_2(\mathbb{F})$. We denote by Z the center of G, by B_u the upper triangular matrices and by B_ℓ the lower triangular matrices. We denote by $\bar{G}$ the quotient group $G/Z = \mathrm{PGL}_2(\mathbb{F})$. Likewise if $x \in \mathrm{GL}_2(\mathbb{F})$ and $S \subset \mathrm{GL}_2(\mathbb{F})$ we denote by $\bar{x}$ and $\bar{S}$ the image of x and S respectively under the quotient map. We denote by $\mathcal{O}$ the ring of integers in $\mathbb{F}$ with maximal prime ideal $\mathfrak{p}$ and by ϖ a prime element in $\mathcal{O}$. We let q be the number of elements in the finite field $\mathcal{O}/\mathfrak{p}$ and we let v be the valuation on $\mathbb{F}^\times$ such that $v(\varpi) = 1$.

5.1 Types and Covers for $\mathbf{PGL}_2(\mathbb{F})$

We have that G has up to conjugacy only two Levi subgroups. The first one is the diagonal matrices that which we denote by $\mathbb{T}$ and the second one is the whole group G. Similarly $\bar{G}$ has up to conjugacy only two Levi subgroups namely $\bar{\mathbb{T}}$ and $\bar{G}$. Let $\bar{\mathfrak{s}} = [\bar{\mathbb{T}}, \phi]_G \in \mathscr{B}(\bar{G})$ (see Bernstein decomposition 2.17). We have then by definition of $\bar{\mathfrak{s}}$ that ϕ is a character for $\bar{\mathbb{T}} \cong \mathbb{F}^\times$. If we consider ϕ as a character of $\mathbb{F}^\times$, we can regard $\phi \otimes \phi^{-1}$ as a character of $\mathbb{T}$ and hence construct an element $\mathfrak{s} \in \mathscr{B}(G)$ by setting $\mathfrak{s} = [\mathbb{T}, \phi \otimes \phi^{-1}]_G$. Later we will show that an $\mathfrak{s}$-type can be related to a $\bar{\mathfrak{s}}$-type.

5.1.1 Types and Covers for $\mathrm{GL}_2(\mathbb{F})$ related to $\mathrm{PGL}_2(\mathbb{F})$

The goal of this subsection is to construct an $\mathfrak{s}$-type for $\mathfrak{s}$ of the form $[\mathbb{T}, \varphi \otimes \varphi^{-1}]_G \in \mathscr{B}(G)$. Let $[\mathfrak{s}]_{\mathbb{T}} = [\mathbb{T}, \varphi \otimes \varphi^{-1}]_{\mathbb{T}}$. It is rather trivial that the pair $({}^\circ\mathbb{T}, \varphi \otimes \varphi^{-1}|_{{}^\circ\mathbb{T}})$ is a $[\mathfrak{s}]_{\mathbb{T}}$-type. We will give a justification of this fact as a warm up. Let (π, V) be an irreducible representation of $\mathbb{T}$. Then $(\pi, V) \in \mathfrak{R}^{[\mathfrak{s}]_{\mathbb{T}}}(\mathbb{T})$ if and only if $\pi = \varphi \otimes \varphi^{-1} \otimes \chi$, where χ is an unramified character i.e trivial in ${}^\circ\mathbb{T}$. Hence the restriction of π to ${}^\circ\mathbb{T}$ is equal to $\varphi \otimes \varphi^{-1}|_{{}^\circ\mathbb{T}}$ showing that $\mathrm{Hom}_{{}^\circ\mathbb{T}}(\varphi \otimes \varphi^{-1}|_{{}^\circ\mathbb{T}}, V) \neq 0$ and therefore showing that $({}^\circ\mathbb{T}, \varphi \otimes \varphi^{-1}|_{{}^\circ\mathbb{T}})$ is a $[\mathfrak{s}]_{\mathbb{T}}$-type.

It follows from the previous paragraph that in order to construct an $\mathfrak{s}$-type it is enough to construct a cover for $({}^\circ\mathbb{T}, \varphi \otimes \varphi^{-1}|_{{}^\circ\mathbb{T}})$. Define the integer $sw(\varphi)$ to be the smallest positive integer n such that $1 + \mathfrak{p}^n \subset \ker \varphi$. Let $J = J_\varphi$ be the compact subgroup of G given by:

$$J = \{[c_{ij}] \in G|\ c_{11}, c_{22} \in \mathcal{O}^\times,\ c_{12} \in \mathcal{O},\ c_{21} \in \mathfrak{p}^{sw(\varphi^2)}\}$$

and define the function $\lambda = \lambda_\varphi$ on J by $\lambda([c_{ij}]) = \varphi^2(c_{11})\varphi^{-1}(\det[c_{ij}])$. Let us check that λ is a one dimensional representation of J.

$$\begin{aligned}
\lambda([a_{ij}] \cdot [b_{ij}]) &= \varphi^2(a_{11}b_{11} + a_{12}b_{21}) \cdot \varphi^{-1}(\det([a_{ij}] \cdot [b_{ij}])) \\
&= \varphi^2(a_{11}b_{11}(1 + a_{12}b_{21}(a_{11}b_{11})^{-1})) \cdot \varphi^{-1}(\det([a_{ij}]) \det([b_{ij}])) \\
&= \varphi^2(a_{11}b_{11}) \cdot \varphi^{-1}(\det([a_{ij}]) \det([b_{ij}])) = \lambda([a_{ij}]) \cdot \lambda([b_{ij}]).
\end{aligned}$$

Only the third equality deserves an explanation and that follows from the fact that $a_{12} \in \mathcal{O}$ and $b_{21} \in \mathfrak{p}^{sw(\varphi^2)}$ then $a_{12}b_{21} \in \mathfrak{p}^{sw(\varphi^2)}$. Since a_{11} and b_{11} are units in $\mathcal{O}$ we get that $1 + a_{12}b_{21}(a_{11}b_{11})^{-1} \in 1 + \mathfrak{p}^{sw(\varphi^2)} \subset \ker \varphi^2$, hence the equality. Note that if $[c_{ij}]$

is a diagonal matrix in J then $\lambda([c_{ij}]) = \varphi^2(c_{11})\varphi^{-1}(\det[c_{ij}]) = \varphi^2(c_{11})\varphi^{-1}(c_{11}c_{22}) = \varphi(c_{11}c_{22}^{-1}) = \varphi\otimes\varphi^{-1}([c_{ij}])$. In other words the restriction of λ_φ to $\mathbb{T}\cap J = {}^\circ\mathbb{T}$ is equal to $\varphi\otimes\varphi^{-1}$.

The group J is compact and the pair $(J_\varphi, \lambda_\varphi)$ is our candidate to be a G-cover for $({}^\circ\mathbb{T}, \varphi\otimes\varphi^{-1}|_{{}^\circ\mathbb{T}})$ and therefore an $\mathfrak{s}$-type. We have only two parabolic subgroups with Levi component $\mathbb{T}$ namely the upper triangular matrices and the lower triangular matrices. For a matrix $[c_{ij}] \in G$, if $c_{11} \neq 0$ we have

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c_{21}c_{11}^{-1} & 1 \end{bmatrix} \begin{bmatrix} c_{11} & 0 \\ 0 & (c_{11}c_{22} - c_{21}c_{12})c_{11}^{-1} \end{bmatrix} \begin{bmatrix} 1 & c_{12}c_{11}^{-1} \\ 0 & 1 \end{bmatrix} \tag{5.1}$$

and if $c_{22} \neq 0$ we have

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 1 & c_{12}c_{22}^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (c_{11}c_{22} - c_{21}c_{12})c_{22}^{-1} & 0 \\ 0 & c_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ c_{12}c_{22}^{-1} & 1 \end{bmatrix} \tag{5.2}$$

It follows at once that for a parabolic subgroup P with Levi component $\mathbb{T}$ we get (J, λ) is decomposed with respect to $(\mathbb{T}, P)$ (See 2.12 for the definition). Indeed, if $P = B_u$ the upper triangular matrices then 5.1 gives us the desired decomposition and if $P = B_\ell$ is the lower triangular matrices then equation 5.2 will do the job. We have remarked before that the restriction of λ_φ to $\mathbb{T}\cap J = {}^\circ\mathbb{T}$ is equal to $\varphi\otimes\varphi^{-1}$. Hence in order to show that $(J_\varphi, \lambda_\varphi)$ is a G-cover for $({}^\circ\mathbb{T}, \varphi\otimes\varphi^{-1}|_{{}^\circ\mathbb{T}})$ we need to prove the existence of an invertible element in $\mathcal{H}(G, \lambda)$ supported on Jz_PJ where z_P is a strongly positive element with respect to (J, P) for every parabolic subgroup P with Levi component $\mathbb{T}$. (see 2.15 if you need to recall the definition of a cover)

Consider the set $\mathcal{I}_G(\lambda) = \{x \in G |\, \mathrm{Hom}_{x^{-1}Jx\cap J}(\lambda^x, \lambda) \neq 0\}$ (definition 2.10).

Let $\mathcal{H}_x(G,\lambda) = \{f \in \mathcal{H}(G,\lambda) | \operatorname{supp}(f) \subset JxJ\}$. We see that for $f \in \mathcal{H}_x(G,\lambda)$ f is determined by its value at x and since λ is one dimensional, $\mathcal{H}_x(G,\lambda)$ has dimension one for $x \in \mathcal{I}_G(\lambda)$. Let $g_x \in \mathcal{H}_x(G,\lambda)$ for $x \in \mathcal{I}_G(\lambda)$ be given by $g_x(x) = 1$. Likewise we can define $g_{\bar{x}} \in \mathcal{H}_{\bar{x}}(\bar{G},\bar{\lambda})$ for $\bar{x} \in \mathcal{I}_{\bar{G}}(\bar{\lambda})$ be given by $g_{\bar{x}}(\bar{x}) = 1$.

Lemma 5.1. *Let G be a connected reductive group over a non archimedean local field. Let J be a compact subgroup and λ a one dimensional representation of J. Let $g_x \in \mathcal{H}_x(G,\lambda)$ for $x \in \mathcal{I}_G(\lambda)$ be given by $g_x(x) = 1$. Then $g_x^* = g_{x^{-1}}$.*

Proof. Let $y = j_1 x^{-1} j_2$ where $j_1, j_2 \in J$. Then $g_x^*(y) = (g_x(y^{-1}))^* = (g_x(j_2^{-1} x j_1^{-1}))^* = (\check{\lambda}(j_2^{-1})\check{\lambda}(j_1^{-1}))^* = (\check{\lambda}(j_1^{-1}))^*(\check{\lambda}(j_2^{-1}))^* = \check{\lambda}(j_1)\check{\lambda}(j_2) = g_{x^{-1}}(y)$. If y is not in $Jx^{-1}J$ then y^{-1} is not in $Jx^{-1}J$ so $g_x^*(y) = (g_x(y^{-1}))^* = 0$.

Let $\Pi = \begin{bmatrix} \varpi & 0 \\ 0 & 1 \end{bmatrix}$. Then for $n \geq 1$, Π^n is easily seen to be strongly positive for (J, B_u) and Π^{-n} is strongly positive for (J, B_ℓ). The elements Π, Π^{-1} are in $\mathcal{I}_G(\lambda)$ hence the existence of g_Π and $g_{\Pi^{-1}}$.

Proposition 5.2. *If $\varphi^2|_{\mathcal{O}^\times} \neq 1$ then the pair $(J_\varphi, \lambda_\varphi) = (J, \lambda)$ is a G-cover for $({}^\circ\mathbb{T}, \varphi \otimes \varphi^{-1}|_{{}^\circ\mathbb{T}})$.*

Proof. We claim that the element $g_\Pi * g_{\Pi^{-1}}$ has support in J. We know that $g_\Pi * g_{\Pi^{-1}}$ has support in $(J\Pi J\Pi^{-1}J) \cap \mathcal{I}_G(\lambda)$. Using the decomposition of $J = J_\ell \cdot J_\mathbb{T} \cdot J_u$ with respect to B_u, we get that $\Pi J \Pi^{-1} = (\Pi J_\ell \Pi^{-1})(\Pi J_\mathbb{T} \Pi^{-1})(\Pi J_u \Pi^{-1}) = \Pi J_\ell \Pi^{-1} J_\mathbb{T} J_u \subset \Pi J_\ell \Pi^{-1} J$. The last equality is true because Π is positive element with respect to (J, B_u), so $\Pi J_u \Pi^{-1} \subset J_u$ and Π commutes with every element of $J_\mathbb{T}$. Let $c(y) = \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}$ and suppose that $c(y) \in (\Pi J_\ell \Pi^{-1} - J)$. We see that $v(y) = sw(\varphi^2) - 1$.

We will prove that $c(y)$ does not intertwine λ. Since $\varphi^2|_{\mathcal{O}^\times} \neq 1$ there exists $a \in \mathcal{O}^\times$ such that $\varphi^2(a) \neq 1$. Let us first consider the case where $sw(\varphi^2) = 1$. Then $y^{-1} \in \mathcal{O}$ and set

$$x_1 = \begin{bmatrix} a & (a-1)y^{-1} \\ 0 & 1 \end{bmatrix},\ x_2 = \begin{bmatrix} 1 & (a-1)y^{-1} \\ 0 & a \end{bmatrix}$$

We have $x_1, x_2 \in J$ and $c(y)x_1c(y)^{-1} = x_2$ but $\lambda(x_1) = \varphi^2(a)\varphi^{-1}(a) = \varphi(a) \neq \varphi(a^{-1}) = \lambda(x_2)$ so $c(y)$ does not intertwine λ.

If $sw(\varphi^2) \geq 2$ take $z \in \mathfrak{p}^{sw(\varphi^2)-1}$ such that $\varphi^2(1+z) \neq 1$. Then $zy^{-1} \in \mathcal{O}$ and $zy \in \mathfrak{p}^{sw(\varphi^2)}$. Consider the elements

$$x_1 = \begin{bmatrix} 1 & -zy^{-1} \\ -zy & 1 \end{bmatrix},\ x_2 = \begin{bmatrix} 1+z & -zy^{-1} \\ 0 & 1-z \end{bmatrix}$$

We have $x_1, x_2 \in J$ and $c(y)x_1c(y)^{-1} = x_2$ but $\lambda(x_1) = \varphi^2(1)\varphi^{-1}(1-z^2)$ and $\lambda(x_2) = \varphi^2(1+z)\varphi^{-1}(1-z^2)$. Since $\varphi^2(1+z) \neq 1$ we obtain $\lambda(x_1) \neq \lambda(x_2)$. This finishes the proof of our claim.

We then get that $g_\Pi * g_{\Pi^{-1}}$ is a constant multiple of the identity. If we show that $g_\Pi * g_{\Pi^{-1}}(1) \neq 0$ we will finish the proof of the proposition. Let us compute $g_\Pi * g_{\Pi^{-1}}(1)$

$$\begin{aligned} g_\Pi * g_{\Pi^{-1}}(1) &= \int_G g_\Pi(x) g_{\Pi^{-1}}(x^{-1}) d\mu_G(x) \\ &= \int_{J\Pi J} g_\Pi(x) g_{\Pi^{-1}}(x^{-1}) d\mu_G(x) \\ &= \mu(J\Pi J) = q \neq 0 \end{aligned}$$

The first equality is just the definition, the second equality follows from the fact that the support of g_Π is contained in $J\Pi J$ and the third equality is true because if $x \in J\Pi J$ then $x = j_1\Pi j_2$ for some $j_1, j_2 \in J$ then $g_\Pi(x)g_{\Pi^{-1}}(x^{-1}) = g_\Pi(j_1\Pi j_2)g_{\Pi^{-1}}(j_2^{-1}\Pi^{-1}j_1^{-1}) = \check{\lambda}(j_1)\check{\lambda}(j_2)\check{\lambda}(j_2^{-1})\check{\lambda}(j_1^{-1}) = 1$. This finishes the proof of the proposition.

We will now consider the case where $\varphi^2|_{\mathcal{O}^\times} = 1$. In this case J_φ is the Iwahori subgroup and we denote it by I. If we let $K = \mathrm{GL}_2(\mathcal{O})$ and $w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ we have the decomposition $K = \mathrm{GL}_2(\mathcal{O}) = I \cup IwI$. The element w intertwines λ, therefore there is an element $g_w \in \mathcal{H}(G, \lambda)$

Proposition 5.3. *If $\varphi^2|_{\mathcal{O}^\times} = 1$ then the pair $(J_\varphi, \lambda_\varphi) = (J, \lambda)$ is a G-cover for $({}^\circ\mathbb{T}, \varphi \otimes \varphi^{-1}|_{{}^\circ\mathbb{T}})$.*

Proof. The support of $g_w * g_w$ is contained in $K = I \cup IwI$, therefore there exists complex numbers a and b, such that $g_w * g_w = ag_w + bg_1$.

$$\begin{aligned} g_w * g_w(1) &= \int_G g_w(x)g_w(x^{-1})d\mu_G(x) \\ &= \int_{IwI} g_w(x)g_w(x^{-1})d\mu_G(x) = \mu_G(IwI) \neq 0 \end{aligned}$$

We get that $b = g_w * g_w(1) \neq 0$. So g_w satisfies a quadratic equation with non-zero constant term and therefore it is invertible. Let $\alpha = \begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}$. Conjugation by α is an automorphism of G that fixes I and also we have that $\lambda(\alpha x\alpha^{-1}) = \lambda(x)$ for all $x \in I$. Therefore there are elements g_α and $g_{\alpha^{-1}}$. We have that $g_\alpha * g_{\alpha^{-1}}$ has support contained in $I\alpha I\alpha^{-1}I = I$. We see that $g_\alpha * g_{\alpha^{-1}} = \mu_G(I\alpha I) \neq 0$. We then conclude that g_α is invertible and since g_w is invertible we get that $g_\alpha * g_w$ is invertible. The support of $g_\alpha * g_w$ lies in $I\alpha IwI = I\alpha wI = I\Pi I$ so we get that the proposition holds true.

5.2 Types and Covers for $\mathbf{GL}_2(\mathbb{F})$ mod the center

We have computed types and covers for G and we mentioned before that we want to relate them to types and covers for $\bar{G}$. Let $P = LN$ be a parabolic subgroup with Levi component L and unipotent subgroup N. Let σ be a supercuspial representation of L trivial on Z. Then $\bar{P} = \bar{L}\bar{N}$ is a parabolic subgroup of $\bar{G}$ with Levi component $\bar{L}$ and unipotent subgroup $\bar{N}$. Let $\mathfrak{s} = [L, \sigma]_G$ and by $\bar{\mathfrak{s}} = [\bar{L}, \bar{\sigma}]_{\bar{G}}$, where $\bar{\sigma}(\bar{x}) = \sigma(x)$ for $x \in L$. Finally, let (ρ, W) be an irreducible representation of a compact open subgroup J of G.

Proposition 5.4. *With the notation above, if (J, ρ) is an $\mathfrak{s}$-type then $\rho(z) = 1$ for all $z \in Z \cap J$ and $(\bar{J}, \bar{\rho})$ is an $\bar{\mathfrak{s}}$-type, where $\bar{\rho}(\bar{x}) = \rho(x)$ for $x \in J$.*

Proof. Let $f \in \mathrm{Ind}_P^G \sigma$, then $z \cdot f(x) = \sigma(z) f(x) = f(x)$ for all $z \in Z$ so $\mathrm{Ind}_P^G \sigma$ has a trivial central character. Since (ρ, W) is an $\mathfrak{s}$-type $\mathrm{Hom}_J(\rho, \mathrm{Ind}_P^G \sigma) \neq 0$ so ρ has also a trivial central character. So makes sense to talk about $\bar{\rho}$ as a representation of $\bar{J} \cong J/(J \cap Z)$. Now let $(\bar{\pi}, V)$ be an irreducible representation of $\bar{G}$ such that $\mathfrak{I}((\bar{\pi}, V)) \in \bar{\mathfrak{s}}$. Consider (π, V) the representation of G such that $\pi(x) = \bar{\pi}(\bar{x})$. Since $\mathfrak{I}((\bar{\pi}, V)) \in \bar{\mathfrak{s}}$ then there exist an unramified character $\bar{\psi}$ of $\bar{L}$ such that $\mathrm{Hom}_{\bar{G}}(\bar{\pi}, \mathrm{Ind}_{\bar{P}}^{\bar{G}} \bar{\sigma}^{\bar{g}} \otimes \bar{\psi}) \neq 0$. Let ψ be the representation of L such that $\psi(x) = \bar{\psi}(\bar{x})$ for $x \in L$ and let $g \in \bar{g}$. For $f \in \mathrm{Ind}_P^G \sigma^g \otimes \psi$ we get $z \cdot f(x) = \sigma(gzg^{-1})\psi(z) f(x) = f(x)$. We can thus consider $\overline{\mathrm{Ind}_P^G \sigma^g \otimes \psi}$ and the map $f \mapsto \bar{f}$ where $\bar{f}(\bar{x}) = f(x)$ is a $\bar{G}$-isomorphism from $\overline{\mathrm{Ind}_P^G \sigma^g \otimes \psi}$ to $\mathrm{Ind}_{\bar{P}}^{\bar{G}} \bar{\sigma}^{\bar{g}} \otimes \bar{\psi}$. Then

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G \sigma^g \otimes \psi) = \mathrm{Hom}_{\bar{G}}(\bar{\pi}, \overline{\mathrm{Ind}_P^G \sigma^g \otimes \psi}) \cong \mathrm{Hom}_{\bar{G}}(\bar{\pi}, \mathrm{Ind}_{\bar{P}}^{\bar{G}} \bar{\sigma}^{\bar{g}} \otimes \bar{\psi}) \neq 0$$

We thus get $\mathfrak{I}((\pi, V)) \in \mathfrak{s}$ and using the fact that (ρ, W) is an $\mathfrak{s}$-type we get that $V_\rho \neq 0$ and note that $V_\rho = \mathrm{Hom}_J(\rho, \pi) = \mathrm{Hom}_{\bar{J}}(\bar{\rho}, \bar{\pi}) = V_{\bar{\rho}} \neq 0$. We can see that all the arguments can be reversed i.e. we can start with the assumption that $V_{\bar{\rho}} \neq 0$ and conclude that $\mathfrak{I}((\bar{\pi}, V)) \in \bar{\mathfrak{s}}$ we then make use of corollary 2.24 to conclude that $(\bar{J}, \bar{\rho})$ is an $\bar{\mathfrak{s}}$-type.

Remark. The proof above does not use the fact that G is $\mathrm{GL}_2(\mathbb{F})$. The same proof works for a general G.

We then have a way to relate types in G to types in $\bar{G}$. We would like to have a similar result for covers. Let $P_u = MN_u$ be a parabolic subgroup of G. Let us denote by N_ℓ the opposite of N_u relative to M. Consider the pair (J, τ) where J is a compact open subgroup and (τ, W) an irreducible representation of J. If the pair (J, τ) is decomposed with respect to (M, P_u), we have by definition of decomposition that $J = J \cap N_\ell \cdot J \cap M \cdot J \cap N_u$. Then we get $\bar{J} = \overline{J \cap N_\ell} \cdot \overline{J \cap M} \cdot \overline{J \cap N_u}$. We know that $Z \cap N_\ell = \{1\}$ and $Z \cap N_u = \{1\}$ therefore the quotient map $x \mapsto \bar{x}$ is injective when restricted to N_ℓ and when restricted to N_u. We then get that $\overline{J \cap N_\ell} = \bar{J} \cap \bar{N}_\ell$ and $\overline{J \cap N_u} = \bar{J} \cap \bar{N}_u$. We certainly have $\overline{J \cap M} \subset \bar{J} \cap \bar{M}$, for the reverse containment we take $\bar{x} \in \bar{J} \cap \bar{M}$ then there is $j \in J$ and $m \in M$ such that $\bar{j} = \bar{m} = \bar{x}$. Therefore $m^{-1}j \in Z \subset M$ so $j \in M$ we conclude that $j \in J \cap M$ and thus $\bar{x} = \bar{j} \in \overline{J \cap M}$. We then have that $\overline{J \cap M} = \bar{J} \cap \bar{M}$. Since the definition of $\bar{\tau}$ in $\bar{J}$ is given by $\bar{\tau}(\bar{x}) = \tau(x)$

for $x \in J$. We see that if $\tau(x) = 1$ then $\bar{\tau}(\bar{x}) = 1$. We can assure that $\bar{J} \cap \bar{N}_\ell$ and $\bar{J} \cap \bar{N}_u$ are contained in $\ker \bar{\tau}$.

Proposition 5.5. *Let J be a compact open subgroup of G and ρ and irreducible representation of J such that $\rho(z) = 1$ for all $z \in Z \cap J$. The map from $\mathcal{H}(G, \rho)$ into $\mathcal{H}(\bar{G}, \bar{\rho})$ given by $f \to \bar{f}$ where $\bar{f}(\bar{x}) = \int_Z f(zx)\mu_Z(z)$ is a homomorphism of involutive algebras such that $supp(\bar{f}) \subset \overline{supp(f)}$.*

Proof. Let $f \in \mathcal{H}(G, \rho)$ if $x \notin \overline{supp(f)}$ we get $f(zx) = 0$ for all $z \in Z$. Therefore

$$\overline{f}(\bar{x}) = \int_Z f(zx) d\mu_Z(z) = 0$$

we conclude that $\bar{x} \notin supp(\bar{f})$ and thus the containment $supp(\bar{f}) \subset \overline{supp(f)}$.

Let $f_1, f_2 \in \mathcal{H}(G, \rho)$ then

$$\begin{aligned}
\overline{f_1 * f_2}(\bar{y}) &= \int_Z f_1 * f_2(zy) d\mu_Z(z) = \int_Z \int_G f_1(x) f_2(x^{-1}zy) d\mu_G(x) d\mu_Z(z) \\
&= \int_G \int_Z f_1(x) f_2(x^{-1}zy) d\mu_G(x) d\mu_Z(z) \\
&= \int_{Z\backslash G} \int_Z \int_Z f_1(ux) f_2(uzx^{-1}y) d\mu_Z(z) d\mu_Z(u) d\mu_{Z\backslash G}(x) \\
&= \int_{Z\backslash G} \int_Z f_1(ux) \bar{f}_2(\bar{x}^{-1}\bar{y}) d\mu_Z(u) d\mu_{Z\backslash G}(x) \\
&= \int_{Z\backslash G} \bar{f}_1(\bar{x}) \bar{f}_2(\bar{x}^{-1}\bar{y}) d\mu_{Z\backslash G}(x) = \bar{f}_1 * \bar{f}_2(\bar{y})
\end{aligned}$$

The first two equalities follow from the definitions. The third equality is an application of Fubini's theorem. The fourth equality is an application of the formula 2.1.2.2. The rest of the equalities are just mere use of the definitions. Let $f \in \mathcal{H}(G, \rho)$ we check that the map $f \mapsto \bar{f}$ commutes with involution.

$$\begin{aligned}\overline{f}^*(\bar{y}) &= (\overline{f}(\bar{y}^{-1}))^* = \left(\int_Z f(zy^{-1})d\mu_Z(z)\right)^* \\ &= \int_Z (f(zy))^* d\mu_Z(z) = \int_Z (f^*(z^{-1}y))d\mu_Z(z) = \overline{f^*}.\end{aligned}$$

We now need to check that the identity goes to the identity. We have that $\bar{1}_{(G,\rho)}$ has support on $\bar{J}$. Take $\bar{k} \in \bar{J}$, we might assume $k \in J$ then

$$\begin{aligned}\bar{1}_{(G,\rho)}(\bar{k}) &= \int_Z 1_{(G,\rho)}(zk)d\mu_Z(z) = \int_{J\cap Z} 1_{(G,\rho)}(zk)d\mu_Z(z) \\ &= \frac{\check{\rho}(k)}{\mu_G(J)}\mu_Z(J\cap Z) = \frac{\check{\rho}(k)}{\mu_{Z\backslash G}(J)} = 1_{(\bar{G},\bar{\rho})}.\end{aligned}$$

This finishes the proof of the proposition.

We have then that if $f \in \mathcal{H}(G,\rho)$ is invertible then into $\bar{f}$ is invertible. If H is a subgroup of G the condition $xHx^{-1} \subset H$ implies $\bar{x}\bar{H}\bar{x}^{-1} \subset \bar{H}$. We see then that if x is a strongly positive element with respect to (J,P) then $\bar{x}$ is strongly positive with respect to $(\bar{J},\bar{P})$. We then have the following corollary.

Corollary 5.6. *Let $L \subset M$ be two Levi components of parabolic subgroups of G. Let $\mathfrak{s}_M \in \mathscr{B}(M)$ and suppose there is a pair $(L,\sigma) \in \mathfrak{s}_M$ such that σ is trivial in Z. If (J_M,τ_M) is a $\mathfrak{s}_M$-type and (J,τ) is a G cover for (J_M,τ_M) then $(\bar{J},\bar{\tau})$ is a $\bar{G}$ cover for $(\bar{J}_M,\bar{\tau}_M)$ and $(\bar{J},\bar{\tau})$ is an $[\bar{\mathfrak{s}}]_{\bar{G}}$-type.*

5.3 Hecke Algebras $\mathcal{H}(\bar{G},\bar{\lambda}_\varphi)$

Let us choose the Haar measure μ_G such that $\mu_G(I) = 1$. Recall the element $w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the existence of the element $g_w \in \mathcal{H}(G,\lambda_\varphi)$ in the case $\varphi^2|_{\mathcal{O}^\times} = 1$.

We have already seen in the proof of proposition 5.3 that $g_w * g_w = ag_w + b1_{(G,\lambda_\varphi)}$ and $b = \mu_G(IwI) = [I : wIw^{-1} \cap I] = q$. Let $x_0, x_1, x_2 \ldots x_{q-1} \in \mathcal{O}$ be different coset representatives of $\mathcal{O}/\mathfrak{p}$, where $x_0 = 0$ and for each x_i, $0 \le i \le q-1$ set $[x_i] = \begin{bmatrix} 1 & x_i \\ 0 & 1 \end{bmatrix}$. Then the matrices $[x_i]$ are a full set of coset representatives of $(wIw^{-1} \cap I)\backslash I$. Let us now compute the complex number a.

$$\begin{aligned} a &= g_w * g_{w^{-1}}(w) = \int_G g_w(x) g_w(x^{-1}w) d\mu_G(x) = \int_{IwI} g_w(x) g_w(x^{-1}w) d\mu_G(x) \\ &= \sum_{i=0}^{q-1} \int_{[x_i]wI} g_w(x) g_w(x^{-1}w) d\mu_G(x) = \varphi(-1)(q-1) \end{aligned} \tag{5.3}$$

Only the last equality deserves explanation. We see that $[x_0] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then if $x \in wI$ say $x = wy$ for $y \in I$ then $x^{-1}w = y^{-1} \in I$ so $g_w(x^{-1}w) = 0$ then $\int_{[x_0]wI} g_w(x) g_w(x^{-1}w) d\mu_G(x) = 0$. Let us now consider the case $x \in [x_i]wI$ for $1 \le i \le q-1$. Let $y \in I$ be such that $x = [x_i]wy$ then $g_w(x)g_w(x^{-1}w) = g_w([x_i]wy) g_w(y^{-1}w[x_i]^{-1}w) = \check{\lambda}_\varphi([x_i])\check{\lambda}_\varphi(y)\check{\lambda}_\varphi(y^{-1}) g_w(w[x_i]^{-1}w) = g_w(w[x_i]^{-1}w) = \varphi(-1)$. The last equality follows easily from the decomposition

$$\begin{aligned} w[x_i]^{-1}w &= \begin{bmatrix} x_i^{-1} & 1 \\ 0 & -x_i \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -x_i \\ 0 & 1 \end{bmatrix}; \\ g_w(w[x_i]^{-1}w) &= \check{\lambda}_\varphi(\begin{bmatrix} x_i^{-1} & 1 \\ 0 & -x_i \end{bmatrix})\check{\lambda}_\varphi(\begin{bmatrix} 1 & -x_i \\ 0 & 1 \end{bmatrix}) \\ &= \varphi^2(x_i^{-1})\varphi^{-1}(-1) = \varphi(-1) \end{aligned}$$

We then get the last equality of 5.3. We make a summary of this in the following lemma.

Lemma 5.7. *Let* $w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *and let* $\varphi^2|_{\mathcal{O}^\times} = 1$. *Then* g_w *is invertible and satisfies the equation* $g_w^2 = (q-1)\varphi(-1)g_w + q1_{(G,\lambda_\varphi)}$.

Proposition 5.8. *Let $\varphi^2|_{\mathcal{O}^\times} = 1$ and fix a Haar measure $\mu_{\bar{G}}$ such that $\mu_{\bar{G}}(\bar{I})$. There is a homomorphism of the involutive algebra $\mathcal{H}(q,1)$ into $\mathcal{H}(\bar{G},\bar{\lambda}_\varphi)$.*

Proof. We know that $\alpha = \begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}$ intertwines λ_φ. Then $\bar{\alpha}$ intertwines $\bar{\lambda}_\varphi$, hence the existence of the element $g_{\bar{\alpha}} \in \mathcal{H}(\bar{G},\bar{\lambda}_\varphi)$. Note that $\alpha^2 \in Z$. We then have that $g_{\bar{\alpha}}^2$ has support on $\bar{I}$. We also have that $g_{\bar{\alpha}}^2(1) = \mu_{\bar{G}}(\bar{I}\bar{\alpha}) = 1$. We conclude that $g_{\bar{\alpha}}^2 = 1_{(\bar{G},\bar{\lambda}_\varphi)}$. We also have that $g_{\bar{\alpha}}^* = g_{\bar{\alpha}^{-1}} = g_{\bar{\alpha}}$. By the discussion above we get that $g_w^2 = \varphi(-1)(q-1)g_w + q1_{(G,\lambda_\varphi)}$. If we let $u = \frac{\varphi(-1)}{q^{1/2}}g_w$, we see that u satisfies the equation $u^2 = (q^{1/2} - q^{-1/2})u + 1$. Since $w = w^{-1}$ we get that $g_w^* = g_w$ and because $\frac{\varphi(-1)}{q^{1/2}}$ is a real number we get $u^* = u$. We then have that $\bar{u}^2 = (q^{1/2} - q^{-1/2})\bar{u} + 1$ and $\bar{u}^* = \bar{u}$. We have then an involutive algebra homorphism from $\mathcal{H}(q,1)$ into $\mathcal{H}(\bar{G},\bar{\lambda}_\varphi)$ by extending the map $s_1 \mapsto \bar{u} =$ and $s_2 \mapsto g_{\bar{\alpha}}$.

Lemma 5.9. *Let $g_x \in \mathcal{H}(G,\lambda_\varphi)$ then $\bar{g}_x$ is a multiple of $g_{\bar{x}}$.*

Proof. We have that that the support of $\bar{g}_x$ is contained in $\bar{J}\bar{x}\bar{J}$ and thus a multiple of $g_{\bar{x}}$. This finishes the proof of this lemma.

Corollary 5.10. *Let $\alpha = \begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}$. Then $\bar{g}_\alpha = \mu_Z(Z \cap I)g_{\bar{\alpha}}$. Let $\Pi = \begin{bmatrix} \varpi & 0 \\ 0 & 1 \end{bmatrix}$ then $\bar{g}_\Pi = \mu_Z(Z \cap I)g_{\bar{\Pi}}$.*

Proof. We can see that the support of $\bar{g}_\alpha$ is contained in $\bar{I}\bar{\alpha}\bar{I} = \bar{I}\bar{\alpha}$.

$$\begin{aligned}
\bar{g_\alpha}(\bar{\alpha}) &= \int_Z g_\alpha(z\alpha)d\mu_Z(z) = \int_{(Z\cap I)} g_\alpha(z\alpha)d\mu_Z(z) \\
&= \int_{(Z\cap I)} g_\alpha(z\alpha)d\mu_Z(z) = \int_{(Z\cap I)} g_\alpha(z\alpha)d\mu_Z(z) \\
&= \int_{(Z\cap I)} \lambda(z) = \mu_Z(Z\cap I)
\end{aligned}$$

We also have that that the support of $\bar{g}_\Pi$ is contained in $\bar{I}\bar{\Pi}\bar{I}$. We have that $z\Pi \in I\Pi I$ implies that $\det(z) \in \mathcal{O}^\times$ so $z \in I$. We thus get that

$$\begin{aligned} \bar{g}_\Pi(\bar{\Pi}) &= \int_Z g_\Pi(z\Pi) d\mu_Z(z) = \int_{(Z\cap I)} g_\Pi(z\Pi) d\mu_Z(z) \\ &= \mu_Z(Z \cap I) \end{aligned}$$

Lemma 5.11. *Let $\varphi^2|_{\mathcal{O}^\times} \neq 1$. We let $g = q^{-1/2} g_{\bar{\Pi}}$ we then have $g^* = g^{-1}$.*

Proof. In the proof of proposition 5.2 was showed that $g_\Pi * g_{\Pi^{-1}}$ is a non-zero multiple of the identity. We also have $\overline{g_\Pi * g_{\Pi^{-1}}} = a \cdot g_{\bar{\Pi}} * g_{\bar{\Pi}^{-1}}$ where a non-zero constant. We then get $g_{\bar{\Pi}} * g_{\bar{\Pi}^{-1}}$ is a multiple of the identity.

$$\begin{aligned} g_{\bar{\Pi}} * g_{\bar{\Pi}^{-1}}(1) &= \int_{\bar{G}} g_{\bar{\Pi}}(\bar{x}) g_{\bar{\Pi}^{-1}}(\bar{x}^{-1}) d\mu_{\bar{G}}(\bar{x}) \\ &= \int_{\bar{J}\bar{\Pi}\bar{J}} g_{\bar{\Pi}}(\bar{x}) g_{\bar{\Pi}^{-1}}(\bar{x}^{-1}) d\mu_{\bar{G}}(\bar{x}) \\ &= [\bar{J} : \bar{J} \cap \bar{\Pi}\bar{J}\bar{\Pi}^{-1}] = q \end{aligned}$$

We thus get that if $g = q^{-1/2} g_{\bar{\Pi}}$ then $g^{-1} = q^{-1/2} g_{\bar{\Pi}^{-1}} = g^*$.

Theorem 5.12. *Let φ be a character of $\mathcal{O}^\times$. Let us fix a Haar measure $\mu_{\bar{\mathbb{T}}}$ such that $\mu_{\bar{\mathbb{T}}}(^\circ\bar{\mathbb{T}}) = 1$. Then $\mathcal{H}(\bar{\mathbb{T}}, \varphi) \cong \mathbb{C}[\mathbb{Z}]$ as Hilbert algebras.*

Proof. We have that $\mathcal{H}(\bar{\mathbb{T}}, \varphi) = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_{\bar{\Pi}^n}(\bar{\mathbb{T}}, \varphi)$. Let $h_{\bar{\Pi}} \in \mathcal{H}(\bar{\mathbb{T}}, \varphi)$ be supported in $^\circ\bar{\mathbb{T}}\bar{\Pi}$ and such that $h_{\bar{\Pi}}(\bar{\Pi}) = 1$. Then for a positive integer n, $h_{\bar{\Pi}}^n$ is supported in $^\circ\bar{\mathbb{T}}\bar{\Pi}^n$. Let $h_{\bar{\Pi}^{-1}} \in \mathcal{H}(\bar{\mathbb{T}}, \varphi)$ be supported in $^\circ\bar{\mathbb{T}}\bar{\Pi}^{-1}$ and such that $h_{\bar{\Pi}^{-1}}(\bar{\Pi}^{-1}) = 1$. The element $h_{\bar{\Pi}} * h_{\bar{\Pi}^{-1}}$ has support in $^\circ\bar{\mathbb{T}}$.

$$\begin{aligned} h_{\bar{\Pi}} * h_{\bar{\Pi}^{-1}}(1) &= \int_{\circ\bar{\mathbb{T}}\bar{\Pi}} h_{\bar{\Pi}}(x) h_{\bar{\Pi}^{-1}}(x^{-1}) d\mu_{\bar{\mathbb{T}}}(x) \\ &= \mu_{\bar{\mathbb{T}}}(^{\circ}\bar{\mathbb{T}}\bar{\Pi}) = 1. \end{aligned}$$

Let $h = h_{\bar{\Pi}}$. We see then see that $h^{-1} = h_{\bar{\Pi}^{-1}}$. Making use of lemma 5.1 we get $h^* = h^{-1}$. We have that h^{-1} has support in $^{\circ}\bar{\mathbb{T}}\bar{\Pi}^{-1}$ and therefore for a positive integer n, h^{-n} has support in $^{\circ}\bar{\mathbb{T}}\bar{\Pi}^{-n}$. Then for n an arbitrary integer we get that h^n spans the space $\mathcal{H}_{\bar{\Pi}^n}(\bar{\mathbb{T}}, \varphi)$. We then get that set $\{h^n\}_{n\in\mathbb{Z}}$ is a basis for $\mathcal{H}(\bar{\mathbb{T}}, \varphi)$, therefore $\mathcal{H}(\bar{\mathbb{T}}, \varphi) = \mathbb{C}[h, h^{-1}]$. Since $h^* = h^{-1}$ we get that $(h^n)^* = (h^*)^n = h^{-n}$. Then for integers n, m we get $[h^n, h^m] = h^n * (h^m)^*(1) = h^{n-m}(1)$. So if $n \neq m$ we get $[h^n, h^m] = 0$ and if $n = m$ we get $[h^n, h^n] = 1$. We have that $\{h^n\}_{n\in\mathbb{Z}}$ is a orthonormal basis. Let $f_n : \mathbb{Z} \longrightarrow \mathbb{C}$ be defined by $f_n(n) = 1$ and $f_n(m) = 0$ for $m \neq n$. It is well known that $f_1^n = f_n$ for all $n \in \mathbb{Z}$ and that $\{f_n\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $\mathbb{C}[\mathbb{Z}]$. We deduce that the map that sends $h \mapsto f_1$ induces an algebra homomorphism $F : \mathcal{H}(\bar{\mathbb{T}}, \varphi) \longrightarrow \mathbb{C}[\mathbb{Z}]$ that sends $h^n \mapsto f_1^n = f_n$. Then F sends an orthonormal basis to an orthonormal basis, so is an isomorphism that preserves the inner-product. In order to finish the proof we need to show that F commutes with the involution. We see that $f_n^*(m) = f_n(-m) = 0$ for $m \neq -n$, and $f_n^*(-n) = f_n(n) = 1$, so $f_n^* = f_{-n}$. Therefore $F((h^n)^*) = F(h^{-n}) = f_{-n} = f_n^* = F(h^n)^*$ for all $n \in \mathbb{Z}$, this implies that $F(x^*) = F(x)^*$ and this finishes the proof of the theorem.

Theorem 5.13. *Let φ be a character of $\mathcal{O}^\times$. The Hecke algebra $\mathcal{H}(\bar{G}, \bar{\lambda}_\varphi) \cong \mathbb{C}[\mathbb{Z}]$ if $\varphi^2|_{\mathcal{O}^\times} \neq 1$ and $\mathcal{H}(\bar{G}, \bar{\lambda}_\varphi) \cong \mathcal{H}(q, 1)$ if $\varphi^2|_{\mathcal{O}^\times} = 1$. Where the isomorphism are* *isomorphism of normalized Hilbert algebras.*

Proof. We claim that the map

$$\mathcal{H}(\bar{\mathbb{T}},\varphi)\otimes_{\mathbb{C}}\mathcal{H}(\bar{K},\bar{\lambda}_\varphi)\longrightarrow\mathcal{H}(\bar{G},\bar{\lambda}_\varphi)$$

$$f\otimes g\longrightarrow f*g$$

is an isomorphism of vector spaces. Indeed, by Theorem 2.30 we get the inequality $\dim_{\mathbb{C}}\mathcal{H}(\bar{K},\bar{\lambda})\leq|\mathbf{W}_{\bar{\mathfrak{s}}}|$ and that the claim holds true if we have equality. We can take an element $\bar{x}\in\bar{G}$ and $\bar{t}\in\bar{\mathbb{T}}$ we can see that the only way that $\bar{x}\bar{t}\bar{x}^{-1}\in\bar{\mathbb{T}}$ is if $\bar{x}\in\bar{\mathbb{T}}\cup\bar{w}\bar{\mathbb{T}}$. Then $|\mathbf{W}_{\bar{\mathfrak{s}}}|\leq|N_{\bar{G}}(\bar{\mathbb{T}})/\bar{\mathbb{T}}|\leq 2$. If $\varphi^2|_{\mathcal{O}^\times}\neq 1$, take $a\in\mathcal{O}^\times$ such that $\varphi(a)\neq\varphi(a)^{-1}$. Let

$$x_1=\begin{bmatrix}a&0\\0&1\end{bmatrix},\text{ and }x_2=\begin{bmatrix}a^{-1}&0\\0&1\end{bmatrix}$$

Then for an unramified character χ of $\bar{\mathbb{T}}$ and an element $\bar{s}\in\bar{\mathbb{T}}$ we get $(\overline{\varphi\otimes\varphi^{-1}}^{\bar{s}}\otimes\chi)(\bar{x}_1)=\overline{\varphi\otimes\varphi^{-1}}(\bar{x}_1)=\varphi(a)\neq\varphi(a^{-1})=\overline{\varphi\otimes\varphi^{-1}}(\bar{x}_2)=\overline{\varphi\otimes\varphi^{-1}}(\bar{w}\bar{x}_1\bar{w}^{-1})=\overline{\varphi\otimes\varphi^{-1}}^{\bar{w}}(\bar{x}_1)$ we then get that $\bar{w}$ does not stabilize $\bar{\mathfrak{s}}$ so $|\mathbf{W}_{\bar{\mathfrak{s}}}|=1$ and therefore $\dim_{\mathbb{C}}\mathcal{H}(\bar{K},\bar{\lambda})=1$ we thus get the claim in the case $\varphi^2|_{\mathcal{O}^\times}\neq 1$. If $\varphi^2|_{\mathcal{O}^\times}=1$ then $\bar{w}$ intertwines $\overline{\varphi\otimes\varphi^{-1}}$ and $\mathcal{I}_{\bar{K}}(\bar{\lambda}_\varphi)=\bar{I}\cup\bar{I}\bar{w}\bar{I}$ we then obtain that $g_{\bar{w}}$ and $1_{(\bar{G},\bar{\lambda}_\varphi)}$ are a basis for $\mathcal{H}(\bar{K},\bar{\lambda})$ so $\dim_{\mathbb{C}}\mathcal{H}(\bar{K},\bar{\lambda})=2=|\mathbf{W}_{\bar{\mathfrak{s}}}|$ this finishes the proof of the claim.

If $\varphi^2\neq 1$ we have that for the upper triangular matrices B then the map $t^i_{\bar{B}}:\mathcal{H}(\bar{\mathbb{T}},\bar{\varphi})\longrightarrow\mathcal{H}(\bar{G},\bar{\lambda}_\varphi)$ is an algebra isomorphism we then get that the map $t^u_{\bar{B}}:\mathcal{H}(\bar{\mathbb{T}},\bar{\varphi})\longrightarrow\mathcal{H}(\bar{G},\bar{\lambda}_\varphi)$ is an isomorphism. We claim that the map $t^u_{\bar{B}}$ preserves the inner product and involution and is therefore an isomorphism of Hilbert algebras. Let $h_{\bar{\Pi}}\in\mathcal{H}(\bar{\mathbb{T}},\varphi)$ be supported in ${}^\circ\bar{\mathbb{T}}\bar{\Pi}$ and such that $h_{\bar{\Pi}}(\bar{\Pi})=1$. Then for a positive integer n, $h^n_{\bar\Pi}$ is supported in $^\circ\bar{\mathbb{T}}\bar\Pi^n$. Let $h_{\bar\Pi^{-1}} \in \mathcal{H}(\bar{\mathbb{T}}, \varphi)$ be supported in $^\circ\bar{\mathbb{T}}\bar\Pi^{-1}$ and such that $h_{\bar\Pi^{-1}}(\bar\Pi^{-1}) = 1$. We have shown in the proof of theorem 5.12. That for $h = h_{\bar\Pi}$, $h^* = h^{-1}$. The way that $t^u_{\bar B}$ was constructed gives us that $t^u_{\bar B}(h) = \delta^{-1/2}(\Pi)g_{\bar\Pi} = q^{-1/2}g_{\bar\Pi}$. Let us set $g = q^{-1/2}g_{\bar\Pi}$. We then have $t^u_{\bar B}((h^n)^*) = (h^{-n}) = g^{-n} = (g^n)^* = t^u_{\bar B}(h^n)^*$. We then get that for all $f \in \mathcal{H}(\bar{\mathbb{T}}, \bar\varphi)$, $t^u_{\bar B}(f^*) = t^u_{\bar B}(f)^*$. We now check it preserves the inner product on the basis elements. Let n be a positive integer since $\bar\Pi^n$ is a positive element and h^n has support in $^\circ\bar{\mathbb{T}}\bar\Pi^n$ then $t^u_{\bar B}(h^n) = g^n$ has support on $\bar J\bar\Pi^n\bar J$, so $g^n(1) = 0$. Also we deduce that $g^n = cg_{\bar\Pi^n}$ so $g^{-n} = (g^n)^* = (cg_{\bar\Pi^n})^* = (c) * g_{\bar\Pi^{-n}}$ so $g^{-n}(1) = 0$. We let n, m be any two integers then

$$\begin{aligned}
[t^u_{\bar B}(h^n), t^u_{\bar B}(h^m)] &= [g^n, g^m] \\
&= g^n * (g^m)^*(1) = g^{n-m}(1)
\end{aligned}$$

We deduce that if $m \neq n$ then $[t^u_{\bar B}(h^n), t^u_{\bar B}(h^m)] = 0$ and if $m = n$ then $[t^u_{\bar B}(h^n), t^u_{\bar B}(h^m)] = 1$ in any case $[t^u_{\bar B}(h^n), t^u_{\bar B}(h^m)] = [h^n, h^m]$. Hence $t^u_{\bar B}$ is an isomorphism of Hilbert algebras showing that that $\mathcal{H}(\bar G, \bar\lambda_\varphi) \cong \mathbb{C}[\mathbb{Z}]$

Let us now consider the case where $\varphi^2|_{\mathcal{O}^\times} = 1$. By the first claim we have that $\mathcal{H}(\bar G, \bar\lambda_\varphi)$ has a basis given by $\{g^n_{\bar\Pi}, g^n_{\bar\Pi} * g_{\bar w}\}$. By proposition 5.8 we get a homomorphism of involutive algebras from $F : \mathcal{H}(q, 1) \longrightarrow \mathcal{H}(\bar G, \bar\lambda_\varphi)$ such that $F(s_1) = \bar u = \frac{\varphi(-1)}{q^{1/2}}\bar g_w$ and $F(s_2) = g_{\bar\alpha}$. We already had mentioned that $g_w * g_\alpha$ is invertible and has support in $I\Pi I$ and is therefore a non-zero multiple of g_Π. Since $g_w * g_\alpha$ is invertible we have that $\overline{g_w * g_\alpha}$ is invertible and thus a non-zero multiple of $g_{\bar\Pi}$. We have that $F(s_1s_2) = \frac{\varphi(-1)}{q^{1/2}}\bar{g}_w * g_{\bar{\alpha}} = \frac{\varphi(-1)}{q^{1/2}}\bar{g}_w * \bar{g}_\alpha = \frac{\varphi(-1)}{q^{1/2}}\overline{g_w * g_\alpha} = \frac{\varphi(-1)}{q^{1/2}}\bar{g}_\Pi = \frac{\varphi(-1)}{q^{1/2}}g_{\bar{\Pi}}$ and hence $F(s_1s_2)$ is a non-zero multiple of $g_{\bar{\Pi}}$. We can say then that for an integer n, $F((s_1s_2)^n) = c_n g_{\bar{\Pi}}^n$ where c_n is a non-zero constant, and from here we can therefore also say that $F((s_1s_2)^n s_1) = \frac{\varphi(-1)}{q^{1/2}}c_n g_{\bar{\Pi}}^n * \bar{g}_w$. This shows that our map F sends a basis of $\mathcal{H}(q,1)$ to a basis of $\mathcal{H}(\bar{G},\bar{\lambda}_\varphi)$ therefore F is bijective.

We see that the set $\mathcal{W} = \{(s_1s_2)^n\}_{n\geq 1} \cup \{(s_1s_2)^n s_1\}_{n\geq 0} \cup \{s_2(s_1s_2)^n\}_{n\geq 0}$ of all words in the letters s_1, s_2 together with the identity forms a basis for $\mathcal{H}(q,1)$. The set $F(\mathcal{W}) = \{g_{\bar{\Pi}}^n\}_{n\geq 1} \cup \{g_{\bar{\Pi}}^n * \bar{u}\}_{n\geq 0} \cup \{g_{\bar{\alpha}} * g_{\bar{\Pi}}^n\}_{n\geq 0}$. We contend that for $f \in F(\mathcal{W})$, $f(\bar{1}) = 0$. The support of $g_{\bar{\Pi}}^n$ is in $\bar{I}\bar{\Pi}^n\bar{I}$ for $n \geq 1$. We have that $z\Pi^n \notin I$ for all $z \in Z$ which implies $\bar{\Pi}^n \notin \bar{I}$, so $\bar{1} \notin \bar{I}\bar{\Pi}^n\bar{I}$, hence $g_{\bar{\Pi}}^n(1) = 0$. The support of $g_{\bar{\Pi}}^n * \bar{u}$ for $n \geq 0$ is contained in $\bar{I}\bar{\Pi}^n\bar{I}\bar{w}$. We get that $\bar{1} \in \bar{I}\bar{\Pi}^n\bar{I}\bar{w}$ if and only if there $\exists\ z \in Z, x_1, x_2 \in I$ such that $x_1 z\Pi^n x_2 w = 1$ but then $z\Pi^n \in \mathrm{GL}_2(\mathcal{O})$ so $n = 0$; then $\bar{I}\bar{\Pi}^n\bar{I}\bar{w} = \bar{I}\bar{w}$ but $\bar{1} \notin \bar{I}\bar{w}$. The support of $g_{\bar{\alpha}} * g_{\bar{\Pi}}^n$ for $n \geq 0$ is contained in $\bar{\alpha}\bar{I}\bar{\Pi}^n\bar{I} = \bar{I}\bar{\alpha}\bar{\Pi}^n\bar{I}$. So $\bar{1} \in \bar{I}\bar{\alpha}\bar{\Pi}^n\bar{I}$ if and only if $\exists\ z \in Z$ such that $z\alpha\Pi^n \in I$; but $\alpha\Pi^n = \begin{bmatrix} 0 & 1 \\ \varpi^{n+1} & 0 \end{bmatrix}$ so if $z = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ we get $z\alpha\Pi^n = \begin{bmatrix} 0 & a \\ a\varpi^{n+1} & 0 \end{bmatrix}$ and this is not in I because the entry in the first row and first column is zero and thus not a unit. We have that the functional $[\ ,\mathbf{1}]$ i.e the inner product with $\mathbf{1}$, is the unique linear functional that is zero in the space spanned by $\mathcal{W}$ and is 1 at $\mathbf{1}$. Therefore the functional $x \mapsto F(x)(1)$ for $x \in \mathcal{H}(q,1)$ is equal to $[\ ,\mathbf{1}]$. We then get that for $x, y \in \mathcal{H}(q,1)$, $[x,y] = [y^*x, \mathbf{1}] = F(y^*x)(1) = F(y)^*F(x)(1) = [F(y)^*, F(x)^*] = [F(x), F(y)]$. This finishes the proof of the theorem.

5.4 Description of the Plancherel Measure

We see that if (π, V) is a supercuspidal representation of $\bar{G}$, then (π, V) is square integrable since the support of the matrix coefficients are compact and thus has positive Plancherel measure. We have by proposition 3.18 that the Plancherel measure for (π, V) is equal to the formal dimension. We hence pay the rest of our attention to the case where the representations are not supercuspidal.

In the proof of Theorem 5.13 we showed that if $\varphi^2|_{\mathcal{O}^\times} \neq 1$ then we have a commutative diagram

$$\begin{tikzcd} \mathcal{H}(\bar{G}, \bar{\lambda}_\varphi) \arrow[r, "g \mapsto f_1"] & \mathbb{C}[\mathbb{Z}] \\ \mathcal{H}(\bar{\mathbb{T}}, \varphi) \arrow[u, "t^u_{\bar{B}}"] \arrow[r, "h \mapsto f_1"] & \mathbb{C}[\mathbb{Z}] \arrow[u, "\mathrm{Id}"'] \end{tikzcd}$$

Let χ_u be the unique unramified character of $\bar{\mathbb{T}}$ such that $\chi_u(\bar{\Pi}) = \varphi(\bar{\Pi})^{-1}$ and let χ_t be the unramified character of $\bar{\mathbb{T}}$ given by $\chi_t(\bar{\Pi}) = q^t \chi_u(\Pi)$. Let $\varphi_t = \chi_t \otimes \varphi$ seen as a representation of $\bar{\mathbb{T}}$ and we denote by $\mathbb{C}_{\varphi_t}$ the vector space where φ_t acts. Let $(\rho_t, V_t) = i_{\bar{B}}^{\bar{G}}(\varphi_t)$. Suppose that (ρ_t, V_t) is a pre-unitary representation of G. We will then denote by $[(\rho_t, V_t)]$ the unitary representation obtained from (ρ_t, V_t) by completion. Let us denote by $\mathbb{C}_t$ the $\mathbb{C}[\mathbb{Z}]$ module given by $f_1 \cdot a = q^t a$ for $a \in \mathbb{C}_t$. Using the theory of types we get that for a representation $(\varphi_t, \mathbb{C}_{\varphi_t})$ corresponds the $\mathcal{H}(\bar{\mathbb{T}}, \varphi)$ module $\mathrm{Hom}_{\mathcal{O}^\times}(\varphi, \varphi_t)$. We can then regard $\mathrm{Hom}_{\mathcal{O}^\times}(\varphi, \varphi_t)$ as a $\mathbb{C}[\mathbb{Z}]$-module by demanding $f_1 \cdot \Phi = h \cdot \Phi$ for all $\Phi \in \mathrm{Hom}_{\mathcal{O}^\times}(\varphi, \varphi_t)$. We contend that $\mathrm{Hom}_{\mathcal{O}^\times}(\varphi, \varphi_t)$ is isomorphic to $\mathbb{C}_t$ as $\mathbb{C}[\mathbb{Z}]$-modules. Indeed, we have that Φ is a linear map from $\mathbb{C}$ to $\mathbb{C}$ and thus is given by multiplication of the complex number $\Phi(1)$. From lemma

2.21 we get that for $s \in \mathbb{C}$

$$\begin{aligned} h \cdot \Phi(s) &= \int_{\bar{\mathbb{T}}} \varphi_t(x)\Phi(\check{h}(x^{-1})s)d\mu_{\bar{\mathbb{T}}}(x) = \Phi(1)s\int_{\bar{\mathbb{T}}} \varphi_t(x)\check{h}(x^{-1})d\mu_{\bar{\mathbb{T}}}(x) \\ &= \Phi(1)s\int_{{}^\circ\bar{\mathbb{T}}\bar{\Pi}} \varphi_t(x)h(x)d\mu_{\bar{\mathbb{T}}}(x) = q^t\Phi(s) \end{aligned}$$

We can then see that representation (ρ_t, V_t) also corresponds to the $\mathbb{C}[\mathbb{Z}]$-module $\mathbb{C}_t$.

Theorem 5.14. *Let us fix a Haar measure such that $\mu_{\bar{G}}(\bar{K}) = 1$. Let φ be a character of $\mathbb{F}^\times$ such that $\varphi^2|_{\mathcal{O}^\times} \neq 1$ and let $\bar{\mathfrak{s}} = [\bar{\mathbb{T}}, \overline{\varphi \otimes \varphi^{-1}}]_{\bar{G}}$. Then the map $t \mapsto [(\rho_{it}, V_{it})]$ gives a Borel isomorphism between the interval $[-\frac{\pi}{\ln q}, \frac{\pi}{\ln q}]$ and ${}_r\widehat{G}(\bar{\mathfrak{s}})$. Moreover, if ν is the Plancherel measure with respect to μ_G and dt is the Lebesgue measure we get for $t \in [-\frac{\pi}{\ln q}, \frac{\pi}{\ln q}]$*

$$d\nu([(\rho_{it}, V_{it})]) = \frac{(q+1)(q^{sw(\varphi^2)-1})\ln q}{2\pi}dt$$

Proof. It is well known that the spectrum of the reduced C^*-algebra of $\mathbb{C}[\mathbb{Z}]$ can be realized as the unit circle with measure given by arc length and where the total measure is 1, see for example [12, p. 88–92]. We then can parametrize the unit circle by sending $t \in [-\frac{\pi}{\ln q}, \frac{\pi}{\ln q}]$ to q^{it}. We also have by the discussion preceding the statement of this theorem that the point q^{it} corresponds to the unitary representation $[(\rho_{it}, V_{it})]$ in ${}_r\widehat{G}(\bar{\mathfrak{s}})$. We get by 4.12 *iv)* the equation

$$d\nu([(\rho_{it}, V_{it})]) = \frac{\dim\lambda_\varphi}{\mu_G(\bar{J}_\varphi)}\frac{\ln q}{2\pi}dt = \frac{\ln q}{\mu_G(\bar{J}_\varphi)2\pi}dt$$

It remains to calculate the number $\mu_G(\bar{J}_\varphi)^{-1}$. We see that $\mu_G(\bar{J}_\varphi)^{-1} = [\bar{K} : \bar{J}_\varphi] = [K : J_\varphi] = (q+1)(q^{sw(\varphi^2)-1})$. This finishes the proof of the theorem.

We now want to give a description of the Plancherel measure restricted to $\widehat{G}(\bar{\mathfrak{s}})$ when $\bar{\mathfrak{s}} = [\bar{G}, \overline{\varphi \otimes \varphi^{-1}}]_{\bar{G}}$ and $\varphi^2|_{\mathcal{O}^\times} = 1$. We first introduce the following result easily deduced from a result stated by Bushnell and Henniart in [4, p. 69].

Lemma 5.15. *Let ϕ be a character of $F^\times$ then there exists an irreducible representation trivial on the center that we denote by St_G and an exact sequence of representations of G.*

$$0 \to \phi \circ \det \cdot St_G \longrightarrow i_B^G(\phi \otimes \phi \cdot \delta_B^{-1/2}) \longrightarrow \phi \circ \det \to 0$$

We see that a necessary and sufficient condition for $\phi \otimes \phi$ and $\phi \circ \det$ to be trivial in Z is that $\phi^2 = 1$. We then have that if $\phi^2 = 1$ we get an exact sequence of representations of $\bar{G}$.

$$0 \to \overline{\phi \circ \det} \cdot \overline{St_G} \longrightarrow i_{\bar{B}}^{\bar{G}}(\overline{\phi \otimes \phi} \cdot \delta_{\bar{B}}^{-1/2}) \longrightarrow \overline{\phi \circ \det} \to 0$$

Let us denote the representation $\overline{\phi \circ \det} \cdot \overline{St_G}$ by $St_{(\bar{G},\phi)}$.

Recall that $d = s_1 s_2 \in \mathcal{H}(q,1)$ and that $D = \mathbb{C}[d, d^{-1}]$. In the proof of Theorem 5.13 we showed that if $\varphi^2|_{\mathcal{O}^\times} = 1$ then we have a commutative diagram

$$\begin{tikzcd} \mathcal{H}(\bar{G},\bar{\lambda}_\varphi) \arrow[r] & \mathcal{H}(q,1) \\ \mathcal{H}(\bar{\mathbb{T}},\varphi) \arrow[u, "t_{\bar{B}}^u"] \arrow[r] & D \arrow[u, "i"'] \end{tikzcd} \tag{5.4}$$

Where the horizontal map at the top of the diagram is given by F and at the bottom by $h \mapsto \varphi(-1)d$ and where i means inclusion.

Let χ_u be the unique unramified character of $\bar{\mathbb{T}}$ such that $\chi_u(\bar{\Pi}) = \varphi(-1)\varphi(\bar{\Pi})^{-1}$ and let χ_t be the unramified character of $\bar{\mathbb{T}}$ given by $\chi_t(\bar{\Pi}) = q^t\chi_u(\bar{\Pi})$. We let

$\varphi_t = \chi_t \otimes \varphi$ seen as a representation of $\bar{\mathbb{T}}$ and we denote by $\mathbb{C}_{\varphi_t}$ the vector space where φ_t acts. Let $(\rho_t, V_t) = i_{\bar{B}}^{\bar{G}}(\varphi_t)$. Suppose that (ρ_t, V_t) is a pre-unitary representation of G. We will then denote by $[(\rho_t, V_t)]$ the unitary representation obtained from (ρ_t, V_t) by completion. Let us denote by $\mathbb{C}_t$ the D module given by $d \cdot a = q^t a$ for $a \in \mathbb{C}_t$. Using the theory of types we get that to a representation $(\varphi_t, \mathbb{C}_{\varphi_t})$ corresponds the $\mathcal{H}(\bar{\mathbb{T}}, \varphi)$ module $\mathrm{Hom}_{\mathcal{O}^\times}(\varphi, \varphi_t)$. We can then regard $\mathrm{Hom}_{\mathcal{O}^\times}(\varphi, \varphi_t)$ as a D-module by demanding $d \cdot \Phi = \varphi(-1)h \cdot \Phi$ for all $\Phi \in \mathrm{Hom}_{\mathcal{O}^\times}(\varphi, \varphi_t)$. We contend that $\mathrm{Hom}_{\mathcal{O}^\times}(\varphi, \varphi_t)$ is isomorphic to $\mathbb{C}_t$ as D-modules. Indeed, we have that Φ is a linear map from $\mathbb{C}$ to $\mathbb{C}$ thus is given by multiplication of the complex number $\Phi(1)$. From lemma 2.21 we get that for $s \in \mathbb{C}$

$$\begin{aligned} d \cdot \Phi &= \varphi(-1)h \cdot \Phi(s) = \varphi(-1) \int_{\bar{\mathbb{T}}} \varphi_t(x)\Phi(\check{h}(x^{-1})s)d\mu_{\bar{\mathbb{T}}}(x) \\ &= \varphi(-1)\Phi(1)s \int_{\bar{\mathbb{T}}} \varphi_t(x)\check{h}(x^{-1})d\mu_{\bar{\mathbb{T}}}(x) = \varphi(-1)\Phi(1)s \int_{\circ\bar{\mathbb{T}}\bar{\Pi}} \varphi_t(x)h(x)d\mu_{\bar{\mathbb{T}}}(x) \\ &= \varphi^2(-1)q^t\Phi(s) = q^t\Phi(s) \end{aligned}$$

We can then see that representation (ρ_t, V_t) corresponds to the $\mathcal{H}(q,1)$-module ind $\mathbb{C}_t$.

We also have two one dimensional unitary representations ρ_1, ρ_2 of $\mathcal{H}(q,1)$ that contribute to the Plancherel measure. These representations are characterized by $\rho_1(d) = q^{-1/2}$ and $\rho_2(d) = -q^{-1/2}$. We will like to know what are the representations of $\bar{G}$ that correspond to ρ_1 and ρ_2. Let (π_1, V_1) be the $\bar{G}$ representation that correspond to ρ_1. Since $\rho_1(d) = q^{-1/2}$ then the restriction of ρ_1 to D is equal to $\mathbb{C}_{-1/2}$. It follows that $\mathrm{Hom}_D(\rho_1, \mathbb{C}_{-1/2}) \neq 0$ so $\mathrm{Hom}_{\mathcal{H}(q,1)}(\rho_1, \mathrm{ind}\ \mathbb{C}_{-1/2}) \neq 0$. We then have that that $\mathrm{Hom}_{\bar{G}}((\pi_1, V_1), i_{\bar{B}}^{\bar{G}}(\varphi_{-1/2})) \neq 0$. We have that φ^2 is an unramified character, and the way that χ_u was defined give us that $\varphi^2\chi_u^2$ is an unramified character trivial on $\bar{\Pi}$. We therefore have that $\varphi^2\chi_u^2 = 1$. It follows that the representation (π_1, V_1) is the representation $St_{(\bar{G},\varphi\chi_u)}$. We can deduce by the same method that the representation that corresponds to ρ_2 is $St_{(\bar{G},\varphi\chi_{-u})}$ where χ_{-u} is the unique unramified character of $\bar{\mathbb{T}}$ such that $\chi_{-u}(\bar{\Pi}) = -\varphi(-1)\varphi(\bar{\Pi})^{-1}$.

Theorem 5.16. *Let us fix a Haar measure such that $\mu_{\bar{G}}(\bar{K}) = 1$. Let φ be a character of $\mathbb{F}^\times$ such that $\varphi^2|_{\mathcal{O}^\times} = 1$ and let $\bar{\mathfrak{s}} = [\bar{\mathbb{T}}, \overline{\varphi \otimes \varphi^{-1}}]_{\bar{G}}$. Then there is a Borel isomorphism from $[0, \frac{\pi}{\ln q}] \cup \{p_1, p_2\}$ into ${}_r\widehat{G}(\mathfrak{s})$ given by $t \mapsto [(\rho_{it}, V_{it})]$ for $t \in [0, \frac{\pi}{\ln q}]$ and $p_1 \mapsto [St_{(\bar{G},\varphi\chi_u)}]$ and $p_2 \mapsto [St_{(\bar{G},\varphi\chi_{-u})}]$. Moreover, if ν is the Plancherel measure with respect to μ_G and dt is the Lebesgue measure. We get for $t \in [0, \frac{\pi}{\ln q}]$*

$$d\nu([(\rho_{it}, V_{it})]) = \frac{\ln(q)(q+1)}{2\pi q} \frac{L(1, 1-2it)L(1, 1+2it)}{L(1, -2it)L(1, 2it)} dt$$

and $\nu([St_{(\bar{G},\varphi\chi_u)}]) = \nu([St_{(\bar{G},\varphi\chi_{-u})}]) = \frac{q-1}{2}$.

Proof. The proof of this theorem is just a restatement of facts that have been proven before. By Corollary 4.19 we have a description of the Plancherel measure for the Hilbert algebra $\mathcal{H}(q, 1)$. Using the corresponding modules from the diagram 5.4 we get the Borel isomorphism from $[0, \frac{\pi}{\ln q}] \cup \{p_1, p_2\}$ into the unitary representations in ${}_r\widehat{G}(\mathfrak{s})$ that come from induction. The discussion before the statement of the theorem shows that the remaining two points with positive measure correspond to the representations $[St_{(\bar{G},\varphi\chi_u)}], [St_{(\bar{G},\varphi\chi_{-u})}]$. The theorem then follows by using Corollary 4.19 except for the fact that we have to rescale the Plancherel measure by the number $[\bar{K} : \bar{I}] = [K : I] = q + 1$. The reason for the rescaling is that the isomorphism of Hilbert algebras

$\mathcal{H}(q,1)$ and $\mathcal{H}(\bar{G},\bar{\lambda}_\varphi)$ is given under the assumption that Haar measure $\mu'_{\bar{G}}(\bar{I}) = 1$ we then have $\mu_{\bar{G}}(\bar{I}) = \frac{1}{[\bar{K}:\bar{I}]}\mu'_{\bar{G}}(\bar{I})$. The theorem follows.

Remark. If we use the notation and the results on section 23 of the book by Bushnell and Henniart [4, p. 143], we see that for an character ψ of $\mathbb{F}$ trivial on $\mathfrak{p}$ but not on $\mathcal{O}$ we have the equality $\gamma(1,s,\psi) = q^{(s-1/2)}\frac{L(1,1-s)}{L(1,s)}$. We also have that the complex conjugate of $\gamma(1,s,\psi)$ is $\gamma(1,-s,\psi) = q^{(-s-1/2)}\frac{L(1,1+s)}{L(1,-s)}$. Letting $s = 2it$ and multiplying $\gamma(1,s,\psi)$ with its conjugate we get the equality $|\gamma(1,s,\psi)|^2 = \frac{1}{q}\frac{L(1,1-2it)L(1,1+2it)}{L(1,-2it)L(1,2it)}$. We may replace then $\frac{1}{q}\frac{L(1,1-2it)L(1,1+2it)}{L(1,-2it)L(1,2it)}$ in the above theorem for the more compact function $|\gamma(1,2it,\psi)|^2$.

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