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# Extensions of Hilbert modules over tensor algebras

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EXTENSIONS OF HILBERT MODULES OVER TENSOR ALGEBRAS

by

Andrew Koichi Greene

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
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Thesis Supervisor: Professor Paul Muhly

## ABSTRACT

This dissertation explores aspects of the representation theory for *tensor algebras*, which are non-selfadjoint operator algebras Muhly and Solel introduced in 1998, by developing a cohomology theory for completely bounded Hilbert Modules. Similar theories have been developed for Banach modules by Johnson in 1970, for operator modules by Paulsen in 1997, and for Hilbert modules over the disc algebra by Carlson and Clark in 1995. The framework presented here was motivated by a desire to further understand the completely bounded representation theory for tensor algebras on Hilbert spaces.

The focal point of this thesis is the first Ext group, $\mathrm{Ext}^1$, which is defined as equivalence classes of short exact sequences of completely bounded Hilbert modules. Alternate descriptions of this group are presented. For general operator algebras, $\mathrm{Ext}^1$ can be realized as the collection of completely bounded derivations equivalent up to an inner derivation. When the operator algebra is a tensor algebra, $\mathrm{Ext}^1$ can be described as a quotient space of intertwining operators, a description analogous to a result of Ferguson in 1996 in the case of the classical disc algebra.

A theorem of Sz.-Nagy and Foiaş from 1967, concerning contractions in triangular form, is applied to analyze derivations that are off-diagonal corners of completely contractive representations. It is proved that, in some cases, this analysis determines when all derivations must be inner or suggests ways to construct non-inner derivations.

In the third chapter, a characterization is given of completely bounded representations of a tensor algebra in terms of similarities of contractive intertwiners. Also proven is that for a $C^*$-correspondence $X$ over a $C^*$-algebra $A$ and the Toeplitz algebra $\mathscr{T}(X)$, $\mathscr{T}(X) \otimes M_n \cong \mathscr{T}(X \otimes M_n)$. The analogous statement for tensor algebras is deduced as a corollary.

In the final chapter, a brief survey of non-abelian category theory is provided. Extensions of completely bounded Hilbert modules over operator algebras are defined. Theorems asserting the projectivity of isometric modules and injectivity of coisometric modules by Carlson, Clark, Foiaş, and Williams in 1995 are generalized to the noncommutative setting of tensor algebras using commutant lifting. A result of Popesecu in 1996 for noncommutative disc algebras is covered in the general framework of this thesis.

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the July 2012 graduation.

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A theorem of Sz.-Nagy and Foiaş from 1967, concerning contractions in triangular form, is applied to analyze derivations that are off-diagonal corners of completely contractive representations. It is proved that, in some cases, this analysis determines when all derivations must be inner or suggests ways to construct non-inner derivations.

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resentations of a tensor algebra in terms of similarities of contractive intertwiners. Also proven is that for a $C^*$-correspondence $X$ over a $C^*$-algebra $A$ and the Toeplitz algebra $\mathscr{T}(X)$, $\mathscr{T}(X) \otimes M_n \cong \mathscr{T}(X \otimes M_n)$. The analogous statement for tensor algebras is deduced as a corollary.

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# CHAPTER 1

# INTRODUCTION AND BACKGROUND

## 1.1 Introduction

This dissertation explores aspects of the representation theory for certain non-selfadjoint operator algebras called *tensor algebras* and introduced by Muhly and Solel in [31]. A cohomology theory for completely bounded Hilbert modules over an operator algebra is developed. The $\mathrm{Ext}^n(H_2, H_1)$ groups are defined to be the collection of equivalence classes of $n$-extensions sequences of $H_1$ by $H_2$. The first Ext group measures the ways in which one prescribed completely bounded Hilbert module $H_1$ can sit inside other completely bounded Hilbert modules such that the resulting quotient is a second prescribed completely bounded Hilbert module $H_2$. Alternatively, $\mathrm{Ext}^1(H_2, H_1)$ can be described as classes of completely bounded derivations that are equivalent up to inner derivations.

Cohomology theory for general algebras began with the work of Hochschild in [20], [21], and [22]. Much research has been done developing homology and cohomology theories for Banach algebras, starting with work of Johnson in [23], and continuing in works of Taylor [54] and Helemski [17]. See also [24], [6], and [50] for cohomology of $*$-algebras.

In much of the theory, cohomology is developed by considering Banach modules [18] or operator modules [38]. Motivated by an interest in the representation theory for operator algebras *on Hilbert spaces*, the emphasis in this dissertation is on Hilbert modules. Douglas and Paulsen developed the theory of Hilbert modules over function algebras in [11]. Muhly and Solel in [30] considered Hilbert modules over general operator algebras. Due to the the particularly tractable theory for completely contractive representations of tensor algebras, cf. [31], the most descriptive results provided in this thesis occur when the Hilbert modules are completely contractive. Applying Paulsen's Similarity Theorem, [37], that says the completely bounded representations are precisely those that are similar to completely contractive ones, completely contractive extensions are readily available in this theory of completely bounded extensions.

Completely contractive representations of tensor algebras may be described in terms of contractive, intertwining operators. This key result in [31] is used in conjunction with the similarity theorem to yield a description of the completely bounded representation theory for tensor algebras in terms of intertwining operators. A result of Sz.-Nagy and Foiaş [51], concerning contractions in triangular form, strengthens the analysis of derivations that are off-diagonal corners of completely contractive representations and, in some cases, is sufficiently powerful to determine when all derivations must be inner or suggest ways to construct non-inner derivations.

## 1.2 Summary

The second chapter of this dissertation reviews basic concepts of Hilbert $C^*$-modules, $C^*$-correspondences, operator algebras, and completely bounded maps. Examples are introduced that will be considered throughout this thesis to illustrate various results.

In the third chapter, the theory of tensor algebras and their representations, following [31], is briefly reviewed. A description is given of the completely bounded representations of a tensor algebra in terms of "similarities" of contractive intertwiners. Also proven is the fact that for a $C^*$-correspondence $X$ over a $C^*$-algebra $A$ and the Toeplitz algebra $\mathscr{T}(X)$, $\mathscr{T}(X) \otimes M_n \cong \mathscr{T}(X \otimes M_n)$. The analogous statement for tensor algebras is deduced as a corollary.

In the fourth chapter, a brief survey of non-abelian category theory is provided, a field that seems to have largely gone unnoticed in the area of operator algebras. Extensions of completely bounded Hilbert modules over operator algebras are defined. Much of that development follows standard machinery of homological algebra, although the relevant category is not abelian. The author's contribution is the application of Paulsen's similarity theorem to connect completely bounded and completely contractive cohomology. Several theorems from [4] and [5] are generalized to the noncommutative setting. New insight to a result of Popesecu, [41], is provided. Conditions are provided in which all derivations are inner. This analysis, on the other hand, inspires constructions of non-inner derivations in some situations in which the previous conditions are not met.

# CHAPTER 2

# PRELIMINARIES AND NOTATION

## 2.1 Introduction

This chapter reviews basic concepts such as Hilbert $C^*$-modules, $C^*$-correspondences, operator algebras and completely bounded maps. Hilbert spaces are denoted by $H, J, K$ or $E$. Bounded, linear operators between $H$ and $K$ are denoted $B(H, K)$. The notation $:=$ signifies a definition. Recall that $T \in B(H, K)$ is bounded if $\|T\| := \sup_h \frac{\|Th\|}{\|h\|} < \infty$, which happens if and only if $T$ is continuous. If $J \subset H$, then the restriction of $T$ to $J$ is denoted by $T|J$. Unless otherwise stated, all representations will be assumed unital.

## 2.2 Hilbert $C^*$-modules and $C^*$-correspondences

Let $A$ be a unital $C^*$-algebra, which is to say, a unital Banach $*$-algebra satisfying the $C^*$-identity:

$$\|a^*a\| = \|a\|^2.$$

Excellent texts on the theory of operator algebras and operator theory include [8], [9], [25], and [47]. For more background on Hilbert $C^*$-modules, the reader is referred to [27]. A (right) Hilbert $C^*$-module $X$ over $A$ is very much like a Hilbert space, with $A$ assuming the role of $\mathbb{C}$.

**Definition 2.1.** A *Hilbert $C^*$-module over $A$* is a right $A$-module equipped with an $A$-valued inner product $\langle \cdot, \cdot \rangle$ on $X \times X$ satisfying, for every $x, y, z \in X, \lambda \in \mathbb{C}$, and every $a \in A$,

1. Linearity:

$$\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle;$$

$$\langle x, \lambda y \rangle = \lambda \langle x, y \rangle;$$

$$\langle x, ya \rangle = \langle x, y \rangle a;$$

2. Hermitian symmetry: $\langle x, y \rangle = \langle y, x \rangle^*$;

3. Positivity: $\langle x, x \rangle \geq 0$ with $\langle x, x \rangle \implies x = 0$;

and such that $X$ is complete with respect to the norm defined by $\|x\| = \|\langle x, x \rangle\|^{1/2}$.

Given Hilbert $C^*$-modules $X$ and $Y$ over $A$, the collection of *adjointable* maps $\mathcal{L}(X, Y)$ is defined as the collection of maps $T : X \to Y$ such that there exists $T^* : Y \to X$ satisfying $\langle Tx, y \rangle_Y = \langle x, T^*y \rangle_X$ for all $x \in X, y \in Y$. It follows from properties of the inner product that $T^*$ is a bounded, linear module map. When $X = Y$, $\mathcal{L}(X, Y)$ is abbreviated as $\mathcal{L}(X)$. A *strictly cyclic* Hilbert $C^*$-module is one that can be expressed as $PA$ for some projection $P \in A$. (In general, $P \in M(A)$, the multiplier algebra of $A$. For the purposes of this thesis, $A$ is unital so $M(A) = A$.)

**Example 2.2.** If $A = \mathbb{C}$, Hilbert $C^*$-modules are the same as Hilbert spaces (with the inner product linear in the second variable) and $\mathcal{L}(H, K) = B(H, K)$, the bounded, i.e., continuous, linear operators from $H$ to $K$.

**Example 2.3.** An example of primary importance in this work is obtained by letting $A$ be a right module over itself with the multiplication action $x \cdot a = xa$ and inner product $\langle a, b \rangle := a^*b$. Assuming $A$ is unital, $\mathcal{L}(A) \cong A$.

Some ways of forming new Hilbert $C^*$-modules from given Hilbert $C^*$-modules are by taking direct sums, internal tensor products, and external tensor products.

**Example 2.4.** (Direct sum) Let $\{X_i\}_{i\in I}$ be an infinite collection of Hilbert $C^*$-modules. Define $\oplus X_i$ to be the set of sequences $x = \{x_i\}$, such that each $x_i \in X_i$ and $\sum_i \langle x_i, x_i\rangle$ converges in $A$. Note that this is a weaker condition than the convergence of $\sum_i \|\langle x_i, x_i\rangle\| = \sum_i \|x\|^2$. The inner product is defined to be

$$\langle \{x_i\}, \{y_i\}\rangle = \sum_i \langle x_i, y_i\rangle.$$

**Example 2.5.** (External tensor product) Let $A$ and $B$ be $C^*$-algebras, $X$ a Hilbert $C^*$-module over $A$ and $Y$ a Hilbert $C^*$-module over $B$. The *exterior tensor product* of $X$ and $Y$ is the completion of the vector space tensor product $X \otimes_{alg} Y$ with respect to the $A \otimes B$-valued inner product

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle = \langle x_1, x_2\rangle \otimes \langle y_1, y_2\rangle \quad x_1, x_2 \in X, y_1, y_2 \in Y$$

and whose right $A \otimes B$ action is given on simple tensors by

$$(x \otimes y) \cdot (a \otimes b) = (xa \otimes yb), \quad x \in X, y \in Y, a \in A, b \in B.$$

Note that $A \otimes B$ refers to the spatial tensor product of $C^*$-algebras. This example will appear later with $B = Y = M_n(\mathbb{C})$.

**Example 2.6.** (Internal tensor product) Let $X$ be a Hilbert $C^*$-module over $A$ and $Y$ a Hilbert $C^*$-module over $B$. Further, let $\phi : A \to \mathcal{L}(Y)$ be a $*$-homomorphism. The homomorphism $\phi$ makes $Y$ a left $A$-module

$$a \cdot y = \phi(a)y \quad a \in A, y \in Y.$$

On the *balanced* algebraic tensor product $X \otimes_A Y$, there is a positive semi-definite inner product satisfying

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle y_1, \phi(\langle x_1, x_2 \rangle) y_2 \rangle \quad x_1, x_2 \in X, y_1, y_2 \in Y$$

that induces a seminorm. A Hilbert $C^*$-module $X \otimes_\phi Y$ (called the *interior tensor product* and sometimes also denoted $X \otimes_A Y$) is obtained by dividing out by the null space

$$N = \{z \in X \otimes_A Y \mid \langle z, z \rangle = 0, \}$$

and completing. Its right $B$-module structure is the obvious one:

$$(x \otimes y) \cdot b = x \otimes (yb).$$

Observe that $Y$ is simultaneously a left $A$-module and a right $B$-module. In fact, it is an $(A - B)$-bimodule, meaning the left and right actions commute. This is evident by exploiting properties of the $B$-valued inner product to show

$$\|(\phi(a)y)b - \phi(a)(yb)\|^2 = 0.$$

With this additional left $A$-module structure, $Y$ is an example of a $C^*$-correspondence from $A$ to $B$.

**Definition 2.7.** A *$C^*$-correspondence $X$ from $A$ to $B$* is a right Hilbert $C^*$-module over a $C^*$-algebra $B$ that is also a left $A$-module whose action is given by a $C^*$-homomorphism $\phi : A \to \mathcal{L}(X)$. In case $A = B$, $X$ is called a *$C^*$-correspondence over $A$.*

**Example 2.8.** Given an automorphism $\alpha : A \to A$ (assumed to be unital and adjoint-preserving), Example 2.3 with $X = A$ becomes a $C^*$-correspondence over $A$ by defining the left action by $a \cdot b = \alpha(a)b$ for $a, b \in A$. This correspondence is denoted by ${}_{\alpha}A$.

**Example 2.9.** Let $\sigma : A \to B(H)$ be a $C^*$-representation and $X$ a right Hilbert $C^*$-module over $A$. Observe that $\sigma$ makes $H$ a $C^*$-correspondence from $A$ to $\mathbb{C}$. Let $X \otimes_\sigma H$ be the internal tensor product. Following Rieffel in [44], $\sigma$ can be *induced* to a representation $\sigma^X : \mathcal{L}(X) \to B(X \otimes_\sigma H)$ defined by $\sigma^X(F)(x \otimes h) = F(x) \otimes h$. In other words, $\sigma^X(F) = F \otimes I_H$.

Alternate notation for the induced representation $\sigma^X$ is

$$X - \operatorname{Ind}_A^{\mathcal{L}(X)} \sigma$$

which is sometimes preferable, for example, when highlighting the algebras $A$ and $\mathcal{L}(X)$ or restricting an induced representation to a subalgebra of $M \subset \mathcal{L}(X)$. The restriction is written as $X - \operatorname{Ind}_A^M \sigma$.

Unless otherwise stated, representations $\sigma : A \to \mathcal{L}(X)$ are assumed to be *nondegenerate*, even when $X = H$ a Hilbert space. This means $\sigma(A)X = X$. One also says that $X$ is *essential*.

## 2.3 Operator Algebras and Completely Bounded Maps

**Definition 2.10.** An *operator algebra* is a norm-closed subalgebra of $B(K)$ for some Hilbert space $K$.

Let $B$ be an operator algebra and $T : B \to B(H)$ be a bounded, linear map.

For each $n \geq 0$ consider the map

$$T \otimes I_n : B \otimes M_n \to B(H) \otimes M_n,$$

where $M_n$ denotes the $n \times n$ complex, matrices and $I_n$ is the identity matrix. After identifying $B \otimes M_n \cong M_n(B)$ and $B(H) \otimes M_n \cong M_n(B(H)) = B(H^n))$, $T \otimes I_n$ equals the map

$$T^{(n)}([b_{ij}]) = [T(b_{ij})].$$

**Definition 2.11.** Let $T : B \to B(H)$ be a bounded, linear map.

1. $T$ is *completely bounded* if

$$\|T \otimes I_n\|_{cb} := \sup_n \|T \otimes I_n\| < \infty.$$

2. $T$ is *completely contractive* if $\|T\|_{cb} \leq 1$.

An important result in the the theory of completely bounded maps, due to Paulsen [37] and given below, is key to connecting completely bounded cohomology with completely contractive cohomology. In fact, this result has influenced this dissertation's usage of bounded module maps in short exact sequences rather than isometries and coisometries, as was done in [30].

**Theorem 2.12.** *If $B$ is an operator algebra and $\rho : B \to B(H)$ is a completely bounded representation, then there exists an invertible $R \in B(H, K)$ so that $R\rho(\cdot)R^{-1} : B \to B(K)$ is completely contractive and $\|\rho\|_{cb} = \|R\| \cdot \|R^{-1}\|$.*

The proof in [36] involved renorming $(H, \|\cdot\|)$ with an equivalent, Hilbert space norm $|\cdot|$ (meaning it comes from an inner product). This is important because the similarity is simply the identify function $(H, \|\cdot\|) \to (H, |\cdot|)$. Such maps are clearly module maps, which will turn out to be a very useful fact.

# CHAPTER 3

# TENSOR ALGEBRAS AND REPRESENTATIONS

## 3.1 Introduction

In this chapter, a characterization is given of completely bounded representations of a tensor algebra in terms of similarities of contractive intertwiners. Also proven is that for a $C^*$-correspondence $X$ over a $C^*$-algebra $A$, $\mathscr{T}(X) \otimes M_n$ and $\mathscr{T}(X \otimes M_n)$ are isomorphic $C^*$-algebras, where we denote the Toeplitz algebras of $X$ and of $X \otimes M_n$ by $\mathscr{T}(X)$ and $\mathscr{T}(X \otimes M_n)$, respectively. The analogous statement for tensor algebras is deduced as a corollary.

## 3.2 Tensor Algebras

Let $X$ be a $C^*$-correspondence over a unital $C^*$-algebra $A$ with left action $\phi : A \to \mathcal{L}(X)$. We let $X^{\otimes n} = X \otimes_A X \otimes_A \cdots \otimes_A X$ be the $n$-fold internal tensor product of $X$ as defined in (2.5). The left $A$-action is given by the $*$-homomorphism $\phi_n : A \to \mathcal{L}(X^{\otimes n})$ satisfying

$$\phi_n(a)(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = (\phi(a)x_1) \otimes x_2 \otimes \cdots \otimes x_n.$$

By convention we declare $X^{\otimes 0} = A$ as a $C^*$-correspondence over itself, as in (2.8), with the automorphism being the identity map $A \to A$.

**Definition 3.1.** The *full Fock space* $\mathscr{F}(X)$ *over* $X$ is the $C^*$-correspondence over $A$ $\oplus_{n=0}^{\infty} X^{\otimes n} = A \oplus X \oplus (X \otimes_A X) \oplus \cdots$. The left $A$-module structure is $\oplus_n \phi_n$ which we denote by $\phi_\infty$. It can be represented by the diagonal matrix

$$\phi_\infty(a) = \begin{bmatrix} a & & & \\ & \phi(a) & & \\ & & \phi_2(a) & \\ & & & \ddots \end{bmatrix}, \quad a \in A$$

where $\phi_n(a)(\xi_1 \otimes \cdots \otimes \xi_n) = (\phi(a)\xi_1) \otimes \cdots \otimes \xi_n$. Looking at the $(1,1)$-entry, it is clear that $\phi_\infty$ is injective. Thus, we will often identify $A$ with its image $\phi_\infty(A)$.

For each $x \in X$, we define the creation operator $T_x \in \mathcal{L}(\mathscr{F}(X))$ by

$$T_x = \begin{bmatrix} 0 & & & & \\ T_x^{(1)} & 0 & & & \\ & T_x^{(2)} & 0 & & \\ & & T_x^{(3)} & 0 & \\ & & & \ddots & \ddots \end{bmatrix}$$

where $T_x^{(k)} : X^{\otimes k} \to X^{\otimes(k+1)}$ is given by the formula

$$T_x^{(k)}(x_1 \otimes \cdots \otimes x_k) = x \otimes x_1 \otimes \cdots \otimes x_k.$$

**Definition 3.2.** Let $X$ be a $C^*$-correspondence over $A$.

1. The *tensor algebra* of $X$, denoted $\mathscr{T}_+(X)$, is the norm closed subalgebra of $\mathcal{L}(\mathscr{F}(X))$ generated by $\phi_\infty(A)$ and $\{T_x \mid x \in X\}$.

2. The *Toeplitz algebra* is the $C^*$-algebra generated by $\mathscr{T}_+(X)$ in $\mathcal{L}(\mathscr{F}(X))$.

Tensor algebras, introduced by Muhly and Solel in [31], are non-selfadjoint subalgebras of the Toeplitz $C^*$-algebras associated to $X$, which, in turn, were originally defined by Pimsner in [39]. Tensor algebras have an attractively tractable completely contractive representation theory expressed in terms of maps defined on the $C^*$-correspondence $X$. The purely algebraic tensor algebras have an analogous property, cf. [7].

**Example 3.3.** If $A = X = \mathbb{C}$, then $\mathscr{F}(\mathbb{C}) \cong H^2$, the classical Hardy space and $\mathscr{T}_+(X) = A(\mathbb{D})$ is the *classical disc algebra.*

**Example 3.4.** In case $A = \mathbb{C}$ and $X = \mathbb{C}^d$, then $\mathscr{T}_+(X) = \mathscr{A}_d$ is Popescu's *noncommutative disc algebra* [40].

**Example 3.5.** Returning to Example 2.8, with $X = {}_\alpha A$, we have $\mathscr{F}(X) \cong \oplus_{\alpha^n} A \cong \ell^2(\mathbb{Z}^+; A)$ Since $A$ is unital with $1 \in A$, $\mathscr{T}_+(X)$ is generated by $\phi_\infty(A)$ and a *single* creation operator $T_1$, which we denote by $S$. Furthermore, $\mathscr{T}_+(X) \cong A \times_\alpha \mathbb{Z}^+$, the analytic (or non-selfadjoint) crossed product of $A$ by $\mathbb{Z}^+$ determined by $\alpha$, an operator algebra whose representations were studied in [29].

### 3.3 Completely Bounded Representations

Representations of $\mathscr{T}(X)$ and $\mathscr{T}_+(X)$ are determined by representations of $A$ and bimodule maps defined from $X$ to $B(H)$. We follow [31] in defining covariant representations of $X$.

**Definition 3.6.** A pair $(V, \sigma)$ is called a *covariant representation* of $X$ on the Hilbert space $H$ if

1. $V : X \to B(H)$ is linear;

2. $\sigma : A \to B(H)$ is a nondegenerate $*$-homomorphism; and

3. $V$ is an $A$-bimodule map:

$$V(xa) = V(x)\sigma(a) \text{ and } V(\phi(a)x) = \sigma(a)V(x) \quad \forall x \in X, a \in A.$$

We say $(V, \sigma)$ is *bounded* (respectively *completely bounded, contractive, completely contractive*) when $V$ is bounded (respectively, completely bounded, contractive, completely contractive) when $X$ is assigned the operator space structure it inherits from the linking algebra

$$\mathfrak{L} = \begin{bmatrix} A & X^* \\ X & K(X) \end{bmatrix}.$$

The linking algebra is an important object in the context of Morita equivalence. For more information, we refer to [2] and [3]. We call $(V, \sigma)$ *isometric* in case $V(x)^*V(y) = \sigma(\langle x, y\rangle)$ for all $x, y \in X$. Note that $V$ is an isometric operator if $\sigma$ is faithful. In [39] Pimsner showed that isometric covariant representations of $X$ are in a bijective correspondence with $C^*$-representations of $\mathscr{T}(X)$. For this reason, isometric covariant representations are often called Toeplitz representations, e.g., in [39] and [14]. For future use, the statement of that theorem is recorded below. Furthermore, Muhly and Solel showed in [31] that the completely contractive covariant representations $(V, \sigma)$ correspond bijectively to completely contractive representations of $\mathscr{T}_+(X)$.

**Proposition 3.7.** *If $X$ is a correspondence over a $C^*$-algebra $A$ and if $(V, \sigma)$ is an isometric covariant representation of $\mathscr{T}(X)$ on a Hilbert space $H$, then the map*

$$\begin{cases} T_x \mapsto V(x), & x \in X, \\ \phi_\infty(a) \mapsto \sigma(a), & a \in A \end{cases}$$

*extends uniquely to a $C^*$-representation of $\mathscr{T}(X)$ on $H$. Conversely, if $\pi : \mathscr{T}(X) \to B(H)$ is a $C^*$-representation, and if $V(x)$ is defined to be $\pi(T_x), x \in X$, while $\sigma(a)$ is defined to be $\pi(\phi_\infty(a)), a \in A$, then $(V, \sigma)$ is an isometric covariant representation of $X$ on $H$.*

A key result in [31], which is also recorded below, uses an isometric dilation construction and Pimsner's theorem to parametrize those completely contractive representations of $\mathscr{T}_+(X)$ that yield $C^*$-representations when restricted to $\phi_\infty(A)$.

**Proposition 3.8.** *To every completely contractive covariant representation, $(V, \sigma)$, of a correspondence $X$ over a $C^*$-algebra $A$, there is a unique completely contractive representation $\rho$ of the algebra $\mathscr{T}_+(X)$ satisfying*

$$\rho(T_\xi) = V(\xi), \xi \in X$$

*and*

$$\rho(a) = \sigma(a), a \in A.$$

*The map $(V, \sigma) \mapsto \rho$ is bijective and onto the set of all completely contractive representations of $\mathscr{T}_+(X)$.*

If $\sigma_i : A \to B(H_i)$ are representations for $i = 1, 2$, then the *intertwining space* of $\sigma_2$ and $\sigma_1$, denoted $\mathcal{I}(\sigma_2, \sigma_1)$, is defined as the space of operators $T \in B(H_2, H_1)$ such that $T\sigma_2(a) = \sigma_1(a)T$ for every $a \in A$. Muhly and Solel [31], provide, yet another portrayal of the completely contractive representations of $\mathscr{T}_+(X)$ in terms of certain contractive intertwiners. Recall that $\sigma^X : \mathcal{L}(X) \to B(X \otimes_\sigma H)$ is the representation of $\sigma : A \to B(H)$ *induced up to* $X$ and satisfies

$$\sigma^X(F)(x \otimes h) = F(x) \otimes h.$$

In fact, $\sigma^X(F) = F \otimes I_H$.

**Definition 3.9.** Let

$$X^\sigma = \mathcal{I}(\sigma^X \circ \phi, \sigma) = \{T \in B(X \otimes_\sigma H, H) \mid T(\phi(a) \otimes I_H) = \sigma(a)T \quad \forall a \in A\}$$

and let $\mathbb{D}(X^\sigma)$ be its open disc.

Then, the points in $\overline{\mathbb{D}(X^\sigma)}$ parametrize those completely contractive representations of $\mathscr{T}_+(X)$ *that equal* $\sigma$ *when restricted to* $\phi_\infty(A)$. Of course, $\overline{\mathbb{D}(X^\sigma)}$, can also be written as $\left(\mathcal{I}(\sigma^X \circ \phi, \sigma)\right)_1$, but the disc notation emphasizes the perspective that elements of $\mathscr{T}_+(X)$ are functions on their space of representations. Indeed, when $X = A = \mathbb{C}$, $\mathscr{T}_+(\mathbb{C}) = \mathcal{A}(\mathbb{D})$ and $\mathbb{D}(X^\sigma) = \mathbb{D}$; the function-theoretic perspective is emphasized to the point of being part of the definition!

Theorem 2.12 enables us to describe the completely bounded representations of $\mathscr{T}_+(X)$ in terms of the discs $\mathbb{D}(X^\sigma)$. Let $\pi : \mathscr{T}_+(X) \to B(H)$ be a completely bounded representation that restricts to $\sigma : \phi_\infty(A) \to B(H)$. Note that $\sigma$ is not assumed to be a $C^*$-representation. By Paulsen's result there exist an invertible $R : H \to \widetilde{H}$ with $R\pi(\cdot)R^{-1} : \mathscr{T}_+(X) \to B(\widetilde{H})$ completely contractive. For convenience, denote $R\pi(\cdot)R^{-1}$ by $\widetilde{\pi}$ and $\widetilde{\pi}|\phi_\infty(A)$ by $\widetilde{\sigma}$. Observe that $R^{-1}\widetilde{\sigma}(a) = \sigma(a)R^{-1}$.

Since $\widetilde{\sigma}$ is a unital, completely contractive representation of a $C^*$-algebra, then it is completely positive. Consequently, $\widetilde{\sigma}$ is a $C^*$-representation. Using Muhly and Solel's characterization, $\widetilde{\pi}$ corresponds to a contraction $\widetilde{T} : X \otimes_{\widetilde{\sigma}} \widetilde{H} \to \widetilde{H}$ satisfying the intertwining equation

$$\widetilde{T}(\widetilde{\sigma}^X \circ \phi(\cdot)) = \widetilde{\sigma}\,(\cdot)\,\widetilde{T}.$$

It is desirable to associate $\widetilde{T}$ with a map $T : E \otimes_\sigma H \to H$. However, balanced tensor products have only been defined when $\sigma$ is a $C^*$-homomorphism; here $\sigma$ is only assumed to be a completely bounded homomorphism. This difficulty is easily dealt with by recognizing that the internal tensor product of $C^*$-correspondences is a special case of the *module Haagerup tensor product* of operator modules, cf. [2]. Thus, $E \otimes_\sigma H$ makes sense by considering $H$ as an operator space with its column operator space structure $C_H$. Then, $CB(H) = B(H)$ and the following map makes sense

$$I_X \otimes R : X \otimes_\sigma H \to X \otimes_{\widetilde{\sigma}} \widetilde{H}.$$

Observe that the property $R\sigma(a) = \widetilde{\sigma}(a)R$ for all $a \in A$, is crucial. Denote by $T$ the operator $R^{-1}\widetilde{T}(I_X \otimes R) : X \otimes_\sigma H \to H$. Observe that

$$\|T\| \leq \|R^{-1}\|\|\widetilde{T}\|\|I_X \otimes R\| \leq \|R^{-1}\|\|R\| = \|\pi\|_{cb}$$

and

$$\begin{aligned} T(\phi(a) \otimes I_H) &= R^{-1}\widetilde{T}(I_X \otimes R)(\phi(a) \otimes I_H) \\ &= R^{-1}\widetilde{T}(\phi(a) \otimes I_H)(I_X \otimes R) \\ &= R^{-1}\widetilde{\sigma}(a)\widetilde{T}(I_X \otimes R) \\ &= \sigma(a)R^{-1}\widetilde{T}(I_X \otimes R) \\ &= \sigma(a)T. \end{aligned}$$

Thus, $T$ is an element of $\mathcal{I}(\sigma^X \circ \phi, \sigma)$ of norm at most $\|\pi\|_{cb}$. Finally,

$$\begin{aligned} T(x \otimes h) &= R^{-1}\widetilde{T}(I_X \otimes R)(x \otimes h) \\ &= R^{-1}\widetilde{T}(x \otimes Rh) \\ &= R^{-1}\widetilde{\pi}(T_x)Rh \\ &= \pi(T_x)h \end{aligned}$$

Consequently, the following theorem has been proved.

**Theorem 3.10.** *Fix a (unital) completely bounded representation $\sigma : A \to B(H)$. Every completely bounded representation $\pi : \mathscr{T}_+(X) \to B(H)$ with $\pi \circ \phi_\infty = \sigma$ is given by an element of $\mathcal{I}(\sigma^X \circ \phi, \sigma)$ of norm less than $\|\pi\|_{cb}$, in the sense that $\pi(T_x)h = T(x \otimes h)$.*

It should be noted that, although an internal tensor product of $C^*$-module and a Hilbert space is itself a Hilbert space, it is not clear whether or not the module Haagerup tensor product $X \otimes_\sigma H$ is a Hilbert space. Blecher, studying metric characterizations of Hilbert $C^*$-modules in [1], gives circumstances under which a module

Haagerup tensor product $X \otimes_\sigma H$ turns out to be a Hilbert space. For example, this is the case when $A$ is represented faithfully and nondegenerately on $H$. In any case, there exists a homomorphism

$$\sigma^X \circ \phi : A \to B(E \otimes_\sigma H),$$

but it need not be a $*$-homomorphism since $\sigma$ was only assumed to be completely bounded.

**Remark 3.11.** The converse of the above theorem is usually false. That is, given an intertwiner $T$, there might *not* be any completely bounded representations $\pi$ with $\pi(T_x)h = T(x \otimes h)$. In the setting of the classical disc algebra, completely bounded representations arise from operators similar to contractions. That is, a representation of the disc algebra is completely bounded if and only if its generator (the operator we denoted by $T$ in Theorem 3.10) is completely polynomially bounded, cf. [37]. This generalizes to the following result.

**Theorem 3.12.** *A point $T \in \mathcal{I}(\sigma^X \circ \phi, \sigma)$ corresponds to a completely bounded representation $\pi$, in the sense that $\pi(T_x)h = T(x \otimes h)$, if there exists an invertible operator $R : H \to K$ such that $R\sigma R^{-1} : A \to B(K)$ is completely contractive and $RT(I_X \otimes R^{-1})$ is contractive, where $\sigma$ refers to $\pi|\phi_\infty(A)$. Furthermore, $\|T\| \leq \|\pi\|_{c.b.}$.*

*Proof.* Note that $R\sigma R^{-1}$ is necessarily a $C^*$-representation, so results of Muhly and Solel apply. Furthermore, $T$ intertwining $\sigma^X \circ \phi$ and $\sigma$ implies $RT(I_X \otimes R^{-1})$ intertwines $(R\sigma R^{-1})^X \circ \phi$ and $R\sigma R^{-1}$. Therefore, $RT(I_X \otimes R^{-1})$ uniquely determines a completely contractive representation of $\mathscr{T}_+(X)$ on $K$. Using the similarity to realize this representation on $H$ yields a completely bounded representation $\pi$ satisfying $\pi(T_x)h = T(x \otimes h)$. Observe that

$$\|T\| = \|R^{-1}RTR^{-1}R\| \leq \|R^{-1}\|\|RTR^{-1}\|\|R\| \leq \|R^{-1}\|\|R\|.$$

Likewise, for any invertible $S : H \to H_S$ such that $S\pi S^{-1}$ is completely contractive, $ST(I_X \otimes S^{-1})$ will be a contraction and we get $\|T\| \leq \|S^{-1}\|\|S\|$. Thus,

$$\|T\| \leq \inf\{\|S\|\|S^{-1}\| \mid S\pi S^{-1} \text{ is completely contractive}\} = \|\pi\|_{cb}.$$

□

**Remark 3.13.** If the correspondence $X$ is strictly cyclic, then contractive representations of $\mathscr{T}_+(X)$ are automatically completely contractive, cf. Theorem 3.13 of [31]. In such cases, the hypotheses above can be replaced with "$R\sigma R^{-1}$ and $RT(I_X \otimes R^{-1})$ are contractive."

Thus, the completely bounded representations $\pi$ of $\mathscr{T}_+(X)$ with $\pi \circ \phi_\infty = \sigma$ are given by the set

$$\bigcup_{R\sigma R^{-1} \text{ c.c.}} \overline{\mathbb{D}\left(X^{R\sigma R^{-1}}\right)},$$

where, as usual, $\mathbb{D}(X^{R\sigma R^{-1}})$ is the disc of the intertwiner space $\mathcal{I}((R\sigma R^{-1})^X \circ \phi, R\sigma R^{-1}) = R\mathcal{I}(\sigma^X \circ \phi, \sigma)R^{-1}$. However, the above presentation involves *much* redundancy. For instance,

$$R\sigma R^{-1} = S\sigma S^{-1} \text{ if and only if } RS^{-1} \in \sigma(A)'$$

where $\sigma(A)'$ denotes the commutant of $\sigma(A)$. Also, a completely bounded $\pi$ may have numerous completely contractive representations to which it is similar. A method for accounting for the redundancy would quite helpful and interesting.

### 3.4 Matricial Toeplitz Algebras

The next result says that the processes of forming a Toeplitz algebra and tensoring with a matrix algebra commute. If $X$ is a $C^*$-correspondence over a $C^*$-algebra $A$ with left action given by $\phi : A \to \mathcal{L}(X)$, then $M_n(X)$ is a $C^*$-correspondence over $M_n(A)$. The right action of $M_n(A)$ on $M_n(X)$ is given by matrix multiplication. The left action is given by the homomorphism $\phi^{(n)} : M_n(A) \to M_n(\mathcal{L}(X))$ defined by

$$\phi^{(n)}\left([a_{ij}]\right) = \left[\phi\left(a_{ij}\right)\right].$$

The $M_n(A)$-valued inner product is defined by

$$\langle [e_{ij}], [f_{ij}] \rangle = \left[\sum_k \langle e_{ki}, f_{kj} \rangle\right].$$

Observe that under the identifications $M_n(X) = X \otimes M_n$ and $M_n(A) = A \otimes M_n$, then the $M_n(A)$-valued inner product is simply that of the exterior tensor product of the Hilbert $A$-module $X$ and the Hilbert $M_n$-module $M_n$.

**Proposition 3.14.**

$$\mathscr{T}(M_n(X)) \cong M_n(\mathscr{T}(X))$$

*Proof.* Consider the Toeplitz representation $(\iota_X, \phi_\infty)$ of the $C^*$-correspondence $X$ over $A$ in the $C^*$-algebra $\mathscr{T}(X)$ that appears in the definition of $\mathscr{T}(X)$. We, then, have the Toeplitz representation $\left(\iota_X^{(n)}, \phi_\infty^{(n)}\right)$ of $M_n(X)$ (a correspondence over $M_n(A)$) in the $C^*$-algebra $M_n(\mathscr{T}(X))$ defined by

$$\iota_X^{(n)}\left([x_{ij}]\right) = [\iota_X(x_{ij})] = \left[T_{x_{ij}}\right]$$

and

$$\phi_\infty^{(n)}\left([a_{ij}]\right) = [\phi_\infty(a_{ij})] = [\phi_\infty(a_{ij})]$$

for $[x_{ij}] \in M_n(X), [a_{ij}] \in M_n(A)$. The following three computations verify that these maps, indeed, constitute a Toeplitz representation.

$$\begin{aligned}
\iota_X^{(n)}([x_{ij}][a_{ij}]) &= \iota_X^{(n)}\left(\left[\sum_k x_{ik}a_{kj}\right]\right) \\
&= \left[\iota_X\left(\sum_k x_{ik}a_{kj}\right)\right] \\
&= \left[\sum_k T_{x_{ik}a_{kj}}\right] \\
&= \left[\sum_k T_{x_{ik}}\phi_\infty(a_{kj})\right] \\
&= \left[T_{x_{ij}}\right][\phi_\infty(a_{ij})] \\
&= \iota_X^{(n)}\left([x_{ij}]\right)\phi_\infty^{(n)}([a_{ij}]).
\end{aligned} \tag{3.1}$$

$$
\begin{aligned}
\iota_X^{(n)}\left(\phi^{(n)}([a_{ij}])[x_{ij}]\right) &= \iota_X^{(n)}\left([\phi(a_{ij})][x_{ij}]\right) \\
&= \iota_X^{(n)}\left(\left[\sum_k \phi(a_{ik})x_{kj}\right]\right) \\
&= \left[\iota_X\left(\sum_k \phi(a_{ik})x_{kj}\right)\right] \\
&= \left[\sum_k T_{\phi(a_{ik})x_{kj}}\right] \qquad (3.2)\\
&= \left[\sum_k \phi_\infty(a_{ik})T_{x_{kj}}\right] \\
&= [\phi_\infty(a_{ij})]\,[\iota_X(x_{ij})] \\
&= \phi_\infty^{(n)}\left([a_{ij}]\right)\iota_X^{(n)}\left([x_{ij}]\right).
\end{aligned}
$$

$$
\begin{aligned}
\iota_X^{(n)}\left([x_{ij}]\right)^*\iota_X^{(n)}\left([y_{ij}]\right) &= \left[\iota_X\left(x_{ij}\right)\right]^*\left[\iota_X\left(y_{ij}\right)\right] \\
&= \left[T_{x_{ij}}\right]^*\left[T_{y_{ij}}\right] \\
&= \left[T_{x_{ji}}^*\right]\left[T_{y_{ij}}\right] \\
&= \left[\sum_k T_{x_{ki}}^* T_{y_{kj}}\right] \qquad (3.3)\\
&= \left[\sum_k \phi_\infty\left(\langle x_{ki}, y_{kj}\rangle\right)\right] \\
&= \phi_\infty^{(n)}\left(\sum_k \langle x_{ki}, y_{kj}\rangle\right) \\
&= \phi_\infty^{(n)}\left(\langle [x_{ij}], [y_{ij}]\rangle\right)
\end{aligned}
$$

By Proposition 3.7 there is a $C^*$-homomorphism $\iota_X^{(n)}\times\phi_\infty^{(n)} = \psi : \mathscr{T}(M_n(X)) \to M_n(\mathscr{T}(X))$. We now use a theorem of Fowler and Raeburn to show $\psi$ is injective. We denote by $\iota_{M_n(A)} : M_n(A) \to \mathscr{T}(M_n(X)$ and $\iota_{M_n(X)} : M_n(X) \to \mathscr{T}(M_n(X))$

the Toeplitz represenation that defines $\mathscr{T}(M_n(X))$. As $\psi$ maps the generators of $\mathscr{T}(M_n(X))$, $\iota_{M_n(X)}([x_{ij}])$ to $[T_{x_{ij}}]$ and $\iota_{M_n(A)}([a_{ij}])$ to $\phi_\infty^{(n)}([a_{ij}])$, we see that the generators of $M_n(\mathscr{T}(X))$ are contained in the image of $\psi$. Combining this with the continuity and after verifying the injectivty of $\psi$ it will follow that $\psi$ is an isomorphism.

It remains to show that $\psi$ is injective. The follow argument is essentially the proof of Corollary 2.2 of [14]. Let $\sigma_0 : A \to B(H_0)$ be a faithful representation. We induce $\sigma_0$ to a representation $\mathscr{F}(X) - \mathrm{Ind}_A^{L(\mathscr{F}(X))} \sigma_0$ of $L(\mathscr{F}(X)$ on the Hilbert space $\mathscr{F}(X) \otimes_{\sigma_0} H_0$, which then restricts to $\mathscr{F}(X) - \mathrm{Ind}_A^{\mathscr{T}(X)} \sigma_0 : \mathscr{T}(X) \to B(\mathscr{F}(X) \otimes_{\sigma_0} H_0)$. This representation induces a representation

$$\left(\mathscr{F}(X) - \mathrm{Ind}_A^{\mathscr{T}(X)} \sigma_0\right)^{(n)} : M_n(\mathscr{T}(X)) \to M_n(B(\mathscr{F}(X) \otimes_{\sigma_0} H_0))$$

in the usual fashion, by applying $\mathscr{F}(X) - \mathrm{Ind}_A^{\mathscr{T}(X)} \sigma_0$ to every entry of an element in $M_n(\mathscr{T}(X))$. Recall that, for any Hilbert space $H$, we can identify $M_n(B(H))$ with $B(\oplus_{i=1}^n H)$. Thus, the representation above acts upon $\oplus_{i=1}^n \mathscr{F}(X) \otimes_{\sigma_0} H_0$. Since $(\iota_X^{(n)}, \phi_\infty^{(n)})$ is a Toeplitz representation, then so is $(Z, \tau)$ defined by

$$Z = \left(\mathscr{F}(X) - \mathrm{Ind}_A^{\mathscr{T}(X)} \sigma_0\right)^{(n)} \circ \iota_X^{(n)} : M_n(X) \to B(\oplus_{i=1}^n \mathscr{F}(X) \otimes_{\sigma_0} H_0)$$

and

$$\tau = \left(\mathscr{F}(X) - \mathrm{Ind}_A^{\mathscr{T}(X)} \sigma_0\right)^{(n)} \circ \phi_\infty^{(n)} : M_n(A) \to B(\oplus_{i=1}^n \mathscr{F}(X) \otimes_{\sigma_0} H_0).$$

Observe that, for $[x_{ij}] \in M_n(X), y_k \otimes h_k \in X^{\otimes m_k}, m_k \geq 0$,

$$
\begin{aligned}
Z\left(([x_{ij}])\right)\begin{bmatrix} y_1 \otimes h_1 \\ y_2 \otimes h_2 \\ \vdots \\ y_n \otimes h_n \end{bmatrix} &= \left(\mathscr{F}(X) - \operatorname{Ind}_A^{\mathscr{T}(X)} \sigma_0\right)^{(n)} \left([T_{x_{ij}}]\right) \begin{bmatrix} y_1 \otimes h_1 \\ y_2 \otimes h_2 \\ \vdots \\ y_n \otimes h_n \end{bmatrix} \\
&= \left[\left(\mathscr{F}(X) - \operatorname{Ind}_A^{\mathscr{T}(X)} \sigma_0\right)\left(T_{x_{ij}}\right)\right] \begin{bmatrix} y_1 \otimes h_1 \\ y_2 \otimes h_2 \\ \vdots \\ y_n \otimes h_n \end{bmatrix} \\
&= [T_{x_{ij}} \otimes id_{H_0}] \begin{bmatrix} y_1 \otimes h_1 \\ y_2 \otimes h_2 \\ \vdots \\ y_n \otimes h_n \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=1}^n T_{x_{1k}} y_k \otimes h_k \\ \sum_{k=1}^n T_{x_{2k}} y_k \otimes h_k \\ \vdots \\ \sum_{k=1}^n T_{x_{nk}} y_k \otimes h_k \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=1}^n x_{1k} \otimes y_k \otimes h_k \\ \sum_{k=1}^n x_{2k} \otimes y_k \otimes h_k \\ \vdots \\ \sum_{k=1}^n x_{nk} \otimes y_k \otimes h_k \end{bmatrix}.
\end{aligned}
$$

Therefore,

$$\overline{Z(M_n(X))\left(\oplus_{i=1}^n \mathscr{F}(X) \otimes_{\sigma_0} H_0\right)} = \oplus_{i=1}^n \left(\oplus_{m=1}^\infty X^{\otimes m} \otimes_{\sigma_0} H_0,\right)$$

which has complement $\oplus_{i=1}^n X^{\otimes 0} \otimes_{\sigma_0} H_0 = \oplus_{i=1}^n A \otimes_{\sigma_0} H_0 \cong \oplus_{i=1}^n H_0$. $M_n(A)$ acts faithfully on this complement via $\sigma_0^{(n)}$ since $\sigma_0$ is faithful. Consequently, it follows from Theorem 2.1 of [14] that $Z \times \tau$ is a faithful representation of $\mathscr{T}(M_n(X))$ on $B(\oplus_{i=1}^n \mathscr{F}(X) \otimes_{\sigma_0} H_0)$. However,

$$Z \times \tau = (\mathscr{F}(X) - \mathrm{Ind}_A^{\mathscr{T}(X)} \sigma_0)^{(n)} \circ (\iota_X^{(n)} \times \phi_\infty^{(n)}) = (\mathscr{F}(X) - \mathrm{Ind}_A^{\mathscr{T}(X)} \sigma_0)^{(n)} \circ \psi.$$

Consequently, $\psi$ is faithful. □

**Corollary 3.15.**

$$\mathscr{T}_+(M_n(X)) \cong M_n(\mathscr{T}_+(X)).$$

*Proof.* Simply restrict the above isomorphism $\psi : \mathscr{T}(M_n(X)) \to M_n(\mathscr{T}(X))$ to $\mathscr{T}_+(M_n(X))$ and observe that its generators are actually mapped into $M_n(\mathscr{T}_+(X))$.

□

**Example 3.16.** Let $A$ be a $C^*$-correspondence over itself with left action given by an automorphism $\alpha$ as in Example 2.8. It follows that $\alpha^{(n)}$, defined by applying $\alpha$ entry-wise, is an automorphism of $M_n(A)$. By (3.15) the algebras $\mathscr{T}_+({}_\alpha A) \otimes M_n$ are analytic cross products $\mathscr{T}_+({}_{\alpha^{(n)}} M_n(A))$ for every $n$.

# CHAPTER 4

# COHOMOLOGY

## 4.1 Introduction

In [11] Douglas and Paulsen cast the Sz-Nagy–Foiaş model theory for contraction operators in the language of modules over functional algebras. As function algebras are subalgebras of commutative $C^*$-algebras, it makes sense to consider similar module-theoretic notions such as projectivity and injectivity in the noncommutative setting. This was the perspective of Muhly and Solel in [30]. In a similar spirit, we extend the theory of Hilbert module extensions developed by Carlson and Clark in [4] for the classical disc algebra to the noncommutative setting.

Relatedly, Paulsen formulates a cohomology theory for operator spaces in [38], but the modules are operator modules and not necessarily Hilbert spaces. At the end of that paper, Paulsen briefly considers an Ext functor, denoted as $Hext$, when operator modules are replaced by *bounded* Hilbert modules. See [4], [5], [12], [13], and [16] for results in *commutative* cases of that setting. Since the completely bounded and completely contractive representation theory is better understood for noncommutative operator algebras, we develop Ext for completely bounded Hilbert modules.

Much of the the following work uses standard machinery from homological algebra, cf. [28, 45, 55]. Category theory was developed to unify common ideas in various homology/cohomology theories. Such theories were commonly abelian categories; however, categories in functional analysis are rarely abelian. The first section, a very brief review of non-abelian category theory, is provided mainly for the purpose of familiarizing functional analysts with some of the terminology and to bring attention to a number of interesting papers in non-abelian category theory.

### 4.2 Non-abelian Categories

The fact that categories in functional analysis are often not abelian makes their homology and cohomology interesting from a purely algebraic point of view. For example, consider the category of completely bounded Hilbert modules over the disc algebra $\mathcal{A}(\mathbb{D})$ with Hilbert $\mathcal{A}(\mathbb{D})$-module maps as morphisms. The first isomorphism theorem is violated as witnessed by the dense embedding of the Hardy space $H^2$ into the Bergman space $B^2$, both of which are contractive (therefore, completely contractive) Hilbert modules.

Helpful references containing the following ideas from (co)homology in non-abelian categories are [26, 42, 43] and especially, [15, 19, 48, 49] which have a functional analytic perspective. Recall that terms like kernel and cokernel have categorial definitions as morphisms satisfying universal properties. Also, an image is defined to be a kernel of a cokernel; dually, a coimage is a cokernel of a kernel. All four of these type of morphisms are frequently identified with their domains or codomains. For the purposes of *this* section, we write $\mathrm{ker}, \mathrm{coker}, \mathrm{im}$, and $\mathrm{coim}$ for the kernel, cokernel, image, and coimage morphisms, respectively. We denote by Ker and Im the domains of ker and im, respectively. Dually, we denote Coker and Coim the codomains of coker and coim, respectively.

**Definition 4.1.** Let $\mathcal{C}$ be a category with objects $Ob(\mathcal{C})$ and morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for $X, Y \in Ob(\mathcal{C})$. We have the following relaxations of abelian categories.

1. The category $\mathcal{C}$ is *pre-additive* if it has a zero object and each set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an abelian group such that addition and composition of morphisms satisfies the distribution laws.

2. A pre-additive category $\mathcal{C}$ is *additive* if contains finite biproducts.

3. An additive category is *pre-abelian* if every morphism has a kernel and a cokernel.

4. A pre-abelian category $\mathcal{C}$ is *semi-abelian* if for every $\beta \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ the canonical morphism $\hat{\beta} : \mathrm{Coim}\,\beta \to \mathrm{Im}\,\beta$ is a *bimorphism*, that is, simultaenously a monomorphism and an epimorphism. Recall that a monomorphism is a left-cancellative morphism and an epimorphism is right-cancellative.

5. A pre-abelian category $\mathcal{C}$ is *quasi-abelian* if its "maximal exact structure" (see [42] for the definition) consists of all kernel-cokernel pairs.

6. A pre-abelian category $\mathcal{C}$ is *abelian* if for every $\beta \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ the morphism $\hat{\beta} : \mathrm{Coim}\,\beta \to \mathrm{Im}\,\beta$ is a *isomorphism*.

**Remark 4.2.** Every quasi-abelian category is semi-abelian, but the converse, known as Raikov's conjecture, has been recently proven false (cf. [46]).

A pullback diagram consists of two morphisms $\beta_i : X_i \to Y$:

$$X_1 \xrightarrow{\beta_1} Y \xleftarrow{\beta_2} X_2 \tag{4.1}$$

The pullback (or fibered product) of this diagram is an object $Z$ along with maps $\gamma_i : Z \to X_i$ satisfying $\beta_1\gamma_1 = \beta_2\gamma_2$ that is a solution to the universal mapping problem: for every $(\hat{Z}, \hat{\gamma_1}, \hat{\gamma_2})$ with $\beta_1\hat{\gamma_1} = \beta_2\hat{\gamma_2}$, there exists a unique morphism $\theta : \hat{Z} \to Z$ making the following diagram commmute.

$$\begin{tikzcd}
\hat{Z} \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "\hat{\gamma_1}"] \arrow[ddr, bend right, "\hat{\gamma_2}"'] & & \\
& Z \arrow[r, "\gamma_1"] \arrow[d, "\gamma_2"'] & X_1 \arrow[d, "\beta_1"] \\
& X_2 \arrow[r, "\beta_1"'] & Y
\end{tikzcd} \tag{4.2}$$

A pushout, also called a fibered sum, is a notion dual to a pullback and is defined by reversing the arrows in 4.1 and 4.2.

**Proposition 4.3.** *Pullbacks and pushouts exist in pre-abelian categories [28].*

*Proof.* Let $X, X_1, X_2 \in Ob(\mathcal{C})$ and let $\iota_{X_i} : X_i \to X_1 \oplus X_2$ and $\pi_{X_i} : X_1 \oplus X_2 \to X_i$, $i = 1, 2$, be the "inclusion" and "projection" maps, respectively. Formally, they are morphisms

$$\pi_{X_i}\iota_{X_i} = 1_{X_i}, \quad \iota_{X_1}\pi_{X_1} + \iota_{X_2}\pi_{X_2} = 1_{X_1 \oplus X_2}$$

and an object $X_1 \oplus X_2$ satisfying a certain universal problem. Let

$$\Delta_X := \iota_1 + \iota_2 : X \to X \oplus X, \quad \nabla_X = \pi_1 + \pi_2 : X \oplus X \to X$$

be the "diagonal" and "codiagonal" maps. We show that the category $\mathcal{C}$ has pullbacks if it has coproducts and kernels. If $\beta_i : X_i \to X$, for $i = 1, 2$, then we set $P =$

$\operatorname{Ker}\nabla_X(\beta_1 \oplus (-\beta_2))$ with $\nabla_X : X \oplus X \to X$ and morphisms $\gamma_i = \pi_{X_i} \ker(\nabla_X(\beta_1 \oplus (-\beta_2))) : P \to X_i$. It follows that $(P, \gamma_1, \gamma_2)$ is the pullback. The dual statement asserting the existence of pushouts is proved using a dual argument. $\square$

**Remark 4.4.** The proof above, whether known abstractly in terms of (co)products and (co)kernels or as a construction in particular categories, is well-known. In abelian categories, pushouts and pullbacks of kernels (cokernels) are also kernels (cokernels). Interestingly, this property is not a necessity in pre-abelian categories In fact, pullbacks of kernels are kernels and pushouts of cokernels are cokernels [43, Theorem 1]. In that paper, the authors provide an example of pushouts of kernels failing even to be monic.

Although some of the papers cited above considered non-abelian categories in functional analysis, most examples consisted of classes of locally convex topological vector spaces that are not Banach spaces. It appears that Hilbert spaces with bounded, linear operators have received little attention, with the exception of [19] There it is proven, among other things, that the monomorphisms are precisely the injective operators and the epimorphisms are the operators with dense range. This result is stated in [18] for any category of locally convex modules. It is easy to show the following morphisms yield correct solutions to the relevant universal problems in the category of Hilbert spaces with bounded, linear maps. Let $\beta \in B(H, K)$ be a bounded, linear operator.

$$\ker \beta = \iota : \beta^{-1}(\{0\}) \to H,$$

$$\operatorname{coker}\beta = \pi : K \to K/\overline{\beta H},$$

$$\operatorname{im}\beta = \ker(\operatorname{coker}\beta) = \iota : \overline{\beta H} \to K,$$

$$\operatorname{coim}\beta = \operatorname{coker}(\ker\beta) = \pi : H \to H/\ker\beta.$$

Note that $\hat{\beta} : \operatorname{Coim}\beta \to \operatorname{Im}\beta$ is the same as $\hat{\beta} : H/\operatorname{Ker}\beta \to \overline{\beta H}$ which is clearly injective and has dense range. Thus, this category is semi-abelian. The determination of whether it is quasi-abelian has yet to be worked out.

## 4.3 Extensions

In this section we develop completely bounded cohomology, largely following [38] but using Hilbert modules rather than the more general category of operator modules. The development below was largely inspired by [4, 31, 30]. Some proofs have been simplified by considering the non-abelian category theory point of view. This more abstract approach via semi-abelian category theory has only recent been discovered by the author. As such it stands as a promising approach that will be pursued in future work. Below, a relative Yoneda presentation is given rather than a derived functor approach because the usual projective resolutions constructed from tensor products of the acting algebra are seemingly unavailable as Hilbert spaces.

**Definition 4.5.** Let $B$ be an operator algebra. A *completely bounded Hilbert module over $B$* is a Hilbert space $H$ along with a completely bounded representation $\rho : B \to B(H)$. If $H_1$ and $H_2$ are Hilbert modules over $B$ with respective representations $\rho_1$ and $\rho_2$, then a *Hilbert $B$-module map* $\beta$ is an operator in $B(H_1, H_2)$ *intertwining* $\rho_1$ and $\rho_2$. This means $\beta\rho_1(b) = \rho_2(b)\beta$ for all $b \in B$. As this dissertation rarely considers representations that are bounded but not completely bounded, we abbreviate the terminology by saying "Hilbert module" rather than writing the adjectives "completely bounded" or "c.b." ad nauseum. Also, we often will refer to $\beta$ simply as a module map.

Let $\mathcal{H}Bmod$ denote the category of completely bounded Hilbert modules over an operator algebra $B$ with bounded, linear $B$-module maps defined above. Observe that the category of Hilbert spaces with bounded, linear maps is the special case of $\mathcal{H}Bmod$ when $B = \mathbb{C}$.

**Definition 4.6.** Let $H = H_0, H_2, \ldots, H_{n+1} = K$ be completely bounded Hilbert modules with module maps $\beta_i : H_{i-1} \to H_i$. The sequence

$$\xi : 0 \longrightarrow H \xrightarrow{\beta_1} H_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{n+1}} K \longrightarrow 0$$

is *exact* if $\ker \beta_i = \operatorname{ran} \beta_{i-1}$. We call such a sequence an *$n$-extension* of $H$ by $K$. (Some authors refer to this as an $n$-extension of $K$ by $H$.) Two $n$-extensions of $H$ by $K$, $\xi$ and $\xi'$ are *related*, denoted $\xi \to \xi'$ or $\xi' \leftarrow \xi$, if there exist module maps $\theta_i : H_i \to H_i'$ such that $\theta_0 = I_H$, $\theta_{n+1} = I_K$ and the following diagram commutes.

$$\begin{tikzcd}
\xi : 0 \arrow[r] & H \arrow[r, "\beta_1"] \arrow[d, equal] & H_1 \arrow[r, "\beta_2"] \arrow[d, "\theta_1"] & \cdots \arrow[r, "\beta_{n+1}"] \arrow[d, phantom, "\cdots"] & K \arrow[r] \arrow[d, equal] & 0 \\
\xi' : 0 \arrow[r] & H \arrow[r, "\beta_1"] & H_1 \arrow[r, "\beta_2"] & \cdots \arrow[r, "\beta_{n+1}"] & K \arrow[r] & 0
\end{tikzcd}$$

This induces an equivalence relation $\approx$ that is described as follows. The $n$-extensions $\xi \approx \xi'$ if and only if there exist $n$-extensions $\eta_1, \ldots, \eta_{2m}$ of $H$ by $K$ such that

$$\xi \to \eta_1 \leftarrow \eta_2 \to \cdots \leftarrow \eta_{2m} \to \xi'.$$

Equivalently, there exists an $n$-extension $\eta$ with $\xi \leftarrow \eta \rightarrow \xi'$ [28, pp. 92, 93]. We denote the equivalence class of $\xi$ by $[\xi]$ and the collection of equivalence classes of $n$-extensions by $\mathrm{Ext}^n(K, H)$. With respect to the Baer sum, $\mathrm{Ext}^n(K, H)$ is an abelian group. Though we do not use the Baer sum in the present work, we mention that it is defined using the direct sum, diagonal, codiagonal, pullback, and pushout. The existence of such things in $\mathcal{H}Bmod$ is demonstrated below.

It is instructive to consider the special case of 1-extensions. Fix Hilbert modules $H_1, H_2$, and $H$ over $B$ and module maps $\beta_1 : H_1 \rightarrow H$ and $\beta_2 : H \rightarrow H_2$. An element $[\xi] \in \mathrm{Ext}^1(H_2, H_1)$ is represented by a short exact sequence

$$\xi : 0 \longrightarrow H_1 \xrightarrow{\beta_1} H \xrightarrow{\beta_2} H_2 \longrightarrow 0.$$

This means $\beta_1$ is injective, $\beta_2$ is surjective, and $\beta_1(H_1) = \ker b_2$. It is common to refer to the module $H$ as an *extension of* $H_1$ *by* $H_2$. Given another extension

$$\eta : 0 \longrightarrow H_1 \xrightarrow{\gamma_1} K \xrightarrow{\gamma_2} H_2 \longrightarrow 0$$

of Hilbert $B$-modules, $\xi$ and $\eta$ are related if there exists a module map $\theta$ making the following diagram commute:

$$\begin{array}{ccccccccc}
\xi : 0 & \longrightarrow & H_1 & \xrightarrow{\beta_1} & H & \xrightarrow{\beta_2} & H_2 & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \theta} & & \| & & \\
\eta : 0 & \longrightarrow & H_1 & \xrightarrow{\gamma_1} & K & \xrightarrow{\gamma_2} & H_2 & \longrightarrow & 0
\end{array}$$

With the existence of $\theta$, it follows that $\xi$ and $\eta$ are actually equivalent. It is an easy consequence that such a map $\theta$ is necessarily invertible with its inverse also a Hilbert $B$-module map. In fact, the proof uses the usual "Short Five Lemma" which is valid in our setting and may be proved either by diagram-chasing or by using pullbacks/pushouts, whose existence we demonstrate below. Observe that the zero element in $\mathrm{Ext}^1(H_2, H_1)$ is the equivalence class of the split extension

$$0 \longrightarrow H_1 \xrightarrow{\beta_1} H_1 \oplus H_2 \xrightarrow{\beta_2} H_2 \longrightarrow 0$$

where $\beta_1(h_1) = (h_1, 0)$ and $\beta_2(h_1, h_2) = h_2$. Furthermore, the $B$-module action on the middle term is given by $f \cdot (h_1, h_2) = (f \cdot h_1, f \cdot h_2)$ for every $f \in B, h_1 \in H_1, h_2 \in H_2$. In other words, if we denote the representations of $B$ on $H_i$ by $\rho_i : B \to B(H_i)$, then the trivial extension is $H_1 \oplus H_2$ along with the representation $\rho_1 \oplus \rho_2$.

Even for extensions

$$0 \longrightarrow H_1 \xrightarrow{\beta_1} H \xrightarrow{\beta_2} H_2 \longrightarrow 0$$

that are not necessarily trivial extensions, the space $H \cong H_1 \oplus H_2$ as *Hilbert spaces*, though not necessarily as $B$-modules. In this dissertation the $\oplus$ symbol, when applied to Hilbert spaces, will always refer to the Hilbert space direct sum. Since $\operatorname{ran} \beta_1 = \ker \beta_2$, $\beta_1$ has closed range and, therefore,

$$H = \operatorname{ran} \beta_1 \oplus (\operatorname{ran} \beta_1)^{\perp}. \tag{4.3}$$

The restriction of $\beta_2$ to $\beta_1(H_1)^{\perp}$ is one-to-one and onto, so by the Open Mapping Theorem, has a bounded, linear inverse $\tilde{\beta} : H_2 \to \beta_1(H_1)^{\perp}$. Thus,

$$H = \operatorname{ran} \beta_1 \oplus \operatorname{ran} \tilde{\beta}_2. \tag{4.4}$$

Before we analyze the representation $\rho$ of $B$ on $H$ with respect to this decomposition, we first prove some properties of $\mathrm{Ext}^1$.

A key property is the functorality of $\mathrm{Ext}^1$. This is accomplished by constructing the pushout and pullback in the category of Hilbert modules. The constructions are standard and can be found in any basic text on Homological Algebra including [45] and [18]. We appeal to results of the previous section.

**Proposition 4.7.** *Pushouts and pullbacks exist in $\mathcal{H}Bmod$.*

*Proof.* Pre-additivity is easy to check. Certainly, $\{0\}$ is trivially a completely bounded Hilbert $B$-module. The spaces of of bounded, intertwiners between Hilbert modules form an abelian group with addition and multiplication satisfying distributive laws. By Proposition 4.3 and duality, it suffices to show that this category contains coproducts and kernels. The kernel of a module map $\beta : H_1 \to H_2$ is the inclusion $\beta^{-1}(\{0_{H_2}\}) \to H$. Its domain is $\beta^{-1}(\{0_{H_2}\})$, which we denote by $\ker \beta$ as is custom despite the notation of the previous section. Observe that it is a closed subspace of $H$, so it is a Hilbert space. Let $\rho_i : B \to B(H_i)$ be the representations of $B$ on $H_i$. Since $\beta$ is a module map, then for all $h$ such that $\beta(h) = 0$ and $f \in B$, $\beta(\rho_1(f)h) = \rho_2(f)\beta(h) = 0$. Thus, $\ker \beta$ is a completely bounded Hilbert $B$-module. Finally, the coproduct of two completely Hilbert $B$-modules is the Hilbert space direct sum endowed with completely bounded $B$-action given by the direct sum of the two original representations. Moreover, the inclusion maps of the direct summands are continuous $B$-module maps. $\square$

**Lemma 4.8.** *Let $H_i$ and $K_i$ be Hilbert modules over $B$ for $i = 1, 2$. Then, module maps $\alpha : K_2 \to H_2$ and $\beta : H_1 \to K_1$ induce "maps" $\alpha^* : \mathrm{Ext}^1(H_2, \cdot) \to \mathrm{Ext}^1(K_2, \cdot)$*

*and* $\beta_* : \mathrm{Ext}^1(\cdot, H_1) \to \mathrm{Ext}^1(\cdot, K_1)$.

*Proof.* Stricly speaking, $\alpha^*$ and $\beta_*$ are not maps as written. To be maps, we ought to "plug in" a Hilbert module $H$ into the "dot." On the other hand, it is possible to interpret them as functors from our Hilbert module category to the category of abelian groups. We choose the former option and fix an auxilary Hilbert module $H$ over $B$. Let $[\eta] \in \mathrm{Ext}^1(H_2, H)$. This equivalence class has as a representative, the short exact sequence

$$\eta : 0 \longrightarrow H \xrightarrow{\gamma_1} J \xrightarrow{\gamma_2} H_2 \longrightarrow 0.$$

Let $(\tilde{J}, \tilde{\gamma_2}, \psi)$ be the pullback of

$$K_2 \xrightarrow{\alpha} H_2 \xleftarrow{\gamma_2} J$$

and let $\tilde{\gamma_1} : H \to \tilde{J}$ be defined by $\tilde{\gamma_1}(h) = (\gamma_1(h), 0)$. We then have the commuting diagram

$$\begin{array}{ccccccccc}
\eta\alpha : 0 & \longrightarrow & H & \xrightarrow{\tilde{\gamma_1}} & \tilde{J} & \xrightarrow{\tilde{\gamma_2}} & K_2 & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \alpha} & & \\
\eta : 0 & \longrightarrow & H & \xrightarrow{\gamma_1} & J & \xrightarrow{\gamma_2} & H_2 & \longrightarrow & 0.
\end{array}$$

Furthermore, if $\eta_1 \approx \eta_2$, then a standard algebraic argument shows $\eta_1\alpha \approx \eta_2\alpha$. Thus, $\alpha$ induces a map $\alpha^* : \mathrm{Ext}^1(H_2, H) \to \mathrm{Ext}^1(K_2, H)$.

Given an extension $\eta : 0 \to H_1 \to J \to H \to 0$, we may define an extension $\beta\eta$ so that we have the diagram

$$\begin{array}{ccccccccc}
\eta : 0 & \longrightarrow & H_1 & \xrightarrow{\gamma_1} & J & \xrightarrow{\gamma_2} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \psi} & & \| & & \\
\beta\eta : 0 & \longrightarrow & K_1 & \xrightarrow{\tilde{\gamma_1}} & \tilde{J} & \xrightarrow{\tilde{\gamma_2}} & H & \longrightarrow & 0.
\end{array}$$

Dually to the previous construction of $\eta\alpha$ above, here $\tilde{J}$ is the pushout of the upper left corner and $\tilde{\gamma}_2 : (K_1 \oplus J)/\overline{W} \to H$ is given by $\tilde{\gamma}_2((k,j) + \overline{W}) = \gamma_2(j)$. Also, we have $\eta_1 \approx \eta_2$ implies $\beta\eta_1 \approx \beta\eta_2$, so $\beta$ induces $\beta_* : \mathrm{Ext}^1(H, H_1) \to \mathrm{Ext}^1(H, K_1)$. $\square$

**Proposition 4.9.** *1. Given $\alpha_i : H_i \to H_{i+1}$ for $i = 1, 2$, it follows $(\alpha_2\alpha_1)^* = \alpha_1^*\alpha_2^*$.*

*2. Given $\beta_i : H_i \to H_{i+1}$ for $i = 2, 1$, it follows $(\beta_2\beta_1)_* = \beta_{2*}\beta_{1*}$.*

*3. Given $\alpha : K_1 \to H_1$ and $\beta : H_2 \to K_2$, we have*

$$\alpha^*\beta_* = \beta_*\alpha^* : \mathrm{Ext}^1(H_1, H_2) \to \mathrm{Ext}^1(K_1, K_2).$$

*Proof.* The proofs of these facts are purely algebraic; straightforward, yet tedious; and standard facts of homological algebra. Thus, we omit them and, instead, refer the reader to the texts cited at the beginning of this section. It should be noted that the third item is guilty of notational abuse. For example, the $\alpha^*$ on the left side of the equation maps $\mathrm{Ext}^1(H_1, K_2) \to \mathrm{Ext}^1(K_1, K_2)$ while the $\alpha^*$ on the right side maps $\mathrm{Ext}^1(H_1, H_2) \to \mathrm{Ext}^1(K_1, H_2)$. Context, specifically the order of composition, fortunately makes the meaning clear. $\square$

**Theorem 4.10.** *Suppose $H_i$ are Hilbert $B$-modules for $i = 1, 2$. Then there exist completely contractive Hilbert $B$-modules $K_i \cong H_i$ with $\mathrm{Ext}^1(H_2, H_1) \cong \mathrm{Ext}^1(K_2, K_1)$.*

*Proof.* We may apply Paulsen's similarity theorem 2.12 and the functorality of $\mathrm{Ext}^1$ to establish a group isomorphism between $\mathrm{Ext}^1(H_2, H_1)$ with $\mathrm{Ext}^1(K_2, K_1)$ where the $K_i$ are completely contractive Hilbert $B$-modules. Let $\rho_i : B \to B(H_i)$ be the completey bounded homomorphisms defining the $B$-module structures on $H_i$. Paulsen's result, then, gives Hilbert spaces $K_i$ and invertible, bounded operators $R_i : H_i \to K_i$ so that $R_i\rho(\cdot)R_i^{-1} : B \to B(K_i)$ are completely contractive representations. In Paulsen's proof, $K_i$ are the same vector spaces as $H_i$ but have been given new, equivalent Hilbert space norms. Importantly, $R_i$ are the identity functions and, therefore, trivially $B$-module maps. Therefore, we have by lemma 4.8,

$$R_{1*} : \mathrm{Ext}^1(\cdot, H_1) \to \mathrm{Ext}^1(\cdot, K_1), \qquad \left(R_1^{-1}\right)_* : \mathrm{Ext}^1(\cdot, K_1) \to \mathrm{Ext}^1(\cdot, H_1),$$

$$R_2^* : \mathrm{Ext}^1(K_2, \cdot) \to \mathrm{Ext}^1(H_2, \cdot), \quad \text{and } \left(R_2^{-1}\right)^* : \mathrm{Ext}^1(H_2, \cdot) \to \mathrm{Ext}^1(K_2, \cdot).$$

Using proposition 4.9, the following equations show that the composition $\left(R_2^{-1}\right)^* R_{1*} : \mathrm{Ext}^1(H_2, H_1) \to \mathrm{Ext}^1(K_2, K_1)$ is an isomorphism:

$$\begin{aligned} \left(\left(R_2^{-1}\right)^* R_{1*}\right)\left(R_2^*\left(R_1^{-1}\right)_*\right) &= \left(R_2^{-1}\right)^* R_2^* R_{1*}\left(R_1^{-1}\right)_* \\ &= \left(R_2 R_2^{-1}\right)^*\left(R_1 R_1^{-1}\right)_* \\ &= \left(id_{K_2}\right)^*\left(id_{K_1}\right)_* \\ &= id_{\mathrm{Ext}^1(K_2,K_1)} \end{aligned}$$

and

$$\begin{aligned} \left(R_2^*\left(R_1^{-1}\right)_*\right)\left(\left(R_2^{-1}\right)^* R_{1*}\right) &= R_2^*\left(R_2^{-1}\right)^*\left(R_1^{-1}\right)_* R_{1*} \\ &= \left(R_2^{-1} R_2\right)^*\left(R_1^{-1} R_1\right)_* \\ &= \left(id_{H_2}\right)^*\left(id_{H_1}\right)_* \\ &= id_{\mathrm{Ext}^1(H_2,H_1)}. \end{aligned}$$

$\square$

The above result allows us, in the context of cohomology, to assume the completely bounded Hilbert modules on the ends of a short exact sequence are completely contractive. An additional application of Paulen's similarity theorem allows us to make the same assumption about the middle term as well.

Let $\xi$ be an extension

$$\xi : 0 \longrightarrow H_1 \xrightarrow{\beta_1} H \xrightarrow{\beta_2} H_2 \longrightarrow 0\cdot$$

As $H$ is a completely bounded Hilbert $B$-module, there is an invertible module map $R : H \to K$ where $K$ is a completely contractive Hilbert $B$-module. Thus, we have the commutative diagram

$$\begin{array}{ccccccccc} \xi : 0 & \longrightarrow & H_1 & \xrightarrow{\beta_1} & H & \xrightarrow{\beta_2} & H_2 & \longrightarrow & 0 \\ & & \Vert & & \downarrow{\scriptstyle R} & & \Vert & & \\ \eta : 0 & \longrightarrow & H_1 & \xrightarrow{R\beta_1} & K & \xrightarrow{\beta_2 R^{-1}} & H_2 & \longrightarrow & 0 \end{array}$$

giving $[\xi] = [\eta]$. In light of this fact, we make the following definition.

**Definition 4.11.** We say that a short exact sequence $0 \to H_1 \to H \to H_2 \to 0$ is a *completely contractive* extension (of $H_1$ by $H_2$) if the middle Hilbert module $H$ is completely contractive.

The ability to replace completely bounded modules with completely contractive ones is especially advantageous in the situation where $B$ has a unital subalgebra $A$ that happens to be a $C^*$-algebra. In this case, restricting a (unital) completely contractive representation $\rho : B \to B(K)$ to $A$ yields a $C^*$-representation $(\rho|_A) : A \to B(K)$, cf. [36]. Below, we specialize our operator algebras to be tensor algebras which have unital $C^*$-subalgebras. As we will see, the fact that representations comprising an extension restrict to $C^*$-representations translates to the $A$-linearity of the associated derivations.

## 4.4 Derivations

We now study the matrix form of the middle representation in the short exact sequence

$$0 \longrightarrow H_1 \xrightarrow{\beta_1} H \xrightarrow{\beta_2} H_2 \longrightarrow 0$$

Recall from (4.4) that we may decompose $H = \operatorname{ran}\beta_1 \oplus \operatorname{ran}\tilde{\beta}_2$. This is a Hilbert space direct sum. While this sequence does split as a sequence of Hilbert (or Banach) *spaces*, we are not asserting that it splits as a sequence of Hilbert *modules* over $B$. The representations on $H_i$ are denoted by $\rho_i : B \to B(H_i)$

The representation $\rho : B \to B(H)$ has the matrix decomposition

$$\rho = \begin{bmatrix} \beta_1\rho_1(\cdot)\tilde{\beta}_1 & \tilde{\Delta}(\cdot) \\ 0 & \tilde{\beta}_2\rho_2(\cdot)\beta_2 \end{bmatrix} : B \to B(H). \tag{4.5}$$

It is customary to identify $\operatorname{ran}\beta_1$ with $H_1$ and $\operatorname{ran}\tilde{\beta}_2$ with $H_2$ so that the the representation on $H_1 \oplus H_2$ becomes

$$\begin{bmatrix} \rho_1(\cdot) & \Delta(\cdot) \\ 0 & \rho_2(\cdot) \end{bmatrix} : B \to B(H_1 \oplus H_2) \tag{4.6}$$

where $\Delta(\cdot) = \tilde{\beta}_1\tilde{\Delta}(\cdot)\beta_2$. This representation has the advantage of its short exact sequence being the standard $0 \to H_1 \to H_1 \oplus H_2 \to H_2 \to 0$ involving the obvious inclusion and quotient maps. Furthermore, the connection between elements of $\operatorname{Ext}^1(H_2, H_1)$ and derivations becomes more clear. However, one drawback is that

(4.6) might not share nice properties that (4.5) may possess such as being completely contractive, forcing us to do some similarity gymnastics reminiscient of Theorem 3.12.

We now derive properties of the $\Delta$ appearing in the corner of $\rho$. The following definition is satisfied by $\Delta$ if and only if

$$\rho = \begin{bmatrix} \rho_1 & \Delta \\ 0 & \rho_2 \end{bmatrix} : B \to B(H_1 \oplus H_2)$$

is a completely bounded representation.

**Definition 4.12.** $\Delta : B \to B(H_2, H_1)$ is a *completely bounded* $(\rho_1, \rho_2)$*-derivation* if, in addition to being a completely bounded, linear map, it satisfies the product formula

$$\Delta(ab) = \rho_1(a)\Delta(b) + \Delta(a)\rho_2(b) \quad \forall a, b \in B.$$

Usually, $\rho_1$ and $\rho_2$ will be clear from context, so we can call $\Delta$ a derivation, for short. The phrase "completely bounded" will be omitted as well, for brevity. It is well-known that linearity implies $\Delta(\mathbb{C}) = 0$ since

$$\Delta(\lambda) = \lambda\Delta(1) = \lambda\Delta(1^2) = \lambda(\rho_1(1)\Delta(1) + \Delta(1)\rho_2(1)) = 2\lambda\Delta(1).$$

We are interested in derivations that are also linear over a $C^*$-subalgebra $A$ of $B$. We note that $B$ is an operator algebra and not assumed to be a $C^*$-algebra, in case the term $C^*$-subalgebra is misleading.

**Proposition 4.13.** *Let*

$$0 \longrightarrow H_1 \xrightarrow{\beta_1} H_1 \oplus H_2 \xrightarrow{\beta_2} H_2 \longrightarrow 0$$

*be an extension of c.c. Hilbert $B$-modules and suppose $A \subseteq B$ is a unital $C^*$-subalgebra. Denoting the representations of $B$ on $H_i$ by $\rho_i : B \to B(H_i)$ and their restrictions to $A$ by $\sigma_i : A \to B(H_i)$, the representation $\rho : B \to B(H_1 \oplus H_2)$ is given by*

$$\rho = \begin{bmatrix} \rho_1(\cdot) & \Delta(\cdot) \\ 0 & \rho_2(\cdot) \end{bmatrix} : B \to B(H_1 \oplus H_2),$$

*where $\Delta : B \to B(H_2, H_1)$ is a derivation. Furthermore, the following are equivalent:*

1. *$\rho|_A$ is a $C^*$-representation.*

2. *$\Delta(A) = 0$.*

3. *$\Delta$ is $A$-linear.*

*Proof.* Given $f, g \in B$, $\rho(fg) = \rho(f)\rho(g)$ since $\rho$ is a homomorphism. Matrix multiplication yields

$$\begin{aligned} \begin{bmatrix} \rho_1(fg) & \Delta(fg) \\ 0 & \rho_2(fg) \end{bmatrix} &= \begin{bmatrix} \rho_1(f) & \Delta(f) \\ 0 & \rho_2(f) \end{bmatrix} \begin{bmatrix} \rho_1(g) & \Delta(g) \\ 0 & \rho_2(g) \end{bmatrix} \\ &= \begin{bmatrix} \rho_1(f)\rho_1(g) & \rho_1(f)\Delta(g) + \Delta(f)\rho_2(g) \\ 0 & \rho_2(f)\rho_2(g) \end{bmatrix}. \end{aligned}$$

Therefore, $\Delta$ is a derivation. It is completely bounded because it is a corner of the completely bounded map $\rho$.

If $\rho|_A$ is a $C^*$-representation, then $\rho(a^{**}) = \rho(a^*)^*$ for every $a \in A$. Thus,

$$\begin{aligned}
\begin{bmatrix} \sigma_1(a) & \Delta(a) \\ 0 & \sigma_2(a) \end{bmatrix} &= \rho(a) \\
&= \rho(a^{**}) \\
&= \rho(a^*)^* \\
&= \begin{bmatrix} \sigma_1(a^*) & \Delta(a^*) \\ 0 & \sigma_2(a^*) \end{bmatrix}^* \\
&= \begin{bmatrix} \sigma_1(a^*)^* & 0 \\ \Delta(a^*)^* & \sigma_2(a^*)^* \end{bmatrix},
\end{aligned}$$

which implies $\Delta(A) = 0$. Conversely, $\Delta(A) = 0$ implies

$$\begin{aligned}
\rho(a)^* &= \begin{bmatrix} \sigma_1(a) & \Delta(a) \\ 0 & \sigma_2(a) \end{bmatrix}^* \\
&= \begin{bmatrix} \sigma_1(a) & 0 \\ 0 & \sigma_2(a) \end{bmatrix}^* \\
&= \begin{bmatrix} \sigma_1(a^*) & 0 \\ 0 & \sigma_2(a^*) \end{bmatrix} \\
&= \begin{bmatrix} \sigma_1(a^*) & \Delta(a^*) \\ 0 & \sigma_2(a^*) \end{bmatrix} \\
&= \rho(a^*).
\end{aligned}$$

Therefore, 1 and 2 are equivalent.

For every $a, b \in A, f \in B$, a couple applications of the product formula give

$$\begin{aligned}\Delta(afb) &= \Delta(a)\rho_2(fb) + \rho_1(a)\Delta(fb) \\ &= \Delta(a)\rho_2(f)\rho_2(b) + \rho_1(a)\Delta(f)\rho_2(b) + \rho_1(a)\rho_1(f)\Delta(b) \\ &= \Delta(a)\rho_2(f)\sigma_2(b) + \sigma_1(a)\Delta(f)\sigma_2(b) + \sigma_1(a)\rho_1(f)\Delta(b).\end{aligned}$$

If $\Delta(A) = 0$, then only the middle term of the right side remains, establishing the $A$-linearity of $\Delta$. Conversely, assume $\Delta$ is $A$-linear. Since $\Delta(1) = 0$, then

$$\Delta(a) = \Delta(a1) = \sigma_2(a)\Delta(1) = 0.$$

Thus, we see 2 and 3 are equivalent. □

Thus, extensions correspond to derivations. We now show that the equivalence classes of extensions $\mathrm{Ext}^1(H_2, H_1)$ are given by equivalence classes of derivations.

**Definition 4.14.** Given $V \in B(H_2, H_1)$, we call $\Delta_V : B \to B(H_2, H_1)$ an *inner derivation* if it satisfies

$$\Delta_V(f) = \rho_1(f)V - V\rho_2(f) \quad \forall f \in B.$$

It is easy to see that $\Delta_V$ satisfies the product formula. Furthermore, $\Delta_V$ is automatically bounded with $\|\Delta_V\| \leq \|V\|(\|\rho_1\| + \|\rho_2\|)$.

**Proposition 4.15.** *Assume $\Delta_V$ is an inner derivation. Then $\Delta_V$ is completely bounded if $\rho_1$ and $\rho_2$ are completely bounded.*

*Proof.* $\Delta_V$ takes values in $B(H_2, H_1)$. The operator space structure on $B(H_2, H_1)$ is obtained using the canoncial shuffle map $M_n(B(H_2, H_1) = B(H_2^n, H_1^n)$ and viewing this collection of operators as a subspace of $B(H_1^n \oplus H_2^n)$. We show that $\Delta_V^{(n)}$ : $M_n(B) \to B(H_2^n, H_1^n)$ is itself an inner derivation and apply the estimate above. We write $V \otimes I_n$ to denote the matrix $diag_n(V, V, \ldots, V)$. Given $[f_{ij}] \in M_n(B)$,

$$\begin{aligned}
\Delta_V^{(n)}([f_{ij}]) &= [\Delta_V(f_{ij})] \\
&= [\rho_1(f_{ij})V - V\rho_2(f_{ij})] \\
&= [\rho_1(f_{ij})]\,(V \otimes I_n) - (V \otimes I_n)[\rho_2(f_{ij})] \\
&= \rho_1^{(n)}([f_{ij}])(V \otimes I_n) - (V \otimes I_n)\rho_2^{(n)}([f_{ij}])
\end{aligned}$$

Thus, $\Delta_V^{(n)} = \Delta_{(V \otimes I_n)}$ is an inner $(\rho_1^{(n)} - \rho_2^{(n)})$-derivation. Therefore,

$$\|\Delta_V^{(n)}\| \leq \|V \otimes I_n\|(\|\rho_1^{(n)}\| + \|\rho_2^{(n)}\| \leq \|V\|(\|\rho_1\|_{cb} + \|\rho_2\|_{cb}) \quad \forall n \geq 0$$

so $\|\Delta_V\|_{cb} < \infty$ □

**Definition 4.16.** We say derivations $\Delta_1, \Delta_2 : B \to B(H_2, H_1)$ are *equivalent* (modulo an inner derivation) if $\Delta_1 - \Delta_2$ is inner.

**Remark 4.17.** It is well-known and easy to prove that this is actually an equivalence relation. Also, observe that by Proposition 4.13, $\Delta_V$ is $A$-linear if and only if $\rho_1(a)V - V\rho_2(a) = 0$ for all $a \in A$ if and only if $V \in \mathcal{I}(\sigma_2, \sigma_1)$. It is worth noting that all $A$-linear derivations will be inequivalent if $\sigma_1$ and $\sigma_2$ are disjoint representations. Also, the collection of inner derivations $\mathrm{Inn}(H_2, H_1)$ forms a subspace of the vector space $\mathrm{Der}(H_2, H_1)$ since $\lambda_1\Delta_{V_1} + \lambda_2\Delta_{V_2} = \Delta_{\lambda_1 V_1 + \lambda_2 V_2}$.

**Theorem 4.18.**

$$\operatorname{Ext}^1(H_2, H_1) = \operatorname{Der}(H_2, H_1)/\operatorname{Inn}(H_2, H_1).$$

*Proof.* We have already seen how extensions give rise to derivations. Furthermore, we have also shown that any extension is simlar to one having the form $0 \to H_1 \oplus H_2 \to H_2 \to 0$. If two such extensions are similar, then the similarity must have the form $\left[\begin{smallmatrix} 1 & V \\ 0 & 1 \end{smallmatrix}\right]$ since $H_1$ is a submodule of $H_1 \oplus H_2$. Denote by $\rho$ and $\rho'$ these two representations on $H_1 \oplus H_2$ and by $\Delta$ and $\Delta'$ their respective derivations. Then an easy computation shows

$$\begin{bmatrix} 1 & V \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_1 & \Delta \\ 0 & \rho_2 \end{bmatrix} = \begin{bmatrix} \rho_1 & \Delta + V\rho_2 \\ 0 & \rho_2 \end{bmatrix}$$

and

$$\begin{bmatrix} \rho_1 & \Delta' \\ 0 & \rho_2 \end{bmatrix} \begin{bmatrix} 1 & -V \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \rho_1 & \rho_1 V + \Delta' \\ 0 & \rho_2 \end{bmatrix}$$

The representations on $H_1 \oplus H_2$ are similar if and only if the left side of the equations are equal if and only if the right sides are equal if and only if $\Delta - \Delta' = \rho_1 V - V\rho_2$. $\square$

## 4.5 Main Theorem

In this section we prove the main theorm of this dissertation which expresses $\operatorname{Ext}^1(H_2, H_1)$ as a quotient of a certain space of intertwining operators from $X \otimes_{\sigma_2} H_2 \to H_1$. Muhly and Solel have shown in [31] how contractive intertwiners parametrize the completely contractive representations of $\mathscr{T}_+(X)$. They advocate an interesting perspective of viewing elements of $\mathscr{T}_+(X)$ as functions on these operator discs.

Our analysis of derivations is greatly enhanced by the following result from Sz.-Nagy and Foiaş, cf. [51], [53], or for a simple matrix proof using Douglas's factorization lemma [10]. Although it is usually stated for an operator on $H_1 \oplus H_2$, the same proof would work for an operator mapping between different spaces $H_1 \oplus H_2$ to $K_1 \oplus K_2$.

**Proposition 4.19.** *An operator*

$$T = \begin{bmatrix} T_1 & D \\ 0 & T_2 \end{bmatrix} \begin{array}{c} H_1 \\ H_2 \end{array} \to \begin{array}{c} K_1 \\ K_2 \end{array}$$

*between the Hilbert spaces $H_1 \oplus H_2 \to K_1 \oplus K_2$ is a contraction if and only if $T_1 \in B(H_1, K_1)$ and $T_2 \in B(H_2, K_2)$ are contractions and $D \in B(H_2, K_1)$ has the form $\sqrt{1 - T_1 T_1^*} L \sqrt{1 - T_2^* T_2}$ where $L$ is a contraction from $\overline{\operatorname{ran}}(1 - T_2^* T_2)$ to $\overline{\operatorname{ran}}(1 - T_1 T_1^*)$. Furthermore, $L$ is unique since we have specified that it maps between the defect spaces of $T_2$ and $T_1^*$.*

For the remainder of this chapter we fix the operator algebra $B = \mathscr{T}_+(X)$ and completely bounded representations $\rho_i : \mathscr{T}_+(X) \to B(H_i)$. We study extensions $0 \to H_1 \to H_1 \oplus H_2 \to H_2 \to 0$ via the derivation $\Delta$. Properties of $\rho_1$ and $\rho_2$ are linked with $\Delta$ through a factorization of the type in Proposition 4.19. We now prove that $\operatorname{Ext}^1(H_2, H_1)$ can be expressed as a quotient of a certain subspace in $\mathcal{I}(\sigma_2^X \circ \phi, \sigma_1) \subset B(X \otimes H_2, H_1)$.

Given an element $[\xi] \in \operatorname{Ext}^1(H_2, H_1)$, Theorem 2.12 provides a representative

$$0 \longrightarrow H_1 \xrightarrow{\beta_1} H \xrightarrow{\beta_2} H_2 \longrightarrow 0$$

in which $\rho : \mathscr{T}_+(X) \to B(H)$ is unital and completely contractive. Let $\widetilde{\sigma}$ be the restriction of $\rho$ to $A$, meaning $\rho \circ \phi_\infty = \widetilde{\sigma}$. Using the decomposition $H = \operatorname{ran} \beta_1 \oplus \operatorname{ran} \tilde{\beta}_2$ with $\tilde{\beta}_2 = (\beta_2 |(\ker \beta_2)^\perp)^{-1}$, then $\rho$ is represented as a matrix

$$\begin{bmatrix} \widetilde{\rho}_1 & \widetilde{\Delta} \\ 0 & \widetilde{\rho}_2 \end{bmatrix}$$

where $\widetilde{\Delta}$ is a completely bounded derivation that is $A$-linear because $\widetilde{\sigma}$ is a $C^*$-representation. Note that this says our sequence splits as a sequence $A$-modules. We also have $\widetilde{\rho}_i$, the compressions of $\rho$,

$$\beta_1 \rho_1(\cdot) \tilde{\beta}_1 : \mathscr{T}_+(X) \to B(\operatorname{ran} \beta_1)$$

and

$$\tilde{\beta}_2 \rho_2(\cdot) \beta_2 : \mathscr{T}_+(X) \to B(\operatorname{ran} \tilde{\beta}_2),$$

are completely contractive. Since they are also unital, their restrictions $\widetilde{\sigma}_i$ to $A$ are $C^*$-representations. Therefore, they correspond to intertwiners $\widetilde{T}_i \in \overline{\mathbb{D}(X^{\widetilde{\sigma}_i})}$, with

$$\widetilde{T}_1(x \otimes \beta_1 h_1) = \beta_1 \rho_1(T_x) \tilde{\beta}_1 \beta_1 h_1 = \beta_1 \rho_1(T_x) h_1 \tag{4.7}$$

and

$$\widetilde{T}_2(x \otimes \tilde{\beta}_2 h_2) = \tilde{\beta}_2 \rho_2(T_x) \beta_2 \tilde{\beta}_2 h_2 = \tilde{\beta}_2 \rho_2(T_x) h_2. \tag{4.8}$$

The representation $\rho$ is given by a point

$$\begin{bmatrix} \widetilde{T}_1 & \widetilde{D} \\ 0 & \widetilde{T}_2 \end{bmatrix} \in \overline{\mathbb{D}\left(X^{\widetilde{\sigma}}\right)}.$$

Note that

$$\widetilde{T}_1 \in \mathcal{I}(\widetilde{\sigma}_1^X \circ \phi, \widetilde{\sigma}_1), \quad \widetilde{T}_2 \in \mathcal{I}(\widetilde{\sigma}_2^X \circ \phi, \widetilde{\sigma}_2), \text{ and } \widetilde{D} \in \mathcal{I}(\widetilde{\sigma}_2^X \circ \phi, \widetilde{\sigma}_1).$$

Since the matrix above is contractive, Proposition 4.19 applies to give us a contraction

$$\widetilde{L} \in B\left(\overline{\operatorname{ran}}\sqrt{1-\widetilde{T}_2^*\widetilde{T}_2}, \overline{\operatorname{ran}}\sqrt{1-\widetilde{T}_1\widetilde{T}_1^*}\right)$$

also intertwining $\widetilde{\sigma}_2^X \circ \phi$ and $\widetilde{\sigma}_1$ and such that

$$\widetilde{D} = \sqrt{1-\widetilde{T}_1\widetilde{T}_1^*}\widetilde{L}\sqrt{1-\widetilde{T}_2^*\widetilde{T}_2}.$$

We can transport everything from $H = \operatorname{ran}\beta_1 \oplus \operatorname{ran}\tilde{\beta}_2$ to $H_1 \oplus H_2$ using the similarity

$$(\tilde{\beta}_1 \oplus \beta_2) : \operatorname{ran}\beta_1 \oplus \operatorname{ran}\tilde{\beta}_2 = H \to H_1 \oplus H_2.$$

$$\begin{bmatrix} \tilde{\beta}_1 & 0 \\ 0 & \beta_2 \end{bmatrix} \rho \begin{bmatrix} \beta_1 & 0 \\ 0 & \tilde{\beta}_2 \end{bmatrix} = \begin{bmatrix} \rho_1 & \Delta \\ 0 & \rho_2 \end{bmatrix}$$

where $\Delta$ is a derivation. It follows that by defining

$$T_1 = \tilde{\beta}_1\widetilde{T}_1(I_X \otimes \beta_1), \quad T_2 = \beta_2\widetilde{T}_2(I_X \otimes \tilde{\beta}_2), \text{ and } D = \tilde{\beta}_1\widetilde{D}(I_X \otimes \tilde{\beta}_2),$$

and using (4.7) and (4.8), we recover

$$T_1(x \otimes h_1) = \tilde{\beta}_1\widetilde{T}_1(I_X \otimes \beta_1)(x \otimes h_1) = \tilde{\beta}_1\widetilde{T}_1(x \otimes \beta_1) = \tilde{\beta}_1\beta_1\rho_1(T_x)h_1 = \rho_1(T_x)h_1$$

and, similarly,

$$T_2(x \otimes h_2) = \rho_2(T_x)h_2.$$

From the definition of $\widetilde{D}$ in terms of $\rho$ and mutiplying out the matrix equation above, we see that

$$\begin{aligned} D(x \otimes h_2) &= \tilde{\beta}_1 \widetilde{D}(I_X \otimes \tilde{\beta}_2)(x \otimes h_2) \\ &= \tilde{\beta}_1 \widetilde{D}(x \otimes \tilde{\beta}_2 h_2) \\ &= \tilde{\beta}_1 \rho(T_x) \tilde{\beta}_2 h_2 \\ &= \Delta(T_x) h_2. \end{aligned}$$

Finally, observe that

$$\|D\| \leq \left\|\tilde{\beta}_1\right\| \left\|\tilde{\beta}_2\right\| \left\|\sqrt{1 - \tilde{T}_1 \tilde{T}_1^*}\right\| \left\|\sqrt{1 - \tilde{T}_2^* \tilde{T}_2}\right\| \leq \left\|\tilde{\beta}_1\right\| \left\|\tilde{\beta}_2\right\|. \tag{4.9}$$

Let $\sigma_1 = \tilde{\beta}_1 \widetilde{\sigma}_1 \beta_1$ and $\sigma_2 = \beta_2 \widetilde{\sigma}_2 \tilde{\beta}_2$. These are completely bounded representations of $A$ and are the restrictions of $\rho_1$ and $\rho_2$, respectively. We have expressed the completely bounded derivation $\Delta$ in terms of an operator

$$D = \tilde{\beta}_1 \widetilde{D}(I_X \otimes \tilde{\beta}_2) \in \tilde{\beta}_1 \mathcal{I}(\widetilde{\sigma}_2^X \circ \phi, \widetilde{\sigma}_1) \tilde{\beta}_2 = \mathcal{I}(\sigma_2^X \circ \phi, \sigma_1)$$

that has defect operators of $\widetilde{T}_2$ and $\widetilde{T}_1^*$ as factors.

It remains to to characterize the inner derivations $\Delta_V$. However, this is simple as $\Delta$ is inner if and only if there exists $V \in B(H_2, H_1)$ such that, for every $x \in X, h_2 \in H_2$,

$$D(x \otimes h_2) = \Delta(T_x) h_2 = \rho_1(T_x) V h_2 - V \rho_2(T_x) h_2 = (T_1(I_X \otimes V) - V T_2)(x \otimes h_2).$$

Let $\mathfrak{D}$ denote the collection of operators $D : X \otimes_{\sigma_2} H_2 \to H_1$ satisfying

$$D = \beta_1^{-1} \sqrt{1 - \widetilde{T}_1 \widetilde{T}_1^*} \widetilde{L} \sqrt{1 - \widetilde{T}_2^* \widetilde{T}_2} (I_X \otimes \beta_2^{-1})$$

where $\beta_1 : H_1 \to K_1$ and $\beta_2 : K_2 \to H_2$ are invertible operators such that, $\widetilde{T}_1 = \beta_1 T_1(I_X \otimes \beta_1^{-1})$ and $\widetilde{T}_2 = \beta_2^{-1} T_2(I_X \otimes \beta_2)$ are contractive intertwiners $X \otimes_{\widetilde{\sigma}_i} K_i \to K_i$, and $\widetilde{L} : \overline{\mathrm{ran}}\sqrt{1 - \widetilde{T}_2^* \widetilde{T}_2} \to \overline{\mathrm{ran}}\sqrt{1 - \widetilde{T}_1 \widetilde{T}_1^*}$ is a contraction intertwining $(\beta_2^{-1}\sigma_2\beta_2)^X \circ \phi$ and $\beta_1\sigma_1\beta_1^{-1}$. We have written $\beta_i^{-1}$ rather than $\tilde{\beta}_i$ because previous issues cease to be relevant; namely, such notation in the preceding proof would have been misleading as $\mathrm{ran}\,\beta_1$ and $\mathrm{ran}\,\tilde{\beta}_2$ were subspaces of $H$.

Let $\mathfrak{I}$ denote the collection of operators $D_V$ in $\mathfrak{D}$ with

$$D_V = T_1(I_X \otimes V) - VT_2$$

where $V \in \mathcal{I}(\sigma_2^X \circ \phi, \sigma_1)$. The converse basically amounts to following the constructions in reverse. Being straightforward, we provide a brief sketch. Given an element of $\mathfrak{D}$, it is possible to implement with $\beta_i$ a similarity that yields a completely contractive representation on $K_1 \oplus K_2$. This representation will be determined by $\begin{bmatrix} \widetilde{T}_1 & \beta_1 D(I_X \otimes \beta_2) \\ 0 & \widetilde{T}_2 \end{bmatrix}$ which shows that the the upper right corner determines a completely bounded derivation $\widetilde{\Delta} : \mathscr{T}_+(X) \to B(K_2, K_1)$. Letting $\Delta = \beta_1^{-1}\widetilde{\Delta}\beta_2^{-1}$, we get a completely bounded derivation with values in $B(H_2, H_1)$. Moreover, it is easy to check $\Delta(T_x)h_2 = D(x \otimes h_2)$. That the equivalence with respect to inner derivations really amounts to equivalence with respect to $\mathfrak{I}$ is clear. Note that we were done proving the converse as soon as we had $\widetilde{\Delta}$ because it corresponds to an extension class. However, it seemed preferrable and natural to return to the original space $H_1 \oplus H_2$.

We have proved the following analogue (for completely bounded extensions) of Proposition 2.2.3 in [4] or Proposition 1.1 in [13]:

**Theorem 4.20.**

$$\mathrm{Ext}^1(H_2, H_1) = \mathfrak{D}/\mathfrak{I}$$

**Corollary 4.21.** *There is a bijective correspondence between completely contractive representations on* $H_1 \oplus H_2$, $\rho = \left[\begin{smallmatrix} \rho_1 & \Delta \\ 0 & \rho_2 \end{smallmatrix}\right]$ *and operators* $D : X \otimes_{\sigma_2} H_2 \to H_1$ *in* $\mathcal{I}(\sigma_2^X \circ \phi, \sigma_1)$ *satisfying*

$$D = \sqrt{1 - T_1 T_1^*} L \sqrt{1 - T_2^* T_2}. \tag{4.10}$$

*where* $L$ *is a contraction*

$$L : \overline{\mathrm{ran}} \sqrt{1 - T_2^* T_2} \to \overline{\mathrm{ran}} \sqrt{1 - T_1 T_1^*} \tag{4.11}$$

*intertwining* $\sigma_2^X \circ \phi$ *and* $\sigma_1$*. Moreover, the operators* $D$ *and* $L$ *correspond bijectively as well.*

*Proof.* Naturally, the above $T_i$ are the operators appearing in the matrix

$$\begin{bmatrix} T_1 & D \\ 0 & T_2 \end{bmatrix} : \begin{matrix} X \otimes_{\sigma_1} H_1 \\ X \otimes_{\sigma_2} H_2 \end{matrix} \to \begin{matrix} H_1 \\ H_2 \end{matrix}$$

which satisfies $\left[\begin{smallmatrix} T_1 & D \\ 0 & T_2 \end{smallmatrix}\right] (x \otimes (h_1 \oplus h_2)) = \rho(T_x)(h_1 \oplus h_2)$. This corollary is actually proven and used in Theorem 4.20 but with $H_1 = \mathrm{ran}\, \beta_1$ and $H_2 = \mathrm{ran}\, \tilde{\beta}_2$. If $\rho$ a completely contractive representation on $H_1 \oplus H_2$, then we have the best of both worlds and no similarity is needed. Results of Muhly and Solel pertaining to completely contractive representations of $\mathscr{T}_+(X)$ and Sz.-Nagy and Foiaş's result immediately give us the corollarly. $\square$

**Remark 4.22.** We highlight the above corollarly since its form is much more simplified than the construction above. Furthermore, it raises an interesting question. Which elements of $\mathrm{Ext}^1(H_2, H_1)$ can be represented by completely contractive $\left[\begin{smallmatrix} \rho_1 & \Delta \\ 0 & \rho_2 \end{smallmatrix}\right]$? To be clear, every extension class is equivalent to a completely contractive extension, but it need not be on $H_1 \oplus H_2$. Identifications of $H_1 = \operatorname{ran} \beta_1$ and $H_2 = \operatorname{ran} \tilde{\beta}_2$ alter the norms involved and may not preserve complete contractivity. In light of these considerations, we make the following definition.

**Definition 4.23.** We define $(\mathrm{Ext}^1(H_2, H_1))_1 \subset \mathrm{Ext}^1(H_2, H_1)$ to be the collection extension classes that may be represented by a completely contractive extension

$$0 \to H_1 \to H_1 \oplus H_2 \to H_2 \to 0,$$

where, as usual, the direct sum is one of Hilbert spaces.

Clearly, the trivial extension is contained in $(\mathrm{Ext}^1(H_2, H_1))_1$ if we start out assuming $H_i$ are completely contractive Hilbert modules. Furthermore, since contractivity and intertwining relations are preserve by convex combinations, $(\mathrm{Ext}^1(H_2, H_1)_1$ is convex. The advantage of looking at $(\mathrm{Ext}^1(H_2, H_1))_1$ is that, by Corollary 4.21, its members *and their differences* have simple factorizations. Indeed, an effective analysis can be done by just considering the factors $L$ in (4.10).

While we have not yet studied the convex structure of $(\mathrm{Ext}^1(H_2, H_1))_1$ in detail, we do have the following result:

**Proposition 4.24.** *Let $D, D_1$, and $D_2$ and their respectively associated $L, L_1$, and $L_2$ be given as in Corollary 4.21. $D$ is an extreme point of* $(\mathrm{Ext}^1(H_2, H_1)_1$ *if and only* *if $L$ is an isometry or a coisometry.*

*Proof.* Observe that for $D = \lambda D_1 + (1-\lambda)D_2$ if and only if $L = \lambda L_1 + (1-\lambda)L_2$. Here the uniqueness of $L, L_1$ and $L_2$ is crucial. Furthermore, $L = 0$ if and only if $D = 0$. Again, using the uniqueness of the $L_i$ we see that $D$ is an extreme point in $(\mathrm{Ext}^1(H_2, H_1)_1$ if and only if $L$ is an extreme point in the operator ball $\left(B\left(\overline{\mathrm{ran}}\sqrt{1-T_2^*T_2}, \overline{\mathrm{ran}}\sqrt{1-T_1T_1^*}\right)\right)_1$. From [47] we know that those are precisely the isometries and coisometries. $\square$

## 4.6 Applications

We now apply our results to a number of cases that, albeit special, are quite common in the literature. The following results generalize theorems in [4], [5], [12] and [41]. We point out the particular connection with [5] in which the notion of a *cramped Hilbert module* over $\mathcal{A}(\mathbb{D})$ is defined as a Hilbert module where multiplication by $z$ is similar to a contraction. This definition corresponds to the completely bounded Hilbert modules over $\mathcal{A}(\mathbb{D})$. However, some of the above references also contain results on *bounded* extensions, not just completely bounded ones. While it is true that contractive modules of $\mathscr{T}_+(X)$ are completely contractive when $X$ is strictly cyclic, it is usually not the case that bounded representations are automatically completely bounded. This does happen, however, with finite-dimensional representations, cf. [36].

**Proposition 4.25.** *Suppose $\rho_i : \mathscr{T}_+(X) \to H_i$ are completely contractive representations for $i = 1, 2$ with associated $T_i \in \overline{\mathbb{D}(X^{\sigma_i})}$, where $\sigma_i = \rho_i|_A$. If $T_1$ and $T_2$ are* *partial isometries, then* $\left(\operatorname{Ext}^1(H_2, H_1)\right)_1$ *equals*

$$\left\{L \in B\left(\ker T_2, \overline{\operatorname{ran} T_1}^{\perp}\right) \mid \|L\| \leq 1 \text{ and } L \in \mathcal{I}(\sigma_2^X \circ \phi, \sigma_1)\right\}.$$

*Proof.* By 4.21 we need only examine $D \in \mathcal{I}(\sigma_2^X \circ \phi, \sigma_1)$ which factors as

$$D = \sqrt{1 - T_1 T_1^*} L \sqrt{1 - T_2^* T_2}$$

with $L$ a contractive intertwiner between the relevant defect spaces, and study the equivalence of such operators $D$ modulo $\mathfrak{I}$.

Since the $T_i$ are partial isometries, $T_2^* T_2 = \operatorname{Proj}(\ker T_2)^{\perp}$ and $T_1 T_1^* = \operatorname{Proj}(\operatorname{ran} T_1)$. Thus, $1 - T_2^* T_2 = \operatorname{Proj}(\ker T_2)$ and $1 - T_1 T_1^* = \operatorname{Proj}(\operatorname{ran} T_1)^{\perp}$. It follows that

$$D = \sqrt{1 - T_1 T_1^*} L \sqrt{1 - T_2^* T_2} = \operatorname{Proj}(\operatorname{ran} T_1)^{\perp} L \operatorname{Proj}(\ker T_2).$$

Furthermore, $D$ intertwines $\sigma_2^X \circ \phi$ and $\sigma_1$ if and only if $L$, considered as an operator from $\ker T_2$ to $(\operatorname{ran} T_1)^{\perp}$ does the intertwining as well.

It remains to show that inner derivations correspond to $L = 0$. To this end, suppose

$$T_1(I_X \otimes V) - V T_2 = \operatorname{Proj}(\operatorname{ran} T_1)^{\perp} L \operatorname{Proj}(\ker T_2) \tag{4.12}$$

holds for some $V \in B(H_2, H_1)$. The left side must equal 0 on $(\ker T_2)^{\perp}$ since the right side has this property. On the other hand, the left side equals $T_1(I_X \otimes V)$ on $\ker T_2$, resulting in an operator mapping into $\operatorname{ran} T_1$. However, the right side has range orthogonal to $\operatorname{ran} T_1$, so the whole thing must equal 0. $\square$

**Corollary 4.26.** *Suppose* $\rho_i : \mathscr{T}_+(X) \to H_i$ *are completely bounded representations for* $i = 1, 2$ *with associated* $T_i \in \mathbb{D}(X^{\sigma_i})$, *where* $\sigma_i = \rho_i|_A$. *Then* $\operatorname{Ext}^1(H_2, H_1) = 0$ *if* $T_2$ *is an isometry or* $T_1$ *is a coisometry.*

*Proof.* Since isometries are injective and coisometries are surjective, the previous corollary says that $(\mathrm{Ext}^1(H_2, H_1))_1 = 0$. Unfortunately, we currently see no way to use that fact to prove $\mathrm{Ext}^1(H_2, H_1) = 0$. However, this does lead to the interesting question of whether it is possible for $(\mathrm{Ext}^1(H_2, H_1))_1 = 0$ without $\mathrm{Ext}(H_2, H_1)$ being trivial as well? We suspect the answer to be no.

First, assume $T_2$ is an isometry and fix a short exact sequence

$$0 \longrightarrow H_1 \longrightarrow H_1 \oplus H_2 \longrightarrow H_2 \longrightarrow 0$$

with representation $\left[\begin{smallmatrix} \rho_1 & \Delta \\ 0 & \rho_2 \end{smallmatrix}\right]$.

. With the factorization (4.10) in mind, it seems compelling to consider an equivalent completely contractive extension.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1 & \longrightarrow & H_1 \oplus H_2 & \longrightarrow & H_2 & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \theta} & & \| & & \\ 0 & \longrightarrow & H_1 & \xrightarrow{\beta_1} & H & \xrightarrow{\beta_2} & H_2 & \longrightarrow & 0 \end{array}$$

As before, $H = \operatorname{ran} \beta_1 \oplus \operatorname{ran} \tilde{\beta}_2$ as Hilbert spaces. The equivalence of extensions is equivalent to the following matrix equation:

$$\begin{bmatrix} \tilde{\beta}_1 & 0 \\ 0 & \beta_2 \end{bmatrix} \begin{bmatrix} \beta_1 T_1 (I_x \otimes \widetilde{\beta}_1) & \beta_1 D (I_X \otimes \beta_2) \\ 0 & \widetilde{\beta}_2 T_2 (I_X \otimes \beta_2) \end{bmatrix} \begin{bmatrix} \beta_1 & 0 \\ 0 & \tilde{\beta}_2 \end{bmatrix} = \begin{bmatrix} T_1 & D \\ 0 & T_2 \end{bmatrix}$$

For convenience, denote the middle matrix on the left side of the equation by $\left[\begin{smallmatrix} \widetilde{T}_1 & \widetilde{D} \\ 0 & \widetilde{T}_2 \end{smallmatrix}\right]$. The operator on the right side involves $T_2$ which is the isometry. However, since the matrix is not assumed to be contractive, the triangular factorization trick does not apply. It does apply to $\left[\begin{smallmatrix} \widetilde{T}_1 & \widetilde{D} \\ 0 & \widetilde{T}_2 \end{smallmatrix}\right]$, but $\widetilde{T}_2$ need not be an isometry.

Instead, we adapt an argument from [5] that worked for $\mathcal{A}(\mathbb{D})$ and works here as well. We have the equation

$$\widetilde{T}_2(I_X \otimes \tilde{\beta}_2) = \tilde{\beta}_2 T_2,$$

which is the same as saying $\tilde{\beta}_2$ is in the commutant of $\tilde{\beta}_2\rho_2(\mathscr{T}_+(X))\beta_2$. Let $U_+$ be the minimal isometric dilation of $\widetilde{T}_2$ on the dilation space

$$K = \tilde{\beta}_2 H_2 \oplus \mathcal{D} \oplus (X \otimes \mathcal{D}) \oplus (X^{\otimes 2} \otimes \mathcal{D}) \oplus \cdots .$$

Here we write $\mathcal{D}$ for $\overline{\operatorname{ran}}\sqrt{1 - \widetilde{T}_2^* \widetilde{T}_2}$. Using a slight modification of the commutant lifting theorem in [31] for tensor algebras, we obtain a lifting $Y$ of $\tilde{\beta}_2$ satisfying $YU_+ = T_2(I_X \otimes Y)$, $\|\beta_2\| = \|Y\|$, cf. [52]. Set $M = Y\beta_2 : \tilde{\beta}_2 H_2 \to K$. $M$ is given by

$$Mh_2 = (h_2, W_0 h_2, W_1 h_2, \cdots)$$

where $W_n : \tilde{\beta}_2 H_2 \to X^{\otimes n} \otimes \mathcal{D}$ and satisfies

$$U_+(I_X \otimes M) = M\widetilde{T}_2.$$

Recall the generalized powers of $\widetilde{T}_1^n : X^{\otimes n} \otimes \beta_1 H_1 \to \beta_1 H_1$ given by

$$\widetilde{T}_1^n(x_1 \otimes \cdots \otimes x_n \otimes \beta_1 h_1) = \widetilde{\rho}_1(T_{x_1} T_{x_2} \cdots T_{x_n})\beta_1 h_1.$$

Thus, we make the definition

$$B\tilde{\beta}_2 h_2 = \sum_{n=0}^{\infty} \widetilde{T}_1^n (I_{X^{\otimes n}} \otimes (1 - \widetilde{T}_1 \widetilde{T}_1^*)^{1/2} \widetilde{L}) W_n \tilde{\beta}_2 h_2.$$

This sum converges in the WOT to an operator in $B(\tilde{\beta}_2 H_2, \beta_1 H_1)$ and

$$B\widetilde{T}_2 - \widetilde{T}_1(I_X \otimes B) = (1 - \widetilde{T}_1 \widetilde{T}_1^*)^{1/2} L (1 - \widetilde{T}_2^* \widetilde{T}_2^*)^{1/2}.$$

We omit the details as they are quite similar to [5]. The equation above results from a telescoping sum and the fact that $W_0 : \tilde{\beta}_2 H_2 \to \mathcal{D}$ satisfies $W_0 \widetilde{T}_2 = \sqrt{1 - \widetilde{T}_2^* \widetilde{T}_2}$. Therefore, $\mathrm{Ext}^1(H_2, H_1) = 0$. The proof for when $T_1$ is coisometry is similar. ☐

The previous corollary, generalizing results in [5] on projective and injective Hilbert modules over the disc algebra $A(\mathbb{D})$, shows that isometric modules are projective and that coisometric modules are injective (in the homological sense.)

Standing in stark contrast to corollary 4.26 is the following result in which noninner derivations do exist.

**Proposition 4.27.** *Suppose there exist unit vectors $w \in \ker T_1^*$ and $v \in \ker T_2$ so that the rank one operator $w \otimes \overline{v} \in B(X \otimes_{\sigma_2} H_2, H_1)$ intertwines $\sigma_2^X \circ \phi$ and $\sigma_1$. Then,* $\mathrm{Ext}^1(H_2, H_1) \neq 0$.

*Proof.* Of course, if $A = \mathbb{C}$, then the intertwining condition is trivially satisfied due to linearity. In that case, the hypothesis can be more simply stated as $T_1^*$ and $T_2$ have nontrivial kernels. Observe that $(1 - T_1 T_1^*)w = w - T_1 T_1^* w = w - 0 = w$. Consequently, $\sqrt{1 - T_1 T_1^*} w = w$ as well. To see this, first assume $1 - T_1 T_1^*$ is invertible. Then, we may express $\sqrt{1 - T_1 T_1^*} = \sum_{n=0}^{\infty} \binom{1/2}{n} (T_1 T_1^*)^n$. All terms, except the constant term which is the identity, map $w$ to 0. If $1 - T_1 T_1^*$ is not invertible, then we may add $\epsilon$, apply the previous argument, and let $\epsilon \to 0$. Thus, $\sqrt{1 - T_1 T_1^*} w = w$. Similarly, $\sqrt{1 - T_2^* T_2} v = v$. Therefore,

$$\sqrt{1 - T_1 T_1^*}(w \otimes \overline{v}) \sqrt{1 - T_2^* T_2} = \left( \sqrt{1 - T_1 T_1^*} w \right) \otimes \overline{\sqrt{1 - T_2^* T_2} v} = w \otimes \overline{v}.$$

Note that $\|w \otimes \overline{v}\| = 1$ because $\|v\| = \|w\| = 1$. Therefore, the Sz.-Nagy–Foiaş result says the operator matrix

$$\begin{bmatrix} T_1 & w \otimes \overline{v} \\ 0 & T_2 \end{bmatrix}$$

is contractive. Furthermore, it intertwines $(\sigma_1 \oplus \sigma_2)^X \circ \phi = (\sigma_1^X \circ \phi) \oplus (\sigma_2^X \circ \phi)$ and $\sigma_1 \oplus \sigma_2$. It follows that this matrix corresponds to a completely contractive representation $\mathscr{T}_+(X) \to B(H_1 \oplus H_2)$. Therefore, we have a completely bounded $A$-derivation $\Delta : \mathscr{T}_+(X) \to B(H_2, H_1)$ such that $\Delta(T_\xi)k = (w \otimes \overline{v})(\xi \otimes k)$.

We now show, by contradiction, that $\Delta$ is non-inner. Suppose there exists $L \in B(H_2, H_1)$ with $\Delta(f) = \pi_1(f)L - L\pi_2(f)$ for every $f \in \mathscr{T}_+(X)$. $\Delta(\phi_\infty(A)) = 0$ implies $L \in \mathcal{I}(\sigma_2, \sigma_1)$ and we also have

$$w \otimes \overline{v} = T_1(I_X \otimes L) - LT_2.$$

Observe that

$$(w \otimes \overline{v})v = \langle v, v\rangle w = \|v\|w = w.$$

On the other hand, $v \in \ker T_2$ implies

$$(T_1(I_X \otimes L) - LT_2)v = T_1(I_X \otimes L)v.$$

This would imply that $w \in \operatorname{ran} T_1$ which contradicts $w \in \ker T_1^*$. Therefore, $\Delta$ is non-inner and $\operatorname{Ext}^1(H_2, H_1) \neq 0$. □

**Remark 4.28.** Popescu calculates the first cohomology group $H^1(\mathcal{A}_d, \mathbb{C})$ for the noncommutative disc algebras $\mathcal{A}_d$ in [41]. Recall, $\mathscr{T}_+(X) \cong \mathcal{A}_d$ when we take $A = \mathbb{C}$

and $X = \mathbb{C}^d$. Popescu calculates cohomology in the setting where

$$\rho_1 = \rho_2 : \mathscr{T}_+(\mathbb{C}^d) \to B(\mathbb{C}) = \mathbb{C}$$

are given by taking $T_1 = T_2 = 0 \in \mathbb{D}(X^{\sigma_i})$. Since $A = \mathbb{C}$, then $\sigma_i = \rho_i|_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$ is the only unital homomorphism that is possible, namely the identity. By identifying $\mathbb{C}^d \otimes_{\sigma_i} \mathbb{C}$ with $\mathbb{C}^d$ and oberving that the intertwining condition is simply linearity over $\mathbb{C}$, we then have $\mathfrak{D}$ equalling $\mathbb{C}^d$ and $\mathfrak{I} = 0$. Thus, $\mathrm{Ext}^1(\mathbb{C}, \mathbb{C}) = \mathbb{C}^d$. Although, $H^1(\mathcal{A}_d, \mathbb{C})$ is defined in that paper as the collection of *bounded* derivations equivalent modulo inner derivations, it coincides with our definition since bounded, maps into $M_n$ are automatically *completely bounded*, cf. Proposition 3.8 of [36]. Furthermore, $\mathrm{Ext}^1(\mathbb{C}, \mathbb{C})$, in this case, is spanned by $d$ inequivalent, non-inner derivations of the type constructed in Proposition 4.27. In more general situations, it would be interesting to find noninner derivations unrelated to those in Proposition 4.27.

## 4.7 Analytic Crossed Product Algebras

This section explores $\mathrm{Ext}^1$ for analytic crossed products. First, we consider an example with commutative $A$.

**Example 4.29.** Let $A = C(X)$ and $\alpha \in \mathrm{Aut}(A)$. We consider $X = A$ as a $C^*$-correspondence over itself as described in Example 2.8. We fix $C^*$-representations $\sigma_i : C(X) \to B(\mathbb{C})$. Of course, each $\sigma_i$ is determined by a point $x_i \in X$ so that $\sigma_i(a) = a(x_i)$ for every $a \in C(X)$. Furthermore, the automorphism $\alpha$ has the form $\alpha(a) = a \circ \Phi^{-1}$ for some homeomorphism $\Phi : X \to X$. An operator $T_i \in B(H_i) \cong \mathbb{C}$ is part of a completely contractive covariant pair $(\sigma_i, T_i)$ if $\|T_i\| \leq 1$ and $\sigma_i(a)T_i =$

$T_i\sigma_i(\alpha(a))$ for every $a \in C(X)$. This covariance equation is the same as $a(x_i)T_i = T_i a(\Phi^{-1}(x_i))$ for every $a \in C(X)$. This can only happen if either $T_i = 0$ or $\Phi(x_i) = x_i$.

Consider each $H_i$ as a completely contractive Hilbert $\mathscr{T}_+({}_\alpha A)$-module with the representation determined by the pair $(\sigma_i, T_i)$. We now compute $(\mathrm{Ext}^1(H_2, H_1))_1$. If $\Delta : \mathscr{T}_+({}_\alpha A) \to B(H_2, H_1)$ is a derivation, then $\Delta = 0$ or $\Phi(x_1) = x_2$. Indeed, $\Delta(S) \in \mathcal{I}(\sigma_2 \circ \alpha, \sigma_1)$ implies

$$a(x_1)\Delta(S) = \Delta(S)a(\Phi^{-1}(x_2)) \quad \forall a \in C(X).$$

Thus, $\Phi^{-1}(x_2) = x_1$ or $\Delta(S) = 0$. As $\Delta$ vanishes on $A$ and satisfies the product rule, $\Delta(S) = 0$ implies $\Delta = 0$. Therefore, $\Delta = 0$ or $\Phi(x_1) = x_2$. It follows that $(\mathrm{Ext}^1(H_2, H_1))_1 = 0$ if $\Phi(x_1) \neq x_2$. Hence, we will assume $\Phi(x_1) = x_2$.

If $T_1 \neq T_2$, then the equation

$$T_1V - VT_2 = (1 - T_1T_1^*)^{1/2}L(1 - T_2^*T_2)^{1/2}$$

can be solved for $V$ since $T_1V - VT_2 = V(T_1 - T_2)$ and $T_1 - T_2$ is invertible. It follows that $(\mathrm{Ext}^1(H_2, H_1))_1 = 0$. If $T_1 = T_2$, then there are no nonzero inner derivations by commutativity in $\mathbb{C}$. Therefore, $(\mathrm{Ext}^1(H_2, H_1))_1$ is given by the ball $B(0; \sqrt{1 - \|T_1\|^2}\sqrt{1 - \|T_2\|^2})$.

The following lemma is an easy consequence of the product formula. We record it now for future use. The $A$-linear derivations of a tensor algebra $\mathscr{T}_+({}_\alpha A)$ are uniquely determined by their value at $S$.

**Lemma 4.30.** *Suppose $H$ and $K$ are Hilbert $\mathscr{T}_+({}_\alpha A)$-modules with representations $\pi : \mathscr{T}_+({}_\alpha A) \to B(H)$ and $\rho : \mathscr{T}_+({}_\alpha A) \to B(K)$, respectively. If $\Delta : \mathscr{T}_+({}_\alpha A) \to$*

*$B(K, H)$ is a derivation, then*

$$\Delta(S^{n+1}) = \sum_{j=0}^{n} \pi(S^{n-j})\Delta(S)\rho(S^j)$$

*for every $n \geq 0$.*

*Proof.* We induct on $n$. The case $n = 0$ is simply the statement that $\Delta(S) = \Delta(S)$. Suppose the result is true for $n = N$. It follows that

$$\begin{aligned} \Delta(S^{N+2}) &= \pi(S)\Delta(S^{N+1}) + \Delta(S)\rho(S^{N+1}) \\ &= \pi(S)\left(\sum_{j=0}^{N} \pi(S^{N-j})\Delta(S)\rho(S^j)\right) + \Delta(S)\rho(S^{N+1}) \\ &= \left(\sum_{j=0}^{N} \pi(S^{N+1-j})\Delta(S)\rho(S^j)\right) + \Delta(S)\rho(S^{N+1}) \\ &= \sum_{j=0}^{N+1} \pi(S^{N+1-j})\Delta(S)\rho(S^j). \end{aligned}$$

Thus, the result is true for all $n \geq 0$. $\square$

Recall that for $X = {}_{\alpha}A$, the Fock space is $\ell^2(\mathbb{Z}^+; A)$. Fix a representation $\psi : A \to B(E)$. From now on we fix the Hilbert $\mathscr{T}_+({}_{\alpha}A)$-module $H$ to be the induced representation space $\ell^2(\mathbb{Z}^+; A) \otimes_{\psi} E$. If $\{e_m\}$ is an orthonormal basis for $E$, then $\{\delta_n \otimes e_m\}$ constitute an orthonormal basis for $\ell^2(\mathbb{Z}^+; A) \otimes_{\psi} E$, where $\delta_n(n') = \delta_{nn'}1$ are the point masses in $\ell^2(\mathbb{Z}^+; A)$.

**Definition 4.31.** Let $\Delta : \mathscr{T}_+({}_{\alpha}A) \to B(K, \ell^2(\mathbb{Z}^+; A) \otimes_{\psi} E)$ be an $A$-linear derivation. We say that a sequence of vectors $k = \{k_m\}$ in $K$ *defines the $A$-linear derivation $\Delta$* if and only if

$$\Delta(S)h = \sum_{m} \langle h, k_m \rangle \delta_0 \otimes e_m \quad \forall h \in K.$$

By $A$-linearity of $\Delta$ and Lemma 4.30, $\Delta$ is uniquely determined by the operator $\Delta(S) \in B(K, \ell^2(\mathbb{Z}^+; A) \otimes_\psi E)$. Thus, the definition above makes sense. Observe that we have the isomorphisms

$$\ell^2(\mathbb{Z}^+; A) \otimes_\psi E \cong (\oplus_{n=0}^\infty {}_{\alpha^n} A) \otimes_\psi E \cong \oplus_{n=0}^\infty \left({}_{\alpha^n} A \otimes_\psi E\right) \cong \oplus_{n=0}^\infty {}_{\psi \circ \alpha^n} E,$$

not just as Hilbert spaces, but also as Hilbert $\mathscr{T}_+({}_\alpha A)$-modules. Indeed, $\phi_\infty(A)$ acts on the summand ${}_{\psi\circ\alpha^n}E$ via

$$a \cdot h = \psi(\alpha^n(a))h$$

and $S \in \mathscr{T}_+({}_\alpha A)$ *shifts* an element $h \in {}_{\psi\circ\alpha^n}E$ to the corresponding vector $h \in {}_{\psi\circ\alpha^{n+1}}E$. The definition above says that the range of the operator $\Delta(S)$ is contained in the subspace ${}_{\psi\circ\alpha^0}E = {}_\psi E$. Since $aS = S\alpha(a)$ in $\mathscr{T}_+({}_\alpha A)$ and $\Delta$ is $A$-linear, then necessarily

$$\sigma_1(a) \sum_m \langle h, k_m \rangle \delta_0 \otimes e_m = \sum_m \langle \sigma_2(\alpha(a))h, k_m \rangle \delta_0 \otimes e_m \tag{4.13}$$

as the following calculation demonstrates:

$$\begin{aligned} \sigma_1(a) \sum_m \langle h, k_m \rangle \delta_0 \otimes e_m &= \sigma_1(a)\Delta(S)h \\ &= \Delta(aS)h \\ &= \Delta(S\alpha(a))h \\ &= = \Delta(S)\sigma_2(\alpha(a))h \\ &= \sum_m \langle \sigma_2(\alpha(a))h, k_m \rangle \delta_0 \otimes e_m \end{aligned}$$

Conversely, it is clear that if $\{k_m\}$ defines a derivation, then the above equation implies $\Delta(S) \in \mathcal{I}(\sigma_2 \circ \alpha, \sigma_1)$. Note that it is trivially satisfied in case $A = \mathbb{C}$. For arbitrary $h \in K$, we have

$$\begin{aligned}
\langle h, k_m \rangle &= \left\langle \sum_{\lambda} \langle h, k_\lambda \rangle \, \delta_0 \otimes e_\lambda, \delta_0 \otimes e_m \right\rangle \\
&= \langle \Delta(S) h, \delta_0 \otimes e_m \rangle \\
&= \langle h, \Delta(S)^* (\delta_0 \otimes e_m) \rangle
\end{aligned}$$

Thus,

$$k_m = \Delta(S)^* (\delta_0 \otimes e_m). \tag{4.14}$$

Since $\Delta(S)^* \in \mathcal{I}(\sigma_1, \sigma_2 \circ \alpha)$, a necessary relationship between the $k_m$, a bit more explicit than (4.13), is readily obtained:

$$\begin{aligned}
\forall a \in A, \quad \sigma_2(\alpha(a)) k_m &= \sigma_2(\alpha(a)) \Delta(S)^* (\delta_0 \otimes e_m) \\
&= \Delta(S)^* \sigma_1(a) (\delta_0 \otimes e_m) \\
&= \Delta(S)^* (\delta_0 \otimes \psi(a) e_m) \\
&= \Delta(S)^* \left( \delta_0 \otimes \left( \sum_{\lambda} \langle \psi(a) e_m, e_\lambda \rangle \right) e_\lambda \right) \\
&= \sum_{\lambda} \langle \psi(a) e_m, e_\lambda \rangle \Delta(S)^* \left( \delta_0 \otimes e_\lambda \right) \\
&= \sum_{\lambda} \langle \psi(a) e_m, e_\lambda \rangle k_\lambda.
\end{aligned}$$

Definition 4.31 is a direct generalization of the notion in [4] of a *vector* $k \in K$ *defining a cocycle* $\sigma : A(\mathbb{D}) \times K \to \mathbb{H}^2$, although the results in that paper are in terms of cocycles rather than derivations. Below, we prove a noncommutative analogue of Proposition 3.1.1 in [4]. A crucial step is realizing a certain summation in the image of a Hankel operator. This is important as the Hankel operator's norm is controlled by an element of $\mathscr{T}_+({}_\alpha A)$. Carlson and Clark accomplish this using Nehari's theorem from [34]. The same proof is equally effective in our setting by appealing to Page's vector-valued analogue of Nehari's theorem, cf. [35].

**Theorem 4.32.** *Let $K$ be a Hilbert $\mathscr{T}_+({}_\alpha A)$-module and suppose that $k = \{k_m\}_{m=0}^{\infty}$ is a sequence in $K$. Then $k$ defines a* bounded *$A$-linear derivation $\Delta : \mathscr{T}_+({}_\alpha A) \to B(K, \ell^2(\mathbb{Z}^+; A) \otimes_\psi E)$ if and only if*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |\langle \rho(S^n)h, k_m\rangle|^2 < \infty \tag{4.15}$$

*and*

$$\sigma_1(a)\sum_m \langle h, k_m\rangle \delta_0 \otimes e_m = \sum_m \langle \sigma_2(\alpha(a))h, k_m\rangle \delta_0 \otimes e_m \tag{4.16}$$

*for all $h \in K$.*

*Proof.* Suppose the sequence $k$ defines a derivation. By the remarks preceeding the theorem, $A$-linearity of $\Delta$ implies (4.16). By Lemma 4.30 $\Delta$ satisfies

$$\begin{aligned}
\Delta(S^{N+1})h &= \sum_{j=0}^{N} \pi(S^{N-j})\Delta(S)\rho(S^j)h \\
&= \sum_{j=0}^{N} \pi(S^{N-j}) \left( \sum_{m=0}^{\infty} \langle \rho(S^j)h, k_m\rangle \delta_0 \otimes e_m \right) \\
&= \sum_{j=0}^{N}\sum_{m=0}^{\infty} \langle \rho(S^j)h, k_m\rangle \delta_{N-j} \otimes e_m
\end{aligned}$$

Since $\Delta$ is bounded,

$$\left\| \sum_{j=0}^{N}\sum_{m=0}^{\infty} \langle \rho(S^j)h, k_m\rangle \delta_{N-j} \otimes e_m \right\| = \|\Delta(S^{N+1})h\| \le \|\Delta\|\|S^{N+1}\|\|h\| = \|\Delta\|\|h\|$$

for arbitrary $N$. Therefore,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |\langle \rho(S^n)h, k_m\rangle|^2 = \left\| \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \langle \rho(S^n)h, k_m\rangle \delta_n \otimes e_m \right\| < \infty.$$

Conversely, suppose (4.15) and (4.16). For notational convenience, we identify $A$ and $\phi_\infty(A)$ and omit the map $\phi_\infty$. We first define $\Delta$ for polynomials in $\mathscr{T}_+({}_\alpha A)$ by setting $\Delta(a) = 0$ for all $a \in A, \Delta(S)h = \sum_m \langle h, k_m\rangle \delta_0 \otimes e_m$ for every $h \in K$, and requiring that the product formula is satisfied:

$$\begin{aligned}
\Delta\left(\sum_{n=0}^{N} a_n S^n\right) h &= \sum_{n=0}^{N} \sigma_1(a_n)\Delta(S^n)h \\
&= \sum_{n=1}^{N} \sigma_1(a_n)\Delta(S^n)h \\
&= \sum_{n=1}^{N}\sum_{j=0}^{n-1} \sigma_1(a_n)\pi(S^{n-1-j})\Delta(S)\rho(S^j)h \\
&= \sum_{n=1}^{N}\sum_{j=0}^{n-1}\sum_{m=0}^{\infty} \sigma_1(a_n)\pi(S^{n-1-j})\langle \rho(S^j)h, k_m\rangle \delta_0 \otimes e_m \\
&= \sum_{n=1}^{N}\sum_{j=0}^{n-1}\sum_{m=0}^{\infty} \sigma_1(a_n)\langle \rho(S^j)h, k_m\rangle \delta_{n-1-j} \otimes e_m \\
&= \sum_{\nu=0}^{N-1}\sum_{j=0}^{N-\nu-1}\sum_{m=0}^{\infty} \sigma_1(a_{j+\nu+1})\langle \rho(S^j)h, k_m\rangle \delta_{\nu} \otimes e_m.
\end{aligned}$$

It is easy to see that this last expression, which is an element of $\ell^2(\mathbb{Z}^+; A) \otimes_\psi E$, has the same norm as

$$\begin{bmatrix} \sigma_1(a_1) & \sigma_1(a_2) & \sigma_1(a_3) & \cdots \\ \sigma_1(a_2) & \sigma_1(a_3) & \sigma_1(a_4) & \cdots \\ \sigma_1(a_3) & \sigma_1(a_4) & \sigma_1(a_5) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{pmatrix} \sum_m \langle h, k_m\rangle \delta_0 \otimes e_m \\ \sum_m \langle \rho(S)h, k_m\rangle \delta_0 \otimes e_m \\ \sum_m \langle \rho(S^2)h, k_m\rangle \delta_0 \otimes e_m \\ \vdots \end{pmatrix},$$

which is an element of $\oplus_{n=0}^{\infty}\ell^2(\mathbb{Z}^+; A) \otimes_\psi E$ and in the image of the Hankel operator $H = \left[\sigma_1(a_{\nu+j+1})\right]$. These norms are finite since $H$ is a bounded operator ($a_n = 0$ for $n > N$) and (4.15) implies the column vector on the right is a vector in $\oplus_{n=0}^{\infty}\ell^2(\mathbb{Z}^+;A)\otimes_\psi E$. Not only is $H$ a bounded operator, but by [35] its norm is controlled by $\|\sum a_n S^n\|$. $\square$

The previous result pertained to *bounded* derivations. However, we are presently interested in *completely bounded* derivations. It was previously hoped that Proposition 3.15 might assist in characterizing the sequences of vectors $\{k_m\}$ giving rise to completely bounded derivations. Unfortunately, it seems that a better understanding of the operator-valued Hankel matrix's norm is needed, specifically relating the Hankel matrix norm to the norm of the "function" in the $\mathscr{T}_+({}_\alpha A)$ having the same "Fourier coefficients." Another approach may involve using the fact that every contractive represention of $\mathscr{T}_+({}_\alpha A)$ is completely contractive.

What follows is a discussion regarding the application of Corollary 3.15. Recall that $\Delta : \mathscr{T}_+({}_\alpha A) \to B(K,H)$ is completely bounded if and only if the norms of $\Delta^{(n)} : M_n(\mathscr{T}_+({}_\alpha A)) \to M_n(B(K,H))$ are uniformly bounded. Furthermore, by the canonical inclusion $B(K,H) \subset B(H \oplus K)$ and Paulsen's "canonical shuffle," cf. [36], $M_n(B(K,H))$ is isometrically isomorphic to $B(K^n, H^n)$. Invoking Corollary 3.15 permits the identification of $\Delta^{(n)}$ with a derivation, which for convenience we also denote by $\Delta^{(n)}$, mapping $\mathscr{T}_+({}_{\alpha^{(n)}}M_n(A))$ to $B(K^n, H^n)$. We now consider the case when $H = \ell^2(\mathbb{Z}^+;A)\otimes_\psi E$.

The $C^*$-representation $\psi : A \to B(E)$ promotes to the $C^*$-representation $\psi^{(n)} : M_n(A) \to B(E^n)$. Likewise, the automorphism $\alpha$ of $A$ promotes to an automorphism $\alpha^{(n)}$ of $M_n(A)$. It is important to remain cognizant of the difference in notation between $\alpha^{(n)}$ and $\alpha^j$ for (nonzero) integers $n$ and $j$. Both are automorphisms, but are defined on different algebras. The latter is a bonafide power, $\alpha \circ \cdots \circ \alpha$, composed $j$ times and defined on $A$; whereas, $\alpha^{(n)}$ is defined on $M_n(A)$ by applying $\alpha$ entry-wise on matrices. Just as we did in Corollary 3.15, we consider $M_n(A)$ as a $C^*$-correspondence over itself with left action skewed by $\alpha^{(n)}$. We induce $\psi^{(n)}$ to the Fock space $\ell^2(\mathbb{Z}^+; M_n(A))$ to obtain a representation of $L(\ell^2(\mathbb{Z}^+; M_n(A))$ on $\ell^2(\mathbb{Z}^+; M_n(A)) \otimes_{\psi^{(n)}} E^n$ which we then restrict to $\mathscr{T}_+({}_{\alpha^{(n)}}M_n(A))$. The resulting representation is unitarily equivalent to $\pi^{(n)}$ which is a representation of $M_n(\mathscr{T}_+({}_\alpha A))$ on $B(\oplus_{i=1}^n \ell^2(\mathbb{Z}^+; A) \otimes_\psi E)$. Note that we are identifying $M_n(\mathscr{T}_+({}_\alpha A))$ and $\mathscr{T}_+({}_{\alpha^{(n)}}M_n(A))$ as Corollary 3.15 permits us to do, in an effort to make the notation less cumbersome. In case the unitary equivalence between the Hilbert spaces is not immediately apparent, observe that it arises from the commuting of direct sums and tensor products and one instance of "fubination" (see the 4th isomorphism below):

$$
\begin{aligned}
\bigoplus_{i=1}^{n} \ell^2(\mathbb{Z}^+; A) \otimes_\psi E &\cong \bigoplus_{i=1}^{n} \left( \left( \bigoplus_{j=0}^{\infty} {}_{\alpha^j} A \right) \otimes_\psi E \right) \\
&\cong \bigoplus_{i=1}^{n} \left( \bigoplus_{j=0}^{\infty} {}_{\alpha^j} A \otimes_\psi E \right) \\
&\cong \bigoplus_{i=1}^{n} \left( \bigoplus_{j=0}^{\infty} {}_{\psi\circ\alpha^j} E \right) \\
&\cong \bigoplus_{j=0}^{\infty} \left( \bigoplus_{i=1}^{n} {}_{\psi\circ\alpha^j} E \right) \\
&\cong \bigoplus_{j=0}^{\infty} {}_{(\psi\circ\alpha^j)^{(n)}} \left( \bigoplus_{i=1}^{n} E \right) \\
&\cong \bigoplus_{j=0}^{\infty} \left( {}_{(\alpha^{(n)})^j} M_n(A) \otimes_{\psi^{(n)}} \left( \bigoplus_{i=1}^{n} E \right) \right) \\
&\cong \left( \bigoplus_{j=0}^{\infty} {}_{(\alpha^{(n)})^j} M_n(A) \right) \otimes_{\psi^{(n)}} \left( \bigoplus_{i=1}^{n} E \right) \\
&\cong \ell^2(\mathbb{Z}^+; M_n(A)) \otimes_{\psi^{(n)}} E^n
\end{aligned}
$$

The derivation $\Delta : \mathscr{T}_+({}_\alpha A) \to B(K, \ell^2(\mathbb{Z}^+; A) \otimes_\psi E)$ will be completely bounded if the associated (after multiple identifications indicated above) $\Delta^{(n)} : \mathscr{T}_+({}_{\alpha^{(n)}} M_n(A)) \to B(K^n, \ell^2(\mathbb{Z}^+; M_n(A)) \otimes_{\psi^{(n)}} E^n)$ are uniformly bounded in norm.

**Theorem 4.33.** *Let $K$ be a Hilbert $\mathscr{T}_+({}_\alpha A)$-module. Every extension class $\zeta \in \operatorname{Ext}(K, , l^2(\mathbb{Z}_+; A) \otimes_\psi E)$ can be represented by a derivation $\Delta_k : \mathscr{T}_+({}_\alpha A) \times K \to l^2(\mathbb{Z}_+; A) \otimes_\psi E)$ defined by for a sequence $k = \{k_m\}_{m=0}^{\infty}$ in $K$.*

*Proof.* Let $\Delta$ be a derivation representative of the extension class $\zeta$ such that the representation $\left[\begin{smallmatrix} \pi & \Delta \\ 0 & \rho \end{smallmatrix}\right]$ is completely contractive. In particular, the operator $\left[\begin{smallmatrix} T_1 & D \\ 0 & T_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \pi(S) & \Delta(S) \\ 0 & \rho(S) \end{smallmatrix}\right]$ is contractive. Equivalently,

$$D = \sqrt{1 - T_1 T_1^*} L \sqrt{1 - T_2^* T_2} \tag{4.17}$$

for some contraction $L : \overline{ran}(1 - T_2^* T_2) \to \overline{ran}(1 - T_1 T_1^*)$ with $L \in \mathcal{I}(\sigma_2 \circ \alpha, \sigma_1)$. Since $T_1$ is an isometry, $T_1 T_1^*$ is the orthogonal projection onto the final space of $T_1$. Hence, $1 - T_1 T_1^* = P_0$, the orthogonal projection onto $(\operatorname{ran} T_1)^\perp$, the closed span of $\{\delta_0 \otimes e_m \mid m \geq 0\}$, which we denote by $\delta_0 \otimes E$. Therefore, (4.17) implies

$$D = P_0 L \sqrt{1 - T_2^* T_2}, \tag{4.18}$$

which says that $\operatorname{ran} D \subset \delta_0 \otimes E$. By defining $k_{n,m} = D^*(\delta_n \otimes e_m)$, we have the following expression, for each $h \in K$:

$$Dh = \sum_n \sum_m \langle h, k_{n,m} \rangle \delta_n \otimes e_m.$$

It follows from $\operatorname{ran} D \subset \delta_0 \otimes E$ that $k_{n,m} = 0$ for every $n \geq 1$. Thus,

$$Dh = \sum_m \langle k_{0,m} \rangle \delta_0 \otimes e_m.$$

Therefore, the sequence $\{k_{0,m}\}$ *defines* $\Delta$. □

We remark that the above result can also be proved by slightly modifying the clever construction in Carlson and Clark's Theorem 3.2.1 of [4]. An additional complication arises from requring that the derivation defined by a constructed sequence $\{k_m\}$ not only is equivalent to the original $A$-linear derivation, but also happens to be $A$-linear as well. The relevant computations are straightforward and this alernate proof, indeed, works. The proof given above exploits the complete boundedness of our setting, and through similarity, benefits from assumptions of complete contractivity.

**Remark 4.34.** Finally, we remark that the above situation involving an induced representation as the module $H_1$ is quite general for isometric $H_1$. By the Wold decomposition for tensor algebras, (cf. [32, Theorem 2.9]) isometric representations of $\mathscr{T}_+(X)$ are a direct sum of a coisometric module and an induced representation. By Proposition 4.9, the functoraility of Ext means the $\mathrm{Ext}^n$ groups split up as direct sums. The projectivity of coisometric modules will make those summands vanish, leaving only the induced representation. Thus, the example provided with $H_1 = \ell^2(\mathbb{Z}^+; A) \otimes_\psi E$ is quite general.

**Remark 4.35.** In the classical model theory of Sz.-Nagy and Foiaş, a contraction splits orthogonally into a unitary operator and a completely non-unitary operator. Furthermore, the latter has a functional model. Future work will include studying the functional model obtained by Muhly and Solel for competely non-coisometric representations of the noncommutative Hardy algebra in [33]. The hope is that it will lead to a projective resolution generalizing the case of classical disc algebra in[5] and resulting in $\mathrm{Ext}^2(H_2, H_1) = 0$ for $n \geq 2$. Interestingly, the tensor algebras in pure algebra also has vanishing cohomology groups for $n \geq 0$, cf. [7].

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