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Applications of deformation rigidity theory in Von Neumann algebras

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APPLICATIONS OF DEFORMATION RIGIDITY THEORY IN VON NEUMANN ALGEBRAS

by

Bogdan Teodor Udrea

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
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Thesis Supervisors: Professor Paul Muhly
Assistant Professor Ionut Chifan

ABSTRACT

This work contains some structural results for von Neumann algebras arising from measure preserving actions by direct products of groups on probability spaces. The technology and the methods we use are a continuation of those used by Chifan and Sinclair in [Ionut Chifan and Thomas Sinclair. *On the structural theory of* II_1 *factors of negatively curved groups*, ArXiv e-prints, March 2011]. By employing these methods, we obtain new examples of strongly solid factors as well as von Neumann algebras with unique or no Cartan subalgebra. We show for instance that every II_1 factor associated with a weakly amenable group in the class $\mathcal{S}$ of Ozawa is strongly solid [Narutaka Ozawa. *Solid von Neumann algebras*, Acta Math., **192** (2004), 111–117]. We also obtain a product version of this result: any maximal abelian *-subalgebra of any II_1 factor associated with a finite direct product of weakly amenable groups in the class $\mathcal{S}$ of Ozawa has an amenable normalizing algebra. Finally, pairing some of these results with Ioana's cocycle superrigidity theorem [Adrian Ioana. *Cocycle superrigidity for profinite actions of property (T) groups*, Duke Math. J., *to appear*], we prove that compact actions by finite products of lattices in $Sp(n,1)$, $n \geq 2$, are virtually W^*-superrigid. The results presented here are joint work with Ionut Chifan and Thomas Sinclair. They constitute the substance of an article which has already been submitted for publication [Ionut Chifan, Thomas Sinclair, and Bogdan Udrea. *On the structural theory of* II_1 *factors of negatively curved groups, ii. actions by product groups*, ArXiv e-prints, August 2011].

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the July 2012 graduation.

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This work contains some structural results for von Neumann algebras arising from measure preserving actions by direct products of groups on probability spaces. The technology and the methods we use are a continuation of those used by Chifan and Sinclair in [Ionut Chifan and Thomas Sinclair. *On the structural theory of* II_1 *factors of negatively curved groups*, ArXiv e-prints, March 2011]. By employing these methods, we obtain new examples of strongly solid factors as well as von Neumann algebras with unique or no Cartan subalgebra. We show for instance that every II_1 factor associated with a weakly amenable group in the class $\mathcal{S}$ of Ozawa is strongly solid [Narutaka Ozawa. *Solid von Neumann algebras*, Acta Math., **192** (2004), 111–117]. We also obtain a product version of this result: any maximal abelian *-subalgebra of any II_1 factor associated with a finite direct product of weakly amenable groups in the class $\mathcal{S}$ of Ozawa has an amenable normalizing algebra. Finally, pairing some of these results with Ioana's cocycle superrigidity theorem [Adrian Ioana. *Cocycle superrigidity for profinite actions of property (T) groups*, Duke Math. J., *to appear*], we prove that compact actions by finite products of lattices in $Sp(n,1)$, $n \geq 2$, are virtually W^*-superrigid. The results presented here are joint work with Ionut Chifan and Thomas Sinclair. They constitute the substance of an article which has already been submitted for publication [Ionut Chifan, Thomas Sinclair, and Bogdan Udrea. *On the structural theory of* II_1 *factors of negatively curved groups, ii. actions by product groups*, ArXiv e-prints, August 2011].

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CHAPTER 1
INTRODUCTION AND PRELIMINARIES

1.1 Preliminaries and Statement of the Problem

The results presented here lie at the intersection of the field of von Neumann algebras with the field of contemporary ergodic theory, understood as the study of the free, ergodic, measure preserving actions of countable, discrete groups on measure spaces. The concept of von Neumann algebras (first called rings of operators) appeared in the late 30's with the papers of Murray and von Neumann [54]. A von Neumann algebra is a self-adjoint subalgebra of $B(\mathcal{H})$, the space of all bounded linear operators on a complex Hilbert space $\mathcal{H}$, that contains the identity and is closed in the so-called strong-operator topology. Of course, $B(\mathcal{H})$ itself is a von Neumann algebra, no matter what the Hilbert space dimension of $\mathcal{H}$ is. When the dimension of $\mathcal{H}$ is finite, say n, then $B(\mathcal{H})$ can be identified with the algebra of all $n \times n$ complex matrices, $\mathcal{M}_n(\mathbb{C})$. Another example is the algebra of all multiplication operators one obtains on $L^2(X, \mu)$ induced by functions in $L^\infty(X, \mu)$. When X is a finite set and μ is the counting measure, then the realization of $L^\infty(X, \mu)$ as multiplication operators is simply the identification of $L^\infty(X, \mu)$ with the algebra of diagonal matrices.

One of the first challenges Murray and von Neumann encountered was that of giving examples of von Neumann algebras beyond those just described. Their first example - class of examples, really - came from free, ergodic, probability-measure-preserving actions of countable discrete groups on measure spaces. At the simplest level, suppose a finite group Γ acts freely and transitively on a set X. The set then must have the same cardinality as Γ, say n, and if we give each point of X measure $\frac{1}{n}$, then we have an example of the set up just described. The action of Γ on X induces an action of Γ on $L^\infty(X)$ via composition. The purely algebraic crossed product determined by the action of Γ on $L^\infty(X,\mu)$, $L^\infty(X,\mu) \rtimes \Gamma$, is the sort of algebra that Murray and von Neumann considered. In the case at hand, when $L^\infty(X,\mu)$ is viewed as the algebra of $n \times n$ diagonal matrix and Γ is viewed as permutation matrices, the algebra $L^\infty(X,\mu) \rtimes \Gamma$ is isomorphic to $\mathcal{M}_n(\mathbb{C})$. The crucial observation here is that the cross-product algebra contains no information about either the group or the action other than the order of the group. Any other finite group of the same order acting freely and transitively on a finite set gives rise to one and the same von Neumann algebra, namely $\mathcal{M}_n(\mathbb{C})$.

When we pass to the class of infinite countable discrete groups acting freely and ergodically in a measure preserving fashion on probability measure spaces we obtain algebras that are *not* isomorphic to $B(\mathcal{H})$. The ergodicity means that the only measurable subsets that are invariant to the action must be either null or co-null. The construction of the cross-product von Neumann algebra associated to such an action is essentially the same as in the finite case. The subalgebra $L^\infty(X,\mu)$ plays a distinguished role in the crossed product much like the diagonal matrices in $\mathcal{M}_n(\mathbb{C})$ and, consequently, it is called the *Cartan subalgebra* of the crossed product. But unlike the finite case, it is not at all clear what features of the group and its action the von Neumann algebra, $L^\infty(X,\mu) \rtimes \Gamma$, “remembers”. Perhaps there is something more in the infinite case other than the cardinalities of the objects considered? This, then, is a fundamental problem: Recover as much information as possible about the group and the action from the crossed product von Neumann algebra.

To understand how to study the action through the prism of its associated cross-product von Neumann algebra, let us introduce some terminology in order to delineate the problem more clearly. Let $\sigma : \Gamma \to Aut(X, \mu)$ and $\alpha : \Lambda \to Aut(Y, \nu)$ be two free, ergodic, measure preserving actions of two countable discrete groups Γ and Λ.

- The actions are called *orbit equivalent* (OE) if there exists a measure space isomorphism $\Delta : X \to Y$ such that $\Delta(\Gamma x) = \Lambda\Delta(x)$, for a.e. $x \in X$, i.e., Δ carries the orbits of one action to the orbits of the other.
- The actions are called *conjugate* if there exists a measure space isomorphism $\Delta : X \to Y$ and a group isomorphism $\delta : \Gamma \to \Lambda$ such that $\Delta(\sigma_\gamma(x)) = \alpha_{\delta(\gamma)}(\Delta(x))$ for a.e. $x \in X$, and for all $\gamma \in \Gamma$.

Clearly conjugacy is much stronger than (OE), since it is in fact an isomorphism of actions.

A (free, ergodic, probability measure preserving) action of a (countable, discrete) group on a measure space is called

- *(OE)-super-rigid* if any other (free, ergodic, probability measure preserving) action that is (OE) to it must actually be conjugate to it.
- W^**-super-rigid* if for any other action such that the asociated cross-product von Neumann algebras are isomorphic it follows that the actions are conjugate.

It is known, due to the work of Singer [88], that two such actions are (OE) if and only if there exists a *-isomorphism between their associated von Neumann algebras $L^\infty(X) \rtimes \Gamma$ and $L^\infty(Y) \rtimes \Lambda$ taking the group-measure space subalgebra $L^\infty(X)$ onto $L^\infty(Y)$. Thus, (OE)-super-rigidity requires that if there exists a *-isomorphism $\Phi : L^\infty(X) \rtimes \Gamma \to L^\infty(Y) \rtimes \Lambda$ that carries the Cartan subalgebra of one algebra to the other (i.e., if $\Phi(L^\infty(X)) = L^\infty(Y)$), then the actions must be conjugate. W^*-super-rigidity requires something more, namely that from the mere existence of such a *-isomorphism, without any information about the mapping of the Cartan subalgebra $L^\infty(X)$, it follows that the actions are conjugate. Hence, W^*-super-rigidity is *a priori* stronger than (OE)-super-rigidity, being the logical sum of the former and the requirement that the von Neumann algebra $L^\infty(X) \rtimes \Gamma$ admits unique group-measure space Cartan subalgebra, up to unitary conjugacy. W^*-super-rigidity provides the possibility of complete recovery of the initial data - the action and the group - from the *-isomorphism class of the associated cross-product von Neumann algebra.

The early years focused mainly on understanding the amenable case. However, even the first examples analyzed revealed a rather surprising lack of rigidity; very different groups such as $\mathbb{Z}$ and $\oplus_{\mathbb{Z}}\mathbb{Z}_2$ admit OE-equivalent actions. It become gradually clear that this phenomenon is characteristic to a quite large class of groups and actions. The seminal work by Dye in [22, 23] showed that *all* free, ergodic actions of *all* countable infinite nilpotent groups are OE-equivalent. In the mid seventies the groundbreaking work of Connes [12] on the classification of injective factors shows in particular that *any* two free, ergodic actions of infinite, amenable groups are indistinguishable under W^*-equivalence. The same striking conclusion holds for OE case by the subsequent work of Connes, Feldman, Ornstein, and Weiss [57], [14]. All these results suggested that in order to find rigidity in von Neumann algebras or orbit equivalence one hope may be to investigate groups as far from being amenable as possible.

The non-amenable case however turned out to be extremely complex and despite signs of rigidity being observed earlier [55, 23, 46, 12], for almost two decades progress was slow, despite several breakthrough discoveries in the eighties [13, 97, 15, 18].

During the past decade however, an entire venue of rigidity results emerged in both orbit equivalence theory and von Neumann algebra theory using either "measured group theory" methods or operator algebraic methods. Particularly important, Sorin Popa proposed in 2001 a novel approach to the rigidity problem now almost unanimously called "deformation-rigidity theory". As this term suggests, his innovative method consists of studying actions of discrete groups $\Gamma \curvearrowright X$ whose von Neumann algebras $L^\infty(X) \rtimes \Gamma = M$ are simultaneously highly deformable and contain sufficiently rigid parts. Then the main underlying idea is to exploit the tension between the deformability (or malleability) properties of M, and the rigidity of certain parts of it, in order to get information about the internal structure of the algebra. The malleability means the existence of nets of normal, unital, completely positive maps from M to itself, that converge pointwise to the identity in the $\|\cdot\|_2$ norm.

The (relative) rigidity of a subalgebra $Q \subset M$ simply means that any such net must converge *uniformly* to the identity in the $\|\cdot\|_2$ on the unit ball of Q. By playing off these properties against each other, usually by various intertwining techniques, one is able to prove results about the position of the subalgebra Q inside M. A concrete example of an action that naturally admits both deformations and rigid parts is the Bernoulli action of $SL_3(\mathbb{Z})$ with base space $[0,1]$. In this case rigidity comes from $SL_3(\mathbb{Z})$ having property (T) of Kazhdan [45] and deformability arises from the Bernoulli type actions having lots of automorphisms. Therefore, applying deformation/rigidity techniques, Popa showed that $SL_3(\mathbb{Z}) \curvearrowright [0,1]^{SL_3(\mathbb{Z})}$ is OE-super rigid, [72, 73]. Subsequently, Ioana showed the uniqueness of the group measure space Cartan subalgebra for the von Neumann algebra associated with this action thus proving the W^*-super rigidity of this action as well, [37].

1.2 Results Obtained

The results presented here are mainly about the uniqueness of Cartan subalgebras in von Neumann algebras coming from profinite[1] actions of direct products of ICC hyperbolic groups, as well as (virtually) W^*-super-rigidity of such actions when the groups additionally have the property (T) of Kazhdan. Let me first provide some context for this. In their breakthrough paper [65] from 2007, Popa and Ozawa found the first examples of von Neumann algebras which have unique Cartan subalgebra, up to unitary conjugacy: von Neumann algebras associated with any profinite action

[1] inverse limit of actions on finite measure spaces

of direct products of non-abelian free groups. In a subsequent paper [66], they generalized this result for profinite actions of a single group that is weakly amenable and admits a proper 1-cocycle into a representation that is weakly contained in the left regular representation. Then in a recent paper [10], Chifan and Sinclair showed that these results can further be extended to the class of hyperbolic groups. And finally, in a sequel [11] of that paper, Chifan, Sinclair and I were able to prove that the same conclusion holds if instead of a single hyperbolic group we allow direct products of finitely many such groups.

Theorem 1.2.1 (Chifan-Sinclair-Udrea). *If Γ_1, Γ_2 are hyperbolic groups with property (T) (e.g. Γ_i lattices in $\mathrm{Sp}(n,1)$ $n \geq 2$), then any free, ergodic, profinite (or more generally compact) action $(\Gamma_1 \times \Gamma_2) \curvearrowright X$ is virtually W^*-superrigid.*

The main technical result that we rely upon in proving this is a classification theorem which describes the possible "positions" of *all* diffuse amenable subalgebras with large (nonamenable) normalizing algebra inside $L^\infty(X) \rtimes (\Gamma_1 \times \Gamma_2)$. Precisely, we have a trichotomy result saying that algebras with large normalizers "live" in either $L^\infty(X)$, $L^\infty(X) \rtimes \Gamma_1$, or $L^\infty(X) \rtimes \Gamma_2$. The proof of this result was accomplished by reinterpreting Ozawa's C^*- algebraic approach of solidity for the group von Neumann algebras of hyperbolic groups in the context of deformation/rigidity theory of Popa. A middle chart between the early works by Ozawa, Popa, Peterson and Ozawa and Popa allowed us to recast the deformation/spectral gap rigidity argument of Popa in the context of actions of products of groups that admit proper quasi-cocycles into the left regular representation [10, 11]. We mention that relative versions of these results are also obtained [11].

Next we mention a couple of other consequences of our technical result, beside the W^*-super rigidity problem. For instance it provides significant insight to the study of structural results of certain group von Neumann algebras. In particular progress is made in understanding the structure of normalizers for MASA's in tensor product factors arising from hyperbolic groups.

Corollary 1.2.2 (Chifan-Sinclair-Udrea). *Let Γ_1, Γ_2 be ICC hyperbolic groups and denote by $M = L\Gamma_1 \bar{\otimes} L\Gamma_2$. Then the normalizing group $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \,|\, uAu^* = A\}$ of* any *maximal abelian subalgebra (MASA) $A \subset M$ generates an amenable von Neumann subalgebra of M, $\mathcal{N}_M(A)''$.*

This result extends previous results by Popa and Ozawa [65, 66] in the case of direct products of non-abelian free groups.

Lastly, our theorem can be successfully used to derive some applications to measured group theory. For instance using the main technical result in combination with an intertwining lemma á la Popa, we obtain new structural results for the groups in the measure equivalence class [26, 27] of a given direct product of non-amenable hyperbolic groups. This result is in the same spirit of previous results by Gaboriau [28] and Monod-Shalom [53].

Corollary 1.2.3 (Chifan-Sinclair-Udrea). *Let Γ_i be weakly amenable groups and let $\pi : \Gamma_i \to \mathcal{U}(\mathcal{H}_\pi)$ be weakly-ℓ^2 representations such that $\mathcal{RA}(\Gamma, \{e\}, \mathcal{H}_\pi) \neq \emptyset$ (e.g. Γ_i are hyperbolic). If $\Gamma_1 \times \Gamma_2 \curvearrowright X$ and $\Lambda \curvearrowright Y$ are any pmp actions such that Λ*

admits an infinite amenable normal subgroup $\Sigma < \Gamma$ *for which the restriction* $\Sigma \curvearrowright Y$ *is still ergodic then* $\Gamma_1 \times \Gamma_2 \curvearrowright X \ncong_{OE} \Lambda \curvearrowright Y$.

Remark 1.2.4. Recently, Sorin Popa and Stefaan Vaes have proved that every (ergodic, free, probability measure preserving) of a non-amenable hyperbolic group, or direct products of such, gives rise to a von Neumann algebra with unique Cartan subalgebra [80, 81].

CHAPTER 2
BACKGROUND

2.1 von Neumann Algebras

A **von Neumann algebra** is simply a self-adjoint subalgebra of $B(\mathcal{H})$ (the space of bounded linear operators on $\mathcal{H}$, for some Hilbert space $\mathcal{H}$), which contains the identity and is closed in the strong operator topology. They were at first called rings of operators by Murray and von Neumann [54]. For basic information, as well as for a systematic and detailed account of the theory of von Neumann algebras, we refer the reader to [20, 42, 43, 44, 84, 89, 90, 91, 92]. In what follows we will only outline a couple of definitions and concepts that will be frequently used in this work.

Definition 2.1.1. *Let $S \subset B(\mathcal{H})$. The* **commutant** *of S, denoted S', is the set $\{x \in B(\mathcal{H}) | xs = sx, \forall s \in S\}$. Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. Then the* **center** *of M is $M' \cap M$. A von Neumann algebra is called a* **factor** *when its center equals the scalar multiples of the identity.*

Definition 2.1.2. *Let M be a von Neumann algebra. A* **trace** *on M is a linear functional τ that is positive, i.e. $\tau(x^*x) \geq 0$, for any $x \in M$, and satisfies $\tau(xy) = \tau(yx)$, for any $x, y \in M$. The trace is moreover called faithful if, for any $x \in M$, $\tau(x^*x) = 0$ implies $x = 0$. A von Neumann algebra M is called* **finite** *if it admits a normal, faithful trace. A type II_1 factor is an infinite dimensional factor which is finite as a von Neumann algebra.*

Throughout this work, we will mainly be concerned with finite von Neumann algebras, always assumed to be represented on separable Hilbert spaces.

Definition 2.1.3. *Let M be a von Neumann algebra, and $N \subset M$ a von Neumann subalgebra. A* **conditional expectation** *of M onto N is a norm one projection of M onto N, i.e. a linear map $E : M \to N$, such that the norm of E as a linear map is 1, and $E(x) = x$, for all $x \in N$.*

It is known that any such map is automatically positive, i.e. $E(x^*x) \geq 0$, for any $x \in M$, and N-bimodular, in the sense that $E(axb) = aE(x)b$, for any $a, b \in N, x \in M$, see for example [90], chapter III. A very useful fact is that for finite von Neumann algebras, the existence of trace preserving conditional expectations onto their subalgebras is automatically satisfied. To be more precise, if (M, τ) is a finite von Neumann algebra and $N \subset M$ a von Neumann subalgebra, then there exists a conditional expectation $E : M \to N$ such that $\tau_N \circ E = \tau$. Here, τ_N is the restriction of τ to N.

Definition 2.1.4. *Let M be a von Neumann algebra. The* **unitary group** *of M, denoted $\mathcal{U}(M)$, is the set $\{u \in M | u^*u = uu^* = 1\}$. Let $B \subset M$ be a von Neumann subalgebra. The* **normalizing group** *of B (in M), denoted by $\mathcal{N}_M(B)$, is the set $\{u \in \mathcal{U}(M) | uBu^* = B\}$.*

It is easy to see that the linear span of the normalizing group of B in M is a *-subalgebra of M.

Definition 2.1.5. *Let M be a finite von Neumann algebra. A von Neumann subalgebra $A \subset M$ is called a* **Cartan subalgebra** *if it is maximal abelian in M, i.e. $A' \cap M = A$, and $\overline{sp\mathcal{N}_M(A)}^w = M$.*

2.2 The Cross-product Construction

Let (N, τ) be a finite tracial von Neumann algebra and Γ a countable discrete group. Let $\sigma : \Gamma \to Aut(N, \tau)$ be an action of Γ by trace-preserving automorphisms of N. To these initial data we can associate a finite von Neumann algebra, called the **cross-product** of N with Γ by the action σ, and denoted $N \rtimes_\sigma \Gamma$. We will omit the symbol σ whenever there is no danger of confusion. The von Neumann algebra $N \rtimes_\sigma \Gamma$ is defined in the following way: consider the Hilbert space $\mathcal{H} = L^2(N, \tau)\overline{\otimes}l^2(\Gamma)$. Every $x \in N$ acts on $\mathcal{H}$ by the formula $x(\xi \otimes \delta_\gamma) = x\xi \otimes \delta_\gamma$. Also, for $g \in \Gamma$, define the unitary operator u_g on $\mathcal{H}$, given by the formula $u_g(\xi \otimes \delta_\gamma) = \sigma_g(\xi) \otimes \delta_{g\gamma}$. Then the **cross-product** $N \rtimes_\sigma \Gamma$ is the von Neumann algebra generated inside $B(\mathcal{H})$ by N and the unitaries u_g, with $g \in \Gamma$. A very important case of this general construction is when Γ acts by measure preserving automorphisms of a standard probability measure space (X, μ), i.e. $\sigma : \Gamma \to Aut(X, \mu)$ is a group homomorphism. This is equivalent to saying that Γ acts by trace preserving automorphisms of the abelian von Neumann algebra $L^\infty(X)$ with the trace given by integration against μ, the connection between the two being described by the formula $\sigma_\gamma(f) = f \circ \sigma_\gamma$, for any $f \in L^\infty(X)$. In this case, the cross-product construction yields the Murray and von Neumann's so-called group-measure space construction [55], which was the source of the first examples of type II_1 factors. Most of the results presented in this work have to do with the possibility of recapturing as much information about the initial data as possible from the cross-product construction. Recall that an action $\sigma : \Gamma \to Aut(X, \mu)$ is **ergodic** if the only σ-invariant measurable subsets of X are either null or co-null. The action is called **free** if $\sigma_\gamma(x) = x$ implies $\gamma = e$, for almost every $x \in X$. When the action is free and ergodic, then the von Neumann algebra $L^\infty(X) \rtimes \Gamma$ is a type II_1 factor, i.e. a von Neumann algebra with trivial center, and which admits a continuous, faithful, normal trace [55]. Moreover, in this case $L^\infty(X)$ is a Cartan subalgebra of $M = L^\infty(X) \rtimes \Gamma$.

2.3 The Basic Construction for an Inclusion of Finite von Neumann Algebras

The **basic construction** is an essential tool in the deformation rigidity theory. We briefly state here the essential facts about it. The interested reader can find more about this in [41]. Let $B \subset N$ be an inclusion of finite von Neumann algebras, and assume that N is endowed with a faithful, normal trace τ. Then there exists a τ-preserving conditional expectation of N onto B, henceforth denoted E_B. The conditional expectation E_B extends to an orthogonal projection e_B from $L^2(N, \tau)$ to $L^2(B, \tau)$. The **basic construction** of N with B, denoted from now on $\langle N, e_B \rangle$, is the von Neumann algebra generated inside $B(L^2(N, \tau))$ by N and e_B. The following relation holds: $e_B x e_B = E_B(x) e_B$, for any $x \in N$. Moreover, we have that $\langle N, e_B \rangle = \overline{sp}^w \{x e_B y | x, y \in N\}$, and that $e_B \langle N, e_B \rangle e_B = B e_B$. Also, if J denotes the canonical conjugation on $L^2(N, \tau)$, then one can check that

$\langle N, e_B \rangle = JBJ' \cap B(L^2(N, \tau))$. This shows, in particular, that $\langle N, e_B \rangle$ is a semi-finite von Neumann algebra. On this semifinite von Neumann algebra, one can define a semi-finite trace in a canonical way, by the formula $\mathrm{Tr}(xe_By) = \tau(xy)$, for any $x, y \in N$.

2.4 Background on Bimodules over von Neumann Algebras

We briefly review here the concept of a (Hilbert) **bimodule** over a pair of von Neumann algebras as well as some related notions. Under the name of correspondence, this concept is essentially due to Connes [16], see also [70].

1. Let M, N, be two von Neumann algebras. An $M - N$ **bimodule** is a Hilbert space $\mathcal{H}$ together with a pair of mutually commuting normal *-representations of M and N^{op} on $\mathcal{H}$. This is equivalent to giving a normal left action of M on $\mathcal{H}$ and a normal right action of N on $\mathcal{H}$, that commute with each other. To every $M - N$ bimodule $\mathcal{H}$, one can canonically associate a *-representation of $M \otimes_{alg} N$ on $\mathcal{H}$, given by $(x \otimes y^{op})\xi = x \cdot \xi \cdot y$, for any $x \in M, y^{op} \in N^{op}, \xi \in \mathcal{H}$.
2. Let $\mathcal{H}$, $\mathcal{K}$ be two $M - N$ bimodules, and denote by $\pi_{\mathcal{H}}$, $\pi_{\mathcal{K}}$ the associated representations of $M \otimes_{alg} N$ on $\mathcal{H}$, $\mathcal{K}$, respectively. We say that $\mathcal{H}$ is **weakly contained** in $\mathcal{K}$ if $\|\pi_{\mathcal{H}}(x)\| \leq \|\pi_{\mathcal{K}}(x)\|$, for any $x \in M \otimes_{alg} N$. We mention that there are equivalent formulations of weak containment of bimodules, see [16, 70, 2].
3. Given an $M - N$ bimodule $\mathcal{H}$ and an $N - P$ bimodule $\mathcal{K}$, one can construct

the **Connes tensor product** $\mathcal{H} \otimes_N \mathcal{K}$, which is an $M-P$ bimodule [16, 17]. The Connes tensor product of bimodules is well-behaved with respect to weak containment of bimodules, in the sense that if $\mathcal{H}_1$ is weakly contained in $\mathcal{H}_2$, then $\mathcal{H}_1 \otimes_N \mathcal{K}$ is weakly contained in $\mathcal{H}_2 \otimes_N \mathcal{K}$, for any $N-P$ bimodule $\mathcal{K}$, and a similar statement holds when tensoring with $\mathcal{K}$ from the left.

4. Examples.

   - The **trivial** $M-M$ bimodule $L^2(M)$, endowed with the obvious left and right actions coming from the fact that M is represented in standard form on $L^2(M)$.

   - The **coarse** $M-M$ bimodule $L^2(M)\bar{\otimes}L^2(M)$, where the left and the right actions are given by $a \cdot \xi \cdot b = (a \otimes 1)\xi(1 \otimes b)$.

   - The basic construction. Let (M, τ) be a finite von Neumann algebra with a faithful trace and let B be a von Neumann subalgebra of M. As seen above, denote by $\langle M, e_B \rangle$ the von Neumann algebra acting on $L^2(M)$, generated by M and the orthogonal projection of $L^2(M)$ onto $L^2(B)$. One can check that $\langle M, e_B \rangle$ equals the commutant of the right action of B on $L^2(M)$. It follows that $\langle M, e_B \rangle$ is a semi-finite von Neumann algebra, in general. There exists a normal, semi-finite, faithful trace on $\langle M, e_B \rangle$, denoted by Tr, uniquely determined by the formula $\mathrm{Tr}(ae_Bb) = \tau(ab)$. We can thus consider the $M-M$ bimodule $L^2(\langle M, e_B \rangle)$ with the obvious left and right actions. Then, as $M-M$ bimodules, we have that $L^2(\langle M, e_B \rangle) \cong L^2(M)\bar{\otimes}_B L^2(M)$.

5. Orthonormal basis for modules over von Neumann algebras. Let (N, τ) be a finite von Neumann algebra and $B \subset N$ a von Neumann subalgebra of N. Let $\mathcal{H} \subset L^2(N)$ be a right B-module. An **orthonormal basis** for $\mathcal{H}$ is a subset η_i of $L^2(N)$ such that $\mathcal{H} = \overline{\sum_i \eta_i B}$, and $E_B(\eta_i^* \eta_j) = \delta_{ij} p_i$, where p_i are projections in B, for all i's, and δ_{ij} is the Kronecker symbol. Under these conditions, it automatically follows that $\xi = \sum_i \eta_i E_B(\eta_i^* \xi)$, for any $\xi \in \mathcal{H}$. Similar statements hold for left modules. Due to a maximality argument, every right Hilbert B-module $\mathcal{H} \subset L^2(N)$ has an orthonormal basis. A right B-module $\mathcal{H} \subset L^2(N)$ is said to be finitely generated if it admits a finite orthonormal basis.

6. The importance of the bimodules in the study of von Neumann algebras is given by the fact that many important properties of the latter are somehow encoded in the structure of their bimodules. For example, Connes established the celebrated result that a finite von Neumann algebra M is approximately finite dimensional, or equivalently injective or amenable, if and only if the trivial $M - M$ bimodule $L^2(M)$ is weakly contained in the $M - M$ coarse bimodule $L^2(M) \bar{\otimes} L^2(M)$ [12]. This further implies that, when B is an amenable von Neumann algebra, the Connes tensor product $M - M$ bimodule $L^2(M) \bar{\otimes}_B L^2(M)$ is weakly contained in the coarse bimodule $L^2(M) \bar{\otimes} L^2(M)$ for every von Neumann algebra M. Moreover, when B is amenable, it follows that $L^2(\langle M, e_B \rangle)$ is weakly contained in $L^2(M) \bar{\otimes} L^2(M)$. The following intertwining criterion of Popa can be seen as further illustration of the above statement, and it also represents the foundation of the deformation - rigidity theory. Its original version

and proof can be found in [72].

Theorem 2.4.1. *Let (M, τ) be a finite von Neumann algebra endowed with a faithful normal trace. Let $B_0 \subset M$, $B \subset M$ be two von Neumann subalgebras of M. Then the following conditions are equivalent:*

(a) There exists $a \in B_0' \cap \langle M, e_B \rangle$ such that $a \geq 0$, $a \neq 0$, and $Tr(a) < \infty$.

(b) There exists a non-zero projection $f \in B_0' \cap \langle M, e_B \rangle$ such that $Tr(f) < \infty$.

(c) There exists a projection $q_0 \in B_0$, a non-zero partial isometry $v \in M$, and a non-zero $\xi \in q_0 L^2(M) vv^*$ such that if we denote $\xi_0 = \xi v$, then $q_0 B_0 q_0 \xi_0 \subset \xi_0 B$.

*(d) There exist non-zero projections $q \in B_0$ and $p \in B$, a *-isomorphism ψ of qB_0q into pBp, and a non-zero partial isometry $v \in M$ such that $vv^* \in (qB_0q)' \cap qMq$, $v^*v \in \psi(qB_0q)' \cap pMp$, and $xv = v\psi(x)$, for any $x \in qB_0q$.*

(e) there exist $a_1, \ldots, a_n \in M$ and $\varepsilon > 0$ such that $\|E_B(a_i u a_j^*)\|_2 \geq \varepsilon$, for any $u \in U(B_0)$ and for any i and j, where $U(B_0)$ denotes the unitary group of B_0.

Notice that (b) amounts to saying that there exists a $B_0 - B$ subbimodule of $L^2(\langle M, e_B \rangle)$ which is finitely generated as a right B-module. The equivalence between (d) and (e) turns out to be particularly useful in applications.

2.5 Popa's Intertwining Techniques

We will briefly review the concept of intertwining two subalgebras inside a von Neumann algebra, along with the main technical tools developed by Popa in [71, 72]. Given N a finite von Neumann algebra, let $P \subset fNf$, $Q \subset N$ be diffuse subalgebras for some projection $f \in N$. We say that *a corner of P can be intertwined into Q inside* N if there exist two non-zero projections $p \in P$, $q \in Q$, a non-zero partial isometry $v \in pNq$, and a $*$-homomorphism $\psi : pPp \to qQq$ such that $v\psi(x) = xv$ for all $x \in pPp$. Throughout this paper we denote by $P \preceq_N Q$ whenever this property holds, and by $P \npreceq_N Q$ otherwise. The partial isometry v is called an intertwiner between P and Q.

Popa established an efficient criterion for the existence of such intertwiners (Theorems 2.1-2.3 in [72]). Particularly useful in concrete applications is the following *analytic* description of absence of intertwiners.

Theorem 2.5.1 (Corollary 2.3 in [72]). *Let N be a von Neumann algebra and let $P \subset fNf$, $Q \subset N$ be diffuse subalgebras for some projection $f \in N$. Then the following are equivalent:*

1. *$P \npreceq_M Q$.*
2. *For every finite set $\mathcal{F} \subset fNf$ and every $\epsilon > 0$ there exists a unitary $v \in \mathcal{U}(P)$ such that*

$$\sum_{x,y\in\mathcal{F}} \|E_Q(xvy^*)\|_2^2 \leq \epsilon.$$

Notice that in the intertwining concept presented above we *a priori* have no control over the image $\psi(pPp)$ inside qQq. When trying to get unitary conjugacy this often becomes a significant issue and additional analysis regarding the position of $\psi(pPp)$ inside qQq is required. Sometimes the $\star$-homorphism ψ can be suitably modified to automatically preserve certain properties from the inclusion $P \subset N$ to the inclusion $\psi(pPp) \subseteq qQq$. For instance, Ioana showed in Lemma 1.5 of [38] that if $P \subset N$ is a m.a.s.a. then ψ can be chosen so that $\psi(pPp) \subseteq qQq$ is again a m.a.s.a. Applying his argument one can show that ψ can be chosen to also preserve the irreducibility of the inclusion $P \subset N$. The precise technical result which will be of essential use to derive some of our main applications is the following:

Proposition 2.5.2. *Let N be a von Neumann algebra together with subalgebras $P, Q \subseteq N$ such that $P' \cap N = \mathbb{C}1$. If we assume that $P \preceq_N Q$ then one can find projections $p \in P$, $q \in Q$, a $\star$-homomorphism $\phi : pPp \to qQq$ and a non-zero partial isometry $v \in qNp$ such that $\phi(x)v = vx$, for all $x \in pPp$, and $\phi(pPp)' \cap qQq = \mathbb{C}q$.*

The proof of this result follows the same strategy as the proof of Lemma 1.5 of [38], so it will be omitted.

We end this section by recalling two important intertwining results from the work of Popa [71, 72]. These results play a very important role in deriving some of our main applications. The first result describes an inclusion of von Neumann algebras where we have complete control over general intertwiners of subalgebras. To properly introduce the statement we need a definition. Given an inclusion of countable groups $\Sigma < \Gamma$ we say that Σ is *malnormal* in Γ if and only if for every $\gamma \in \Gamma \setminus \Sigma$ we have $\gamma\Sigma\gamma^{-1} \cap \Sigma$ is finite.

Proposition 2.5.3 (Theorem 3.1 in [72]). *Let $\Sigma < \Gamma$ be a malnormal group, let $\Gamma \curvearrowright A$ be a trace preserving action and denote by $M = A \rtimes \Gamma$ the corresponding crossed product von Neumann algebra. Also let $p \in A \rtimes \Sigma$ be a projection and suppose that $P \subseteq p(A \rtimes \Sigma)p$ is a diffuse subalgebra such that $P \npreceq_{A \rtimes \Sigma} A$. If there exist elements $x, x_1, x_2, \ldots, x_n \in M$ such that $Px \subseteq \sum_i x_i P$ then $x \in A \rtimes \Sigma$.*

The second result which will be needed in the sequel is Popa's unitary conjugacy criterion for Cartan subalgebras.

Theorem 2.5.4 (Appendix 1 in [71]). *Let N be a II_1 factor and $A, B \subset N$ two semiregular m.a.s.a. 's (i.e., their normalizing algebras $\mathcal{N}_N(A)''$ and $\mathcal{N}_N(B)''$ are subfactors of N). If $B_0 \subset B$ is a von Neumann subalgebra such that $B_0' \cap N = B$, and $B_0 \preceq_N A$, then there exists a unitary $u \in N$ such that $uAu^* = B$.*

CHAPTER 3
MACHINERY

3.1 Relative Arrays and Relative Quasi-cocycles

In this section we consider relative versions of the notions of arrays [10] and quasi-cocycles [51, 47, 48] for groups. This will allow us to generalize, from the viewpoint of deformation/rigidity theory, the structural results obtained in [10]. After introducing the definitions, we summarize a few useful properties, relating these with other concepts extant in the literature. In the last part of the section we will present several examples, some of them arising naturally from geometric group theory.

3.1.1 Relative arrays

Assume that Γ is a countable, discrete group together with $\mathcal{G} = \{\Sigma_i\}_i$, a family of subgroups of Γ and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, a unitary representation. We say that a group Γ admits *a proper array relative to $\mathcal{G}$ into $\mathcal{H}$* if there exists a map $r : \Gamma \to \mathcal{H}$ which satisfies the following conditions:

1. $\pi_\gamma(r(\gamma^{-1})) = -r(\gamma)$ for all $\gamma \in \Gamma$;
2. for every $\gamma \in \Gamma$ we have
$$\sup_{\delta \in \Gamma} \|r(\gamma\delta) - \pi_\gamma(r(\delta))\| = C(\gamma) < \infty;$$
3. the map $\gamma \to \|r(\gamma)\|$ is proper with respect to $\mathcal{G}$, i.e. for every $C > 0$ there exist finite subsets $\mathcal{F} \subset \mathcal{G}$ and $K, L \subset \Gamma$ such that

$$\{\gamma \in \Gamma \,|\, \|r(\gamma)\| \leq C\} \subseteq \cup_{\Sigma\in\mathcal{F}} K\Sigma L.$$

Given a map $\phi : \Gamma \to \mathbb{R}$ we denote by $\lim_{\gamma\to\infty/\mathcal{G}} \phi(\gamma) = L$, if for every $\epsilon > 0$ there exist finite sets $K, L \subset \Gamma, \mathcal{F} \subset G$ such that $|\phi(\gamma) - L| < \epsilon$ for all $\gamma \notin K\mathcal{F}L$.

From now on the set of all such relative arrays will be denoted by $\mathcal{RA}(\Gamma, \mathcal{G}, \mathcal{H}_\pi)$. Notice that when $\mathcal{G}$ consists of the trivial subgroup only, one recovers the notion of arrays as defined in [10]. For further discussions on array the reader may consult Section 1 in [10].

When considering exact groups, the above notion of relative array into the left regular representation is closely related with the notion of bi-exactness introduced by Ozawa (Definition 15.1.2 in [6]). We are indebted to Narutaka Ozawa for kindly demonstrating to us the direct implication in the following result.

Proposition 3.1.1. *Let Γ be an exact group together with $\mathcal{G}$ a family of subgroups. Then $\mathcal{RA}(\Gamma, \mathcal{G}, \ell^2(\Gamma)) \neq \emptyset$ if and only if Γ is bi-exact with respect to $\mathcal{G}$.*

Remark 3.1.2. A recent result of Popa and Vaes [**?**] establishes the same result under the weaker assumption that $\mathcal{RA}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \neq \emptyset$ for some weakly-ℓ^2 representation π.

Proof. The reverse implication can be shown using the same method as in [10] and therefore we only prove the direct implication. So let $r : \Gamma \to \ell^2(\Gamma)$ an array relative to the family $\mathcal{G}$ and denote by $\pi : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ the left regular representation. Let $\mathrm{Prob}(\Gamma)$ be the set of positive borel probability measures on Γ. For any $f \in \ell^\infty(\Gamma)$ we let $\phi(f)$ the natural "diagonal" operator acting by pointwise multiplication.

Then we define the map $\mu : \Gamma \to \mathrm{Prob}(\Gamma)$ by letting

$$\langle \mu(\gamma), f \rangle = \frac{1}{\|r(\gamma)\|^2} \langle \phi(f) r(\gamma), r(\gamma) \rangle,$$

for all $\gamma \in \Gamma$ and $f \in \ell^\infty(\Gamma)$. Also if we fix $s, t \in \Gamma$ then denoting by $C_s = \sup_{\gamma \in \Gamma} \|r(s\gamma) - \pi_s(r(\gamma))\|$ and using the triangle inequality together with the anti-symmetry of the array r we have that

$$\begin{aligned} \|r(s\gamma t) - \pi_s(r\gamma)\| &\leq \|r(s\gamma t) - r(s\gamma)\| + \|r(s\gamma) - \pi_s(r(\gamma))\| \\ &= \| - \pi_{s\gamma t}(r(t^{-1}\gamma^{-1}s^{-1})) + \pi_{s\gamma}(r(\gamma^{-1}s^{-1}))\| + C_s \\ &= \| - \pi_t(r(t^{-1}\gamma^{-1}s^{-1})) + r(\gamma^{-1}s^{-1})\| + C_s \\ &\leq C_t + C_s, \end{aligned} \tag{3.1}$$

for all $\gamma \in \Gamma$.

In the remaining part we will use this estimate to show that for all $s, t \in \Gamma$ we have

$$\lim_{\gamma \to \infty / \mathcal{G}} \|\mu(s\gamma t) - s.\mu(\gamma)\| = 0, \tag{3.2}$$

which in turn will give the desired conclusion.

To see this we fix $s, t, \gamma \in \Gamma$ and $f \in \ell^\infty(\Gamma)$. Then applying the triangle inequality in combination with (3.1) and the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
& |\langle \mu(s\gamma t), f\rangle - \langle s.\mu(\gamma), f\rangle| \\
\leq\ & \frac{1}{\|r(s\gamma t)\|^2} \left|\langle \phi(f) r(s\gamma t), r(s\gamma t) - \pi_s(r(\gamma))\rangle\right| + \\
& + \left| \left(\frac{1}{\|r(s\gamma t)\|^2} - \frac{1}{\|r(\gamma)\|^2} \right) \langle \phi(f) r(s\gamma t), \pi_s(r(\gamma))\rangle \right| + \\
& + \frac{1}{\|r(\gamma)\|^2} \left|\langle \phi(f) r(s\gamma t) - \pi_s(r(\gamma)), r(\gamma)\rangle\right| \\
\leq\ & \frac{C_s + C_t}{\|r(s\gamma t)\|^2} \|\phi(f) r(s\gamma t)\| + \left| \frac{1}{\|r(s\gamma t)\|^2} - \frac{1}{\|r(\gamma)\|^2} \right| \|\phi(f) r(s\gamma t)\| \|r(\gamma)\| + \\
& + \frac{1}{\|r(\gamma)\|} \|\phi(f) r(s\gamma t) - \pi_s(r(\gamma))\| \\
\leq\ & 2(C_s + C_t)\|f\| \left(\frac{1}{\|r(s\gamma t)\|} + \frac{1}{\|r(\gamma)\|} \right).
\end{aligned}
$$

Since r is assumed to be proper with respect to the set $\mathcal{G}$ then $\lim_{\gamma\to\infty/\mathcal{G}} \|r(s\gamma t)\| = \infty$, $\lim_{\gamma\to\infty/\mathcal{G}} \|r(\gamma)\| = \infty$ and thus taking the limit in the previous inequality we get (3.2).

3.1.2 Relative quasi-cocycles

In the same spirit, if Γ is a group together with a family of subgroups $\mathcal{G} = \{\Sigma_i\}_i$ and a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, we say that pair $(\Gamma, \mathcal{G})$ admits *a relative quasi-cocycle into* $\mathcal{H}$ if there exists a map $r : \Gamma \to \mathcal{H}$ satisfying the following condition:

1. there exists a constant $C > 0$ such that

$$\sup_{\gamma,\delta\in\Gamma} \|r(\gamma\delta) - \pi_\gamma(r(\delta)) - r(\gamma)\| \leq C.$$

2. the map $\gamma \to \|r(\gamma)\|$ is proper relative to $\mathcal{G}$.

From now on, the set of all such relative quasi-cocycles we will denoted by $\mathcal{RQ}(\Gamma, \mathcal{G}, \mathcal{H}_\pi)$. Using the terminology from [93], it is clear that $\mathcal{RQ}(\Gamma, \mathcal{G}, \mathcal{H}_\pi)$ is a subset of $\mathcal{Q}H^1(\Gamma, \mathcal{H}_\pi)$ which is stable under scalar multiplication and translation by uniformly bounded maps, without being in general a vector subspace. It is also straightforward that every relative quasi-cocycle is a relative array, i.e., we always have $\mathcal{RQ}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \subseteq \mathcal{RA}(\Gamma, \mathcal{G}, \mathcal{H}_\pi)$. The next proposition summarizes a few basic properties which follow directly from definitions.

Proposition 3.1.3. *For each $n \in \mathbb{N}$ let $\mathcal{G}_n$ be a family of subgroups of Γ together with $\pi_n : \Gamma \to \mathcal{U}(\mathcal{H}_n)$ a unitary representation. Then we have the following:*

1. *If $\mathcal{G}_1 \subset \mathcal{G}_2$ then $\mathcal{RA}(\Gamma, \mathcal{G}_1, \mathcal{H}_{\pi_1}) \subseteq \mathcal{RA}(\Gamma, \mathcal{G}_2, \mathcal{H}_{\pi_1})$;*
2. *If $r \in \mathcal{RA}(\Gamma, \mathcal{G}_1, \mathcal{H}_1)$ and $c : \Gamma \to \mathcal{H}_1$ is a uniformly bounded map then $r + c \in \mathcal{RA}(\Gamma, \mathcal{G}, \mathcal{H})$;*
3. *If $\mathcal{G}_n = \mathcal{G}_1$ and $\pi_n = \pi_1$ for all n and there exists a sequence $r_n \in \mathcal{RA}(\Gamma, \mathcal{G}_1, \mathcal{H}_{\pi_1})$ with uniformly bounded defects such that r_n converges to r uniformly then $r \in \mathcal{RA}(\Gamma, \mathcal{G}_1, \mathcal{H}_{\pi_1})$;*
4. *Denote by $\wedge_n \mathcal{G}_n = \{\Sigma_1 \cap \bigcap_{j \neq 1} s_j \Sigma_j s_j^{-1} \,|\, \Sigma_1 \in \mathcal{G}_1, \Sigma_j \in \mathcal{G}_j, s_j \in \Gamma\}$. If for every $n \in \mathbb{N}$ there exists $c_n > 0$ and $r_n \in \mathcal{RA}(\Gamma, \mathcal{G}, \mathcal{H}_{\pi_n})$ satisfying $\sum_n c_n^2 \|r_n(\gamma)\|^2 < \infty$ for all $\gamma \in \Gamma$, then*

$$\mathcal{RA}(\Gamma, \wedge_n \mathcal{G}_n, \oplus \mathcal{H}_{\pi_n}) \neq \emptyset.$$

Cocycles, quasi-cocycles, and arrays combine both geometric and representation-theoretical data in a way that can be used to efficiently extract information about a group's internal structure. For instance, by the same proof as in Proposition 1.5.3 of [10] we can locate centralizers of certain subgroups and, in some cases, even normalizers. This property, generically called the "spectral gap rigidity principle", is the main intuition for the von Neumann algebraic structural results obtained in the subsequent sections.

Proposition 3.1.4. *Let* Γ *be a countable group,* $\mathcal{G}$ *be a family of subgroups, and* $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ *a representation such that* $\mathcal{RA}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \neq \emptyset$*. If* $\Lambda < \Gamma$ *is a subgroup such that* $1_\Lambda \not\prec \pi$ *then there exists* $h \in \Gamma$ *and* $\Sigma \in \mathcal{G}$ *such that its centralizer* $C_\Gamma(\Lambda)$ *satisfies* $[C_\Gamma(\Lambda) : h\Sigma h^{-1} \cap C_\Gamma(\Lambda)] < \infty$*.*

Proof. Let $q : \Gamma \to \mathcal{H}_\pi$ be an array. Since $1_\Lambda \not\prec \pi$ there exists a finite, symmetric subset $S \subset \Lambda$ and $K' > 0$ such that

$$\|\xi\| \leq K' \sum_{s \in S} \|\pi_s(\xi) - \xi\|, \text{ for all } \xi \in \mathcal{H}_\pi. \tag{3.3}$$

Since q is an array there exists $K'' > o$ such that $\|q(s\gamma) - \pi_s(q(\gamma))\| \leq K''$ for all $s \in S$ and $\gamma \in \Gamma$. Set $K = \max K', K''$. Then, for every $\gamma \in C_\Gamma(\Lambda)$ we have that

$$\begin{aligned}
\|q(\gamma)\| &\leq K \sum_{s \in S} \|\pi_s(q(\gamma)) - q(\gamma)\| \\
&\leq K \sum_{s \in S} \|q(s\gamma)) - q(\gamma)\| + K^2|S| \\
&\leq K \sum_{s \in S} \|q(\gamma s)) - q(\gamma)\| + K^2|S| \\
&\leq K \sum_{s \in S} \|\pi_{\gamma s} q(s^{-1}\gamma^{-1})) - \pi_\gamma(q(\gamma^{-1}))\| + K^2|S| \\
&= K \sum_{s \in S} \|\pi_s q(s^{-1}\gamma^{-1})) - q(\gamma^{-1})\| + K^2|S| \\
&\leq 2K^2|S|
\end{aligned}$$

This shows that q is bounded on $C_\Gamma(\Lambda)$ and since q is proper with respect to the family $\mathcal{G}$ it follows that $C_\Gamma(\Lambda)$ is small with respect to the family $\mathcal{G}$ this means that there exists a finite collection of groups $\Sigma_i \in \mathcal{G}$ and a finite set of elements $h_i, k_i \in \Gamma$ such that $C_\Gamma(\Lambda) \subseteq \bigcup_i h_i\Sigma_i k_i$. Therefore, if we denote by $\Omega_i = h_i\Sigma_i h_i^{-1}$, there exists a finite set of elements $\ell_i \in \Gamma$ such that $C_\Gamma(\Lambda) \subseteq \bigcup_i \Omega_i\ell_i$. In particular this implies that $C_\Gamma(\Lambda) = \bigcup_i (\Omega_i\ell_i \cap C_\Gamma(\Lambda))$. After dropping all the empty intersections we can assume that $\Omega_i\ell_i \cap C_\Gamma(\Lambda) \neq \emptyset$, for all i. Hence there exists $s_i \in \Omega_i$ such that $r_i = s_i\ell_i \in C_\Gamma(\Lambda)$ and we obviously have that

$$C_\Gamma(\Lambda) = \bigcup_i (\Omega_i r_i \cap C_\Gamma(\Lambda)) = \bigcup_i (\Omega_i \cap C_\Gamma(\Lambda))\, r_i.$$

Finally, by Lemma 4.1 in [56], the previous relation implies that $\Omega_i \cap C_\Gamma(\Lambda)$ have finite index in $C_\Gamma(\Lambda)$ and we are done.

Here are two concrete situations when this happens: Λ has property (T) and the restriction $\pi|_\Lambda$ has no invariant vectors; Λ is not co-amenable with respect to a subgroup $\Sigma < \Gamma$ and π is the left semi-regular representation $\ell^2(\Gamma/\Sigma)$.

Moreover, if Γ is weakly amenable (see f. Section 3), Λ is amenable, and the normalizing group satisfies $1_{N_\Gamma(\Lambda)} \not\prec \pi$ then Λ is small with respect to $\mathcal{G}$. This is a easy consequence of Corollary 3.2 in the sequel but in the in this form can be shown by direct arguments similar to the above proof.

Examples 3.1.5. There are many examples of groups that admit relative quasi-cocycles (arrays) into various representations. First we analyze a few examples arising from canonical group constructions:

A. Exact sequences. Let L, K, Γ be groups such that $0 \to L \to K \to \Gamma \to 0$ is a short exact sequence. If $\mathcal{RA}(\Gamma, \{e\}, \ell^2(\Gamma)) \neq \emptyset$ then we have $\mathcal{RA}(K, \{L\}, \ell^2(K/L)) \neq \emptyset$.

B. Product groups. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be a collection of groups, and denote by $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$. For every $i = 1, \ldots, n$ we denote by $\hat{\Gamma}_i$ the subgroup of the direct product Γ which consists of all elements whose i^{th} coordinate is trivial. Assume that $\mathcal{G}_i$ is family of subgroups of Γ_i and denote by $\mathcal{G} = \bigcup_i \{\Lambda \times \hat{\Gamma}_i \,|\, \Lambda \in \mathcal{G}_i\}$. If $\mathcal{RA}(\Gamma_i, \mathcal{G}_i, \mathcal{H}_i) \neq \emptyset$ for all $i = 1, \ldots, n$, then $\mathcal{RA}(\Gamma, \mathcal{G}, \otimes_i \mathcal{H}_i) \neq \emptyset$. For the proof of this fact see Proposition 1.10 in [10]. In particular $\Gamma_1 \times \Gamma_2$ admits an array into $\ell^2(\Gamma_1 \times \Gamma_2)$ which is proper with respect to $\{\Gamma_1, \Gamma_2\}$ whenever Γ_1 and Γ_2 admit proper array into their left regular representations.

C. Semidirect products. Let Γ and A be countable discrete groups together with $\mathcal{G}$ a family of subgroups of Γ and assume that $\rho : \Gamma \to Aut(A)$ is an action by group automorphisms. Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ be a unitary representation and define $\tilde{\pi} : A \rtimes_\rho \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ by letting $\tilde{\pi}_{a\gamma}(\xi) = \pi_\gamma(\xi)$ for every $a \in A$, $\gamma \in \Gamma$ and $\xi \in \mathcal{H}_\pi$. If $c \in \mathcal{RA}(\Gamma, \mathcal{G}, \mathcal{H}_\pi)$ then the formula $\tilde{c}(a\gamma) = c(\gamma)$ defines an array which belongs to $\mathcal{RA}(A \rtimes_\rho \Gamma, \{A \rtimes_\rho \Sigma \,|\, \Sigma \in \mathcal{G}\}, \mathcal{H}_{\tilde{\pi}})$.

We now look at semidirect products by finite groups. So let Γ be a countable discrete group together with a family of subgroups $\mathcal{G}$, Λ be a finite group, and $\rho : \Lambda \to Aut(\Gamma)$ be an action by automorphisms. It is an exercise for the reader to check that for any $r \in \mathcal{RA}(\Gamma, \mathcal{G}, \ell^2(\Gamma))$, the map $r'(\gamma\alpha) = \frac{1}{|\Lambda|^{\frac{1}{2}}} \sum_{\delta \in \Lambda} \lambda_\delta(r(\rho_{\delta^{-1}}(\gamma)))$ defines an array belonging to $\mathcal{RA}(\Gamma \rtimes_\rho \Lambda, \mathcal{G}, \ell^2(\Gamma \rtimes_\rho \Lambda))$, where $\gamma \in \Gamma, \alpha \in \Lambda$ and λ is the left regular representation on $\ell^2(\Gamma \rtimes_\rho \Lambda)$. The defect of r' will not exceed the defect of r.

D. Free products. Let $\{\Gamma_n\}_{1\leq i\leq n}$ be a finite collection of groups. Denote by $\Gamma = \star_i \Gamma_i$ their free product, and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ a unitary representation. If for every $1 \leq i \leq n$ we have $\mathcal{RA}(\Gamma_i, \{e\}, \mathcal{H}_\pi) \neq \emptyset$, then the proof of Lemma 5.1 and Theorem 5.3 in [93] show that $\mathcal{RA}(\Gamma, \{e\}, \mathcal{H}_\pi^{\oplus n}) \neq \emptyset$. Note that the when considering arrays proper with respect to families of subgroups, it is not clear whether the resulting array is proper to any canonical family of subgroups—rather just a finite length subsets over the families of groups we started with. However, if we assume that $\Sigma \lhd \Gamma_i$ is a common normal subgroup, $\Gamma = \star_\Sigma \Gamma_i$ is the amalgamated free product over Σ, and for every $1 \leq i \leq n$ we have $\mathcal{RA}(\Gamma_i/\Sigma, \{e\}, \ell^2(\Gamma_i/\Sigma)) \neq \emptyset$ then $\mathcal{RA}(\Gamma, \{\Sigma\}, \ell^2(\Gamma/\Sigma)) \neq \emptyset$.

E. HNN-extensions. Denote by $\Gamma = (H, L, \theta)$ the HNN-extension associated with a given inclusion groups $L < H$ and a monomorphism $\theta : L \to H$. We also assume that $K \lhd H$ is a normal subgroup which contains L and $\theta(L)$ and from now on we will denote by $L_1 = L$, $L_{-1} = \theta(L)$. The group Γ may be presented as $\{H, t \,|\, \theta(\ell) = t\ell t^{-1}, \ell \in L\}$. By Britton's Lemma, every element $\gamma \in \Gamma$ has a canonical reduced form $\gamma = \gamma_0 t^{\varepsilon_1}\gamma_1 t^{\varepsilon_2} \ldots \gamma_{n-1} t^{\varepsilon_n}\gamma_n$, where $\gamma_i \in H$, $\varepsilon_i \in \{-1, 1\}$ and whenever $\varepsilon_i \neq \varepsilon_{i+1}$ we have that $\gamma_i \notin L_{\varepsilon_i}$, for all $i = 1, \ldots, n-1$.

Assume that $q : H/K \to \ell^2(H/K)$ is an array. By the construction in the first example there exists an array $c : H \to \ell^2(H/K)$ which vanishes on K and moreover $c \in \mathcal{RA}(H, \{K\}, \ell^2(H/K))$ whenever q is proper. We can define a map $r : \Gamma \to \ell^2(\Gamma/K)$ in the following way: for every $\gamma = \gamma_0 t^{\varepsilon_1}\gamma_1 t^{\varepsilon_2} \ldots \gamma_{n-1} t^{\varepsilon_n}\gamma_n$ and $s = 0, 1$ we let

$$\begin{aligned} r_q^s(\gamma) &= \lambda_{\gamma_0 t^{\varepsilon_1}\gamma_1 t^{\varepsilon_2}\ldots\gamma_{n-1}t^{\varepsilon_n}}c(\gamma_n) + \lambda_{\gamma_0 t^{\varepsilon_1}\gamma_1 t^{\varepsilon_2}\ldots\gamma_{n-1}}d^s(t^{\varepsilon_n}) + \\ &+\lambda_{\gamma_0 t^{\varepsilon_1}\gamma_1 t^{\varepsilon_2}\ldots\gamma_{n-1}t^{\varepsilon_{n-1}}}c(\gamma_{n-1}) + \lambda_{\gamma_0 t^{\varepsilon_1}\gamma_1 t^{\varepsilon_2}\gamma_{n-2}}d^s(t^{\varepsilon_{n-1}}) + \\ &+ \ldots + \lambda_{\gamma_0}d^s(t^{\varepsilon_1}) + c(\gamma_0), \end{aligned}$$

where $d^1(t^\varepsilon) = \delta_{t^\varepsilon K}$ and $d^0(t^\varepsilon) = 0$ for all $\varepsilon \in \{-1, 1\}$. Here λ denotes the left semi-regular representation $\ell^2(H/K)$. It is a straightforward exercise to see that this map is well defined and it satisfies the array relation. Moreover, when $q = 0$, the map is actually a 1-cocycle.

Therefore, applying part (4) in Proposition 3.1.3 we have that $r_0^1 \oplus r_q^0$ is an array into $\ell^2(\Gamma/K) \oplus \ell^2(\Gamma/K)$. If q is assumed proper it follows that $r_0^1 \oplus r_q^0$ is proper with respect to various *subsets* of Γ, e.g. sets of words with finite length over t's whose letters from H are "small" over K. However to have properness with respect to subgroups we need to impose additional assumptions on K. For instance, one may assume that L and $\theta(L)$ have finite index in K, in which case we would have $r_0^1 \oplus r_q^0 \in \mathcal{RA}(\Gamma, \{K\}, \ell^2(\Gamma/K) \oplus \ell^2(\Gamma/K))$.

F. Inductive limits. Let $\Gamma_n \nearrow \Gamma$ be an inductive limit of groups and for each $n \in \mathbb{N}$ let $\mathcal{G}_n$ be a family of subgroups of Γ_n such that $\mathcal{G}_n \subseteq \mathcal{G}_{n+1}$. Assume that for each n, there exists $r_n \in \mathcal{RA}(\Gamma_n, \mathcal{G}_n, \ell^2(\Gamma_n))$ so that:

1. $\sup_{\gamma\in\Gamma_{\min(n,m)}} \|r_n(\gamma) - r_m(\gamma)\| < \infty$, for every $n, m \in \mathbb{N}$;
2. $\sup_{n\in\mathbb{N}} \|C_n(\gamma)\| < \infty$, for every $\gamma \in \Gamma$;
3. for every $C > 0$ there exists $n_C \in \mathbb{N}$ such that for all $n \geq n_C$ we have $\{\gamma \in$

$\Gamma_{n+1} \mid \|r_{n+1}(\gamma)\| \leq C\} \subset \Gamma_n$.

For every $\gamma \in \Gamma$ we define a map $r : \Gamma \to \ell^2(\Gamma)$ by letting $r(\gamma) = r_n(\gamma)$, where n is chosen to be the smallest natural number such that $\gamma \in \Gamma_n$. The above properties then imply that $r \in \mathcal{RA}(\Gamma, \cup_n \mathcal{G}_n, \ell^2(\Gamma))$.

Doing some simple calculations the reader may verify that this above construction together with Proposition 1.10 in [10] shows that if there exists a sequence $r_n \in \mathcal{RA}(\Gamma_n, \{e\}, \ell^2(\Gamma_n))$ with uniform bounded equivariance then $r \in \mathcal{RA}(\oplus_n \Gamma_n, \{e\}, \ell^2(\oplus \Gamma_n))$. In particular we have that if $\mathcal{RA}(\Gamma, \{e\}, \ell^2(\Gamma)) \neq \emptyset$ then $\mathcal{RA}(\Gamma^{\oplus\infty}, \{e\}, \ell^2(\Gamma^{\oplus\infty})) \neq \emptyset$.

As expected, to obtain relative quasi-cocycles we have to impose stronger assumptions. For example, if there exist relative quasi-cocycles $r_n \in \mathcal{RQ}(\Gamma_n, \mathcal{G}_n, \ell^2(\Gamma_n))$ satisfying $\sup_{m,n} \sup_{\gamma \in \Gamma_{\min(n,m)}} \|r_n(\gamma) - r_m(\gamma)\| < \infty$, $\sup_{n \in \mathbb{N}} D_n < \infty$, and condition (3), the same construction as before shows that $\mathcal{RQ}(\Gamma, \cup_n \mathcal{G}_n, \ell^2(\Gamma)) \neq \emptyset$. Also we notice that by a basic rescaling procedure the same conclusion follows if we completely drop the uniform boundedness on the defects D_n, keep condition (3), and replace the first condition by the following: there exists a sequence $K_n \geq D_n$ such that

$$\sup_{m,n} \sup_{\gamma \in \Gamma_{\min(n,m)}} \|\frac{1}{K_n} r_n(\gamma) - \frac{1}{K_m} r_m(\gamma)\| < \infty.$$

The examples presented above arise more or less from canonical algebraic constructions. More interestingly, relative quasi-cocycles on groups can be constructed naturally from purely geometric considerations. Below we single out a class of such examples which are intensely studied in geometric group theory.

G. Relative hyperbolic groups. The results in [51, 47, 48] imply that every Gromov-hyperbolic group Γ admits a proper quasi-cocycle into a multiple of $\ell^2(\Gamma)$ (Lemma 4.2 in [93]). Using a similar reasoning we will show a relative version of this result for the relatively hyperbolic groups in the sense of Bowditch [4].

Briefly, given a group Γ together with a family of subgroups $\mathcal{G}$, we say that Γ is hyperbolic relative to $\mathcal{G}$ if there exists a graph $\mathcal{K}$ on which Γ acts such that the following conditions are satisfied: a) Γ and every $\Sigma \in \mathcal{G}$ are finitely generated, b) $\mathcal{K}$ is fine (see (1) in Definition 2 from [4]) and has thin triangles, c) there are finitely many orbits and each edge stabilizer is finite, d) the infinite vertex stabilizers are precisely the elements of $\mathcal{G}$ and their conjugates.

Here are some examples of relatively hyperbolic groups: a free product is relatively hyperbolic with respect to its factors; if Γ is hyperbolic relative to a family of subgroups $\mathcal{G}$ and $\alpha : \Sigma_1 \to \Sigma_2$ is a monomorphism with $\Sigma_i \in \mathcal{G}$, then the HNN extension $\Gamma\star_\alpha$ is hyperbolic with respect to $\mathcal{G} \setminus \{\Sigma_1, \Sigma_2\}$ [19]; geometrically finite Kleinian groups are hyperbolic with respect to their cusp subgroups [24]; the fundamental group of a complete hyperbolic manifold of finite volume is hyperbolic relative to its cusp subgroups [24]; Sela's limit groups are hyperbolic relative to their maximal noncyclic abelian subgroups [19].

Mineyev and Yaman [49] showed that whenever Γ is hyperbolic relative to a finite set $\mathcal{G}$ of subgroups, there exists an ideal hyperbolic tuple $(\Gamma, \mathcal{G}, X, \nu')$ (Definition 42 in [49]). Furthermore, using this in combination with the machinery developed in [47], they constructed a homological $\mathbb{Q}$-bicombing in X which is Γ-equivariant, anti-symmetric, quasi-geodesic, and has bounded area (Theorem 47 in [49]). Therefore, applying the same arguments as in the proof of Theorem 7.13 of [52], we see that this bicombing gives rise naturally to relative quasi-cocycles for Γ into a multiple of the left semi-regular representations with respect to some conjugates of elements in $\mathcal{G}$. In effect, the bounded area together with anti-symmetry will imply the quasi-cocycle relation and being quasi-geodesic will imply properness with respect to the family $\mathcal{G}$.

Proposition 3.1.6. *If a group Γ is hyperbolic relative to a finite family of subgroups $\mathcal{G}$ in the sense of Bowditch [4], then we have that $\mathcal{RQ}(\Gamma, \mathcal{G}, \oplus_{i,j}\ell^2(\Gamma/\gamma_j\Sigma_i\gamma_j^{-1})) \neq \emptyset$, for some $g_j \in \Gamma$ and $\Sigma_i \in \mathcal{G}$.*

3.2 Weak amenability for groups and von Neumann algebras

The notion of weak amenability for groups was introduced by Cowling and Haagerup in [18]. There are several equivalent definitions ([5, 18]) and for reader's convenience we recall the following:

Definition 3.2.1. *A countable discrete group Γ is said to be weakly amenable with constant C if there exists a sequence of finitely supported functions $\phi_n : \Gamma \to \mathbb{C}$ such that $\phi_n \to 1$ pointwise and $\limsup_n \|\widehat{\phi}_n\|_{cb} \leq C$, where $\|\widehat{\phi}_n\|_{cb}$ denotes the (completely bounded) norm of the Schur multiplier on $\mathfrak{B}(\ell^2(\Gamma))$ associated with the kernel $\widehat{\phi}_n : \Gamma \times \Gamma \to \mathbb{C}$ given by $\widehat{\phi}_n(\gamma, \delta) = \phi_n(\gamma^{-1}\delta)$.*

The Cowling-Haagerup constant $\Lambda_{cb}(\Gamma)$ is defined to be the infimum of all C for which such a sequence (ϕ_n) exists. If Γ is not weakly amenable then we write

$\Lambda_{cb}(\Gamma) = \infty$.

Below we summarize some families of groups known to be weakly amenable, also specifying their Cowling-Haagerup constants:

1. all amenable groups ($\Lambda_{cb}(\Gamma) = 1$);
2. all lattices in $SO(n, 1)$ and $SU(n, 1)$ ($\Lambda_{cb}(\Gamma) = 1$) or lattices in $Sp(n, 1)$ ($\Lambda_{cb}(\Gamma) = 2n - 1$), [18];
3. Coxeter groups ($\Lambda_{cb}(\Gamma) = 1$) [40];
4. more generally, all groups which act properly on finite dimensional CAT(0)-cube complexes ($\Lambda_{cb}(\Gamma) = 1$), [30, 50];
5. all hyperbolic groups (in this case no explicit constants were computed), [61].
6. all limit groups in the sense of Sela ($\Lambda_{cb}(\Gamma) = 1$); this is an observation due to Ozawa based on a result from [**?**].

Groups which are not weakly amenable include $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$, [34] (see also, [21]), lattices in higher-rank simple Lie groups, and any non-amenable wreath products of the form $\mathbb{Z} \wr \Sigma$, [63].

The class of weakly amenable groups is closed under taking subgroups, cartesian products, co-amenable extensions, measure equivalence [63], and inductive limits of groups with uniformly bounded Cowling-Haagerup constants. However, it is not known whether weak amenability is closed under taking a free product of two groups except in the case that the Cowling–Haagerup constants of both groups are one [83].

By analogy with the group case discussed above, one can define a similar approximation property for von Neumann algebras. The precise formulation is the following.

Definition 3.2.2. *A von Neumann algebra M is said to have the* weak* completely bounded approximation property, *abreviated W*CBAP, if there is a sequence of ultraweakly-continuous finite-rank maps (ϕ_n) on M such that $\phi_n \to \mathrm{id}_M$ in the point-ultraweak topology and* $\limsup_n \|\phi_n\|_{cb} < \infty$.

In [65] Ozawa and Popa discovered that the presence of this finite-dimensional approximation (with constant one) on a group imposes a certain type of "rigidity" on its internal structure. More precisely, they showed that if $\Lambda_{cb}(\Gamma) = 1$ then for any amenable subgroup $\Omega < \Gamma$ with non-amenable normalizing group $N_\Gamma(\Omega)$ there exists an $\Omega \rtimes N_\Gamma(\Omega)$ invariant state on $\ell^\infty(\Omega)$, where the semidirect product $\Omega \rtimes N_\Gamma(\Omega)$ acts on Ω by $(\gamma, a) \cdot x = \gamma a x \gamma^{-1}$. In other words, the natural action of the normalizer $N_\Gamma(\Omega)$ on Ω is fairly "small"; for instance, it cannot be of Bernoulli type. Later, Ozawa showed that in fact *all* weakly amenable groups satisfy this property, [63]. In fact, this rigidity even manifests in the von Neumann–algebraic context, as follows:

Theorem 3.2.3 (Ozawa and Popa [65], Ozawa [63]). *Let M be a von Neumann algebra which has W*CBAP and let $P \subset M$ be a diffuse amenable subalgebra. Then the natural action by conjugation of the normalizer $\mathcal{N}_M(P) \curvearrowright P$ is* weakly compact, *i.e., there exists a net of unit vectors $(\eta_n)_{n\in\mathbf{N}}$ in $L^2(M)\bar{\otimes}L^2(\bar{M})$ such that:*

1. *$\|\eta_n - (v \otimes \bar{v})\eta_n\| \to 0$, for all $v \in \mathcal{U}(P)$;*
2. *$\|[u \otimes \bar{u}, \eta_n]\| \to 0$, for all $u \in \mathcal{N}_M(P)$;*
3. *$\langle (x \otimes 1)\eta_n, \eta_n\rangle = \tau(x) = \langle (1 \otimes \bar{x})\eta_n, \eta_n\rangle$, for all $x \in M$.*

In combination with deformation techniques, weak compactness turned out to be an powerful tool for obtaining many important structural results for group–measure space factors [65, 66, 32, 87].

3.3 The Gaussian Construction, Bimodules and Weak Containment

Given an orthogonal representation $\pi : \Gamma \to \mathcal{O}(\mathcal{H}_{\mathbb{R}})$ of a discrete, countable group there exists a way of associating to it a p.m.p. action of Γ on a measure space such that the induced Koopman representation is unitarily equivalent to the infinite direct sum of the symmetric tensor powers of π (see the proof of Lemma 3.5 in [94]). This is called the Gaussian construction associated to $(\Gamma, \pi, \mathcal{H}_{\mathbb{R}})$. We briefly describe this construction here, indicating how it can be extended to measure preserving actions by product groups.

If $\pi : \Gamma \to \mathcal{O}(\mathcal{H}_{\mathbb{R}})$ is an orthogonal representation, the Gausssian construction as described in [69] or [87] provides a probability measure space (Y_π, ν) and a family $\omega(\xi)_{\xi \in \mathcal{H}}$ of unitaries in $L^\infty(Y_\pi)$ such that $L^\infty(Y_\pi)$ is generated as a von Neumann algebra by the $\omega(\xi)$'s and the following relations hold:

1. $\omega(0) = 1$, $\omega(\xi_1 + \xi_2) = \omega(\xi_1)\omega(\xi_2)$, $\omega(\xi)^* = \omega(-\xi)$ for all $\xi, \xi_1, \xi_2 \in \mathcal{H}_{\mathbb{R}}$
2. $\tau(\omega(\xi)) = \exp(-\|\xi\|^2)$ where τ is the trace on $L^\infty(Y_\pi)$ given by integration.

The action σ of Γ on $L^\infty(Y_\pi)$ is given by $\sigma_g(\omega(\xi)) = \omega(\pi_g(\xi))$, for all $\xi \in \mathcal{H}_{\mathbb{R}}$.

Suppose now that $\Gamma_1 \times \Gamma_2$ acts in a trace preserving manner on an abelian von Neumann algebra (A, τ) and denote by $M = A \rtimes (\Gamma_1 \times \Gamma_2)$ the corresponding crossed product von Neumann algebra.

For each $i = 1, 2$ let $\pi_i : \Gamma_i \to \mathcal{O}(\mathcal{H}_i)$ be an orthogonal representation which is weakly contained in the (real) left regular representation of Γ_i. Let $L^2(Y_{\pi_i})_0 = L^2(Y_{\pi_i}) \ominus \mathbb{C}1$ be the Koopman representation of the Gaussian action corresponding to π_i which, by the assumptions, it is also weakly contained in the left regular representation. Consider the Hilbert space $\mathcal{K} = L^2(A)\bar{\otimes}L^2(Y_{\pi_1})_0\bar{\otimes}L^2(Y_{\pi_2})_0\bar{\otimes}\ell^2(\Gamma_1 \times \Gamma_2)$ with the M-bimodular structure defined as

$$(au_g) \cdot (\xi \otimes \xi_1 \otimes \xi_2 \otimes \delta_k) \cdot (bu_h) = (a\sigma_g(\xi)\sigma_{gk}(b)) \otimes (\pi_g(\xi_1 \otimes \xi_2)) \otimes (\delta_{gkh}),$$

for every $a, b \in A$, $\xi \in L^2(A)$, $\xi_1 \in L^2(Y_{\pi_1})_0$, $\xi_2 \in L^2(Y_{\pi_2})_0$, and $g, k, h \in \Gamma_1 \times \Gamma_2$. Here $\pi = \pi_1 \otimes \pi_2$.

One of the key ingredients needed in the proof of Theorem 4.1.2 is that whenever A is amenable the above M-bimodule is weakly contained in the coarse M-bimodule.

Lemma 3.3.1 (Fell's absorption principle). *As an M-bimodule, $\mathcal{K}$ is isomorphic with a multiple of $L^2(\langle M, A\rangle, Tr)$. In particular, when A is amenable, it follows that $\mathcal{K}$ is weakly contained in the coarse bimodule, $L^2(M)\bar{\otimes}L^2(M)$.*

Proof. First we notice that when π_i is weakly contained in ρ_i then the bimodule associated to the pair (π_1, π_2) is weakly contained in the bimodule associated with the pair (ρ_1, ρ_2). It is therefore enough to prove the statement in the case when π_i is the (real) left representation of Γ_i.

For simplicity, throughout this proof, we denote by $\Gamma = \Gamma_1 \times \Gamma_2$. Since $\mathcal{K}$ is canonically identified with $L^2(A)\bar{\otimes}\ell^2(\Gamma)\bar{\otimes}\ell^2(\Gamma)$, we will obtain the desired conclusion by showing that the map

$$L^2(A)\bar{\otimes}\ell^2(\Gamma)\bar{\otimes}\ell^2(\Gamma) \ni \xi \otimes \delta_g \otimes \delta_h \to \xi u_g e_A u_{g^{-1}h} \in L^2(\langle M, A\rangle, Tr)$$

implements an isomorphism between the two bimodules.

To this purpose it suffices to show that

$$\begin{aligned} & \langle (au_s)\cdot(\xi\otimes\delta_g\otimes\delta_h)\cdot(bu_t), (a'u_{s'})\cdot(\xi'\otimes\delta_{g'}\otimes\delta_{h'})\cdot(b'u_{t'})\rangle \qquad (3.4) \\ = \quad & \langle au_s(\xi u_g e_A u_{g^{-1}h})bu_t, a'u_{s'}(\xi' u_{g'} e_A u_{g'^{-1}h'})b'u_{t'}\rangle_{Tr}, \end{aligned}$$

for all $a, a', b, b' \in A$, $\xi, \xi' \in L^2(A)$, and $s, t, g, h, s', t', g', h' \in \Gamma$.

On the one hand, by definitions, the left side in the previous equation is equal to

$$\begin{aligned} & \langle (au_s)\cdot(\xi\otimes\delta_g\otimes\delta_h)\cdot(bu_t), (a'u_{s'})\cdot(\xi'\otimes\delta_{g'}\otimes\delta_{h'})\cdot(b'u_{t'})\rangle \\ = \quad & \langle (a\sigma_s(\xi)\sigma_{sh}(b))\otimes\delta_{sg}\otimes\delta_{sht}, (a'\sigma_{s'}(\xi')\sigma_{s'h'}(b'))\otimes\delta_{s'g'}\otimes\delta_{s'h't'}\rangle \\ = \quad & \delta_{sg,s'g'}\delta_{sht,s'h't'}\langle (a\sigma_s(\xi)\sigma_{sh}(b)), (a'\sigma_{s'}(\xi')\sigma_{s'h'}(b'))\rangle \\ = \quad & \delta_{sg,s'g'}\delta_{sht,s'h't'}\tau(\sigma_{s'h'}(b'^*)\sigma_{s'}(\xi'^*)a'^*a\sigma_s(\xi)\sigma_{sh}(b)). \end{aligned}$$

On the other hand, using basic computations and $\tau(\sigma_{s'g'}(x)) = \tau(x)$ for all $x \in A$ we see that the right side of (3.4) is equal to

$$
\begin{aligned}
&\langle au_s(\xi u_g e_A u_{g^{-1}h})bu_t, a'u_{s'}(\xi' u_{g'} e_A u_{g'^{-1}h'})b'u_{t'}\rangle_{Tr} \\
= \;& Tr(u_{t'^{-1}} b'^* u_{h'^{-1}g'} e_A u_{g'^{-1}} \xi'^* u_{s'^{-1}} a'^* a u_s \xi u_g e_A u_{g^{-1}h} b u_t) \\
= \;& Tr(e_A u_{g'^{-1}} \xi'^* u_{s'^{-1}} a'^* a u_s \xi u_g e_A u_{g^{-1}h} b u_{tt'^{-1}} b'^* u_{h'^{-1}g'} e_A) \\
= \;& \tau(E_A(u_{g'^{-1}} \xi'^* u_{s'^{-1}} a'^* a u_s \xi u_g) E_A(u_{g^{-1}h} b u_{tt'^{-1}} b'^* u_{h'^{-1}g'})) \\
= \;& \tau(E_A(\sigma_{g'^{-1}}(\xi'^*)\sigma_{g'^{-1}s'^{-1}}(a'^* a \sigma_s(\xi)) u_{g'^{-1}s'^{-1}sg}) E_A(\sigma_{g^{-1}h}(b)\sigma_{g^{-1}htt'^{-1}}(b'^*) u_{g^{-1}htt'^{-1}h'^{-1}g'})) \\
= \;& \delta_{g'^{-1}s'^{-1}sg,e}\delta_{g^{-1}htt'^{-1}h'^{-1}g',e}\tau(\sigma_{g'^{-1}}(\xi'^*)\sigma_{g'^{-1}s'^{-1}}(a'^* a\sigma_s(\xi))\sigma_{g^{-1}h}(b)\sigma_{g^{-1}htt'^{-1}}(b'^*) \\
= \;& \delta_{sg,s'g'}\delta_{g^{-1}ht,g'^{-1}h't'}\tau(\sigma_{(s'g')^{-1}}(\sigma_{s'}(\xi'^*)a'^* a\sigma_s(\xi)\sigma_{s'g'g^{-1}h}(b)\sigma_{s'g'g^{-1}htt'^{-1}}(b'^*)) \\
= \;& \delta_{sg,s'g'}\delta_{sht,s'h't'}\tau(\sigma_{s'}(\xi'^*)a'^* a\sigma_s(\xi)\sigma_{sh}(b)\sigma_{s'h'}(b'^*)).
\end{aligned}
$$

This establishes (3.4) and hence the conclusion of the lemma.

3.4 A Path of Automorphisms of the Extended Roe Algebra

Let $\Gamma = \Gamma_1 \times \Gamma_2 \curvearrowright^{\sigma} X$ be a measure preserving action of Γ on a measure space X. Assume we are given orthogonal representations $\pi_i : \Gamma_1 \to \mathcal{O}(\mathcal{H}_i)$. As shown in the previous section, to these representations we can associate the Gaussian actions $\Gamma_i \curvearrowright^{\pi_i} (Y_{\pi_i}, \nu_i)$ (in a slight abuse of notation we will denote the Gaussian action by the same letter). Next we consider the product action $\Gamma \curvearrowright^{\pi_1 \otimes \pi_2} (Y_{\pi_1} \times Y_{\pi_2}, \nu_1 \times \nu_2)$ and the diagonal action of Γ on $(X \times Y_{\pi_1} \times Y_{\pi_2}, \mu \times \nu_1 \times \nu_2)$. To this action, following [10], we can associate the extended Roe algebra $C^*_u(\Gamma \curvearrowright Z)$ (where the action is the one above and $Z = X \times Y_{\pi_1} \times Y_{\pi_2}$).

Additionally, given any pair of quasi-cocycles $q_i : \Gamma_i \to \mathcal{H}_i$ for the respective representations π_i, $i = 1, 2$, we can construct a one-parameter family $(\alpha_t)_{t\in\mathbb{R}}$ of $*$-automorphisms of $C^*_u(\Gamma \curvearrowright Z)$, by exponentiating the q_i's. This traces back to the construction of a malleable deformation of $L\Gamma$ from a cocycle b as carried out in §3 of [87]. Moreover, this family will be pointwise continuous with respect to the uniform norm as $t \to 0$ (Theorem 3.4.3).

Given the quasi-cocycles $q_i : \Gamma_i \to \mathcal{H}_i$, one can construct, following section §1.2 of [87], two one-parameter families of maps $\upsilon^i_t : \Gamma_i \to \mathcal{U}(L^\infty(Y_{\pi_i}, \nu_i))$ defined by the formula $\upsilon^i_t(\gamma_i)(x) = \exp(\sqrt{-1}tq_i(\gamma_i)(x))$, where $\gamma_i \in \Gamma_i$, $x \in Y_{\pi_i}$, respectively. To understand this formula, the reader must think about $\mathcal{H}_i$ as being identified with a subspace of $L^2(Y_{\pi_i}, \nu_i)$, viewing the elements $q_i(\gamma_i)$ as functions on Y_{π_i}. The same computations as in [69, 87] show the following:

Proposition 3.4.1. *Assuming the same notations as above, we have that:*

1. *If the representation π_i is weakly-ℓ^2, $i = 1, 2$, then the (tensor) product of Koopman representations $\pi_1 \otimes \pi_{2|L^2_0(Y_{\pi_1})\otimes L^2_0(Y_{\pi_2})}$ is also weakly-ℓ^2;*
2. *$\int_Y^{\pi_i} \upsilon^i_t(\gamma_i)(y)\upsilon^i_t(\delta_i)^*(y)d\mu^{\pi_i}(y) = \kappa^i_t(\gamma_i, \delta_i)$, $i = 1, 2$, and $\gamma_i, \delta_i \in \Gamma_i$.*

   Here, $\kappa^i_t(\gamma_i, \delta_i) = \exp(-t\|q_i(\gamma_i) - q_i(\delta_i)\|)$.

With the help of these maps we can construct a path of unitary operators $V_t \in \mathfrak{B}(L^2(Z)\bar{\otimes}\ell^2(\Gamma)) = \mathfrak{B}(L^2(Y_{\pi_1})\bar{\otimes}L^2(Y_{\pi_2})\bar{\otimes}L^2(X)\bar{\otimes}\ell^2(\Gamma))$ by letting $V_t(\xi_1 \otimes \xi_2 \otimes \eta \otimes \delta_{(\gamma_1,\gamma_2)}) = \upsilon^1_t(\gamma_1)\xi_1 \otimes \upsilon^2_t(\gamma_2)\xi_2 \otimes \eta \otimes \delta_{(\gamma_1,\gamma_2)}$ for every $\eta \in L^2(X)$, $\xi_i \in L^2(Y_{\pi_i})$, and $\gamma_i \in \Gamma_i$, where $i = 1, 2$. The computations in [10] show that the V_t enjoy the following basic properties.

Proposition 3.4.2. *For every $t, s \in \mathbb{R}$ we have that:*

1. $V_t V_s = V_{t+s}$, $V_t V_t^* = V_t^* V_t = 1$
2. *If the array is anti-symmetric we have $JV_tJ = V_t$ and if it is symmetric we have $JV_tJ = V_{-t}$. Here we denoted by $J : L^2(Z)\bar{\otimes}\ell^2(\Gamma) \to L^2(Z)\bar{\otimes}\ell^2(\Gamma)$ is Tomita's conjugation.*

The unitary V_t implements an inner $\star$-automorphism α_t on $\mathfrak{B}(L^2(Z)\bar{\otimes}\ell^2(\Gamma))$ by letting $\alpha_t(x) = V_t x V_t^*$ for all $x \in \mathfrak{B}(L^2(Z)\bar{\otimes}\ell^2(\Gamma))$. The α_t then restricts to a family of inner automorphisms of the uniform Roe algebra. Moreover, when restricting to the uniform Roe algebra one can recover from α_t the multipliers introduced in section 2 of [10] by the formula $\mathfrak{m}_t([x_{\gamma,\delta}]) = ([\kappa_t(\gamma,\delta)x_{\gamma,\delta}])$. Precisely, we have $E_M \circ \alpha_t(x) = \mathfrak{m}_t(x)$ for all $x \in C_u^*(\Gamma)$. The same computations as in [10] can be used to show that, α_t, when restricted to the Roe algebra, is a C^*-deformation, i.e., it is pointwise-$\|\cdot\|_\infty$ continuous.

Theorem 3.4.3. *Let q be any symmetric or anti-symmetric array. Assuming the notations above, for every $x \in L^\infty(X) \rtimes_{\sigma,r} \Gamma$ we have*

$$\|(\alpha_t(x) - x) \circ e\|_\infty \to 0 \text{ as } t \to 0; \tag{3.5}$$

$$\|(\alpha_t(JxJ) - JxJ) \circ e\|_\infty \to 0 \text{ as } t \to 0. \tag{3.6}$$

where $\|\cdot\|_\infty$ denotes the operatorial norm in $\mathfrak{B}(L^2(X)\bar{\otimes}\ell^2(\Gamma))$. Here e denotes the orthogonal projection from $L^2(Z)\bar{\otimes}\ell^2(\Gamma)$ onto $L^2(X)\bar{\otimes}\ell^2(\Gamma)$.

CHAPTER 4
MAIN RESULTS

4.1 Proofs of the main results involving product of groups

We start by proving the main technical result of the paper which involves product of groups. Specifically, we obtain a result describing all weakly compact embeddings in the crossed product von Neumann algebras arising from actions of products of hyperbolic groups (Theorem 4.1.2). Our approach follows the general outline of the proof of Theorem B in [66] and Theorem B in [10]. However, it is based on a new ingredient which allows us to treat the more general case of arrays rather than just quasi-cocycles as proved in [10]. This was influenced by the approach taken in [8].

To properly state the theorem we need to introduce first the following definition.

Definition 4.1.1. *Let Γ be a countable group and let $\Sigma < \Gamma$ be a subgroup. Then we denote by $\mathcal{QN}_\Gamma(\Sigma)$ the set of all elements $\gamma \in \Gamma$ for which there exists a finite set $\mathcal{F} \subset \Gamma$ such that $\gamma\Sigma \subseteq \cup_{s\in\mathcal{F}}\Sigma s$ and $\Sigma\gamma \subseteq \cup_{s\in\mathcal{F}} s\Sigma$. It can be easily checked that $\mathcal{QN}_\Gamma(\Sigma)$ is a subgroup of Γ containing Σ and it is called the* quasi-normalizer *of Σ inside Γ. One can see that this group contains $\mathcal{N}_\Gamma(\Sigma)$, the normalizing group of Σ in Γ, in many instances being much larger. When $\mathcal{QN}_\Gamma(\Sigma) = \Gamma$ we say that Σ is* quasi-normal *in Γ.*

Theorem 4.1.2. *For $i = 1, 2$ let Γ_i be an exact group with a quasi-normal subgroup $\Sigma_i < \Gamma_i$ and let $\pi_i : \Gamma_i \to \mathcal{U}(\mathcal{H}_{\pi_i})$ be a weakly-ℓ^2 representation such that $\mathcal{RA}(\Gamma_i, \{\Sigma_i\}, \mathcal{H}_{\pi_i}) \neq \emptyset$. Let $\Gamma_1 \times \Gamma_2 \curvearrowright X$ be a measure-preserving action on a probability space and denote by $M = L^\infty(X) \rtimes (\Gamma_1 \times \Gamma_2)$. If $P \subset M$ is a weakly compact embedding (cf. [65]), then one can find projections p_0, p_1, p_2, $p_3 \in \mathcal{Z}(\mathcal{N}_M(P)' \cap M)$ with $p_0 + p_1 + p_2 + p_3 = 1$ such that the following hold:*

1. *$\mathcal{N}_M(P)'' p_0$ is amenable;*
2. *$Pp_1 \preceq_M L^\infty(X) \rtimes (\Gamma_1 \times \Sigma_2)$;*
3. *$Pp_2 \preceq_M L^\infty(X) \rtimes (\Sigma_1 \times \Gamma_2)$;*
4. *$Pp_3 \preceq_M L^\infty(X) \rtimes (\Sigma_1 \times \Sigma_2)$.*

Proof. For simplicity we establish the following notations: $M_1 = L^\infty(X) \rtimes (\Gamma_1 \times \Sigma_2)$, $M_2 = L^\infty(X) \rtimes (\Sigma_1 \times \Gamma_2)$, $N = \mathcal{N}_M(P)''$ and $\mathcal{Z} = \mathcal{Z}(N' \cap M)$. Let $p_0 \in \mathcal{Z}$ be the maximal projection such that Np_0 is amenable. Let $p_1 \in (P' \cap M)(1-p_0)$ be a maximal projection satisfying (2) together with $Pq_1 \npreceq_M L^\infty(X) \rtimes (\Sigma_1 \times \Sigma_2)$ for all $0 \neq q_1 \leq 1 - p_1$ (obtained via a standard maximality argument). Similarly let $p_2 \in (P' \cap M)(1-p_0)$ be a maximal projection satisfying (3) together with $Pq_2 \npreceq_M L^\infty(X) \rtimes (\Sigma_1 \times \Sigma_2)$ for all $0 \neq q_2 \leq 1 - p_2$. Also we denote by $p_3 \in (P' \cap M)(1 - p_0)$ be a maximal projection satisfying (4). By maximality, we must have that $p_1, p_2, p_3 \in \mathcal{Z}(P' \cap M)$. Moreover, we have that $p_1, p_2, p_3 \in \mathcal{Z}$. Indeed, if $u \in \mathcal{N}_M(P)$, let $\tilde{p}_3 = up_3u^*(p_0 - p_3)$. Then $p_3 + \tilde{p}_3$ also satisfies (4) and by the maximality of p_3, we get that $\tilde{p}_3 = 0$. Thus $p_3 = up_3u^*$, for every $u \in \mathcal{U}(P)$, hence $p_3 \in \mathcal{Z}$.

Notice that from the definitions we have that $p_1p_3 = p_2p_3 = 0$. Further, we claim that $p_1' := p_1p_2 = 0$. Otherwise, $Pp_1' \preceq_M M_1$, hence we can find projections $p \in P$, $p' \in P' \cap M$, a non-zero partial isometry v and a $*$-homomorphism $\psi : pPpp_1' \to M_1$ such that $v^*v = pp'p_1'$ and $\psi(x)v = vx$, for all $x \in pPpp_1'$. Let z denote the central support of p in P and set $p_2' = zp'p_1' \in P' \cap M$. By using the fact that $pPpp'p_1' \subset v^*M_1v$, for every $\varepsilon > 0$, we can find a finite set $W \subset \Gamma_2$ such that $\|x - \sum_{g\in W} E_{M_1}(xu_g^*)u_g\|_2 \leq \varepsilon$, for all $x \in Pp_2'$ with $\|x\| \leq 1$.

Since $p_2' \leq p_2$ then $Pp_2' \npreceq_M L^\infty(X) \rtimes (\Sigma_1 \times \Sigma_2)$ and combining this with the above it follows that $Pp_2' \npreceq_M M_2$, contradicting the fact that $0 \neq p_2' \leq p_2$.

Therefore, to prove the theorem, we only need to show that $p_0 + p_1 + p_2 + p_3 = 1$. By contradiction, assume that $p := 1 - (p_0 + p_1 + p_2 + p_3) \neq 0$. Note that $Pp \npreceq_M M_1$ and $Pp \npreceq_M M_2$. Indeed, if $Pp \preceq_M M_1$, then by reasoning as in the previous paragraph we can find a non-zero projection $\tilde{p}_1 \in (P' \cap M)p$ such that $P\tilde{p}_1 \preceq_M M_1$. Thus, $P(p_1 + \tilde{p}_1) \preceq_M M_1$, which contradicts the maximality of p_1. Also, note that Np has no amenable direct summand. If $N\tilde{p}_0$ is amenable for some non-zero projection $\tilde{p}_0 \in \mathcal{Z}p$, it follows that $N(p_0 + \tilde{p}_0)$ is also amenable, contradicting the maximality of p_0.

By assumption $P \subset M$ is weakly compact, so there exists a net of unit vectors $(\eta_n)_{n\in\mathbf{N}}$ in $L^2(M)\bar{\otimes}L^2(\bar{M})$ such that:

1. $\|\eta_n - (v \otimes \bar{v})\eta_n\| \to 0$, for all $v \in \mathcal{U}(P)$;
2. $\|[u \otimes \bar{u}, \eta_n]\| \to 0$, for all $u \in \mathcal{N}_M(P)$;

3. $\langle (x \otimes 1)\eta_n, \eta_n \rangle = \tau(x) = \langle (1 \otimes \bar{x})\eta_n, \eta_n \rangle$, for all $x \in M$.

As in the previous section we consider the von Neumann algebra

$$\tilde{M} = (L^\infty(X) \bar{\otimes} L^\infty(Y_{\pi_1}) \bar{\otimes} L^\infty(Y_{\pi_2})) \rtimes (\Gamma_1 \times \Gamma_2),$$

together with the following subalgebras:

$$\tilde{M}_1 = (L^\infty(X) \bar{\otimes} L^\infty(Y_{\pi_1}) \bar{\otimes} \mathbb{C}1) \rtimes (\Gamma_1 \times \Gamma_2)$$

$$\tilde{M}_2 = (L^\infty(X) \bar{\otimes} \mathbb{C}1 \bar{\otimes} L^\infty(Y_{\pi_2})) \rtimes (\Gamma_1 \times \Gamma_2)$$

For every $1 \le i \le 2$ we denote by e_i the orthogonal projection from $L^2(\tilde{M}) \bar{\otimes} L^2(\bar{M})$ onto $L^2(\tilde{M}_i) \bar{\otimes} L^2(\bar{M})$ and notice that e_1 and e_2 commute. It follows that $e = (1 - e_1)(1 - e_2)$ is a projection and we denote its image by $\mathcal{K} = e\left(L^2(\tilde{M}) \bar{\otimes} L^2(\bar{M})\right)$. Since e_i is M-bimodular we have that $\mathcal{K}$ is an $M \bar{\otimes} \bar{M}$-subbimodule of $L^2(\tilde{M}) \bar{\otimes} L^2(\bar{M})$.

Fixing $t > 0$ we consider the unitary V_t associated with $q_i \in \mathcal{RA}(\Gamma_i, \{\Sigma_i\}, \mathcal{H}_{\pi_i})$ as defined in the previous section and we then denote by

$$\tilde{\eta}_{n,t} = (V_t \otimes 1)(p \otimes 1)\eta_n \in L^2(\tilde{M}) \bar{\otimes} L^2(\bar{M}).$$

Using these notations we show next the following inequality:

Lemma 4.1.3.

$$\operatorname*{Lim}_n \|e(\tilde{\eta}_{n,t})\| \ge \frac{1}{2}\|p\|_2,$$

where "Lim" is a generalized Banach limit.

Proof. We argue by contradiction, so passing to a subsequence we assume that

$$\|e(\tilde{\eta}_{n,t})\| < \frac{1}{2}\|p\|_2 \text{ for all } n. \tag{4.1}$$

Denoting by $\zeta_n = (p \otimes 1)\eta_n$ we observe that $\|\tilde{\eta}_{n,t}\|_2 = \|\zeta_n\|_2 = \|p\|_2$. Since by construction $e + (e_1 - e_1e_2) + e_2 = 1$, we get that

$$\|e(\tilde{\eta}_{n,t})\|_2^2 + \|e_1(\tilde{\eta}_{n,t})\|_2^2 + \|e_2(\tilde{\eta}_{n,t})\|_2^2 \geq \|\tilde{\eta}_{n,t}\|_2^2 = \|p\|_2^2. \tag{4.2}$$

Since we assumed that $\|e(\tilde{\eta}_{n,t})\|_2 \leq \frac{1}{2}\|p\|_2$, for all $n \geq n_0$, after passing to a subsequence and without any loss of generality, we may assume that

$$\|e_1(\tilde{\eta}_{n,t})\|_2 \geq \sqrt{\frac{3}{8}}\|p\|_2 \text{ for all } n. \tag{4.3}$$

Throughout the proof, for any subset $\mathcal{F} \subset \Gamma_1$, we denote by $P_{\mathcal{F}}$ the orthogonal projection from $L^2(M)\bar{\otimes}L^2(\bar{M})$ onto the closed linear span of the set $\{(M_2)u_h)\bar{\otimes}\bar{M} \mid h \in \mathcal{F}\}$.

The main strategy is to prove that relation (4.3) together with the assumption $Pp \nprec_M M_2$ will enable us to construct an infinite sequence $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots$ of finite subsets of Γ such that $\Sigma_1\mathcal{F}_i$ are disjoint and which satisfies the following property: for every $k \in \mathbb{N}$ we can find $n_k \in \mathbb{N}$ such that for all $i \leq k$ and $n \geq n_k$ we have

$$\|P_{\mathcal{F}_i}(\zeta_n)\| \geq \frac{1}{10}\|p\|_2. \tag{4.4}$$

First we briefly explain how this claim leads to a contradiction, thus finishing the proof of the lemma. Since the sets $\Sigma_1\mathcal{F}_i$ are disjoint, relation (4.4) implies $\|p\|_2^2 = \|\zeta_n\|^2 \geq \sum_{i=1}^{k} \|P_{\mathcal{F}_i}(\zeta_n)\|_2^2 \geq k\left(\frac{1}{10}\|p\|_2\right)^2$, for all $k \in \mathbb{N}$ and $n \geq n_k$. This is obviously impossible when letting k be sufficiently large.

So we are left to prove (4.4). To show this we will proceed by induction on k.

First we prove case $k = 1$. Since $\zeta_n \in L^2(M)\bar{\otimes}L^2(\bar{M})$, we write $\zeta_n = \sum_{g\in\Gamma_1} \zeta_g^n\delta_g$, where $\zeta_g^n \in L^2(X) \rtimes (1 \times \Gamma_2)\bar{\otimes}L^2(\bar{M})$. Then, using the definition of

V_t, a straight forward computation shows that

$$\|e_1(\tilde{\eta}_{n,t})\|^2 = \sum_{g\in\Gamma_1} \exp(-2t^2\|q_1(g)\|^2)\|\zeta_n^g\|_2^2, \text{ for all } n.$$

When combined with (4.3) this formula implies that, for all n we have

$$\sum_{g\in\Gamma_1} \exp(-2t^2\|q_1(g)\|^2)\|\zeta_n^g\|_2^2 > \frac{3}{8}\|p\|_2^2. \tag{4.5}$$

Since the map $\Gamma_1 g \to \|q_1(g)\|$ is a proper relative to $\{\Sigma_1\}$ and Σ_1 is quasi-normal in Γ_1, then the set $\{g \in \Gamma_1 \mid \exp(-t^2\|q_1(g\|^2) \geq \frac{1}{4}\}$ is contained in $\Sigma_1\mathcal{F}$ for some finite set $\mathcal{F} \subset \Gamma$ and, using the inequality (4.5), we further deduce that

$$\frac{3}{8}\|p\|_2^2 < \frac{1}{16}\sum_{g\in\Gamma_1\setminus\Sigma_1\mathcal{F}} \|\zeta_n^g\|_2^2 + \sum_{g\in\Sigma_1\mathcal{F}} \|\zeta_n^g\|_2^2, \text{ for all } n.$$

By basic algebraic manipulations, the above inequality gives that $\sum_{g\in\Sigma_1\mathcal{F}} \|\zeta_n^g\|_2^2 > \frac{1}{3}\|p\|_2^2$ for all n which implies

$$\|P_{\mathcal{F}}(\zeta_n)\| > \frac{1}{\sqrt{3}}\|p\|_2. \tag{4.6}$$

So case $k = 1$ follows by letting $\mathcal{F}_1 = \mathcal{F}$ and $n_1 = 1$.

Next we show the induction step, i.e., assuming that we have constructed the sets $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k \subset \Gamma$ and $n_k \in \mathbb{N}$, we indicate how to construct $\mathcal{F}_{k+1} \subset \Gamma$ and $n_{k+1} \in \mathbb{N}$ satisfying (4.4).

Consider the set $\mathcal{G} = \cup_{i=1}^k(\mathcal{F}_i\mathcal{F}^{-1}) \subset \Gamma$. Since $\mathcal{G}_1$ is finite and $Pp \not\preceq_M M_2$, by Popa's intertwining techniques, there exist a unitary $v \in \mathcal{U}(P)$, a finite set $\mathcal{K} \subset \Gamma_1$, and an element v' in the linear span of $\{M_2 u_h \mid h \in \mathcal{K}\}$ such that

$$\Sigma_1\mathcal{K} \cap \Sigma_1\mathcal{G} = \emptyset; \tag{4.7}$$

$$\|v' - vp\|_2 \leqslant \frac{1}{40|\mathcal{F}|}\|p\|_2. \tag{4.8}$$

Next, we show that for $n \in \mathbb{N}$ and $z \in M$ we have

$$\|(z \otimes 1)P_{\mathcal{F}}(\zeta_n)\| \leq |\mathcal{F}|\|z\|_2. \tag{4.9}$$

Fix n and denote by P the orthogonal projection onto $L^2(M_2)\bar{\otimes}L^2(\bar{M})$, i.e. $P = P_{\emptyset}$. We have $P_{\mathcal{F}}(\zeta_n) = \sum_{h\in\mathcal{F}} P(\zeta_n(u_h^* \otimes 1))(u_h \otimes 1)$ and by the Cauchy-Schwarz inequality we deduce

$$\|(z \otimes 1)P_{\mathcal{F}}(\zeta_n)\|^2 \leqslant |\mathcal{F}| \sum_{h\in\mathcal{F}} \|(z \otimes 1)P(\zeta_n(u_h^* \otimes 1))\|^2 \tag{4.10}$$

Now we let E_{M_2} to be the conditional expectation from M onto M_2 and we denote by $a = E_{M_2}(z^*z)^{\frac{1}{2}}$. Using the formulas $\langle (x \otimes 1)P(\zeta), P(\zeta)\rangle = \langle (E_{M_2}(x) \otimes 1)P(\zeta), P(\zeta)\rangle$ and $\|(x \otimes 1)\eta_n\| = \|x\|_2$, for all $\zeta \in L^2(M)\bar{\otimes}L^2(\bar{M})$ and $x \in M$, we obtain the following:

$$\begin{aligned}
&\|(z \otimes 1)P(\zeta_n(u_h^* \otimes 1))\|^2 \\
&= \langle (z^*z \otimes 1)P(\zeta_n(u_h^* \otimes 1)), P(\zeta_n(u_h^* \otimes 1))\rangle \\
&= \langle (E_{M_2}(z^*z) \otimes 1)P(\zeta_n(u_h^* \otimes 1)), P(\zeta_n(u_h^* \otimes 1))\rangle \\
&= \|(a \otimes 1)P(\zeta_n(u_h^* \otimes 1))\|_2^2 = \|P((a \otimes 1)\zeta_n(u_h^* \otimes 1))\|^2 \\
&\leq \|(a \otimes 1)\zeta_n\|^2 = \|ap\|_2^2 \leq \|a\|_2^2 = \|z\|_2^2.
\end{aligned}$$

It is clear that the last inequalities combined with (4.10) give (4.9).

To this end, applying the triangle inequality, for all $v \in \mathcal{U}(P)$ and all $n \in \mathbb{N}$, we have

$$\|\zeta_n - (v' \otimes \bar{v})P_{\mathcal{F}}(\zeta_n)\| \leq \tag{4.11}$$

$$\|\zeta_n - (vp \otimes \bar{v})\zeta_n\| + \|\zeta_n - P_{\mathcal{F}}(\zeta_n)\| + \|((v' - vp) \otimes 1)P_{\mathcal{F}}(\zeta_n)\|$$

Since p and v commute, we have $\zeta_n - (vp \otimes \bar{v})\zeta_n = (p \otimes 1)(\eta_n - (v \otimes \bar{v})\eta_n)$. Thus, since $\lim_{n\to\infty} \|\eta_n - (v \otimes \bar{v})\eta_n\|_2 = 0$, we can find $n_{k+1} \geq n_k$ such that for all $n \geq n_{k+1}$ we have

$$\|\zeta_n - (vp \otimes \bar{v})\zeta_n\| \leq \frac{1}{40}\|p\|_2. \tag{4.12}$$

Using (4.9) for $z = vp - v'$ in combination with (4.8) for all n we have

$$\|((v' - vp) \otimes 1)P_{\mathcal{F}}(\zeta_n)\| \leq \frac{1}{40}\|p\|_2. \tag{4.13}$$

Altogether, (4.11), (4.12), (4.13), and (5.5) show that that for all $n \geq n_{k+1}$ we have

$$\|\zeta_n - (v' \otimes \bar{v})P_{\mathcal{F}}(\zeta_n)\| \leq \frac{1}{20}\|p\|_2 + \|(\zeta_n - P_{\mathcal{F}}(\zeta_n))\| < \frac{3 + 20\sqrt{6}}{60}\|p\|_2. \tag{4.14}$$

Finally we let $\mathcal{F}_{k+1} = \mathcal{K}\mathcal{F}$ and by (4.7) we see that $\Sigma_1\mathcal{F}_{k+1}$ is disjoint from $\Sigma_1\mathcal{F}, \Sigma_1\mathcal{F}_2, \ldots, \Sigma_1\mathcal{F}_k$. Moreover, we have that $(v' \otimes v)P_{\mathcal{F}}(\zeta_n)$ belongs to the closed linear span of $\{(M_2 u_h) \otimes \bar{M} \mid h \in \mathcal{F}_{k+1}\}$. Thus, $P_{\mathcal{F}_{k+1}}((v' \otimes \bar{v})P_{\mathcal{F}}(\zeta_n)) = (v' \otimes \bar{v})P_{\mathcal{F}}(\zeta_n)$ and (4.14) together with the triangle inequality give that

$$\|\zeta_n - P_{\mathcal{F}_{k+1}}(\zeta_n)\| < \frac{3 + 20\sqrt{6}}{60}\|p\|_2, \text{ for all } n \geq n_{k+1}.$$

Hence, for all $n \geq n_{k+1}$, we have

$$\|P_{\mathcal{F}_{k+1}}(\zeta_n)\| > \left(1 - \left(\frac{3 + 20\sqrt{6}}{60}\right)^2\right)^{\frac{1}{2}} \|p\|_2 > \frac{1}{10}\|p\|_2,$$

which ends the proof of (4.4).

We also notice that from Lemma 3.3.1 it follows that, as an M-bimodule, $\mathcal{K}$ is weakly contained in the coarse bimodule. Following the same argument as in Theorem B of [66] we define a state ψ_t on $\mathcal{N} = \mathfrak{B}(\mathcal{K}) \cap \rho(M^{op})'$. Explicitly, if we denote by $\xi_{n,t} = e(\tilde{\eta}_{n,t})$ we let $\psi_t(x) = \mathrm{Lim}_n \frac{1}{\|\xi_{n,t}\|^2}\langle (x \otimes 1)\xi_{n,t}, \xi_{n,t}\rangle$ for every $x \in \mathcal{N}$. Next we recall that from [10] we have the following two lemmas

Lemma 4.1.4 (Lemma 4.3 in [10]). *For every $\varepsilon > 0$ and every finite set $K \subset L^\infty(X) \rtimes_{\sigma,r} (\Gamma_1 \times \Gamma_2)$ with $dist_{\|\cdot\|_2}(y, (N)_1) \leq \varepsilon$ for all $y \in K$ one can find $t_\varepsilon > 0$ and a finite set $L_{K,\varepsilon} \subset \mathcal{N}_M(P)$ such that*

$$|\langle ((yx - xy) \otimes 1)\xi_{n,t}, \xi_{n,t}\rangle| \leq 10\varepsilon + 2 \sum_{v \in L_{K,\varepsilon}} \|[v \otimes \bar{v}, \eta_n]\|, \tag{4.15}$$

for all $y \in K$, $\|x\|_\infty \leq 1$, $t_\varepsilon > t > 0$, and n.

Lemma 4.1.5 (Lemma 4.4 in[10]). *For every $\varepsilon > 0$ and any finite set $F_0 \subset \mathcal{U}(N)$ there exist a finite set $F_0 \subset F \subset M$, a c.c.p. map $\varphi_{F,\varepsilon} : span(F) \to L^\infty(X) \rtimes_{\sigma,r} (\Gamma_1 \times \Gamma_2)$, and $t_\varepsilon > 0$ such that*

$$|\psi_{t_\varepsilon}(\varphi_{F,\varepsilon}(up)^* x \varphi_{F,\varepsilon}(up)) - \psi_{t_\varepsilon}(x)| \leq 116\varepsilon, \tag{4.16}$$

for all $u \in F_0$ and $\|x\|_\infty \leq 1$.

For the remaining part of the proof we mention that one can use Haagerup criterion to show that Np is amenable. In fact the reasoning in Theorem B in [66] applies verbatim in our case and we leave the details to the reader.

We notice that even though we choose to state the result only for one quasi-normal subgroup, the same method can be used ti treat the case of arbitrary families of quasi-normal subgroups. This proof can be upgraded to work for any families of subgroups [**?**].

4.2 Applications of the main result

Proof of Corollary 1.2.2. Applying the previous theorem for $A = \mathbb{C}1$ and $\Sigma_i = e$ there exist p_0, p_1, $p_2 \in \mathcal{Z}$ with $p_0 + p_1 + p_2 = 1$ such that $p_0\mathcal{N}_M(P)''$ is amenable, $p_1 B \preceq_M L\Gamma_1$, and $p_2 B \preceq_M L\Gamma_2$. Therefore the conclusion follows if we show that $p_0 = 1$. Assuming this is not the case one can find $p_1 \neq 0$ such that $p_1 B \preceq_M L\Gamma_1$. Then Remark 3.8 in [95] implies that $L\Gamma_1' \cap M \preceq_M p_1 B' \cap M$ and since $L\Gamma_1' \cap M = L\Gamma_2$ then we have $L\Gamma_2 \preceq_M p_1 B' \cap M$. This however is a contradiction because $L\Gamma_2$ is a non-amenable factor while $p_1 B' \cap M$ is assumed to be an amenable algebra.

Proof of Theorem 1.2.1. Assume that $\Lambda \curvearrowright Y$ is a free, ergodic action which is W^*-equivalent to $\Gamma_1 \times \Gamma_2 \curvearrowright X$. This amounts to the existence of an $\star$-isomorphism $\psi : L^\infty(Y) \rtimes \Lambda \to L^\infty(X) \rtimes (\Gamma_1 \times \Gamma_2)$. For simplicity we will denote by $A = L^\infty(X)$, $B = L^\infty(Y)$, $M = A \rtimes (\Gamma_1 \times \Gamma_2)$, $M_1 = A \rtimes \Gamma_1$, and $M_2 = A \rtimes \Gamma_2$.

Below we will prove that there exists a unitary $x \in \mathcal{U}(M)$ such that $x\psi(B)x^* = A$. Notice that since $C = \psi(B)$ Cartan in M its normalizing algebra is non-amenable so by Theorem 4.1.2 we can assume that $C \preceq_M M_1$. Therefore one can find nonzero projections $p \in C$, $q \in M_1$, a partial isometry $v \in M$, and a $\star$-homomorphism $\phi : Cp \to qM_1q$ such that for all $x \in Cp$ we have

$$\phi(x)v = vx. \tag{4.17}$$

Since C is a maximal abelian subalgebra of M then by Lemma 1.5 in [38] we can assume that $\phi(Cp) \subset qM_1q$ is also a maximal abelian subalgebra. Fixing $u \in \mathcal{N}_{pMp}(Cp)$ we can easily see that for all $x \in Cp$ we have

$$vuv^*\phi(x) = vuxv^* = vuxu^*uv^* = \phi(uxu^*)vuv^*. \tag{4.18}$$

Notice that $vuv^*vu^*v^* = \phi(uv^*vu^*)vv^*$ is a projection and hence vuv^* is a partial isometry. Also, applying the conditional expectation E_{qM_1q} to equation (4.18), we obtain that for all $x \in Cp$ we have

$$E_{qM_1q}(vuv^*)\phi(x) = \phi(uxu^*)E_{qM_1q}(vuv^*).$$

Taking the polar decomposition $E_{qM_1q}(vuv^*) = w_u|E_{qM_1q}(vuv^*)|$, the previous equation entails that $|E_{qM_1q}(vuv^*)| \in \phi(Cp)' \cap qM_1q = \phi(Cp)$ and for all $x \in Cp$ we have

$$w_u\phi(x) = \phi(uxu^*)w_u.$$

This implies in particular that $w_uw_u^*, w_u^*w_u \in \phi(Cp)' \cap qM_1q = \phi(Cp)$ and therefore $w_u \in \mathcal{GN}_{qM_1q}(\phi(Cp))$, the normalizing groupoid of $\phi(Cp)$ in qM_1q. Altogether, we have shown that

$$E_{qM_1q}(vuv^*) \subseteq \mathcal{GN}_{qM_1q}(\phi(Cp))''.$$

By [23], we have that $\mathcal{GN}_{qM_1q}(\phi(Cp))'' = \mathcal{N}_{qM_1q}(\phi(Cp))''$ and since the above containment holds for every $u \in \mathcal{N}_{pMp}(Cp)''$ and $\mathcal{N}_{pMp}(Cp)'' = pMp$ we have that

$$E_{qM_1q}(vMv^*) \subseteq \mathcal{N}_{qM_1q}(\phi(Cp))'',$$

and hence $vv^*M_1vv^* \subseteq \mathcal{N}_{qM_1q}(\phi(Cp))''$. This shows in particular that $\mathcal{N}_{qM_1q}(\phi(Cp))''$ is non-amenable; therefore, by Theorem B in [10] we have that $\phi(Cp) \preceq_{M_1} A$. By

Remark 3.8 in [95] this further implies that $C \preceq_M A$. Finally, by Theorem 2.5.4, one can find a unitary $x \in \mathcal{U}(M)$ such that $x\phi(B)x^* = xCx^* = A$.

In particular, our claim shows that the actions $\Gamma_1 \times \Gamma_2 \curvearrowright X$ and $\Lambda \curvearrowright Y$ are orbit equivalent. Note that, since Γ_1 and Γ_2 have property (T) then so is the product $\Gamma_1 \times \Gamma_2$, so it follows from Ioana's Cocycle Superrigidity Theorem [36] that the actions $\Gamma_1 \times \Gamma_2 \curvearrowright X$ and $\Lambda \curvearrowright Y$ are virtually conjugate.

Corollary 4.2.1. *Let Γ_i be weakly amenable groups and let $\pi : \Gamma_i \to \mathcal{U}(\mathcal{H}_\pi)$ be weakly-ℓ^2 representations such that $\mathcal{RA}(\Gamma, \{e\}, \mathcal{H}_\pi) \neq \emptyset$ (e.g. Γ_i are hyperbolic). If $\Gamma_1 \times \Gamma_2 \curvearrowright X$ and $\Lambda \curvearrowright Y$ are any pmp actions such that Λ admits an infinite amenable normal subgroup $\Sigma < \Gamma$ for which the restriction $\Sigma \curvearrowright Y$ is still ergodic then $\Gamma_1 \times \Gamma_2 \curvearrowright X \ncong_{OE} \Lambda \curvearrowright Y$.*

Proof. We will assume that $\Gamma_1 \times \Gamma_2 \curvearrowright X \cong_{OE} \Lambda \curvearrowright Y$ and then show that this leads to a contradiction. Thus there exists a $\star$-isomorphism $\psi : L^\infty(Y) \rtimes \Lambda \to L^\infty(X) \rtimes (\Gamma_1 \times \Gamma_2)$. We will also denote by $A = L^\infty(X)$, $B = L^\infty(Y)$, $P = \psi(L^\infty(Y) \rtimes \Sigma)$, $M = A \rtimes (\Gamma_1 \times \Gamma_2)$, $M_1 = A \rtimes \Gamma_1$, $M_2 = A \rtimes \Gamma_2$, and notice that $\psi(B) = A$.

Since the Cowling-Haagerup constant is an ME-invariant it follows that $\psi(L\Sigma)$ is a weakly compact embedding in M. Since Σ is normal in Λ, then applying Theorem 4.1.2, we can assume that $\psi(L\Sigma) \preceq_M M_1$ and since $\psi(B) = A$ we conclude that $P \preceq_M M_1$. Therefore, one can find nonzero projections $p \in P$, $q \in M_1$, a partial isometry $v \in M$, and a $\star$-homomorphism $\phi : pPp \to qM_1q$ such that for all $x \in pPp$ we have

$$\phi(x)v = vx. \tag{4.19}$$

Since P is an irreducible subfactor of M, by Proposition 2.5.2 we can assume that $\phi(pPp) \subset qM_1q$ is also a irreducible subfactor. Fixing $u \in \mathcal{N}_{pMp}(pPp)$ we can easily see that for all $x \in pPp$ we have

$$vuv^*\phi(x) = vuxv^* = vuxu^*uv^* = \phi(uxu^*)vuv^*. \tag{4.20}$$

Notice that $vuv^*vu^*v^* = \phi(uv^*vu^*)vv^*$ is a projection and hence vuv^* is a partial isometry. Also, applying the conditional expectation E_{qM_1q} to equation (4.20), we obtain that for all $x \in pPp$ we have

$$E_{qM_1q}(vuv^*)\phi(x) = \phi(uxu^*)E_{qM_1q}(vuv^*).$$

Taking the polar decomposition $E_{qM_1q}(vuv^*) = w_u|E_{qM_1q}(vuv^*)|$, the previous equation entails that $|E_{qM_1q}(vuv^*)| \in \phi(pPp)' \cap qM_1q = \mathbb{C}q$ and for all $x \in pPp$ we have

$$w_u\phi(x) = \phi(uxu^*)w_u.$$

This implies in particular that $w_uw_u^*, w_u^*w_u \in \phi(pPp)' \cap qM_1q = \mathbb{C}q$ and therefore w_u is a scalar multiple of a normalizing unitary in $\in \mathcal{N}_{qM_1q}(\phi(pPp))$. Altogether, we have shown that

$$E_{qM_1q}(vuv^*) \subseteq \mathcal{N}_{qM_1q}(\phi(pPp))''.$$

Since the above containment holds for every $u \in \mathcal{N}_{pMp}(pPp)$ and $\mathcal{N}_{pMp}(pPp)'' = pMp$ we have that

$$E_{qM_1q}(vMv^*) \subseteq \mathcal{N}_{qM_1q}(\phi(pPp))'',$$

and hence $vv^*M_1vv^* \subseteq \mathcal{N}_{qM_1q}(\phi(pPp))''$. This shows in particular that $\mathcal{N}_{qM_1q}(\phi(pPp))''$

is non-amenable; therefore, by Theorem B in [10] we have that $\phi(pPp) \preceq_{M_1} A$. By Remark 3.8 in [95] this would imply that $P \preceq_M A$, which is an obvious contradiction.

CHAPTER 5
FURTHER RESULTS AND FINAL REMARKS

5.1 Further Results and Final Remarks

The main purpose of this section is to point out that many of the structural results obtained in [10] can be pushed forward in the context of groups which admit arrays that are proper with respect to certain families of subgroups. For instance, we have the following result is a generalization of Theorem 3.2 and Theorem 4.1 from [10]. Our proof largely overlaps with the proof of Theorem 4.1. [10], but there is a key step which will allow us to upgrade the result from quasi-cocycles as presented in [10] to arrays. We include a proof below which addresses only this step. The technical argument used is essentially the same as in the Lemma 6.2 above in the case of a single factor. We reproduce it here for the reader's convenience .

Theorem 5.1.1. *Let Γ be an exact group together with a family of subgroups $\mathcal{G}$, and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ be a weakly-ℓ^2 representation. Also, let $\Gamma \curvearrowright X$ be a free, ergodic action and denote by $M = L^\infty(X) \rtimes \Gamma$ the corresponding crossed-product von Neumann algebra.*

1. *If $\mathcal{RA}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \neq \emptyset$ and $P \subseteq M$ is diffuse subalgebra, then either $A' \cap M$ is amenable or there exists a group $\Sigma \in \mathcal{G}$ such that $P \preceq_M L^\infty(X) \rtimes \Sigma$.*

2. *If $\mathcal{RQ}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \neq \emptyset$ and $P \subseteq M$ is a weakly compact embedding with P diffuse, then either the normalizing algebra $\mathcal{N}_M(P)''$ is amenable or there exists $\Sigma \in \mathcal{G}$ such that $P \preceq_M L^\infty(X) \rtimes \Sigma$.*

3. *Assume that $\mathcal{G}$ is a family of quasi-normal subgroups of* Γ. *If* $\mathcal{RA}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \neq \emptyset$ *and* $P \subseteq M$ *is a weakly compact embedding with* P *diffuse, then either the normalizing algebra* $\mathcal{N}_M(P)''$ *is amenable or there exists* $\Sigma \in \mathcal{G}$ *such that* $P \preceq_M L^\infty(X) \rtimes \Sigma$.

Proof. As stated, the first part is Theorem 3.2 in [10] while the second part follows exactly as in the proof of Theorem 4.1 in [10]. Indeed the only ingredient needed for this is to adapt Proposition 2.6 in [10] to the case of quasi-cocycles that are proper with respect to a family of subgroups. One can see however that this is a straight forward exercise and we leave it to the reader. So we only prove the third part.

For simplicity we will denote by $N = \mathcal{N}_M(P)''$ and then we fix $p \in \mathcal{Z}(N' \cap M)$ a projection. Also to not complicate the notations we assume that $\mathcal{G}$ consists of a single subgroup of Γ, i.e., $\mathcal{G} = \{\{\Sigma\}\}$. Then the general strategy of the proof is to show that the assumption $P \not\preceq_M L^\infty(X) \rtimes \Sigma$ implies that Np is amenable. By assumption $P \subset M$ is weakly compact, so there exists a net of positive unit vectors $(\eta_n)_{n\in\mathbf{N}}$ in $L^2(M) \otimes L^2(\bar{M})$ such that

1. $\|\eta_n - (v \otimes \bar{v})\eta_n\| \to 0$, for all $v \in \mathcal{U}(P)$;
2. $\|[u \otimes \bar{u}, \eta_n]\| \to 0$, for all $u \in \mathcal{N}_M(P)$; and
3. $\langle (x \otimes 1)\eta_n, \eta_n\rangle = \tau(x) = \langle (1 \otimes \bar{x})\eta_n, \eta_n\rangle$, for all $x \in M$.

From here on the proof follows the same argument as in Lemma 6.2 in the previous section. So let $\mathcal{H} = L^2_0(Y^\pi) \otimes L^2(X) \otimes \ell^2(\Gamma)$ which as we remarked before is weakly contained as an M-bimodule in the coarse bimodule. Fixing $t > 0$ we consider the unitary V_t associated with an array q as defined in the previous sections. Next denote by $\tilde{\eta}_{n,t} = (V_t \otimes 1)(p \otimes 1)\eta_n$, $\zeta_{n,t} = (e \otimes 1)\tilde{\eta}_{n,t} = (e \cdot V_t \otimes 1)(p \otimes 1)\eta_n$, and

$\xi_{n,t} = \tilde{\eta}_{n,t} - \zeta_{n,t} = (e^{\perp} \otimes 1)\tilde{\eta}_{n,t} \in \mathcal{H} \otimes L^2(M)$.

Using these notations we show next the following inequality:

Lemma 5.1.2.

$$\operatorname*{Lim}_{n} \|\xi_{n,t}\| \geq \frac{1}{16}\|p\|_2.$$

Proof. We argue by contradiction, so passing to a subsequence we assume that

$$\|\xi_{n,t}\| < \frac{1}{16}\|p\|_2 \text{ for all } n. \tag{5.1}$$

Denoting by $\zeta_n = (p \otimes 1)\eta_n$ we have $\|\tilde{\eta}_{n,t}\| = \|\zeta_n\| = \|p\|_2$ and using the identity $\|(e \otimes 1)(\tilde{\eta}_{n,t})\|^2 + \|(e^{\perp} \otimes 1)(\tilde{\eta}_{n,t})\|^2 = \|\tilde{\eta}_{n,t}\|^2 = \|p\|_2^2$ in combination with (5.1) we have

$$\|(e \otimes 1)(\tilde{\eta}_{n,t})\| > \frac{15}{16}\|p\|_2, \text{ for all } n. \tag{5.2}$$

Throughout the proof, for any subset $\mathcal{F} \subset \Gamma$, we denote by $P_{\mathcal{F}}$ the orthogonal projection from $L^2(M)\bar{\otimes}L^2(\bar{M})$ onto the closed linear span of the set $\{((L^\infty(X) \rtimes \Sigma)u_h)\bar{\otimes}\bar{M} \mid h \in \mathcal{F}\}$.

The main strategy is to prove that relation (5.2) together with the assumption $Pp \nprec_M L^\infty(X) \rtimes \Sigma$ will enable us to construct an infinite sequence $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots$ of finite subsets of Γ such that $\Sigma\mathcal{F}_i$ are disjoint and which satisfies the following property: for every $k \in \mathbb{N}$ we can find $n_k \in \mathbb{N}$ such that for all $i \leq k$ and $n \geq n_k$ we have

$$\|P_{\mathcal{F}_i}(\zeta_n)\| \geq \frac{1}{1000}\|p\|_2. \tag{5.3}$$

First we briefly explain how this claim leads to a contradiction, thus finishing the proof of the lemma. Since the sets $\mathcal{F}_i\Sigma\mathcal{F}_i$ are disjoint, relation (5.3) implies

$\|p\|_2^2 = \|\zeta_n\|^2 \geq \sum_{i=1}^{k} \|P_{\mathcal{F}_i}(\zeta_n)\|_2^2 \geq k\left(\frac{1}{1000}\|p\|_2\right)^2$, for all $k \in \mathbb{N}$ and $n \geq n_k$. This is obviously impossible when letting k be sufficiently large.

So we are left to prove (5.3). To show this we will proceed by induction on k.

First we prove case $k = 1$. Since $\zeta_n \in L^2(M)\bar{\otimes}L^2(\bar{M})$, we write $\zeta_n = \sum_{g\in\Gamma} \zeta_g^n \delta_g$, where $\zeta_g^n \in L^2(X)\bar{\otimes}L^2(\bar{M})$. Then, using the definition of V_t, a straight forward computation shows that

$$\|e(\tilde{\eta}_{n,t})\|^2 = \sum_{g\in\Gamma} \exp(-2t^2\|q(g\|^2)\|\zeta_n^g\|^2, \text{ for all } n.$$

When combined with (5.2) this formula implies that, for all n we have

$$\sum_{g\in\Gamma} \exp(-2t^2\|q(g\|^2)\|\zeta_n^g\|^2 > \frac{211}{256}\|p\|_2^2. \tag{5.4}$$

Since the map $g \to \|q(g)\|$ is a proper relatively to $\{\Sigma\}$, then the set $\{g \in \Gamma \mid \exp(-t^2\|q(g\|^2) \leq \frac{1}{2}\}$ is contained in $\Sigma\mathcal{F}$ for some finite set $\mathcal{F} \subset \Gamma$ and, using the inequality (5.4), we further deduce that

$$\frac{211}{256}\|p\|_2^2 < \frac{1}{4}\sum_{g\in\Gamma\setminus\Sigma\mathcal{F}} \|\zeta_n^g\|^2 + \sum_{g\in\Sigma\mathcal{F}} \|\zeta_n^g\|^2, \text{ for all } n.$$

By basic algebraic manipulations, the above inequality gives that $\sum_{g\in\Sigma\mathcal{F}} \|\zeta_n^g\|^2 > \frac{49}{64}\|p\|_2^2$ for all n which implies

$$\|P_{\mathcal{F}}(\zeta_n)\| > \frac{7}{8}\|p\|_2. \tag{5.5}$$

So case $k = 1$ follows by letting $\mathcal{F}_1 = \mathcal{F}$ and $n_1 = 1$.

Next we show the induction step, i.e., assuming that we have constructed the sets $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k \subset \Gamma$ and $n_k \in \mathbb{N}$, we indicate how to construct $\mathcal{F}_{k+1} \subset \Gamma$ and $n_{k+1} \in \mathbb{N}$ satisfying (5.3).

Consider the set $\mathcal{G} = \cup_{i=1}^{k}(\mathcal{F}_i\mathcal{F}^{-1}) \subset \Gamma$. Since $\mathcal{G}$ is finite and $Pp \nprec_M L^\infty(X)$, by Popa's intertwining techniques, there exist a unitary $v \in \mathcal{U}(P)$, a finite set $\mathcal{K} \subset \Gamma$, and an element v' in the linear span of $\{(L^\infty(X) \rtimes \Sigma)u_h \mid h \in \mathcal{K}\}$ such that

$$\Sigma\mathcal{K} \cap \Sigma\mathcal{G} = \emptyset; \tag{5.6}$$

$$\|v' - vp\|_2 \leqslant \frac{1}{1000|\mathcal{F}|}\|p\|_2. \tag{5.7}$$

Next, we show that for $n \in \mathbb{N}$ and $z \in M$ we have

$$\|(z \otimes 1)P_{\mathcal{F}}(\zeta_n)\| \leq |\mathcal{F}|\|z\|_2. \tag{5.8}$$

Fix n and denote by P the orthogonal projection onto $L^2(X)\bar{\otimes}L^2(\bar{M})$, i.e. $P = P_\emptyset$. We have $P_{\mathcal{F}}(\zeta_n) = \sum_{h\in\mathcal{F}} P(\zeta_n(u_h^* \otimes 1))(u_h \otimes 1)$ and by the Cauchy-Schwarz inequality we deduce

$$\|(z \otimes 1)P_{\mathcal{F}}(\zeta_n)\|^2 \leqslant |\mathcal{F}| \sum_{h\in\mathcal{F}} \|(z \otimes 1)P(\zeta_n(u_h^* \otimes 1))\|^2 \tag{5.9}$$

Now we let $E_{L^\infty(X)}$ to be the conditional expectation from M onto $L^\infty(X)$ and we denote by $a = E_{L^\infty(X)}(z^*z)^{\frac{1}{2}}$. Using the formulas $\langle (x \otimes 1)P(\zeta), P(\zeta)\rangle = \langle (E_{L^\infty(X)}(x) \otimes 1)P(\zeta), P(\zeta)\rangle$ and $\|(x \otimes 1)\eta_n\| = \|x\|_2$, for all $\zeta \in L^2(M)\bar{\otimes}L^2(\bar{M})$ and $x \in M$, we obtain the following:

$$\|(z\otimes 1)P(\zeta_n(u_h^*\otimes 1))\|^2$$

$$= \langle (z^*z\otimes 1)P(\zeta_n(u_h^*\otimes 1)), P(\zeta_n(u_h^*\otimes 1))\rangle$$

$$= \langle (E_{L^\infty(X)}(z^*z)\otimes 1)P(\zeta_n(u_h^*\otimes 1)), P(\zeta_n(u_h^*\otimes 1))\rangle$$

$$= \|(a\otimes 1)P(\zeta_n(u_h^*\otimes 1))\|_2^2 = \|P((a\otimes 1)\zeta_n(u_h^*\otimes 1))\|^2$$

$$\leq \|(a\otimes 1)\zeta_n\|^2 = \|ap\|_2^2 \leq \|a\|_2^2 = \|z\|_2^2.$$

It is clear that the last inequalities combined with (5.9) give (5.8).

To this end, applying the triangle inequality, for all $v\in\mathcal{U}(P)$ and all $n\in\mathbb{N}$, we have

$$\|\zeta_n - (v'\otimes\bar{v})P_{\mathcal{F}}(\zeta_n)\| \leq \tag{5.10}$$

$$\|\zeta_n - (vp\otimes\bar{v})\zeta_n\| + \|\zeta_n - P_{\mathcal{F}}(\zeta_n)\| + \|((v'-vp)\otimes 1)P_{\mathcal{F}}(\zeta_n)\|$$

Since p and v commute, we have $\zeta_n - (vp\otimes\bar{v})\zeta_n = (p\otimes 1)(\eta_n - (v\otimes\bar{v})\eta_n)$. Thus, since $\lim_{n\to\infty}\|\eta_n - (v\otimes\bar{v})\eta_n\|_2 = 0$, we can find $n_{k+1}\geq n_k$ such that for all $n\geq n_{k+1}$ we have

$$\|\zeta_n - (vp\otimes\bar{v})\zeta_n\| \leq \frac{1}{1000}\|p\|_2. \tag{5.11}$$

Using (5.8) for $z = vp - v'$ in combination with (5.7) for all n we have

$$\|((v'-vp)\otimes 1)P_{\mathcal{F}}(\zeta_n)\| \leq \frac{1}{1000}\|p\|_2. \tag{5.12}$$

Altogether, (5.10), (5.11), (5.12), and (5.5) show that that for all $n\geq n_{k+1}$ we have

$$\|\zeta_n - (v' \otimes \bar{v}) P_{\mathcal{F}}(\zeta_n)\| \leq \frac{1}{500}\|p\|_2 + \|(\zeta_n - P_{\mathcal{F}}(\zeta_n))\| < \frac{8+500\sqrt{15}}{4000}\|p\|_2. \tag{5.13}$$

Finally we let $\mathcal{F}_{k+1} = \mathcal{K}\mathcal{F}$ and by (5.6) we see that $\Sigma\mathcal{F}_{k+1}$ is disjoint from $\Sigma\mathcal{F}, \Sigma\mathcal{F}_2, \ldots, \Sigma\mathcal{F}_k$. Moreover, we have that $(v' \otimes v) P_{\mathcal{F}}(\zeta_n)$ belongs to the closed linear span of $\{(L^\infty(X)u_h) \otimes \bar{M} \mid h \in \mathcal{F}_{k+1}\}$. Thus, $P_{\mathcal{F}_{k+1}}((v' \otimes \bar{v}) P_{\mathcal{F}}(\zeta_n)) = (v' \otimes \bar{v}) P_{\mathcal{F}}(\zeta_n)$ and (5.13) together with the triangle inequality give that

$$\|\zeta_n - P_{\mathcal{F}_{k+1}}(\zeta_n)\| < \frac{8+500\sqrt{15}}{2000}\|p\|_2, \text{ for all } n \geq n_{k+1}.$$

Hence, for all $n \geq n_{k+1}$, we have

$$\|P_{\mathcal{F}_{k+1}}(\zeta_n)\| > \left(1 - \left(\frac{8+500\sqrt{15}}{2000}\right)^2\right)^{\frac{1}{2}} \|p\|_2 > \frac{1}{1000}\|p\|_2,$$

which ends the proof of (5.3).

Next we briefly explain how to use the previous lemma in order to get the proof of the theorem. First notice that, as an M-bimodule, $\mathcal{H}$ is weakly contained in the coarse bimodule. Then following the same argument as in Theorem B of [66] we define a state ψ_t on $\mathcal{N} = \mathfrak{B}(\mathcal{H}) \cap \rho(M^{op})'$. Explicitly, if we denote by $\xi_{n,t} = e(\tilde{\eta}_{n,t})$ we let $\psi_t(x) = \mathrm{Lim}_n \frac{1}{\|\xi_{n,t}\|^2}\langle (x \otimes 1)\xi_{n,t}, \xi_{n,t}\rangle$ for every $x \in \mathcal{N}$.

To get the proof, from here on, one can proceed exactly as explained in Theorem 4.1 in [10] or Theorem 4.1.2 from the previous section. Namely we use the same Lemmas 4.3 and 4.4 from [10] and the final part in the proof of Theorem B in [66] to conclude that Np is amenable. We leave the details to the reader.

The first part of the conclusion is a restatement (via Proposition 3.1.1) of a well-known theorem due to Ozawa (Theorem in [6]), our contribution being the second and the third part. We should mention that in the light of Proposition 1.10 in [10] it is not very difficult to see that there is a way to deduce Theorem 6.1 from the third part of Theorem 7.1.

Finally, we note that very recently Popa and Vaes extended the third part to arbitrary families of subgroups, or in the case that Γ is weakly amenable, arbitrary free ergodic actions [**?**].

In the remaining part of the section we explain how the second and third part can be successfully exploited to produce new examples of von Neumann algebras with either unique Cartan subalgebra or no Cartan subalgebras. With this purpose in mind, we introduce the following definition.

Definition 5.1.3. *A subgroup $\Sigma < \Gamma$ is called* weakly malnormal *if there exist finitely many elements $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$ such that*

$$\left| \bigcap_{i=1}^{n} \gamma_i \Sigma \gamma_i^{-1} \right| < \infty.$$

Therefore, when the second and the third part in the intertwining theorem above is combined with Corollary 7 from [33] we immediately obtain the following uniqueness (absence) of Cartan subalgebra statement.

Corollary 5.1.4. *Let Γ be a weakly amenable group and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ be a weakly-ℓ^2 representation such that one of the following cases holds:*

1. *$\mathcal{RQ}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \neq \emptyset$ for a family of weakly malnormal subgroups $\mathcal{G}$ of Γ, or*

2. $\mathcal{RA}(\Gamma, \{e\}, \mathcal{H}_\pi) \neq \emptyset$.

Also let $\Gamma \curvearrowright X$ *be a weakly compact, free action. If* Γ *is as in the first case (1) above we assume in addition that the restrictions* $\Sigma \curvearrowright X$ *are ergodic for all* $\Sigma \in \mathcal{G}$.

Then $L^\infty(X) \rtimes \Gamma$ *has unique Cartan subalgebra. If in addition* Γ *is i.c.c. and* $\mathcal{G}$ *is a family of malnormal groups then* $L\Gamma$ *has no Cartan subalgebra.*

Employing the same strategy as in the proof of Corollary B.2 from [10] and using the fact that the class of weakly amenable groups is closed under ME-subgroups, we obtain new structural results for measure equivalence of groups.

Corollary 5.1.5. *Let* Γ *be a weakly amenable group and let* $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ *be a weakly-*ℓ^2 *representation such that one of the following holds: either* $\mathcal{RQ}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \neq \emptyset$ *for a family of amenable, malnormal subgroups* $\mathcal{G}$*, or* $\mathcal{RQ}(\Gamma, \{e\}, \mathcal{H}_\pi) \neq \emptyset$. *If* Λ *is any ME-subgroup of* Γ *then* $L\Lambda$ *is strongly solid i.e., given any diffuse amenable subalgebra* $A \subseteq L\Lambda$ *its normalizing algebra* $\mathcal{N}_{L\Lambda}(A)''$ *is still amenable. In particular, every amenable subgroup of* Λ *has amenable normalizer.*

Examples 5.1.6. The following are examples of groups that satisfy the conditions required in the above corollary: any weakly amenable group that is in the class $\mathcal{S}$ of Ozawa [60]–in particular any weakly amenable group Γ that is hyperbolic relative to a family of amenable subgroups (e.g. Sela's limit groups which are weakly amenable and hyperbolic with respect to their noncyclic maximal abelian subgroups [19]); any weakly amenable HNN extension $\Gamma\star_\alpha$ of a group Γ, where $\alpha : \Sigma_1 \to \Sigma_2$ is a monomorphism with $\Sigma_i \in \mathcal{G}$; any infinite free product $\star_{n\in\mathbb{N}}\Gamma_n$ where Γ_n is hyperbolic relative to a finite family $\mathcal{G}_n$ of malnormal groups, $\Lambda_{cb}(\Gamma_n) = 1$ and $\mathcal{G}_n = \{e\}$ for all but finitely many n's – in this case we choose $\mathcal{G} = \cup_n \mathcal{G}_n$.

Next we discuss another application of the main intertwining theorem: let Γ be a hyperbolic group and let Λ be an abelian inductive limit of finite groups. Assume that $\rho : \Lambda \to Aut(\Gamma)$ is an action by automorphism with finite orbits such that for every $\lambda \in \Lambda$ and every finite index subgroup $\Gamma_0 < \Gamma$ the restriction $\rho_\lambda|_{\Gamma_0}$ is not the conjugacy by an element in Γ. Then applying part (2) in Theorem 5.1.1 we obtain that every type II, amenable subalgebra $A \subset L(\Gamma \rtimes \Lambda)$ has amenable normalizing algebra $\mathcal{N}_{L(\Gamma\rtimes\Lambda)}(A)''$. Also if we assume that $A \subset L(\Gamma \rtimes \Lambda)$ is a semiregular MASA containing $L\Lambda$ with non-amenable normalizing algebra then any other semiregular MASA in $L(\Gamma \rtimes \Lambda)$ is either unitary conjugated to A or it has amenable normalizing algebra. Finally, we notice that if we assume Λ is i.c.c. and $L(\Lambda)' \cap L(\Gamma \rtimes \Lambda)$ has no amenable summand then every MASA in $L(\Gamma \rtimes \Lambda)$ have amenable normalizing algebra.

In the remaining part of the section, we discuss some insight provided by Theorem 5.1.1 regarding the structure of equivalence relations induced by certain p.m.p. actions. Suppose $\Gamma \curvearrowright X$ is any free ergodic action with Γ satisfying the hypothesis from part (1) of Theorem 5.1.1 and let $\mathcal{G}$ be a family of weakly malnormal subgroups of Γ. If $\mathcal{R} = \mathcal{R}_{\Gamma \curvearrowright X}$ denotes the equivalence relation on $X \times X$ induced by $\Gamma \curvearrowright X$ then $\mathcal{R} \ncong \mathcal{R}_{\Lambda_1 \times \Lambda_2 \curvearrowright Y}$, for any free action $\Lambda_1 \times \Lambda_2 \curvearrowright Y$ with Λ_i infinite.

To see this, we assume by contradiction that there exists an $\star$-isomorphism between the von Neumnan algebras $\phi : L^\infty(Y) \rtimes (\Lambda_1 \times \Lambda_2) \to L^\infty(X) \rtimes \Gamma = M$

which preserves the Cartan subalgebras. Since Γ is non-amenable, we may assume without loosing any generality that Λ_1 is also non-amenable. Then applying first part of Theorem 5.1.1 we have that $\phi(L\Lambda_2) \preceq_M L^\infty(X) \rtimes \Sigma$, for some $\Sigma \in \mathcal{G}$. Since ϕ preserves the Cartan subalgebras, we also have that $\phi(L^\infty(X) \rtimes \Lambda_2) \preceq_M L^\infty(X) \rtimes \Sigma$. Furthermore, since $\phi(L^\infty(X) \rtimes \Lambda_2)$ is regular in M and Σ is weakly malnormal in Γ, then Corollary 7 in [33] implies that $\phi(L^\infty(X) \rtimes \Lambda_2) \preceq_M L^\infty(X)$, which is obviously a contradiction because Λ_2 is infinite.

This expands upon some indecomposability results for equivalence relations from [1]. We mention that this result also follows from Ozawa's earlier results, [6]. It is very likely that this indecomposability property still holds without the exactness assumption on Γ. In this case however the proof should be more ergodic theoretic in nature, as the present von Neumann algebras techniques rely heavily on exactness.

If in addition we assume that the action $\Gamma \curvearrowright X$ is weakly compact and Γ is a hyperbolic group with $\mathcal{G} = \{e\}$ then by the second part in Theorem 5.1.1 we have the following: for any infinite, amenable sub-equivalence relation $\mathcal{S} \subset \mathcal{R}$ its normalizing equivalence relation $\mathcal{N}_{\mathcal{R}}(\mathcal{S})$ in $\mathcal{R}$ is still amenable. Notice that this partially recovers Corollary 6.2 from [1]. However, there is also a more general version of this result.

Corollary 5.1.7. *Let Γ be a weakly amenable group together with a family of malnormal subgroups $\mathcal{G}$ and a weakly-ℓ^2 representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ such that $\mathcal{RQ}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \neq \emptyset$. Let $\Gamma \curvearrowright X$ be a weakly compact, free, ergodic, p.m.p. action and denote by $\mathcal{R}$ the induced equivalence relation. Then given any free infinite, amenable sub-equivalence relation $\mathcal{S} \subset \mathcal{R}$ its normalizing equivalence relation $\mathcal{N}_{\mathcal{R}}(\mathcal{S})$ in $\mathcal{R}$ is still amenable.*

The proof follows from Theorem 5.1.1 and Proposition 2.5.3 by applying the same argument as in the proof of Corollary 1.2.1.

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