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Thin position, bridge structure, and monotonic simplification of knots

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THIN POSITION, BRIDGE STRUCTURE, AND MONOTONIC SIMPLIFICATION OF KNOTS

by

Alexander Martin Zupan

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

July 2012

Thesis Supervisors: Professor Maggy Tomova
Professor Charles Frohman

ABSTRACT

Since its inception, the notion of thin position has played an important role in low-dimensional topology. Thin position for knots in the 3-sphere was first introduced by David Gabai in order to prove the Property R Conjecture. In addition, this theory factored into Cameron Gordon and John Luecke's proof of the knot complement problem and revolutionized the study of Heegaard splittings upon its adaptation by Martin Scharlemann and Abigail Thompson.

Let $h : S^3 \to \mathbb{R}$ be a Morse function with two critical points. Loosely, thin position of a knot K in S^3 is a particular embedding of K which minimizes the total number of intersections with a maximal collection of regular level sets, where this number of intersections is called the width of the knot. Although not immediately obvious, it has been demonstrated that there is a close relationship between a thin position of a knot K and essential meridional planar surfaces in its exterior $E(K)$.

In this thesis, we study the nature of thin position under knot companionship; namely, for several families of knots we establish a lower bound for the width of a satellite knot based on the width of its companion and the wrapping or winding number of its pattern. For one such class of knots, cable knots, in addition to finding thin position for these knots, we establish a criterion under which non-minimal bridge positions of cable knots are stabilized. Finally, we exhibit an embedding of the unknot whose width must be increased before it can be simplified to thin position.

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the July 2012 graduation.

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ABSTRACT

Since its inception, the notion of thin position has played an important role in low-dimensional topology. Thin position for knots in the 3-sphere was first introduced by David Gabai in order to prove the Property R Conjecture. In addition, this theory factored into Cameron Gordon and John Luecke's proof of the knot complement problem and revolutionized the study of Heegaard splittings upon its adaptation by Martin Scharlemann and Abigail Thompson.

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In this thesis, we study the nature of thin position under knot companionship; namely, for several families of knots we establish a lower bound for the width of a satellite knot based on the width of its companion and the wrapping or winding number of its pattern. For one such class of knots, cable knots, in addition to finding thin position for these knots, we establish a criterion under which non-minimal bridge positions of cable knots are stabilized. Finally, we exhibit an embedding of the unknot whose width must be increased before it can be simplified to thin position.

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CHAPTER 1
INTRODUCTION

The concepts of thin position and width for knots in the 3-sphere were first introduced by David Gabai in [7] and have received considerable attention in the last 25 years. Since its inception, thin position has proven to be an adaptable tool which is useful in a wide variety of contexts. Gabai utilized thin position arguments to prove the Property R Conjecture for knots in S^3. Cameron Gordon and John Luecke incorporated similar ideas into their proof of the Knot Complement Problem [9], as did Mark Culler, Gordon, Luecke, and Peter Shalen in their pivotal work about Dehn surgery [5].

Following Gabai, Martin Scharlemann and Abigail Thompson reworked thin position in the context of 3-manifolds, developing the important notion of a generalized Heegaard splitting [23]. In addition, newer versions of thin position for a knot in an arbitrary 3-manifold have been presented by Chuichiro Hayashi and Koya Shimokawa [11] and Maggy Tomova [31]. Further, Scott Taylor and Tomova have adjusted these definitions to study thin position of knotted graphs in S^3 [29]. Most recently, Jesse Johnson has axiomatized ideas from thin position theory in order to generalize the many different versions appearing in the literature [14]. Another good reference for background material regarding thin position may be found in [12].

In some sense, thin position of knots may be viewed as a refinement of the study of bridge presentations, introduced by Horst Schubert in [25]. If a knot complement contains no essential surfaces, work of Thompson demonstrates that the bridge number of a knot may be recovered from its width [30]. On the other hand, recent work of Ryan Blair and Tomova confirms the existence of knots whose thin position does not realize minimal bridge number [1].

In most cases, width is used to measure the global complexity of a given decomposition of a space into topologically simple pieces, and thin position is some decomposition that minimizes this complexity. Quite often the thin position of an object reveals an interesting embedded surface; for instance, Thompson shows that for a knot K in S^3, either thin position coincides with minimal bridge position or the exterior of K contains an essential planar surface [30]. Ying-Qing Wu goes further to show that if thin position is not minimal bridge position, then a thinnest level surface corresponding to the thin position of K must be essential [33].

In this thesis, we will also study the interaction between thin position and essential surfaces. In particular, we study the nature of knot companionship and knot width, manipulating essential tori in order to carry out specific calculations. In [26], Jennifer Schultens uses the singular foliations induced by a Morse function to provide an updated proof of Schubert's classical result relating bridge number and companionship [25]. Although the work of Blair and Tomova [1] reveals that one cannot hope for knot width to be as nicely behaved as bridge number, in Chapter 2 we use some of Schultens' machinery to gain control over the companion torus. As a result, we are able to bound the width of satellite knots with 2-bridge companion and to compute the width of a cable knots using the width of the companion. In Chapter 3, we examine a different class of satellite knots, cable knots, in order bound the width of these knots from below.

With a measure of complexity such as knot width or bridge number, a natural question to ask is whether any position can be simplified via isotopy to a position of minimal complexity with the additional restriction that the complexity is non-increasing. To this end, Jean-Pierre Otal demonstrates that if K is the unknot or a 2-bridge knot, then any non-minimal bridge position of K may be simplified monotonically until one arrives at a minimal (necessarily 1- or 2-bridge) position for K [16]. Makoto Ozawa has shown that the same statement holds if K is a torus knot [18]. In Chapter 4 we use ideas from Chapter 3 in order to prove that if J is a knot such that $E(J)$ contains no essential meridional planar surface and and any non-minimal bridge position of J is stabilized, then every cable knot K of J has exhibits the same properties. In particular, iterated torus knots and iterated cables of 2-bridge knots have these properties.

The question of monotonic simplification with respect to width and thin position appears in several forms in the literature. In his treatment of the subject [21], Scharlemann asks if any position of the unknot of non-minimal width can be simplified to thin position by an isotopy through which width in non-increasing. More generally, Schultens has developed the notion of width complexes to study the relationships between every possible position for a fixed knot, and she poses the question as to whether for an arbitrary knot K any position of K admits monotonic simplification [27]. These questions are motivated by natural properties of thin position and generalized Heegaard splittings of 3-manifolds; namely, it follows from the work of

Friedhelm Waldhausen [32], Francis Bonahan-Otal [2], and Scharlemann-Thompson [23] that if M is S^3 or a lens space, then any non-minimal splitting of M can be simplified monotonically into a Heegaard splitting of minimal genus. In Chapter 5 we exhibit an embedding of the unknot that resists monotonic simplification, and we use this embedding to construct for any knot K infinitely many embeddings which are not thin position yet which cannot be simplified without first increasing knot width.

Finally, in Chapter 6 we consider future research questions stemming from the material contained in this thesis.

1.1 Thin position of knots in S^3

We begin with a definition of thin position for knots in S^3, due to Gabai [7]. In order to impose the additional structure we will require later, we work in the context of Morse functions. Fix a Morse function $h : S^3 \to \mathbb{R}$ such that h has exactly two critical points, a maximum and a minimum which we denote $\pm\infty$. Now, fix a knot K and let $\mathcal{K}$ denote the set of all embeddings k of S^1 into S^3 isotopic to K and such that $h|_k$ is Morse. For each such k, enumerate the critical values $c_0 < \cdots < c_p$ of $h|_k$. Choose regular values $c_0 < r_1 < \cdots < r_p < c_p$ of h, and for each i, let X_i denote the regular level surface $X_i = h^{-1}(r_i)$. Setting $x_i = |k \cap X_i|$, we associate the tuple $(x_1, \ldots, x_p)$ of even integers to k. Define the width of k to be

$$w(k) = \sum x_i$$

and the bridge number of k, $b(k)$, to be the number of maxima (or minima) of $h|_k$. A related complexity, the trunk of k, is defined as

$$\text{trunk}(k) = \max\{x_i\}.$$

Finally, the width $w(K)$, and bridge number $b(K)$ (first introduced by Schubert [25]), and trunk of the knot K (from Ozawa [18]) are defined as

$$\begin{aligned} w(K) &= \min_{k \in \mathcal{K}} w(k), \\ b(K) &= \min_{k \in \mathcal{K}} b(k), \\ \text{trunk}(K) &= \min_{k \in \mathcal{K}} \text{trunk}(k). \end{aligned}$$

Any $k \in \mathcal{K}$ satisfying $w(k) = w(K)$ is a called a thin position of K, and any k with all maxima occurring above all minima is called a bridge position of K. If k is a bridge position and $b(k) = b(K)$, we call k a minimal bridge position of K.

For any $k \in \mathcal{K}$ with associated tuple $(x_1, \ldots, x_p)$, we say a regular level X_i is a thick level if $x_i > x_{i-1}, x_{i+1}$; similarly, X_i is a thin level if $x_i < x_{i-1}, x_{i+1}$, where $2 \le i \le p-1$. Enumerate the thick level surfaces $\hat{A}_1, \ldots, \hat{A}_n$ and the thin level surfaces $\hat{B}_1, \ldots, \hat{B}_{n-1}$, noting that there will always be at least one thick surface and one fewer thin than thick surfaces. An embedding k is a bridge position if and only if it has one thick surface $\hat{A}_1$ and no thin surfaces. Define $a_i = |k \cap \hat{A}_i|$ and $b_i = |k \cap \hat{B}_i|$, so that we may associate a thick/thin tuple of integers $(a_1, b_1, a_2, \ldots, b_{n-1}, a_n)$ to k.

From [22],

$$w(k) = \frac{1}{2}\left(\sum a_i^2 - \sum b_i^2\right). \tag{1.1}$$

Observe that any thin position k of a knot K must have some thick level $\hat{A}_i$ such that $a_i = |k \cap \hat{A}_i|$ is at least as large as the trunk of K; thus the above yields

$$w(K) \geq \frac{1}{2}\text{trunk}(K)^2. \tag{1.2}$$

The definitions in the next two paragraphs are taken from [27]. We say that two embeddings k and k' in $\mathcal{K}$ are equivalent if there is an isotopy of S^3 taking k to k' which carries thick surfaces to thick surfaces to thick surfaces and thin surfaces to thin surfaces. More rigorously, if $\hat{A}_i$ and $\hat{B}_i$ are defined as above, then k' is equivalent to k if there exists an isotopy f_t of S^3 such that $f_0 = \text{Id}$, $f_1(k) = k'$, and $\{f_1(\hat{A}_1), \dots, f_1(\hat{A}_n)\}$ and $\{f_1(\hat{B}_1), \dots, f_1(\hat{B}_{n-1})\}$ correspond to complete collections of thick and thin surfaces for k'. As such, it is clear that in this case $w(k') = w(k)$.

Given an embedding k, there may exist certain isotopies that decrease $w(k)$, as determined by the intersections of strict upper and lower disks. An upper disk for k at a thick surface $\hat{A}_i$ is an embedded disk D such that ∂D consists of two arcs, one arc in k and one arc in $\hat{A}_i$, where the arc in k does not intersect any thin surfaces and contains exactly one maximum. A strict upper disk is an upper disk whose interior contains no critical points with respect to h. A lower disk and strict lower disk are defined similarly.

If there is a pair (D, E) of strict upper and lower disks for k at $\hat{A}_i$ that intersect in a single point contained in k, we can cancel out the maximum and minimum of k contained in D and E, and $w(k)$ decreases by $2a_i - 2$. We call this a type I move. If D and E are disjoint, we can slide the minimum above the maximum with respect to h, decreasing $w(k)$ by 4. This is called a type II move. If k is a bridge position a admits a type I move at $\hat{A}_1$, we say k is stabilized.

1.2 Essential Surfaces

One method employed to study the topology of knots in S^3 is to classify those embedded surfaces with meaningful topological and algebraic properties, which are known as "essential surfaces." In general, suppose M is a compact 3-manifold and S is a properly embedded surface in M. A compressing disk for S in M is an embedded disk D such that $D \cap S = \partial D$ and such that ∂D does not bound a disk in S. If $\partial M, \partial S \neq \emptyset$, a ∂-compressing disk is an embedded disk Δ such that $\partial\Delta$ is the endpoint union of arcs α and β, where $\Delta \cap S = \alpha$ and $\Delta \cap \partial M = \beta$. We say S is incompressible (resp. ∂-incompressible) if there does not exist compressing disk (resp. ∂-compressing disk) for S in M.

It follows from the Loop Theorem that an orientable surface S is incompressible if and only if the map $i_* : \pi_1(S) \to \pi_1(M)$ induced by the inclusion map is injective (see [13], for instance). If S is isotopic to a subsurface in ∂M, we say S is ∂-parallel. Finally, we define S to be essential if it is incompressible and is not ∂-parallel. It is well known that if ∂M is a collection of tori, then an incompressible surface S with boundary is ∂-parallel if and only if it is ∂-compressible.

For the remainder of this thesis, we will let $\eta(\cdot)$ denote an open regular neighborhood in the ambient manifold. For $K \subset S^3$, the exterior $E(K)$ of K is defined by $E(K) = S^3 \setminus \eta(K)$. By the above discussion, if a surface S is essential in $E(K)$ then it is also ∂-incompressible. There is a close relationship between thin position and a certain class of essential surfaces contained in $E(K)$. We say that a properly embedded surface $S \subset E(K)$ with nonempty boundary is meridional if ∂S consists of meridian curves of $\eta(K)$. The following surprising result was proved by Thompson:

Theorem 1.1. *[30] If a thin position for K contains a thin level, then $E(K)$ contains an essential meridional planar surface.*

This has since been strengthened by Wu, who shows that such a surface will be well-behaved with respect to the Morse function h on S^3:

Theorem 1.2. *[33] If a thin position for K contains a thin level, then a thinnest thin level is essential in $E(K)$.*

It will be useful to separate knots into classes whose exteriors contain certain essential surfaces. We say that K is

1. small if $E(K)$ contains no closed essential surface,
2. meridionally small (or m-small) if $E(K)$ contains no essential meridional surface, and
3. meridionally-planar small (or mp-small) if $E(K)$ contains no essential meridional planar surface.

We let $\mathcal{S}$ denote the collection of small knots, $\mathcal{M}$ the collection of m-small knots, and $\mathcal{MP}$ the collection of mp-small knots. By [5], $\mathcal{S} \subset \mathcal{M}$ and it is clear from the definitions that $\mathcal{M} \subset \mathcal{MP}$. The collection $\mathcal{S}$ contains many familiar classes of knots; for instance, all alternating knots and all torus knots are small [15]. An easy consequence of Theorem 1.2 is

Corollary 1.1. *If K is meridionally-planar small, then any thin position is a minimal bridge position of K.*

1.3 Satellite Knots

As demonstrated above, there are strong connections between the combinatorics determining thin position of a knot K and the topology of its exterior $E(K)$. In this vein, we will examine knots whose complements contain closed essential tori in an attempt to extract information about thin position of such knots K. Such knots are called "satellite knots" and are defined here:

Suppose $\hat{K}$ is a knot contained in a solid torus V with core C such that every meridian disk of V intersects $\hat{K}$, and let J be any nontrivial knot in S^3. In addition, let $\varphi : V \to S^3$ be an embedding such that $\varphi(C)$ is isotopic to J. We say that $K = \varphi(\hat{K})$ is a satellite knot with companion J and pattern $\hat{K}$. We call $\partial\varphi(V)$ the companion torus corresponding to J.

We will be concerned with two related measures of complexity of a satellite knot, winding and wrapping numbers. Using the terminology above, the wrapping number, or index, of a pattern $\hat{K}$ is the minimal geometric intersection (counted without sign) of any meridian disk of V with $\hat{K}$. Then winding number of $\hat{K}$ is the algebraic intersection number (counted with sign) of any meridian disk of V with $\hat{K}$. Equivalently, let $r : V \to C$ be a retract which takes the solid torus to its core. Then the winding number of $\hat{K}$ is also the degree of $r|_{\hat{K}}$.

As an example consider Figure 1.1 below. The left-hand side contains a depiction of a pattern of wrapping and winding number three. This pattern is called a braid; that is, there exists a foliation of V by meridian disks such that every meridian disk intersects $\hat{K}$ the same number of times. In other words, the map $r|_{\hat{K}}$ contains no critical points. The right-hand side is a picture of a satellite knot K with trefoil companion and pattern $\hat{K}$.

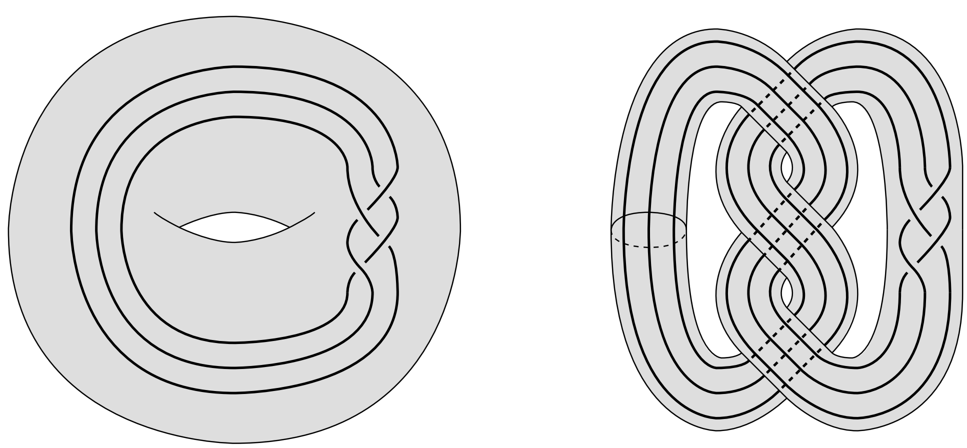

Figure 1.1: At left, a braid pattern in a solid torus and at right, a satellite knot with trefoil companion

Observe that the winding number of $\hat{K}$ is always bounded above by its wrapping number. The following result relating bridge numbers of satellite knots and wrapping numbers was first proved by Schubert [25], with an updated proof provided by Schultens [26]:

Theorem 1.3. *Suppose K is a satellite knot with companion J and whose pattern has wrapping number n. Then*

$$b(K) \geq n \cdot b(J).$$

As an example, consider the knot K shown in Figure 1.1. The above theorem implies that $b(K) \geq 6$; thus $b(K) = 6$ and the figure depicts a minimal bridge position of K.

Using the material found in the proof of Theorem 1.3, both authors also show that for knots K_1 and K_2, we have $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$, and a minimal bridge position for $K_1 \# K_2$ can be constructed in an obvious way using minimal positions for K_1 and K_2. The analogous question for knot width is the following:

Question 1.1. *Is it true that*

$$w(K_1 \# K_2) = w(K_1) + w(K_2) - 2$$

for all knots K_1 and K_2 in S^3?

By stacking thin positions for K_1 and K_2, it is quite trivial to show that $w(K_1 \# K_2) \leq w(K_1) + w(K_2) - 2$, and Question 1.1 has been answered affirmatively by Yo'av Rieck and Eric Sedgwick in the case that $K_1, K_2 \in \mathcal{MP}$ [19]. In the direction of this question, Scharlemann and Schultens establish a lower bound for $w(K_1\#K_2)$, namely

Theorem 1.4. *[22] For knots $K_1, K_2 \in S^3$,*

$$w(K_1\#K_2) \geq \max\{w(K_1), w(K_2)\}.$$

Scharlemann-Thompson provide examples for which the inequality in Theorem 1.4 is conjecturally sharp, and Blair-Tomova confirm these examples, proving

Theorem 1.5. *[1] There exist knots $K_1, K_2 \in S^3$ such that*

$$w(K_1\#K_2) = \max\{w(K_1), w(K_2)\} < w(K_1) + w(K_2) - 2.$$

The above reveals that the behavior knot width is somewhat more pathological than bridge number under the connected sum operation. However, we may still consider the more general properties of width under companionship, to which end we make the following conjecture as an analogue of Theorem 1.3:

Conjecture 1.6. *Suppose K is a satellite knot with companion J and whose pattern has wrapping number n. Then*

$$w(K) \geq n^2 \cdot w(J).$$

A weaker form of Conjecture 1.6 is

Conjecture 1.7. *Suppose K is a satellite knot with companion J and whose pattern has winding number n. Then*

$$w(K) \geq n^2 \cdot w(J).$$

We will prove Conjecture 1.7 for any satellite knot with 2-bridge companion. Further, we will prove Conjecture 1.6 for a special class of knots known as cable knots. Cable knots are a well-studied group of knots; for instance, it is conjectured that if a knot K admits a reducible Dehn surgery, then K is a cable knot. This is the well-known Cabling Conjecture.

We say that a satellite knot K is a cable knot if its pattern $\hat{K}$ is a torus knot. In this case, there is an isotopy which carries $\hat{K}$ into the companion torus $T = \partial\varphi(V)$, and so we may suppose without loss of generality that K is contained in T. This allows us much greater control over determining the positions K can occupy in S^3 than in the general case.

If K is a cable whose pattern is a (p, q)-torus knot and companion is J, we say that K is a (p, q)-cable of J. We follow the convention of [8], where a (p, q)-torus knot in T has homology class $p[\mu]+q[\lambda]$ with μ a meridian and λ a longitude of V. Observe that a (p, q)-torus knot has algebraic intersection number q with any meridian of V; hence the wrapping and winding number of a (p, q)-cable is q. See Figure 1.2 below for an example.

If K is a (p, q)-cable of J, then by taking an "obvious" cabling of J, we may

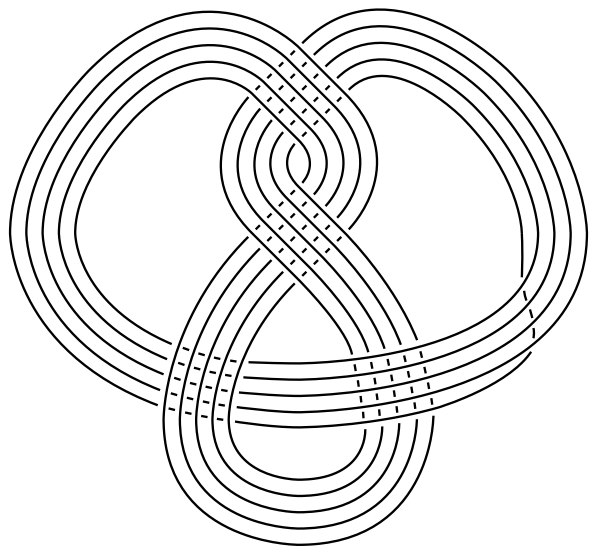

Figure 1.2: A $(1, 5)$-cable of the figure eight knot with the blackboard framing

exhibit a bridge position k of K satisfying $b(k) = q \cdot b(J)$. In this case, the inequality from Theorem 1.3 is sharp and k is a minimal bridge position. We will prove a special case of Conjecture 1.6 that a similar statement holds for knot width:

Theorem 1.8. *[36] Suppose J is a knot in S^3, and K is a (p, q)-cable of J. Then*

$$w(K) = q^2 \cdot w(J).$$

This theorem implies that the embedding pictured in Figure 1.2 is a thin position for the $(1, 5)$-cable of the figure eight knot.

CHAPTER 2
THE WIDTH OF SATELLITE KNOTS WITH 2-BRIDGE COMPANIONS

We will devote this chapter to the proof of a special case of Conjecture 1.6. To this end, we show that if K is a satellite knot whose pattern has winding number n, then

$$w(K) \geq 8n^2.$$

Much of the material from this chapter appears in [34]. Note the for any 2-bridge knot J, thin position is a minimal bridge position and thus $w(J) = 8$. Thus, Theorem 2.7 establishes the conjecture for knots with 2-bridge companion. If in addition the pattern is a braid, this bound is sharp; that is, for such K, $w(K) = 8n^2$. We rely heavily on the essential companion torus to bound the width from below.

2.1 Manipulating the companion torus

For the remainder of the chapter, K will denote a satellite knot with companion J and pattern $\hat{K}$. We let T denote the companion torus $\partial\varphi(V)$ (notation as in the Chapter 1). Observe that we may write $E(K) = E(J) \cup_T (V \setminus \eta(\hat{K}))$. Since every meridian of V must intersect $\hat{K}$ assuming K is a nontrivial satellite knot, T must be incompressible and not boundary parallel in $V \setminus \eta(\hat{K})$. Further, if T were compressible in $E(J)$, it would imply that J were the unknot, a contradiction. Thus, T is essential in $E(K)$.

Fix $k \in \mathcal{K}$, and let T denote the companion torus as it relates to this particular embedding of k. If necessary, perturb T so that $h|_T$ is a Morse function. We wish to restrict our investigation to tori T with only certain types of saddle points. In this vein, we follow [26], from which the next definition is taken. Consider the singular foliation, F_T, of T induced by $h|_T$. Let σ be a leaf corresponding to a saddle point. Then one component of σ is the wedge of two circles s_1 and s_2. If either is inessential in T, we say that σ is an inessential saddle. Otherwise, σ is an essential saddle.

The next lemma is the Pop Over Lemma from [26]:

Lemma 2.1. *If F_T contains inessential saddles, then after a small isotopy of T, there is an inessential saddle σ in T such that*

1. *s_1 bounds a disk $D_1 \subset T$ such that F_T restricted to D_1 contains only one maximum or minimum,*

2. *for L the level surface of h containing σ, D_1 cobounds a 3-ball B with a disk $\tilde{D}_1 \subset L$ such that B does not contain $\pm\infty$ and such that s_2 lies outside of $\tilde{D}_1$.*

In the following lemma, we mimic Lemma 2 of [26] with a slight modification to preserve the height function h on k:

Lemma 2.2. *There exists an isotopy $f_t : S^3 \to S^3$ such that $f_0 = Id$, $h = h \circ f_1$ on k, and the foliation of $f_1(T)$ contains no inessential saddles.*

Proof. Suppose that F_T contains an inessential saddle, σ, lying in the level 2-sphere L. By the previous lemma, we may suppose that σ is as described above, and suppose without loss of generality that D_1 contains only one maximum. By slightly pushing

D_1 into $\text{int}(B)$, we can create another 3-ball B' such that $B' \cap D_1 = \emptyset$ and $(k \cup T) \cap \text{int}(B) \subset B'$. First, we construct an isotopy which pushes B' below L into a small neighborhood of $\tilde{D}_1$ and then cancels the maximum of D_1 with the saddle point σ.

Now, there exists a monotone increasing arc beginning at the maximum of B', ascending through the disk $\tilde{D}_2 \subset L$ bounded by s_2, intersecting only maxima of T, and disjoint from k. Thus, we may construct another isotopy which pushes B' upward through a regular neighborhood of α, increasing the heights of maxima of T if necessary, until the heights of maxima and minima of $k \cap \text{int}(B')$ are the same as before any of the above isotopies. We see that after performing both isotopies T has one fewer inessential saddle and no new critical points have been created. See Figure 2.1. Repeating this process, we eliminate all inessential saddles via isotopy. $\square$

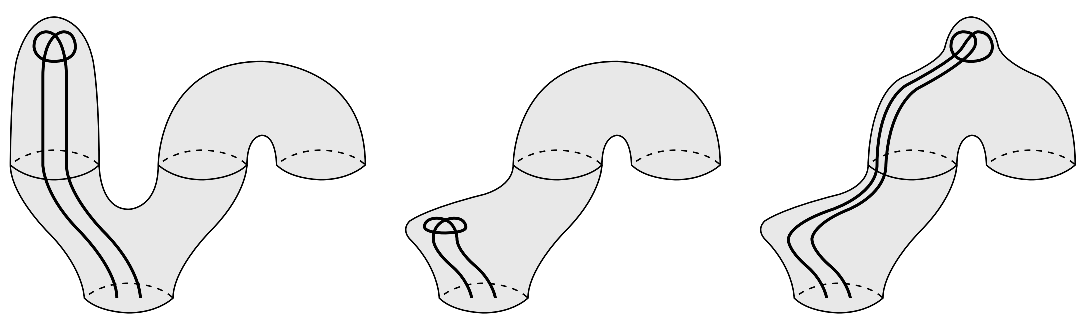

Figure 2.1: An illustration of the process of eliminating an inessential saddle described in the proof of Lemma 2.2

Thus, from this point forward, we may replace any $k \in \mathcal{K}$ with $f_1(k)$ from the lemma, where $w(k) = w(f_1(k))$. In addition, we may suppose that the foliation F_T

contains no inessential saddles. It follows that if γ is a loop contained in a level 2-sphere that bounds a disk $D \subset T$, then $h|_D$ has exactly one critical point, a minimum or a maximum. For if not, D would contain a saddle point, which would necessarily be inessential.

2.2 The connectivity graph

For each regular value r of $h|_{T,k}$, we have that $h^{-1}(r)$ is a level 2-sphere and $h^{-1}(r) \cap T$ is a collection of simple closed curves. Let $\gamma_1, \ldots, \gamma_n$ denote these curves.

A bipartite graph is a graph together with a partition of its vertices into two sets $\mathcal{A}$ and $\mathcal{B}$ such that no two vertices from the same set share an edge. We will create a bipartite graph Γ_r from $h^{-1}(r)$ as follows: Cut the 2-sphere $h^{-1}(r)$ along $\gamma_1, \ldots, \gamma_n$, splitting $h^{-1}(r)$ into a collection of planar regions $R_1, \ldots, R_m$. The vertex set $\{v_1, \ldots, v_m\}$ of Γ_r corresponds to the regions $R_1, \ldots, R_m$, and the edges correspond to the curves $\gamma_1, \ldots, \gamma_n$ that do not bound disks in T. For each such γ_i, make an edge between v_j and v_k if $\gamma_i = R_j \cap R_k$ in $h^{-1}(r)$. To see that Γ_r is bipartite, we create two vertex sets $\mathcal{A}_r$ and $\mathcal{B}_r$, letting $v_i \in \mathcal{A}_r$ if $R_i \subset V$, and $v_i \in \mathcal{B}_r$ otherwise. We call Γ_r the essential connectivity graph with respect to the regular value r of h, where the word "essential" emphasizes the fact that edges correspond to only those γ_i that are essential in T. Note that since each γ_i separates $h^{-1}(r)$, the graph Γ_r must be a tree. An endpoint of Γ_r is a vertex that is incident to exactly one edge.

We remark that the term connectivity graph also appears in [22], but the two notions are not related. In the above definition, the essential connectivity graph

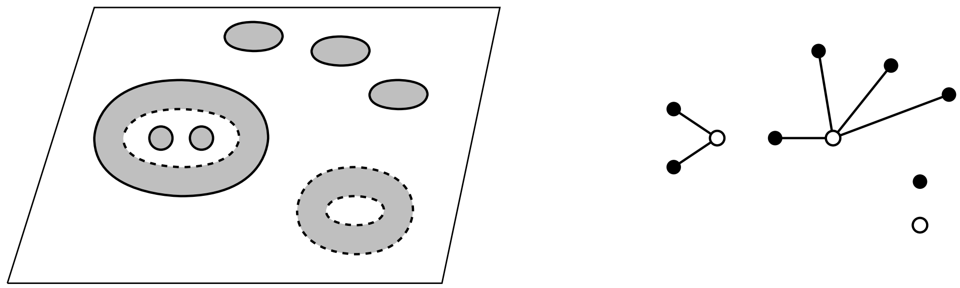

Figure 2.2: A level 2-sphere at left and its corresponding essential connectivity graph at right. Note that dotted curves on the left correspond to curves bounding disks in T.

represents adjacencies between components of intersections of $\varphi(V)$ (the solid torus whose core is isotopic to the companion J) and $\overline{S^3 \setminus \varphi(V)}$ with a single level 2-sphere. In [22], the connectivity graph is more global, representing adjacencies between components of a 3-manifold cut along certain level surfaces.

For instance, Figure 2.2 depicts a possible level 2-sphere and corresponding essential connectivity graph. Observe that since $\varphi(V)$ is a knotted solid torus, T is only compressible on one side as a surface in S^3, and every compression disk for T is a meridian of $\varphi(V)$. This leads to the third lemma:

Lemma 2.3. *If $v_i \in \Gamma_r$ is an endpoint, then $v_i \in \mathcal{A}_r$.*

Proof. Suppose R_i is the region in $h^{-1}(r)$ corresponding to v_i. Then ∂R_i contains exactly one essential curve in T, call it γ, and some (possibly empty) set of curves that bound disks in T. Since each of these disks contains only one maximum or minimum by the discussion above, any two such disks must be pairwise disjoint. Thus, we can glue each disk to R_i to create an embedded disk D such that $\partial D = \gamma$. Now, push each glued disk into a collar of T in V, so that $T \cap \mathrm{int}(D) = \emptyset$, and thus D is a compression disk for T. We conclude $D \subset V$ and $\mathrm{int}(R_i) \cap \mathrm{int}(D) \neq \emptyset$, implying $R_i \subset V$ and $v_i \in \mathcal{A}_r$. □

Using similar arguments, we prove the next lemma:

Lemma 2.4. *Suppose that* $v_1, \ldots, v_n \subset \Gamma_r$ *are endpoints corresponding to regions* $R_1, \ldots, R_n \subset h^{-1}(r)$. *Then* $\gamma_1, \ldots, \gamma_n$ *bound meridian disks* $D_1, \ldots, D_n \subset V$ *such that* $k \cap D_i \subset R_i$ *for all* i.

Proof. The existence of the disks $D_1, \ldots, D_n$ is given in the proof of Lemma 2.3. Thus, suppose that Δ is a disk glued to R_i to construct D_i. When we push Δ into a collar of T, we can choose this collar to be small enough so that it does not intersect k. Thus, we may suppose that $\Delta \cap k = \emptyset$ for every such Δ, which implies that all intersections of k with D_i must be contained in R_i. □

We note that the Lemmas 2.3 and 2.4 are inspired by the proof of Theorem 1.9 of [18]. Essentially, Lemma 2.4 demonstrates that even though the set of meridian disks $D_1, \ldots, D_n$ may not be level, we may assume they are level for the purpose of bounding below the number of intersections of k with $h^{-1}(r)$, since any intersection of k with one of these disks occurs in one of the level regions R_i. Let r be a regular value of $h|_{T,k}$. We define the trunk of the level 2-sphere $h^{-1}(r)$, denoted $\mathrm{trunk}(r)$, to be the number of endpoints of Γ_r.

For example, if r is the regular value whose essential connectivity graph is pictured in Figure 2.2, then $\mathrm{trunk}(r) = 6$. We are now in a position to use the winding number of the pattern $\hat{K}$.

Lemma 2.5. *Let r be a regular value of $h|_{T,k}$, and let n denote the winding number of $\hat{K}$.*

- *If trunk(r) is even, then $|k \cap h^{-1}(r)| \geq n \cdot \text{trunk}(r)$;*
- *if trunk(r) is odd, then $|k \cap h^{-1}(r)| \geq n \cdot [\text{trunk}(r) + 1]$.*

Proof. First, suppose that $m = \text{trunk}(r)$ is even. Since each meridian of V has algebraic intersection $\pm n$ with K, we know that each meridian must intersect k in at least n points. Let $v_1, \ldots, v_m$ be endpoints of Γ_r corresponding to regions $R_1, \ldots, R_m$. By Lemma 2.4, $|k \cap R_i| = |k \cap D_i| \geq n$ for each i. Further, since these regions are pairwise disjoint, it follows that $|k \cap h^{-1}(r)| \geq n \cdot m$, completing the first part of the proof.

Now, suppose that m is odd. If N_1 is the algebraic intersection number of k with $R = \cup R_i$, we have that

$$N_1 = \sum_{i=1}^{m} \pm n.$$

In particular, as m is odd it follows that $|N_1| \geq n$. Let $R' = \overline{h^{-1}(r) \setminus R}$. Then $R' \cap R \subset T$, so K does not intersect $R' \cap R$. Let N_2 denote the algebraic intersection number of K with R'. Since $h^{-1}(r)$ is a 2-sphere which bounds a ball in S^3, $h^{-1}(r)$ is homologically trivial, implying that the algebraic intersection of k with $h^{-1}(r)$ is zero. This means $N_1 + N_2 = 0$, so $|N_2| \geq n$ and thus $|k \cap R'| \geq n$. Lastly,

$$|k \cap h^{-1}(r)| = |k \cap R| + |k \cap R'| = \sum_{i=1}^{m} |k \cap R_i| + |k \cap R'| \geq n \cdot (m + 1).$$

□

2.3 Bounding the width of satellite knots

We will use the trunk of the level surfaces to impose a lower bound on the trunk of k, which in turn forces a lower bound on the width of the k. We need the following lemma, which is Claim 2.4 in [18]:

Lemma 2.6. *Let S be a torus embedded in S^3, and if necessary perturb S so that $h|_S$ is Morse. Suppose that for every regular value r of $h|_S$, all curves in $h^{-1}(r) \cap S$ that are essential in S are mutually parallel in $h^{-1}(r)$. Then S bounds solid tori V_1 and V_2 in S^3 such that $V_1 \cap V_2 = T$.*

As a result of this lemma, we have

Corollary 2.1. *There exists a regular value r of $h|_{T,k}$ such that trunk$(r) \geq 3$.*

Proof. Suppose not, and let r be any regular value of $h|_{T,k}$ such that $h^{-1}(r)$ contains essential curves in T. Such a regular value must exist; otherwise T could not contain a saddle point. By assumption, trunk$(r) \leq 2$, so Γ_r has exactly two endpoints, v_1 and v_2. But this implies that Γ_r is a path, and thus all essential curves in $h^{-1}(r)$ are mutually parallel. As this is true for every such regular value r, we conclude by Lemma 2.6 that $\varphi(V)$ is an unknotted solid torus, contradicting the fact that K is a satellite knot with nontrivial companion J. □

This brings us to our main theorem of the chapter, which establishes Conjecture 1.7 for a satellite knot with 2-bridge companion.

Theorem 2.7. *Suppose K is a satellite knot whose pattern $\hat{K}$ has winding number n. Then*

$$w(K) \geq 8n^2.$$

Proof. Let k be a thin position of K, with T defined as above. By Corollary 2.1 above, there exists a regular value r of $h|_{T,k}$ such that $\text{trunk}(r) \geq 3$. From Lemma 2.5, it follows that $|k \cap h^{-1}(r)| \geq 4n$. It follows that $\text{trunk}(K) = \text{trunk}(k) \geq 4n$. Finally, using the lower bound for width based on trunk (Equation 1.2),

$$w(K) \geq \frac{\text{trunk}(K)^2}{2} \geq 8n^2,$$

as desired. □

Corollary 2.2. *Suppose K is a satellite knot, with pattern $\hat{K}$ and companion J. If $\hat{K}$ is a braid with winding number n and J is a 2-bridge knot, then $w(K) = 8n^2$ and any thin position for K is a minimal bridge position.*

Proof. For such K we can exhibit an embedding $k \in \mathcal{K}$ satisfying $w(k) = 8n^2$, $b(k) = 2n$, and $\text{trunk}(k) = 4n$. By [26], $b(K) = b(k)$, so k is both a bridge and thin position for k, and further every minimal bridge position k' for K satisfies $w(k') = 8n^2$ and is also thin. It follows from the proof of the above theorem that $\text{trunk}(K) = 4n$, so any $k \in \mathcal{K}$ that is not a minimal bridge position satisfies $w(k) > 8n^2$. □

The above corollary shows that the embeddings pictured in Figures 1.1 (right) and 1.2 are thin positions, since both are satellite knots with 2-bridge companions.

CHAPTER 3
THE WIDTH OF CABLE KNOTS

Here we push the methods introduced in Chapter 2 further in order to determine the width of a cable knot with arbitrary companion. This establishes Conjecture 1.6 in the case that the pattern $\hat{K}$ is a torus knot. Most of the material in this chapter appears in [35].

3.1 Simplification of cable knots

Suppose that S is any closed surface properly embedded in S^3. As noted above, h induces a singular foliation, F_S provided that that $h|_S$ is Morse. As mentioned in Section 1.3, when we restrict our investigation from a satellite knot to a cable knot K, we achieve greater control over the behavior of K with respect to h by assuming that K is embedded in the companion torus T and studying F_T.

One might hope that the type I and type II moves defined in Section 1.1 would suffice to simplify any position of a cable knot, but unfortunately this is not the case. To this end, we define one additional move, a disk slide, which does not change width but can allow us to slightly modify the position of K with respect to T: Suppose that some embedding k of a knot K is contained in a surface S such that $h|_S$ is Morse. Suppose further that $D \subset S$ is a disk satisfying the following conditions:

1. ∂D is contained in some level surface $h^{-1}(r)$;

2. $\text{int}(D)$ contains exactly one minimum or maximum;

3. some component of $k \cap D$ is an outermost arc γ that contains exactly one critical point of $h|_k$ and cobounds a disk Δ with an arc in ∂D, where $\text{int}(\Delta)$ contains the critical point of D.

Then there is an isotopy through Δ supported in a neighborhood of Δ that takes γ to an arc γ' that cobounds a disk Δ' with an arc in ∂D such that $\text{int}(\Delta')$ contains no critical points and $\text{int}(\Delta') \cap k = \emptyset$. Additionally, γ' can be chosen so that it contains exactly one critical point occurring at the same height as the critical point of γ. We call the isotopy that replaces γ with γ' a disk slide, pictured in Figure 3.1. Note that by requiring that the critical point of γ' occur on the same level as the

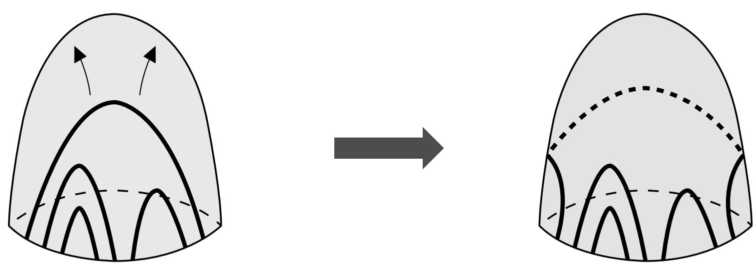

Figure 3.1: A disk slide

critical point of γ, we ensure that if k' is the result of performing a disk slide on k, we have $w(k') = w(k)$, although the width of k may temporarily increase during some intermediate step of the disk slide.

3.2 Efficient position of k

From this point forward, we set the convention that K is a (p, q)-cable knot with companion J and companion torus T. As noted above, for any $k \in \mathcal{K}$, we may assume there is a torus isotopic to T containing k. In order to study each embedding $k \in \mathcal{K}$ with respect to various representatives in the isotopy class of the companion torus T, we define the following collection:

$$\mathcal{KT} = \{(k, T_k) : k \in \mathcal{K},\, T_k \sim T,\, k \subset T_k,\ \text{and } h|_{T_k} \text{ is Morse}\}\,.$$

For an arbitrary element (k, T_k) of $\mathcal{KT}$, the torus T_k is compressible in S^3 only on one side, and we let V_k denote the solid torus in S^3 bounded by T_k. We may also assume that the critical points of k and T_k occur at different levels.

The goal of this section is to show that any embedding k can be deformed to lie on a torus isotopic to T in an efficient way and without increasing $w(k)$. To this end, consider an arc γ of k embedded in an annulus A, where both boundary components of A are level and $\text{int}(A)$ contains no critical points with respect to h. Suppose that γ is an essential arc containing critical points, and let x denote the lowest maximum of γ. There are two points $y, z \in \gamma$ corresponding to minima (one could be an endpoint) such that γ contains a monotone arc connecting x to y and x to z. Without loss of generality, suppose that $h(y) > h(z)$. Then level arc components of $A \cap h^{-1}(h(y) + \epsilon)$ cobound disks $D, E \subset A$ with arcs in γ such that ∂D contains exactly one maximum (x), ∂E contains exactly one minimum (y), and $\text{int}(D) \cap \gamma = \text{int}(E) \cap \gamma = \emptyset$. Further,

$D \cap E$ is a single point contained in γ. Refer to Figure 3.2.

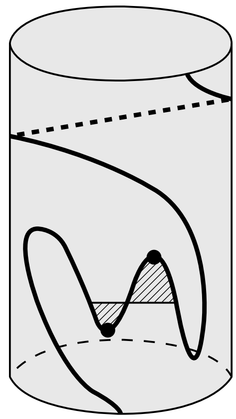

Figure 3.2: An essential arc γ containing critical points, shown with upper and lower disks D and E described above.

If there is exactly one thick surface between x and y, then k admits a type I move canceling the two critical points. If there is more than one such thick surface, we can slide x down along D, performing some number of type II moves, until there is one such thick surface, after which k admits a type I move. A similar argument is valid if γ is inessential in A or if γ is embedded in a disk $\hat{D}$ with level boundary and one critical point, and γ contains more than one critical point. We will use these ideas in the proof of the next lemma, which is very similar to Lemma 2.2 modified slightly to accommodate the fact that for any $(k, T_k) \in \mathcal{KT}$, we have $k \subset T_k$.

Lemma 3.1. *For any $k \in \mathcal{K}$, there exists $(k', T_{k'}) \in \mathcal{KT}$ such that $w(k') \leq w(k)$ and the foliation of $T_{k'}$ by h contains no inessential saddles.*

Proof. Suppose that $(k, T_k) \in \mathcal{KT}$ and F_{T_k} contains an inessential saddle. By Lemma

2.1, we can modify T_k so that there exists a saddle σ of F_{T_k} with corresponding critical value c such that some component P of $T_k \cap h^{-1}([c-\epsilon, c+\epsilon])$ is a pair of pants, with boundary components s_1, s_2, s_3 satisfying

1. s_1 and s_2 are contained in the same level surface L of h,

2. s_1 bounds a disk $D_1 \subset T_k$ such that F_{T_k} restricted to D_1 contains only one maximum or minimum,

3. D_1 co-bounds a 3-ball B with a disk $\tilde{D}_1 \subset L$ such that B does not contain $\pm\infty$ and such that s_2 lies outside of $\tilde{D}_1$.

Without loss of generality, let D_1 contain one maximum. By the above argument, we can perform type I and II moves so that each component of $k \cap D_1$ contains exactly one maximum. Let $A = P \cup D_1$, so that A is an annulus with boundary components s_2 and s_3. We can choose P so that $k \cap P$ consists only of vertical arcs. Then if $k \cap D_1 \neq \emptyset$, $k \cap A$ contains inessential arcs with exactly one critical point and boundary in s_3. Let γ denote an outermost arc in A, so that γ cobounds a disk Δ with an arc in s_3 and $\text{int}(\Delta) \cap k = \emptyset$. If $\text{int}(\Delta)$ contains a critical point, we can perform a disk slide to remove it, after which can remove γ from $k \cap A$, possibly by type II moves.

Thus, after some combination of type I moves, type II moves, and disk slides, we may assume that $k \cap D_1 = \emptyset$. By the proof of Lemma 2.2, we can cancel the saddle σ with the maximum contained in D_1 without increasing $w(k)$. Repeating this process finitely many times finishes the proof. □

Hence, for the purpose of finding thin position we may assume that for any $(k, T_k) \in \mathcal{KT}$, the foliation F_{T_k} contains no inessential saddles, and so each disk $D \subset T_k$ with level boundary must contain exactly one critical point. We distinguish between two types of saddles: Let $(k, T_k) \in \mathcal{KT}$, with σ a saddle point of F_{T_k} and c the corresponding critical value. Then some component P of $T \cap h^{-1}([c-\epsilon, c+\epsilon])$ is a pair of pants. We say that σ is a <u>lower saddle</u> if $P \cap h^{-1}(c+\epsilon)$ has two components; otherwise, σ is an <u>upper saddle</u>.

Next, for $(k, T_k) \in \mathcal{KT}$, we suppose that the foliation of T_k contains only essential saddles and decompose T_k into annuli as follows: for each critical value c_i corresponding to a saddle, some component of $T_k \cap h^{-1}([c_i - \epsilon, c_i + \epsilon])$ is a pair of pants, call it P_i. If P_i is a lower saddle, then $P_i \cap h^{-1}(c - \epsilon)$ is a simple closed curve that bounds a disk D_i in S with exactly one minimum. If P_i is an upper saddle, there exists a similar disk D_i with exactly one maximum. In either case, $A_i = P_i \cup D_i$ is an annulus whose foliation contains exactly one saddle and one minimum or maximum, called a <u>lower annulus</u> or an <u>upper annulus</u>, respectively.

Now $\overline{T_k \setminus \cup A_i}$ is a collection of vertical annuli whose foliations contain no critical points, and we have decomposed S into a collection of lower, upper, and vertical annuli, which we denote $\{A_1, \ldots, A_r\}$. In addition, these r annuli as a collection have r distinct boundary components, which we denote $\{\delta_1, \ldots, \delta_r\}$. Let L be any level 2-sphere containing some δ_i. Of all curves in $T_k \cap L$ that are essential in T_k, consider a curve α that is innermost in L. Thus α bounds a disk D in L containing no other essential simple closed curves in $T_k \cap L$. Potentially, D contains some inessential curves in $T_k \cap L$, but these curves bound disks in T_k, so after some gluing operations, we see that α bounds a compressing disk for T_k. Since T_k is compressible only on one side, it follows that α bounds a meridian disk of the solid torus V_k; hence, δ_i also bounds a meridian disk since it is parallel to α in T_k (this is also shown in Lemma 2.3). As K is a (p, q)-cable, each δ_i (oriented properly) has algebraic intersection q with k, which implies k has at least $q \cdot r$ intersections with the collection $\{\delta_1, \ldots, \delta_r\}$.

We would like k to be contained in the companion torus as efficiently as possible; for this purpose, we define efficient position, where we decompose T_k into annuli as describe above. We say that $(k, T_k) \in \mathcal{KT}$ is an efficient position if $k \cap A_i$ is a collection of essential arcs in A_i for every i. Further, if A_i is a vertical annulus, we require that each arc of $k \cap A_i$ contain no critical points, and if A_i is an upper or lower annulus, we require that each arc of $k \cap A_i$ contains exactly one minimum or maximum.

Note that if (k, T_k) is an efficient position, it is implicit in the above definition that the foliation of T_k contains only essential saddles. Of course, given an arbitrary element $(k, T_k) \in \mathcal{KT}$, we may not necessarily assume that (k, T_k) is an efficient position, but we may employ the next lemma.

Lemma 3.2. *For any $k \in \mathcal{K}$, there exists $(k', T_{k'}) \in \mathcal{K}$ such that $w(k') \leq w(k)$ and $(k', T_{k'})$ is an efficient position.*

Proof. Let $(k, T_k) \in \mathcal{KT}$ such that F_{T_k} contains no inessential saddles and decompose T_k into annuli as described above. Suppose that $\sum |k \cap \delta_i|$ is minimal up to isotopies that do not increase width. By previous arguments, we suppose that after a series of type I and type II moves, we have for every vertical annulus A_i, $k \cap A_i$ consists of monotone essential arcs and inessential arcs with exactly one critical point, and for every lower or upper annulus A_i, $k \cap A_i$ consists of essential and inessential arcs with exactly one critical point. As determined above, $\sum |k \cap \delta_i| \geq q \cdot r$. If $\sum |k \cap \delta_i| = q \cdot r$, then every oriented intersection of k with δ_i must occur with the same sign, so every arc of $k \cap A_i$ is essential and (k, T_k) is an efficient position.

Conversely, if every arc of $k \cap A_i$ is essential for all i, then all oriented intersections of k with δ_i occur with the same sign and $\sum |k \cap \delta_i| = q \cdot r$. Thus, if $\sum |k \cap \delta_i| > q \cdot r$, there exists A_j such that $k \cap A_j$ contains an inessential arc. Let γ be an inessential arc that is outermost in A_j, so that γ cobounds a disk Δ with a level arc in ∂A_j such that $\text{int}(\Delta) \cap k = \emptyset$. Possibly after a disk slide if A_j is upper or lower, we can slide γ along Δ to remove two points of some $k \cap \delta_i$. This isotopy does not increase the width of k, contradicting the minimality assumption above. □

As a result of the proof of Lemma 3.2, if (k, T_k) is an efficient position, we have that $k \cap A_i$ consists of q essential arcs for each i. See Figure 3.4.

3.3 The width of the companion torus and the width of cable knots

Neglecting K for the moment, let L be any knot in S^3 and $\mathcal{L}$ the set of embeddings of S^1 isotopic to L. In this section, we define the width of a torus in S^3. However, instead of modifying the standard definition that counts the number of intersections of a knot with level 2-spheres, we will keep track of the order in which the upper and lower saddles occur in the foliation of the torus by h. In the case of

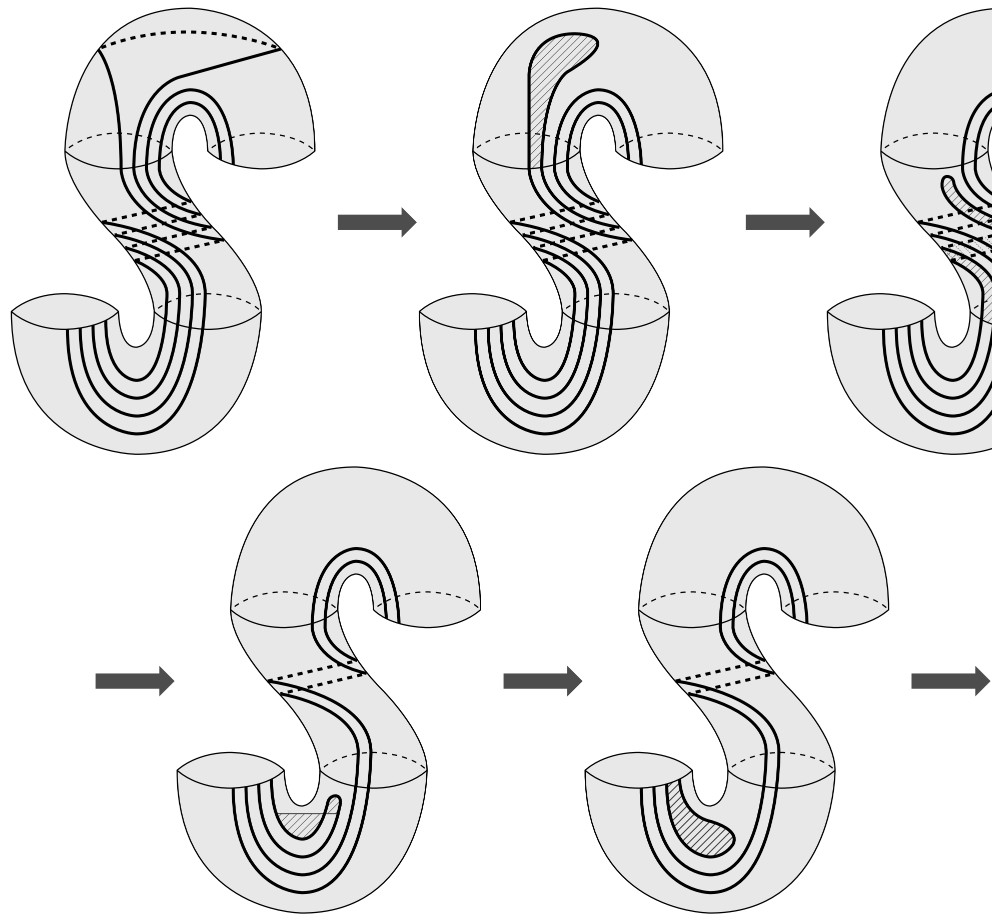

Figure 3.3: An example of the simplification suggested by Lemma 3.2. At upper right, an inessential arc γ is contained in an upper annulus. After a disk slide, we remove γ from the upper annulus, and then remove it from the vertical annulus. Now the inessential arc contains three critical points, so after a Type I move, we can remove it from the lower annulus, continuing as necessary until an efficient position is attained.

knots, observe that we can calculate the width of any embedding $l \in \mathcal{L}$ if we know the order in which all minima and maxima of l occur with respect to h. To formalize this notion, let Z be the free monoid generated by two elements, m and M. Now, define $\hat{Z}$ to be those elements $\sigma = m^{\alpha_1} M^{\beta_1} \ldots m^{\alpha_n} M^{\beta_n} \in Z$ such that

1. $\alpha_i, \beta_i \neq 0$ for all i,

2. $\sum_{i=1}^{j} \alpha_i > \sum_{i=1}^{j} \beta_i$ for all $j < n$, and

3. $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i$.

We define a map from $\mathcal{L}$ to $\hat{Z}$, $l \mapsto \sigma_l$, by the following: Let $c_0 < \cdots < c_p$ be the critical values of $h|_l$. We create a word σ_l consisting of $p+1$ letters by mapping the tuple $(c_0, \ldots, c_p)$ to a word by assigning m to each minimum and M to each maximum. Next, we define the width of an element $\sigma = m^{\alpha_1} M^{\beta_1} \ldots m^{\alpha_n} M^{\beta_n} \in \hat{Z}$ by

$$w(\sigma) = 2\left(\sum \alpha_i\right)^2 - 4\sum_{i>j} \alpha_i \beta_j.$$

The following lemma should be expected:

Lemma 3.3. *For any $l \in \mathcal{L}$, $w(l) = w(\sigma_l)$.*

Proof. Let $\sigma_l = m^{\alpha_1} M^{\beta_1} \ldots m^{\alpha_n} M^{\beta_n}$. Then the thick/thin tuple for l is $(2\alpha_1, 2(\alpha_1 - \beta_1), 2(\alpha_1 - \beta_1 + \alpha_2), \ldots, 2(\alpha_1 - \beta_1 + \alpha_2 - \cdots + \alpha_n))$. From the width formula given by (1.1),

$$\begin{aligned} w(k) &= \frac{1}{2}[(2\alpha_1)^2 - (2(\alpha_1 - \beta_1))^2 + (2(\alpha_1 - \beta_1 + \alpha_2))^2 - \ldots \\ &\quad \cdots + (2(\alpha_1 - \beta_1 + \alpha_2 - \cdots + \alpha_n))^2] \\ &= 2[\alpha_1^2 + \alpha_2^2 + 2\alpha_2(\alpha_1 - \beta_1) + \ldots \\ &\quad \cdots + \alpha_n^2 + 2\alpha_n(\alpha_1 - \beta_1 + \alpha_2 - \cdots - \beta_{n-1})] \\ &= 2\left(\sum \alpha_i\right)^2 - 4\sum_{i>j} \alpha_i \beta_j, \end{aligned}$$

as desired. □

As an example, consider the embedding l pictured in Figure 3.4. A simple verification shows that $\sigma_l = m^3 MmM^3$ and $w(l) = w(\sigma_l) = 28$.

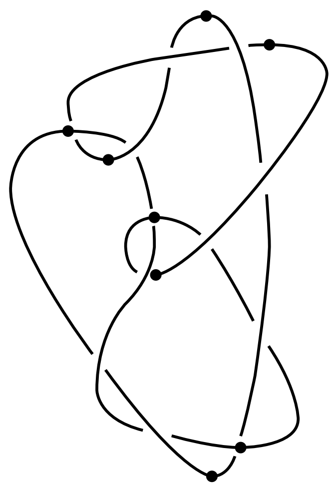

Figure 3.4: An embedding l with $\sigma_l = m^3 MmM^3$.

Next, we consider the collection of possible companion tori for a satellite knot with companion L. Define $\mathcal{S}_L$ to be the collection of embedded tori S such that

1. $h|_S$ is Morse,

2. the foliation F_S induced by h contains no inessential saddles, and

3. S is isotopic to the boundary of a regular neighborhood of L.

Now, as with $\mathcal{L}$, we can define a map from $\mathcal{S}_L$ to $\hat{Z}$, $S \mapsto \sigma_S$, by the following: Let $c_0 < \cdots < c_p$ be the critical values corresponding to the saddles of F_S. We create a word σ_S consisting of $p+1$ letters by mapping the tuple $(c_0, \ldots, c_p)$ to a word by assigning m to each lower saddle and M to each upper saddle. It is clear that $\sigma_S \in Z$, and slightly more difficult to see that $\sigma_S \in \hat{Z}$. The proof of this fact is left to the reader.

Using this association, we define the width of any torus $S \in \mathcal{S}_L$ by $w(S) = w(\sigma_S)$, and similar to the definition of knot width, we say the neighborhood width of the knot L, $nw(L)$, is given by

$$nw(L) = \min_{S \in \mathcal{S}_L} w(S).$$

As a knot invariant, neighborhood width is not new; it is equivalent to knot width by the next lemma.

Lemma 3.4. *For any knot L, $nw(L) = w(L)$.*

Proof. First, let l_0 be an embedding of L such that $w(l_0) = w(L)$, and let S_0 be the boundary of a regular neighborhood of l_0 in S^3 such that every saddle of S_0 occurs slightly above a minimum or slightly below a maximum of l_0. All such saddles are easily seen to be essential, so $S_0 \in \mathcal{S}_L$. Further, $w(S_0) = w(\sigma_{S_0}) = w(\sigma_{l_0}) = w(L)$; hence

$$nw(L) \leq w(L).$$

For the reverse inequality, let $S \in \mathcal{S}_L$. Since S is isotopic to a regular neighborhood of L, we have that any longitude of S is isotopic to L. Decompose S into a collection of upper, lower, and vertical annuli $\{A_1, \ldots, A_r\}$ as in Section 3.2. For each lower or upper annulus A_i, let l_i be an essential arc in A_i passing through the saddle with exactly one minimum or maximum. For each vertical annulus A_i, let l_i be a monotone arc connecting the two endpoints of the arcs contained in the annuli adjacent to A_i (these must contain saddles). Lastly, let l be the union of all the arcs l_i, so that l is a simple closed curve.

As shown above, the collection of boundary components $\{\delta_i\}$ of the annuli $\{A_i\}$ bound meridian disks for the solid torus bounded by S, and since l intersects each δ_i once, l is a longitude of S. By construction $w(S) = w(\sigma_S) = w(\sigma_l) = w(l) \geq w(L)$, and since this is true for all $S \in \mathcal{S}_L$, we have

$$nw(L) = w(L),$$

completing the proof. □

Finally, we have all the necessary tools to find the width of a cable knot.

Theorem 3.5. *Suppose that K is a (p,q)-cable of a nontrivial knot J. Then*

$$w(K) = q^2 \cdot w(J).$$

Proof. Let j be an embedding of J such that $w(j) = w(J)$, and let T_j be the boundary of a regular neighborhood of J such that every saddle of T_j occurs slightly above a minimum or slightly below a maximum of j. Suppose that $\sigma_j = m^{\alpha_1}M^{\beta_1}\ldots m^{\alpha_n}M^{\beta_n}$. We can cable j along T_j, creating an embedding $k \in \mathcal{K}$ such that $\sigma_k =$

$m^{q\alpha_1}M^{q\beta_1}\dots m^{q\alpha_n}M^{q\beta_n}$. By Lemma 3.3,

$$w(K) \leq w(\sigma_k) = q^2 \cdot w(\sigma_j) = q^2 \cdot w(J).$$

We call k an "obvious" cabling of j.

On the other hand, suppose $k' \in \mathcal{K}$ is a given thin position. By Lemmas 3.1 and 3.2, there exists a torus $T_{k'}$ such that $(k', T_{k'}) \in \mathcal{KT}$ is an efficient position. Let $\sigma_{T_{k'}} = m^{\alpha'_1}M^{\beta'_1}\dots m^{\alpha'_{n'}}M^{\beta'_{n'}}$. Note that $T_{k'} \in \mathcal{S}_J$, and thus $w(T_{k'}) \geq w(J)$ by Lemma 3.4. We decompose $T_{k'}$ into a collection of upper, lower, and vertical annuli $\{A_1, \dots, A_r\}$ as above in Section 3.2. For each upper annulus A_i, $k \cap A_i$ consists of q essential arcs, each containing one maximum. Suppose c_i is the critical value corresponding to the saddle in A_i. Then there is an isotopy of k supported in A_i taking the q arcs to arcs in $h^{-1}((c_i, c_i + \epsilon])$, and this isotopy does not increase $w(k)$. A similar statement is true for each lower annulus A_i. As all critical points of k are contained in upper or lower annuli, we can compute $\sigma_k = m^{q\alpha'_1}M^{q\beta'_1}\dots m^{q\alpha'_{n'}}M^{q\beta'_{n'}}$, and thus

$$w(k) = w(m^{q\alpha'_1}M^{q\beta'_1}\dots m^{q\alpha'_{n'}}M^{q\beta'_{n'}}) = q^2 \cdot w(T_{k'}) \geq q^2 \cdot w(J),$$

completing the proof. □

The proof of the theorem reveals that if k is an "obvious" cabling of a thin position of J, then it is a thin position of K.

Corollary 3.1. *For a knot J in S^3, the following are equivalent:*

(a) Thin position of J is a minimal bridge position.

(b) Every cable of J has the property that thin position is a minimal bridge position.

(c) There exists a cable of J with the property that thin position is a minimal bridge position.

CHAPTER 4
BRIDGE POSITIONS OF CABLE KNOTS

In this chapter, we use the techniques introduced in Chapter 3 to analyze the bridge structure of cable knots. The study of bridge structure is motivated by the rich body of material concerning Heegaard structure of 3-manifolds. Analogous to a bridge splitting of a knot, a genus g Heegaard splitting of a 3-manifold M is the decomposition of M as $M = V \cup_\Sigma W$, where V and W are handlebodies and $\Sigma = V \cap W = \partial V = \partial W$ is a closed genus g surface. It is well known that every 3-manifold admits a Heegaard splitting and that any Heegaard splitting induces generic splittings of higher genus via a process known as stabilization.

Conversely, if a Heegaard splitting can be simplified to one of smaller genus, the splitting is called stabilized. As with bridge positions, we are often concerned with splittings of least complexity, and the Heegaard genus $g(M)$ of a 3-manifold M is the smallest g such that M admits a genus g splitting. In studying the Heegaard structure of 3-manifolds, an important question is the following:

Question 4.1. *What 3-manifolds M have the property that every non-minimal genus Heegaard splitting of M is stabilized?*

Waldhausen demonstrated that non-minimal genus splittings of S^3 are stabilized [32], and Bonahan and Otal show the same for lens spaces [2]. It should be noted that not all manifolds have this property; for example, Casson and Gordon have produced examples of manifolds M with infinitely many non-minimal genus splittings which are not stabilized [3].

Recall that a bridge position is said to be stabilized if the corresponding unique thick sphere admits a type I move. In this case, the bridge position reduces to one of smaller bridge number. In the language of bridge positions, Question 4.1 becomes

Question 4.2. *Which knots $K \subset S^3$ have the property that all non-minimal bridge positions of K are stabilized?*

In this direction, Otal reveals that if K is the unknot or a 2-bridge knot, then every non-minimal bridge position of K can be reduced in this way, and Ozawa has recently proved a similar statement for torus knots [17]. On the other hand, Ozawa and Takao have discovered a knot with a non-minimal bridge position which is not stabilized [20], which demonstrates that not all knots yield an affirmative answer to Question 4.2.

We devote this chapter to the proof of the fact that if J is an mp-small knot whose non-minimal bridge positions are stabilized and K is a (p, q)-cable of J, then K exhibits the same behavior. In particular, using the results of Otal and Ozawa, this shows that iterated cables of 2-bridge knots and iterated torus knots have this property.

4.1 Meridional smallness of cable knots

In the previous chapter, we showed that the property of thin position coinciding with minimal bridge position is preserved by cabling. Here we wish to show the same is true for meridional smallness and mp-smallness, defined in Chapter 1. We recall that to show a surface S with boundary is essential in a knot exterior $E(K)$, it suffices to show S is incompressible and not a ∂-parallel annulus.

Once again, we will let K denote a (p, q)-cable of J, with pattern $\hat{K}$ a (p, q)-torus knot contained in a solid torus V. In this setting, we will once again assume that $\hat{K} \subset \text{int}(V)$ despite the fact that K is a cable knot. Letting $C_{p,q} = V \setminus \eta(\hat{K})$, we observe that we can decompose $E(K)$ as $E(J) \cup C_{p,q}$, where the attaching map depends on the framing $\varphi : V \to S^3$ that maps a core of V to J. Following [8], we call $C_{p,q}$ a (p, q)-cable space. See Figure 4.1. A cable space falls into a well-behaved class

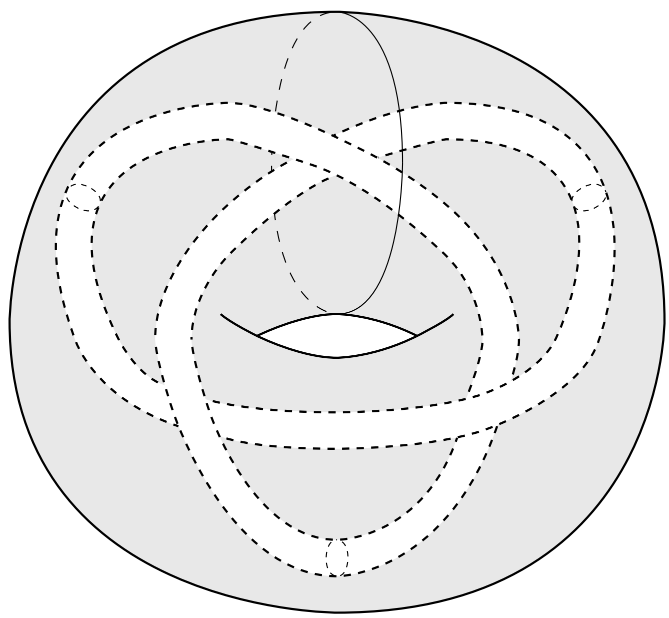

Figure 4.1: The cable space $C_{3,2}$.

of 3-manifolds known as Seifert fibered spaces, which loosely are 3-manifolds which can be decomposed as the disjoint union of simple closed curves. For a detailed treatment of Seifert fibered spaces, see for example [10] or [28].

Note that $C_{p,q}$ has a Seifert fibering with one exceptional fiber (a core of V), and it has two torus boundary components, ∂V (the outer boundary, denoted $\partial_+ C_{p,q}$) and $\partial\,\eta(\hat{K})$ (the inner boundary, denoted $\partial_- C_{p,q}$). In order to understand essential meridional surfaces in $E(K)$, we require the next lemma. We note that this lemma is implied by Lemma 3.1 of [8], but we do not need its full generality and so we provide an elementary proof. A surface in a Seifert fibered space is horizontal if it is transverse to all fibers and vertical if it is a union of fibers.

Lemma 4.1. *Suppose S is a connected incompressible surface in $C_{p,q}$, where $S \cap \partial_- C_{p,q} \neq \emptyset$ and each component of $S \cap \partial_- C_{p,q}$ is a meridian curve. Then S is a disk with q punctures, and $S \cap \partial_+ C_{p,q}$ is a meridian curve of V.*

Proof. Since $C_{p,q}$ is Seifert fibered, every incompressible surface is either horizontal or vertical by [10]. By assumption S has meridional boundary in $\partial_- C_{p,q}$, so S is horizontal. We can extend S to a horizontal surface S' in a Seifert fibered solid torus by attaching meridian disks to each component of $S \cap \partial_- C_{p,q}$ and gluing $\eta(\hat{K})$ back into $C_{p,q}$. But every connected horizontal surface in such a torus is a meridian disk, so S' is a meridian disk, from which it follows that S is a disk with q punctures and $\partial S \cap \partial_+ C_{p,q} = \partial S'$ is a meridian curve. □

On the other hand, let S' be a meridian disk of V intersecting $\hat{K}$ minimally, with $S = S' \cap C_{p,q}$. Then $C_{p,q}$ cut along S is homeomorphic to $S \times I$, from which it follows that S is incompressible in $C_{p,q}$. We use these facts in the proof of the next theorem.

Theorem 4.2. *For a knot J in S^3, the following are equivalent:*

(a) J is meridionally small.

(b) Every cable of J is meridionally small.

(c) There exists a cable of J that is meridionally small.

Proof. First we prove that (c) implies (a) or, equivalently, if J is not meridionally small and K is a cable of J, then K is not meridionally small. If J is not meridionally small then $E(J)$ contains an essential meridional surface S, and let K be a (p, q)-cable of J, so that $E(K) = E(J) \cup_T C_{p,q}$, where $T = \partial E(J)$. Then every component of ∂S bounds a disk with q punctures in $C_{p,q}$, and thus we can construct a meridional surface S' in $E(K)$ by gluing such a disk D to each boundary component of ∂S. We claim that S' is essential in $E(K)$. By the above, $S' \cap C_{p,q}$ is incompressible in $C_{p,q}$. Note that $|\partial S'| = q \cdot |\partial S| > 2$, so S' is not a ∂-parallel annulus. Suppose Δ is a compressing disk for S' in $E(K)$. If $\Delta \cap T = \emptyset$, then either $\Delta \subset C_{p,q}$, which is ruled out by the argument above, or $\Delta \subset E(J)$, which contradicts the incompressibility of S.

Thus, $\Delta \cap T \neq \emptyset$, and suppose Δ is chosen so that $|\Delta \cap T|$ is minimal. Since T is incompressible in $E(K)$, $\Delta \cap T$ contains no simple closed curves. Let α be an arc in $\Delta \cap T$ that is outermost in Δ, so that α cobounds a disk $\Delta' \subset \Delta$ with an arc $\beta \subset \partial\Delta$ such that $\mathrm{int}(\Delta') \cap T = \emptyset$. There are two cases to consider: First, suppose that $\Delta' \subset C_{p,q}$. Then both endpoints of α are contained in the same component of $S' \cap T$, which means that α is inessential in an annular component of $\overline{T \setminus S'}$ and cobounds a disk $\Delta'' \subset T$ with an arc $\gamma \subset S' \cap T$. Gluing Δ' to Δ'' along α yields a disk Δ^* with $\partial\Delta^* \subset S' \cap C_{p,q}$, and by the incompressibility of $S' \cap C_{p,q}$ in $C_{p,q}$, $\partial\Delta^* = \beta \cup \gamma$ bounds a disk D contained in $S' \cap C_{p,q}$. Sliding β along D, we can remove at least one intersection of Δ with T, contradicting the minimality of $|\Delta \cap T|$.

In the second case, suppose that $\Delta' \subset E(K)$. If α is an inessential arc in some annular component of T, the above argument holds. If α is essential, then Δ' is a ∂-compressing disk for S in $E(J)$, contradicting the assumption that S is essential. We conclude that S' is an essential meridional surface in $E(K)$, showing that (c) implies (a).

Now we show that (a) implies (b) or, equivalently, if some cable of J is not meridionally small then J is not meridionally small. Suppose that there exists a (p, q)-cable K of J such that $E(K)$ contains an essential meridional surface R. As above, $E(K) = E(J) \cup_T C_{p,q}$. Assume $|R \cap T|$ is minimal up to isotopy. We claim that $R \cap C_{p,q}$ is incompressible in $C_{p,q}$. Suppose not. Then there is a compressing disk $\Delta \subset C_{p,q}$ for $R \cap C_{p,q}$. Since R is incompressible in $E(K)$, $\partial\Delta$ bounds a disk $\Delta' \subset R$, and by isotopy we can replace Δ' with Δ and reduce the number of intersections of R with T, a contradiction. By Lemma 4.1, each component of $R \cap C_{p,q}$ is a punctured disk, and it follows that $R' = R \cap E(J)$ is a meridional surface. If R' is a ∂-parallel annulus in $E(J)$, we can lower $|R \cap T|$ via isotopy, contradicting the minimality of $|R \cap T|$. Likewise, if R' is compressible in $E(J)$, we can lower $|R \cap T|$ as in the case of $R \cap C_{p,q}$ above. Thus, R' is an essential meridional surface in $E(J)$, and (a) implies (b). Clearly, (b) implies (c), completing the proof. □

It should be noted that in the above proof, if either surface S or R is planar, then the induced surface S' or R' is also planar, yielding:

Theorem 4.3. *For a knot J in S^3, the following are equivalent:*

(a) J is mp-small.

(b) Every cable of J is mp-small.

(c) There exists a cable of J that is mp-small.

4.2 Destabilizing non-minimal bridge positions of cables

Recall from Chapter 1 that any bridge position has a unique thick sphere and no thin spheres, and we say two bridge positions l and l' with corresponding thick spheres $\hat{A}$ and $\hat{A}'$ are equivalent if there exists an isotopy f_t, called a bridge isotopy, taking l to l' which carries $\hat{A}$ to $\hat{A}'$. In this context a type I move is also called a destabilization. Since a type I move cancels a minimum and a maximum, it is clear that every minimal bridge position is not stabilized. For K a (p, q)-cable of J, we have the following:

Lemma 4.4. *Suppose that k is a bridge position. Then there exists $(k', T_{k'}) \in \mathcal{KT}$ such that k' is equivalent to k and $(k', T_{k'})$ is an efficient position, or k is stabilized.*

Proof. Let $(k, T_k) \in \mathcal{KT}$. By Lemma 2.4 of [17], there exists $(k', T_{k'}) \in \mathcal{KT}$ such that k' is equivalent to k and all saddles in the foliation of $T_{k'}$ by h are essential. Note that any disk slide of k' is supported in a neighborhood away from a bridge sphere;

thus, if k'' is the result of a disk slide on k', k'' is equivalent to k'. Now, as in Section 3.2, we split $T_{k'}$ into a collection $\{A_1, \ldots, A_r\}$ of vertical, upper, and lower annuli, where each component of $k' \cap A_i$ is an arc containing at most one critical point, or else a minimum and maximum of some arc can be canceled and k' is stabilized. If A_i is vertical and some arc component α of $k' \cap A_i$ is inessential, outermost in A_i, and contains a maximum, we can push α off A_i and onto the adjacent lower annulus A_j, where its maximum cancels a minimum of an arc component of $k' \cap A_j$. A similar argument holds if α contains a minimum or if α is an inessential arc in an upper or lower annulus, although in this case we may require a disk slide before pushing α. See Figure 3.4. We conclude either $(k', T_{k'})$ is an efficient position, or k' admits a destabilization. □

In [17], Makoto Ozawa shows that if K is a torus knot, then every non-minimal bridge position of K is stabilized. We essentially follow his proof, which we will summarize here, to show that if J is mp-small and every non-minimal bridge position of J is stabilized, then every cable K of J has the same property. Ozawa utilizes the following theorem, proved by Hayashi and Shimokawa [11] and later by Tomova [31]:

Theorem 4.5. *If $k \in \mathcal{K}$ is a bridge position admitting a type II move, then either k is stabilized or $E(K)$ contains an essential meridional planar surface.*

Thus, if K is mp-small, any bridge position admitting a type II move is stabilized.

From this point on, suppose J is mp-small. By Theorem 4.3, K is also mp-small. We recall that for $(k, T_k) \in \mathcal{KT}$, the solid torus bounded by T_k is denoted V_k.

Although [17] deals specifically with torus knots, some results apply in the setting of mp-small cable knots. Lemmas 3.2, 3.3, and 3.4 of [17] together imply

Lemma 4.6. *Suppose k is a bridge position. Then there exists $(k', T_{k'}) \in \mathcal{KT}$ an efficient position such that k' is equivalent to k and a bridge sphere $\hat{A} = h^{-1}(a_1)$ such that all upper saddles of $T_{k'}$ occur above $\hat{A}$, all lower saddles occur below $\hat{A}$, and $V_{k'} \cap \hat{A}$ is a collection of disks, or k is stabilized.*

Thus, h foliates the solid torus $V_{k'}$ by disks, and any such $(k', T_{k'})$ induces a bridge position j of J by taking j a longitude of $T_{k'}$ as constructed in Lemma 3.3. This fact also ensures that any upper or lower disk for j can be chosen to miss $\text{int}(V_{k'})$ and thus can easily be modified to an upper or lower disk for k'. Finally, we have

Theorem 4.7. *Suppose K is a (p, q)-cable with companion J, where J is mp-small.*

(a) If every non-minimal bridge position of J is stabilized, then every non-minimal bridge position of K is stabilized.

(b) The cardinality of the collection of minimal bridge positions of K does not exceed the cardinality of the collection of minimal bridge positions of J.

Proof. (a) Suppose $k \in \mathcal{K}$ is a bridge position. By Lemmas 4.4 and 4.6, we may pass to an equivalent bridge position and assume there exists $(k, T_k) \in \mathcal{KT}$ an efficient position such that h foliates V_k by disks, or else k is stabilized. Let $\hat{A}$ be a bridge sphere for k guaranteed by Lemma 4.6, and note that the induced bridge position j of J satisfies $b(j) = \frac{1}{2}|\hat{A} \cap T_k|$. There are two cases to consider: First, suppose that $|\hat{A} \cap T_k| = 2 \cdot b(J)$. Since each upper annulus contains q maxima, we have $b(k) = q \cdot b(J)$. By Theorem 1 of [26], $b(K) \geq q \cdot b(J)$, and thus k is a minimal bridge position of K. On the other hand, suppose that $|\hat{A} \cap T_k| > 2 \cdot b(J)$. Then j is a non-minimal bridge position for J, and by assumption j is stabilized. It follows that k is stabilized, finishing the first part of the proof.

(b) For each minimal bridge position j of J, assign an obvious cabling $k \subset \partial\eta(j)$ of j with $q \cdot b(J)$ maxima. It suffices to prove that the association $\varphi : j \mapsto k$ is surjective. First, suppose that k' is another obvious cabling of a bridge position of j' equivalent to j with $q \cdot b(J)$ maxima. Then there is a bridge isotopy taking j' to j, which induces an isotopy from $\partial\eta(j')$ to $\partial\eta(j)$ which is also a bridge isotopy of k'. Since k is isotopic to k' in $\partial\eta(j)$, we can remove intersections of k and k' and then push k' onto k by a bridge isotopy, and so φ is well-defined. Now, if k is a minimal bridge position of K, we may pass to an equivalent bridge position and assume there exists (k, T_k) such that h foliates V_k by disks by Lemma 4.6. Thus, the induced minimal bridge position j for J from Lemma 3.3 satisfies $\varphi(j) = k$, as desired. □

From [16] and [17], we have

Corollary 4.1. *If K is an n-fold cable of a torus knot or a 2-bridge knot, then any non-minimal bridge position of K is stabilized. Additionally, if K is an n-fold cable of a torus knot, it has a unique minimal bridge position, and if K is an n-fold cable of a 2-bridge knot, it has at most two minimal bridge positions.*

In [4], Coward proves that if J is a hyperbolic knot, then J has finitely many minimal bridge positions. Hence

Corollary 4.2. *If K is a (p, q)-cable of J, where J is hyperbolic and mp-small, then K has finitely many minimal bridge positions.*

CHAPTER 5
WIDTH COMPLEXES FOR KNOTS

Evidenced by the investigation in Chapter 4, a major component of cataloguing the bridge surfaces of a given knot is to determine the relationships between all possible bridge surfaces. The same can be said for Heegaard splittings, as well as for the Morse positions of a knot used to search for thin position. To this end, Schultens has defined the width complex of a knot in order to better understand these isotopies and the various positions a given knot can occupy in S^3 [27]. Specifically, she asks the following two questions:

Question 5.1. *Can the width complex of a knot have local minima that are not global minima?*

Question 5.2. *Is every vertex of the width complex of a knot connected to one of the global minima of this complex by a monotonically decreasing path?*

Schultens also defines a similar width complex for 3-manifolds, and her Theorem 13 from [27] provides a positive answer to the 3-manifold version of Question 5.1, namely that there exist 3-manifolds whose width complexes contain local minima that are not global minima. On the other hand, combining the results of [2], [21], and [32], we see that if M is S^3 or a lens space, then the width complex of M has a unique minimum, corresponding to a minimal genus Heegaard splitting. Thus, it seems reasonable to expect that the simplest knots might share this property. This is further suggested by Otal's proof that non-minimal bridge positions of the unknot and 2-bridge knots are stabilized [16] and Ozawa's recent proof of the same statement for torus knots [18].

Schultens compares Question 5.1 to one answered by Goeritz in 1934. Goeritz produced a nontrivial diagram of the unknot 0_1 such that any Reidemeister move increases the diagram's crossing number. As an analogue to Goeritz' result, Scharlemann poses the next question in his comprehensive treatment of thin position:

Question 5.3. *[21] Suppose $K \subset S^3$ is the unknot. Is there an isotopy of K to thin position (i.e. a single minimum and maximum) via an isotopy during which the width is never increasing?*

In this chapter, we draw on material from [36] to provide an answer, finding a nontrivial embedding of 0_1 such that any isotopy must increase the width of the embedding. As a result, we give an affirmative answer to Schultens' first question, which shows that the answer to the second question must be no. In fact, we show the surprising and much stronger result that for every knot K, the width complex of K has infinitely many local minima that are not global minima.

5.1 The Width Complex of K

Under a slight abuse of notation, we fix a knot K consider to $\mathcal{K}$ to be the collection of embeddings k of K up to the equivalence defined in Chapter 1. Now, we use the collection $\mathcal{K}$ and pairs of strict upper and lower disks to define the width complex of K, a directed graph Γ whose vertices correspond to elements of $\mathcal{K}$. We will introduce several more terms inspired by the study of Heegaard splittings: If a bridge position k admits a type II move, we call k weakly reducible. If k admits either a type I or type II move, we say k is reducible, and if not, k is strongly irreducible.

Elements of $k \in \mathcal{K}$ with reducible thick surfaces will be at the tail of directed edges in the width complex of K. If $k \in \mathcal{K}$ has a thick surface $\hat{A}_i$ which admits a type I move, recall there exist lower and upper disks D and E intersecting in a single point in k and along which we may cancel a minimum and maximum, changing k to $k' \in \mathcal{K}$ such that $w(k') = w(k) - (2|A_i \cap k| - 2)$. If k has a thick surface $\hat{A}_i$ admitting a type II move, there are disjoint lower and upper disks D and E along which we can construct an isotopy transforming k to $k' \in \mathcal{K}$, where $w(k') = w(k) - 4$. In either case, we call (D, E) a pair of reducing disks at $\hat{A}_i$ and we make a directed edge from k to k' in Γ. The next theorem, Theorem 1 from [27], is important to our understanding of the width complex:

Theorem 5.1. *The width complex of a knot is connected.*

This theorem says that given $k, k' \in \mathcal{K}$, there is a series of level isotopies and type I and type II moves and their inverses taking k to k'. Schultens' width complex also contains higher dimensional cells, but we need only consider the one-skeleton of the complex in this context.

We call $k \in \mathcal{K}$ a local minimum of the width complex if there are no directed edges leaving k in Γ. The position k is called a local minimum because any isotopy that changes k to $k' \in \mathcal{K}$ must increase $w(K)$. Let $\hat{\mathcal{K}} \subset \mathcal{K}$ denote the set of local minima of the width complex of K. It is clear that any thin position k for K must come from $\hat{\mathcal{K}}$; otherwise there is an isotopy decreasing $w(k)$. We also have the following, the proof of which is clear from the definition of the width complex:

Lemma 5.2. *An element $k \in \mathcal{K}$ is in $\hat{\mathcal{K}}$ if and only if every thick level of k is strongly irreducible.*

Using the definitions of this section, we can reformulate Schultens' questions as follows:

Question 5.4. *Is there a knot K with $k \in \hat{\mathcal{K}}$ such that $w(k) > w(K)$?*

Question 5.5. *Given $k \in \mathcal{K}$, is there a directed path in Γ starting at k and ending at a thin position for K?*

Explicitly, a directed path is a sequence of vertices $k = k_0, k_1, \ldots, k_n$ such that there is a directed edge from k_i to k_{i+1} for each $i < n$.

5.2 A local minimum in the width complex of the unknot

Let K be the unknot in S^3, and let $k \in \mathcal{K}$ be the position of the unknot depicted in Figure 1, where h is the standard height projection onto a vertical axis. We will label the thick/thin levels of k as $A_0, B_1, A_1, B_2, A_2, B_3, A_3, B_4, A_4$, as shown (dropping the $\hat{}$ notation for simplicity).

Theorem 5.3. *The pictured embedding k of the unknot is a local minimum in the width complex.*

Proof. By Lemma 5.2, it suffices to show that every thick surface of k is strongly irreducible. Observe that the regions between consecutive thin surfaces around thick

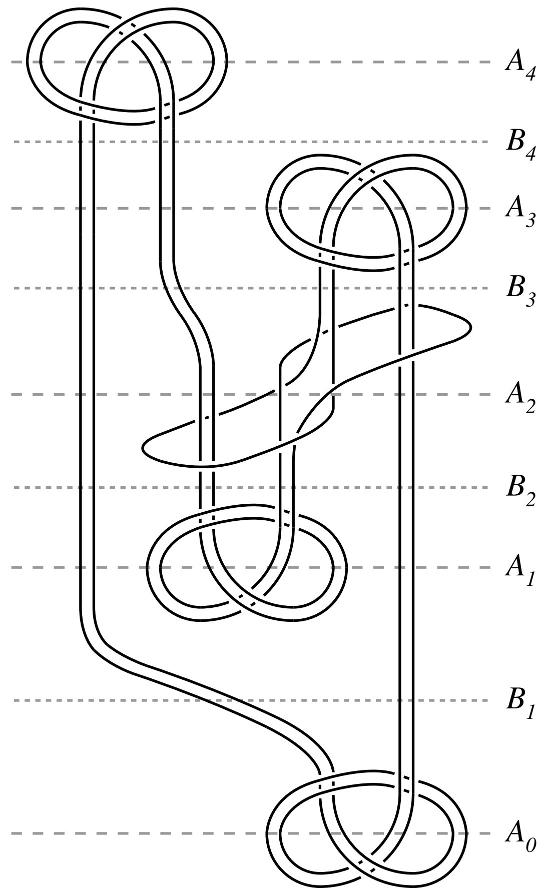

Figure 5.1: A troublesome embedding k of the unknot, shown with thick/thin levels

surfaces A_0, A_1, A_3, A_4 are identical except for the extra vertical segments passing through A_1 and A_3. Thus, we need only show that A_0 and A_2 are strongly irreducible.

Claim 1: A_0 is strongly irreducible. Suppose not. Then there is a pair of reducing disks (D, E) at A_0. Let b_1 denote the regular value corresponding to the thin level B_1. If we restrict our attention to $k^* = k \cap h^{-1}(-\infty, b_1]$, we can easily see that by adding two arcs to the four intersection points of k with B_1, we can complete k^* to a $(p, 2)$-cable of the trefoil for some p, whose thin position is minimal bridge position by Corollary 3.1, and such that A_0 becomes a bridge sphere. Thus, the pair

(D, E) of reducing disks at A_0 is also a pair of reducing disks at the bridge sphere A_0 of the trefoil's cable, a contradiction to the fact that this cable is in thin position. We conclude that A_0 and thus A_1, A_3, and A_4 are strongly irreducible.

Claim 2: A_2 is strongly irreducible. Let b_2 and b_3 be the regular values corresponding to B_2 and B_3, respectively. Then $\mathcal{A}_2 = h^{-1}([b_2, b_3])$ is homeomorphic to $S^2 \times I$, and $k' = k \cap \mathcal{A}_2$ has exactly one maximum contained in an arc κ_1 and exactly one minimum contained in an arc κ_2 properly embedded in $\mathcal{A}_2$. Note that $\mathcal{A}_2$ intersects six additional vertical segments, two of which extend from B_1 to B_4, call these γ_1 and γ_1', two of which extend from B_2 to B_4, call these γ_2 and γ_2', and two of which extend from B_1 to B_3, call these γ_3 and γ_3'.

As above, we will suppose A_2 is reducible and add extra arcs along the endpoints of components of k' to derive a contradiction. If A_2 is reducible, there is a pair of reducing disks (D, E) for k at A_2, where D contains the maximum of κ_1 and E contains the minimum of κ_2. Note that $\kappa_1 \cap \kappa_2 = \emptyset$, implying $D \cap E = \emptyset$. Thus by extending D down to B_2 and E up to B_3, we can find disjoint disks D' and E' such that $\partial D' = \kappa_1 \cup \delta$ for some level arc $\delta \subset B_2$ and $\partial E' = \kappa_2 \cup \eta$ for some level arc $\eta \subset B_3$. Note further that each pair of vertical arcs γ_i and γ_i' cobounds a rectangle R_i with level arcs $\iota_i \subset B_2$ and $\iota_i' \subset B_3$. After isotopy, we may assume that R_1, R_2, R_3, D', and E' are pairwise disjoint.

Now, as pictured in Figure 2, we add four arcs to $k' \cap B_2$ and four arcs to $k' \cap B_3$ to get a link k'', which is an unlinked square knot (the connected sum of two trefoils, one left-hand and one right-handed) and two unknots. In addition, we may attach the arcs so that two arcs of the square knot component cobound a rectangle R with ι_i and δ and two other arcs of this component cobound a rectangle R' with ι_i' and η, where these rectangles are disjoint from $\text{int}(R_1)$, $\text{int}(D')$, and $\text{int}(E')$. But this implies that the square knot component of k'' bounds a disk $D' \cup R \cup R_1 \cup R' \cup E'$, a contradiction. We conclude that A_2 is strongly irreducible, completing the proof.

□

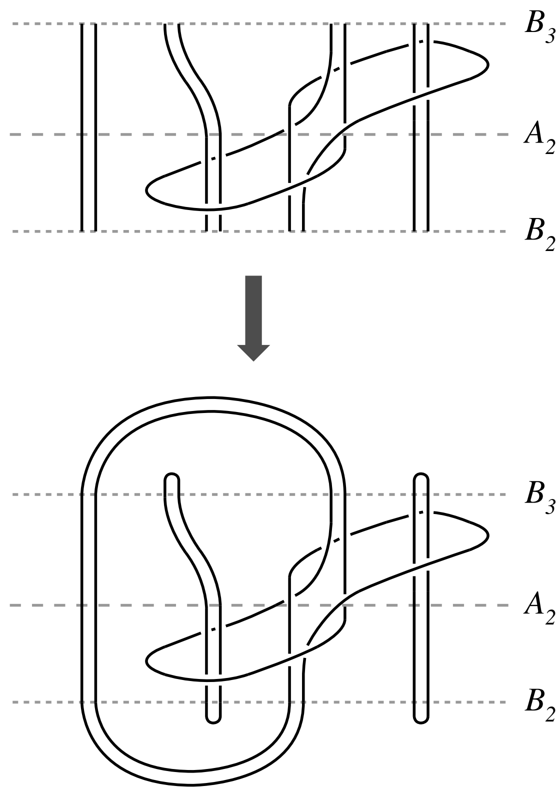

Figure 5.2: The tangle k' is shown at top, and the link k'' is shown at bottom. If A_2 is reducible, the square knot component of k'' is isotopic to the unknot.

5.3 Local minima in the width complex of an arbitrary knot

Suppose that k_1 and k_2 are embeddings representing local minima in the width complexes of knots K_1 and K_2. Then we can find an embedding k of $K_1 \# K_2$ by connecting the highest maximum of k_1 to the lowest minimum of k_2. Observe that this creates a new thin surface but does not interfere with the reducibility of the thick surfaces of k_1 and k_2. Thus, every thick surface of k is strongly irreducible, and by Lemma 5.2, k represents a local minimum in the width complex of $K_1 \# K_2$.

For instance, consider the projection of the figure eight knot 4_1 shown in Figure 3. Note that minimal bridge position is thin position for 4_1 by [30]. Here we have taken k_1 to be minimal bridge position of the figure eight knot and k_2 to be the unknot projection shown above, creating a new projection k of 4_1. Since every thick sphere is strongly irreducible, this projection is a local minimum in the knot's width complex. This suggests the following:

Corollary 5.1. *The width complex of every knot contains infinitely many local minima.*

Proof. Let K be an arbitrary knot, with embedding k representing a local minimum in the width complex of K. For any such k, we exhibit another local minimum k' of the width complex of K with $w(k') > w(k)$, showing that there are infinitely many such embeddings. Let K_0 denote the unknot, and let k_0 be the embedding representing the local minimum of the width complex in Theorem 5.3. Since $K \# K_0 = K$, we can attach k to k_0 by connecting the highest maximum of k to the lowest minimum of k_0 to get a new embedding k' of K with $w(k') > w(k)$. By the above argument, every

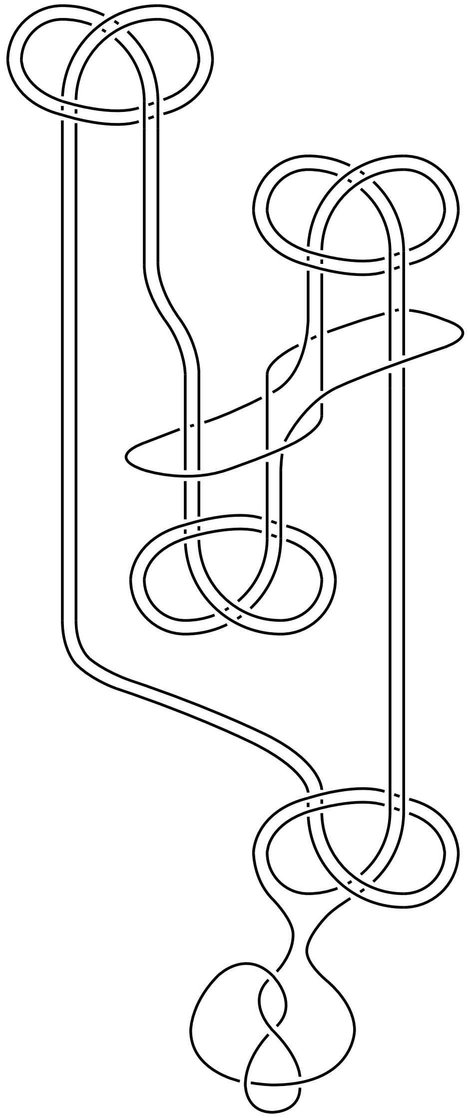

Figure 5.3: A local minimum in the width complex of the figure eight knot

thick sphere of k' is strongly irreducible, so k' is another local minimum in the width complex. $\square$

CHAPTER 6
FUTURE RESEARCH

The work described above has great potential to be expanded to other problems in the future. In this chapter I will briefly describe some problems that I have considered and will continue to consider stemming from the research contained in this thesis.

First, I would like to prove Conjecture 1.6, which is certainly not an easy task. Considerable work was required to prove the special cases above. We note that a proof of Conjecture 1.6 would imply Theorem 1.4, which itself comes from a complicated argument contained in a lengthy paper. In this direction, the idea of a height-preserving reimbedding is attractive; in fact, developing a height-preserving reimbedding was crucial in Scharlemann and Schultens' proof of Theorem 1.4. Roughly, a height-preserving reimbedding of a knot k is an embedding k' of a different knot such that minima and maxima occur of k and k' occur at the same critical values.

I will seek to answer the following question:

Question 6.1. *Suppose k is a thin position of a satellite knot K with companion J and pattern whose wrapping number is n. Does there exist a height-preserving reimbedding k' of k such that k' is a $(*, n)$-cable of J?*

This would prove Conjecture 1.6, as such a k and k' must satisfy

$$w(k) = w(k') \geq n^2 \cdot w(J).$$

One way to approach Question 6.1 is to first answer it in the case that the pattern $\hat{K}$ is a braid, possibly by finding a level meridian disk, cutting along it, and reattaching in a way that "unwinds" the braid while at the same time preserving height. Even in this case, we are not guaranteed that a level meridian disk necessarily intersects the knot minimally, which would be required for this sort of proof.

A second approach involves trying to find an isotopy that pushes k into the companion torus T without affecting $h(k)$; however, the counterexamples of Blair and Tomova show that this is not always possible [1]. In any case, Question 6.1 is both interesting and difficult.

The next question is also of interest:

Question 6.2. *Under what conditions does a knot K have a unique irreducible bridge position?*

By Ozawa, torus knots have such a bridge position [17], and by Theorem 4.1, iterated cables of torus knots have the same property. However, not every knot has a unique non-stabilized bridge position; in fact, Ozawa and Takao have released a preprint exhibiting a knot which has a non-minimal irreducible bridge position [20]. This relates to

Question 6.3. *For any n, is there an example of a knot K which has at least n irreducible bridge positions?*

I plan to attack a generalization of this question which involves generalized (g, b)-bridge splittings. A <u>(g, b)-bridge splitting</u> of a knot K in a manifold M is the expression of (M, K) as

$$(M, K) = (V, \alpha) \cup_\Sigma (W, \beta),$$

where V and W are genus g handlebodies and α and β are collections of b trivial arcs in V and W. A collection of arcs in a handlebody V is trivial if each is isotopic into ∂V. A bridge position with b bridges is easily seen to be a $(0, b)$-bridge splitting.

We consider two bridge splitting surfaces Σ and Σ' for a knot K to be equivalent if there is an isotopy of K which carries Σ to Σ'. In this context, question 6.3 generalizes to

Question 6.4. *For any n, is there an example of a knot K which has at least n irreducible bridge splitting surfaces?*

Here, a bridge splitting is irreducible if there does not exist a simplification of Σ which reduces either the genus of Σ or the number of arcs in α or β, or a simplification called meridional destabilization which transforms a (g, b)-surface to a $(g-1, b+1)$ surface. Question 6.4 may be approached by examining iterated cables of torus knots and bounding the genus g bridge numbers introduced by Doll [6] below by adapting machinery developed to deal with a special class of Heegaard surfaces known as strongly irreducible surfaces.

I will hopefully be able to use the machinery to answer the following question:

Question 6.5. *Let K be a (p, q)-torus knot in S^3. How many irreducible bridge splitting surfaces does K have?*

Conjecturally, the answer is two: A $(0, \min\{p, q\})$-splitting surface and a $(1, 0)$-

splitting surface. Compare this to a result of Scharlemann and Tomova which states that a 2-bridge knot has a unique irreducible bridge splitting surface [24].

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