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Two varieties of tunnel number subadditivity

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TWO VARIETIES OF TUNNEL NUMBER SUBADDITIVITY

by

Trenton Frederick Schirmer

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

July 2012

Thesis Supervisor: Associate Professor Maggy Tomova

ABSTRACT

Knot theory and 3-manifold topology are closely intertwined, and few invariants stand so firmly in the intersection of these two subjects as the tunnel number of a knot, denoted $t(K)$. We describe two very general constructions that result in knot and link pairs which are subbaditive with respect to tunnel number under connect sum. Our constructions encompass all previously known examples and introduce many new ones. As an application we describe a class of knots $K \subset S^3$ such that, for every manifold M obtained from an integral Dehn filling of $E(K)$, $g(E(K)) > g(M)$.

Abstract Approved: ______________________________

Thesis Supervisor

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the July 2012 graduation.

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ABSTRACT

Knot theory and 3-manifold topology are closely intertwined, and few invariants stand so firmly in the intersection of these two subjects as the tunnel number of a knot, denoted $t(K)$. We describe two very general constructions that result in knot and link pairs which are subbaditive with respect to tunnel number under connect sum. Our constructions encompass all previously known examples and introduce many new ones. As an application we describe a class of knots $K \subset S^3$ such that, for every manifold M obtained from an integral Dehn filling of $E(K)$, $g(E(K)) > g(M)$.

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CHAPTER 1
INTRODUCTION

Knot theory and the theory of 3-manifolds are closely interrelated fields, and today one cannot be an expert in one without having at least some knowledge of the other. For the combinatorial 3-manifold topologist, knot and link complements constitute an important and interesting class of objects to study, especially in light of the classic Dehn-Lickorish-Wallace theorem which says that all closed 3-manifolds can be obtained via Dehn fillings of link complements in S^3 [38],[16]. Together with triangulations and Heegaard diagrams, link surgery diagrams form one of the three most common ways of constructing and describing 3-manifolds.

The study of Dehn fillings and link surgery diagrams has long been a major area of research in 3-manifold topology, with many interesting and important results being acheived, perhaps most notably being Gabai's proof of the "Property R" conjecture[6], and Kronheimer and Mrowka's proof of the "Property P" conjecture [15]. Questions surrounding Dehn surgery continue to be a major driver of research in the field today, most notably the Berge conjecture. The importance of surgery diagrams also stems from the fact that they allow a gateway to 3-manifolds for powerful algebraic and diagrammatic knot theoretic tools, since an invariant which can be computed algorithmically from a link diagram can sometimes be modified to produce an invariant which is invariant under the moves of the Kirby calculus, and thus become a 3-manifold invariant.

Heegaard diagrams, first defined by their namesake [9], have a history that goes back even further–Poincare's famous first example of a homology sphere was described by means of a Heegaard diagram, and they also play an important role in Lickorish's proof of the Dehn-Lickorish-Wallace theorem discussed above. Some classical results on the Heegaard structure of 3-manifolds include Haken's lemma [8] and Waldhausen's theorem that every positive genus Heegaard splitting of the 3-sphere is stabilized [37]. A particularly important advance occurs in Casson and Gordon's paper [3], in which the notion of weakly reducible and strongly irreducible splittings are introduced, and it is shown that an irreducible manifold which admits a weakly reducible minimal genus Heegaard splitting must contain an incompressible surface. Subsequently this work was expanded upon by Scharlemann, Schultens, Thompson and many others, who developed the theory of generalized Heegaard splittings and an associated notion of "thin position" to accompany it. Excellent expositions of the theory can be found in [31] and [29].

The above paragraphs of course only touch very briefly and incompletely on what has been done with Dehn fillings and Heegaard splittings, but their importance ought to be clear. Continuing in this vein, few knot invariants stand so firmly in the intersection of knot theory proper and combinatorial 3-manifold topology as the tunnel number of a knot K, typically denoted $t(K)$. First defined in Clark's 1980 paper [5], where it is used in what is today an obvious way to obtain upper bounds on the Heegaard genus of manifolds obtained by Dehn fillings of K and cables of K, the tunnel number of a knot K is now typically defined as the minimal number of arcs one can attach to the knot so that the exterior of the resulting graph is a handlebody (one thinks of the arcs as tracing out tunnels drilled through the exterior of K). It is not difficult to see that $t(K) + 1$ is just the Heegaard genus of the exterior of K, and herein lies its significance. This invariant has since received substantial attention from many researchers.

Because unknotting tunnels occur naturally when studying Heegaard splittings of knot exteriors, many significant results relating to tunnel number were proved before the concept was formally defined, such as Birman, Hilden [1] and Viro's [36] result on the existence, for any tunnel number one knot $K \subset M$, of an involution of M that fixes K setwise. Likewise, after the invention of tunnel number but without reference to it, Norwood proved that all tunnel number one knots are prime by proving that all knots having a 2-generator fundamental group are prime [25]. Scharlemann, in one of the earliest papers to use the term "tunnel number"[30], soon followed up Norwood's result by showing that tunnel number one knots are, moreover,"doubly prime" and in fact satisfy property R (this was before Gabai gave the final word on the subject of property R).

Since then there has been a small avalanche of research on the subject of tunnel number. The majority of research achieved up to this point can be roughly placed within three categories. First, there are theorems that classify isotopy classes of tunnels for knots, which include Kobayashi's classification of the unknotting tunnels of two-bridge knots [13], the"leveling" theorem of Goda-Scharlemann-Thompson [7], and most recently the work of Cho and McCullough [4]. Second, there are theorems that relate the tunnel number of a knot to the Heegaard genus of 3-manifolds obtained by

Dehn fillings of the knot; here Moriah-Rubinstein [19], Rieck [26], and Rieck-Sedgwick [27] are especially important, and will be discussed in more detail below. Finally, the subject of the behavior of tunnel number under the operation of connected sum has been actively studied. Major contributers include Morimoto, Moriah-Rubinstein, Rieck and Kobayashi (separately and as a pair), and Scharlemann and Schultens. It is this last topic–the behavior of tunnel number under connect sum–that will be our focus in this thesis.

It is known that tunnel number can be additive $t(K_1 \# K_2) = t(K_1) + t(K_2)$, subadditive $t(K_1 \# K_2) < t(K_1) + t(K_2)$, or superadditive $t(K_1 \# K_2) > t(K_1) + t(K_2)$. It is not difficult to see that for superadditivity one has $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$ at most. The first published proof of the existence of superadditive pairs of knots appears in Morimoto, Sakuma, and Yokota's paper [22], where they describe a class of tunnel number 1 knots, any pair of which is superadditive. Their proof relied heavily on results from earlier papers by Morimoto [21], in which a necessary and sufficient condition for superadditivity is given in terms of the "one-bridge genus" of tunnel number one knots, and Yokota [39], in which quantum $SU(2)$ invariants are used to place lower bounds on one-bridge genus. Morimoto's result in [21] only applied to tunnel number one knots, and has since been shown not to hold in general for knots of higher tunnel number [14], although an extension of Morimoto's result to prime knots of higher tunnel number remains a possibility.

Another proof of the existence of superadditive pairs of knots appeared soon after [22] in a paper by Moriah and Rubinstein [19], and used very different methods.

The main result of [19] was, roughly speaking, that the Heegaard structure of Dehn filled hyperbolic manifolds eventually stabilizes as the numerator and denominator of the surgery grow large. As an application of this result, they prove that, for any pair of integers n_1, n_2, there are knots K_1, K_2 with $t(K_i) = n_i$ and $t(K_1 \# K_2) = n_1 + n_2 + 1$. Rieck [26], and later Rieck-Sedgwick [27] generalized and sharpened the main result of Moriah and Rubinstein using purely topological techniques, and later Kobayashi and Rieck [14] considerably extended Moriah and Rubinstein's result on superadditive knot pairs.

Going the other direction, Morimoto was the first to find a subadditive pair of knots in [23], and Kobayashi soon after found that the "degeneration" $t(K_1) + t(K_2) - t(K_1 \# K_2)$ can be arbitrarily large [12]. One essential ingredient that allowed Kobayashi to extend Morimoto's result was his invention and skilled use of <u>free tangle decompositions</u>, which are fundamental to our own work below. Free tangle decompositions subsequently appear in several papers, such as [20], and [24]. To explain the significance of these papers, however, we must discuss the notion of <u>degeneration ratio</u>.

Scharlemann and Schultens used the machinery of generalized Heegaard splittings (and plenty of elbow grease) to find lower bounds on tunnel number subadditivity. The first major result of theirs in this area [33] states that $t(K_1 \# \cdots \# K_n) \geq n$. In a second paper [32] they defined the "degeneration ratio" $d(K_1, K_2) = 1 - \frac{t(K_1 \# K_2)}{t(K_1) + t(K_2)}$ of a pair of knots K_1, K_2. There they proved, among other things, that for prime K_i, $d(K_1, K_2) \leq \frac{3}{5}$, and if a minimal genus Heegaard splitting of the exterior of $K_1 \# K_2$

is strongly irreducible, $d(K_1, K_2) \leq \frac{1}{2}$.

To this day the lower bounds found by Scharlemann and Schultens remain the best known, although it is not known whether they are best possible over the class of all prime pairs of knots. The first examples of tunnel number degeneration found by Morimoto satisfied $d(K_1, K_2) = \frac{1}{3}$, and this was the best known example for knots until Nogueira's thesis appeared [24]. There Nogueira gives examples of knot pairs achieving a degeneration ratio of $\frac{2}{5}$.

The present thesis is devoted to giving new examples of tunnel number subadditivity and applications. Subsequent sections of the present chapter are devoted to an exposition of the fundamental tools required for our later work. Chapter 2 is then devoted to a construction of subadditive knot pairs using the theory of generalized Heegaard splittings and the results of Kobayashi and Rieck [14]. The class of pairs K_1, K_2 produced by this construction are of arbitrarily high tunnel number with the property that the degeneration ratio asymptotically approaches $1/3$ as $t(K_i)$ grows large. What sets these examples apart is that the resulting Heegaard surface for the exterior of $K_1 \# K_2$ is always weakly reducible.

In Chapter 3 we adapt the machinery of Morimoto from [20] to a much broader class of free decompositions than he worked with. This adaption requires new techniques and results in the construction of a collection of links L which, in conjunction with certain kinds of knots, asymptotically achieve a degeneration ratio of $3/7$ as $t(L)$ grows large. Moreover, the links L can be modified in a straightforward manner to produce knots K which we conjecture to achieve the same degeneration ratio (one worries only about a loss of tunnel number when the switch is made from L to K). If our conjecture is correct, the resulting class of knots, which in the simplest case correspond to Nogueira's, would asymptotically achieve the highest known tunnel number degeneration under connect sum.

We start Chapter 4 with a refinement of results in Chapter 3 that bound tunnel number from above in terms of the complexity of free decompositions. As an application we show that Nogueira's knots from [24] have the property that the Heegaard genus of any manifold obtained by an integral Dehn filling of K is one less than the Heegaard genus of the exterior of K. This reveals yet another surprising connection between the behavior of Heegaard genus under Dehn fillings of knots and the behavior of tunnel number under connect sum. We conclude with a series of general observations and conjectures.

1.1 Preliminaries

We assume a familiarity with the basic concepts and terminology of 3-manifold topology and knot theory as presented in [10],[11],[2], and [28], but we review a bit of it here. Throughout we will be working in the piecewise linear category, which by a famous theorem of Moise [17] imposes no real restriction beyond the desirable one of forcing our links and surfaces to be tame.

A closed regular neighborhood of a polyhedron Y embedded in a manifold X is denoted $N(Y, X)$, and if $Y \subset X \subset Z$ it is to be understood that our choices of $N(Y, X)$ and $N(Y, Z)$ will satisfy the equation $N(Y, X) = N(Y, Z) \cap X$, unless stated otherwise. If X is the largest manifold involved in our discussion (with respect to inclusion), we will write simply $N(Y)$ for $N(Y, X)$. With analogous conventions let $E(Y, X) = \overline{X \setminus N(Y, X)}$ denote the exterior of Y in X, and define the scar set $Sc(Y)$ of Y in $E(Y, X)$ to be $N(Y, X) \cap E(Y, X)$, which lies in the boundary of $E(Y, X)$.

We assume that embedded polyhedra meet one another tranversely and that all embeddings are proper unless described otherwise. For sets $Y, Z \subset X$, $|Y \cap Z|$ denotes the number of connected components of $Y \cap Z$. Also, the following abuse of notation will occur frequently to ease the exposition: If $X = \{X_1, \cdots, X_n\}$ is a set of disjointly embedded submanifolds, we will often also use the same symbol X to refer to $X_1 \cup \cdots \cup X_n$ when ambiguity cannot arise.

Finally, if X and Y are topological spaces, $X' \subset X$ and $f : X' \to Y$ is any function, $X \cup_f Y$ will denote the quotient space of the disjoint union of X and Y modulo the smallest equivalence relation $\sim$ on $X \cup Y$ satisfying $x \sim f(x)$, and we will refer to $X \cup_f Y$ as X glued to Y via f. If $\pi : X \cup Y \to X \cup_f Y$ is the quotient map and $Z \subset X \cup Y$, then $\pi(Z)$ will be referred to as the image of Z after gluing.

A surface F of positive genus properly embedded in a 3-manifold M is compressible if there is a disk D embedded in M such that $D \cap F = \partial D$ and ∂D does not bound a disk in F. In this case D is called a compression disk. If F is a sphere or disk then it is said to be compressible if one component of $E(F, M)$ is a ball. A surface which is not compressible is said to be incompressible. Similarly, if a surface F satisfies $\partial F \neq \emptyset$ and $\chi(F) < 1$ then it is said to be ∂-compressible if there exists an embedded disk $D \subset M$ such that $\partial D = \alpha \cup \beta$, $\partial\alpha = \partial\beta = \alpha \cap \beta$, $\alpha = D \cap F$, $\beta \subset \partial M$, and such that the closure of neither component of $F \setminus \alpha$ is a disk. In this case D is said to be a boundary compression disk. If F is not boundary compressible, it is ∂-incompressible.

A 3-manifold M is said to be irreducible if it contains no incompressible 2-spheres and it is said to be ∂-irreducible if ∂M is incompressible, otherwise it is said to be reducible and ∂-reducible, respectively.

The connected sum $M_1 \# M_2$ of two oriented 3-manifolds M_1, M_2 is obtained by removing any pair of 3-balls $B_i \subset int(M_i)$ and attaching $M_1 \setminus int(B_1)$ to $M_2 \setminus int(B_2)$ via an orientation reversing homeomorphism $h : \partial B_1 \to \partial B_2$. It is known that this operation does not depend on the choice of B_i or h.

1.2 Heegaard Splittings

In this section M and M_i are always compact, connected, orientable 3-manifolds.

Definition 1.1. A compression body V is a connected manifold obtained by taking a collection of 3-balls B and thickened orientable closed surfaces of positive genus $F_1 \times I, \cdots F_n \times I$ and attaching orientable one handles along $\partial B \cup F_1 \times \{1\} \cup \cdots \cup F_n \times \{1\}$. We let $\partial_- V = F_1 \times \{0\} \cup \cdots \cup F_n \times \{0\}$ denote the negatative boundary of V, and $\partial_+ V = \partial V \setminus \partial_- V$ the positive boundary.

Note that V is a handlebody if $\partial_- V = \emptyset$, and that our definition counts the 3-ball and all connected orientable closed positive genus thickened surfaces as compression bodies.

Definition 1.2. A connected separating closed surface F embedded in M is said to be a Heegaard surface for M if the closure of each component V_1, V_2 of $M \setminus F$

is a compression body and $\partial_+\overline{V_1} = \partial_+\overline{V_2} = F$. In this case, (M, V_1, V_2) is called a Heegaard splitting of M whose genus is defined to be the genus of F.

For good proofs of the remaining propositions in this section, see [31] and [29].

Proposition 1.3. *Every compact orientable manifold with no* 2*-sphere boundary components admits a Heegaard splitting.*

Definition 1.4. The Heegaard genus of M, denoted $g(M)$, is the minimum genus of a Heegaard splitting for M.

Definition 1.5. Let (M, V_1, V_2) be a Heegaard splitting of M with Heegaard surface F. The splitting is said to be

- stabilized if there are compressing disks $D_1 \subset V_1$, $D_2 \subset V_2$ such that $|D_1 \cap D_2| = 1$.
- reducible if there are compressing disks $D_1 \subset V_1$, $D_2 \subset V_2$ such that $\partial D_1 = \partial D_2$.
- weakly reducible if there are compressing disks $D_1 \subset V_1$, $D_2 \subset V_2$ such that $D_1 \cap D_2 = \emptyset$
- strongly irreducible if, for every pair of compressing disks $D_1 \subset V_1$, $D_2 \subset V_2$, $D_1 \cap D_2 \neq \emptyset$

With a bit of imagination it is easy to see that a stabilized splitting is reducible, and a reducible splitting is weakly reducible, thus in fact for a strongly irreducible splitting (M, V_1, V_2) and any pair of compressing disks $D_i \subset V_i$, we have $|D_1 \cap D_2| > 1$. The following proposition is Haken's lemma.

Proposition 1.6. *[8] If M is reducible then every Heegaard splitting of M is reducible.*

As a consequence we have the following.

Proposition 1.7. *Heegaard genus behaves additively with respect to connect sum, i.e. $g(M_1 \# M_2) = g(M_1) + g(M_2)$ for all M_1, M_2.*

For the purposes of this thesis, we need only the definition of a generalized Heegaard splitting and a single proposition regarding them.

Definition 1.8. A generalized Heegaard splitting of M is a decomposition of M into compression bodies $V_1, \cdots, V_n$ such that $M = V_1 \cup \cdots \cup V_n$, $\partial M \subset \partial_- V_1 \cup \cdots \partial_- V_n$, and, for all $1 \leq i, j \leq n$, $i \neq j$, either $V_i \cap V_j = \partial_+ V_i = \partial_+ V_j$ or $V_i \cap V_j = \partial_- V_i \cap \partial_- V_j$.

The "or" in that definition is meant to be exclusive, and note that it will often be the case that $V_i \cap V_j = \partial_- V_i \cap \partial_- V_j = \emptyset$. The following proposition is widely known, but we include a brief sketch of the proof for the sake of descriptiveness.

Proposition 1.9. *Suppose (M, V_1, V_2, W_1, W_2) is a generalized Heegaard splitting of M with $\partial_+ V_1 = \partial_+ V_2$, $\partial_+ W_1 = \partial_+ W_2$, $\partial_- V_1 \cup \partial_- W_2 \subset \partial M$, and $\partial_- V_2 \cap \partial_- W_1 = F$, where F is a connected surface. Then one can obtain a Heegaard surface P for M of genus $g(\partial_+ V_1) + g(\partial_+ V_2) - g(F)$ which induces a Heegaard splitting (M, Y_1, Y_2) satisfying $\partial_- Y_i = (\partial_- V_i \cup \partial_- W_i) \setminus F$.*

Proof. The generalized Heegaard splitting in question can be made to arise from a Morse function f. The index i critical points of f will correspond to the 1-handles of V_i and W_i which are attached to $\partial_- V_i$ and $\partial_- W_i$, respectively, for $i = 1, 2$. So after an isotopy of f which amounts to sliding its index 1 critical points below its index 2 points, we obtain a new Morse function f' whose index 1 critical points all lie below its index 2 critical points, and if s is any regular value between the index 1 and index 2 critical values, $f^{-1}(s) = P$ will be a surface of the kind we claimed to exist. $\square$

The splitting coming from the surface F in the proof is said to be obtained via amalgamation of the original generalized splitting. In general, if M is a manifold and $F \subset M$ is an embedded surface, the connectivity graph $\Gamma(F)$ is defined to have one vertex $v(N)$ for each component N of $M \setminus F$, and one edge connecting connecting $v(N)$ to $v(N')$ for every component of F lying in $\overline{N} \cap \overline{N'}$. The following proposition is easily obtained from the previous one via induction.

Proposition 1.10. *Let $M = V_1 \cup \cdots \cup V_n$ be a generalized Heegaard splitting with $F_- = \partial_- V_1 \cup \cdots \cup \partial_- V_n$. Then if $\Gamma(F_-)$ is a tree, M admits an amalgamated Heegaard splitting.*

1.3 Knot and Link Basics

We now set out our elementary notation and definitions regarding knots and links.

Definition 1.11. An oriented link L of n components in an oriented 3-manifold M is the oriented image of an embedding of n copies of S^1 into M. A knot is a link of one component. Two oriented links are equivalent if there is an orientation preserving homeomorphism $h : (M, L_1) \to (M, L_2)$.

From now on we always assume that our links are oriented and that the 3-manifolds M in which our links are embedded are compact and orientable with no sphere boundary components. However, we will rarely refer to the orientations in question because the invariants we work with below do not depend on the orientation of the links involved. Indeed the only reason we have bothered with an oriented notion of link equivalence is to ensure that the notion of connected sum is well defined with respect to it.

Compressing disks for solid tori are called meridian disks, and the boundaries of such disks are called meridians, which are unique up to isotopy. A regular neighborhood of a component of a link in an orientable manifold is always a solid torus, and a meridion disk D for this torus is also called a meridian disk for L, and likewise ∂D is known as a meridian of L.

An arc α properly embedded in a ball B is unknotted if it cobounds an embedded disk $D \subset M$ with an arc on ∂B. More generally, if α is properly embedded in a manifold M it is said to be unknotted if, for every ball $B \subset M$ such that $\alpha \cap B$ is a single properly embedded arc in B, $\alpha \cap B$ is unknotted. We are now ready for the following important definition.

Definition 1.12. Let $L \subset M$ be a link of n components $L_1, \cdots L_n$ and let $L' \subset M'$ be a link of m components $L'_1, \cdots L'_m$. Let $S \subset M$, $S' \subset M'$ be separating 2-spheres bounding balls B, B' respectively, such that $L \cap B = L_i \cap B$ and $L' \cap B' = L'_j \cap B'$ are unknotted arcs in B and B', respectively. Let $h : (S, S \cap L) \to (S', S' \cap L')$ be an orientation reversing homeomorphism and let $\tilde{M} = M \setminus int(B)$, $\tilde{M}' = M' \setminus int(B')$.

Then the connect sum along (L_i, L_j), denoted $L\#_{i,j}L'$, is defined to be the link in $M\#M'$ obtained by gluing $\tilde{M}$ to $\tilde{M}'$ via h.

In the case that one or both elements of the connected sum are knots we drop the corresponding subscripts from our notation, so that the connect sum of a knot K with a link L along its ith component will be denoted $K\#_iL$, and the connect sum of two knots K and K' is written $K\#K'$ as usual. If $K_1, \cdots K_n$ are knots and L is an n component link, we write $L\#(K_1, \cdots, K_n)$ for $(\cdots((L\#_1K_1)\#_2K_2)\cdots)\#_nK_n$.

The image $S \subset M_1\#M_2$ of the spheres $S_i \subset M_i$ after gluing will be called the decomposing sphere of the connected sum, and the corresponding annulus $A = S \cap E(L\#_{i,j}L')$ in $E(L\#_{i,j}L')$ will be called the decomposing annulus. In fact, $E(L\#_{i,j}L')$ can be obtained by gluing $E(L)$ to $E(L')$ along a pair of meridional annuli lying on their respective boundaries, and the image of these annuli after the gluing is a single annulus which is isotopic to the decomposing annulus. See, for example, [32] for a more detailed account.

Yet another well known way of describing the connect sum of a pair of links can be given via the "satellite" construction described, e.g. in [2]. Here, let $L_1, \cdots L_n = L \subset M$, and $L'_1, \cdots L'_m = L' \subset M'$, let μ be a meridion of L_i, and let $V = E(N(\mu), M)$, so that $L \subset V$, and, for some compressing disk D of ∂V, $|D \cap L| = |D \cap L_i| = 1$. One then glues V to $E(L'_j, M')$ via any homeomorphism $h : \partial V \to \partial E(L'_j, M') \setminus \partial M'$ that sends ∂D to a meridian curve on $\partial N(L'_j)$. The link $L\#_{i,j}L'$ is then the image after gluing of $L'_1 \cup \cdots \cup L'_{j-1} \cup h(L) \cup L'_{j+1} \cup \cdots \cup L'_m$ in $M\#M'$. In this case we call the image after gluing of ∂V a swallow-follow torus, which in this case "swallows" L and "follows" L'_j.

Definition 1.13. A bridge surface F for a link $L \subset M$ is a Heegaard surface for M such that each arc of $L \cap E(F)$ cobounds an embedded disk with an arc in $\partial E(F)$. The pair (L, F) is said to be a bridge presentation of L.

Given any Heegaard surface F for M, and any link L in M, it is possible to perturb L so that (L, F) becomes a bridge presentation.

Definition 1.14. The genus g bridge number of $L \subset M$, denoted $b_g(L)$, is defined to be $\min\{|F \cap L|/2\}$, where the minimum is taken over all genus g Heegaard surfaces of M. If M is the 3-sphere, we denote $b_0(L)$ simply as $b(L)$.

Of course $b(L)$ is just the standard bridge number first defined by Schubert [34] and proved by him to be additive with respect to connect sum in knots. We now come to the formal definition of tunnel number.

Definition 1.15. An unknotting system for L is a collection of arcs $T = \{t_1 \cup \cdots t_n\}$ properly embedded in $E(L)$ such that $E(L \cup t_1 \cup \cdots t_n)$ is a handlebody. The minimal cardinality of an unknotting system for L shall be denoted $t(L)$, the tunnel number of L.

Here is the formal definition of the degeneration ratio.

Definition 1.16. Let L be a link of n components $L_1, \cdots, L_n$, let $K_1, \cdots, K_n$ be a collection of knots, and let $L\#(K_1, \cdots, K_n)$ be the connect sum taken so that K_i connects along L_i. The degeneration ratio is then given as follows:

$$d_L(K_1, \cdots, K_n) = \frac{t(L)+t(K_1)+\cdots+t(K_n)-t(L\#(K_1,\cdots,K_n))}{t(L)+t(K_1)+\cdots+t(K_n)}$$

In the case that $L = K$ is a knot, we use the notation $d(K, K')$ for $d_K(K')$.

We conclude with a brief description of Dehn surgery, for the sake of simplicity we restrict our attention to knots.

Definition 1.17. A manifold M' is said to be obtained via a Dehn surgery along a knot $K \subset M$ if $M' \cong E(K) \cup_h V$, where V is a solid torus and $h : \partial V \to \partial E(K)$ is a homeomorphism.

We will also call M' a Dehn filling of $E(K)$, and V the filling torus. If D is a meridian disk of V, and if $h(\partial D)$ and $h'(\partial D)$ both represent the same element of $H_1(\partial E(K))$, then $E(K) \cup_h V \cong E(K) \cup_{h'} V$. Moreover, if γ is any simple closed curve embedded in a torus T, and $[\gamma] = a\mu + b\lambda \in H_1(T)$, where μ and λ are any basis of $H_1(T)$, then a and b are relatively prime. Thus after fixing a basis μ, λ for $H_1(\partial E(K))$, we may index the manifolds obtained by Dehn surgeries along K by elements of $\mathbb{Q}$, as follows: If $\frac{a}{b} = q$ we let $M_K(q)$ denote the manifold obtained by a Dehn surgery along K such that $[h(\partial D)] = a\mu + b\lambda$. We let $K(q) \subset M_K(q)$ denote the core of V after gluing.

In this thesis, we always assume that the generator μ in our preferred basis of $H_1(\partial N(K))$ is represented by a meridian of $N(K)$. This means that the integral Dehn fillings of $E(K)$ are those obtained by attaching the meridian of V to a curve on $\partial N(K)$ that intersects some meridian of $N(K)$ exactly once.

Of course, all of the definitions and terminology surrounding Dehn fillings of knot complements will also apply in the natural and obvious way to Dehn fillings of link complements more generally.

1.4 Cutting and Pasting

In all of our work below the notion of a spine is crucial.

Definition 1.18. If V is a handlebody, an embedded graph X in V is said to be a spine of V if $E(X)$ is homeomorphic to $(\partial V)\times I$, where $I = [0,1] \subset \mathbb{R}$. A subgraph X' of a spine X in V will be called a subspine if it contains no contractible components, and in the special case that X' is a collection of loops it will be called a core of V.

A compression body can be regarded as a handlebody with an open regular neighborhood of a subspine removed.

Definition 1.19. A collection of simple closed curves $\mathbf{C} = \{C_1, \cdots, C_n\}$ embedded in ∂V is called primitive if there exists a disjoint collection of compressing disks $\mathbf{D} = \{D_1, \cdots, D_n\}$ for V such that $|C_i \cap D_j| = \delta_{ij}$ (where δ_{ij} is the Kronecker delta). The disks $\mathbf{D}$ are said to be dual to $\mathbf{C}$.

Proposition 1.20. *A primitive collection* $\mathbf{C} = \{C_1, \cdots, C_n\}$ *of curves embedded in the boundary of a handlebody* V *is isotopic to a core of* V*.*

Proof. It suffices to show that $\mathbf{C}$ can be isotoped into V so that $E(\mathbf{C})$ becomes a compression body. Let $\mathbf{D} = \{D_1, \cdots, D_n\}$ be a collection of disks dual to $\mathbf{C}$. Then $\mathbf{E} = \overline{\partial N(C_1 \cup \cdots \cup C_n \cup D_1 \cup \cdots \cup D_n) \setminus \partial V}$ is a collection of compressing disks which cut V into a handlebody $\tilde{V}$ and a collection of solid tori T_i, each having a core isotopic to exactly one element $C_i \in \mathbf{C}$. Thus, we see that $E(\mathbf{C})$ is obtained by attaching a collection of thickened tori $\overline{T_i \setminus N(C_i)}$ to a handlebody $\tilde{V}$ along disks, and so must be a compression body. $\square$

The following proposition is well known and occurs, e.g., in [32]. We sketch the proof here.

Proposition 1.21. *If* $\mathbf{C}$ *is a primitive collection of simple closed curves on the boundary of a collection of handlebodies* V *with dual disk collection* $\mathbf{D}$*, and* $h : N(\mathbf{C}) \cap \partial V \to \partial V \setminus (\mathbf{C} \cup \mathbf{D})$ *is an orientation reversing embedding, then* $V/(x \sim h(x))$ *is a handlebody.*

Proof. If $H = N(\mathbf{C} \cup \mathbf{D})$, $V' = H \cup_h E(H, V)$ is homeomorphic to $E(\mathbf{D}, V)$ and thus is a handlebody. But $V/(x \sim h(x))$ is obtained from V' by attaching one-handles along the image of the disks $\overline{\partial H \setminus \partial V}$ after gluing. $\square$

It follows from this proposition that the attachment of a 2-handle along a primitive annulus on the positive boundary of a compression body always yields another compression body of lower genus. This fact plays a role in our study of stabilizing arcs.

Definition 1.22. An arc α properly embedded in a manifold M is said to be <u>stabilizing</u> if and only if there exists a disk D transversely embedded in M with $\partial D = \alpha \cup \beta$, where β is an arc embedded in ∂M and $\partial \alpha = \partial \beta$. In this case we say that α <u>cobounds</u> D with the arc β.

For the sake of descriptiveness, if α is any arc properly embedded in M, then we will often call the scar set $Sc(\alpha) \subset \partial E(\alpha, M)$ the annulus associated with α. An arc α embedded in M is stabilizing if and only if its associated annulus is primitive in $E(\alpha, M)$, in the sense that there exists a compressing disk D for $E(\alpha, M)$ satisfying $|D \cap Sc(\alpha)| = 1$.

Proposition 1.23. *Suppose α is a stabilizing arc in M. Then M is a handlebody if and only if $E(\alpha, M)$ is.*

Proof. First suppose that M is a handlebody. Then there is a collection of disks $\mathbf{D}$ such that $E(\mathbf{D}, M) = B$ is a 3-ball. Then since α is stabilizing it cobounds a disk D with an arc in ∂M. One can then easily construct an isotopy of α in M via embeddings $h_t : (I, \partial I) \to (D, \beta)$ so that $h_0(I) = \alpha$, $h_1(I) = \alpha'$ cobounds a small disk $D' \subset D$ with a small subarc of β satisfying $D' \cap \mathbf{D} = \emptyset$.

It follows that $E(\alpha', B) = V$ is a solid torus and, after choosing a meridian disk D'' for V that is disjoint from the scar set $Sc(\mathbf{D}) \subset \partial V$, we obtain the collection $\{D''\} \cup \mathbf{D}$ of compressing disks for $E(\alpha', M)$ which cuts it into a 3-ball, thus showing it to be a handlebody as well. Since $E(\alpha', M) \cong E(\alpha, M)$ the first half of the proof is done.

For the second half, simply note that M is obtained from $E(\alpha, M)$ via a two-handle attachment along the annulus associated with α, which, as noted in the remark preceding this proposition, is primitive. As observed in the remark following Proposition 1.21, the result of this handle attachment must therefore be a handlebody, thus completing the proof.

□

If $\alpha_1, \cdots, \alpha_n$ is a disjoint collection of stabilizing arcs in M, then routine innermost and outermost disk arguments similar to above allow us to choose disjoint embedded disks $D_1, \cdots, \cdots D_n$, where each α_i cobounds D_i with an arc in ∂M. Thus we can generalize Proposition 1.23 using the same argument.

Proposition 1.24. *If $\alpha_1, \cdots \alpha_n$ is a disjoint collection of stabilizing arcs embedded in M, then M is a handlebody if and only if $E(\alpha_1 \cup \cdots \cup \alpha_n, M)$ is.*

CHAPTER 2
WEAKLY REDUCIBLE SUBADDITIVE PAIRS

In the following chapter we construct knots of arbitrarily high tunnel number which experience proportionally high degeneration via weakly reducible Heegaard splittings. Asymptotically, the knot pairs we construct will approach a degeneration ratio of 1/3 as the tunnel number grows large.

2.1 μ Primitivity

We say that a Heegaard splitting $V_1 \cup V_2$ of a knot exterior $E(K)$ is $\mu(n)$-primitive, if there is a disjoint collection of n disks, each intersecting K once, and whose boundary curves form a pair of cores C_1, C_2 of V_1 and V_2 respectively. For a knot K then define $\mu(K)$ to be the largest n for which a minimal genus Heegaard splitting is $\mu(n)$-primitive. This can be seen as a slight generalization of the concept of μ-primitivity [18]; if $\mu(K) > 0$ then K is said to be μ-primitive.

Proposition 2.1. *Suppose* $\mu(K) = n$, *and let* $K_1, \cdots, K_n$ *be any collection of* n *knots. Then* $t(K\#K_1\#\cdots\#K_n) \leq t(K)+t(K_1)+\cdots+t(K_n) = m$, *and* $E(K\#K_1\#\cdots\#K_n)$ *admits a weakly reducible* $m+1$ *genus Heegaard splitting.*

Proof. Let F be a minimal genus Heegaard splitting of $E(K)$ with complementary compression bodies V_1, V_2 in M such that $\partial_- V_1 = \partial(N(K))$, and let $C_1 \cup C_2 = C$ be a primitive pair of cores realizing $\mu(K)$. Then the connect sum $K\#K_1\#\cdots\#K_n$ can be taken via the satellite construction along the collection of tori $\partial(N(C))$ in $E(K)$ (see the discussion following Definition 1.12). Given any minimal genus Heegaard splittings of the K_i, the connect sum taken along the cores C naturally induces a generalized Heegaard splitting of $E(K\#K_1\#\cdots\#K_{n-1}\#K_n)$ which can be amalgamated to a splitting of genus $t(K)+t(K_1)+\cdots+t(K_n)+1$, yielding the desired result.

$\square$

As a consequence we have the well known fact that, for any knot K whose connect sum with another knot K' is superadditive with respect to tunnel number, $\mu(K)=0$. The next proposition gives a useful criterion for a knot to satisfy $\mu(K)\geq n+1$.

Proposition 2.2. *If $K\subset S^3$ is an $n+1$ bridge knot with $t(K)=n$ then $\mu(K)\geq n+1$.*

Proof. If S is a sphere in S^3 realizing $b(K)=n+1$, then K intersects the two complementary balls B_1, B_2 of $E(S)$ trivially, that is, each arc of $K\cap B_i$ cobounds a disk in B with an arc in ∂B, and all of these disks can be chosen disjoint. If $s_1,\cdots,s_{n+1}$ are the arcs of intersection of K with B_1, then there is a collection $t_1,\cdots,t_n$ of unknotted arcs embedded in B_1 disjoint from the bridge disks in question, such that the endpoints of t_i lie in s_i and s_{i+1}, $1\leq i\leq n$.

Since $E(s_1\cup\cdots\cup s_{n+1}\cup t_1\cup\cdots\cup t_n, B_1)$ can be isotoped onto a small collar neighborhood of $E(K\cap\partial B_1,\partial B_1)$, $E(K\cup t_1\cup\cdots\cup t_n)$ is homeomorphic to $E(K\cap B_2, B_2)$. Thus the t_i form a minimal unknotting system for K and the disks cobounded by the arcs of $K\cap B_2$ in B_2 correspond to a complete collection of compressing disks for the handlebody $E(K\cup t_1\cup\cdots\cup t_n)$ which are dual to the core C consisting of one

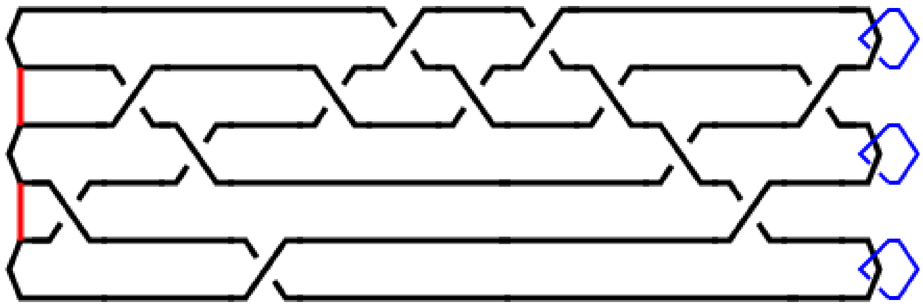

Figure 2.1: Bridge tunnels and core indicating $\mu(K) \geq 3$ for the tunnel number 2 knot $K = 8_{16}$

essential curve on each annulus associated with the arcs of the tangle $(B_2, K \cap B_2)$. This proves our claim. $\square$

2.2 The First Construction

To construct our examples, we require some theorems from the literature. The first result is a classic of Schubert's.

Proposition 2.3. *[34] If K_1 and K_2 are knots in S^3, $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$.*

The second result is a more recent result of Scharlemann and Schultens.

Proposition 2.4. *[33] Given any collection $K_1, \cdots, K_n$ of knots in S^3, $t(K_1 \# \cdots \# K_n) \geq n$.*

From these two propositions we deduce the existence of $(n+1)$-bridge, tunnel number n knots for arbitrary $n > 0$:

Proposition 2.5. *If $K_1, \cdots K_n$ is any collection of 2-bridge knots in S^3 and $K = K_1 \# \cdots \# K_n$, then $b(K) = n + 1$ and $t(K) = n$*

Proof. The fact that $b(K) = n+1$ follows from Schubert's theorem, and using bridge tunnels as in Proposition 2.2 we see that $t(K) \leq b(K) - 1 = n$. The theorem of Scharlemann and Schultens now gives $t(K) = n$. $\square$

The knots K of this proposition will constitute one half of the superadditive pairs we will construct. It seems likely that they can be chosen prime, and certainly there are known examples of prime knots satisfying $t(K) = n$, $b(K) = n+1$ for $n \leq 3$. To construct the second kind of knot that occurs in our pairs, we require the following (here somewhat abridged) theorem of Kobayashi and Rieck.

Proposition 2.6. *[14] For any collection of positive integers $\{m_1, \cdots, m_n\}$ there exists a collection of knots $\{K_1, \cdots, K_n\}$ in S^3 satisfying $t(K_i) = m_i$ and $t(K_1 \# \cdots \# K_n) = n - 1 + \sum_i m_i$.*

We are now ready to prove this chapter's main result.

Proposition 2.7. *For arbitrary $n > 0$ there exist knots $K, K' \subset S^3$ with $t(K) = n$, $t(K') = 2n+1$, and $t(K \# K') \leq 2n+1$. Moreover $E(K \# K')$ admits a weakly reducible genus $2n+2$ Heegaard splitting.*

Proof. By Proposition 2.5 we can find an $n+1$ bridge, tunnel number n knot K, and by Proposition 2.6 we have a knot $K' = K_1 \# \cdots \# K_{n+1}$ where $t(K_i) = 1$ for all $1 \leq i \leq n+1$ and $t(K') = 2n+1$. The result now follows from Proposition 2.1. $\square$

As an immediate corollary we obtain:

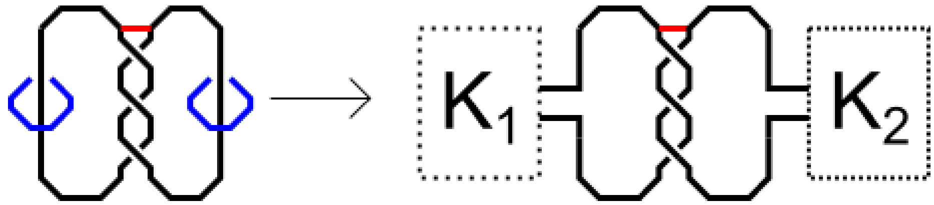

Figure 2.2: Taking the connect sum of the Trefoil with a superadditive pair of knots K_1, K_2 along a primitive core

Proposition 2.8. *The knots K, K' of Proposition 2.7 satisfy* $\frac{n}{3n+1} \leq d(K, K') \leq \frac{2n-1}{3n+1}$.

Proof. The lower bound is Proposition 2.7, the upper bound comes from Proposition 2.4. $\square$

CHAPTER 3
SUBADDITIVITY VIA FREE DECOMPOSITIONS

In this section we construct our second, more complex, family of subadditive knot-link pairs which utilizes free tangle decompositions. This construction allows for greater degeneration, and as we discuss in the next chapter, can be utilized to reveal some interesting connections with Dehn surgery.

3.1 Free Tangle Decompositions

Free tangle decompositions were introduced into the literature by Kobayashi [12] in order to prove the existence of knot pairs whose tunnel number experiences arbitrarily large degeneration under connect sum. In this section we define a more general notion of free decompositions and prove some useful propositions about them.

Definition 3.1. Let M be a compact orientable 3 manifold with non-empty boundary, and let $T = \{t_1, \cdots, t_n\}$ be a collection of arcs properly embedded in M. The pair (M, T) is called a tangle in M, and it is free if $E(T, M)$ is a handlebody.

Observe that an unknotting system for L is a free tangle in $E(L, M)$. It is useful to specialize this definition as follows.

Definition 3.2. A tangle (M, T) is said to be trivial if (M, T) is free and every arc of T is stabilizing in M.

A tangle (M, T') is said to be a subtangle of (M, T) if $T' \subset T$

Definition 3.3. A subtangle (M, T') of a free tangle (M, T) is said to be trivializing if $(E(T', M), T \setminus T')$ is trivial.

As an immediate consequence of Proposition 1.24 we then deduce

Proposition 3.4. *If (M, T') is a trivializing subtangle of a free tangle (M, T), then (M, T') is also free.*

If $S \subset M$ is an embedded surface and N_1 and N_2 are two connected components of $E(S, M)$, we say that N_1 is adjacent to N_2 if and only if there is an edge connecting $v(N_1)$ to $v(N_2)$ in the connectivity graph $\Gamma(S)$ (see the discussion following Proposition 1.9). In the next definition, the condition that the surface $S \subset M$ be strongly separating means that each component of $E(S, M)$ can be labeled with a $+$ or $-$ in such a way that no adjacent pair of connected components shares a common sign.

Definition 3.5. Let S be a strongly separating closed surface in a closed orientable 3-manifold M, and let L be a link in M transverse to S. If, for each component V_i of $E(S, M)$, the tangle $(V_i, L \cap V) = (V_i, T_i)$ is free, S is said to be a free decomposing surface for L, and $(S, (V_1, T_1), \cdots, (V_n, T_n))$ is a free tangle decomposition of L.

The following somewhat technical lemma is of central importance to the work that follows; it can be viewed as an extension of Proposition 3.5 of Kobayashi's paper [12] to our more general setting:

Proposition 3.6. *Suppose L is a link in a closed orientable 3-manifold M with components $L_1, \cdots, L_n$ and a free decomposition $(S, (V_1, T_1), \cdots, (V_k, T_k))$. Suppose* *further that there is a collection of arcs* $Y = \{y_1, \cdots y_n\}$ *in* $T_1 \cup \cdots \cup T_k$ *satisfying the following conditions.*

- *For all* $1 \leq i \leq n$, y_i *is a subarc of* L_i.
- *For all* $1 \leq j \leq k$, $(V_j, T_j \setminus Y)$ *is trivializing in* (V_j, T_j).

Further, let $\{K_i\}_{i=1}^n$ *be a collection of knots in* M_i *with bridge surfaces* $\{F_i\}_{i=1}^n$ *satisfying* $|K_i \cap F_i| = |L_i \cap S|$ *for each* $1 \leq i \leq n$.

Then the connect sum $L' = L\#(K_1, \cdots, K_n)$ *can be taken in such a way that the surface* S' *which is the image after gluing of* $\overline{S \setminus N(L)} \cup \overline{F_1 \setminus N(K_1)} \cup \cdots \cup \overline{F_n \setminus N(K_n)}$ *becomes a closed connected Heegaard surface of* $E(L')$.

Proof. For each K_i, let $E(F_i) = W_i^1 \cup W_i^2$, and choose any component w_i of $K_i \cap W_i^1$. Let R_i^1 be the sphere $\partial N(K_i \setminus w_i)$, and R_i^2 be the sphere $\partial N(L_i \setminus y_i)$. Then the connect sum in question can be obtained by attaching $E(K_i)$ to $E(L_i)$ via an orientation reversing map $h_i : (R_i^1, F_i \cap R_i^1) \to (R_i^2, S \cap R_i^2)$, which ensures that all of the meridional surfaces in question get glued into a single closed surface in $E(L')$.

We assumed in Definition 3.3 that S strongly separates M into non-adjacent $+$ and $-$ components. Without loss of generality suppose $\mathbf{V}_+ = \{V_1, \cdots V_l\}$ and $\mathbf{V}_- = \{V_{l+1}, \cdots, V_k\}$ are the sets of $+$ and $-$ components, respectively. If, for a given i, y_i lies in a $+$ component of $E(S)$, then h_i glues W_i^1 only to $+$ components, and W_i^2 only to $-$ components. The situation is reversed if y_i lies in a $-$ component, but in either case W_i^1 and W_i^2 can be labeled consistently with the components of $E(S)$ to which they are connected, and so the complementary components of the resulting

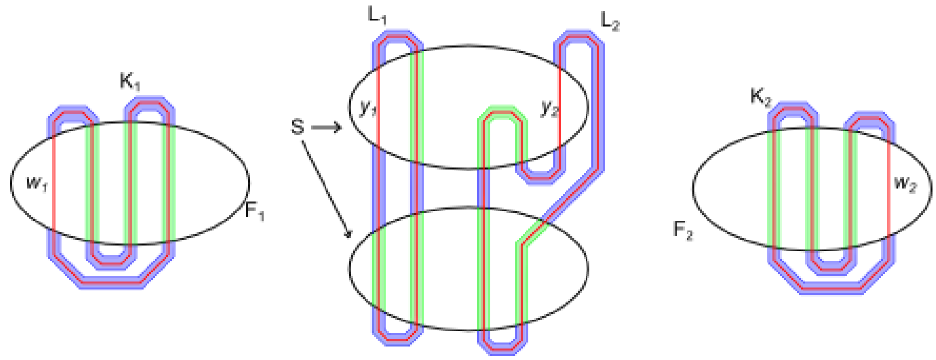

Figure 3.1: Schematic diagram for the case $n = 2$. The ovals represent the free decomposing surfaces. The knots and links are represented by the curves winding through the surfaces, with the spheres R_i^j tubed along most of their length.

surface S' can be labeled with $+$ and $-$ as required for S' to be strongly separating in M'.

We will next show that each component of $E(S')$ is a handlebody. This automatically implies that S' is connected, otherwise some component of $E(S')$ would have multiple boundary components, contrary to the fact that it is a handlebody.

By hypothesis, $E(T_j \setminus Y, V_j)$ is trivializing in the handlebody $E(T_j, V_j)$, and so by Proposition 3.4 $E(T_j \setminus Y, V_j)$ is itself a handlebody for all $1 \leq j \leq k$. Likewise, for all $1 \leq i \leq n$, the tangles (W_i^1, T_i^1) and (W_i^2, T_i^2) are free and in fact trivial, where T_i^1 is the collection of arcs in $(K \cap W_i^1) \setminus w_i$ and T_i^2 is the collection of arcs in $(K \cap W_i^2)$.

It follows that the scar sets $Sc(T_i^1), Sc(T_i^2)$ form a primitive collection of annuli on the boundaries of the handlebodies W_i^1, W_i^2, respectively, for all $1 \leq i \leq n$. Since U_1 and U_2 are obtained by gluing these handlebodies to the handlebodies $E(T_j \setminus Y, V_j)$, Proposition 1.21 implies that each of them is a handlebody, as required.

All that remains is to show that the collection of curves $L' \cap U_i$ forms a core of Y_i for $i = 1, 2$. We prove this for U_1, the proof for U_2 being identical. Without loss of generality suppose that $L' \cap U_1 = L'_1 \cup \cdots \cup L'_m$, where L'_i is the image in U_1 of $y_i \cup w_i$ after gluing for all $1 \leq i \leq m$.

By hypothesis, for each i, w_i cobounds a disk D_i with an arc on F_i and y_i cobounds a disk D'_i with an arc on $\partial E(T_j \setminus Y, V_j)$ for some j, and in such a way that the disks in the resulting collection $\{D'_i\}$ are pairwise disjoint. Moreover, after a small isotopy of these disks we can ensure that $\alpha_i^1 = D_i \cap R_i^1$ and $\alpha_i^2 = D'_i \cap R_i^2$ each consists of a pair of arcs, with $h_i(\alpha_i^1) = \alpha_i^2$.

Furthermore, since the tangle $(W_i^1, K_i \cap W_i^1)$ is trivial, we can find a disjoint collection of compressing disks $D_i''^1 \cup \cdots \cup D_i''^{p(i)}$ (some pairs of which may be parallel), each of which intersects the collection of annuli $Sc(K_i \cap W_i^1) \subset \partial E(K_i \cap W_i^1, W_i^1)$ in exactly one essential arc, and such that $h_i^{-1}(\partial D'_i \cap Sc(T_j \setminus Y)) = (\partial D_i''^1 \cup \cdots \partial D_i''^{p(i)}) \cap Sc(K_i \cap W_i^1)$.

By our construction, then, the image of $D_i \cup D'_i \cup D_i''^1 \cup \cdots \cup D_i''^{p(i)}$ after gluing along h_i will be an annulus A_i in U_1 having L'_i as one boundary component, and a simple closed curve in S' as the other component. Moreover, the annuli A_i, A_j are disjoint for $i \neq j$, so it follows that $L' \cap U_1$ can be isotoped along these annuli onto ∂U_1.

Finally note that for each component L'_i, the closure of either of the two open disk components of $R_i^1 \setminus F_i$ will become a compressing disk E_i for U_1 that intersects A_i in exactly one essential arc. Thus in fact $L' \cap U_1$ has been isotoped onto a primitive collection of simple closed curves on ∂U_1, which by Proposition 1.20 implies it is a core of U_1 as required.

$\square$

Free decompositions of links place upper bounds on their tunnel number, as was shown by Morimoto in the case when the decomposing surface is a single sphere in [20]. His methods do not extend to our more general case. However a bound does exist in terms of arbitrary free decomposing surfaces.

Proposition 3.7. *Let $L \subset M$ be a link with a free decomposing surface S. Then $g(E(L)) \leq 1 + |L \cap S| - \frac{\chi(S)}{2}$.*

Proof. For each component L_i, pick a single small subarc γ_i of some component of $L_i \setminus S$, and add the spheres $S_i = \partial N(\gamma_i))$ to S to obtain a new free decomposing surface $\tilde{S}$ satisfying the hypothesis of Proposition 3.6. Corresponding to each L_i, let K_i be the unknot in S^3 together with an $|L_i \cap \tilde{S}|/2$ bridge sphere F_i.

Taking the connect sum of L with the K_i as in Proposition 3.6 yields back L again, together with a Heegaard surface for S' for $E(L)$. We now compute

$$\chi(S') = \chi(\tilde{S} \setminus L) + \sum_i \chi(F_i \setminus K_i)$$

and since

$$\chi(\tilde{S} \setminus L) = \chi(S \setminus L),\ \chi(F_i \setminus K_i) = 2 - |K_i \cap F_i| = -|L_i \cap S|$$

we obtain

$$g(S') = 1 - \tfrac{\chi(S')}{2} = 1 - \tfrac{\chi(S \setminus L) - |L \cap S|}{2}$$

and, since $\chi(S \setminus L) = \chi(S) - |L \cap S|$, we deduce the desired inequality.

$\square$

The small spheres added to S to obtain $\tilde{S}$ can always be chosen so that every component of L lies on the same side of S'. So we can in fact conclude $t(L) \leq |L \cap S| - \frac{\chi(S)}{2}$.

3.2 The Second Construction

The links we find below that admit high degeneration are of the following kind:

Definition 3.8. A free decomposing surface S for a link L is optimal if $t(L) = |L \cap S| - \frac{\chi(S)}{2}$.

Proposition 3.9. *If an n-component link L admits an optimal free decomposing surface S, then there exists a collection of knots $K_1, \cdots, K_n$ such that*

$$d_L(K_1, \cdots, K_n) \geq \frac{|S \cap L|}{3|S \cap L| - \chi(S)}$$

Proof. Follow the proof of Proposition 3.7 exactly, except let your K_i be $|L_i \cap \tilde{S}|/2$ bridge, tunnel number $|L_i \cap \tilde{S}|/2 - 1$ knots in S^3, where $\tilde{S}$ is the same modified surface described there.

By hypothesis $t(L) = |L \cap S| - \frac{\chi(S)}{2}$ and $t(K_i) = |L_i \cap S|/2$, while the remark following Proposition 3.7 shows that $t(L\#(K_1, \cdots, K_n) \leq |L \cap S| - \frac{\chi(S)}{2}$. This yields the desired result.

$\square$

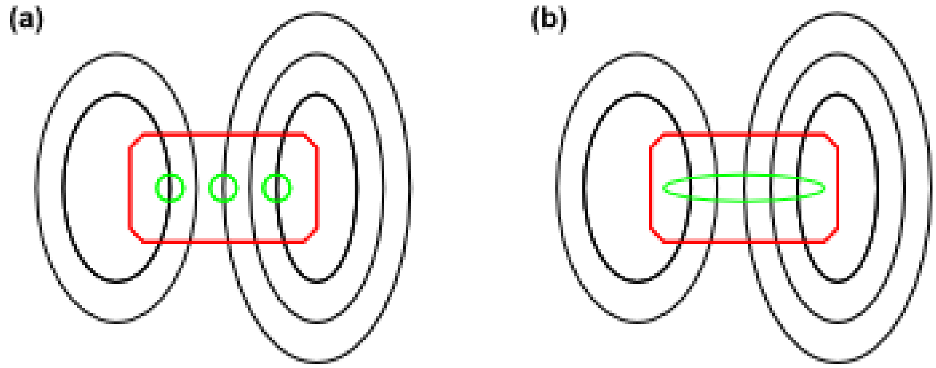

Figure 3.2: Schematic diagram depicting the links constructed in Propositions 3.10 and 3.11.

Proposition 3.10. *For all integers $n > 0$, there exist $n+1$ component links L and knots $K_1, \cdots, K_{n+1}$ in S^3 such that*

$$d_L(K_1, \cdots, K_{n+1}) \geq \tfrac{3n-1}{7n-2}$$

Proof. Let $J_1, \cdots, J_{2n}$ be a collection of knots in S^3 satisfying $t(J_i) = 1$ for all i and $t(J_1 \# \cdots \# J_{2n}) = 4n-1$, which exist by Proposition 2.6. Let $S = S_1 \cup \cdots \cup S_{2n-1}$ be a collection of decomposing spheres for the connect sum $J = J_1 \# \cdots \# J_{2n}$ satisfying the property that, for each i, S_i bounds a ball B satisfying $B \cap S = S_1 \cup \cdots \cup S_{i-1}$, i.e. let the S_i be nested as in Figure 3.2. Let $W_1, \cdots, W_{2n}$ be the closures of the components of $E(J) \setminus S$, labeled so that $E(J_i) \cong W_i$ and $W_i \cap W_{i+1} = \overline{S_i \setminus N(J)}$.

Since $t(J_i) = 1$, each W_i admits an arc t_i such that (W_i, t_i) is free. Moreover, for each odd i the arcs t_i and t_{i+1} can be properly isotoped in W_i and W_{i+1} respectively so that $\partial t_i = \partial t_{i+1}$. The result will be that the union $t_i \cup t_{i+1} = L_{(i+1)/2}$, forms a closed loop in S^3 for each $i \equiv 1 \pmod 2$, see Figure 3.2(a). Let L be the link $J \cup L_1 \cup \cdots \cup L_n$.

Clearly $t(L) \geq t(J)$, since any Heegaard splitting for L is also a Heegaard splitting for K, and by Proposition 3.7, $t(L) \leq t(J)$ as well, since S is a free decomposing surface for L, in fact an optimal one. The inequality now follows from Proposition 3.9.

□

Proposition 3.11. *For all integers $n > 0$ there are two component links $L \subset S^3$ with $t(L) = 3n$ and pairs of knots $K_1, K_2 \subset S^3$ such that $d_L(K_1, K_2) \geq 2/5$.*

Proof. The construction is nearly identical to that of Proposition 3.11, except we start by taking the connect sum $J = J_1 \# \cdots \# J_{n+1}$ with $t(J_1) = t(J_{n+1}) = 1$, $t(J_i) = 2$ for $1 < i < n+1$, and $t(J) = 3n$. The difference is that after decomposing the connect sum along nested spheres as in Proposition 3.11, the tunnels in W_i can be slid together to form a single loop instead of many, as in Figure 3.2(b).

□

3.3 Knots instead of Links

Let L be a link with free decomposing surface S. Then we may regard $N(S) \cap L$ as a collection of trivial braids in $N(S) \cong S \times I$ (one for each component of $S \times I$). Substituting an arbitrary collection of braids for $N(S) \cap L$ yields another link for which S is also a free decomposing surface, see Figure 3.3 for an example.

Conjecture 3.12. *There exist knots K obtained from the links of Propositions 3.10 and 3.11 by braid substitutions which are optimal.*

Nogueira [24] has already given an affirmative answer to Conjecture 3.12 in the case $n = 1$ of Propositions 3.10 and 3.11 (which coincide). If an optimal knot

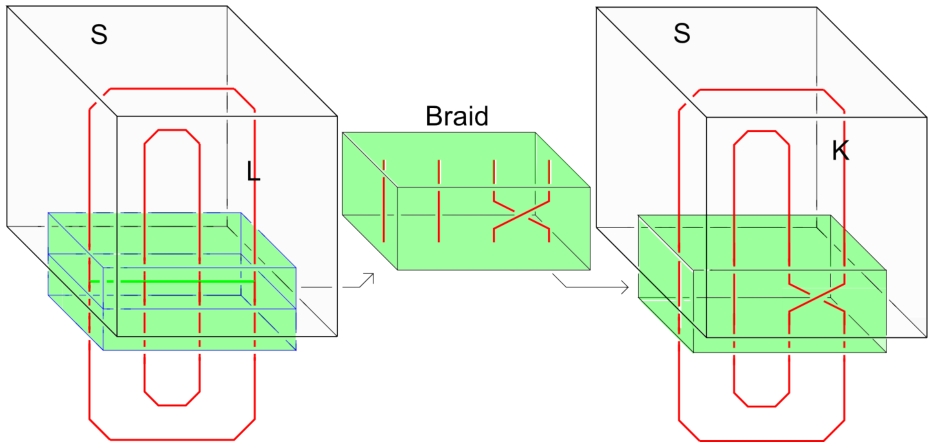

Figure 3.3: Braid Substitution

of the kind described existed for $n > 1$, then together with any knot K' satisfying $b(K') = t(K') + 1 = 3n$, it would achieve the highest degeneration ratio of any pair of knots found to date.

A somewhat surprising fact is that, as it stands, our constructions with links allow us to rediscover the subadditive knot pairs of Chapter 2. The Heegaard surface for $E(L\#(K_1, \cdots, K_n))$ in Proposition 3.10 is also a Heegaard surface for the exterior of each of its components, which includes the knot $J\#K_1$, where recall that $t(J) = 4n - 1$, and K_1 is any $2n$ bridge, tunnel number $2n - 1$ knot. Thus we can deduce $d(J, K) \geq \frac{2n-1}{6n-2}$, and though we constructed the Heegaard surfaces differently to prove it, this is just the lower bound found in Proposition 2.8.

CHAPTER 4
APPLICATIONS, OBSERVATIONS, AND CONJECTURES

In this final chapter we discuss the connection between certain classes of optimal knots and integral Dehn fillings. We then discuss the generality of the constructions described here, which encompass all known examples of tunnel number subadditivity. We conclude with some conjectures.

4.1 Dehn Fillings

This section is devoted to showing that, if a link L is optimal with respect to a connected free decomposing surface F, then every integral Dehn filling of any m components of L results in a manifold M with the property that $g(M) \leq g(E(L))-m$. This proposition actually arises in the course of a proof that, with some refinements, free decompositions can in fact be used to place even stronger upper bounds on tunnel number that those derived in Chapter 3.

The following definition first occurs in [20].

Definition 4.1. A free tangle (M,T) is a type (n,d) tangle if $|T| = n$ and d is the minimum cardinality of $|T'|$ among all trivializing subtangles T' of T.

We always consider T to trivialize itself so an (n,n) tangle is one which has no proper trivializing subtangles. If T is already trivial then it is trivialized by the empty set and is thus type $(n,0)$.

The following proposition is a substantial generalization of a proposition stated by Morimoto in [20], and the proof required is fundamentally different from the one he sketched there. Recall that, by definition, we require all free decomposing surfaces $F \subset M$ to be strongly separating, so that the connected components of $M \setminus F$ can be labeled with a $+$ or $-$ in such a way that no two components with the same label are adjacent.

Proposition 4.2. *Let F be a free decomposing surface for a k-component link $L = L_1 \cup \cdots \cup L_k \subset M$, and let W be the closure of the union of the positively marked components of $M \setminus F$. Then if $(W, W \cap L) = (W, T)$ is a type (n, d) tangle, $g(E(L, M)) \leq 1 + n + d - \frac{\chi(F)}{2}$.*

Proof. First, let $M' = E(L) \cup_h (V_1 \cup \cdots \cup V_k)$ be any integral Dehn filling of $E(L)$, and let $L' = L'_1 \cup \cdots \cup L'_k$ be the image of the cores of the filling solid tori $V_1, \cdots, V_k$. It suffices to bound the Heegaard genus of $E(L', M')$, since it is homeomorphic to $E(L, M)$. Our strategy will be to construct a surface $F' = F'_1 \cup \cdots \cup F'_k$, with $F'_i \subset V_i$ for $1 \leq i \leq k$, so that the image F'' of $(F \setminus N(L)) \cup F'$ in M' after gluing is a Heegaard surface for M' with L' as a core.

Since the filling is an integral one and F is a meridional surface, we can parameterize $V_i \cong S^1 \times D_2$ in such a way that $h(F \cap \partial N(L_i))$ is a collection of $2m$ curves on $S^1 \times \partial D_2$ of the form $S^1 \times *$, i.e. they will appear as longitudes. More precisely, let $\Gamma = \gamma_1 \cup \cdots \cup \gamma_{2m}$ denote this collection of curves, where we can assume that $\gamma_j = S^1 \times \{e^{\frac{j}{m}\pi\sqrt{-1}}\}$ (here of course D_2 is regarded as the unit disk in the complex plane and S^1 is the unit circle). The next four paragraphs will describe the surface F'_i we need in detail.

Let α_j denote the subarc of ∂D_2 bounded by $e^{\frac{j}{m}\pi\sqrt{-1}}$ and $e^{\frac{j+1}{m}\pi\sqrt{-1}}$, and let

$A_j = S^1 \times \alpha_j$, $1 \leq j \leq 2m$. Then we may assume $h(Sc(T \cap L_i)) = A_1 \cup A_3 \cup \cdots \cup A_{2m-1}$.

Let ω_j denote the straight line segment in D_2 joining $e^{\frac{2j}{m}\pi\sqrt{-1}}$ and $e^{\frac{2j+1}{m}\pi\sqrt{-1}}$, $1 \leq j \leq m$, and let Y_j denote the solid torus bounded by the annuli $S^1 \times \omega_j$ and A_{2j} in V_i. Finally, for each ω_j with midpoint m_j, let a_j denote a point on ω_j between $e^{\frac{2j}{m}\pi\sqrt{-1}}$ and m, and let b_j denote a point on ω_j between m and $e^{\frac{2j+1}{m}\pi\sqrt{-1}}$.

If (W, T') is a trivializing subtangle of (W, T) with $|T'| = d$, then $h(Sc(T' \cap L_i))$ appears as a subcollection $\mathbf{A}'_T$ of the annuli $h(Sc(T \cap L_i)) = A_1 \cup A_3 \cup \cdots \cup A_{2m-1}$. The boundary components of each annulus A in $\mathbf{A}'_T$ will lie on an adjacent pair of tori Y_j and Y_{j+1} for some j; in this case let $\beta(A)$ denote the straight line segment in D_2 joining b_j to a_{j+1}. Let B denote the union of the arcs $\{1\} \times \beta(A) \subset S^1 \times D_2$ taken over all A in $\mathbf{A}'_T$. Intuitively speaking, B can be described as a set of "tubing arcs" for F'_i.

Set $X'_i = N(B) \cup Y_1 \cup \cdots \cup Y_m$, $X_i = \overline{V_i \setminus X'_i}$, and $F'_i = \partial X_i \cap \partial X'_i$. I claim that the image of $F \cup F'_1 \cup \cdots F'_k$ after gluing is a Heegaard surface for M'.

For each i, note first that both X_i and X'_i are handlebodies, with X_i and X'_i being glued to the collections of handlebodies $H = E(L \cap W, W)$ and $H' = \overline{E(L, M) \setminus H}$, respectively. By construction, the annuli along which X'_i is glued to H' are all primitive, and thus by Proposition 1.21 $H' \cup (X'_1 \cup \cdots \cup X'_k)$ is a collection of handlebodies (we will see later that it is in fact a single handlebody).

We prove that $H \cup_h (X_1 \cup \cdots \cup X_k)$ is a collection of handlebodies by applying Proposition 1.21 iteratively. If T' is the trivializing subtangle, then by our construction the annuli $h(Sc(T'))$ are primitive in X_i, with a dual disk collection $\mathbf{D}$ that is disjoint from the annuli in $h(Sc(T \setminus T'))$ (this was the purpose of tubing F_i' along the arcs B). Thus $H'' = H \cup_{h|Sc(T')} (X_1 \cup \cdots \cup X_k)$ is a collection of handlebodies by Proposition 1.21.

I claim further that the image of $Sc(T \setminus T')$ after gluing becomes a primitive collection of annuli on $\partial H''$ with dual disk collection $\mathbf{D}''$ disjoing from $h(Sc(T \setminus T'))$, so that Proposition 1.21 can again be applied to show that $H''/(x \sim h_{|Sc(T \setminus T')}) \cong H \cup_h (X_1 \cup \cdots \cup X_k)$ is a collection of handlebodies.

Since $T \setminus T'$ is trivial in $E(T', W)$, we can find a collection of compressing disks $\mathbf{D}'$ in $E(T, W)$ which intersect $Sc(T)$ only in essential arcs and are dual to $Sc(T \setminus T')$. Just as in the proof of Proposition 3.6, we can form a disjoint collection $\mathbf{E}$ of compressing disks in X_i (consisting of disks from $\mathbf{D}$ defined above, possibly with multiple parallel copies of some) and isotope them slightly to ensure that $h(\mathbf{D}' \cap Sc(T')) = \mathbf{E} \cap h(Sc(T'))$. The image in H'' of $\mathbf{D}' \cup \mathbf{E}$ after gluing is then the dual disk collection $\mathbf{D}''$ promised above.

It follows that both components of $E(F'', M')$ are collections of handlebodies, and it is easy to see that F'' is strongly separating, so just as in the proof of Proposition 3.6 we deduce that F'' is connected and thus in fact a Heegaard surface for M' as required.

All that is left is to show that the core of each V_i (i.e. the component L_i' of L') can be isotoped to a primitive curve on F'', thus proving F'' to be a Heegaard splitting of $E(L', M')$. Using the same notation as in the first part of the proof, note that the core $S^1 \times \{0\}$ of V_i can be isotoped onto one of the punctured annuli

$(S^1 \times \omega_j) \setminus N(B)$. Notice now that $(\{1\} \times D_2) \cap X_i$ is a collection of disks, one of them containing the point in the center of D_2, call that one D_c. If $D_c \cap \partial D_2 = \emptyset$, then D_c is already dual to L_i', and we are done. If not then we can extend D_c to a compressing disk for F'' using the same extension technique as earlier in this proof and in the proof of Proposition 3.6, in fact the disk we want is the image of $D_c \cup \mathbf{D}''$ after possibly isotoping D_c slightly.

To compute the genus of F'', note first that

$$\chi(F'') = \chi(F) - |F \cap L| + \chi(F_1' \cup \cdots \cup F_k')$$

However, since $\chi(F_1' \cup \cdots \cup F_k') = -2m$, $|F \cap L| = 2n$, and $g(F'') = 1 - \frac{\chi(F'')}{2}$, we obtain

$$g(F'') = 1 + n + m - \tfrac{\chi(F)}{2}$$

□

We remark that, in the case that (W, T) is a type (n, n) tangle, the bound achieved here reduces to the same one found in Proposition 3.7, although here we have to content ourselves with a bound on $g(E(L))$ instead of a bound on $t(L)$, because we no longer have control over which side of our Heegaard surface the various components of L will fall into.

The proof of Proposition 4.2 has one more interesting upshot, however. In the case that (W, T) is a type (n, n) tangle, for each $1 \le i \le k$ we in fact use one more tube to construct F_i' than we need to in order to ensure that the eventual surface F'' is a Heegaard surface for M' (in the notation of that proof, a surface with one fewer tube still suffices to make the annuli $A_1 \cup A_3 \cup \cdots \cup A_{2m-1}$ primitive). The extra tubes merely serve to ensure that F'' is a Heegaard surface for $E(L', M')$. Thus we deduce the following:

Proposition 4.3. *If $L \subset M$ is a k-component link that admits a free decomposing surface F, and M' is any manifold obtained via integral Dehn filling of each component of L, then $g(M') \leq 1 + |L \cap F| - k - \frac{\chi(F)}{2}$.*

Definition 4.4. A free decomposing surface F for a link L is called Heegaard optimal if $g(E(L)) = 1 + |L \cap F| - \frac{\chi(F)}{2}$

This definition coincides with Definition 3.8 in the case that L is a knot. Moreover, all of the links we constructed in Chapter 3 were Heegaard optimal, and Nogueira has proved that the knots of Conjecture 3.12 are Heegaard optimal in the simplest case. Using Proposition 4.3 we may conclude the following about these knots and links.

Proposition 4.5. *If L is a k component, Heegaard optimal link, then for every manifold M' obtained by integral surgery on L, $g(E(L)) \geq g(M') + k$.*

4.2 Conclusions

Several remarks are in order to put the work done here into perspective. To begin with, all known examples of knots and links which experience degneration are constructed more or less as in Chapters 2 or 3.

As we saw in Chapter 2, degeneration can occur via weakly reducible splittings of $E(K_1 \# K_2)$. This construction does not appear to have any connections to surgery. Moreover, with one exception, it does not achieve very good degeneration when compared with the method of Chapter 3.

The exception involves Hopf links. Indeed, if K is a tunnel number n knot with a minimal genus splitting F that admits a core of n components lying on the same side of F as K, then just as in Proposition 2.1 we may take the connect sum of K with n Hopf links $L_1, \cdots, L_n$ along that core, and since the exterior of a Hopf link admits a genus one Heegaard surface parallel to both of its boundary components, amalgamation just yields back the surface F, and every component of the resulting link $K\#L_1\#\cdots\#L_n$ will lie on the same side of F. Thus $t(K\#L_1\#\cdots\#L_n) = t(K) = n$, and since $t(L_i) = 1$ for each $1 \leq i \leq n$, we obtain a degeneration ratio of 1/2.

Thus it appears that the construction of Chapter 2 dwarfs that of Chapter 3 in terms of creating degeneration in links. However it is not difficult to see that, for the hopf link L, $t(L)+1 > g(E(L))$, whereas typicaly $t(L)+1 = g(E(L))$, and so this degeneration phenomenon is actually just due to the pathological behavior of tunnel number with respect to Heegaard genus. If one switches over to the 3-dimensionally more natural link invariants $g(E(L))$ or $g(E(L))-1$, this degeneration becomes much less impressive or disappears entirely.

On the other hand the construction in Chapter 3 is very general. In addition to recovering the subbaditive knot pairs found in Chapter 2, Morimoto's original class of examples in [23] can easily be recovered and expanded using Proposition 3.6 as follows.

Proposition 4.6. *[20] Let $K \subset S^3$ be a knot which admitting a free decomposition $(S, (B_1, T_1), (B_2, T_2))$, where S is a sphere, (B_1, T_1) is type $(1, 1)$, and (B_2, T_2) is nontrivial. Then $t(K) = 2$ and, for any two bridge knot K', $t(K \# K') = 2$.*

Proof. The fact that $t(K) = 2$ follows from Proposition 4.2 and a result in Scharlemann's paper [30] that tunnel number one knots are doubly prime, since the existence of S implies that K is not doubly prime. The fact that $t(K \# K') \leq 2$ then follows from Proposition 3.6, and equality follows from the fact that tunnel number one knots are prime [25]. □

Indeed, the construction of Chapter 3 appears to be somewhat generic, as Proposition 4.8 below will attest to. To prove it, we need some outside help from Schultens.

Proposition 4.7. *[35] Let F be an incompressible surface properly embedded in a compression body H with $\partial F \subset \partial_+ H$. Then every component of $E(F)$ is a compression body.*

An annulus A properly embedded in a compression body H is said to be <u>spanning</u> if one component of ∂A lies on $\partial_+ H$ and the other on $\partial_- H$.

Proposition 4.8. *Let $K_1 \subset M_1$, $K_2 \subset M_2$ be a pair of knots in the closed 3-manifolds M_1, M_2 which contain no nonseparating spheres, and suppose F is a Heegaard surface for $E(K_1 \# K_2) \subset M_1 \# M_2$ which intersects the decomposing annulus A only in curves that are essential in A. Then every component of $E(A \cup F, E(K_1 \# K_2))$ is a handlebody.*

Proof. Let $E(F, E(K_1 \# K_2)) = H_1 \cup H_2$, where H_1 is the compression body and H_2 the handlebody. Then since neither of M_1 or M_2 contains a nonseparating sphere, the collection of annuli $A \cap H_i$ is incompressible in H_i for $i = 1, 2$. Moreover, every component of $E(A \cap H_2, H_2)$ is a compression body by Proposition 4.7 and, having connected boundary, is in fact a handlebody. On the other hand there are two spanning annuli A_1, A_2 in $A \cap H_1$, so we can only deduce that $E((A \cap H_1) \setminus (A_1 \cup A_2), H_1)$ consists of compression bodies. All of them will have connected boundary except for one special component V which has $\partial_- H_1 = \partial N(K_1 \# K_2)$ in its boundary, and thus all except V are handlebodies. Thus we will be done with our proof once we show that the components of $E(A_1 \cup A_2, V)$ are handlebodies.

Since $E((A_1 \cup A_2) \cap \partial_- H_1, \partial_- H_1)$ consists of exactly two annuli components A_1' and A_2', and since $A \cap H_1$ separates H_1, it must be the case that $E(A_1 \cup A_2, V)$ consists of exactly two components V_1 and V_2, which we label so that $A_1' \subset \partial V_1$ and $A_2' \subset \partial V_2$.

Let $T \times I$ be small collar neighborhood of $\partial_- H_1$ with $T \times \{0\} = \partial_- H_1$. Then $T \times \{1\} \setminus (A_1 \cup A_2)$ also consists of two (open) annuli components; label their respective closures T_1, T_2 in such a way that $T_i \cap V_{3-i} = \emptyset$.

Let $T_i' = T_i \cup ((A_1 \cup A_2) \setminus T \times I)$. Then $E(T_i', V)$ consists of two components, one of them homeomorphic to V_i. Moroever, T_i' is an incompressible annulus properly embedded in $\partial_+ V$ (as above its compressibility would imply the existence of a non-separating sphere in M_i, contrary to our assumption), thus Proposition 4.7 tells us that each complementary component is a compression body. Thus V_i, having connected boundary, is in fact a handlebody for $i = 1, 2$. The proof is finished.

□

What this proposition tells us is that, after identifying the two components of $E(A, E(K_1 \# K_2))$ with $E(K_1)$ and $E(K_2)$ and attaching meridian disks of $N(K_i)$ to $F_i = F \cap E(K_i) \subset M_i$ along its boundary, we get a free decomposing surface for K_i, $i = 1, 2$. Thus, for knots pairs K_1, K_2 which have minimal genus Heegaard splittings that intersect the decomposing annulus essentially, something like the construction of Chapter 3 must occur for there to be degeneration. Based on this we conjecture the following.

Conjecture 4.9. *If $K_1 \# K_2$ admits a minimal genus Heegaard splitting which intersects the decomposing annulus only in essential circles, $t(K_1 \# K_2) \geq \max(t(K_1), t(K_2))$.*

In particular, when $E(K_1 \# K_2)$ admits a strongly irreducible splitting the Heegaard surface can always be made to intersect the decomposing annulus essentially (assuming neither of the manifolds in which the K_i are embedded contains non-separating spheres, for in that case the decomposing annulus may not be incompressible).

Throughout this thesis we have taken a geometric approach in order to bring to light the connection between free decompositions, tunnel number, and Dehn surgery at the end. However it is intriguing that many of the results in Chapter 3 could have been obtained instead by group theoretic means alone. The rank of a knot or link L, denoted $rk(L)$ is the minimal number of generators required to generate its fundamental group. We allow ourselves to speculate wildly here without giving more precise reasons:

Conjecture 4.10. *For optimal links L, $g(E(L)) > rk(L)$*

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