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UNIVERSAL DEFORMATION RINGS OF MODULES FOR ALGEBRAS OF DIHEDRAL TYPE OF POLYNOMIAL GROWTH

by

Shannon Nicole Talbott

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

July 2012

Thesis Supervisor: Professor Frauke Bleher

ABSTRACT

Deformation theory studies the behavior of mathematical objects, such as representations or modules, under small perturbations. This theory is useful in both pure and applied mathematics and has been used in the proof of many long-standing problems. In particular, in number theory Wiles and Taylor used universal deformation rings of Galois representations in the proof of Fermat's Last Theorem. The main motivation for determining universal deformation rings of modules for finite dimensional algebras is that deep results from representation theory can be used to arrive at a better understanding of deformation rings. In this thesis, I study the universal deformation rings of certain modules for algebras of dihedral type of polynomial growth which have been completely classified by Erdmann and Skowroński using quivers and relations.

More precisely, let k be an algebraically closed field and let Λ be a k-algebra of dihedral type which is of polynomial growth. In this thesis, I first classify all Λ-modules whose stable endomorphism ring is isomorphic to k and which are given combinatorially by strings, and then I determine the universal deformation ring of each of these modules.

Abstract Approved: ______________________________

Thesis Supervisor

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Title and Department

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Date

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A thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
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Thesis Supervisor: Professor Frauke Bleher

Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Shannon Nicole Talbott

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the July 2012 graduation.

Thesis Committee: ______________________________
Frauke Bleher, Thesis Supervisor

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Dan Anderson

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Frederick Goodman

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Paul Muhly

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Victor Camillo

To Papa and Gram
both of whom I wish to be when I grow up

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Throughout graduate school, there have been plenty of rough times. There have been late night study sessions, frustrating homework with a solution almost within my grasp, barriers and breakthroughs in research, etc. My friends with whom I have taken this journey, thank you for sharing these times as well as all of the successes which were made all the sweeter by the work required to achieve them. I want to thank you for letting me borrow your strength when I did not have my own.

ABSTRACT

Deformation theory studies the behavior of mathematical objects, such as representations or modules, under small perturbations. This theory is useful in both pure and applied mathematics and has been used in the proof of many long-standing problems. In particular, in number theory Wiles and Taylor used universal deformation rings of Galois representations in the proof of Fermat's Last Theorem. The main motivation for determining universal deformation rings of modules for finite dimensional algebras is that deep results from representation theory can be used to arrive at a better understanding of deformation rings. In this thesis, I study the universal deformation rings of certain modules for algebras of dihedral type of polynomial growth which have been completely classified by Erdmann and Skowroński using quivers and relations.

More precisely, let k be an algebraically closed field and let Λ be a k-algebra of dihedral type which is of polynomial growth. In this thesis, I first classify all Λ-modules whose stable endomorphism ring is isomorphic to k and which are given combinatorially by strings, and then I determine the universal deformation ring of each of these modules.

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CHAPTER 1
INTRODUCTION

1.1 Motivation

Deformation theory studies the behavior of mathematical objects, such as representations or modules, under small perturbations. The motivation for this work is to use methods from the representation theory of finite dimensional algebras to study universal deformation rings of modules over such algebras.

More precisely, let k be an algebraically closed field. In [5, Corollary 4.7], Erdmann and Skowroński classified all finite-dimensional k-algebras Λ of dihedral type which are of polynomial growth. The goal of this thesis is to find all Λ-modules whose stable endomorphism rings are isomorphic to k, and then to determine the universal deformation ring for each of these modules.

1.2 Overview

In Chapter 2, some necessary definitions and results are given which will be used in the remainder of the thesis. We define symmetric algebras, syzygies, stable endomorphism rings, quivers, path algebras, Auslander-Reiten quivers, special biserial algebras, and universal deformation rings.

In Chapter 3, we consider all algebras $\Lambda = kQ/I$ of dihedral type which are of polynomial growth. Each quiver Q has 1, 2, or 3 vertices. After giving an introduction in section 3.1, we concentrate in section 3.2 on one particular algebra $\Lambda = kQ/I$ for which Q has 3 vertices. We find all string modules $M(S)$ for Λ with stable endomorphism ring k and determine the universal deformation ring $R(\Lambda, M(S))$ for $M(S)$. In sections 3.3 and 3.4, we perform the same calculations for one particular algebra $\Lambda = kQ/I$ for which Q has 2 vertices, respectively 1 vertex. In section 3.5, we provide for each of the remaining algebras Λ of dihedral type of polynomial growth a list of all string modules $M(S)$ whose stable endomorphism rings are isomorphic to k and for each such $M(S)$ its universal deformation ring $R(\Lambda, M(S))$.

CHAPTER 2
DEFINITIONS AND BACKGROUND

Let k be an algebraically closed field and let Λ be a finite dimensional k-algebra. All modules will be finitely generated left modules. Denote the category of all finitely generated Λ-modules by Λ-mod.

2.1 Frobenius and Symmetric Algebras

Definition 2.1. The algebra Λ is said to be a Frobenius algebra if there is a linear map $\lambda : \Lambda \to k$ such that

(i) $Ker(\lambda)$ has no non-zero left or right ideal.

We say Λ is symmetric if it satisfies (i) together with

(ii) For all $a, b \in \Lambda$, $\lambda(ab) = \lambda(ba)$.

We say Λ is self-injective if it is injective as a module over itself, that is if ${}_\Lambda\Lambda$ is an injective Λ-module.

Proposition 2.2. *(i) If Λ is a Frobenius algebra over k then Λ is self-injective.*

(ii) Suppose Λ is self-injective and let M be a finitely generated Λ-module. The following are equivalent:

(a) M is projective.

(b) M is injective.

(c) $Hom_k(M, k)$ is projective.

(d) $Hom_k(M, k)$ is injective.

Proof. See [2, Proposition 1.6.2] □

2.2 Module Theory

Definition 2.3. Let M be a Λ-module.

(i) The socle of M, denoted by $soc(M)$, is the submodule of M generated by all semisimple submodules of M.

(ii) The radical of M, denoted by $rad(M)$, is the intersection of all the maximal submodules of M.

(iii) The top of M, denoted by $top(M)$, is the quotient module $M/rad(M)$.

Definition 2.4. (i) An epimorphism $f : A \to B$ in Λ-mod is called an essential epimorphism if $Ker(f) \subseteq rad(A)$.

(ii) Let M be a Λ-module. A projective cover of M is a projective Λ-module P together with an essential epimorphism $f : P \to M$, denoted by (P, f).

Theorem 2.5. *Every finitely generated Λ-module M has a projective cover (P, f). Moreover, any two projective covers (P_1, f_1) and (P_2, f_2) are isomorphic, in the sense that there exists a Λ-module isomorphism $\pi : P_1 \to P_2$ with $f_2 \circ \pi = f_1$.*

Proof. See [1, Theorem I.4.2]. □

Remark 2.6. If M is a Λ-module, then (P, f) is a projective cover of M if and only if the epimorphism $P/rad(P) \to M/rad(M)$ induced by f is an isomorphism (see [1, Proposition I.4.3]).

Definition 2.7. Let M be a finitely generated Λ-module. The first syzygy of M, denoted by $\Omega(M)$, is the kernel of a projective cover $f : P \to M$. By Theorem 2.5, $\Omega(M)$ is unique up to isomorphism.

Definition 2.8. Let M and N be Λ-modules.

(i) Define $PHom_\Lambda(M, N)$ to be the k-subspace of $Hom_\Lambda(M, N)$ consisting of those Λ-module homomorphisms which factor through a projective Λ-module.

(ii) Define the k-vector space of stable homomorphisms from M to N to be the quotient space $\underline{Hom}_\Lambda(M, N) = Hom_\Lambda(M, N)/PHom_\Lambda(M, N)$.

If $M = N$ then $\underline{Hom}_\Lambda(M, M)$ is a k-algebra denoted by $\underline{\mathrm{End}}_\Lambda(M)$.

(iii) Define Λ-$\underline{\mathrm{mod}}$ to be the category with Ob(Λ-$\underline{\mathrm{mod}}$)=Ob(Λ-mod) and morphisms $Mor_{\Lambda-\underline{mod}}(M, N) = \underline{Hom}_\Lambda(M, N)$ for all objects M, N. The category Λ-$\underline{\mathrm{mod}}$ is called the stable module category.

Lemma 2.9. *There exists a functor $\Omega : \Lambda$-$\underline{mod} \to \Lambda$-$\underline{mod}$ such that $\Omega(M)$ is the first syzygy for all Λ-modules M. This functor is called the syzygy functor. If Λ is self-injective, then Ω is an equivalence.*

Proof. See [1, pp. 124-126]. □

Corollary 2.10. *If Λ is a self-injective algebra, then $\underline{\mathrm{End}}_\Lambda(M) \cong \underline{\mathrm{End}}_\Lambda(\Omega(M))$ for all finitely generated Λ-modules M.*

Definition 2.11. Let M and N be Λ-modules, and let

$$\cdots \to P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\epsilon} M \to 0$$

be a projective resolution of M. Applying $Hom_\Lambda(-,N)$ to the projective resolution of M, we obtain a sequence of k-vector spaces

$$0 \to Hom_\Lambda(P_0,N) \xrightarrow{\delta_1^\star} Hom_\Lambda(P_1,N) \xrightarrow{\delta_2^\star} Hom_\Lambda(P_2,N) \xrightarrow{\delta_3^\star} \cdots$$

For all $n \geq 0$, define $Ext^n_\Lambda(M,N) = Ker(\delta^\star_{n+1})/Im(\delta^\star_n)$. Since $Hom_\Lambda(-,N)$ is left exact, $Ext^0_\Lambda(M,N) = Ker(\delta_1^\star) = Im(\epsilon^\star) \cong Hom_\Lambda(M,N)$.

Theorem 2.12. *If Λ is self-injective, and M and N are Λ-modules, then*

$$Ext^n_\Lambda(M,N) \cong \underline{Hom}_\Lambda(\Omega^n(M),N)$$

for all $n \geq 0$.

Proof. See [6, Theorem 2.19]. □

2.3 Morita Equivalence

Definition 2.13. Let Λ_0 be another finite dimensional k-algebra. We say Λ and Λ_0 are Morita equivalent, written $\Lambda \sim_M \Lambda_0$, if Λ-mod and Λ_0-mod are equivalent categories.

Definition 2.14. Recall that k is an algebraically closed field. The k-algebra Λ is said to be a basic algebra if all simple Λ-modules are one dimensional over k.

Theorem 2.15. *Two basic algebras Λ and Λ_0 are Morita equivalent if and only if they are isomorphic.*

Proof. See [4, Lemma I.2.6]. □

Theorem 2.16. *There exists a unique basic algebra Λ_0 such that $\Lambda \sim_M \Lambda_0$.*

Proof. See [4, Corollary I.2.7]. □

2.4 Quivers and Path Algebras

Definition 2.17. A quiver is a directed graph $Q = (Q_0, Q_1, s, e)$ consisting of a set of vertices Q_0, a set of arrows Q_1, and two maps $s, e : Q_1 \to Q_0$ where s associates to each arrow $\alpha \in Q_1$ the vertex $s(\alpha)$ at which α begins, and e associates to each arrow $\alpha \in Q_1$ the vertex $e(\alpha)$ at which α ends. A quiver Q is said to be finite if Q_0 and Q_1 both are finite sets.

Definition 2.18. Let $Q = (Q_0, Q_1, s, e)$ be a quiver.

(i) A path of length $l \geq 1$ in Q from vertex a to vertex b is a sequence $(\alpha_l, \cdots, \alpha_2, \alpha_1)$ where $\alpha_j \in Q_1$ for all $1 \leq j \leq l$, and where $s(\alpha_1) = a$, $e(\alpha_l) = b$, and $e(\alpha_j) = s(\alpha_{j+1})$ for all $1 \leq j \leq l-1$. We denote this path by $\alpha_l \cdots \alpha_2\alpha_1$. We also associate a path of length $l = 0$ to each vertex $i \in Q_0$, which we call the trivial path at i and which we denote by e_i.

(ii) The path algebra kQ of Q is defined to be the k-vector space whose k-basis is the set of all paths in Q. The product of two paths $\alpha_l \cdots \alpha_2\alpha_1$ and $\beta_k \cdots \beta_2\beta_1$ in kQ is $\alpha_l \cdots \alpha_2\alpha_1\beta_k \cdots \beta_2\beta_1$ if $e(\beta_k) = s(\alpha_1)$ and 0 otherwise. This defines a k-algebra structure on kQ.

(iii) Let J be the ideal of the path algebra kQ which is generated by all paths of length 1. An ideal I of kQ is said to be admissible if there exists some $s \geq 2$ such that $J^s \subseteq I \subseteq J^2$.

Proposition 2.19. *Let Q be a quiver and let I be an admissible ideal of kQ. Then the image $\overline{J}$ of J in kQ/I is equal to $rad(kQ/I)$.*

Proof. See [1, Proposition III.1.6]. □

Definition 2.20. Let Q be a finite quiver. A representation $\mathcal{V} = (V_i, \phi_\alpha)$ of Q over k is defined as follows:

(a) To each vertex $i \in Q_0$ a k-vector space V_i is associated.

(b) To each arrow $\alpha : i \to j$ in Q_1 a k-linear map $\phi_\alpha : V_i \to V_j$ is associated.

The representation $\mathcal{V}$ is said to be finite dimensional if each vector space V_i is finite dimensional.

Let $\mathcal{V} = (V_i, \phi_\alpha)$ and $\mathcal{V}' = (V_i', \phi_\alpha')$ be two representations of Q over k. A morphism of representations $f : \mathcal{V} \to \mathcal{V}'$ is a family of k-linear maps $f_i : V_i \to V_i'$ for each $i \in Q_0$ such that $\phi_\alpha' \circ f_a = f_b \circ \phi_\alpha$ for each arrow $\alpha : a \to b$ in Q_1.

Let $Rep_k(Q)$ be the category of finite dimensional representations of Q over k.

Theorem 2.21. *The categories $Rep_k(Q)$ and $kQ - mod$ are equivalent categories.*

Proof. See [1, Theorem III.1.5]. □

Theorem 2.22. *(Gabriel) Suppose Λ is basic. Then there exists a unique quiver Q and an admissible ideal I such that $\Lambda \cong kQ/I$.*

Proof. See [1, Corollary III.1.10]. □

2.5 Auslander-Reiten Quivers

Definition 2.23. Let A, B, C be Λ-modules.

(i) A morphism $f : A \to B$ in Λ-mod is said to be left almost split if it is not a split monomorphism and any monomorphism $A \to Y$ in Λ-mod which is not split factors through f.

(ii) A morphism $g : B \to C$ in Λ-mod is said to be right almost split if it is not a split epimorphism and any epimorphism $X \to C$ in Λ-mod which is not split factors through g.

(iii) An exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in Λ-mod is called an almost split sequence if f is left almost split and g is right almost split.

Theorem 2.24. *(a) If C is an indecomposable non-projective Λ-module, then there exists an almost split sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in Λ-mod.*

(b) If A is an indecomposable non-injective Λ-module, then there exists an almost split sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in Λ-mod.

Proof. See [1, Theorem V.1.15]. □

Definition 2.25. Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an almost split sequence in Λ-mod. Define the Auslander translate τC of C to be $\tau C = A$, and define $\tau^{-1} A = C$.

Proposition 2.26. *If Λ is symmetric, then for each non-projective Λ-module C we have $\tau C \cong \Omega^2(C)$.*

Proof. See [1, Proposition IV.3.8]. □

Definition 2.27. A morphism $g : B \to C$ in Λ-mod is called irreducible if it is neither a split monomorphism nor a split epimorphism, and if $g = t \circ s$ for certain

$s : B \to X$ and $t : X \to C$ in Λ-mod, then s is a split monomorphism or t is a split epimorphism.

Theorem 2.28. *Let A and C be indecomposable Λ-modules.*

(i) A morphism $f : A \to B$ in Λ-mod is irreducible if and only if there exists a morphism $f' : A \to B'$ in Λ-mod such that the induced morphism $\begin{pmatrix} f \\ f' \end{pmatrix} : A \to B \coprod B'$ is a minimal left almost split morphism.

(ii) A morphism $g : B \to C$ in Λ-mod is irreducible if and only if there exists a morphism $g' : B' \to C$ in Λ-mod such that the induced morphism $(g, g') : B \coprod B' \to C$ is a minimal right almost split morphism.

Proof. See [1, Theorem V.5.3]. □

Definition 2.29. (i) The Auslander Reiten quiver of Λ is the quiver $\Gamma(\Lambda)$ whose vertices are the isomorphism classes of indecomposable Λ-modules. Denoting the vertex corresponding to an indecomposable Λ-module M by $[M]$, there is an arrow $[M] \to [N]$ between two vertices if and only if there is an irreducible morphism $M \to N$.

(ii) The stable Auslander Reiten quiver of Λ is the quiver $\Gamma_S(\Lambda)$ which is obtained from $\Gamma(\Lambda)$ by removing all vertices $[\tau^{-i}P]$ and $[\tau^i I]$ and all adjacent arrows for all projective Λ-modules P and all injective Λ-modules I, and all $i \geq 0$. In the special case when Λ is self-injective, one only has to remove the vertices $[P]$ for P projective and the adjacent arrows.

Remark 2.30. Suppose $\Lambda = kQ/I$ is a basic algebra. Define

$$L = \{i \in Q_0 | \Lambda e_i \text{ is injective}\}$$

and let $S = \bigoplus_{i \in L} soc(\Lambda e_i)$. Then the indecomposable Λ-modules are given by the indecomposable Λ/S-modules together with Λe_i, $i \in L$. Moreover, the Auslander-Reiten quiver $\Gamma(\Lambda/S)$ is obtained from $\Gamma(\Lambda)$ by removing the modules Λe_i, $i \in L$. If Λ is self-injective, then $S = soc(\Lambda)$ and $\Gamma(\Lambda/S)$ is the stable Auslander-Reiten quiver $\Gamma_S(\Lambda)$ (see [4, I.8.11]).

2.6 Special Biserial Algebras

Definition 2.31. The algebra Λ is said to be a special biserial algebra if its basic algebra has the form kQ/I, for a unique quiver Q and an admissible ideal I, satisfying the following conditions:

(i) At most two arrows start at any vertex i of Q, and at most two arrows end at any vertex i of Q.

(ii) Given an arrow β there is at most one arrow γ such that $s(\beta) = e(\gamma)$ and $\beta\gamma \notin I$, and given an arrow ϵ there is at most one arrow δ such that $s(\delta) = e(\epsilon)$ and $\delta\epsilon \notin I$.

Definition 2.32. The algebra Λ is called a string algebra if Λ is special biserial and its basic algebra has the form kQ/I where I is generated by paths.

Definition 2.33. Let $\Lambda = kQ/I$ be a basic string algebra. Given an arrow β of Q, let β^{-1} denote the formal inverse of β, and define $s(\beta^{-1}) = e(\beta)$, $e(\beta^{-1}) = s(\beta)$, and

$(\beta^{-1})^{-1} = \beta$. Define an alphabet for Λ by taking as letters the arrows of Q and their formal inverses.

(i) A word is defined to be a sequence $w = w_1 w_2 \cdots w_n$ where each w_i is either an arrow or a formal inverse, and where $s(w_i) = e(w_{i+1})$ for $1 \leq i \leq n-1$. We define $s(w) = s(w_n)$, $e(w) = e(w_1)$, and $w^{-1} = w_n^{-1} \cdots w_2^{-1} w_1^{-1}$. For each vertex i of Q, we define an empty word of length 0, denoted by e_i, where $s(e_i) = e(e_i) = i$ and $e_i^{-1} = e_i$.

(ii) We define an equivalence relation $\sim$ on the set of all words by $w \sim w'$ if $w = w'$ or $w^{-1} = w'$. We define an equivalence relation $\sim_r$ on the set of all words w of length at least 1 which satisfy $s(w) = e(w)$ by $w \sim_r w'$ if either w or w^{-1} is obtained from w' by a cyclic rotation.

(iii) Let $\mathcal{S}$ be a complete set of representative words $w = w_1 w_2 \cdots w_n$ under the relation $\sim$ such that either w is an empty word, or $w_i \neq w_{i+1}^{-1}$ for $1 \leq i \leq n-1$, and no subpath of w or w^{-1} belongs to I. The elements of $\mathcal{S}$ are called strings.

(iv) Let $\mathcal{B}$ be a complete set of representative words $w = w_1 w_2 \cdots w_n$ with $n \geq 1$ and $s(w) = e(w)$ under the relation $\sim_r$ such that $w_i \neq w_{i+1}^{-1}$ for $1 \leq i \leq n-1$, $w_n \neq w_1^{-1}$, w is not a power of a smaller word, and no subword of w^m belongs to I for all $m \geq 1$. The elements of $\mathcal{B}$ are called bands.

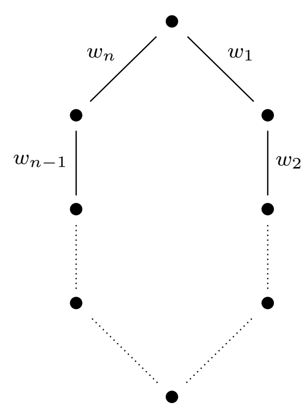
Definition 2.34. Let Λ be a string algebra.

(i) Let $w = w_1 w_2 \cdots w_n$ or $w = e_i$ be a string of length $n \geq 0$. Let Q_w be the quiver with underlying graph

where the edge labeled w_i points to the left if w_i is an arrow and to the right otherwise. Define a functor $G_w : k - mod \to \Lambda - mod$ as follows:

If V is a k-vector space, define $G_w(V)$ to be the Λ-module which is defined by the representation of Q_w, which assigns to each vertex of Q_w the vector space V and to each arrow in Q_w the identity linear transformation.

(ii) Let $w = w_1 w_2 \cdots w_n$ be a band, and assume, without loss of generality, that w_1 is an arrow. Let Q_w be the circular quiver



where the edge labeled w_i points counterclockwise if w_i is an arrow and clockwise otherwise. Define a functor $G_w : k[x, x^{-1}] - mod \to \Lambda - mod$ as follows:

If V is an object in $k[x, x^{-1}] - mod$, define $G_w(V)$ to be the Λ-module which is defined by the representation of Q_w, which assigns to each vertex of Q_w the vector space V and to the arrow w_1 the linear transformation x and to all other arrows the identity linear transformation.

Theorem 2.35. *Let Λ be a string algebra. Then the modules $G_w(V)$, where w ranges over all strings and bands, and $V = k$ if w is a string, and V ranges over all indecomposable $k[x, x^{-1}]$-modules if w is a band, form a complete set of representatives of indecomposable Λ-modules.*

Proof. See [3, Theorem on p. 161]. $\square$

Definition 2.36. Let Λ be a string algebra.

(i) If C is a string, then we define the string module $M(C)$ to be $G_C(k)$.

(ii) If B is a band, $\lambda \in k^\star$, and n is a positive integer, then we define the band module $M(B, \lambda, n)$ to be $G_B(V_n(\lambda))$ where $V = V_n(\lambda)$ is the $k[x, x^{-1}]$-module of k-dimension n on which x acts as the $n \times n$ Jordan block $J_n(\lambda)$.

2.7 Homomorphisms and Almost Split Sequences for String Modules

Let $\Lambda = kQ/I$ be a string algebra.

Definition 2.37. Let S and T be strings for Λ. Suppose C is a substring of both S and T such that the following conditions are satisfied:

(i) $S \sim BCD$, where B is a substring which is either of length 0 or $B = B'\tau$ for an arrow τ, and D is a substring which is either of length 0 or $D = \phi^{-1}D'$ for an arrow ϕ. In other words,

$$S \sim B' \xleftarrow{\tau} C \xrightarrow{\phi} D'.$$

(ii) $T \sim ECF$, where E is a substring which is either of length 0 or $E = E'\epsilon^{-1}$ for an arrow ϵ, and F is a substring which is either of length 0 or $F = \mu F'$ for an arrow μ. In other words,

$$S \sim E' \xrightarrow{\epsilon} C \xleftarrow{\mu} F'.$$

Then there exists a canonical Λ-module homomorphism

$$\alpha_C : M(S) \twoheadrightarrow M(C) \hookrightarrow M(T).$$

Theorem 2.38. *Each $f \in Hom_\Lambda(M(S), M(T))$ can be written uniquely as a k-linear combination of canonical Λ-module homomorphisms as in Definition 2.37. In particular, if $M(S) = M(T)$, then the canonical endomorphisms generate $End_\Lambda(M(S))$.*

Proof. See [7]. □

Definition 2.39. Let $C = c_1c_2 \cdots c_n$ be a string of length $n \geq 0$.

(i) We say that C starts on a peak if there is no arrow β such that $C\beta$ is a string, and we say that C ends on a peak if there is no arrow β such that $\beta^{-1}C$ is a string.

(ii) We say that C starts in a deep if there is no arrow γ such that $C\gamma^{-1}$ is a string, and we say that C ends in a deep if there is no arrow γ such that γC is a string.

(iii) We call C a directed string if all c_j are arrows, and we call C an inverse string if all c_j are formal inverses of arrows.

Definition 2.40. Let C be a string.

(i) If C does not start on a peak, then there is an arrow β such that $C\beta$ is a string. There is a unique directed string D such that $C\beta D^{-1}$ is a string starting in a deep. We use the notation $C_h = C\beta D^{-1}$ and say C_h is obtained from C by adding a hook on the right.

(ii) If C does not end on a peak, then there is an arrow β such that $\beta^{-1}C$ is a string. There is a unique directed string D such that $D\beta^{-1}C$ is a string ending in a deep. We use the notation ${}_hC = D\beta^{-1}C$ and say ${}_hC$ is obtained from C by adding a hook on the left.

(iii) If C does not start in a deep, then there is an arrow γ such that $C\gamma^{-1}$ is a string. There is a unique directed string D such that $C\gamma^{-1}D$ is a string starting on a peak. We use the notation $C_c = C\gamma^{-1}D$ and say C_c is obtained from C by adding a cohook on the right.

(iv) If C does not end in a deep, then there is an arrow γ such that γC is a string. Then there is a unique directed string D such that $D^{-1}\gamma C$ is a string ending on a peak. We use the notation ${}_cC = D^{-1}\gamma C$ and say ${}_cC$ is obtained from C by adding a cohook on the left.

Definition 2.41. We call the following exact sequences "canonical" exact sequences:

(i) Let α be an arrow starting at vertex i. The almost split sequence ending in $V_\alpha = \Lambda e_i/\Lambda\alpha$ is of the form

$$0 \to \tau(V_\alpha) \to N_\alpha \to V_\alpha \to 0$$

where N_α is a string module $M(S)$ such that S is a string of the form $S = D^{-1}\beta C$ and C and D are maximal directed strings. (See [3, section 1]).

(ii) If C is a string which neither starts nor ends on a peak, then C_h, ${}_hC$, and ${}_hC_h$ exist, and we have an exact sequence

$$0 \to M(C) \to M({}_hC) \oplus M(C_h) \to M({}_hC_h) \to 0.$$

(iii) If C is a string which does not start on a peak but ends on a peak, then C_h exists. We can write $C = {}_cD$ for some string D not starting on a peak, and hence D_h exists. We have an exact sequence

$$0 \to M(C) \to M(D) \oplus M(C_h) \to M(D_h) \to 0.$$

(iv) If C is a string which starts on a peak but does not end on a peak, then ${}_hC$ exists. We can write $C = D_c$ for some string D not ending on a peak, and hence ${}_hD$ exists. We have an exact sequence

$$0 \to M(C) \to M({}_hC) \oplus M(D) \to M({}_hD) \to 0.$$

(v) If C is a string which both starts on a peak and ends on a peak, then we can write $C = {}_cD_c$ for some string D. We have an exact sequence

$$0 \to M(C) \to M(D_c) \oplus M({}_cD) \to M({}_cD_c) \to 0.$$

Theorem 2.42. *The canonical exact sequences are the almost split sequences containing string modules.*

Proof. See [3, Proposition on p. 172]. □

Remark 2.43. Let Λ be a self-injective basic k-algebra, let $S = soc(\Lambda)$, and define $\Lambda_0 = \Lambda/S$. By Remark 2.30, the non-projective indecomposable Λ-modules are precisely the indecomposable Λ_0-modules, and $\Gamma(\Lambda_0)$ is the stable Auslander-Reiten quiver $\Gamma_S(\Lambda)$.

Suppose now that Λ_0 is a string algebra. Then we call the string modules for Λ_0 also string modules for Λ. In particular, all of the above definitions and results about the homomorphisms and almost split sequences for string modules for Λ_0 remain valid when we view these modules as Λ-modules by inflation.

2.8 Universal Deformation Rings

Let $\hat{\mathcal{C}}$ be the category of all complete local commutative Noetherian k-algebras with residue field k, where the morphisms are continuous k-algebra homomorphisms inducing the identity on k. Let Λ be a finite dimensional k-algebra, and let V be a finitely generated Λ-module. Then V is finite dimensional as a k-vector space, say $dim_k V = m$ which means that we can identify $V = k^m$ as a k-vector space. Moreover, the Λ-module structure of V is given by the k-algebra homomorphism $\Lambda \xrightarrow{\alpha_V} End_k(V) = Mat_{m\times m}(k)$ with $\alpha_V(x) = A_x$ for all $x \in \Lambda$, where $A_x v = xv$ for all $v \in V$. Let $R \in Ob(\hat{\mathcal{C}})$ and let $\pi : R \to k$ be the natural surjection onto the residue field. Define $R\Lambda = R \otimes_k \Lambda$.

Definition 2.44. (i) A lift of V over R is a free R-module $M = R^m$, where $m = dim_k V$ as above, together with a k-algebra homomorphism $\Lambda \xrightarrow{\alpha_M} End_R(M) =$

$Mat_{m\times m}(R)$ such that $\pi \circ \alpha_M = \alpha_V$.

Put differently, a lift of V over R is a finitely generated $R\Lambda$-module M, which is free as an R-module, together with a Λ-module isomorphism $\phi : k \otimes_R M \to V$. We denote this lift of V by (M, ϕ).

(ii) Two lifts (M, ϕ) and (M', ϕ') of V over R are isomorphic if there exists an $R\Lambda$-module isomorphism $f : M \to M'$ such that $\phi = \phi' \circ (id \otimes f)$.

(iii) The isomorphism class of the lift (M, ϕ) is called a deformation of V over R and it is denoted by $[M, \phi]$. The set of deformations of V over R is denoted by $Def_\Lambda(V, R)$.

Definition 2.45. We define a covariant functor $F : \hat{\mathcal{C}} \to Sets$ as follows:

For $R \in Ob(\hat{\mathcal{C}})$, let $F(R) = Def_\Lambda(V, R)$.

For $\alpha : R \to R'$ in $\hat{\mathcal{C}}$, define $F(\alpha) : Def_\Lambda(V, R) \to Def_\Lambda(V, R')$ by $F(\alpha)([M, \phi]) = [R' \otimes_{R,\alpha} M, \phi_\alpha]$ where ϕ_α is the composition $k \otimes_{R'} (R' \otimes_{R,\alpha} M) \xrightarrow{\cong} k \otimes_R M \xrightarrow{\phi} V$. We call F the deformation functor associated to V.

Definition 2.46. Let F be the deformation functor associated to V. If F is representable, then there exists a ring $R(\Lambda, V)$ in $\hat{\mathcal{C}}$ and a deformation $[U, \phi_U]$ of V over $R(\Lambda, V)$ such that for every R in $\hat{\mathcal{C}}$ and every lift (M, ϕ) of V over R, there exists a unique morphism $\alpha : R(\Lambda, V) \to R$ in $\hat{\mathcal{C}}$ such that $F(\alpha)([U, \phi_U]) = [M, \phi]$. In this case, we call $R(\Lambda, V)$ the universal deformation ring of V and $[U, \phi_U]$ the universal deformation of V over $R(\Lambda, V)$.

Remark 2.47. In general, there exists a ring $R(\Lambda, V)$ in $\hat{\mathcal{C}}$ and a deformation $[U, \phi_U]$ of V over $R(\Lambda, V)$ such that for every R in $\hat{\mathcal{C}}$ and every lift (M, ϕ) of V over R, there exists a morphism $\alpha : R(\Lambda, V) \to R$ in $\hat{\mathcal{C}}$, which is not necessarily unique, such that $F(\alpha)([U, \phi_U]) = [M, \phi]$. In this general situation, we call $R(\Lambda, V)$ the versal deformation ring of V and $[U, \phi_U]$ the versal deformation of V over $R(\Lambda, V)$.

Theorem 2.48. *Let F be the deformation functor associated to V. Then F is representable if one of the following is true:*

(a) $End_\Lambda(V) \cong k$

(b) Λ is self-injective and $\underline{\mathrm{End}}_\Lambda(V) \cong k$

Proof. See [9, Theorem 3.5.3]. □

Theorem 2.49. *Let Λ be a Frobenius algebra, and suppose $\underline{\mathrm{End}}_\Lambda(V) \cong k$. Then $\underline{\mathrm{End}}_\Lambda(\Omega(V)) \cong k$, and $R(\Lambda, V) \cong R(\Lambda, \Omega(V))$.*

Proof. See [9, Lemma 3.6.1 and Theorem 3.6.7]. □

Theorem 2.50. *Suppose the deformation functor F associated to V is representable. If $dim_k Ext^1_\Lambda(V, V) = r$, then there exists a surjective homomorphism*

$$\lambda : k[[t_1, \cdots, t_r]] \to R(\Lambda, V)$$

in $\hat{\mathcal{C}}$, and r is minimal with this property.

Proof. Let $k[\epsilon]$ be the ring of dual numbers, i.e. $k[\epsilon] \cong k[u]/(u^2)$. Because F is representable by assumption, it follows that $F(k[\epsilon]) \cong Hom_{\hat{\mathcal{C}}}(R(\Lambda, V), k[\epsilon])$. Since by [9, Lemma 3.4.1], $F(k[\epsilon]) \cong Ext^1_\Lambda(V,V)$ as a k-vector space, this implies the theorem. $\square$

CHAPTER 3
ALGEBRAS OF DIHEDRAL TYPE OF POLYNOMIAL GROWTH

3.1 Introduction

Suppose k is an algebraically closed field. In [5, Cor. 4.7], Erdmann and Skowroński classified all finite dimensional k-algebras of dihedral type which are of polynomial growth. Algebras of dihedral type are symmetric algebras which play an important role in Erdmann's classification in [4] of all tame blocks of group algebras with dihedral defect groups. A finite dimensional k-algebra is said to be of polynomial growth if it is tame, i.e. its indecomposable modules of any given k-dimension d can be parameterized using only a finite number $\mu(d)$ of one-parameter families, and if there is a natural number m such that $\mu(d) \leq d^m$ for all $d \geq 1$.

Let now $\Lambda = kQ/I$ be an algebra of dihedral type which is of polynomial growth. The classification in [5] shows that there are exactly 8 possibilities for the quiver Q and gives a list of ideals I for each Q, which determines each Λ uniquely up to Morita equivalence. Note that each quiver Q has either 1, 2, or 3 vertices. Moreover, $\Lambda/soc(\Lambda)$ is always a special biserial algebra, so all indecomposable Λ-modules are either string or band modules (see section 2.6). In the next four sections, each of these algebras Λ will be considered. The goal is to find all string modules whose stable endomorphism ring is k and then to determine the universal deformation ring for each of these modules.

To organize the string modules for each Λ, we consider the connected components of the stable Auslander-Reiten quiver $\Gamma_S(\Lambda)$. We use the combinatorial description of these components given by hooks and cohooks (see section 2.7). To ease the notational difference between adding, respectively subtracting, hooks and cohooks, we use the following conventions when displaying connected components of $\Gamma_S(\Lambda)$ containing string modules:

(1) Instead of displaying a string module $M(C)$, we just display its underlying string C.

(2) If C is a string and we add a hook or subtract a cohook on the left (respectively right), we denote the resulting string by ${}_LC$ (respectively C_R). We denote the resulting string module by ${}_LM(C)$ (respectively $M(C)_R$).

For a selected algebra Λ, the connected components of $\Gamma_S(\Lambda)$ containing string modules can be viewed in section 3.2.3. Moreover, the universal deformation rings of the Λ-modules which were calculated are listed in the respective component diagrams.

3.2 A Particular Algebra With Three Isomorphism Classes of Simple Modules

Let Λ be the symmetric basic algebra of type $\mathrm{D}(3\mathcal{A})_2$ (following the labeling by Erdmann and Skowroński in [5, Cor. 4.7]). Hence $\Lambda = kQ/I$ where

$$Q = \begin{tikzcd} \underset{\bullet}{0} \arrow[r, shift left, "\beta"] & \underset{\bullet}{1} \arrow[l, shift left, "\gamma"] \arrow[r, shift left, "\delta"] & \underset{\bullet}{2} \arrow[l, shift left, "\eta"] \end{tikzcd}$$

and I is the ideal of the path algebra kQ generated by the set of relations

$$\left\{\delta\beta, \gamma\eta, (\beta\gamma)^2 - (\eta\delta)^2\right\}.$$

There exist precisely three simple Λ-modules, up to isomorphism, corresponding to the three vertices of Q, denoted by S_0, S_1, S_2. Their projective covers P_0, P_1, P_2 can be visualized as follows:

$$P_0\colon\ \begin{array}{c}0\\1\\0\\1\\0\end{array} \qquad\qquad P_1\colon\ \begin{array}{ccc} & 1 & \\ 2 & & 0\\ 1 & & 1\\ 2 & & 0\\ & 1 & \end{array} \qquad\qquad P_2\colon\ \begin{array}{c}2\\1\\2\\1\\2\end{array}$$

The stable Auslander-Reiten quiver $\Gamma_S(\Lambda)$ has three connected components consisting of string modules:

- Component 1, which contains the simple module S_0 and the string modules corresponding to the maximal directed strings $\delta\eta\delta$, $\eta\delta\eta$;
- Component 2, which contains the simple module S_1 and the string modules corresponding to the directed strings β, γ, δ, η, $\gamma\beta$, $\delta\eta$;
- Component 3, which contains the simple module S_2 and the string modules corresponding to the maximal directed strings $\gamma\beta\gamma$, $\beta\gamma\beta$.

Components 1 and 3 are 3-tubes, and component 2 is a non-periodic component of the form $\mathbb{Z}\tilde{A}_{3,3}$, see [4, Theorem II.7.1]. Note that component 3 can be obtained from component 2 by applying the syzygy functor Ω.

3.2.1 Stable Endomorphism Rings

We first want to determine all string modules $M(S)$ for Λ whose stable endomorphism ring is isomorphic to k. Since $\underline{\mathrm{End}}_\Lambda(M(S))$ is Ω-invariant by Corollary

2.10, we can concentrate on components 2 and 3. The following remark gives a full set of representatives of the Ω-orbits for each of the components 2 and 3.

Remark 3.1. (a) Component 3.

The simple module $S_2 = M(e_2)$ and the string modules $M(\gamma\beta\gamma)$ and $M(\beta\gamma\beta)$ are in the same Ω^2-orbit and lie at the end of component 3. As described in section 2.7, all modules in component 3 are string modules whose corresponding strings can be obtained from e_2, $\gamma\beta\gamma$, $\beta\gamma\beta$, respectively, by either adding hooks or subtracting cohooks on the left. Thus a full set of representatives of the Ω-orbits of component 3 is given as follows, where we use the notation introduced in section 3.1:

$$M(e_2), {}_LM(e_2), {}_{LL}M(e_2), \cdots .$$

(b) Component 2.

As noted above, component 2 has the form $\mathbb{Z}\tilde{A}_{3,3}$. Using section 2.7, we see that there are precisely 3 Ω-orbits of string modules in component 2 with representatives:

$$S_1 = M(e_1), M(\gamma), M(\eta).$$

Lemma 3.2. *We have* $\underline{\mathrm{End}}_\Lambda(M(S)) \cong k$ *for*

$$S \in \{e_0, e_1, e_2, \gamma, \eta\} \textit{ or } M(S) \in \{{}_LM(e_2), {}_{LL}M(e_2)\}.$$

Proof. By Schur's lemma, $End_\Lambda(M(S_i)) \cong k$ for $i \in \{0, 1, 2\}$, which implies

$$\underline{\mathrm{End}}_\Lambda(M(S_i)) \cong k \text{ for } i \in \{0, 1, 2\}.$$

Next, consider $M(\gamma)$. Since the only Λ-module endomorphisms of $M(\gamma)$ are scalar multiples of the identity (see Theorem 2.38), it follows that $End_\Lambda(M(\gamma)) \cong k$, which implies $\underline{\mathrm{End}}_\Lambda(M(\gamma)) \cong k$. Similarly, it follows that $\underline{\mathrm{End}}_\Lambda(M(\eta)) \cong k$.

Next, consider ${}_LM(e_2)$. By Theorem 2.38, a k-basis for $End_\Lambda({}_LM(e_2))$ is given by the identity morphism together with the morphism

$$\begin{array}{ccccccccccc}
 & & & \boxed{1_{b_1}} & & & & & & 1_{b_1} & \\
 & & \boxed{0_{b_2}} & & 2_{b_0} & \longrightarrow & & & 0_{b_2} & & 2_{b_0} \\
 & 1_{b_3} & & & & & & \boxed{1_{b_3}} & & & \\
0_{b_4} & & & & & & \boxed{0_{b_4}} & & & &
\end{array}$$

which sends the basis element b_1 of ${}_LM(e_2)$ to the basis element b_3 and the basis element b_2 to the basis element b_4 and all the other basis elements to zero. Hence $dim_k End_\Lambda({}_LM(e_2)) = 2$. The morphism $(b_1, b_2) \to (b_3, b_4)$ factors through the projective Λ-module P_0 as follows:

$$\begin{array}{ccccccccccccc}
 & & & & & & 0_{c_0} & & & & & & \\
 & & & \boxed{1_{b_1}} & & & \boxed{1_{c_1}} & & & & & 1_{b_1} & \\
 & & \boxed{0_{b_2}} & & 2_{b_0} & \xrightarrow{\alpha} & \boxed{0_{c_2}} & \xrightarrow{\beta} & & & 0_{b_2} & & 2_{b_0} \\
 & 1_{b_3} & & & & & 1_{c_3} & & & \boxed{1_{b_3}} & & & \\
0_{b_4} & & & & & & 0_{c_4} & & \boxed{0_{b_4}} & & & &
\end{array}$$

where α maps $b_1 \to c_1$, $b_2 \to c_2$, $b_3 \to c_3$, $b_4 \to c_4$, and $b_0 \to 0$, and β maps $c_0 \to b_2$, $c_1 \to b_3$, $c_2 \to b_4$, and all other basis elements of P_0 to 0. Thus, $\underline{\mathrm{End}}_\Lambda({}_LM(e_2)) \cong k$.

Next, consider ${}_{LL}M(e_2)$. By Theorem 2.38, a k-basis for $End_\Lambda({}_{LL}M(e_2))$ is given by the identity morphism together with the morphism

$$\begin{array}{ccccccc} & \boxed{1_{b_1}} & & & & 1_{b_1} & \\ 0_{b_2} & & 2_{b_0} & \longrightarrow & 0_{b_2} & & 2_{b_0} \\ 1_{b_3} & & & & \boxed{1_{b_3}} & & \end{array}$$

which sends the basis element b_1 of ${}_{LL}M(e_2)$ to the basis element b_3 and all other basis elements to zero. Hence $dim_k End_\Lambda({}_{LL}M(e_2)) = 2$. The morphism $b_1 \to b_3$ factors through the projective Λ-module P_1 as follows:

$$\begin{array}{ccccccccccc} & & & & & 1_{c_0} & & & & & \\ & \boxed{1_{b_1}} & & & 0_{c_1} & & 2_{c_7} & & & 1_{b_1} & \\ 0_{b_2} & & 2_{b_0} & \xrightarrow{\alpha} & \boxed{1_{c_2}} & & 1_{c_6} & \xrightarrow{\beta} & 0_{b_2} & & 2_{b_0} \\ 1_{b_3} & & & & 0_{c_3} & & 2_{c_5} & & \boxed{1_{b_3}} & & \\ & & & & & 1_{c_4} & & & & & \end{array}$$

where α maps $b_1 \to c_2$, $b_2 \to c_3$, $b_3 \to c_4$, and $b_0 \to 0$, and β maps $c_0 \to b_1$, $c_1 \to b_2$, $c_2 \to b_3$, $c_7 \to b_0$, and all other basis elements of P_1 to 0. Thus, $\underline{\mathrm{End}}_\Lambda({}_{LL}M(e_2)) \cong k$. $\square$

Lemma 3.3. *If $M(S)$ is a string module for Λ and $M(S)$ does not lie in the Ω-orbit of either S_0, S_1, S_2 or $M(\gamma)$ or $M(\eta)$ or ${}_LM(e_2)$ or ${}_{LL}M(e_2)$ then $\underline{\mathrm{End}}_\Lambda(M(S)) \ncong k$.*

Proof. By Remark 3.1, we only have to show that $\underline{\mathrm{End}}_\Lambda(M(S)) \ncong k$ for $M(S)$ of the form ${}_{L\cdots LL}M(e_2)$.

For ${}_{LLL}M(e_2)$, the endomorphism

$$\begin{array}{ccccccccccc} & & & 1 & & & & & & 1 & \\ \boxed{2} & & 0 & & 2 & \longrightarrow & 2 & & 0 & & \boxed{2} \\ & 1 & & & & & & 1 & & & \end{array}$$

does not factor through any projective Λ-module. Since this morphism is k-linearly independent from the identity morphism, it follows that $dim_k \underline{\mathrm{End}}_\Lambda({}_{LLL}M(e_2)) \geq 2$.

In general, we obtain that ${}_{L\cdots LL}M(e_2)$ is as in one of the following three cases:

$$\text{(i)} \quad \begin{array}{ccccccccc} & & & 1 & & & & 1 & \\ 2 & & 0 & & \cdots & & 0 & & 2 \\ & 1 & & & & 1 & & & \end{array}$$

$$\text{(ii)} \quad \begin{array}{cccccccccccc} & & & 1 & & & & & & & 1 & \\ & & 0 & & 2 & & & 2 & & 0 & & 2 \\ & 1 & & & & 1 & \cdots & & 1 & & & \\ 0 & & & & & & & & & & & \end{array}$$

$$\text{(iii)} \quad \begin{array}{ccccccccccc} & & 1 & & & & & & & 1 & \\ & 0 & & 2 & & \cdots & 2 & & 0 & & 2 \\ 1 & & & & 1 & & & 1 & & & \end{array}$$

In each of these three cases, it is enough to find an endomorphism α of ${}_{L\cdots LL}M(e_2)$ such that α is not a scalar multiple of the identity and such that α does not factor through a projective Λ-module.

In case (i), α can be taken as the endomorphism

$$\begin{array}{ccccccccccccccccccc}
&&&1&&&&1&&&&&&1&&&&1& \\
\boxed{2}&&0&&\cdots&&0&&2&\longrightarrow&2&&0&&\cdots&&0&&\boxed{2} \\
&1&&&&1&&&&&&1&&&&1&&&
\end{array}$$

In case (ii), α can be taken as the endomorphism

$$\begin{array}{ccccccccccccccccccccccccc}
&&&\boxed{1}&&&&&&&1&&&&&&1&&&&&&&\boxed{1}& \\
&&\boxed{0}&&\boxed{2}&&\cdots&2&&0&&2&\longrightarrow&&&0&&2&&\cdots&2&&\boxed{0}&&\boxed{2} \\
&\boxed{1}&&&&1&&&1&&&&&&1&&&&1&&&\boxed{1}&&& \\
0&&&&&&&&&&&&&0&&&&&&&&&&&
\end{array}$$

In case (iii), α can be taken as the endomorphism

$$\begin{array}{ccccccccccccccccccccccc}
&&\boxed{1}&&&&&&&1&&&&&1&&&&&&&\boxed{1}& \\
&\boxed{0}&&\boxed{2}&&\cdots&2&&0&&2&\longrightarrow&&0&&2&&\cdots&2&&\boxed{0}&&\boxed{2} \\
\boxed{1}&&&&1&&&1&&&&&1&&&&1&&&\boxed{1}&&&
\end{array}$$

□

Corollary 3.4. *Let M be a string module for Λ. Then $\underline{\mathrm{End}}_\Lambda(M) \cong k$ if and only if M lies in the Ω-orbit of S_0, S_1, S_2 or $M(\gamma)$ or $M(\eta)$ or ${}_LM(e_2)$ or ${}_{LL}M(e_2)$.*

3.2.2 Universal Deformation Rings

In this section, we determine the universal deformation ring $R(\Lambda, M(S))$ for all string modules $M(S)$ whose stable endomorphism ring is isomorphic to k. By Theorem 2.50, if $dim_k Ext^1_\Lambda(M(S), M(S)) = r$, then $R(\Lambda, M(S))$ is isomorphic to a quotient algebra of a power series algebra $k[[t_1, \cdots, t_r]]$ and r is minimal with this property.

Theorem 3.5. *If $M(S) \in \{S_0, S_1, S_2, {}_LM(e_2)\}$ then $Ext^1_\Lambda(M(S), M(S)) = 0$ and*

$R(\Lambda, M(S)) \cong k.$

Proof. Since for all these modules $M(S)$, $Hom_\Lambda(\Omega(M(S)), M(S)) = 0$, it follows by Theorem 2.12 that $Ext^1_\Lambda(M(S), M(S)) = 0$. Therefore, $R(\Lambda, M(S)) \cong k$. □

Theorem 3.6. *If $S \in \{\gamma, \eta\}$ then $dim_k Ext^1_\Lambda(M(S), M(S)) = 1$ and $R(\Lambda, M(S)) \cong k[[t]]/(t^2)$.*

Proof. We consider $S = \gamma$, the case of $S = \eta$ being similar. We have

$$\Omega(M(\gamma)) = \begin{array}{ccccc} & & & 2 & \\ 1 & & & & 1 \\ & 0 & & 2 & \\ & & 1 & & \end{array}$$

By Theorem 2.38, there is one basis element for $Hom_\Lambda(\Omega(M(\gamma), M(\gamma))$:

$$\begin{array}{ccccc} & & & 2 & \\ \boxed{1} & & & & 1 \\ & \boxed{0} & & 2 & \\ & & 1 & & \end{array} \longrightarrow \begin{array}{cc} & \boxed{1} \\ \boxed{0} & \end{array}$$

Since this morphism does not factor through a projective Λ-module, we have

$$dim_k Ext^1_\Lambda(M(\gamma), M(\gamma)) = dim_k \underline{Hom}_\Lambda(\Omega(M(\gamma)), M(\gamma)) = 1.$$

Therefore, $R(\Lambda, M(\gamma))$ is a quotient algebra of $k[[t]]$.

Considering $M(\gamma\beta\gamma)$, we have a short exact sequence of Λ-modules:

$$0 \to M(\gamma) \xrightarrow{\iota} M(\gamma\beta\gamma) \xrightarrow{\tau} M(\gamma) \to 0$$

where ι and τ are the following canonical Λ-module homomorphisms:

$$\begin{array}{ccccccc} & & & & & & 1 \\ & \boxed{1} & & & & 0 & \\ \boxed{0} & & \xrightarrow{\iota} & & \boxed{1} & & \\ & & & \boxed{0} & & & \end{array} \quad \text{and} \quad \begin{array}{ccccccc} & & & \boxed{1} & & & \\ & & \boxed{0} & & & & \boxed{1} \\ & 1 & & & \xrightarrow{\tau} & \boxed{0} & \\ 0 & & & & & & \end{array}$$

Since $Im(\iota) = Ker(\tau)$, τ induces an isomorphism $\overline{\tau} : M(\gamma\beta\gamma)/Im(\iota) \to M(\gamma)$.

Let $R_0 = k[[t]]/(t^2)$. We obtain an R_0-module structure on $M(\gamma\beta\gamma)$ by letting t act as $\iota \circ \tau$. Let $\{\overline{z_0}, \overline{z_1}\}$ be a k-basis for $M(\gamma)$. Lift $\overline{z_0}, \overline{z_1}$ to elements $z_0, z_1 \in M(\gamma\beta\gamma)$ with $\tau(z_i) = \overline{z_i}$ for $i \in \{0, 1\}$. Since $z_0, z_1 \notin Im(\iota \circ \tau)$, we see that $\{tz_0, tz_1\}$ is a k-basis for $Im(\iota)$. Hence $\{z_0, z_1, tz_0, tz_1\}$ is a k-basis for $M(\gamma\beta\gamma)$, which implies that $M(\gamma\beta\gamma)$ is a free R_0-module of rank 2 with R_0-basis $\{z_0, z_1\}$.

Since $R_0/tR_0 \cong k$, there is a short exact sequence $0 \to tR_0 \to R_0 \to k \to 0$ of R_0-modules. Tensoring with $M(\gamma\beta\gamma)$ over R_0, we obtain a short exact sequence

$$0 \to tM(\gamma\beta\gamma) \to M(\gamma\beta\gamma) \to k \otimes_{R_0} M(\gamma\beta\gamma) \to 0.$$

Since $Im(\iota) = tM(\gamma\beta\gamma)$, the isomorphism $\overline{\tau}$ defines a Λ-module isomorphism $\rho : k \otimes_{R_0} M(\gamma\beta\gamma) \to M(\gamma)$. Hence, $(M(\gamma\beta\gamma), \rho)$ is a lift of $M(\gamma)$ over $R_0 = k[[t]]/(t^2)$, and $[M(\gamma\beta\gamma), \rho]$ is a deformation of $M(\gamma)$ over R_0.

Let $[U, \phi]$ be the universal deformation of $M(\gamma)$ over $R(\Lambda, M(\gamma))$. Since $[U, \phi]$ is the universal deformation, there exists a unique k-algebra homomorphism $\theta : R(\Lambda, M(\gamma)) \to R_0$ such that $F(\theta)([U, \phi]) = [M(\gamma\beta\gamma), \rho]$, where F is the deformation functor associated to $M(\gamma)$. Recall that $F(\theta)([U, \phi]) = [R_0 \otimes_{R(\Lambda, M(\gamma)),\theta} U, \phi_\theta] = [M(\gamma\beta\gamma), \rho]$ where ϕ_θ is the composition

$$k \otimes_{R_0} (R_0 \otimes_{R(\Lambda,M(\gamma)),\theta} M(\gamma\beta\gamma)) \xrightarrow{\cong} k \otimes_{R(\Lambda,M(\gamma))} M(\gamma\beta\gamma) \xrightarrow{\phi} M(\gamma).$$

We now prove that θ is an isomorphism, which then implies that $R(\Lambda, M(\gamma)) \cong R_0$ and $[U, \phi] = [M(\gamma\beta\gamma), \rho]$.

Since the trivial lift of $M(\gamma)$ over R_0 is $(R_0 \otimes_k M(\gamma), \alpha)$ where α is the natural isomorphism $k \otimes_{R_0} (R_0 \otimes_k M(\gamma)) \overset{\cong}{\to} M(\gamma)$, it follows that $(M(\gamma\beta\gamma), \rho)$ is not the trivial lift of $M(\gamma)$ over R_0. Therefore, θ is surjective.

If θ is not an isomorphism, then we can find a surjective k-algebra homomorphism $\theta_1 : R(\Lambda, M(\gamma)) \to R_1 = k[[t]]/(t^3)$ such that $\theta = \Pi \circ \theta_1$, where $\Pi : R_1 \to R_0$ is the canonical surjection.

Let $M_1 = R_1 \otimes_{R(\Lambda, M(\gamma)), \theta_1} U$. Tensoring the short exact sequence of R_1-modules $0 \to t^2 R_1 \to R_1 \overset{\Pi}{\to} R_0 \to 0$ with M_1 over R_1, we obtain a short exact sequence of Λ-modules $0 \to t^2 M_1 \to M_1 \to R_0 \otimes_{R_1, \Pi} M_1 \to 0$. Since $\theta = \Pi \circ \theta_1$, we obtain

$$M(\gamma\beta\gamma) \cong R_0 \otimes_{R(\Lambda, M(\gamma)), \theta} U \cong R_0 \otimes_{R_1, \Pi} (R_1 \otimes_{R(\Lambda, M(\gamma)), \theta} U) \cong R_0 \otimes_{R_1, \Pi} M_1.$$

Hence, $M(\gamma\beta\gamma) \cong M_1/t^2 M_1$ as $R_1\Lambda$-modules. Since $R_1/tR_1 \cong k$, it follows that $M_1/tM_1 \cong k \otimes_{R_1} M_1 \cong M(\gamma)$. Define $g : M_1 \to t^2 M_1$ by $g(x) = t^2 x$ for all $x \in M_1$. Since M_1 is free over R_1, it follows that $Ker(g) = tM_1$. Thus $t^2 M_1 \cong M_1/tM_1 \cong M(\gamma)$. We obtain a short exact sequence of $R_1\Lambda$-modules

$$0 \to M(\gamma) \to M_1 \to M(\gamma\beta\gamma) \to 0. \tag{3.1}$$

We now show that (3.1) does not split as a sequence of Λ-modules. Suppose (3.1) splits as a sequence of Λ-modules. Then, $M_1 \cong M(\gamma) \oplus M(\gamma\beta\gamma)$ as Λ-modules. Identify $M_1 = M(\gamma) \oplus M(\gamma\beta\gamma)$, and let $\begin{pmatrix} u \\ v \end{pmatrix} \in M(\gamma) \oplus M(\gamma\beta\gamma)$. Recall that t

acts on $M(\gamma)$ as 0, and t acts on $M(\gamma\beta\gamma)$ as $\iota \circ \tau$. Hence, t acts on $\begin{pmatrix} u \\ v \end{pmatrix} \in M_1$ as the matrix $U_t = \begin{pmatrix} 0 & \epsilon \\ 0 & \iota \circ \tau \end{pmatrix}$, where $\epsilon : M(\gamma\beta\gamma) \to M(\gamma)$ is a surjective Λ-module homomorphism. Note ϵ is surjective because M_1 is free as an R_1-module. By Theorem 2.38, it follows that τ generates $Hom_\Lambda(M(\gamma\beta\gamma), M(\gamma))$ as a k-vector space. Since $\epsilon \in Hom_\Lambda(M(\gamma\beta\gamma), M(\gamma))$ is surjective, there exists $c \in k^\star$ such that $\epsilon = c\tau$. Because $M(\gamma) \cong t^2 M_1$ is non-zero, there exists an element $\begin{pmatrix} z \\ m \end{pmatrix} \in M(\gamma) \oplus M(\gamma\beta\gamma) = M_1$ such that $U_{t^2} \begin{pmatrix} z \\ m \end{pmatrix} \neq 0$. Since $\epsilon = c\tau$, it follows that $Ker(\epsilon) = tM(\gamma\beta\gamma)$. Consider

$$(U_t)^2 \begin{pmatrix} z \\ m \end{pmatrix} = \begin{pmatrix} c\tau(t(m)) \\ t^2(m) \end{pmatrix}.$$

Since $Ker(c\tau) = tM(\gamma\beta\gamma)$ and $m \in M(\gamma\beta\gamma)$, we have $c\tau(t(m)) = 0$ and $t^2(m) = 0$. Thus $(U_t)^2 \begin{pmatrix} z \\ m \end{pmatrix} = 0$, which is a contradiction. Therefore, (3.1) does not split as a sequence of Λ-modules. However,

$$Ext^1_\Lambda(M(\gamma\beta\gamma), M(\gamma)) \cong \underline{Hom}_\Lambda(\Omega(M(\gamma\beta\gamma)), M(\gamma)) \cong \underline{Hom}_\Lambda(M(\eta\delta\eta), M(\gamma)) = 0.$$

Thus, the short exact sequence (3.1) must split as a sequence of Λ-modules, which is a contradiction. This implies that $\theta : R(\Lambda, M(\gamma)) \to R_0 = k[[t]]/(t^2)$ is an isomorphism.

$\square$

Theorem 3.7. *If* $M(S) = {}_{LL}M(e_2)$, *then*

$$dim_k Ext^1_\Lambda(M(S), M(S)) = 1$$

and $R(\Lambda, M(S)) \cong k[[t]]$.

Proof. We have $\Omega({}_{LL}M(e_2)) = \begin{array}{ccc} & 1 & \\ 0 & & 2 \\ & 1 & \end{array}$. By Theorem 2.38, there is one basis element for $Hom_\Lambda(\Omega({}_{LL}M(e_2)),{}_{LL} M(e_2))$:

$$\begin{array}{ccc} & \boxed{1} & \\ 0 & & 2 \\ & 1 & \end{array} \longrightarrow \begin{array}{ccc} & 1 & \\ 0 & & 2 \\ & \boxed{1} & \end{array}$$

Since this morphism does not factor through a projective Λ-module, we have

$$dim_k Ext^1_\Lambda({}_{LL}M(e_2),{}_{LL} M(e_2)) = dim_k \underline{Hom}_\Lambda(\Omega({}_{LL}M(e_2)),{}_{LL} M(e_2)) = 1.$$

Therefore, $R(\Lambda,{}_{LL} M(e_2))$ is a quotient algebra of $k[[t]]$.

Let $C = \beta\gamma\delta^{-1}$. Then ${}_{LL}M(e_2) = M(C)$, where we label the k-basis elements as follows: $\begin{array}{ccccc} & & 1_{b_{1,1}} & & \\ & 0_{c_{1,1}} & & 2_{a_{1,1}} & \\ 1_{d_{1,1}} & & & & \end{array}$.

For all $i \geq 1$, let $T_i = C(\eta^{-1}C)^{i-1}$, and label the k-basis elements of $M(T_i)$ as follows:

$$\begin{array}{cccccccccccccc} & & 1_{b_{1,i}} & & & 1_{b_{2,i}} & & & & 1_{b_{i-1,i}} & & & 1_{b_{i,i}} & \\ & 0_{c_{1,i}} & & 2_{a_{1,i}} & 0_{c_{2,i}} & & 2_{a_{2,i}} & \cdots & 0_{c_{i-1,i}} & & 2_{a_{i-1,i}} & 0_{c_{i,i}} & & 2_{a_{i,i}} \\ 1_{d_{1,i}} & & & 1_{d_{2,i}} & & & & 1_{d_{i-1,i}} & & & 1_{d_{i,i}} & & & \end{array}$$

In particular, $T_1 = C$ and $M(T_1) =_{LL} M(e_2)$.

For $i \geq 1$, let $R_i = k[[t]]/(t^i)$. Thus, $R_1 = k$. Fix $i \geq 2$. Consider the following Λ-module homomorphisms $\pi_{i,i-1} : M(T_i) \to M(T_{i-1})$ and $\iota_{i-1,i} : M(T_{i-1}) \to M(T_i)$, where $\pi_{i,i-1}$ sends $x_{j,i}$ to $x_{j,i-1}$ for $1 \leq j \leq i-1$ and $x_{i,i}$ to 0 for $x \in \{a, b, c, d\}$, and $\iota_{i-1,i}$ sends $x_{j,i-1}$ to $x_{j+1,i}$ for $1 \leq j \leq i-1$ and $x \in \{a, b, c, d\}$. Define the Λ-module endomorphism $\tau_i : M(T_i) \to M(T_i)$ by $\tau_i = \iota_{i-1,i} \circ \pi_{i,i-1}$. Then $Ker(\tau_i) \cong M(C)$ and $Im(\tau_i) \cong M(T_{i-1})$. Thus, $Im(\tau_i^{i-1}) \cong M(C)$ and $Im(\tau_i^i) = 0$.

The Λ-module $M(T_i)$ is naturally an R_i-module by letting t act as τ_i. Let $B = \{a_{1,i}, b_{1,i}, c_{1,i}, d_{1,i}\}$. Then $\{x + tM(T_i) | x \in B\}$ is a k-basis for $M(T_i)/tM(T_i) \cong M(C)$. Applying powers of τ_i to the elements in B, it follows that $x_{j,i} = t^{j-1}x_{1,i}$ for $2 \leq j \leq i$ and $x \in \{a, b, c, d\}$. Hence, $\{t^{j-1}x | x \in B, 2 \leq j \leq i\}$ is a k-basis of $tM(T_i)$. Therefore, $M(T_i)$ is a free R_i-module with R_i-basis B.

We have a short exact sequence $0 \to tR_i \to R_i \to k \to 0$. Tensoring this sequence with $M(T_i)$ over R_i, we obtain a short exact sequence of $R_i\Lambda$-modules $0 \to tM(T_i) \to M(T_i) \to k \otimes_{R_i} M(T_i) \to 0$. Since $M(C) \cong M(T_i)/tM(T_i)$, this means that there exists a Λ-module isomorphism $\psi_i : k \otimes_{R_i} M(T_i) \to M(C)$. Therefore, $(M(T_i), \psi_i)$ is a lift of $M(C) = {}_{LL}M(e_2)$ over R_i.

For all $i \geq 1$, the $R_i\Lambda$-module $M(T_i)$ is a $k[[t]]\Lambda$-module via the natural projections $p_i : k[[t]] \to R_i$. Also, the $k[[t]]\Lambda$-modules $M(T_i)$ form an inverse system $(\{M(T_i)\}_{i \in \mathbb{Z}^+}, \{\pi_{ji}\}_{j \geq i})$ where $\pi_{ji} : M(T_j) \to M(T_i)$ is the composition $\pi_{ji} = \pi_{i+1,i} \circ \cdots \circ \pi_{j,j-1}$. Let $N = \varprojlim M(T_i)$, and let $\overleftarrow{\tau_i} : N \to N$ be the inverse limit of the morphism $\tau_i : M(T_i) \to M(T_i)$. Then N is a $k[[t]]\Lambda$-module where the action of t

is given by $\overleftarrow{\tau_i}$. In particular, $N/tN \cong M(C)$.

Let $a, b, c, d \in N$ be such that $\{a + tN, b + tN, c + tN, d + tN\}$ is a k-basis for $N/tN \cong M(C)$. By Nakayama's lemma, $\{a, b, c, d\}$ generates N as a $k[[t]]$-module. By considering k-bases for $M(T_i)$ for all $i \geq 1$ and using that $N = \varprojlim M(T_i)$, we see that $\{t^j a, t^j b, t^j c, t^j d | j \geq 0\}$ is a k-basis for N. This implies that N is a free $k[[t]]$-module with $k[[t]]$-basis $\{a, b, c, d\}$.

Consider the short exact sequence $0 \to (t) \to k[[t]] \to k \to 0$. Tensoring with N over $k[[t]]$, we obtain a short exact sequence of $k[[t]]\Lambda$-modules

$$0 \to tN \to N \to k \otimes_{k[[t]]} N \to 0.$$

Hence $M(C) \cong N/tN \cong k \otimes_{k[[t]]} N$; that is, there exists a Λ-module isomorphism $\psi : k \otimes_{k[[t]]} N \to M(C)$. Thus, (N, ψ) is a lift of $M(C)$ over $k[[t]]$.

Let $[U, \phi]$ be the universal deformation of $M(C)$ over $R(\Lambda, M(C))$. There exists a unique k-algebra homomorphism $\theta : R(\Lambda, M(C)) \to k[[t]]$ such that $F(\theta)([U, \phi]) = [N, \psi]$, where F is the deformation functor associated to $M(C)$. In particular, $N \cong k[[t]] \otimes_{R(\Lambda, M(C)), \theta} U$.

We now prove that θ is an isomorphism, which then implies that $R(\Lambda, M(C)) \cong k[[t]]$. Since $R(\Lambda, M(C)) \cong k[[t]]/I$ for some ideal $I \subseteq (t^2)$ of $k[[t]]$, it suffices to show that θ is surjective.

Let $m_{R(\Lambda, M(C))}$ and $m_{k[[t]]}$ be the maximal ideals of $R(\Lambda, M(C))$ and $k[[t]]$, respectively. By [8, Lemma 1.1], it follows that θ is surjective if and only if the induced morphism $\overline{\theta} : \frac{R(\Lambda, M(C))}{m^2_{R(\Lambda, M(C))}} \to \frac{k[[t]]}{m^2_{k[[t]]}} \cong k[[t]]/(t^2)$ is surjective. But this is the case if and only if $p_2 \circ \theta : R(\Lambda, M(C)) \to k[[t]]/(t^2) = R_2$ is surjective, where $p_2 : k[[t]] \to R_2$ is the natural surjection. Since $R_2 \otimes_{R(\Lambda,M(C)),p_2\circ\theta} U \cong R_2 \otimes_{k[[t]],p_2} (k[[t]] \otimes_{R(\Lambda,M(C)),\theta} U) \cong R_2 \otimes_{k[[t]],p_2} N \cong N/t^2N \cong M(T_2)$ and $M(T_2)$ is an indecomposable Λ-module, it follows that the lift of $M(C)$ corresponding to $p_2 \circ \theta$ is not the trivial lift. This implies that $p_2 \circ \theta$, and hence $\overline{\theta}$, is surjective. Therefore, θ is an isomorphism and $R(\Lambda, M(C)) \cong k[[t]]$ where $M(C) =_{LL} M(e_2)$. $\square$

3.2.3 Components of the Stable Auslander-Reiten Quiver

In this subsection, we display the three connected components of the stable Auslander-Reiten quiver $\Gamma_S(\Lambda)$ consisting of string modules. Recall that components 1 and 3 are 3-tubes, and component 2 is a non-periodic component of the form $\mathbb{Z}\tilde{A}_{3,3}$. This means that given any module M belonging to component 2, the component around M looks like

$$\begin{tikzcd}
X \arrow[dr] & & & \\
& \bullet \arrow[dr] & & \\
& & \bullet \arrow[dr] & \\
& & & M \\
& & \bullet \arrow[ur] & \\
& \bullet \arrow[ur] & & \\
X \arrow[ur] & & &
\end{tikzcd}$$

Moreover, every M has precisely two predecessors and two successors in component 2.

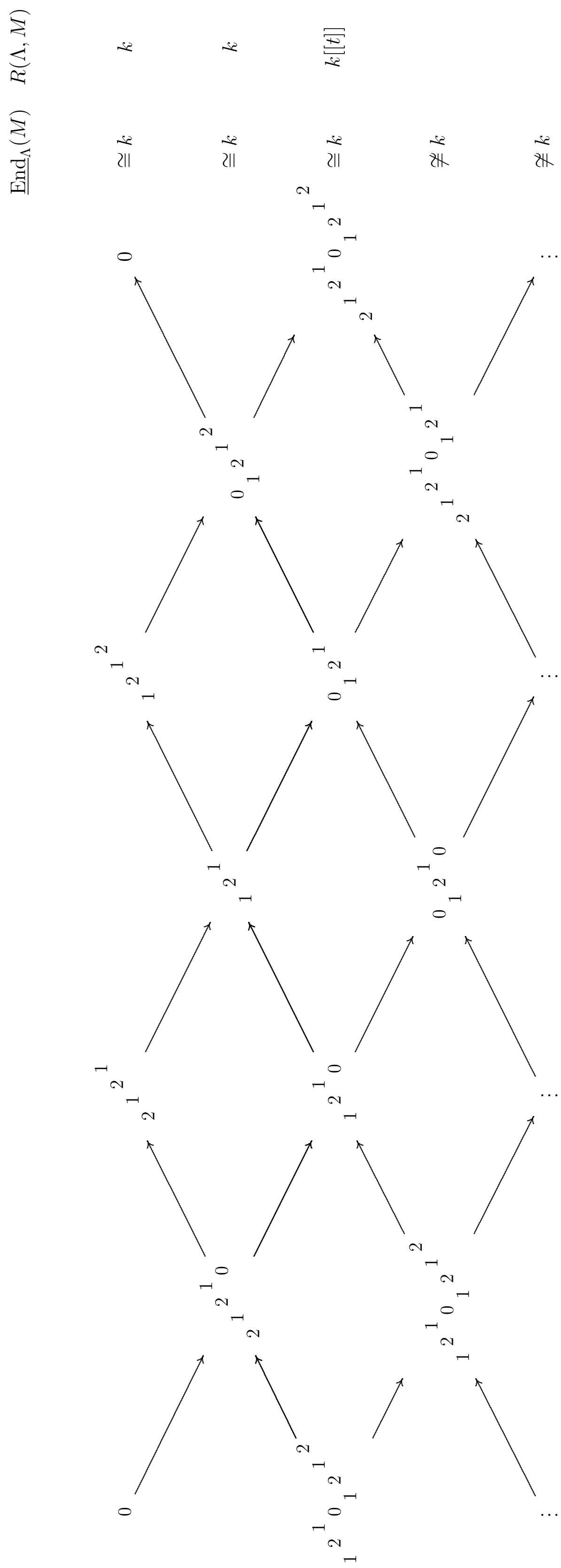


Figure 3.1: Component 1

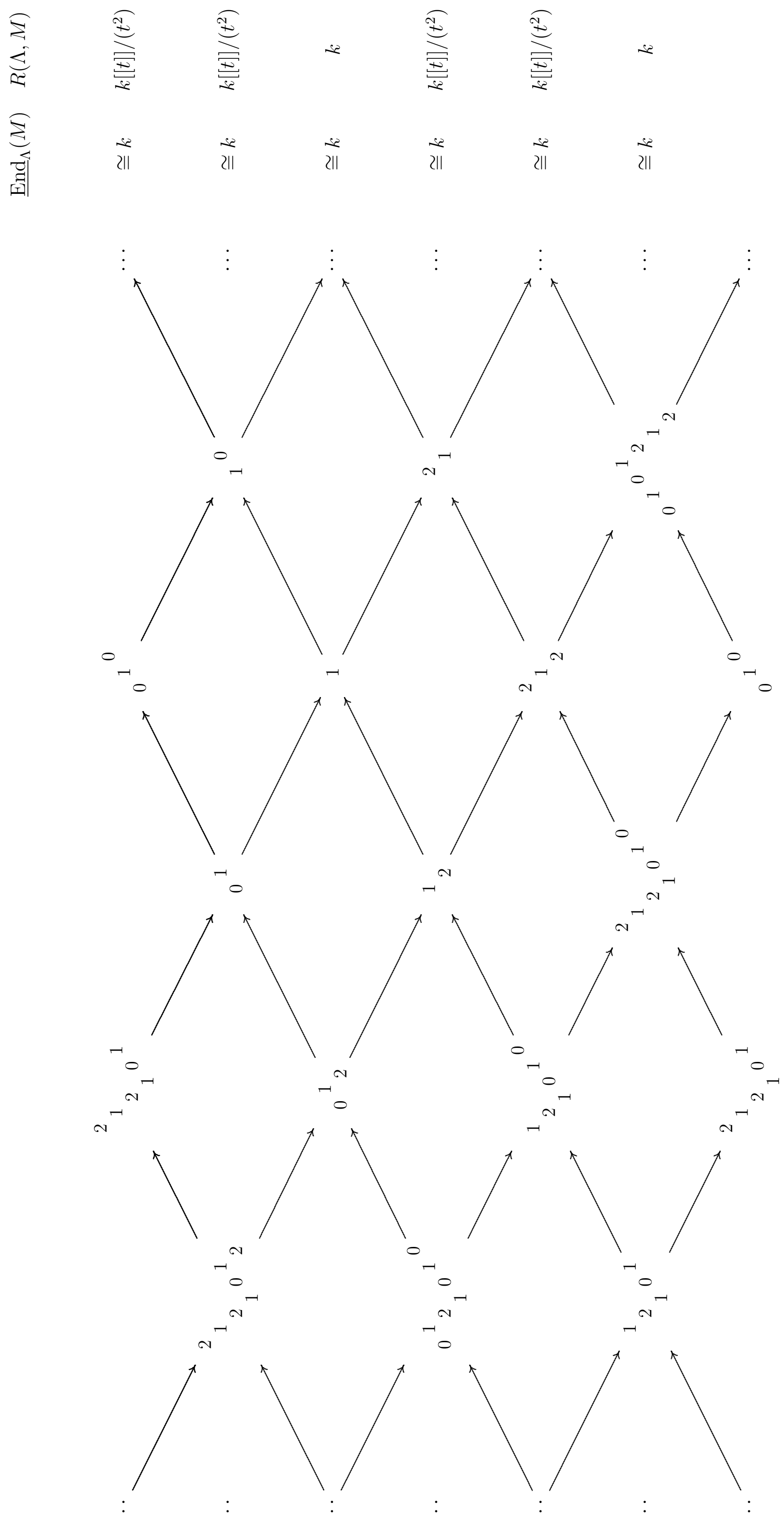


Figure 3.2: Component 2

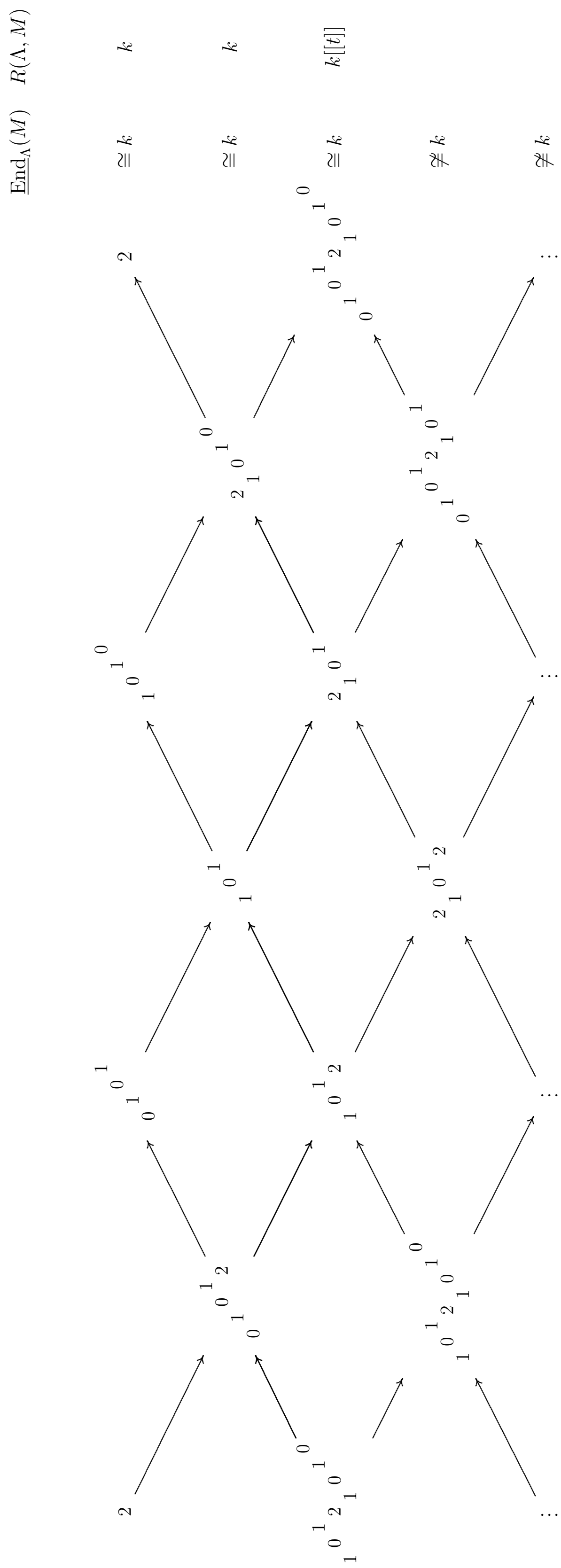


Figure 3.3: Component 3

3.3 A Particular Algebra With Two Isomorphism Classes of Simple Modules

Let Λ be the symmetric basic algebra of type D($2\mathcal{A}$) (following the labeling by Erdmann and Skowroński in [5, Cor. 4.7]). Hence $\Lambda = kQ/I$ where

$$Q = \begin{tikzcd} \overset{0}{\bullet} \arrow[loop left, "\alpha"] \arrow[r, shift left, "\beta"] & \overset{1}{\bullet} \arrow[l, shift left, "\gamma"] \end{tikzcd}$$

and I is the ideal of the path algebra kQ generated by the set of relations

$$\left\{\beta\gamma, \alpha^2 - c\gamma\beta\alpha, \gamma\beta\alpha - \alpha\gamma\beta\right\}.$$

Here $c \in \{0, 1\}$ if $\mathrm{char}(k) = 2$ and $c = 0$ otherwise.

For the remainder of this section, let $\mathrm{char}(k) = 2$ and $c = 1$. There exist precisely two simple Λ-modules, up to isomorphism, corresponding to the two vertices of Q, denoted by S_0, S_1. Their projective covers P_0, P_1 can be visualized as follows:

$$P_0 : \quad \begin{array}{ccc} & 0 & \\ 0 & & 1 \\ 1 & & 0 \\ & 0 & \end{array} \qquad\qquad P_1 : \quad \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array}$$

The stable Auslander-Reiten quiver $\Gamma_S(\Lambda)$ has three connected components consisting of string modules:

- Component 1, which contains the simple module S_0 and the string modules corresponding to the directed strings γ, β;
- Component 2, which contains the simple module S_1 and the string modules corresponding to the directed string α and the maximal directed strings $\alpha\gamma$, $\beta\alpha$;
- Component 3, which contains the string module corresponding to the maximal directed string $\gamma\beta$.

Component 1 is a non-periodic component of the form $\mathbb{Z}\tilde{A}_{3,1}$, component 2 is a 3-tube, and component 3 is a 1-tube.

3.3.1 Stable Endomorphism Rings

We first want to determine all string modules $M(S)$ for Λ whose stable endomorphism ring is isomorphic to k. The following remark gives a full set of representatives of the Ω-orbits for each of the components 1, 2 and 3.

Remark 3.8. (a) Component 1.

As noted above, component 1 has the form $\mathbb{Z}\tilde{A}_{3,1}$. Using section 2.7, we see that there are precisely 2 Ω-orbits of string modules in component 1 with representatives:

$$S_0 = M(e_0), M(\beta).$$

(b) Component 2.

The simple module $S_1 = M(e_1)$ and the string modules $M(\alpha\beta)$, $M(\gamma\alpha)$ are in the same Ω^2-orbit and lie at the end of component 2. As described in section 2.7, all modules in component 2 are string modules whose corresponding strings can be obtained from e_1, $\alpha\beta$, $\gamma\alpha$, respectively, by either adding hooks or subtracting cohooks on the left. Thus a full set of representatives of the Ω-orbits of component 2 is given as follows, where we use the notation introduced in section 3.1:

$$M(e_1), {}_LM(e_1), {}_{LL}M(e_1), \cdots .$$

(c) Component 3.

The string module $M(\gamma\beta)$ lies at the end of component 3. As described in section 2.7, all modules in component 3 are string modules whose corresponding strings can be obtained from $\gamma\beta$ by either adding hooks or subtracting cohooks on the left. Thus a full set of representatives of the Ω-orbits of component 3 is given as follows, where we use the notation introduced in section 3.1:

$$M(\gamma\beta), {}_LM(\gamma\beta), {}_{LL}M(\gamma\beta), \cdots .$$

Lemma 3.9. *We have* $\underline{\mathrm{End}}_\Lambda(M(S)) \cong k$ *for* $S \in \{e_0, e_1, \beta, \gamma\beta\}$.

Proof. By Schur's lemma, $End_\Lambda(M(S_i)) \cong k$ for $i \in \{0, 1\}$, which implies

$$\underline{\mathrm{End}}_\Lambda(M(S_i)) \cong k \text{ for } i \in \{0, 1\}.$$

Next, consider $M(\beta)$. Since the only Λ-module endomorphisms of $M(\beta)$ are scalar multiples of the identity (see Theorem 2.38), it follows that $End_\Lambda(M(\beta)) \cong k$, which implies $\underline{\mathrm{End}}_\Lambda(M(\beta)) \cong k$.

Next, consider $M(\gamma\beta)$. By Theorem 2.38, a k-basis for $End_\Lambda(M(\gamma\beta))$ is given by the identity morphism together with the morphism

$$\begin{array}{ccccccc} & & \boxed{0_{b_0}} & & & & 0_{b_0} \\ & 1_{b_1} & & \longrightarrow & & 1_{b_1} & \\ 0_{b_2} & & & & \boxed{0_{b_2}} & & \end{array}$$

which sends the basis element b_0 of $M(\gamma\beta)$ to the basis element b_2 and all the other basis elements to zero. Hence $dim_k End_\Lambda(M(\gamma\beta)) = 2$. The morphism $b_0 \to b_2$ factors through the projective Λ-module P_0 as follows:

$$\begin{array}{ccccccccccc} & & 0_{b_0} & & & 0_{c_0} & & & & & 0_{b_0} \\ & & & \xrightarrow{\alpha} & & & & \xrightarrow{\beta} & & & \\ & 1_{b_1} & & & 0_{c_1} & & 1_{c_5} & & & 1_{b_1} & \\ 0_{b_2} & & & & 1_{c_2} & & 0_{c_4} & & 0_{b_2} & & \\ & & & & & 0_{c_3} & & & & & \end{array}$$

where α maps $b_0 \to c_1 - c_4$, $b_1 \to c_2$, $b_2 \to c_3$, and β maps $c_0 \to b_0$, $c_5 \to b_1$, $c_4 \to b_2$, and all other basis elements of P_0 to 0. Thus, $\underline{\mathrm{End}}_\Lambda(M(\gamma\beta)) \cong k$. $\square$

Lemma 3.10. *If $M(S)$ is a string module for Λ and $M(S)$ does not lie in the Ω-orbit of either S_0, S_1 or $M(\beta)$ or $M(\gamma\beta)$ then $\underline{\mathrm{End}}_\Lambda(M(S)) \ncong k$.*

Proof. By Remark 3.8, we only have to show that $\underline{\mathrm{End}}_\Lambda(M(S)) \ncong k$ for $M(S)$ of the form ${}_LM(e_1), {}_{LL}M(e_1), \cdots$ or of the form ${}_LM(\gamma\beta), {}_{LL}M(\gamma\beta), \cdots$.

For ${}_LM(e_1)$, the endomorphism

$$\begin{array}{cccc} & & \boxed{0_{b_1}} & \\ & 0_{b_2} & & \boxed{1_{b_0}} \\ 1_{b_3} & & & \end{array} \longrightarrow \begin{array}{cccc} & & 0_{b_1} & \\ & \boxed{0_{b_2}} & & 1_{b_0} \\ \boxed{1_{b_3}} & & & \end{array}$$

which sends the basis element b_1 to the basis element b_2 and the basis element b_0 to the basis element b_3 and all other basis elements to zero does not factor through any projective Λ-module. Since this morphism is k-linearly independent from the identity morphism, it follows that $dim_k \underline{\mathrm{End}}_\Lambda({}_L M(e_1)) \geq 2$.

For ${}_{LL}M(e_1)$, the endomorphism

$$\begin{array}{ccc} & \boxed{0_{b_1}} & \\ 0_{b_2} & & 1_{b_0} \end{array} \longrightarrow \begin{array}{ccc} & 0_{b_1} & \\ \boxed{0_{b_2}} & & 1_{b_0} \end{array}$$

which sends the basis element b_1 to the basis element b_2 and all other basis elements to zero does not factor through any projective Λ-module. Since this morphism is k-linearly independent from the identity morphism, it follows that

$$dim_k \underline{\mathrm{End}}_\Lambda({}_{LL}M(e_1)) \geq 2.$$

For ${}_{LLL}M(e_1)$, the endomorphism

$$\begin{array}{cccc} \boxed{1_{b_3}} & & 0_{b_1} & \\ & 0_{b_2} & & 1_{b_0} \end{array} \longrightarrow \begin{array}{cccc} 1_{b_3} & & 0_{b_1} & \\ & 0_{b_2} & & \boxed{1_{b_0}} \end{array}$$

which sends the basis element b_3 to the basis element b_0 and all other basis elements to zero does not factor through any projective Λ-module. Since this morphism is k-linearly independent from the identity morphism, it follows that

$$dim_k \underline{\mathrm{End}}_\Lambda({}_{LLL}M(e_1)) \geq 2.$$

In general, we obtain that ${}_{L\cdots LL}M(e_1)$ is as in one of the following three cases:

(i) $$\begin{array}{cccccccc} & & 0 & & & & 0 & \\ & 0 & & 1 & \cdots & 0 & & 1 \\ 1 & & & & & & & \end{array}$$

(ii) $$\begin{array}{cccccccc} 1 & & 0 & & & & 0 & \\ & 0 & & 1 & \cdots & 0 & & 1 \end{array}$$

(iii) $$\begin{array}{ccccccc} & 0 & & & & 0 & \\ 0 & & 1 & \cdots & 0 & & 1 \end{array}$$

In each of these three cases, it is enough to find an endomorphism α of ${}_{L\cdots LL}M(e_1)$ such that α is not a scalar multiple of the identity and such that α does not factor through a projective Λ-module.

In case (i), α can be taken as the endomorphism

$$\begin{array}{cccccccc} & & 0 & & & & \boxed{0} & \\ & 0 & & 1 & \cdots & 0 & & \boxed{1} \\ 1 & & & & & & & \end{array} \longrightarrow \begin{array}{cccccccc} & & 0 & & & & 0 & \\ & \boxed{0} & & 1 & \cdots & 0 & & 1 \\ \boxed{1} & & & & & & & \end{array}$$

In case (ii), α can be taken as the endomorphism

$$\begin{array}{cccccccc} \boxed{1} & & 0 & & & & 0 & \\ & 0 & & 1 & \cdots & 0 & & 1 \end{array} \longrightarrow \begin{array}{cccccccc} 1 & & 0 & & & & 0 & \\ & 0 & & 1 & \cdots & 0 & & \boxed{1} \end{array}$$

In case (iii), α can be taken as the endomorphism

$$\begin{array}{ccccccc} & \boxed{0} & & & & 0 & \\ 0 & & 1 & \cdots & 0 & & 1 \end{array} \longrightarrow \begin{array}{ccccccc} & 0 & & & & 0 & \\ \boxed{0} & & 1 & \cdots & 0 & & 1 \end{array}$$

Finally, we look at $M(S)$ of the form ${}_LM(\gamma\beta), {}_{LL}M(\gamma\beta), {}_{LLL}M(\gamma\beta), \cdots$. In

general, all these modules have the following form:
$$\begin{array}{ccccccccc} &&&&&& 0 \\ &&&& 0 & \cdots & 1 \\ && 1 &&&& 0 \\ & 0 \end{array}$$

It is enough to find an endomorphism α of such a module $M(S)$ such that α is not a scalar multiple of the identity and such that α does not factor through a projective Λ-module. We can take α as the endomorphism

$$\begin{array}{ccccc} &&& 0 \\ && \boxed{0} & \cdots & 1 \\ & \boxed{1} &&& 0 \\ \boxed{0} \end{array} \longrightarrow \begin{array}{ccccc} &&& \boxed{0} \\ && 0 & \cdots & \boxed{1} \\ & 1 &&& \boxed{0} \\ 0 \end{array}$$
□

Corollary 3.11. *Let M be a string module for Λ. Then $\underline{\mathrm{End}}_\Lambda(M) \cong k$ if and only if M lies in the Ω-orbit of S_0, S_1 or $M(\beta)$ or $M(\gamma\beta)$.*

3.3.2 Universal Deformation Rings

In this section, we determine the universal deformation ring $R(\Lambda, M(S))$ for all string modules $M(S)$ whose stable endomorphism ring is isomorphic to k. By Theorem 2.50, if $dim_k Ext^1_\Lambda(M(S), M(S)) = r$, then $R(\Lambda, M(S))$ is isomorphic to a quotient algebra of a power series algebra $k[[t_1, \cdots, t_r]]$ and r is minimal with this property.

Theorem 3.12. *If $S = e_1$ then $Ext^1_\Lambda(M(S), M(S)) = 0$ and $R(\Lambda, M(S)) \cong k$.*

Proof. Since $Hom_\Lambda(\Omega(M(e_1)), M(e_1)) = 0$, it follows by Theorem 2.12 that

$$Ext^1_\Lambda(M(e_1), M(e_1)) = 0.$$

Therefore, $R(\Lambda, M(e_1)) \cong k$. $\square$

Theorem 3.13. *If $S \in \{e_0, \beta\}$ then $dim_k Ext^1_\Lambda(M(S), M(S)) = 1$ and $R(\Lambda, M(S)) \cong k[[t]]/(t^2)$.*

Proof. We consider $S = \beta$, the case of $S = e_0$ being similar, and argue similarly to the proof of Theorem 3.6. We have

$$\Omega(M(\beta)) = \begin{array}{ccc} & & 0 \\ 0 & 1 & \\ & 0 & \end{array}$$

By Theorem 2.38, there is one basis element for $Hom_\Lambda(\Omega(M(\beta), M(\beta))$:

$$\begin{array}{ccc} & & \boxed{0} \\ 0 & \boxed{1} & \\ & 0 & \end{array} \quad \longrightarrow \quad \begin{array}{cc} & \boxed{0} \\ \boxed{1} & \end{array} ,$$

which does not factor through a projective Λ-module. Therefore, $R(\Lambda, M(\beta))$ is a quotient algebra of $k[[t]]$.

Considering $M(\beta\alpha^{-1}\beta^{-1})$, we have a short exact sequence of Λ-modules:

$$0 \to M(\beta) \xrightarrow{\iota} M(\beta\alpha^{-1}\beta^{-1}) \xrightarrow{\tau} M(\beta) \to 0$$

where ι and τ are canonical Λ-module homomorphisms.

Let $R_0 = k[[t]]/(t^2)$. We obtain an R_0-module structure on $M(\beta\alpha^{-1}\beta^{-1})$ by letting t act as $\iota \circ \tau$. Using similar arguments as in the proof of Theorem 3.6, it follows that $M(\beta\alpha^{-1}\beta^{-1})$ is a free R_0-module and that there is a lift $(M(\beta\alpha^{-1}\beta^{-1}), \rho)$ of $M(\beta)$ over R_0. Let $[U, \phi]$ be the universal deformation of $M(\beta)$ over $R(\Lambda, M(\beta))$.

There exists a unique k-algebra homomorphism $\theta : R(\Lambda, M(\beta)) \to R_0$ such that $F(\theta)([U, \phi]) = [M(\beta\alpha^{-1}\beta^{-1}), \rho]$, where F is the deformation functor associated to $M(\beta)$. Since $(M(\beta\alpha^{-1}\beta^{-1}), \rho)$ is not the trivial lift over R_0, it follows that θ is surjective. Because $Ext^1_\Lambda(M(\beta\alpha^{-1}\beta^{-1}), M(\beta)) = 0$ and the only surjective Λ-module homomorphisms $\epsilon : M(\beta\alpha^{-1}\beta^{-1}) \to M(\beta)$ are of the form $\epsilon = c\tau$ for some $c \in k^\star$, we can use similar arguments as in the proof of Theorem 3.6 to show that $\theta : R(\Lambda, M(\beta)) \to R_0 = k[[t]]/(t^2)$ is an isomorphism. $\square$

Theorem 3.14. *If $S = \gamma\beta$, then $dim_k Ext^1_\Lambda(M(S), M(S)) = 1$ and $R(\Lambda, M(S)) \cong k[[t]]$.*

Proof. We argue similarly to the proof of Theorem 3.7. Since $\Omega(M(\gamma\beta)) = M(\gamma\beta)$, $\underline{Hom}_\Lambda(\Omega(M(\gamma\beta)), M(\gamma\beta)) = \underline{\mathrm{End}}_\Lambda(M(\gamma\beta)) \cong k$ by Lemma 3.9. Hence we have $dim_k Ext^1_\Lambda(M(\gamma\beta), M(\gamma\beta)) = dim_k \underline{Hom}_\Lambda(\Omega(M(\gamma\beta)), M(\gamma\beta)) = 1$, which means that $R(\Lambda, M(\gamma\beta))$ is a quotient algebra of $k[[t]]$.

Let $C = \gamma\beta$ so that $M(\gamma\beta) = M(C)$. For all $i \geq 1$, let $T_i = C(\alpha^{-1}C)^{i-1}$, and let $R_i = k[[t]]/(t^i)$. Similarly to the proof of Theorem 3.7, it follows that for all $i \geq 2$, $M(T_i)$ defines a lift $(M(T_i), \psi_i)$ of $M(\gamma\beta)$ over R_i. Taking the inverse limit $N = \varprojlim M(T_i)$, we obtain a lift (N, ψ) of $M(\gamma\beta)$ over $k[[t]]$. We can use similar arguments as in the proof of Theorem 3.7 to show that $R(\Lambda, M(\gamma\beta)) \cong k[[t]]$.

$\square$

3.4 A Particular Algebra With One Isomorphism

Class of Simple Modules

Let Λ be the symmetric basic algebra of type D(1) (following the labeling by Erdmann and Skowroński in [5, Cor. 4.7]). Hence $\Lambda = kQ/I$ where

$$Q = \begin{tikzcd} \overset{0}{\bullet} \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}$$

and I is the ideal of the path algebra kQ generated by the set of relations

$$\left\{\alpha^2 - c\beta\alpha, \beta^2, \beta\alpha - \alpha\beta\right\}.$$

Here $c \in \{0, 1\}$ if $\mathrm{char}(k) = 2$ and $c = 0$ otherwise.

For the remainder of this section, let $\mathrm{char}(k) = 2$ and $c = 1$. There exists precisely one simple Λ-module, up to isomorphism, corresponding to the unique vertex of Q, denoted by S_0. The projective cover P_0 can be visualized as follows:

$$P_0 = \begin{tikzcd} & 0 \arrow[dl, dash, "\alpha"'] \arrow[dr, dash, "\beta"] & \\ 0 \arrow[dr, dash, "\beta"'] \arrow[dr, dash, bend left, "\alpha"] & & 0 \arrow[dl, dash, "\alpha"] \\ & 0 & \end{tikzcd}$$

The stable Auslander-Reiten quiver $\Gamma_S(\Lambda)$ has three connected components consisting of string modules:

- Component 1, which contains the simple module S_0;
- Component 2, which contains the string module corresponding to the maximal directed string α;

- Component 3, which contains the string module corresponding to the maximal directed string β.

Component 1 is a non-periodic component of the form $\mathbb{Z}\tilde{A}_{1,1}$, and components 2 and 3 are 1-tubes.

3.4.1 Stable Endomorphism Rings

We first want to determine all string modules $M(S)$ for Λ whose stable endomorphism ring is isomorphic to k. The following remark gives a full set of representatives of the Ω-orbits for each of the components 1, 2 and 3.

Remark 3.15. (a) Component 1.

As noted above, component 1 has the form $\mathbb{Z}\tilde{A}_{1,1}$. Using section 2.7, we see that there is precisely 1 Ω-orbit of string modules in component 1 with representative:

$$S_0 = M(e_0).$$

(b) Component 2.

The string module $M(\alpha)$ lies at the end of component 2. As described in section 2.7, all modules in component 2 are string modules whose corresponding strings can be obtained from α by either adding hooks or subtracting cohooks on the left. Thus a full set of representatives of the Ω-orbits of component 2 is given as follows, where we use the notation introduced in section 3.1:

$$M(\alpha), {}_LM(\alpha), {}_{LL}M(\alpha), \cdots .$$

(c) Component 3.

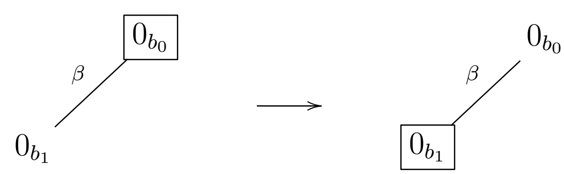
The string module $M(\beta)$ lies at the end of component 3. As described in section 2.7, all modules in component 3 are string modules whose corresponding strings can be obtained from β by either adding hooks or subtracting cohooks on the left. Thus a full set of representatives of the Ω-orbits of component 3 is given as follows, where we use the notation introduced in section 3.1:

$$M(\beta), {}_LM(\beta), {}_{LL}M(\beta), \cdots .$$

Lemma 3.16. *We have* $\underline{\mathrm{End}}_\Lambda(M(S)) \cong k$ *for* $S \in \{e_0, \beta\}$.

Proof. By Schur's lemma, $End_\Lambda(M(S_0)) \cong k$, which implies $\underline{\mathrm{End}}_\Lambda(M(S_0)) \cong k$.

Next, consider $M(\beta)$. By Theorem 2.38, a k-basis for $End_\Lambda(M(\beta))$ is given by the identity morphism together with the morphism



which sends the basis element b_0 of $M(\beta)$ to the basis element b_1 and the basis element b_1 to zero. Hence $dim_k End_\Lambda(M(\beta)) = 2$. The morphism $b_0 \to b_1$ factors through the projective Λ-module P_0 as follows:

$$\begin{tikzcd}
 & 0_{b_0} & & & 0_{c_0} \arrow[dr, dash, "\beta"] & & & & 0_{b_0} \\
0_{b_1} \arrow[ur, dash, "\beta"] & {} \arrow[rr, "\delta"] & & 0_{c_1} \arrow[ur, dash, "\alpha"] \arrow[dr, dash, "\beta"'] \arrow[dr, dash, bend left, "\alpha"] & & 0_{c_3} \arrow[rr, "\gamma"] & & 0_{b_1} \arrow[ur, dash, "\beta"] & \\
 & & & & 0_{c_2} \arrow[ur, dash, "\alpha"'] & & & &
\end{tikzcd}$$

where δ maps $b_0 \to c_1 - c_3$ and $b_1 \to c_2$, and γ maps $c_0 \to b_0$, $c_3 \to b_1$, and all other basis elements of P_0 to 0. Thus, $\underline{\mathrm{End}}_\Lambda(M(\beta)) \cong k$. $\square$

Lemma 3.17. *If $M(S)$ is a string module for Λ and $M(S)$ does not lie in the Ω-orbit of either S_0 or $M(\beta)$ then $\underline{\mathrm{End}}_\Lambda(M(S)) \ncong k$.*

Proof. By Remark 3.15, we only have to show that $\underline{\mathrm{End}}_\Lambda(M(S)) \ncong k$ for $M(S)$ of the form ${}_LM(\beta), {}_{LL}M(\beta), {}_{LLL}M(\beta), \cdots$ or of the form $M(\alpha), {}_LM(\alpha), {}_{LL}M(\alpha), \cdots$.

For $M(\alpha)$, the endomorphism

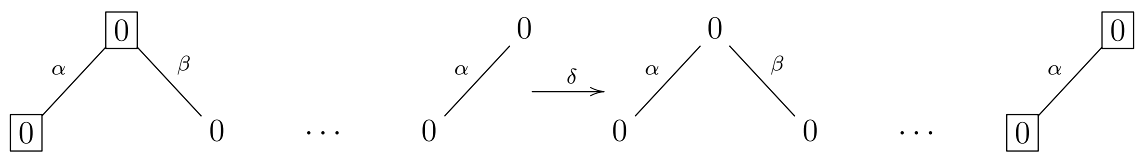
$$\begin{tikzcd}
 & \boxed{0_{b_0}} & & & 0_{b_0} \\
0_{b_1} \arrow[ur, dash, "\alpha"] & {} \arrow[r] & {} & \boxed{0_{b_1}} \arrow[ur, dash, "\alpha"] &
\end{tikzcd}$$

does not factor through the projective Λ-module P_0. Since this morphism is k-linearly independent from the identity morphism, it follows that $dim_k \, \underline{\mathrm{End}}_\Lambda(M(\alpha)) \geq 2$.

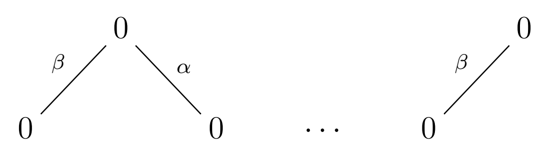
In general, we obtain that ${}_{L\cdots L}M(\alpha)$ has the following form:

$$\begin{tikzcd}
 & 0 \arrow[dr, dash, "\beta"] & & & & 0 \\
0 \arrow[ur, dash, "\alpha"] & & 0 & \cdots & 0 \arrow[ur, dash, "\alpha"] &
\end{tikzcd}$$

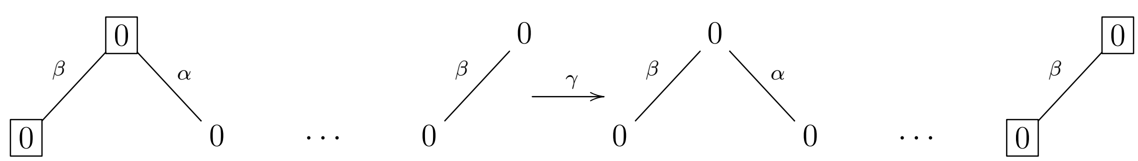
It is enough to find an endomorphism δ of ${}_{L\cdots L}M(\alpha)$ such that δ is not a scalar multiple of the identity and such that δ does not factor through the projective Λ-module P_0. We can take δ to be the endomorphism



Similarly, in general, we obtain that ${}_{L\cdots L}M(\beta)$ has the following form:



It is enough to find an endomorphism γ of ${}_{L\cdots L}M(\beta)$ such that γ is not a scalar multiple of the identity and such that γ does not factor through the projective Λ-module P_0. We can take γ to be the endomorphism



$\square$

Corollary 3.18. *Let M be a string module for Λ. Then $\underline{\mathrm{End}}_\Lambda(M) \cong k$ if and only if M lies in the Ω-orbit of S_0 or $M(\beta)$.*

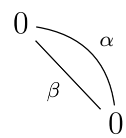
3.4.2 Universal Deformation Rings

In this section, we determine the universal deformation ring $R(\Lambda, M(S))$ for all string modules $M(S)$ whose stable endomorphism ring is isomorphic to k. By

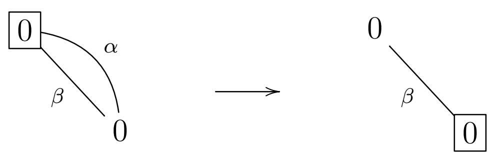
Theorem 2.50, if $dim_k Ext^1_\Lambda(M(S), M(S)) = r$, then $R(\Lambda, M(S))$ is isomorphic to a quotient algebra of a power series algebra $k[[t_1, \cdots, t_r]]$ and r is minimal with this property.

Theorem 3.19. *If $S = \beta$, then $dim_k Ext^1_\Lambda(M(S), M(S)) = 1$ and $R(\Lambda, M(S)) \cong k[[t]]$.*

Proof. We argue similarly to the proof of Theorem 3.7. We have

$\Omega(M(\beta)) =$  .

There is only one basis element of $Hom_\Lambda(\Omega(M(\beta)), M(\beta))$:



Since this morphism does not factor through a projective Λ-module, we have

$$dim_k Ext^1_\Lambda(M(\beta), M(\beta)) = dim_k \underline{Hom}_\Lambda(\Omega(M(\beta)), M(\beta)) = 1.$$

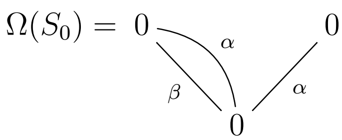
Therefore, $R(\Lambda, M(\beta))$ is a quotient algebra of $k[[t]]$.

Let $C = \beta$ so that $M(\beta) = M(C)$. For all $i \geq 1$, let $T_i = C(\alpha^{-1}C)^{i-1}$, and let $R_i = k[[t]]/(t^i)$. Similarly to the proof of Theorem 3.7, it follows that for all $i \geq 2$, $M(T_i)$ defines a lift $(M(T_i), \psi_i)$ of $M(\beta)$ over R_i. Taking the inverse limit $N = \varprojlim M(T_i)$, we obtain a lift (N, ψ) of $M(\beta)$ over $k[[t]]$. We can use similar arguments as in the proof of Theorem 3.7 to show that $R(\Lambda, M(\beta)) \cong k[[t]]$. $\square$

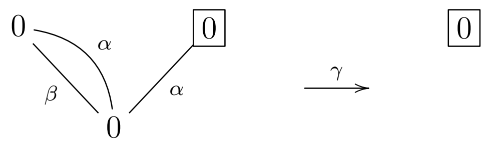
Theorem 3.20. *For $S_0 = M(e_0)$, we have*

$$dim_k Ext^1_\Lambda(S_0, S_0) = 2 \text{ and } R(\Lambda, S_0) \cong \Lambda \cong k[[t_1, t_2]]/(t_1^2, t_2^2 - t_1 t_2).$$

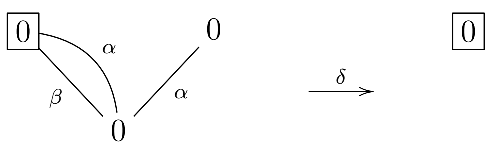
Proof. We have



There are two basis elements for $Hom_\Lambda(\Omega(S_0), S_0)$:



and



Since these morphisms do not factor through the projective Λ-module P_0, we have $dim_k Ext^1_\Lambda(S_0, S_0) = dim_k \underline{Hom}_\Lambda(\Omega(S_0), S_0) = 2$. Therefore, $R(\Lambda, S_0)$ is a quotient algebra of $k[[t_1, t_2]]$.

To prove that $R(\Lambda, S_0) \cong \Lambda$, we consider the representations of Λ corresponding to lifts of S_0. More precisely, as in section 2.8, let $\hat{\mathcal{C}}$ be the category of all complete local commutative Noetherian k-algebras with residue field k, where the morphisms are continuous k-algebra homomorphisms inducing the identity on k. For each ring R in $\hat{\mathcal{C}}$, let $\pi_R : R \to k$ denote its natural surjection onto the residue field. Note that Λ is a ring in $\hat{\mathcal{C}}$ with $\pi_\Lambda : \Lambda \to k$ given by $\pi_\Lambda(c_1 e_0 + c_2 \alpha + c_3 \beta + c_4 \alpha\beta) = c_1$ for all $c_1, c_2, c_3, c_4 \in k$.

We have $S_0 = k$ as a k-vector space and the Λ-action on S_0 is given by $\alpha_{S_0} : \Lambda \to k$ with $\alpha_{S_0} = \pi_\Lambda$. Let R be a ring in $\hat{\mathcal{C}}$. Following Definition 2.44, a lift of S_0 over R is given by a free R-module $M = R$ and a k-algebra homomorphism $\alpha_M : \Lambda \to R$ with $\pi_R \circ \alpha_M = \alpha_{S_0}$. This implies that α_M is a continuous k-algebra homomorphism and $\pi_R \circ \alpha_M = \pi_\Lambda$, meaning that α_M induces the identity on k. In other words, $\alpha_M : \Lambda \to R$ is a morphism in $\hat{\mathcal{C}}$. If we have another lift of S_0 over R, given by $M' = R$ and $\alpha_{M'} : \Lambda \to R$ in $\hat{\mathcal{C}}$, then M and M' are isomorphic as lifts if and only if there exists $r \in R^\star$ with $\pi_R(r) = 1$ and $r\alpha_{M'}r^{-1} = \alpha_M$. Since this implies $\alpha_{M'} = \alpha_M$, it follows that the set of isomorphism classes of lifts of S_0 over R contains exactly one element.

We now define a lift U of S_0 over Λ by letting $U = \Lambda$ as a Λ-module and defining $\alpha_U : \Lambda \to \Lambda$ in $\hat{\mathcal{C}}$ to be the identity morphism, i.e. $\alpha_U = id_\Lambda$. This certainly defines a lift of S_0 over Λ since $\pi_\Lambda \circ \alpha_U = \pi_\Lambda = \alpha_{S_0}$. We want to show that α_U defines the universal lift of S_0 over Λ. Let $R \in Ob(\hat{\mathcal{C}})$ and let M be a lift of S_0 over R given by $\alpha_M : \Lambda \to R$. As seen above, α_M is a morphism in $\hat{\mathcal{C}}$. We need to show that there exists a unique morphism $\tau : \Lambda \to R$ in $\hat{\mathcal{C}}$ such that the lift of S_0 over R defined by $\tau \circ \alpha_U$ is isomorphic to the lift defined by α_M. By our above discussion of isomorphism classes of lifts, this means we need to show that there exists a unique morphism $\tau : \Lambda \to R$ in $\hat{\mathcal{C}}$ with $\tau \circ \alpha_U = \alpha_M$. Since $\alpha_U = id_\Lambda$ and α_M is a morphism in $\hat{\mathcal{C}}$, it follows that $\tau = \alpha_M$ is the unique choice. This implies that $R(\Lambda, S_0) \cong \Lambda$.

$\square$

3.5 Remaining Cases

3.5.1 The Symmetric Basic Algebra of Type $\mathrm{D}(3\mathcal{A})_1$

Let Λ be the symmetric basic algebra of type $\mathrm{D}(3\mathcal{A})_1$ (following the labeling by Erdmann and Skowroński in [5, Cor. 4.7]). Hence $\Lambda = kQ/I$ where

$$Q = \begin{tikzcd} \overset{0}{\bullet} \arrow[r, shift left, "\beta"] & \overset{1}{\bullet} \arrow[l, shift left, "\gamma"] \arrow[r, shift left, "\delta"] & \overset{2}{\bullet} \arrow[l, shift left, "\eta"] \end{tikzcd}$$

and I is the ideal of the path algebra kQ generated by the set of relations

$$\{\gamma\beta, \delta\eta, \eta\delta\beta\gamma - \beta\gamma\eta\delta\}\,.$$

There exist precisely three simple Λ-modules, up to isomorphism, corresponding to the three vertices of Q, denoted by S_0, S_1, S_2. Their projective covers P_0, P_1, P_2 can be visualized as follows:

$$P_0\colon \begin{array}{c} 0\\1\\2\\1\\0 \end{array} \qquad P_1\colon \begin{array}{ccc} & 1 & \\ 0 & & 2\\ 1 & & 1\\ 2 & & 0\\ & 1 & \end{array} \qquad P_2\colon \begin{array}{c} 2\\1\\0\\1\\2 \end{array}$$

Lemma 3.21. *Let M be a string module for Λ. Then $\underline{\mathrm{End}}_\Lambda(M) \cong k$ if and only if M lies in the Ω-orbit of S_0, S_1, S_2 or $M(\beta)$ or $M(\eta)$.*

Proof. The proof uses similar arguments as the proofs of Lemma 3.2 and Lemma 3.3. □

Theorem 3.22. *If $S \in \{e_0, e_1, e_2, \beta, \eta\}$ then*

$$Ext^1_\Lambda(M(S), M(S)) = 0 \text{ and } R(\Lambda, M(S)) \cong k.$$

Proof. The proof uses similar arguments as the proof of Theorem 3.5. □

3.5.2 The First of Two Symmetric Basic Algebras of Type D($3\mathcal{B}$)$_2$

Let Λ be the symmetric basic algebra of type D($3\mathcal{B}$)$_2$ (following the labeling by Erdmann and Skowroński in [5, Cor. 4.7]). Hence $\Lambda = kQ/I$ where

$$Q = \begin{tikzcd} \overset{0}{\bullet} \arrow[loop left, "\alpha"] \arrow[r, shift left, "\beta"] & \overset{1}{\bullet} \arrow[l, shift left, "\gamma"] \arrow[r, shift left, "\delta"] & \overset{2}{\bullet} \arrow[l, shift left, "\eta"] \end{tikzcd}$$

and I is the ideal of the path algebra kQ generated by the set of relations

$$\left\{\alpha\gamma, \beta\alpha, \delta\beta, \gamma\eta, (\gamma\beta)^k - \alpha^2, (\beta\gamma)^k - (\eta\delta)^l\right\}.$$

Here $k, l \in \{1, 2\}$.

For the remainder of this subsection, let $k = 1$, $l = 2$. The case $k = 2$, $l = 1$ is considered in Section 3.5.3. There exist precisely three simple Λ-modules, up to isomorphism, corresponding to the three vertices of Q, denoted by S_0, S_1, S_2. Their projective covers P_0, P_1, P_2 can be visualized as follows:

$$P_0: \begin{array}{ccc} & 0 & \\ 0 & & 1 \\ & 0 & \end{array} \qquad P_1: \begin{array}{ccc} & 1 & \\ 2 & & \\ 1 & & 0 \\ 2 & & \\ & 1 & \end{array} \qquad P_2: \begin{array}{c} 2 \\ 1 \\ 2 \\ 1 \\ 2 \end{array}$$

Lemma 3.23. *Let M be a string module for Λ. Then $\underline{\mathrm{End}}_\Lambda(M) \cong k$ if and only if M lies in the Ω-orbit of S_0, S_1, S_2 or $M(\eta)$ or ${}_LM(e_2)$ or ${}_{LL}M(e_2)$.*

Proof. The proof uses similar arguments as the proofs of Lemma 3.2 and Lemma 3.3. $\square$

Theorem 3.24. *If $S \in \{e_1, e_2\}$ or $M(S) = {}_LM(e_2)$ then $Ext^1_\Lambda(M(S), M(S)) = 0$ and $R(\Lambda, M(S)) \cong k$.*

Proof. The proof uses similar arguments as the proof of Theorem 3.5. $\square$

Theorem 3.25. *If $S \in \{e_0, \eta\}$ then $dim_k Ext^1_\Lambda(M(S), M(S)) = 1$ and*

$$R(\Lambda, M(S)) \cong k[[t]]/(t^2).$$

Proof. We consider $S = \eta$, the case of $S = e_0$ being similar. Similarly to the proof of Theorem 3.6, we show that

$$dim_k Ext^1_\Lambda(M(S), M(S)) = dim_k \underline{Hom}_\Lambda(\Omega(M(S), M(S)) = 1,$$

which implies that $R(\Lambda, M(S))$ is a quotient algebra of $k[[t]]$.

Letting $L_0 = M(\eta\delta\eta)$, we have a short exact sequence of Λ-modules:

$$0 \to M(S) \xrightarrow{\iota} L_0 \xrightarrow{\tau} M(S) \to 0$$

where ι and τ are canonical Λ-module homomorphisms. Let $R_0 = k[[t]]/(t^2)$. We obtain an R_0-module structure on L_0 by letting t act as $\iota \circ \tau$. Similarly to the proof of Theorem 3.6, it follows that L_0 defines a lift (L_0, ρ) of $M(S)$ over R_0 and that the corresponding unique k-algebra homomorphism $\theta : R(\Lambda, M(S)) \to R_0$ is an isomorphism. $\square$

Theorem 3.26. *If $M(S) = {}_{LL}M(e_2)$ then*

$$dim_k Ext^1_\Lambda(M(S), M(S)) = 1 \text{ and } R(\Lambda, M(S)) \cong k[[t]].$$

Proof. We consider $M(S) = {}_{LL}M(e_2)$. Similarly to the proof of Theorem 3.7, we show that $dim_k Ext^1_\Lambda(M(S), M(S)) = dim_k \underline{Hom}_\Lambda(\Omega(M(S)), M(S)) = 1$, which implies that $R(\Lambda, M(S))$ is a quotient algebra of $k[[t]]$.

Let $C = \beta\alpha^{-1}\gamma\delta^{-1}$, and let for all $i \geq 1$, $T_i = C(\eta^{-1}C)^{i-1}$. Similarly to the proof of Theorem 3.7, it follows that for all $i \geq 2$, $M(T_i)$ defines a lift $(M(T_i), \psi_i)$ of $M(S)$ over $k[[t]]/(t^i)$. Taking the inverse limit $N = \varprojlim M(T_i)$, we obtain a lift (N, ψ) of $M(S)$ over $k[[t]]$. Arguing similarly as in the proof of Theorem 3.7, it follows that $R(\Lambda, M(S)) \cong k[[t]]$. $\square$

3.5.3 The Second of Two Symmetric Basic Algebras of Type D(3$\mathcal{B}$)$_2$

Let Λ be the symmetric basic algebra of type D(3$\mathcal{B}$)$_2$ (following the labeling by Erdmann and Skowroński in [5, Cor. 4.7]). Hence $\Lambda = kQ/I$ where

$$Q = \begin{tikzcd} \overset{0}{\bullet} \arrow[loop left, "\alpha"] \arrow[r, shift left, "\beta"] & \overset{1}{\bullet} \arrow[l, shift left, "\gamma"] \arrow[r, shift left, "\delta"] & \overset{2}{\bullet} \arrow[l, shift left, "\eta"] \end{tikzcd}$$

and I is the ideal of the path algebra kQ generated by the set of relations

$$\left\{\alpha\gamma, \beta\alpha, \delta\beta, \gamma\eta, (\gamma\beta)^k - \alpha^2, (\beta\gamma)^k - (\eta\delta)^l\right\}.$$

Here $k, l \in \{1, 2\}$.

For the remainder of this subsection, let $k = 2$, $l = 1$. The case $k = 1$, $l = 2$ is considered in Section 3.5.2. There exist precisely three simple Λ-modules, up to isomorphism, corresponding to the three vertices of Q, denoted by S_0, S_1, S_2. Their projective covers P_0, P_1, P_2 can be visualized as follows:

$$P_0: \begin{array}{ccc} & 0 & \\ 1 & & \\ 0 & & 0 \\ 1 & & \\ & 0 & \end{array} \qquad P_1: \begin{array}{ccc} & 1 & \\ 0 & & \\ 1 & & 2 \\ 0 & & \\ & 1 & \end{array} \qquad P_2: \begin{array}{c} 2 \\ 1 \\ 2 \end{array}$$

Lemma 3.27. *Let M be a string module for Λ. Then $\underline{\mathrm{End}}_\Lambda(M) \cong k$ if and only if M lies in the Ω-orbit of S_0, S_1, S_2 or $M(\gamma)$ or $M(\beta\gamma)$ or $M(\gamma\beta)$.*

Proof. The proof uses similar arguments as the proofs of Lemma 3.2 and Lemma 3.3. □

Theorem 3.28. *If $S \in \{e_1, e_2, \beta\gamma\}$ then $Ext^1_\Lambda(M(S), M(S)) = 0$ and*

$$R(\Lambda, M(S)) \cong k.$$

Proof. The proof uses similar arguments as the proof of Theorem 3.5. □

Theorem 3.29. *If $S \in \{e_0, \gamma\}$ then $dim_k Ext^1_\Lambda(M(S), M(S)) = 1$ and*

$$R(\Lambda, M(S)) \cong k[[t]]/(t^2).$$

Proof. We consider $S = \gamma$, the case of $S = e_0$ being similar. Similarly to the proof of Theorem 3.6, we show that

$$dim_k Ext^1_\Lambda(M(S), M(S)) = dim_k \underline{Hom}_\Lambda(\Omega(M(S), M(S)) = 1,$$

which implies that $R(\Lambda, M(S))$ is a quotient algebra of $k[[t]]$.

Letting $L_0 = M(\gamma\beta\gamma)$, we have a short exact sequence of Λ-modules:

$$0 \to M(S) \xrightarrow{\iota} L_0 \xrightarrow{\tau} M(S) \to 0$$

where ι and τ are canonical Λ-module homomorphisms. Let $R_0 = k[[t]]/(t^2)$. We obtain an R_0-module structure on L_0 by letting t act as $\iota \circ \tau$. Similarly to the proof of Theorem 3.6, it follows that L_0 defines a lift (L_0, ρ) of $M(S)$ over R_0 and that the corresponding unique k-algebra homomorphism $\theta : R(\Lambda, M(S)) \to R_0$ is an isomorphism. □

Theorem 3.30. *If $S = \gamma\beta$ then $dim_k Ext^1_\Lambda(M(S), M(S)) = 1$ and*

$$R(\Lambda, M(S)) \cong k[[t]].$$

Proof. We consider $S = \gamma\beta$. Similarly to the proof of Theorem 3.7, we show that

$$dim_k Ext^1_\Lambda(M(S), M(S)) = dim_k \underline{Hom}_\Lambda(\Omega(M(S)), M(S)) = 1,$$

which implies that $R(\Lambda, M(S))$ is a quotient algebra of $k[[t]]$.

Let $C = S = \gamma\beta$, and let for all $i \geq 1$, $T_i = C(\alpha^{-1}C)^{i-1}$. Similarly to the proof of Theorem 3.7, it follows that for all $i \geq 2$, $M(T_i)$ defines a lift $(M(T_i), \psi_i)$ of $M(S)$ over $k[[t]]/(t^i)$. Taking the inverse limit $N = \varprojlim M(T_i)$, we obtain a lift (N, ψ) of $M(S)$ over $k[[t]]$. Arguing similarly as in the proof of Theorem 3.7, it follows that $R(\Lambda, M(S)) \cong k[[t]]$. □

3.5.4 The Symmetric Basic Algebra of Type D$(3\mathcal{D})_2$

Let Λ be the symmetric basic algebra of type D$(3\mathcal{D})_2$ (following the labeling by Erdmann and Skowroński in [5, Cor. 4.7]). Hence $\Lambda = kQ/I$ where

$$Q = \begin{tikzcd} \overset{0}{\bullet} \arrow[loop left, "\alpha"] \arrow[r, shift left, "\beta"] & \overset{1}{\bullet} \arrow[l, shift left, "\gamma"] \arrow[r, shift left, "\delta"] & \overset{2}{\bullet} \arrow[l, shift left, "\eta"] \arrow[loop right, "\xi"] \end{tikzcd}$$

and I is the ideal of the path algebra kQ generated by the set of relations

$$\left\{\alpha\gamma, \beta\alpha, \delta\beta, \gamma\eta, \xi\delta, \eta\xi, \beta\gamma - \eta\delta, \gamma\beta - \alpha^2, \delta\eta - \xi^2\right\}.$$

There exist precisely three simple Λ-modules, up to isomorphism, corresponding to the three vertices of Q, denoted by S_0, S_1, S_2. Their projective covers P_0, P_1,

P_2 can be visualized as follows:

$$P_0\colon\ \begin{array}{ccc} & 0 & \\ 0 & & 1 \\ & 0 & \end{array} \qquad\qquad P_1\colon\ \begin{array}{ccc} & 1 & \\ 0 & & 2 \\ & 1 & \end{array} \qquad\qquad P_2\colon\ \begin{array}{ccc} & 2 & \\ 2 & & 1 \\ & 2 & \end{array}$$

Lemma 3.31. *Let M be a string module for Λ. Then $\underline{\mathrm{End}}_\Lambda(M) \cong k$ if and only if M lies in the Ω-orbit of S_0, S_1, S_2 or $M(\gamma)$ or ${}_LM(\gamma)$ or ${}_{LL}M(\gamma)$.*

Proof. The proof uses similar arguments as the proofs of Lemma 3.2 and Lemma 3.3. $\square$

Theorem 3.32. *If $S \in \{e_1, \gamma\}$ or $M(S) = {}_LM(\gamma)$ then $Ext^1_\Lambda(M(S), M(S)) = 0$ and $R(\Lambda, M(S)) \cong k$.*

Proof. The proof uses similar arguments as the proof of Theorem 3.5. $\square$

Theorem 3.33. *If $S \in \{e_0, e_2\}$ then $dim_k Ext^1_\Lambda(M(S), M(S)) = 1$ and*

$$R(\Lambda, M(S)) \cong k[[t]]/(t^2).$$

Proof. We consider $S = e_2$, the case of $S = e_0$ being similar. Similarly to the proof of Theorem 3.6, we show that

$$dim_k Ext^1_\Lambda(M(S), M(S)) = dim_k \underline{Hom}_\Lambda(\Omega(M(S), M(S)) = 1,$$

which implies that $R(\Lambda, M(S))$ is a quotient algebra of $k[[t]]$.

Letting $L_0 = M(\xi)$, we have a short exact sequence of Λ-modules:

$$0 \to M(S) \xrightarrow{\iota} L_0 \xrightarrow{\tau} M(S) \to 0$$

where ι and τ are canonical Λ-module homomorphisms. Let $R_0 = k[[t]]/(t^2)$. We obtain an R_0-module structure on L_0 by letting t act as $\iota \circ \tau$. Similarly to the proof of Theorem 3.6, it follows that L_0 defines a lift (L_0, ρ) of $M(S)$ over R_0 and that the corresponding unique k-algebra homomorphism $\theta : R(\Lambda, M(S)) \to R_0$ is an isomorphism. $\square$

Theorem 3.34. *If* $M(S) = {}_{LL}M(\gamma)$ *then*

$$dim_k Ext^1_\Lambda(M(S), M(S)) = 1 \text{ and } R(\Lambda, M(S)) \cong k[[t]].$$

Proof. We consider $M(S) = {}_{LL}M(\gamma)$. Similarly to the proof of Theorem 3.7, we show that $dim_k Ext^1_\Lambda(M(S), M(S)) = dim_k \underline{Hom}_\Lambda(\Omega(M(S)), M(S)) = 1$, which implies that $R(\Lambda, M(S))$ is a quotient algebra of $k[[t]]$.

Let $C = \xi\eta^{-1}\beta\alpha^{-1}\gamma$, and let for all $i \geq 1$, $T_i = C(\delta^{-1}C)^{i-1}$. Similarly to the proof of Theorem 3.7, it follows that for all $i \geq 2$, $M(T_i)$ defines a lift $(M(T_i), \psi_i)$ of $M(S)$ over $k[[t]]/(t^i)$. Taking the inverse limit $N = \varprojlim M(T_i)$, we obtain a lift (N, ψ) of $M(S)$ over $k[[t]]$. Arguing similarly as in the proof of Theorem 3.7, it follows that $R(\Lambda, M(S)) \cong k[[t]]$. $\square$

3.5.5 The Symmetric Basic Algebra of Type D(3$\mathcal{K}$)

Let Λ be the symmetric basic algebra of type D(3$\mathcal{K}$) (following the labeling by Erdmann and Skowroński in [5, Cor. 4.7]). Hence $\Lambda = kQ/I$ where

$$Q = \begin{tikzcd} \underset{0}{\bullet} \arrow[rr, shift right, "\beta"'] \arrow[dr, shift left, "\kappa"] & & \underset{1}{\bullet} \arrow[ll, shift right, "\gamma"'] \arrow[dl, shift right, "\delta"'] \\ & \underset{2}{\bullet} \arrow[ul, shift left, "\lambda"] \arrow[ur, shift right, "\eta"'] & \end{tikzcd}$$

and I is the ideal of the path algebra kQ generated by the set of relations

$$\{\delta\beta, \lambda\delta, \beta\lambda, \kappa\gamma, \eta\kappa, \gamma\eta, \gamma\beta - \lambda\kappa, \beta\gamma - \eta\delta, \delta\eta - \kappa\lambda\}\,.$$

There exist precisely three simple Λ-modules, up to isomorphism, corresponding to the three vertices of Q, denoted by S_0, S_1, S_2. Their projective covers P_0, P_1, P_2 can be visualized as follows:

$$P_0:\ \begin{array}{ccc} & 0 & \\ 1 & & 2 \\ & 0 & \end{array} \qquad P_1:\ \begin{array}{ccc} & 1 & \\ 2 & & 0 \\ & 1 & \end{array} \qquad P_2:\ \begin{array}{ccc} & 2 & \\ 0 & & 1 \\ & 2 & \end{array}$$

Lemma 3.35. *Let M be a string module for Λ. Then $\underline{\mathrm{End}}_\Lambda(M) \cong k$ if and only if M lies in the Ω-orbit of S_0, S_1, S_2 or $M(\beta)$ or $M(\gamma)$.*

Proof. The proof uses similar arguments as the proofs of Lemma 3.2 and Lemma 3.3. $\square$

Theorem 3.36. *If $S \in \{e_0, e_1, e_2, \beta, \gamma\}$ then*

$$Ext^1_\Lambda(M(S), M(S)) = 0 \text{ and } R(\Lambda, M(S)) \cong k.$$

Proof. The proof uses similar arguments as the proof of Theorem 3.5. $\square$

3.5.6 The Symmetric Basic Algebra of Type D($3\mathcal{L}$)

Let Λ be the symmetric basic algebra of type D($3\mathcal{L}$) (following the labeling by Erdmann and Skowroński in [5, Cor. 4.7]). Hence $\Lambda = kQ/I$ where

$$Q = \begin{tikzcd} \overset{0}{\bullet} \arrow[loop left, "\alpha"] \arrow[rr, "\beta"] & & \overset{1}{\bullet} \arrow[dl, "\delta"] \\ & \overset{2}{\bullet} \arrow[ul, "\lambda"] & \end{tikzcd}$$

and I is the ideal of the path algebra kQ generated by the set of relations

$$\left\{\beta\alpha, \alpha\lambda, (\lambda\delta\beta)^2 - \alpha^2, \delta(\beta\lambda\delta)^2\right\}.$$

There exist precisely three simple Λ-modules, up to isomorphism, corresponding to the three vertices of Q, denoted by S_0, S_1, S_2. Their projective covers P_0, P_1, P_2 can be visualized as follows:

$$\begin{array}{ccccccccc}
 & 0 & & \qquad & & 1 & \qquad & & 2 \\
 & & 1 & & & 2 & & & 0 \\
 & & 2 & & & 0 & & & 1 \\
0 & & 0 & & P_1: & 1 & & P_2: & 2 \\
 & & 1 & & & 2 & & & 0 \\
 & & 2 & & & 0 & & & 1 \\
 & 0 & & & & 1 & & & 2
\end{array}$$

Lemma 3.37. *Let M be a string module for Λ. Then $\underline{\mathrm{End}}_\Lambda(M) \cong k$ if and only if M lies in the Ω-orbit of S_0, S_1, S_2 or $M(\beta)$ or $M(\delta\beta)$ or ${}_LM(e_2)$ or ${}_{LL}M(e_2)$.*

Proof. The proof uses similar arguments as the proofs of Lemma 3.2 and Lemma 3.3. □

Theorem 3.38. *If $S \in \{e_1, e_2, \beta\}$ or $M(S) = {}_LM(e_2)$ then $Ext^1_\Lambda(M(S), M(S)) = 0$ and $R(\Lambda, M(S)) \cong k$.*

Proof. The proof uses similar arguments as the proof of Theorem 3.5. □

Theorem 3.39. *If $S \in \{e_0, \delta\beta\}$ then $dim_k Ext^1_\Lambda(M(S), M(S)) = 1$ and*

$$R(\Lambda, M(S)) \cong k[[t]]/(t^2).$$

Proof. We consider $S = \delta\beta$, the case of $S = e_0$ being similar. Similarly to the proof of Theorem 3.6, we show that

$$dim_k Ext^1_\Lambda(M(S), M(S)) = dim_k \underline{Hom}_\Lambda(\Omega(M(S), M(S)) = 1,$$

which implies that $R(\Lambda, M(S))$ is a quotient algebra of $k[[t]]$.

Letting $L_0 = M(\delta\beta\lambda\delta\beta)$, we have a short exact sequence of Λ-modules:

$$0 \to M(S) \xrightarrow{\iota} L_0 \xrightarrow{\tau} M(S) \to 0$$

where ι and τ are canonical Λ-module homomorphisms. Let $R_0 = k[[t]]/(t^2)$. We obtain an R_0-module structure on L_0 by letting t act as $\iota \circ \tau$. Similarly to the proof of Theorem 3.6, it follows that L_0 defines a lift (L_0, ρ) of $M(S)$ over R_0 and that the corresponding unique k-algebra homomorphism $\theta : R(\Lambda, M(S)) \to R_0$ is an isomorphism. $\square$

Theorem 3.40. *If $M(S) = {}_{LL}M(e_2)$ then*

$$dim_k Ext^1_\Lambda(M(S), M(S)) = 1 \text{ and } R(\Lambda, M(S)) \cong k[[t]].$$

Proof. We consider $M(S) = {}_{LL}M(e_2)$. Similarly to the proof of Theorem 3.7, we show that $dim_k Ext^1_\Lambda(M(S), M(S)) = dim_k \underline{Hom}_\Lambda(\Omega(M(S)), M(S)) = 1$, which implies that $R(\Lambda, M(S))$ is a quotient algebra of $k[[t]]$.

Let $C = \alpha\beta^{-1}\delta^{-1}$, and let for all $i \geq 1$, $T_i = C(\lambda^{-1}C)^{i-1}$. Similarly to the proof of Theorem 3.7, it follows that for all $i \geq 2$, $M(T_i)$ defines a lift $(M(T_i), \psi_i)$ of $M(S)$ over $k[[t]]/(t^i)$. Taking the inverse limit $N = \varprojlim M(T_i)$, we obtain a lift (N, ψ) of $M(S)$ over $k[[t]]$. Arguing similarly as in the proof of Theorem 3.7, it follows that $R(\Lambda, M(S)) \cong k[[t]]$. $\square$

3.5.7 The Symmetric Basic Algebra of Type D(3$\mathcal{Q}$)

Let Λ be the symmetric basic algebra of type D(3$\mathcal{Q}$) (following the labeling by Erdmann and Skowroński in [5, Cor. 4.7]). Hence $\Lambda = kQ/I$ where

$$Q = \begin{tikzcd} \overset{0}{\bullet} \arrow[loop left, "\alpha"] \arrow[rr, "\beta"] & & \overset{1}{\bullet} \arrow[loop right, "\rho"] \arrow[dl, "\delta"] \\ & \underset{2}{\bullet} \arrow[ul, "\lambda"] & \end{tikzcd}$$

and I is the ideal of the path algebra kQ generated by the set of relations

$$\left\{\beta\alpha, \alpha\lambda, \rho\beta, \delta\rho, \lambda\delta\beta - \alpha^2, \beta\lambda\delta - \rho^2\right\}.$$

There exist precisely three simple Λ-modules, up to isomorphism, corresponding to the three vertices of Q, denoted by S_0, S_1, S_2. Their projective covers P_0, P_1, P_2 can be visualized as follows:

$$P_0: \begin{array}{ccc} & 0 & \\ 0 & & 1 \\ & & 2 \\ & 0 & \end{array} \qquad P_1: \begin{array}{ccc} & 1 & \\ 1 & & 2 \\ & & 0 \\ & 1 & \end{array} \qquad P_2: \begin{array}{c} 2 \\ 0 \\ 1 \\ 2 \end{array}$$

Lemma 3.41. *Let M be a string module for Λ. Then $\underline{\mathrm{End}}_\Lambda(M) \cong k$ if and only if M lies in the Ω-orbit of S_0, S_1, S_2 or $M(\delta)$ or ${}_LM(e_2)$ or ${}_{LL}M(e_2)$.*

Proof. The proof uses similar arguments as the proofs of Lemma 3.2 and Lemma 3.3. □

Theorem 3.42. *If $S \in \{e_2, \delta\}$ or $M(S) = {}_LM(e_2)$ then $Ext^1_\Lambda(M(S), M(S)) = 0$ and $R(\Lambda, M(S)) \cong k$.*

Proof. The proof uses similar arguments as the proof of Theorem 3.5. □

Theorem 3.43. *If $S \in \{e_0, e_1\}$ then $dim_k Ext^1_\Lambda(M(S), M(S)) = 1$ and*

$$R(\Lambda, M(S)) \cong k[[t]]/(t^2).$$

Proof. We consider $S = e_1$, the case of $S = e_0$ being similar. Similarly to the proof of Theorem 3.6, we show that

$$dim_k Ext^1_\Lambda(M(S), M(S)) = dim_k \underline{Hom}_\Lambda(\Omega(M(S), M(S)) = 1,$$

which implies that $R(\Lambda, M(S))$ is a quotient algebra of $k[[t]]$.

Letting $L_0 = M(\rho)$, we have a short exact sequence of Λ-modules:

$$0 \to M(S) \xrightarrow{\iota} L_0 \xrightarrow{\tau} M(S) \to 0$$

where ι and τ are canonical Λ-module homomorphisms. Let $R_0 = k[[t]]/(t^2)$. We obtain an R_0-module structure on L_0 by letting t act as $\iota \circ \tau$. Similarly to the proof of Theorem 3.6, it follows that L_0 defines a lift (L_0, ρ) of $M(S)$ over R_0 and that the corresponding unique k-algebra homomorphism $\theta : R(\Lambda, M(S)) \to R_0$ is an isomorphism. □

Theorem 3.44. *If $M(S) = {}_{LL}M(e_2)$ then*

$$dim_k Ext^1_\Lambda(M(S), M(S)) = 1 \text{ and } R(\Lambda, M(S)) \cong k[[t]].$$

Proof. We consider $M(S) = {}_{LL}M(e_2)$. Similarly to the proof of Theorem 3.7, we show that $dim_k Ext^1_\Lambda(M(S), M(S)) = dim_k \underline{Hom}_\Lambda(\Omega(M(S)), M(S)) = 1$, which implies that $R(\Lambda, M(S))$ is a quotient algebra of $k[[t]]$.

Let $C = \alpha\beta^{-1}\rho\delta^{-1}$, and let for all $i \geq 1$, $T_i = C(\lambda^{-1}C)^{i-1}$. Similarly to the proof of Theorem 3.7, it follows that for all $i \geq 2$, $M(T_i)$ defines a lift $(M(T_i), \psi_i)$ of

$M(S)$ over $k[[t]]/(t^i)$. Taking the inverse limit $N = \varprojlim M(T_i)$, we obtain a lift (N, ψ) of $M(S)$ over $k[[t]]$. Arguing similarly as in the proof of Theorem 3.7, it follows that $R(\Lambda, M(S)) \cong k[[t]]$. $\square$

3.5.8 The Symmetric Basic Algebra of Type D(2$\mathcal{A}$)

Let Λ be the symmetric basic algebra of type D(2$\mathcal{A}$) (following the labeling by Erdmann and Skowroński in [5, Cor. 4.7]). Hence $\Lambda = kQ/I$ where

$$Q = \begin{tikzcd} \overset{0}{\bullet} \arrow[loop left, "\alpha"] \arrow[r, shift left, "\beta"] & \overset{1}{\bullet} \arrow[l, shift left, "\gamma"] \end{tikzcd}$$

and I is the ideal of the path algebra kQ generated by the set of relations

$$\left\{\beta\gamma, \alpha^2 - c\gamma\beta\alpha, \gamma\beta\alpha - \alpha\gamma\beta\right\}.$$

Here $c \in \{0, 1\}$ if $\mathrm{char}(k) = 2$ and $c = 0$ otherwise.

For the remainder of this section, let $c = 0$. The case $c = 1$ and $\mathrm{char}(k) = 2$ is considered in Section 3.3. There exist precisely two simple Λ-modules, up to isomorphism, corresponding to the two vertices of Q, denoted by S_0, S_1. Their projective covers P_0, P_1 can be visualized as follows:

$$P_0\colon \begin{array}{ccc} & 0 & \\ 0 & & 1 \\ 1 & & 0 \\ & 0 & \end{array} \qquad\qquad P_1\colon \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array}$$

Lemma 3.45. *Let M be a string module for Λ. Then $\underline{\mathrm{End}}_\Lambda(M) \cong k$ if and only if M lies in the Ω-orbit of S_0, S_1 or $M(\beta)$.*

Proof. The proof uses similar arguments as the proofs of Lemma 3.9 and Lemma 3.10. □

Theorem 3.46. *If $S = e_1$ then $Ext^1_\Lambda(M(S), M(S)) = 0$ and $R(\Lambda, M(S)) \cong k$.*

Proof. The proof uses similar arguments as the proof of Theorem 3.5. □

Theorem 3.47. *If $S \in \{e_0, \beta\}$ then $dim_k Ext^1_\Lambda(M(S), M(S)) = 1$ and*

$$R(\Lambda, M(S)) \cong k[[t]]/(t^2).$$

Proof. We consider $S = \beta$, the case of $S = e_0$ being similar. Similarly to the proof of Theorem 3.6, we show that

$$dim_k Ext^1_\Lambda(M(S), M(S)) = dim_k \underline{Hom}_\Lambda(\Omega(M(S), M(S)) = 1,$$

which implies that $R(\Lambda, M(S))$ is a quotient algebra of $k[[t]]$.

Letting $L_0 = M(\beta\alpha^{-1}\beta^{-1})$, we have a short exact sequence of Λ-modules:

$$0 \to M(S) \xrightarrow{\iota} L_0 \xrightarrow{\tau} M(S) \to 0$$

where ι and τ are canonical Λ-module homomorphisms. Let $R_0 = k[[t]]/(t^2)$. We obtain an R_0-module structure on L_0 by letting t act as $\iota \circ \tau$. Similarly to the proof of Theorem 3.6, it follows that L_0 defines a lift (L_0, ρ) of $M(S)$ over R_0 and that the corresponding unique k-algebra homomorphism $\theta : R(\Lambda, M(S)) \to R_0$ is an isomorphism. □

3.5.9 The Symmetric Basic Algebra of Type D(1)

Let Λ be the symmetric basic algebra of type D(1) (following the labeling by Erdmann and Skowroński in [5, Cor. 4.7]). Hence $\Lambda = kQ/I$ where

$$Q = \begin{tikzcd} \overset{0}{\bullet} \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}$$

and I is the ideal of the path algebra kQ generated by the set of relations

$$\left\{\alpha^2 - c\beta\alpha, \beta^2, \beta\alpha - \alpha\beta\right\}.$$

Here $c \in \{0, 1\}$ if $\mathrm{char}(k) = 2$ and $c = 0$ otherwise.

For the remainder of this section, let $c = 0$. The case $c = 1$ and $\mathrm{char}(k) = 2$ is considered in Section 3.4. There exists precisely one simple Λ-module, up to isomorphism, corresponding to the unique vertex of Q, denoted by S_0. The projective cover P_0 can be visualized as follows:

$$P_0 = \begin{tikzcd}[column sep=small] & 0 \arrow[dl, dash, "\alpha"'] \arrow[dr, dash, "\beta"] & \\ 0 \arrow[dr, dash, "\beta"'] & & 0 \arrow[dl, dash, "\alpha"] \\ & 0 & \end{tikzcd}$$

Lemma 3.48. *Let M be a string module for Λ. Then $\underline{\mathrm{End}}_\Lambda(M) \cong k$ if and only if M lies in the Ω-orbit of S_0.*

Proof. The proof uses similar arguments as the proofs of Lemma 3.16 and Lemma 3.17. $\square$

Theorem 3.49. *For $S_0 = M(e_0)$, we have*

$$dim_k Ext^1_\Lambda(S_0, S_0) = 2 \text{ and } R(\Lambda, S_0) \cong \Lambda \cong k[[t_1, t_2]]/(t_1^2, t_2^2).$$

Proof. Similarly to the proof of Theorem 3.20, we show that $dim_k Ext^1_\Lambda(S_0, S_0) = dim_k \underline{Hom}_\Lambda(\Omega(S_0), S_0) = 1$, which implies that $R(\Lambda, S_0)$ is a quotient algebra of $k[[t_1, t_2]]$.

To prove that $R(\Lambda, S_0) \cong \Lambda$, we consider the representations of Λ corresponding to lifts of S_0. Note that Λ is a ring in $\hat{\mathcal{C}}$ with $\pi_\Lambda : \Lambda \to k$ given by $\pi_\Lambda(c_1 e_0 + c_2\alpha + c_3\beta + c_4\alpha\beta) = c_1$ for all $c_1, c_2, c_3, c_4 \in k$.

We have $S_0 = k$ as a k-vector space and the Λ-action on S_0 is given by $\alpha_{S_0} : \Lambda \to k$ with $\alpha_{S_0} = \pi_\Lambda$. Let R be a ring in $\hat{\mathcal{C}}$. Following Definition 2.44, a lift of S_0 over R is given by a free R-module $M = R$ and a k-algebra homomorphism $\alpha_M : \Lambda \to R$ with $\pi_R \circ \alpha_M = \alpha_{S_0}$. This implies that α_M is a morphism in $\hat{\mathcal{C}}$.

We define a lift U of S_0 over Λ by letting $U = \Lambda$ as a Λ-module and defining $\alpha_U : \Lambda \to \Lambda$ in $\hat{\mathcal{C}}$ to be the identity morphism. We want to show that α_U defines the universal lift of S_0 over Λ. Let $R \in Ob(\hat{\mathcal{C}})$ and let M be a lift of S_0 over R given by $\alpha_M : \Lambda \to R$. As seen above, α_M is a morphism in $\hat{\mathcal{C}}$. Let $\tau = \alpha_M$, so τ is in $\hat{\mathcal{C}}$. Similarly to the proof of Theorem 3.20, we see that the lift of S_0 over R defined by $\tau \circ \alpha_U$ is isomorphic to the lift defined by α_M, and τ is unique in $\hat{\mathcal{C}}$ with this property. It follows that $R(\Lambda, S_0) \cong \Lambda$.

□

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