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Equivariant cohomology and local invariants of Hessenberg varieties

Erik Andrew Insko
University of Iowa

EQUIVARIANT COHOMOLOGY AND LOCAL INVARIANTS OF HESSENBERG VARIETIES

by

Erik Andrew Insko

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

July 2012

Thesis Supervisor: Professor Julianna Tymoczko

ABSTRACT

Nilpotent Hessenberg varieties are a family of subvarieties of the flag variety, which include the Springer varieties, the Peterson variety, and the whole flag variety. In this thesis I give a geometric proof that the cohomology of the flag variety surjects onto the cohomology of the Peterson variety; I provide a combinatorial criterion for determing the singular loci of a large family of regular nilpotent Hessenberg varieties; and I describe the equivariant cohomology of any regular nilpotent Hessenberg variety whose cohomology is generated by its degree two classes.

Abstract Approved: ______________________________
Thesis Supervisor

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Title and Department

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Date

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Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Erik Andrew Insko

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the July 2012 graduation.

Thesis Committee: ______________________________
Julianna Tymoczko, Thesis Supervisor

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Charles Frohman

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Richard Randell

____
Maggy Tomova

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Daniel Anderson

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ABSTRACT

Nilpotent Hessenberg varieties are a family of subvarieties of the flag variety, which include the Springer varieties, the Peterson variety, and the whole flag variety. In this thesis I give a geometric proof that the cohomology of the flag variety surjects onto the cohomology of the Peterson variety; I provide a combinatorial criterion for determing the singular loci of a large family of regular nilpotent Hessenberg varieties; and I describe the equivariant cohomology of any regular nilpotent Hessenberg variety whose cohomology is generated by its degree two classes.

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CHAPTER 1
INTRODUCTION

1.1 Motivation

This thesis deals with the cohomology and singular loci of Hessenberg varieties. Hessenberg varieties are a family of closed subvarieties of the flag variety, which are of importance to combinatorists, geometers, and representation theorists.

Hessenberg varieties were introduced by DeMari-Shayman in order to efficiently compute the eigenvalues and eigenspaces of a linear operator X in questions relating to numerical analysis [DeMSha88]. They were defined as follows: Given a Lie algebra $\mathfrak{g}$ with Borel subalgebra $\mathfrak{b}$. Let H be a subspace of $\mathfrak{g}$ containing $\mathfrak{b}$. Let G and B be the linear algebraic groups corresponding to $\mathfrak{g}$ and $\mathfrak{b}$ respectively. The Hessenberg variety parameterized by the linear operator X and the space H is the set of all elements $gB \in G/B$ for which $g^{-1}Xg \in H$.

We classify Hessenberg varieties by the type of linear operator X used to define them. For instance, we say that a Hessenberg variety is *regular semisimple* when X is diagonalizable with distinct eigenvalues. We call a Hessenberg variety *nilpotent* if X is nilpotent and *regular nilpotent* if X is nilpotent with one Jordan block.

One of the seminal examples of Hessenberg varieties is the family the Springer varieties. (These are the Hessenberg varieties for which $H = \mathfrak{b}$.) The Springer variety for a nilpotent operator X is defined as the set of flags fixed by X. In 1976 Springer constructed irreducible representations of the Weyl groups on the cohomology of Springer varieties [Spr76]. More recently, this representation has been studied in knot theoretic contexts by Khovanov [Kho04], Russell [Rus09], and Russell-Tymoczko [RusTym11]. Borho and MacPherson generalized Springer varieties to the larger class of nilpotent Hessenberg varieties where $H = \mathfrak{p}$ is a parabolic subalgebra. They showed that the intersection cohomology of these Hessenberg varieties can also be viewed as representations of the Weyl group [BorMac83]. Related Weyl group representations for regular semisimple Hessenberg varieties have been identified by Teff [Tef11]. Shareshian and Wachs have studied representations arising in combinatorial contexts which are conjectured to be isomorphic with those studied by Teff [ShaWac11].

The geometry of Springer varieties has also attracted great interest to geometers and combinatorists. Spaltenstein identified the irreducible components of Springer varieties in Lie type-A_{n-1}. He proved that they are paved by affines and gave a combinatorial description of the cells in this paving [Spa76, Spa77, Spa82]. Vargas proved that Springer varieties can have singular irreducible components [Var79]. Recently, Fresse and Melnikov have classified which irreducible components of Springer varieties are singular in certain special cases [Fre10, FreMel10].

The findings of Borho-MacPherson and Teff suggest that the cohomology rings of more general Hessenberg varieties carry interesting Weyl group representations. In order to identify these representations, we first need to understand the cohomology of the Hessenberg varieties. The first step in understanding a space's cohomology is to identify the Betti numbers of the cohomology ring. One way to find these Betti numbers is to identify a nice cellular decomposition of the space. De Mari-Procesi-Shayman provided a cell decomposition of $Hess(X, H)$ when X is regular semisimple by utilizing a Białynicki-Birula decomposition [DeMProSha92]. Tymoczko extended this result by describing a cell decomposition for any linear operator X in classical Lie types [Tym07]. We have since proven that all regular nilpotent Hessenberg varieties are paved by affines [InsTym12].

These cellular decompositions tell us the Betti numbers of the cohomology groups, but they do not necessarily tell us anything about the ring structure of the cohomology. Identifying the cohomology of regular nilpotent Hessenberg varieties has been an open question for the past 10 years. Several special cases have been computed. Brion and Carrell proved that the S^1-equivariant cohomology of the Peterson variety is the coordinate ring of a particular affine curve [BriCar04]. More recently, Harada and Tymoczko provided the first known explicit computation of the S^1-equivariant cohomology of the Peterson varieties with generators and relations [HarTym09]. Bayegan-Harada described the S^1-equivariant cohomology for one other family of Hessenberg varieties [BayHar10]. Mbirika used truncated elementary symmetric functions to describe a ring which is conjectured to be isomorphic to the cohomology ring of regular nilpotent Hessenberg varieties [Mbi10]. However, an explicit description of the cohomology ring of regular nilpotent Hessenberg varieties has remained unattainable for all but these few cases.

Identifying the equivariant cohomology of regular nilpotent Hessenberg varieties is one of the main goals of this paper. Chapter 3 uses intersection theory to prove that the cohomology of the flag variety surjects onto the cohomology of the Peterson variety. Chapter 5 describes the $\mathbb{C}^*$-equivariant cohomology ring of any regular nilpotent Hessenberg variety for which the cohomology of the flag variety surjects onto its cohomology.

In addition to the Springer varieties, there is another family of Hessenberg varieties that has received much attention in the past 15 years. These varieties have become known as Peterson varieties; named in honor of Dale Peterson. In a lecture series at MIT, Peterson announced that the quantum cohomology of the full flag variety is isomorphic to a dense open subset of this Hessenberg variety. This remarkable discovery led Kostant to study the geometric structure of the Peterson variety [Pet97]. Kostant partially identified the singular locus of the Peterson variety and proved that Peterson varieties are generally not normal [Kos96, Theorem 14]. Joint work with Yong gave a combinatorial description of the singular locus of the Peterson variety and its Peterson-Schubert subvarieties [InsYon11]. This description generalizes Konstant's work and is much easier to compute. Chapter 4 of this thesis generalizes the joint work with Yong to a much larger family of Hessenberg varieties.

1.2 Overview

In Chapter 2, we provide definitions and background information about Hessenberg varieties, torus actions, and Białynicki-Birula decompositions. We also include quick introductions on algebraic geometry and equivariant cohomology.

In Chapter 3, we describe a cell decomposition of regular nilpotent Hessenberg varieties. We use this decomposition to calculate products in the cohomology ring of the flag variety using intersection theory. These calculations partially compute the expansions of the homology classes of Peterson-Schubert varieties in the basis of Schubert classes. Using Poincaré duality, these intersection products also compute the cohomology classes of Peterson-Schubert varieties in terms of the Schubert class basis. These partial expansions allow us to prove that cohomology ring of the flag variety surjects onto the cohomology ring of the Peterson variety. (Chapter 3 presents joint work with Julianna Tymoczko.)

In Chapter 4, we define patch ideals and use them to compute the singular loci of many regular nilpotent Hessenberg varieties. The description of the singular locus is combinatorial and uses the cell decomposition described in Chapter 3. We then prove some new combinatorial formulas relating the geometry of regular nilpotent Hessenberg varieties to diagrams on an $n \times n$ grid. (Section 4.1 presents joint work with Alex Yong.)

In Chapter 5 we give a combinatorial description of the equivariant cohomology ring of regular nilpotent Hessenberg varieties whose cohomology is generated by its degree two classes. (We conjecture that every regular nilpotent Hessenberg variety has this property.) The heart of the proof is a combinatorial description of the fixed points that are contained in each Hessenberg variety.

CHAPTER 2
DEFINITIONS AND BACKGROUND

This chapter contains the preliminaries needed to prove the main results of Chapters 3-5. We begin by introducing the general vocabulary of varieties and schemes in Section 2.1. In Sections 2.2 and 2.3 we discuss Hessenberg varieties and Schubert cells in general Lie type. We treat torus actions and Białynicki-Birula decompositions in Section 2.5. In Section 2.6 we finish with an exposition of equivariant cohomology.

2.1 A primer on algebraic varieties and schemes

This section contains a brief introduction to some of the vocabulary of algebraic geometry. We assume that the reader has skimmed the first two chapters of Hartshorne at some point in his or her life. In particular, we assume that the reader knows what a sheaf and a stalk are.

2.1.1 Affine varieties

A polynomial f in the ring $\mathbb{C}[x_1, ..., x_n]$ can be viewed as $\mathbb{C}$-valued function on $\mathbb{C}^n$ by evaluating f at the points in $\mathbb{C}^n$. For each set S of polynomials in $\mathbb{C}[x_1, ..., x_n]$ define the *zero-locus* $Z(S)$ as the set of points in $\mathbb{C}^n$ on which the functions in S all simultaneously vanish, namely:

$$Z(S) = \{x \in \mathbb{C}^n : f(x) = 0 \text{ for all } f \in S\}.$$

We call a subset V of $\mathbb{C}^n$ an *affine algebraic set* if $V = Z(S)$ for some S. A nonempty affine algebraic set V is called *irreducible* if it cannot be written as the union of two proper algebraic subsets. We call an affine algebraic set an *affine variety.* (Many authors, including Hartshorne, assume that affine varieties are also irreducible, but we do not make this distinction here. In particular, Springer varieties are not irreducible.) Affine varieties are endowed with the *Zariski topology* by declaring the closed sets to be precisely the affine algebraic sets. Given a subset V of $\mathbb{C}^n$ we define $I(V)$ to be the ideal of all functions vanishing on V:

$$I(V) = \{f \in \mathbb{C}[x_1, \ldots, x_n] : f(x) = 0 \text{ for all } x \in V\}.$$

For any affine algebraic set V the coordinate ring of V is the quotient of the polynomial ring by this ideal.

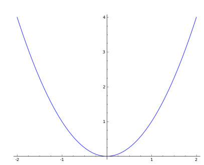


Figure 2.1: Zero-locus of $y - x^2$

Example 2.1. The zero-locus of the polynomial $g(x, y) = y - x^2$ in $\mathbb{C}[x, y]$ defines an affine algebraic variety. The real part of that zero-locus is the parabola defined by $y = x^2$ in $\mathbb{R}^2$. The coordinate ring of that variety defined by $V = Z(g)$ is

$\mathbb{C}[x,y]/\langle y-x^2\rangle$. ⊞

Example 2.2. The polynomial $f(x,y) = xy$ in $\mathbb{C}[x,y]$ defines a reducible affine algebraic set. It is the union of the x and y axes. The polynomial $f(x,y) = x^2 - y^3$ in $\mathbb{C}[x,y]$ defines a singular variety. The singularity occus at $x = y = 0$. ⊞

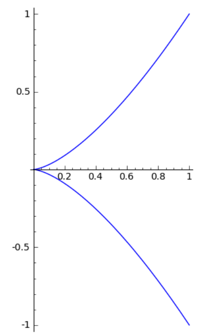


Figure 2.2: Zero-locus of $x^2 - y^3$

2.1.2 Projective varieties

Let $\mathbb{CP}^n$ denote the n-dimensional projective space over $\mathbb{C}$. Let $f \in \mathbb{C}[x_0,\ldots,x_n]$ be a homogeneous polynomial of degree d, i.e. all of the terms in f have degree d. Because f is homogeneous, $f(\lambda x_0,\ldots,\lambda x_n) = \lambda f(x_0,\ldots,x_n)$. Hence it makes sense to ask whether f vanishes at a point $[x_0 : \cdots : x_n]$. For each set S of homogeneous polynomials, define the zero-locus of S to be the set of points in $\mathbb{CP}^n$ on which the functions in S vanish:

$$Z(S) = \{x \in \mathbb{C}^n : f(x) = 0 \text{ for all } f \in S\}.$$

A subset V of $\mathbb{CP}^n$ is called a projective algebraic set if $V = Z(S)$ for some set S of homogeneous polynomials. A projective algebraic set is also called a *projective variety.* Projective varieties are equipped with the Zariski topology by declaring all projective algebraic sets to be closed. Given a subset V of $\mathbb{CP}^n$, let $I(V)$ be the ideal generated by all homogeneous polynomials vanishing on V. The coordinate ring of a projective algebraic set V is the quotient of the polynomial ring by this ideal.

2.1.3 Schemes

A *ringed space* $(V, \mathcal{O}_V)$ is a topological space V together with a sheaf of rings $\mathcal{O}_V$ on V. We refer to V as the *underlying topological space* associated with the ringed space $(V, \mathcal{O}_V)$. The sheaf of rings $\mathcal{O}_V$ is called the *structure sheaf* of V. A *locally ringed space* is a ringed space $(V, \mathcal{O}_V)$ such that all stalks of $\mathcal{O}_V$ are local rings (i.e. they have unique maximal ideals). Colloquially, a *ringed space* is a space together with a collection of commutative rings. The elements of these rings are "functions" on each open set of the space.

The *prime spectrum* $\mathrm{Spec}(R)$ of a ring R is the collection of all prime ideals. An *affine scheme* is a locally ringed space $(V, \mathcal{O}_V)$ whose underlying topological space V is isomorphic to $\mathrm{Spec}\left(\frac{\mathbb{C}[x_1,\ldots,x_n]}{I}\right)$ for some ring $\frac{\mathbb{C}[x_1,\ldots,x_n]}{I}$. A *scheme* is a locally ringed space $(V, \mathcal{O}_V)$ admitting a covering by open sets U_i such that the restriction of the structure sheaf $\mathcal{O}_V$ to each U_i is an affine scheme. Just as a smooth manifold is covered by smooth "coordinate charts," we may think of a scheme as being covered by "coordinate charts" or "coordinate patches" of affine schemes.

Recall that a ring R is called a *reduced ring* if it has no non-zero nilpotent elements. A scheme $(V, \mathcal{O}_V)$ is a called *reduced* if all of the rings $\mathcal{O}_V(U)$ are reduced rings. Equivalently, none of its rings of sections $\mathcal{O}_V(U)$ for any open subset U of V has any nonzero nilpotent element. The underlying topological space of an affine scheme defines an affine variety. There are many schemes which have the same underlying topological space as illustrated in Example 2.3. However, there is exactly one reduced scheme $(V, \mathcal{O}_V)$ with a particular affine variety V as its underlying topological space.

Example 2.3. The following schemes all have the same underlying topological space:

- Spec $\left(\frac{\mathbb{C}[x]}{\langle x \rangle}\right)$
- Spec $\left(\frac{\mathbb{C}[x]}{\langle x^2 \rangle}\right)$
- and Spec $\left(\frac{\mathbb{C}[x]}{\langle x^3 \rangle}\right)$.

Their underlying topological space is the origin 0 in Spec$(\mathbb{C}[x]) \cong \mathbb{C}$. However, the schemes Spec $\left(\frac{\mathbb{C}[x]}{\langle x^2 \rangle}\right)$ and Spec $\left(\frac{\mathbb{C}[x]}{\langle x^3 \rangle}\right)$ are not reduced schemes. The ring $\frac{\mathbb{C}[x]}{\langle x^2 \rangle}$ contains the nilpotent element x. The ring $\frac{\mathbb{C}[x]}{\langle x^3 \rangle}$ contains the nilpotent elements x and x^2. $\boxplus$

There are two major lessons to take away from this discussion:

- Non-reduced schemes are one of the major generalizations from varieties to schemes, and
- When discussing the coordinate ring $\mathbb{C}[V]$ of an affine variety V, we assume that $\mathbb{C}[V]$ is in fact a reduced ring.

The main results of Chapter 4 will concern the following terms. An affine variety $V = \mathrm{Spec}\left(\frac{\mathbb{C}[x_1,\ldots,x_n]}{I(V)}\right)$ of dimension m is called a *complete intersection* if the ideal $I(V)$ is generated by $n-m$ polynomial generators. In other words, the codimension of the variety V in $\mathbb{C}^n$ is equal to the number of generators of the ideal $I(V)$. A *local complete intersection* is a variety for which each local ring is a complete intersection. An affine scheme is called *generically reduced* if its localization at any minimal prime ideal is reduced.

2.2 Hessenberg varieties

We begin with the original definition of a Hessenberg variety $Hess(X, H)$ given by DeMari-Shayman [DeMSha88]: Given a Lie algebra $\mathfrak{g}$ with a Borel subalgebra $\mathfrak{b}$, a *Hessenberg space* H is a $\mathfrak{b}$-submodule of $\mathfrak{g}$ which contains $\mathfrak{b}$. For a fixed element X in $\mathfrak{g}$, and let $G = exp(\mathfrak{g})$ be the linear algebraic group associated to $\mathfrak{g}$. We consider the group elements $g \in G$ such that $\mathrm{Ad}(g^{-1})(X)$ is contained in H. This gives a subset $G(X, H)$ of the linear algebraic group G. Since the Hessenberg space H is closed under conjugation by the elements of the Borel subgroup B corresponding to $\mathfrak{b}$, the subset $G(X, H)$ is closed under right multiplication by elements of B. Thus the image of $G(X, H)$ in the flag variety G/B is a closed projective subvariety $Hess(X, H)$ of G/B. This subvariety $Hess(X, H)$ is the *Hessenberg variety* corresponding to X and H.

Much of this thesis will consider only Hessenberg varieties in type A_{n-1}. In type A_{n-1}, a coset gB can be associated to a *flag*, a nested sequence of vector subspaces $F_\bullet = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n$, by taking F_i to be the span of the first i columns of g. In these terms, the definition of a Hessenberg variety given above is equivalent to the following. Let $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a weakly increasing function satisfying $h(i) \geq i$ for all i. Then the Hessenberg variety is the set of all flags such that X maps the ith subspace into the $h(i)$th subspace:

$$Hess(X, h) = \{F_\bullet : Flags(\mathbb{C}^n) : XF_i \subset F_{h(i)} \text{ for all } 1 \leq i \leq n\}.$$

We will use the notation $Hess(X, H)$ when referring to Hessenberg varieties in general Lie type and use $Hess(X, h)$ when we wish to restrict to type A_{n-1}. We will deal exclusively with regular nilpotent Hessenberg varieties. In type A_{n-1} that means the linear operator X is nilpotent with one Jordan block.

2.3 Schubert cells in general Lie type

The following is a brief recap of the vocabulary regarding Schubert cells in G/B where G is any linear algebraic group. It assumes the reader has prior knowledge of Borel subgroups and root space decompositions of linear algebraic groups. Fix a pair of opposite Borel subgroups B and B^- of G so that $B \cap B^- = T$ is a maximal torus. Let G/B denote the flag variety and let $W = N_G(T)/T$ denote the Weyl group of G. (The reader who is unfamiliar with the general theory of linear algebraic groups can take G to be $GL_n(\mathbb{C})$, $B \subset G$ to be the subgroup of upper-triangular matrices, $T \subset B$ to be the diagonal invertible matrices, and $W \subset G$ to be the subgroup of permutation matrices.)

The Bruhat decomposition theorem states that for any Borel subgroup $B \subset G$ the group G can be partitioned into double cosets $G = \coprod_{w \in W} BwB$. Passing to the quotient G/B one obtains a decomposition of the flag variety G/B into cosets $G/B = \coprod_{w \in W} BwB/B$ called *Schubert cells*. The closures $X_w = \overline{BwB/B}$ are the *Schubert subvarieties* of the flag variety.

Let $\mathfrak{g}, \mathfrak{b}$, and $\mathfrak{t}$ denote the Lie algebras of G, B and T respectively. We denote the adjoint action of $g \in G$ on $X \in \mathfrak{g}$ by

$$Ad(g)(X) = gXg^{-1}.$$

Let Φ denote the roots of $\mathfrak{g}$ and Φ^+ the roots corresponding to $\mathfrak{b}$. There is a partial order on Φ^+ given by the rule that $\alpha > \beta$ if and only if $\alpha - \beta$ is a sum of positive roots. The subalgebra $\mathfrak{b}$ determines a base $\Delta \subseteq \Phi^+$ of simple positive roots $\alpha_i \in \Delta$. We denote the root space corresponding to α by $\mathfrak{g}_\alpha$. We denote the root subgroup corresponding to α by U_α. The maximal unipotent subgroups of B and B^- are denoted U and U^- respectively. For each root subgroup $U_\alpha \subseteq U$ we fix a group homomorphism denoted $u_\alpha : \mathbb{C} \to U_\alpha$.

We may choose a set of coset representatives for the Schubert cell BwB/B using the subgroup

$$U_w = \{u \in U : w^{-1}uw \in U^-\}.$$

Lemma 2.1. *[Hum64, Theorems 28.3, 28.4] The Schubert cell BwB/B is isomorphic to U_w*

Lemma 2.1 implies that each Schubert cell BwB/B is isomorphic to the affine space $U_w \cong \mathbb{C}^{\ell(w)}$ where $\ell(w)$ is the number of positive roots α that are mapped to negative roots by w^{-1}. In 1958, Chevalley proved that the Schubert cells $\{BwB/B\}$ in G/B form a CW-decomposition of the flag variety [Che58]. While Hessenberg varieties do not always have nice CW-decompositions, they do have a more general cellular decomposition called a *paving by affines.*

Definition 2.2. A *paving* of an algebraic variety V is an ordered partition into disjoint cells $V_0, V_1, V_2, \ldots$, such that every finite ordered union

$$\coprod_{i=0}^{j} V_i$$

is Zariski-closed in V. A *paving by affines* is a paving where each cell V_i is homeomorphic to affine space $V_i \cong \mathbb{C}^{d_i}$.

A paving by affines of a space X is useful because the cells of the paving correspond to the generators of the cohomology ring $H^*(X;\mathbb{Z})$. In joint work with Tymoczko, we proved that regular nilpotent Hessenberg varieties are paved by affines in all Lie types [InsTym12]. The cells in these pavings come from intersecting Schubert cells with Hessenberg varieties.

2.4 Simple reflections, parabolic subgroups, and Bruhat order

For any linear algebraic group G and maximal algebraic torus $T \subset G$, the corresponding Weyl group $W = N_G(T)/T$ is generated by *simple reflections.* The *simple reflection* s_α of W is the lone element which maps the positive simple root α to $-\alpha$ and maps all other positive roots to positive roots. There is one such simple reflection for each simple root $\alpha \in \Delta^+$. Every element $w \in W$ can be written as a product (or word) of simple reflections. We say the word of simple reflections for w is *reduced* when it can be written with no fewer simple reflections. We call the number of simple reflections in a reduced word for w, the *length* of w. While, the number of simple reflections in a reduced word for w is unique, there are often several reduced words for a given Weyl group element w. For instance $s_1 s_3 = s_3 s_1$ are both reduced words for the permutation $w = 2143$.

A *parabolic subgroup* P of G is any subgroup which contains a Borel subgroup B. Let $\mathfrak{b}$ and $\mathfrak{p}$ denote the Lie algebras of B and P respectively. Then each parabolic Lie algebra $\mathfrak{p}$ has a root space decomposition of the form

$$\mathfrak{p} = \mathfrak{b} \bigoplus_{\alpha \in J} \mathfrak{g}_\alpha$$

where $J \subset \Delta^-$ is a collection of negative simple roots. For this reason, a subgroup $W_J \leq W$ generated by the simple reflections s_α for α in J is called a *parabolic subgroup of W*.

The reader may find it interesting that not even the experts know how parabolic subgroups got their name. There is a post on Mathoverflow.net discussing two different conjectures:

1. It has something to do with parabolic elements of $SL(2, \mathbb{R})$.

2. "Parabolic" is short for "para-Borelic," meaning "containing a Borel subgroup."

After a lengthy discussion, Jim Humphreys, Benoît Kloeckner, Timothy Chow, and Gjergji Zaimi decided that both seem quite plausible.

The *Bruhat order* on W is defined by stipulating that $u \leq w$ if any reduced word for u is a subword of any reduced word for w. Bruhat order has an interesting geometric interpretation. The closure of a Schubert cell $\overline{BwB/B}$ is the union of all Schubert cells BuB/B where $u \leq w$ in Bruhat order

$$X_w = \overline{BwB/B} = \coprod_{u \leq w} BuB/B.$$

There are two important elements of each parabolic subgroup W_J. The *longest word*, denoted w_J in W_J and w_0 in W, is the unique element of maximal length with respect to a chosen generating set J. A *Coxeter element* is a product of all simple reflections. In this paper we will focus attention on the particular Coxeter element of W_J with $s_{i_1} s_{i_2} \cdots s_{i_k}$ where $i_1 < i_2 < \cdots < i_k$ and denote this Coxeter element by v_J.

2.5 Torus Actions and Białynicki-Birula decompositions

Let $T = B \cap B^- \cong (\mathbb{C}^*)^n$ denote the maximal torus in G. The torus T acts on G/B by left multiplication, and this action restricts to an action on each Schubert subvariety X_w of G/B. The T-fixed point set in G/B is the set of flags $\{wB : w \in W\}$ corresponding to the elements of the Weyl group. The T-fixed points in X_w are in the set $\{u \in W : u \leq w\}$ in the Bruhat order.

On the other hand, the T-action on G/B does not preserve the regular nilpotent Hessenberg varieties, as Examples (2.4) and (2.5) demonstrate. Not all is lost; Harada-Tymoczko showed that there are one-dimensional subgroups $S \cong \mathbb{C}^* \subset T$ of the torus that act on regular nilpotent Hessenberg varieties in all Lie types, and that the S-fixed points in $Hess(X, H)$ are precisely the T-fixed points in G/B which are contained in $Hess(X, H)$. In other words, $G/B^T \cap Hess(X, H) = Hess(X, H)^S$ [HarTym10, Lemma 5.1].

Example 2.4. A sufficient condition for S to preserve a Hessenberg variety. The action of a torus S will preserve the Hessenberg variety $Hess(X, h)$ if

$$Ad(s^{-1})(N) = c \cdot N$$

for all $s \in S$ and constant $c \in \mathbb{C}^*$. ⊞

Example 2.5. Hessenberg varieties are not preserved by the full torus action. Consider the regular nilpotent Hessenberg variety $Hess(X, h)$ corresponding to the Hessenberg function $h(1, 2, 3) = (2, 3, 3)$. Let T be the set of diagonal matrices $T = \{(t_1, t_2, t_3) : t_i \in \mathbb{C}^*\}$, and consider the action of T on the big cell $Bw_0B/B \cap Hess(X, h)$ in the Hessenberg variety $Hess(X, h)$:

$$\mathbf{t} \cdot gB = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \cdot \begin{pmatrix} a & b & 1 \\ b & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B = \begin{pmatrix} \left(\frac{t_1}{t_3}\right) a & \left(\frac{t_1}{t_2}\right) b & 1 \\ \left(\frac{t_2}{t_3}\right) b & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B.$$

For $\mathbf{t} \in T$ if $\frac{t_1}{t_2} \neq \frac{t_2}{t_3}$ then $\frac{t_2}{t_3} b \neq \frac{t_1}{t_2} b$. This implies that X times the first column of g is not contained the span of the first two columns of g. Hence the point $\mathbf{t} \cdot gB$ is not contained in the Hessenberg variety. $\boxplus$

Torus actions are useful in the study of complex projective varieties for a number of reasons. The following result describes how a torus action leads to a decomposition, called a *Białynicki-Birula decomposition*, of a projective T-variety into disjoint subsets.

Theorem 2.3. *[Bia-Bir76, Theorem 4.4] Let T be a torus that acts on a smooth projective variety X with isolated fixed points $\{p_1, \ldots, p_k\}$. Let $\lambda : \mathbb{C}^* \to T$ be a group homomorphism. Define the cells*

$$BB^+_{p_i} = \left\{ x \in X \mid \lim_{z \to 0} \lambda(z)x = p_i \right\}$$

and

$$BB^-_{p_i} = \left\{ x \in X \mid \lim_{z \to \infty} \lambda(z)x = p_i \right\}$$

as the collection of points lying on $\mathbb{C}^*$-invariant curves that limit into or out of p_i respectively. Then there are two canonical decompositions of X as $X = \coprod BB^+_{p_i}$ and $X = \coprod BB^-_{p_i}$. These cells are is isomorphic to affine spaces $BB^+_{p_i} \cong \mathbb{C}^{\ell_i}$ (respectively $BB^-_{p_i} \cong \mathbb{C}^{m_i}$) and each cell contains exactly one $\mathbb{C}^*$-fixed point.

Colloquially, the action of $\mathbb{C}^* \subset T$ on a complex projective variety X induces a vector flow on the variety. Theorem 2.3 says that when X is a smooth variety, X can be partitioned into affine cells, where each cell consists of points whose vectors flow toward a given $\mathbb{C}^*$-fixed point.

The following result says that if one picks the appropriate $\mathbb{C}^*$-action on the flag variety, then the Białynicki-Birula decomposition agrees with the Schubert cell decomposition.

Proposition 2.4. *[Aky81, p. 546] Suppose $\lambda : \mathbb{C}^* \to T$ is a group homomorphism which acts on G/B with $\langle \alpha, \lambda \rangle > 0$ for all $\alpha \in \Phi^+$. Then $(G/B)^{\mathbb{C}^*} = (G/B)^T$ and the Białynicki-Birula decomposition $\{BB_w^- : w \in W\}$ agrees with the Schubert cell decomposition, in the sense that $BB_w^- = B^- wB/B$.*

It turns out that the one-dimensional subtorus $S \subseteq T$ identified by Harada and Tymoczko has a group homomorphism $\lambda : S \to T$ which satisfies $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Phi^+$. Thus, the Białynicki-Birula decomposition of S agrees with the Schubert cell decomposition $BB_w^+ = BwB/B$ [HarTym10, Lemma 5.1]. This implies that the paving by affines identified in Theorem 3.2 is a generalization of the Białynicki-Birula decomposition to the regular nilpotent Hessenberg varieties.

2.6 Equivariant cohomology and equivariant formality

In this section we assume V is a topological space with group action denoted by $G \cdot V \to V$. The G-equivariant cohomology $H_G^*(V)$ is a cohomology theory which encodes both topological data about the space V and information on the group action of G.

We say that a group G acts *freely* on V when the stabilizer of every point in V is the subgroup $\{e\}$ of G. We say that G acts *transitively* on V if V is the orbit $Gx = \{gx : g \in G\}$ for any point $x \in V$. When G acts freely on V the equivariant cohomology $H^*_G(V)$ is defined as the singular cohomology of the quotient space $H^*(V/G)$. For instance, suppose $V = S^1 \cong \{e^{i\theta} \in \mathbb{C}\}$ is the unit circle sitting inside of the complex plane $\mathbb{C}$ and $G = S^1 \cong \{e^{i\beta} \in \mathbb{C}\}$ is another unit circle acting on V by $e^{i\beta} \cdot e^{i\theta} = e^{i(\beta+\theta)}$. In this case

$$H^i_G(V;\mathbb{C}) = H^i(S^1/S^1;\mathbb{C}) = H^i(pt;\mathbb{C}) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i > 0 \end{cases}$$

Thus the equivariant cohomology reflects the fact that the quotient space V/G is a point.

For a more complicated example, let $V = S^1 \times S^1$ and $G = S^1$, and let $G \times V \to V$ be defined by $e^{i\beta} \cdot (e^{i\theta_1}, e^{i\theta_2}) = (e^{i\theta_1}, e^{i(\beta+\theta_2)})$. In this instance,

$$H^i_G(V) = H^i(S^1 \times S^1/S^1) \cong H^i(S^1) \cong \begin{cases} \mathbb{C} & i = 0, 1 \\ 0 & i > 1. \end{cases}$$

The equivariant cohomology encodes the information that $V = S^1 \times S^1$ is the union of an S^1-worth of G-orbits.

In contrast with the last example, let $G = S^1$ and $V = S^1 \times S^1$. Define $G \cdot V \to V$ by $e^{i\beta} \cdot (e^{i\theta_1}, e^{i\theta_2}) = (e^{i(\beta+\theta_1)}, e^{i(\sqrt{2}\beta+\theta_2)})$. In this case the action of G is transitive. This is reflected by the fact that the equivariant cohomology is

$$H^i_G(V) = H^i(pt) \cong \begin{cases} \mathbb{C} & i = 0 \\ \\ 0 & i > 0. \end{cases}$$

Thus we see how the G-equivariant cohomology of a space V encodes information the topology of V and the action of G on V.

Of course most group actions are not free. An example of such an action is the circle S^1 rotating the sphere S^2 around an axis. This action has two fixed points at the north and south poles of this axis. In this instance, the topological quotient S^2/S^1 is a line segment. Thus its cohomology is trivial. Our original definition of equivariant cohomology fails to capture any meaningful information about the group action and the space.

Borel generalized this the definition of equivariant cohomology to other group actions [Bor53]. The idea is to construct a space that is both homotopically equivalent to V and carries a free action. To do this, we take a contractible space EG on which G acts freely and form the space $V \times EG$. Since G acts freely on EG it acts freely on $V \times EG$ via the diagonal action. Let $V \times_G EG$ denote the quotient space $(V \times EG)/G$ under the diagonal action of G. The equivariant cohomology of V is defined as the singular cohomology of the space $V \times_G EG$, namely

$$H^*_G(V) := H^*(V \times_G EG).$$

If G acts freely on V, then $V \times_G EG \cong V$, and Borel's construction of equivariant cohomology agrees with our previous convention for free actions.

In this paper, we only consider the case when $G = T \cong (\mathbb{C}^*)^n$ is an algebraic torus. So we explicitly construct ET for this case. Let $T = \mathbb{C}^*$ be the one-dimensional algebraic torus. We know that $\mathbb{C}^*$ acts freely on $(\mathbb{C}^*)^n$ by left multiplication on each factor. However $(\mathbb{C}^*)^n$ is not a contractible space. If $(\mathbb{C}^*)^n$ is embedded in $(\mathbb{C}^*)^p$ with $p > n$, in the $x_1, \ldots, x_n$-plane for instance, then $(\mathbb{C}^*)^n$ can be contracted down to a point while avoiding the origin.

Thus we take ET to be the union $\cup_{n=1}^{\infty}(\mathbb{C}^*)^n = (\mathbb{C}^*)^\infty$ to obtain a contractible space which carries a free $\mathbb{C}^*$ action. Since $(\mathbb{C}^*)^n/\mathbb{C}^* \cong \mathbb{CP}^{n-1}$, we see that the quotient space $(\mathbb{C}^*)^\infty/\mathbb{C}^* = \cup_{n=1}^{\infty}\mathbb{CP}^n = \mathbb{CP}^\infty$. Let $BT = ET/T$ denote the quotient space. It is commonly called the classifying space of T.

We include a basic example to demonstrate this construction.

Example 2.6. Let $V = \{pt\}$ and $T = \mathbb{C}^*$ act trivially on V. Then

$$H_T^*(V) = H^*(V \times_T ET/T) = H^*(ET/T) \cong H^*(\mathbb{CP}^\infty) \cong \mathbb{C}[t]$$

where t is a cohomology generator of degree 2. ⊞

For $T = (\mathbb{C}^*)^n$ we simply take ET to be the n-fold product $(\mathbb{C}^*)^\infty \times \cdots \times (\mathbb{C}^*)^\infty$. In this case, the equivariant cohomology of a point is

$$H_T^*(pt) = \mathbb{C}[t_1] \otimes \cdots \otimes \mathbb{C}[t_n] \cong \mathbb{C}[t_1, \ldots, t_n]$$

by the (equivariant) Künneth formula.

The following technical property is extremely useful when studying equivariant cohomology. We will give the formal definition and then state some simple conditions that imply it.

Definition 2.5. The T action on a space V is said to be *equivariantly formal* if the Leray-Serre spectral sequence for the fibration $h : V \times_T ET \to BT$ collapses. That means $E^2 = E^\infty$.

An action of T on a space V is *equivariantly formal* if:

1. The cohomology $H^*(V;\mathbb{C})$ vanishes in odd degrees.

2. There is a T-invariant paving by affines.

3. The cohomology $H^*(V;\mathbb{C})$ is generated by its degree two classes.

Equivariant formality is an incredibly useful property for a space to have. The following proposition emphasizes its significance: if the action of T on V is equivariantly formal, then $H_T^*(V)$ is a free module over $H_T^*(pt) \cong \mathbb{C}[t_1, \ldots, t_n]$.

Proposition 2.6. *If the action of T on V is equivariantly formal, then there is an $H_T^*(pt)$-module isomorphism $H_T^*(V) \cong H^*(V) \otimes H_T^*(pt)$.*

In fact the singular cohomology $H^*(V)$ can easily be recovered from the equivariant cohomology $H_T^*(V)$ for any equivariantly formal T-space V. Denote by $I = \langle t_1, t_2, \ldots, t_n\rangle$ the ideal in $H_T^*(V)$ generated by $t_1, \ldots, t_n$. When V is an equivariantly formal T-space, then we have the following ring isomorphism

$$H^*(V) \cong \frac{H_T^*(V)}{I \cdot H_T^*(V)}.$$

Informally, the singular cohomology of V is recovered from the equivariant cohomology $H_T^*(V)$ by setting the variables $t_1, t_2, \ldots, t_n$ equal to zero. What is even more remarkable is that the equivariant cohomology of an equivariantly formal space is sometimes easier to calculate than the singular cohomology. In fact, there are two Hessenberg varieties for which the equivariant cohomology $H_S^*(Hess(X,h))$ was calculated before the singular cohomology $H^*(Hess(X,h))$! These were the regular nilpotent Hessenberg varieties $Hess(X,h)$ corresponding to the Hessenberg functions

- $h(i) = i+1$ for $1 \leq i \leq n-1$ and
- $h(1) = 3$ and $h(i) = i+1$ for $2 \leq i \leq n-1$.

The first case was calculated by Harada-Tymoczko [HarTym09], and the second was carried out by Bayegan-Harada [BayHar10]. Once the equivariant cohomology of $H_S^*(Hess(X,h))$ was calculated, the singular cohomology was recovered by taking the quotient by the ideal $I = \langle t \rangle$. So in fact the calculation of the equivariant cohomology only preceded that of the singular cohomology by mere minutes!

CHAPTER 3
CELL DECOMPOSITIONS IN OTHER LIE TYPES AND EXPANSIONS OF COHOMOLOGY CLASSES

This chapter uses intersection theory to expand the classes induced by Hessenberg-Schubert varieties inside the homology of the flag variety in the basis of Schubert classes. Section 3.1 introduces a new result of Insko-Tymoczko which proves that Hessenberg-Schubert cells form a paving by affines of the Hessenberg varieties [InsTym12]. Section 3.2 proves that the homology classes of the closures of the Hessenberg-Schubert cells form a basis for the homology groups in $H_*(Hess(X, H); \mathbb{Z})$. Section 3.3 contains two important lemmas about intersections of S-invariant subspaces of G/B. Section 3.4 calculates the intersections of Hessenberg-Schubert varieties and opposite Schubert varieties in the flag varieties, first in Type A_{n-1} and then in the other Lie types. Section 3.7 partially computes the expansions of the Hessenberg-Schubert classes in the basis of Schubert classes, and proves that the cohomology of the flag variety surjects onto the cohomology of the flag variety.

3.1 Pavings in general Lie type

In this section, we describe our result about paving regular nilpotent Hessenberg varieties by affines.

Lemma 3.1. *[DeMProSha92, Lemma 1] The datum associated with a Hessenberg space H is equivalent to a set of negative roots $M_H \subset \Phi^-$ satisfying the closure relation:*

- *If $\beta \in M_H$ and $\beta + \alpha \in \Phi^-$ for some positive simple root $\alpha \in \Delta^+$ then $\beta + \alpha \in M_H$.*

The bijection between the two sets is given by decomposing the Hessenberg space

$$H = \mathfrak{b} \oplus \left(\bigoplus_{\alpha \in M_H} \mathfrak{g}_\alpha \right).$$

Let $\mathfrak{u}_w$ denote the Lie algebra of U_w and let Φ_w denote the set of roots appearing in the root space decomposition of $\mathfrak{u}_w$. In other words

$$\mathfrak{u}_w = \bigoplus_{\alpha \in \Phi_w} \mathfrak{g}_\alpha.$$

Define the subset $U_{w,H} \subset U_w$ to be $U_{w,H} = \{u \in U_w : \mathrm{Ad}(u^{-1})(X) \in wHw^{-1}\}$. Consider the regular nilpotent Hessenberg variety $Hess(X, H)$. The work of Tymoczko in classical types and Insko-Tymoczko in any Lie type proved that this subset is homeomorphic to the Schubert cell intersected with Hessenberg variety, namely $U_{w,H} \cong (BwB/B \cap Hess(X, H))$ [Tym07, InsTym12]. It also proved that each of these sets is isomorphic to affine space $\mathbb{C}^d$ for some d. Thus the cells $BwB/B \cap Hess(X, H)$ form a paving by affines in all Lie types for any regular nilpotent Hessenberg variety $Hess(X, H)$.

Theorem 3.2. *[Tym07, InsTym12] If $X = \sum_{\alpha_i \in \Delta} E_{\alpha_i}$ then*

1. *the variety $U_{w,H} = \{u \in U_w : Ad(u^{-1})(X) \in wHw^{-1}\}$ is isomorphic to affine space $\mathbb{C}^d$ of dimension $d = |\Phi_w \cap wM_H|$ and*

2. this set is nonempty if and only if $w^{-1}(\alpha_j) \in M_H$ *for all* j*.*

3.2 Cohomology classes of Hessenberg-Schubert varieties

In this section we show that the closures of the affine cells $BwB/B \cap Hess(X, H)$ from Theorem 3.2 correspond bijectively with a basis of homology classes in $H_*(Hess(X, H); \mathbb{Z})$. We begin with a result of Carrell and Goresky about Białynicki-Birula decompositions.

If V is a closed subvariety of Y then we will denote the class induced by a variety V in $H^*(Y; \mathbb{Z})$ by $[V]$. The following proposition was originally stated in a much more general context. We have restricted the statement to highlight our case of a $\mathbb{C}^*$-invariant subspace of the flag variety, since that is what we deal with in this thesis.

Proposition 3.3 (Carrell and Goresky Theorem 1' [CarGor83])**.** *Suppose we are given a smooth algebraic* $\mathbb{C}^*$*-action on* G/B *with a finite number of isolated fixed points. Let* V *be a* $\mathbb{C}^*$*-invariant closed subspace of* G/B *with fixed point set*

$$V^{\mathbb{C}^*} = \{w_1 B, w_2 B, \ldots, w_r B\} \subset G/B^{\mathbb{C}^*}.$$

Denote the Białynicki-Birula plus decomposition of V *with respect to* $\mathbb{C}^*$ *by* $V = \bigcup_{j=1}^{r} BB^+_{w_j}$*. If each* $BB^+_{w_j}$ *is homeomorphic to affine space* $\mathbb{C}^{m_j}$ *then the classes of the closures* $[\overline{BB^+_{w_j}}]$ *freely generate* $H_*(V; \mathbb{Z})$*.*

Let $X_{w,H}$ denote the closure of the cell $\overline{BwB/B \cap Hess(X, H)}$ in $Hess(X, H)$. We call $X_{w,H}$ a *Hessenberg-Schubert variety.* The paving by affines of the Hessenberg varieties $Hess(X, H)$ satisfy all of the criteria in Proposition 3.3. Hence we obtain the following result.

Corollary 3.1. *The homology classes $[X_{w,H}]$ freely generate the integral homology $H_*(Hess(X, H); \mathbb{Z})$.*

Proof. The subtorus $S \cong \mathbb{C}^* \subseteq T$ acts on $Hess(X, H)$ smoothly with a finite fixed point set. Moreover $Hess(X, H)^S \subset G/B^T$. The Schubert cells BwB/B are the Białynicki-Birula plus cells BB^+_{wB} of the flag variety G/B under this S-action. Thus the cells $BwB/B \cap Hess(X, H)$ are the Białynicki-Birula plus cells of the S-action in $Hess(X, H)$. Theorem 3.2 proved these cells $BwB/B \cap Hess(X, H)$ are affine. Hence they satisfy the conditions of Proposition 3.3. The result now follows from that proposition. $\square$

3.3 Intersections of closed S-invariant varieties

The Schubert varieties and Hessenberg varieties are closed S-invariant subvarieties of G/B. In this section we prove a useful criterion for calculating intersections of S-stable subvarieties of G/B.

Lemma 3.4. *Let $\gamma : S \to T$ be a group homomorphism as in Theorem 2.3 and let z denote an element of $S \cong \mathbb{C}^*$. Let V and Y be two S-invariant closed subvarieties of G/B. Suppose $gB \in V \cap Y$. Then the limits $\lim_{z \to \infty} \gamma(z)gB$ and $\lim_{z \to 0} \gamma(z)gB$ are also contained in $V \cap Y$. If gB is not fixed by S then $V \cap Y$ contains two distinct fixed points.*

Proof. Since the varieties V and Y are closed S-invariant subspaces of G/B their intersection is a closed S-invariant subspace of G/B. Being S-invariant means that if gB is contained in $V \cap Y$ then the point $\lambda(z)gB$ is contained in $V \cap Y$ for all $z \in \mathbb{C}^*$. Since $V \cap Y$ is a closed subvariety of G/B we know that the limits $\lim_{z\to\infty} \gamma(z)gB$ and $\lim_{z\to 0} \gamma(z)gB$ are contained in $V \cap Y$. If gB is not fixed by S then the limits $\lim_{z\to\infty} \gamma(z)gB$ and $\lim_{z\to 0} \gamma(z)gB$ are distinct fixed points in the flag variety. Thus they are distinct points in $V \cap Y$. This proves the lemma. $\square$

Corollary 3.2. *Let V and Y be two S-invariant closed subvarieties of G/B. If $V \cap Y$ contains only one S-fixed point wB then that is the only point in the intersection $V \cap Y$. Furthermore, if $V \cap Y$ contains no fixed points then $V \cap Y$ is empty.*

Proof. Suppose that $V \cap Y$ contains a point gB that is not fixed by S. Lemma 3.4 proves that the limits $\lim_{z\to\infty} \gamma(z)gB$ and $\lim_{z\to 0} \gamma(z)gB$ are distinct S-fixed points in $V \cap Y$. We conclude that if $V \cap Y$ contains any other point than a fixed point wB then it must contain two distinct fixed points. On the other hand, if $V \cap Y$ contains no fixed point, then it can not contain a non-fixed point gB. Hence it is empty. $\square$

3.4 Intersections of Peterson-Schubert varieties and Schubert varieties

In this section we restrict our attention to the Peterson variety, namely the regular nilpotent Hessenberg variety for which M_H is the set of negative simple roots $M_H = \Delta^-$. We calculate the intersections of Peterson-Schubert varieties and Schubert varieties. Subsection 3.5 carries the calculation out in Type A_{n-1}. Subsection 3.6 generalizes these calculations to all other Lie types.

First we set some notation which we will use throughout this chapter. As in Section 2.4 let $X_w = \overline{BwB/B} = \coprod_{u \leq w} BuB/B$ denote the Schubert variety that is the closure of the Schubert cell BwB/B in G/B. Also, let

$$X^w = \overline{B^- wB/B} = \coprod_{u \geq w} B^- uB/B$$

denote the opposite Schubert variety which is the closure of the opposite Schubert cell $B^- wB/B$ in G/B. Let $X_{w,\mathfrak{P}}$ denote the closure of the cell $BwB/B \cap \mathrm{Pet}_n$. We call the variety $X_{w,\mathfrak{P}} = \overline{BwB/B \cap \mathrm{Pet}_n}$ a *Peterson-Schubert variety.*

Following the notation set in Subsection 2.4, let J and K denote subsets of simple positive roots Φ^+ and let W_J and W_K denote the subgroups of W generated by the sets $\{s_i : i \in J\}$ and $\{s_i : i \in K\}$ respectively. Let w_J denote the longest word in the subgroup W_J. Harada and Tymoczko showed that a cell $BwB/B \cap \mathrm{Pet}_n$ is nonempty if and only if w is the longest word w_J for some parabolic subgroup $W_J \subset W$ [HarTym10, Proposition 5.8].

Let v_J and u_J denote the Coxeter elements of the form

$$v_J = s_{i_{|J|}} s_{i_{|J|-1}} \cdots s_{i_1} \text{ and } u_J = s_{i_1} s_{i_2} \cdots s_{i_{|J|}}, \text{ where } i_1 < i_2 < \cdots < i_{|J|}.$$

If $J = \Delta$ then we write w_0 for w_J and v_Δ for v_J. The statements in this section refer to v_J but the same results are true for u_J.

Recall the following fact about Schubert varieties.

Proposition 3.5. *[Ful97, Chapter 10 Proposition 7] The T-fixed points in X_w consist* *of all flags vB with $v \leq w$ in Bruhat order, and the T-fixed points in X^u consist of all flags vB with $u \leq v$.*

Suppose that V, Y are subvarieties of G/B. We say that V intersects Y *properly* in G/B if the codimension of $V \cap Y$ in G/B is equal to the codimension of V in G/B plus the codimension of Y in G/B. In the remainder of this section, we will show that X^{v_J} intersects $X_{w_J,\mathfrak{P}}$ properly in just one point. We start by proving this fact in Type A_{n-1} and then generalize the argument to other Lie types.

3.5 Intersections in type A_{n-1}

In this section we calculate the intersections between $X_{w_J,\mathfrak{P}}$ and X^{v_J} in Type A_{n-1}. We show the intersection is proper and transverse.

Lemma 3.6. *The dimension of $Bw_JB/B \cap \mathrm{Pet}_n$ is $|J|$.*

Proof. Suppose that $\alpha_i, \alpha_{i+1}, \ldots, \alpha_{i+k}$ is a string of consecutive simple roots appearing in J. Suppose that this string is maximal in the sense that that α_{i-1} and α_{i+k+1} do not appear in J. The permutation w_J has a subword

$$(s_i s_{i+1} \cdots s_{i+k})(s_i s_{i+1} \cdots s_{i+k-1}) \cdots (s_i s_{i+1}) s_i.$$

Thus the permutation matrix w_J has a diagonal block of size $k+1$ with 1's along the antidiagonal corresponding to the string $\alpha_i, \ldots, \alpha_{i+k}$. In this block, the nonzero variables in $Bw_JB/B \cap \mathrm{Pet}_n$ repeat along the k-antidiagonals lying to the left of the main antidiagonal. Hence the dimension of each block is k. Adding up the dimension of each block, we see that $Bw_JB/B \cap \mathrm{Pet}_n$ has dimension $|J|$. □

Example 3.1. Consider the Peterson-Schubert cell Bw_JB/B in the Peterson variety in $GL_6(\mathbb{C})/B$ where the permutation $w_J = 432165 = (s_1s_2s_3)(s_1s_2)s_1s_5$. The set J is $J = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5\}$ and the cell Bw_JB/B has the form

$$Bw_JB/B \cap \mathrm{Pet}_n = \left\{ \begin{pmatrix} a & b & c & 1 & 0 & 0 \\ b & c & 1 & 0 & 0 & 0 \\ c & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \right\}.$$

We see that the cell Bw_JB/B has a block corresponding to the string $\alpha_1, \alpha_2, \alpha_3$ of dimension 3 and a block corresponding to α_5 of dimension 1. ⊞

Algorithm 3.7. *Woo and Yong described a simple algorithm for identifying explicit coset representatives for a neighborhood of the fixed point wB in X^v in Type A_{n-1} [WooYon08, Section 3.2]. We give a brief summary of the process with a small working example to clarify each step. The set $B^-B/B \cong U^-$ is an affine neighborhood for $(id)B$ in $GL_n(\mathbb{C})/B$. Thus one can take the set of lower-triangular unipotent matrices U^- as a set of coset representatives of a neighborhood of $(id)B$ in $GL_n(\mathbb{C})/B$. Translating this set by w, one sees that the set*

$$wU^- := \{wu : u \in U^-\}$$

is a set of coset representatives for the affine open neighborhood of wB in $GL_n(\mathbb{C})/B$.

Let $w = w_J = s_1s_3s_4s_3$ and $v = v_J = s_1s_3s_4$. Then

$$w_JU^- \cong \left\{\begin{pmatrix} x_{11} & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ x_{31} & x_{32} & x_{33} & x_{34} & 1 \\ x_{41} & x_{42} & x_{43} & 1 & 0 \\ x_{51} & x_{52} & 1 & 0 & 0 \end{pmatrix}\right\} \text{ and } v_J = \left\{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}\right\}$$

Then one uses the following algorithm to identify the neighborhood of wB in X^v. Start with the set wU^-. Mark the positions of the pivots in v with ()'s. Underline the nonzero positions that are strictly north and west of the pivots in their respective row and column.

$$\left\{\begin{pmatrix} x_{11} & (1) & 0 & 0 & 0 \\ (1) & 0 & 0 & 0 & 0 \\ x_{31} & x_{32} & x_{33} & x_{34} & (1) \\ x_{41} & x_{42} & (x_{43}) & 1 & 0 \\ x_{51} & x_{52} & 1 & (0) & 0 \end{pmatrix}\right\} \left\{\begin{pmatrix} \underline{x_{11}} & (1) & 0 & 0 & 0 \\ (1) & 0 & 0 & 0 & 0 \\ x_{31} & x_{32} & \underline{x_{33}} & \underline{x_{34}} & (1) \\ x_{41} & x_{42} & (x_{43}) & 1 & 0 \\ x_{51} & x_{52} & 1 & (0) & 0 \end{pmatrix}\right\}$$

Place braces {} around the positions that are the furthest south and east of any connected subset of underlined positions. Let $r_{\{(i,j)\}}$ denote the number of pivots of v_J that are northwest of the braced positions $\{(i,j)\}$. In the neighborhood of wB in X^v, the $(r_{\{(i,j)\}}+1) \times (r_{\{(i,j)\}}+1)$ minors northwest of a braced position $\{(i,j)\}$

must all vanish. Hence, we define the patch ideal I_{w,X^v} *to be the ideal generated by all* $(r_{\{(i,j)\}}+1)\times(r_{\{(i,j)\}}+1)$ *minors northwest of a braced position* $\{(i,j)\}$*. (We consider minors containing the position* $\{(i,j)\}$ *to be northwest of* $\{(i,j)\}$ *too.)*

In our example, the determinantal conditions are that all 1×1 *minors northwest of the underlined position* $\{(1,1)\}$ *and all* 3×3 *minors northwest of the underlined position* $\{(3,4)\}$ *must vanish. These conditions require the matrix entries* x_{11}, x_{33}*, and* x_{34} *to be zero. Thus the neighborhood of* w_J *in* X^{v_J} *is isomorphic to the set of matrices below on the right.*

$$\left\{\begin{pmatrix} \{\underline{x_{11}}\} & (1) & 0 & 0 & 0 \\ (1) & 0 & 0 & 0 & 0 \\ x_{31} & x_{32} & \underline{x_{33}} & \{\underline{x_{34}}\} & (1) \\ x_{41} & x_{42} & (x_{43}) & 1 & 0 \\ x_{51} & x_{52} & 1 & (0) & 0 \end{pmatrix}\right\} \left\{\begin{pmatrix} \{\underline{0}\} & (1) & 0 & 0 & 0 \\ (1) & 0 & 0 & 0 & 0 \\ x_{31} & x_{32} & \underline{0} & \{\underline{0}\} & (1) \\ x_{41} & x_{42} & (x_{43}) & 1 & 0 \\ x_{51} & x_{52} & 1 & (0) & 0 \end{pmatrix}\right\}$$

Remark 3.1. Where w_J has a diagonal block with 1's along the antidiagonal, the permutation v_J has a block with a 1 in the top right corner and 1's along the main diagonal in the remaining rows:

$$\text{Block in } w_J = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \ddots & 1 & 0 & 0 \\ 0 & \vdots & 1 & 0 & 0 & 0 \\ \vdots & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \text{Block in } v_J = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Due to the special form of v_J the top row of each block in $w_J U^- \cap X^{v_J}$ has the form:

$$\begin{pmatrix} \underline{x_{11}} & \underline{x_{12}} & \underline{x_{13}} & \cdots & \{\underline{x_{1n}}\} & (1) \\ (x_{21}) & x_{22} & x_{23} & \cdots & 1 & 0 \\ x_{31} & (x_{32}) & x_{33} & 1 & 0 & 0 \\ \vdots & \vdots & (1) & 0 & 0 & 0 \\ \vdots & 1 & 0 & (0) & 0 & 0 \\ 1 & 0 & \cdots & 0 & (0) & 0 \end{pmatrix}$$

The determinantal conditions defining $I_{w_J, X^{v_J}}$ stipulate that each of the x_{ij} in the top row of this block must vanish. In other words, where w_J has a diagonal block, the corresponding block in $w_J U^- \cap X^{v_J}$ has the form

$$\begin{pmatrix} \underline{0} & \underline{0} & \underline{0} & \cdots & \{\underline{0}\} & 1 \\ (x_{21}) & x_{22} & x_{23} & \cdots & 1 & 0 \\ x_{31} & (x_{32}) & x_{33} & 1 & 0 & 0 \\ \vdots & \vdots & (1) & 0 & 0 & 0 \\ \vdots & 1 & 0 & (0) & 0 & 0 \\ 1 & 0 & \cdots & 0 & (0) & 0 \end{pmatrix}$$

$\boxplus$

Lemma 3.8. *The codimension of X^{v_J} in G/B is $|J|$.*

Proof. The dimension X_{v_J} is $\ell(v_J) = |J|$. It is a well-known fact that

$$\dim(X_{v_J}) + \dim(X^{v_J}) = \dim(G/B)$$

[Ful97, Proposition 10.2.2]. Thus the codimension of X^{v_J} in G/B must be $|J|$. $\square$

We now prove that the varieties X^{v_J} and $X_{w_J,\mathfrak{P}}$ intersect in the lone fixed point $w_J B$.

Lemma 3.9. *The varieties $X_{w_J,\mathfrak{P}}$ and X^{v_J} intersect properly at the point $w_J B$.*

Proof. We start by showing that the neighborhood $Bw_J B/B \cap \mathrm{Pet}_n$ of $w_J B$ in $X_{w_J,\mathfrak{P}}$ and the neighborhood $wB^- B/B \cap X^{v_J}$ of $w_J B$ in X^{v_J} only intersect at $w_J B$. We will then argue that this is the only fixed point in the intersection. Then we cite Lemma

3.2 to prove that this is the only point in the intersection. Finally, we argue that the intersection is proper by showing the codimensions of $X_{w_J,\mathfrak{P}}$ and X^{v_J} in $GL_n(\mathbb{C})/B$ sum to give the codimension of $X_{w_J,\mathfrak{P}} \cap X^{v_J}$ in $GL_n(\mathbb{C})/B$.

By Remark 3.1, for any point in the intersection of $Bw_JB/B \cap \mathrm{Pet}_n$ and the neighborhood $wB^-B/B \cap X^{v_J}$ of w_JB in X^{v_J} we may choose a coset representative with $x_{ij} = 0$ across the top row of each diagonal block. The only point in Bw_JB/B with a coset representative which has $x_{ij} = 0$ across the top row of each diagonal block is w_JB. This is the only point in the intersection $Bw_JB/B \cap \mathrm{Pet}_n \cap wB^-B/B \cap X^{v_J}$.

The closure of the cell $Bw_JB/B \cap \mathrm{Pet}_n$ consists of cells $Bw_KB/B \cap \mathrm{Pet}_n$ where $K \subset J$ is a proper subset. None of the permutations $w_K < w_J$ satisfy $w_K \geq v_J$. Lemma 3.5 says that none of these w_K intersect X^{v_J}. We conclude that w_JB is the only fixed point in $X_{w_J,\mathfrak{P}} \cap X^{v_J}$.

Next we show that $X_{w_J,\mathfrak{P}} \cap X^{v_J} = w_JB$ is proper. Lemma 3.6 says that $\dim(X_{w_J,\mathfrak{P}}) = |J|$. Lemma 3.8 says that $\dim(X^{v_J}) = \dim(GL_n(\mathbb{C})/B) - |J|$. Thus $X_{w_J,\mathfrak{P}} \cap X^{v_J}$ is proper provided it has dimension 0. This is clearly the case, so is $X_{w_J,\mathfrak{P}} \cap X^{v_J} = w_JB$ is a proper intersection. $\square$

Example 3.2. Consider the Peterson variety in $GL_4(\mathbb{C})/B$. The big Peterson-

Schubert cell has the form

$$Bw_0B/B \cap \mathrm{Pet}_n = \left\{ \begin{pmatrix} a & b & c & 1 \\ b & c & 1 & 0 \\ c & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

The closure of this cell contains the S-fixed points wB for w in the set

$$\{1234, 2134, 1324, 1243, 2143, 3214, 1432, 4321\}.$$

The cell $Bw_0B/B \cap X^{v_\Delta}$ has the form

$$Bw_0B/B \cap X^{v_\Delta} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 1 \\ x & y & 1 & 0 \\ z & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

By Proposition 3.5 the closure of this cell contains the S-fixed points uB for $u \geq v_J$ in Bruhat order, namely uB for u in the set $\{2341, 2431, 3241, 3421, 4231, 4321\}$. Hence we can see that the varieties $X_{w_J,\mathfrak{P}}$ and X^{v_Δ} have only one fixed point 4321 in common. Since the intersection must be S-closed, we deduce that this is the only point of intersection between the two subvarieties in $GL_4(\mathbb{C})/B$. $\boxplus$

Theorem 3.10. *In type A_{n-1} the tangent spaces satisfy*

$$T_{w_J} X_{w_J,P} \oplus T_{w_J} X^{v_J} = T_{w_J} GL_n(\mathbb{C})/B.$$

Proof. The cell $Bw_JB/B \cap \mathrm{Pet}_n$ is block diagonal, and in each block, the entries repeat along each antidiagonal. In particular, the conditions defining the cell $Bw_JB/B \cap \mathrm{Pet}_n$ in Bw_JB/B are all linear. This implies that the cell $Bw_JB/B \cap \mathrm{Pet}_n$ is homeomorphic to the tangent space $T_{w_J} X_{w_J,\mathfrak{P}}$. It has dimension $|J|$.

The neighborhood of w_JB in X^{v_J} is a smooth affine cell with 0's in the top row of each diagonal block in w_J. In particular, it too is defined by only linear conditions. So the neighborhood of w_JB in X^{v_J} is homeomorphic to the tangent space $T_{w_JB} X^{v_J}$. This neighborhood has dimension $\dim(GL_n(\mathbb{C})/B) - |J|$.

If two linear spaces intersect in a single point, then the intersection is transverse. We deduce that $T_{w_J} X_{w_J,P} \oplus T_{w_J} X^{v_J} = T_{w_J} GL_n(\mathbb{C})/B$. □

Example 3.3. We see that the cell $Bw_JB/B \cap \mathrm{Pet}_n$ is a smooth neighborhood of w_JB in $X_{w_J,\mathfrak{P}}$

$$\left\{ \begin{pmatrix} a & b & 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & d & 1 \\ 0 & 0 & 0 & d & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$$

It is a linear subspace that is homeomorphic to the tangent space $T_{w_JB}X_{w_J,\mathfrak{P}}$.

The neighborhood of w_JB in X^{v_J} has the form

$$\mathcal{N}_{w_J,GL_n(\mathbb{C})/B} = \left\{ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ x_{21} & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ x_{41} & x_{42} & x_{43} & 0 & 0 & 1 \\ x_{51} & x_{52} & x_{53} & x_{54} & 1 & 0 \\ x_{61} & x_{62} & x_{63} & 1 & 0 & 0 \end{pmatrix} \right\}.$$

It too is a linear space homeomorphic to the tangent space $T_{w_JB}X^{v_J}$. These two linear spaces intersect in the lone fixed point w_JB. If two linear spaces intersect in a single point and their dimensions add up to the dimension of the ambient space, then their intersection is transverse. We see that this intersection is transverse. $\boxplus$

3.6 Intersections in other Lie types

In this section, we identify the intersections $X_{w_J,\mathfrak{P}} \cap X^{v_J}$ in other Lie types, generalizing the results from the previous section. Our first result associates to each Peterson-Schubert variety $X_{w_J,\mathfrak{P}}$ an opposite Schubert variety X^{v_J} which intersects $X_{w_J,\mathfrak{P}}$ properly in a point. We then show that the intersection $X_{w_J,\mathfrak{P}} \cap X^{v_K}$ is empty whenever $|J| = |K|$ but $J \neq K$. We end the section with an example that shows the intersection $X_{w_J,\mathfrak{P}} \cap X^{v_J}$ is usually not transverse in Lie types other than A_{n-1}.

Lemma 3.11. *The varieties $X_{w_J,\mathfrak{P}}$ and X^{v_J} only intersect at the point w_JB.*

Proof. The S-fixed points in $X_{w_J,\mathfrak{P}}$ all have the form $w_K B$ with $K \subseteq J$. By contrast, the fixed points in X^{v_J} are wB with $w \geq v_J$. Hence, if $w \in X^{v_J} \cap X_{w_J,\mathfrak{P}}$ then $w = w_K$ with $J \subseteq K \subseteq J$. We conclude $w_J B$ is the only point in $X_{w_J,\mathfrak{P}} \cap X^{v_J}$. By Corollary 3.2 this is the only point in $X_{w_J,\mathfrak{P}} \cap X^{v_J}$. □

The next result is an immediate corollary of Theorem 3.2 stated in the terminology of this section.

Corollary 3.3. *The Peterson-Schubert variety* $X_{w_J,\mathfrak{P}} = X_{w_J} \cap \mathrm{Pet}_n$ *has dimension* $|J|$.

Proof. The cell $Bw_J B/B \cap \mathrm{Pet}_n$ is the largest cell in the intersection $X_{w_J} \cap \mathrm{Pet}_n$. Thus the dimension of $X_{w_J} \cap \mathrm{Pet}_n$ is $|\Phi_{w_J} \cap w_J M_P|$ by Theorem 3.2. The set of roots M_P defining the Hessenberg space for the Peterson variety consists of the negative simple roots. The element w_J of W_J sends every negative simple root $\alpha_j \in -J$ to a positive simple root $-\alpha_i \in J$ and sends the negative roots $\alpha_k \notin -J$ to a negative root [Hum90, p.15]. Hence the intersection $\Phi_{w_J} \cap w_J M_P = J$ proving that the dimension of $X_{w_J} \cap \mathrm{Pet}_n$ is $|J|$. □

We now prove that the intersection $X_{w_J,\mathfrak{P}} \cap X^{v_J}$ is proper in any Lie type.

Proposition 3.12. *The subvarieties* $X_{w_J,\mathfrak{P}}$ *and* X^{v_J} *of* G/B *intersect properly at the* S*-fixed point* $w_J B$.

Proof. The dimension of G/B is $|\Phi^+|$. The dimension of X_{v_J} is $\ell(v_J) = |J|$. Schubert varieties and opposite Schubert varieties have the property that

$$\dim(X^{v_J}) + \dim(X_{v_J}) = \dim(G/B)$$

[Ful97, Proposition 10.2.2]. Hence $\dim\left(X^{v_J}\right) = |\Phi^+| - |J|$. Corollary 3.3 showed that $X_{w_J,\mathfrak{P}}$ has dimension $|J|$. Hence

$$\dim(G/B) = \dim\left(X_{w_J,\mathfrak{P}}\right) + \left(\dim X^{v_J}\right).$$

Thus the intersection $X_{w_J,\mathfrak{P}} \cap X^{v_J}$ is proper if and only if it is 0-dimensional. Lemma 3.11 showed that the intersection $X_{w_J,\mathfrak{P}} \cap X^{v_J}$ is the point $w_J B$. Hence the intersection is indeed proper. $\square$

Example 3.4. The flag variety $GL_6(\mathbb{C})/B$ has complex dimension $|\Phi^+| = \binom{6}{2} = 15$. The simple positive roots are $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. Let $J = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5\}$. The longest word in W_J is

$$w_J = 321654 = s_1 s_2 s_1 s_4 s_5 s_4.$$

In this example we can write all of the cells explicitly. The cell $Bw_JB/B \cap \mathrm{Pet}_n$ has the form

$$\left\{ \begin{pmatrix} a & b & 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & d & 1 \\ 0 & 0 & 0 & d & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$$

By inspection we see it has dimension $4 = |J|$.

On the other hand, the largest cell in X^{v_J} is $B^- v_J B/B$ and has the form

$$\left\{ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ x_{31} & 1 & 0 & 0 & 0 & 0 \\ x_{41} & x_{42} & x_{43} & 0 & 0 & 1 \\ x_{51} & x_{52} & x_{53} & 1 & 0 & 0 \\ x_{61} & x_{62} & x_{63} & x_{64} & 1 & 0 \end{pmatrix} \right\}.$$

It has complex dimension 11.

As highlighted in Lemma 3.11 these varieties intersect only in the S-fixed point $w_J B = (321654)B$. Their dimensions add up to the dimension of the full flag variety

$$\dim\left(X_{w_J,\mathfrak{P}}\right) + \dim\left(X^{v_J}\right) = 4 + 11 = 15 = \dim\left(GL_6(\mathbb{C})/B\right).$$

Hence the intersection is proper. ⊞

Lemma 3.11 says that the intersection $X_{w_J,\mathfrak{P}} \cap X^{v_K}$ is nonempty when $J = K$. The next lemma shows that when $|J| = |K|$ the intersection $X_{w_J,\mathfrak{P}} \cap X^{v_K}$ is empty when $J \neq K$. The proof follows from Proposition 3.5 and the following fact.

Lemma 3.13. *Let J and K be two distinct sets of simple roots in Δ with $|J| = |K|$. Then the intersection $X_{w_J,\mathfrak{P}} \cap X^{v_K}$ is empty.*

Proof. By Corollary 3.2 if $X_{w_J,\mathfrak{P}} \cap X^{v_K}$ contains no fixed point then it is empty. We argue that there are no fixed points in $X_{w_J,\mathfrak{P}} \cap X^{v_K}$. Suppose $J \neq K$. The fixed-point uB is contained in $X_{w_J} \cap X^{v_K}$ if and only if $v_K \leq u \leq w_J$. The element $v_k > s_k$ and $s_K \notin J$. Hence $v_K \not\leq w_J$. We conclude that there are no fixed points in $X_{w_J} \cap X^{v_K}$. Thus the intersection $X_{w_J} \cap X^{v_K}$ is empty, which implies $X_{w_J,\mathfrak{P}} \cap X^{v_K}$ is empty as well. □

Example 3.5. Suppose $J = \{\alpha_1, \alpha_2, \alpha_3\}$ and $K = \{\alpha_1, \alpha_3, \alpha_4\}$ in Type A_4. Then $w_J = s_1 s_2 s_1 s_3 = 31254$ and $v_K = s_1 s_3 s_4 = 21453$. None of the S-fixed points wB in $X_{w_J,\mathfrak{P}}$ have the property that $w \geq s_4$. Every fixed point uB in X^{v_K} has the property $u \geq s_1 s_3 s_4$. Since the two varieties have no S-fixed points in common, they can not intersect at all. ⊞

Example 3.6. The intersection $X_{w_J,\mathfrak{P}} \cap X^{v_J}$ is often not transverse in Lie types other than type A_{n-1}, as we demonstrate in type B_2. For the purposes of this example, let

x_{ij} denote the exponentional of $\mathfrak{g}_{i\alpha_1+j\alpha_2}$. The Schubert cell Bw_0B/B is

$$\left\{\begin{pmatrix} (x_{01}x_{11}-x_{12})x_{10}-1/2x_{11}^2 & -x_{01}x_{11}+x_{12} & x_{11} & x_{10} & 1 \\ 1/2x_{01}^2x_{10}-x_{12} & -1/2x_{01}^2 & x_{01} & 1 & 0 \\ x_{01}x_{10}-x_{11} & -x_{01} & 1 & 0 & 0 \\ -x_{10} & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}\right\}.$$

The tangent space

$$T_{w_0,G/B} = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2}$$

The polynomials defining $Bw_0B/B \cap Hess(X,H)$ in Bw_0B/B are $x_{01}-x_{10}=0$ and $-1/2x_{01}^2+x_{11}=0$. The tangent space $T_{w_0,Hess(X,H)}$ is

$$\{(x,y) \in \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} : x = y\} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}.$$

The tangent space $T_{w_0,X^{v_\Delta}}$ is

$$\mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2}.$$

We see that $T_{w_0,Hess(X,H)} \oplus T_{w_0,X^{v_\Delta}}$ is only a three dimensional subspace of $T_{w_0,G/B}$. Thus the intersection is not transverse. $\boxplus$

3.7 Intersection theory computations in the cohomology ring of the flag variety

Let $i : \mathrm{Pet}_n \hookrightarrow G/B$ denote the (proper) embedding of Pet_n as a closed subvariety of G/B. Let

$$i_* : H_*(\mathrm{Pet}_n; \mathbb{Z}) \to H_*(G/B; \mathbb{Z})$$

denote the homological push-forward and

$$i^* : H^*(G/B; \mathbb{Z}) \to H^*(\mathrm{Pet}_n; \mathbb{Z})$$

denote the pull-back in cohomology induced from this map.

We now review some fundamentals of the intersection theory of the flag variety. All of these facts can be found in Appendix B of Fulton's *Young Tableaux* [Ful97].

Every Schubert variety $X_w \subseteq G/B$ induces a class $[X_w] \in H_*(G/B; \mathbb{Z})$. Chevalley proved that these Schubert classes form an orthonormal additive $\mathbb{Z}$-basis of $H_*(G/B; \mathbb{Z})$ [Che58]. The flag variety satisfies Poincaré duality because it is a compact orientable differentiable manifold. Using Poincaré duality we can identify the $H_*(G/B; \mathbb{Z})$ with $H^*(G/B; \mathbb{Z})$. Under this identification, the cup product of cohomology classes corresponds with the proper intersection of their Poincaré dual homology classes. Suppose the varieties X and Y intersect properly in a subvariety Z and m is the intersection multiplicity. The standard convention is to write $[X] \cdot [Y] = m[Z]$ when this is the case [Ful97, Appendix B]. The only case we use is when $Z = \{pt\}$.

For each Peterson-Schubert class $[X_{w_J,\mathfrak{P}}]$ in $H_*(\mathrm{Pet}_n;\mathbb{Z})$ the push-forward $i_*([X_{w_J,\mathfrak{P}}])$ defines a fundamental class in $H_*(G/B;\mathbb{Z})$.

Theorem 3.14. *Let $i_*([X_{w_J,\mathfrak{P}}])$ denote the class in $H_*(G/B;\mathbb{Z})$ induced by a Peterson-Schubert variety $X_{w_J,\mathfrak{P}}$ and $[X_w]$ denote the fundamental class of a Schubert variety in $H_*(G/B;\mathbb{Z})$. Write the class of the Peterson-Schubert variety in the Schubert basis classes as*

$$i_*([X_{w_J,\mathfrak{P}}]) = \sum a_u[X_u].$$

The coefficients a_{v_J} satisfy

1. *$a_{v_J} \neq 0$ in the expansion of $i_*([X_{w_J,\mathfrak{P}}])$, and*

2. *$a_{v_J} = 0$ in the expansion for any $i_*([X_{w_K,\mathfrak{P}}])$ with $K \neq J$.*

Proof. The intersection product gives a ring structure to $H_*(G/B;\mathbb{Z})$. We will use this ring structure to determine the coefficients a_u. Proposition 3.12 states that the closed irreducible subvarieties $X_{w_J,\mathfrak{P}}$ and X^{v_J} intersect properly in a point. Thus the product $i_*([X_{w_J,\mathfrak{P}}]) \cdot [X^{v_J}] = a_{v_J}[w_J B]$ for some integer multiplicity $a_{v_J} > 0$ [Ful97, Appendix B, Equation 31]. Moreover $[X_u] \cdot [X^u] = 1$ and $[X_u] \cdot [X^v] = 0$ if $u \neq v$ [Ful97, p. 160 Equation (1)]. We conclude that the coefficient of $[X_{v_J}]$ in the expansion $i_*([X_{w_J,\mathfrak{P}}]) = \sum a_u[X_u]$ is a_{v_J}. On the other hand, Lemma 3.13 says that $X_{w_J,\mathfrak{P}} \cap X^{v_K} = \emptyset$ when $K \neq J$. It follows that $i_*([X_{w_J,\mathfrak{P}}]) \cdot [X^{v_K}] = 0$.

From the fact that $i_*([X_{w_J,\mathfrak{P}}]) \cdot [X^{v_J}] = a_{v_J}$ we deduce that

$$i_*([X_{w_J,\mathfrak{P}}]) = a_{v_J}[X_{v_J}] + \text{ other terms.}$$

In contrast, $i_*([X_{w_J,\mathfrak{P}}]) \cdot [X^{v_K}] = 0$ for $K \neq J$. Thus

$$i_*([X_{w_J,\mathfrak{P}}]) = \sum_{|K|=|J|, K\neq J} 0[X_{v_K}] + a_{v_J}[X_{v_J}] + \text{ other terms.}$$

$\square$

Theorem 3.14 gives a partial decomposition of the Peterson-Schubert homology classes in terms of the Schubert basis classes. Our next result uses this partial decomposition to show that the homology of the Peterson variety injects into the homology of the flag variety under the push-forward of the inclusion map.

Theorem 3.15. *The push-forward $i_* : H_*(\mathrm{Pet}_n; \mathbb{Z}) \to H_*(G/B; \mathbb{Z})$ is an injection in all Lie types.*

Proof. Every Peterson-Schubert fundamental class $i_*([X_{w_J,\mathfrak{P}}])$ can be written as a $\mathbb{Z}$-linear combination of the Schubert classes

$$i_*([X_{w_J,\mathfrak{P}}]) = \sum a_u[X_u].$$

Theorem 3.14 says that in the expansion for $i_*([X_{w_J,\mathfrak{P}}])$ the coefficient $a_{v_J} \neq 0$. It also says that $a_{v_k} = 0$ in the expansion of $i_*([X_{v_J,\mathfrak{P}}])$ for $K \neq J$ with $|K| = |J|$.

For each $0 \leq k \leq n$ let $i_{2k} : H_{2k}(\mathrm{Pet}_n; \mathbb{Z}) \to H_{2k}(G/B; \mathbb{Z})$ be the transition matrix. Let b_{2k} be the $2k$-th Betti number of $H_*(\mathrm{Pet}_n; \mathbb{Z})$. Up to reordering of the bases $\{[X_{w_J,\mathfrak{P}}]\}$ and $\{[X_u]\}$ the matrix of i_{2k} has a $b_{2k} \times b_{2k}$ submatrix that is diagonal. This is the submatrix corresponding to the homology basis elements

$\{i_*([X_{w_J,\mathfrak{P}}])\}$ and $\{[X_{v_J}]\}$. The diagonal entry corresponding to $i_*([X_{w_J,\mathfrak{P}}])$ and $[X_{v_J}]$ of this submatrix is a_{v_J}. Thus the map i_{2k} is an injective linear transformation for all k with $0 \leq 2k \leq \dim_{\mathbb{R}}(\mathrm{Pet}_n)$. ☐

Corollary 3.4. *The pull-back $i^* : H^*(G/B;\mathbb{Z}) \to H^*(\mathrm{Pet}_n;\mathbb{Z})$ is a surjection in all Lie types.*

Proof. We know from Theorem 3.15 that $i_* : H_*(\mathrm{Pet}_n, \mathbb{Z}) \hookrightarrow H_*(G/B, \mathbb{Z})$ is injective. Cohomology is a left exact contravariant functor, which implies that $i^* : H^*(G/B;\mathbb{Z}) \to H^*(\mathrm{Pet}_n;\mathbb{Z})$ is a surjection whenever i_* is injective. ☐

The following result is useful for computing equivariant cohomology; in the language of Section 5, it proves that the Peterson variety is a GM-space.

Corollary 3.5. *The cohomology of the Peterson variety $H^*(\mathrm{Pet}_n;\mathbb{Z})$ is generated by its degree 2 cohomology classes in all Lie types.*

Proof. Since the pull-back of the natural inclusion map i^* is a surjection, we know

$$H^*(\mathrm{Pet}_n;\mathbb{Z}) \cong \frac{H^*(G/B;\mathbb{Z})}{ker(i^*)}.$$

The Borel presentation of the cohomology ring of the flag variety says that $H^*(G/B;\mathbb{Z})$ is isomorphic to a ring which is generated by its degree 2 cohomology classes [Bor53]. The map i^* induced by the canonical embedding $i : \mathrm{Pet}_n \hookrightarrow G/B$ is a degree preserving map. Therefore, the quotient $\frac{H^*(G/B;\mathbb{Z})}{ker(i^*)}$ must also be generated by its degree 2 classes. This means that the cohomology ring of the Peterson variety is generated by its degree 2 cohomology classes as well. ☐

CHAPTER 4
PATCH IDEALS, CELL DECOMPOSITIONS, AND SINGULAR LOCI IN TYPE A_{N-1}

The previous chapter described the topology of regular nilpotent Hessenberg varieties by giving a cellular decomposition. We now turn our attention to questions about the local features of their geometry. We prove that they are local complete intersections, and we characterize where they look locally like smooth varieties.

4.1 Patches in flag varieties

Since the flag variety $GL_n(\mathbb{C})/B$ is a smooth complex variety, it is obtained by gluing together smooth affine coordinate patches. Each patch in the flag variety is actually a collection of B-cosets. We will identify explicit representatives for these cosets. This gives explicit coordinates for each patch. In order to perform local computations for Hessenberg varieties, we then identify polynomial equations that define how the Hessenberg variety is "cut out" of each patch.

First we identify the patches in the flag variety $GL_n(\mathbb{C})/B$ with translates of the big opposite Schubert cell $B^-(id)B/B \cong U^-$. The cell $B^-(id)B/B \cong U^-(id)B/B$ is an affine open neighborhood of the flag $(id)B$. Translating by w we find that the space $wU^-(id)B/B$ is an affine open neighborhood of wB in $GL_n(\mathbb{C})/B$. We denote this neighborhood by $\mathcal{N}_w := wU^-(id)B/B$ and call it a *patch* in the flag variety. The set of patches $\{\mathcal{N}_w\} = \{wU^-(id)B/B\}$ covers the flag variety. Indeed, each patch $\mathcal{N}_w = wU^-(id)B/B$ contains the Schubert cell $U_w(w)B/B$ and the Schubert cells partition the flag variety.

Similarly, we define a *patch* in the Hessenberg variety $Hess(X, h)$ to be

$$\mathcal{N}_{w,h} = wB^{-}B/B \cap Hess(X, h).$$

To obtain explicit coordinates for the patch $\mathcal{N}_w$ we use Lemma 2.1 to identify $\mathcal{N}_{(id)} \cong U^{-} \cong \mathbb{A}^{\binom{n}{2}}$. It follows that the each patch $\mathcal{N}_w$ can be identified with the set of invertible matrices $\{wu : u \in U^{-}\} \cong \mathbb{A}^{\binom{n}{2}}$. With the identification $wU^{-} \cong \mathcal{N}_w$ in mind, we refer to the set $\mathcal{M}_w = wU^{-}$ as the *matrix patch.* Figure 4.1 gives a diagram of the flag variety GL_3/B and the six matrix patches $\mathcal{M}_w$ that cover it.

4.2 Matrix patches in the Hessenberg variety

To understand what a Hessenberg variety looks like locally near a point wB we study the patch $\mathcal{N}_{w,h} \cong (wB^{-}(id)B/B \cap Hess(X, h))$ obtained by intersecting the patch $\mathcal{N}_w$ with the Hessenberg variety. In terms of matrix patches this amounts to finding the matrices $M \in \mathcal{M}_w = wU^{-}$ for which the regular nilpotent matrix X maps the jth column M_j into the span of the first $h(j)$ columns of M for all $1 \leq j \leq n$. The following algorithm explicitly describes how to compute the matrix patch $\mathcal{M}_{w,h}$ corresponding to the patch $\mathcal{N}_{w,h} \cong (wB^{-}(id)B/B \cap Hess(X, h))$.

Algorithm 4.1 (Patch ideal algorithm for Hessenberg varieties)**.** *This algorithm identifies the matrices M in $\mathcal{M}_w$ such that X maps the jth column of M into the span of the first $h(j)$ columns of M. Let $M = (x_{ij})_{i,j}$ denote a generic matrix in*

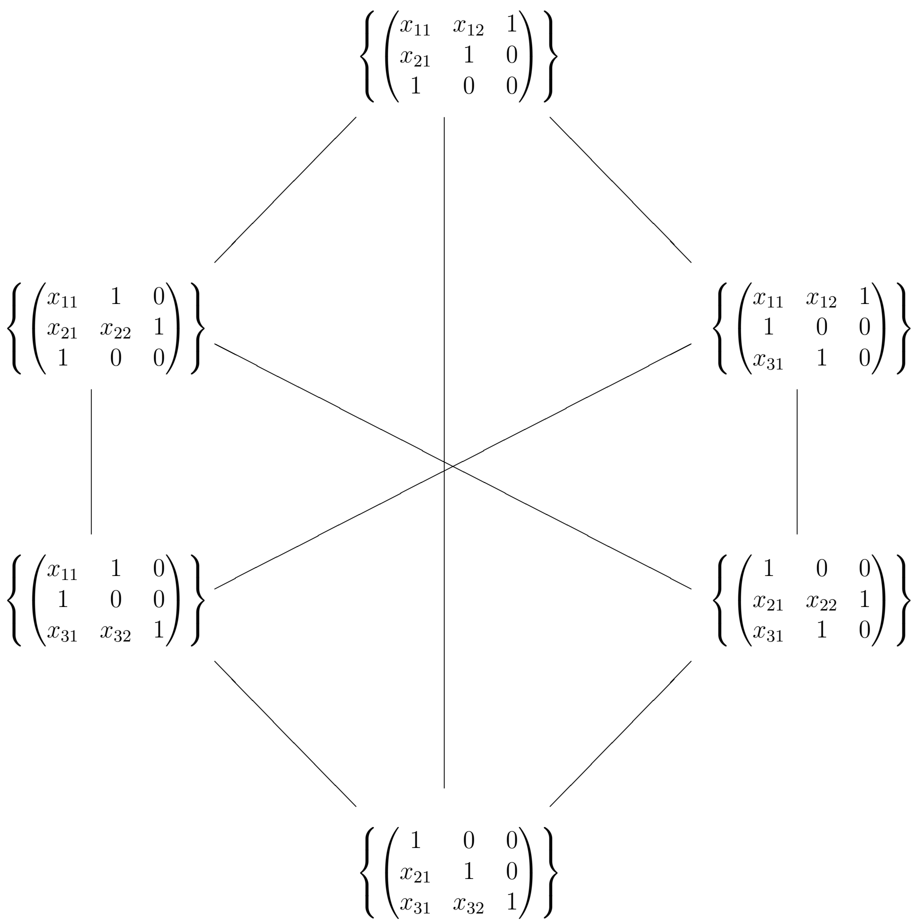


Figure 4.1: Matrix Patches for $GL_3(\mathbb{C})/B$

$\mathcal{M}_w \cong wU^-$. *Note that*

$$x_{w(j),j} = 1 \text{ and } x_{ij} = 0 \text{ if } j > w^{-1}(i). \tag{4.1}$$

Let $\boldsymbol{x} = \{x_{ij}\}$ *denote the collection of variables which have not been set equal to* 0 *or* 1 *by (4.1). To be brief, let* $\mathbb{C}[\boldsymbol{x}]$ *denote the polynomial ring with these variables, i.e.,* $\mathbb{C}[\boldsymbol{x}]$ *denotes the coordinate ring of the matrix patch* $\mathcal{M}_w$*. Let* X *be the regular nilpotent* $n \times n$ *matrix consisting of* 1*'s on the main superdiagonal and* 0*'s elsewhere.*

The product $X \cdot M_j$ *is contained in the span of the first* $h(j)$ *columns if there exist unknowns* $\{\alpha_{j,\ell}\}$ *satisfying the following equation:*

$$X \cdot M_j = \alpha_{j,1} M_1 + \alpha_{j,2} M_2 + \cdots + \alpha_{j,h(j)} M_{h(j)} \qquad \text{for } 1 \leq j \leq n. \tag{4.2}$$

We can solve for these unkowns $\{\alpha_{j,\ell}\}$ *in terms of the variables* $x_{i,j}$*. Look at the* k*th row of equation (4.2). This is the following equation in the variables* $\{x_{k+1,j}, x_{k,1}, \ldots, x_{k,h(j)}\}$ *with coefficients* $\alpha_{j,1}, \alpha_{j,2}, \ldots, \alpha_{j,h(j)}$ *in the polynomial ring* $\mathbb{C}[\boldsymbol{x}]$*.*

$$\alpha_{j,1} x_{k,1} + \cdots + \alpha_{j,h(j)} x_{k,h(j)} = x_{k+1,j} \tag{4.3}$$

where we use the convention $x_{n+1,j} = 0$*. We then solve for the unknowns* $\alpha_{j,i}$ *using the rows*

$w(1), w(2), \ldots, w(h(j))$*. We use the symbol* $\overline{\alpha_{j,i}}$ *to denote the fact that we have already found the polynomial value of* $\alpha_{j,i}$ *and substituted it into the next equation. We see that the* $w(1)$*th row has the form* $x_{w(1)+1,j} = \alpha_{j,1}$*. The* $w(2)$*th row has the form* $x_{w(2)+1,j} = \overline{\alpha_{j,1}} x_{w(2),1} + \alpha_{j,2}$*. In general the* $w(l)$*th row will have the form*

$$x_{w(l)+1,j} = \overline{\alpha_{j,1}} x_{w(l),1} + \overline{\alpha_{j,2}} x_{w(l),2} + \cdots + \overline{\alpha_{j,l-1}} x_{w(l),l-1} + \alpha_{j,l}. \tag{4.4}$$

Remark 4.1. From this construction we see that each of the unknowns $\overline{\alpha_{j,1}}, \ldots, \overline{\alpha_{j,h(j)}}$ is a polynomial in the variables from columns $1, \ldots, h(j) - 1$. In particular, they do not contain any variable $x_{*,h(j)}$ from the $(h(j))$th column. ⊞

There are $n - h(j)$ rows which we did not use to find the $\alpha_{j,1}, \alpha_{j,2}, \ldots, \alpha_{j,h(j)}$. These are the kth rows, where $k \neq w(\ell)$ for any $1 \leq \ell \leq h(j)$. After we have solved for all of $\overline{\alpha_{j,1}}, \ldots, \overline{\alpha_{j,h(j)}}$ we substitute these expressions into the remaining rows to obtain polynomials of the form

$$g_{k,j} = \overline{\alpha_{j,1}} x_{k,1} + \cdots + \overline{\alpha_{j,h(j)}} x_{k,h(j)} - x_{k+1,j} \in \mathbb{C}[\boldsymbol{x}].$$

Define the ideal

$$I_{w,Hess(X,h)} = \langle g_{k,j} : k \neq w(\ell), 1 \leq \ell \leq h(j) \rangle \subseteq \mathbb{C}[\boldsymbol{x}]. \tag{4.5}$$

Then $\mathcal{M}_{w,h} \cong \operatorname{Spec}\left(\frac{\mathbb{C}[\boldsymbol{x}]}{I_{w,Hess(X,h)}}\right)$. We call the ideal $I_{w,Hess(X,h)}$ the patch ideal of the Hessenberg variety $Hess(X,h)$ at wB. It describes the intersection of the Hessenberg variety and the patch $\mathcal{N}_w$ as a set.

Remark 4.2. This construction shows that the polynomial g_{ij} is a generator of $I_{w,Hess(X,h)}$ if and only if $h(j) < w^{-1}(i)$. When $h(j) \geq w^{-1}(i)$ the ith row is used to solve for one of the unknowns α_{jk} with $1 \leq k \leq h(j)$. $\boxplus$

From the definition of the generators $g_{k,j}$ can prove the following result, which will be used in the proof of Proposition 4.8

Lemma 4.2. *Suppose that $h(i) > h(j)$ for all $j < i$. Then the variable $x_{l,h(i)}$ does not appear in any of the polynomial generators $g_{k,i}$ for $k < l$ or $g_{l,j}$ for $j < i$.*

Proof. The generators $g_{k,i}$ for $k < l$ have the form

$$g_{k,i} = \overline{\alpha_{i,1}} x_{k,1} + \cdots + \overline{\alpha_{i,h(i)}} x_{k,h(i)} - x_{k+1,i}.$$

Remark 4.1 pointed out that none of the $\overline{\alpha_{i,1}}, \ldots, \overline{\alpha_{i,h(i)}}$ contain the variable $x_{l,h(i)}$. Thus, none of the $g_{k,i}$ for $k < l$ contains the variable $x_{l,h(i)}$. In addition, the generators $g_{l,j}$ for $1 \leq j < i$ are polynomials in the variables from columns $1, \ldots, h(j)$. Since $h(j) < h(i)$, we conclude that $x_{l,h(i)}$ does not appear in any $g_{l,j}$ with $j < i$. $\square$

Following the defintions through, we also prove the following result which is used to prove Theorem 4.10.

Lemma 4.3. *Each g_{kj} has the form*

$$g_{kj} = x_{k,w^{-1}(w(j)-1)} - x_{k+1,j} + (\textit{terms of degree} > 2).$$

Proof. Let $l = w^{-1}(w(j) - 1)$. Then we have $x_{w(l)+1,j} = x_{w(j),j} = 1$. Substituting this into Equation 4.4 we see that

$$1 = \overline{\alpha_{j,1}} x_{w(l),1} + \overline{\alpha_{j,2}} x_{w(l),2} + \cdots + \overline{\alpha_{j,l-1}} x_{w(l),l-1} + \alpha_{j,l}.$$

Solving for $\alpha_{j,l}$ we find $\alpha_{j,l} = 1 + \sum_{i<l} \overline{\alpha_{j,i}} x_{w(l),i}$. This is the only unknown $\alpha_{j,*}$ that contains a linear term. The rest are polynomials in $\{x_{ij}\}$ of degree ≥ 1. Hence each

g_{kj} has the form

$$g_{kj} = x_{k,w^{-1}(w(j)-1)} - x_{k+1,j} + (\text{terms of degree} > 2).$$

$\square$

Remark 4.3. From this set up we obtain a set-theoretic isomorphism $\mathcal{M}_{w,h} \cong \mathcal{N}_{w,h}$. However, Example 4.2 shows that the patch ideal $I_{w,Hess(X,h)}$ may not be reduced, and thus $\mathcal{M}_{w,h}$ may differ from $\mathcal{N}_{w,h}$ scheme-theoretically. In fact, the scheme defined by the patch ideal $I_{w,Hess(X,h)}$ will not be reduced if $h(i) = i$ for any $1 \leq i \leq n$. $\boxplus$

Example 4.1. We calculate the patch ideal for the point wB in $Hess(X,h)$ where $h(1,2,3,4) = (2,4,4,4)$ and

$$w = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \mathcal{M}_w = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{21} & x_{22} & 1 & 0 \\ x_{31} & 1 & 0 & 0 \\ x_{41} & x_{42} & x_{43} & 1 \end{pmatrix} \right\}.$$

The matrix patch $\mathcal{M}_{w,h}$ is the set

$$\mathcal{M}_{w,h} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{21} & x_{22} & 1 & 0 \\ x_{31} & 1 & 0 & 0 \\ x_{41} & x_{42} & x_{43} & 1 \end{pmatrix} : \begin{pmatrix} x_{21} \\ x_{31} \\ x_{41} \\ 0 \end{pmatrix} = \alpha_{11} \begin{pmatrix} 1 \\ x_{21} \\ x_{31} \\ x_{41} \end{pmatrix} + \alpha_{12} \begin{pmatrix} 0 \\ x_{22} \\ 1 \\ x_{42} \end{pmatrix} \right\}.$$

Here $\overline{\alpha_{11}} = x_{21}$ and $\overline{\alpha_{12}} = x_{41} - \overline{\alpha_{11}}x_{31} = x_{41} - x_{21}x_{31}$. The polynomials

$$g_{21} = x_{31} - \overline{\alpha_{11}}x_{21} + \overline{\alpha_{12}}x_{22} = x_{31} - x_{21}^2 + (x_{41} - x_{21}x_{31})x_{22}$$

$$g_{41} = \overline{\alpha_{11}}x_{41} + \overline{\alpha_{12}}x_{42} = x_{21}x_{41} + (x_{41} - x_{21}x_{31})x_{42}$$

generate the patch ideal

$$I_{w,Hess(X,h)} = \langle g_{21}, g_{41} \rangle = \langle x_{31} - x_{21}^2 + (x_{41} - x_{21}x_{31})x_{22}, x_{21}x_{41} + (x_{41} - x_{21}x_{31})x_{42} \rangle.$$

$\boxplus$

Example 4.2. Patch ideals may not be reduced. Let $h(1,2,3) = (1,3,3)$ and N be the regular nilpotent operator. Then the matrix patch

$$\mathcal{M}_{(id),h} \cong \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x_{21} & 1 & 0 \\ x_{31} & x_{32} & 1 \end{pmatrix} : \begin{matrix} x_{31} = x_{21}^2 \\ x_{31} = 0 \end{matrix} \right\}.$$

The matrix patch $M_{w,Hess(X,h)} \cong \operatorname{Spec}\left(\frac{\mathbb{C}[x_{21},x_{31},x_{32}]}{\langle x_{31}-x_{21}^2, x_{31}\rangle}\right)$ is not a reduced scheme. $\boxplus$

4.3 Previous work concerning patch ideals

The study of patch ideals in flag varieties appeared implicitly in the work of Fulton, Knutson-Miller, and Woo-Yong [Ful91, KnuMil05, WooYon08]. The first study of patch ideals in the Hessenberg varieties appeared in our work with Alex Yong where we focused on the Peterson variety [InsYon11]. We proved that the patches in the Peterson variety are reduced schemes and complete intersections.

Theorem 4.4 (Insko-Yong 2011, Theorem 1.6)**.** *The matrix patch* $\mathcal{M}_{w,\mathrm{Pet}_n}$ *is a reduced scheme and is isomorphic to the affine neighborhood*

$$\mathcal{N}_{w_J,\mathrm{Pet}_n} = w_J B^- B/B \cap \mathrm{Pet}_n \text{ of } w_J B \in \mathrm{Pet}_n.$$

In studying the Toda lattice and the quantum cohomology of the flag variety, Kostant partially identified the singular locus of the Peterson variety [Kos96, Theorem 6]. This classification of the singular locus requires prior knowledge of the Toda lattice. In contrast, our second main result gives a complete, combinatorial description of the singular locus of the Peterson variety in terms of pattern containment of permutations.

Before stating the result, we recall the notion of pattern containment. Let $w(i)$ denote the ith number in the one-line notation of w. Then the one-line notation of a permutation $w = w(1)w(2)\cdots w(n) \in S_n$ *contains* the pattern

$$u = u(1)u(2)\cdots u(m) \in S_m$$

if there exists indices $i_1 < i_2 < \ldots < i_m$ such that the numbers $w(i_1), w(i_2), \ldots, w(i_m)$ have the same relative order as $u(1), u(2), \ldots, u(m)$. For instance 45123 contains the pattern 3412 in the sequences 4512, 4513, and 4523.

Theorem 4.5 (Insko-Yong 2011, Theorem 1.4)**.** *The singular locus of* Pet_n *is given* *by*

$$Sing(\mathrm{Pet}_n) = \coprod_{w_J} (Bw_JB/B \cap \mathrm{Pet}_n),$$

where the union is over all w_J satisfying any of the equivalent conditions $(I)-(III)$ below:

(I) w_JB is singular in Pet_n

(II) w_J is not one of the permutations:

- $n\ n-1\ n-2\ \cdots\ 3\ 2\ 1$;
- $1\ n\ n-1\ \cdots\ 3\ 2$; or
- $n-1\ n-2\ \cdots\ 1\ n$.

(III) w_J contains at least one of the patterns 123 *or* 2143*.*

Example 4.3. The permutations $\underline{123}4$, $\underline{124}3$, $\underline{13}2\underline{4}$, and $2\underline{134}$ all contain the pattern 123 in the underlined positions. In addition, 2143 obviously contains the pattern 2143. Thus, the singular cells $BwB/B \cap Hess(X, h)$ in the Hessenberg variety with $h = (2, 3, 4, 4)$ are those corresponding to the permutations 1234, 1243, 1324, 2134 and 2143. The singular locus is circled in the Figure 4.10. Since the cell $B_{2143}B/B \cap \mathrm{Pet}_n$ is $\mathbb{C}$ -codimension 1 in the Peterson variety, this example shows that the Peterson variety in $GL_4(\mathbb{C})/B$ is not normal.

$\boxplus$

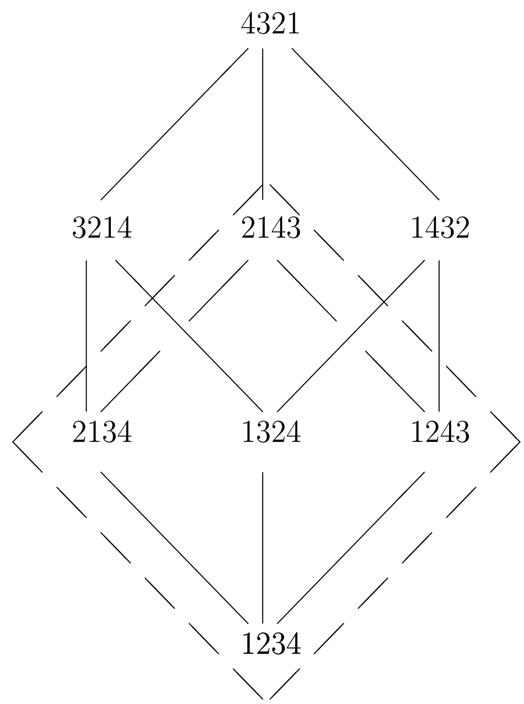


Figure 4.2: The Singular Locus of $\mathfrak{Pet}_4$

4.4 Patches are reduced complete intersections

Following the notation of Section 2.1.3, let

$$V = \mathrm{Spec}\left(\frac{\mathbb{C}[x_1, \ldots, x_n]}{I}\right)$$

be an affine variety. Recall that V is a *complete intersection* if the ideal I can be generated by $n - m$ generators; in other words, V is a complete intersection if the number of generators for I is equal to the codimension of V in $\mathbb{C}^n$. A projective variety V is called a *local complete intersection* if every point $x \in V$ has an affine neighborhood which is a complete intersection. An affine scheme is called *generically* *reduced* if it is irreducible and reduced at a general point. Local complete intersections have the nice property that they are reduced if and only if they are generically reduced.

In this section, we prove that the matrix patches $\mathcal{M}_{w,h}$ are complete intersections for all regular nilpotent Hessenberg varieties. Then we generalize Theorem 4.4 by proving that the matrix patches $\mathcal{M}_{w,h}$ are reduced schemes for all regular nilpotent Hessenberg varieties with strictly increasing Hessenberg functions.

Since the number of generators in the patch ideal is equal to the codimension of the Hessenberg variety we obtain the following lemma.

Lemma 4.6. *The matrix patch $\mathcal{M}_{w,h}$ is a complete intersection.*

Proof. Let $m(h) = \sum_{i=1}^{n}(n-h(i))$. In defining the patch ideal $I_{w,h}$ we noted that there are $m(h)$ generators for $I_{w,h}$. The work of Anderson and Tymoczko proved that the $\mathbb{C}$-dimension of a regular nilpotent Hessenberg variety $Hess(X,h)$ is $\sum_{i=1}^{n}(h(i)-i)$ [AndTym08, Lemma 7.1]. The $\mathbb{C}$-dimension of the full flag variety is $\sum_{i=1}^{n}(n-i)$. Thus the $\mathbb{C}$-codimension of the regular nilpotent Hessenberg variety $Hess(X,h)$ is

$$\left(\sum_{i=1}^{n}(n-i)\right) - \left(\sum_{i=1}^{n}(h(i)-i)\right) = \sum_{i=1}^{n}(n-h(i)) = m(h).$$

That means the number of generators for the patch ideal $I_{w,h}$ equals the $\mathbb{C}$-codimension of the regular nilpotent Hessenberg variety $\mathcal{M}_{w,h}$ in $\mathcal{M}_w$. We conclude that $\mathcal{M}_{w,h}$ is a complete intersection. □

Lemma 4.7. *Let $h(1,2,\ldots,n) = (j_1, j_2, \ldots, j_k, n, \ldots, n)$ be a Hessenberg function satisfying $1 < j_1 < j_2 < \cdots < j_k < n$. Let $Hess(X,h)$ be the regular nilpotent Hes-*

senberg variety associated with h. Let wB be any fixed point contained in $Hess(X,h)$. Let $\Psi(\boldsymbol{x}) = \prod_{1\le i\le k} \overline{\alpha_{i,h(i)}}$. (Recall that we use the symbol $\overline{\alpha_{ij}}$ to denote the polynomial value of α_{ij} we found in Algorith 4.1.) The set of points $p \in \mathcal{M}_{w,h}$ such that $\Psi(p) \neq 0$ is dense in $\mathcal{M}_{w,h}$.

Proof. Since $h = (j_1, j_2, \ldots, j_k, n, \ldots, n)$ with $1 < j_1 < j_2 < \ldots < j_k$ the set of points

$$\{p \in \mathcal{M}_{w,h} : \overline{\alpha_{j_i,h(j_i)}}(p) = 0\}$$

is contained in a matrix patch $\mathcal{M}_{w,h'}$ associated to a Hessenberg variety $Hess(X,h')$ with

$$h' = (j_1, j_2, \ldots, j_{i-1}, j_i - 1, j_{i+1}, \ldots, j_k, n, \ldots, n).$$

The variety $Hess(X,h')$ is codimension 1 in $Hess(X,h)$. So the patch $\mathcal{M}_{w,h'}$ is codimension 1 in $\mathcal{M}_{w,h}$.

The set of points p in $\mathcal{M}_{w,h}$ where $\Psi(p) = 0$ is the finite union of matrix patches $\coprod_{1\le i\le k} \mathcal{M}_{w,h'_i}$. Each of these $\mathcal{M}_{w,h'_i}$ is an algebraic set of codimension 1 in $\mathcal{M}_{w,h}$. Therefore, the set of points $p \in \mathcal{M}_{w,h}$ where $\Psi(p) \neq 0$ is dense in $\mathcal{M}_{w,h}$. □

Proposition 4.8. *Let $h(1,2,\ldots,n) = (j_1, j_2, \ldots, j_k, n, \ldots, n)$ be a Hessenberg function with $1 < j_1 < j_2 < \cdots < j_k < n$. Let $Hess(X,h)$ be the regular nilpotent Hessenberg variety associated with h. Let wB be any fixed point contained in $Hess(X,h)$. The set of points $p \in \mathcal{M}_{w,h}$ where the Jacobian evaluated at p has maximal rank*

$$\mathrm{rk}(\mathrm{Jac}(I_{w,H})(p)) = m(h)$$

is dense in $\mathcal{M}_{w,h}$. In other words, a general point in $\mathcal{M}_{w,h}$ is smooth.

Proof. Let $m(h) = \sum_{i=1}^{n}(n-h(i))$. Order the rows of the Jacobian $\left[\frac{\partial g_{l,i}}{\partial x_{j,k}}\right]$ by favoring smaller l first and breaking ties by favoring smaller i. Order the columns by favoring smaller j first and breaking ties by favoring smaller k. Let p be any point in $\mathcal{M}_{w,h}$ with $\Psi(p) \neq 0$. We argue that the Jacobian

$$\left[\frac{\partial g_{l,i}(p)}{\partial x_{j,k}}\right]$$

evaluated at p has full rank by showing there is an $m(h) \times m(h)$ submatrix that is lower-triangular with respect to this ordering.

By Lemma 4.2, the variable $x_{l,h(i)}$ does not appear in any of the polynomial generators $g_{k,i}$ for $k < l$ or $g_{l,j}$ for $j < i$. We conclude that the column of the Jacobian corresponding to $\partial/\partial x_{l,h(i)}$ contains only 0 before the row corresponding to $g_{l,i}$.

Next we prove that the column corresponding to $x_{l,h(i)}$ has a nonzero entry in the row corresponding to the polynomial $g_{l,i}$. Remark 4.1 notes that the $\overline{\alpha_{i,j}}$ for $1 \leq j \leq h(i)$ are polynomials in the variables from columns $1, 2, \ldots, h(i) - 1$. In particular, the variable $x_{l,h(i)}$ does not appear in any of the $\overline{\alpha_{i,j}}$ for $1 \leq j \leq h(i)$. From this we see that $\overline{\alpha_{i,h(i)}}x_{l,h(i)}$ is the unique summand in

$$g_{l,i} = x_{l+1,i} - \overline{\alpha_{i,1}}x_{l,1} + \cdots + \overline{\alpha_{i,h(i)}}x_{l,h(i)}$$

involving $x_{l,h(i)}$. This implies that $\frac{\partial g_{l,i}}{\partial x_{l,h(i)}} = \frac{\partial \overline{\alpha_{i,h(i)}}x_{l,h(i)}}{\partial x_{l,h(i)}} = \overline{\alpha_{i,h(i)}}$. We conclude that the row of the Jacobian corresponding to $g_{l,i}$ has $\overline{\alpha_{i,h(i)}}$ in the column corresponding to $\frac{\partial}{\partial x_{l,h(i)}}$.

Since $\overline{\alpha_{i,h(i)}}(p) \neq 0$ for all $1 \leq i \leq k$ we conclude that the $m(h) \times m(h)$ submatrix

$$\left[\frac{\partial g_{l,i}}{\partial x_{l,h(i)}}\right]$$

of the Jacobian is lower-triangular when evaluated at p. We conclude the Jacobian has rank $m(h)$. Therefore, any point p where $\Psi(p) \neq 0$ is a smooth point in $\mathcal{M}_{w,h}$. $\square$

Our next theorem shows that the matrix patch is reduced and $\mathcal{M}_{w,h} \cong \mathcal{N}_{w,h}$ for all regular nilpotent Hessenberg varieties with strictly increasing Hessenberg functions. It proves that these Hessenberg varieties are local complete intersections.

Theorem 4.9. *Let $h(1, 2, \ldots, n) = (j_1, j_2, \ldots, j_k, n, \ldots, n)$ be a Hessenberg function satisfying $1 < j_1 < j_2 < \cdots < j_k < n$. Let $Hess(X, h)$ be the regular nilpotent Hessenberg variety associated with h. Let wB be any fixed point contained in $Hess(X, h)$. The matrix patch $\mathcal{M}_{w,h}$ is a reduced and hence $\mathcal{M}_{w,h} \cong \mathcal{N}_{w,h}$.*

Proof. Since the patch ideal $I_{w,h}$ defines a complete intersection (by Lemma 4.6), it follows that the scheme it defines is Cohen-Macaulay [Eis95, Prop 18.13,Cor 21.19]. A Cohen-Macaulay scheme is reduced if and only if it is generically reduced [Eis95, Exercise 18.9].

We now show that the matrix patch $\mathcal{M}_{w,h}$ is generically reduced. The work of Anderson and Tymoczko showed that every regular nilpotent Hessenberg variety is irreducible [AndTym08, Lemma 7.1]. The patch $\mathcal{N}_{w,h}$ is obtained by intersecting a dense open algebraic subset of $GL_n(\mathbb{C})/B$ with the irreducible subvariety $Hess(X, h)$.

Hence $\mathcal{N}_{w,h}$ is a dense open algebraic subset of $Hess(X, h)$. Open algebraic subsets of irreducible varieties are irreducible. Hence $\mathcal{N}_{w,h}$ is irreducible. The matrix patch $\mathcal{M}_{w,h}$ is set-theoretically isomorphic to $\mathcal{N}_{w,h}$. So it too is irreducible.

Proposition 4.8 showed that a general point in $\mathcal{M}_{w,h}$ is smooth. Since it is smooth (regular) outside a subvariety of codimension 0 the matrix patch $\mathcal{M}_{w,h}$ satisfies Serre's R_0 condition. Hence the matrix patch $\mathcal{M}_{w,h}$ is generically reduced. The fact that $\mathcal{M}_{w,h}$ is generically reduced and Cohen-Macaulay implies that it is reduced. Since $\mathcal{M}_{w,h}$ is set-theoretically isomorphic to $\mathcal{N}_{w,h}$ and a reduced scheme, we conclude that $\mathcal{M}_{w,h} \cong \mathcal{N}_{w,h}$ is a scheme-theoretic isomorphism. $\square$

The following example may shed some light on the proof.

Example 4.4. Let $h = (3, 4, 5, 5, 5)$ and $w = 21345$. Here $k = 3$ and $m(h) = 3$. Recall that we use the notation $\overline{\alpha_{i,j}}$ to denote the fact that we have found the polynomial value for $\alpha_{i,j}$. The matrix patch in this case is

$$\mathcal{M}_{w,h} = \left\{ \begin{pmatrix} x_{11} & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ x_{31} & x_{32} & 1 & 0 & 0 \\ x_{41} & x_{42} & x_{43} & 1 & 0 \\ x_{51} & x_{52} & x_{53} & x_{54} & 1 \end{pmatrix} : \begin{array}{c} x_{51} = \overline{\alpha_{11}}x_{41} + \overline{\alpha_{12}}x_{42} + \overline{\alpha_{13}}x_{43} \\ 0 = \overline{\alpha_{11}}x_{51} + \overline{\alpha_{12}}x_{52} + \overline{\alpha_{13}}x_{53} \\ 0 = \overline{\alpha_{21}}x_{51} + \overline{\alpha_{22}}x_{52} + \overline{\alpha_{23}}x_{53} + \overline{\alpha_{24}}x_{54} \end{array} \right\}$$

The Jacobian is

$\partial/\partial x_{41}$	$\partial/\partial x_{42}$	$\partial/\partial x_{43}$	$\partial/\partial x_{51}$	$\partial/\partial x_{52}$	$\partial/\partial x_{53}$	$\partial/\partial x_{54}$	
$\overline{\alpha_{11}}$	$\overline{\alpha_{12}}$	$\overline{\alpha_{13}}$	1				g_{41}
			$\overline{\alpha_{11}}$	$\overline{\alpha_{12}}$	$\overline{\alpha_{13}}$		g_{51}
			$\overline{\alpha_{21}}$	$\overline{\alpha_{22}}$	$\overline{\alpha_{23}}$	$\overline{\alpha_{24}}$	g_{52}

⊞

4.5 Singular loci of Hessenbergs

This section gives a combinatorial description of the singular loci of regular nilpotent Hessenberg varieties.

Theorem 4.10. *The cell $BwB/B \cap Hess(X,h)$ is singular in $Hess(X,h)$ if and only if there exists an integer k such that all of the following inequalities hold:*

$$h(w^{-1}(k)) < w^{-1}(n)$$

$$h(w^{-1}(k-1)) < w^{-1}(n-1),$$

$$\vdots$$

$$h(w^{-1}(1)) < w^{-1}(n-k+1).$$

Proof. Remark 4.2 noted that the ideal $I_{w,Hess(X,h)}$ is generated by the polynomials g_{ij} for $h(j) < w^{-1}(i)$. From Lemma 4.3 each g_{ij} has the form

$$g_{ij} = x_{i,w^{-1}(w(j)-1)} - x_{i+1,j} + (\text{terms of degree} > 2).$$

Consequently, the row corresponding to g_{ij} in the Jacobian will have a 1 in the column corresponding to $\partial/\partial x_{i+1,j}$ and -1 in the column corresponding to $\partial/\partial x_{i,w^{-1}(w(j))-1))}$.

The row corresponding to g_{ij} will have a 0 everywhere else when the Jacobian is evaluated at the origin $\mathbf{x} = 0$.

There are two exceptions where the linear terms $x_{i+1,j}$ or $x_{i,w^{-1}(w(j)-1)}$ may not appear:

- $x_{n+1,j} = 0$ in $g_{n,j}$ and
- $x_{i,w^{-1}(w(w^{-1}(1))-1)} = x_{i,0} = 0$ in $g_{i,w^{-1}(1)}$.

Order the $g_{i,j}$ and the $x_{i,j}$ according to the rules:

- $(i_1, j_1) < (i_2, j_2)$ when $i_1 < i_2$ and
- $(i_1, j_1) < (i_1, j_2)$ if $j_1 < j_2$.

This implies that the Jacobian will have the block form

	$\partial/\partial x_{i,w^{-1}(1)}$	$\partial/\partial x_{i,w^{-1}(2)}$	$\partial/\partial x_{i,w^{-1}(3)}$	$\cdots$	$\partial/\partial x_{i,w^{-1}(n)}$
$g_{1,w^{-1}(i)}$	-N	0	0	0	0
$g_{2,w^{-1}(i)}$	I	-N	0	0	0
$g_{3,w^{-1}(i)}$	0	I	-N	0	0
$\vdots$	0	0	$\ddots$	$\ddots$	0
$g_{n,w^{-1}(i)}$	0	0	0	I	-N

where N will be a block matrix of the form

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and I is the identity matrix.

If the polynomials $g_{n,w^{-1}(k)}, g_{n-1,w^{-1}(k-1)}, \ldots, g_{n-k+1,w^{-1}(1)}$ are all in $I_{w,Hess(X,h)}$ for some $1 \leq k \leq n$ then the rank of those rows will be less than k. Hence the Jacobian will fail to have maximal rank. This happens precisely when there is some k with $1 \leq k \leq n - h(1)$ such that all of the following inequalities hold:

$$\begin{gathered} h(w^{-1}(k)) < w^{-1}(n) \\ h(w^{-1}(k-1)) < w^{-1}(n-1) \\ \vdots \\ h(w^{-1}(1)) < w^{-1}(n-k+1). \end{gathered}$$

When the Jacobian fails to have full rank, the rank of the Jacobian does not equal to the codimension of $\mathcal{N}_{w,h}$ in $\mathcal{N}_w$. Thus the point wB is singular in $Hess(X,h)$.

On the other hand, if no such k exists, there is no cancellation between the entries in the I blocks and the entries in the $-N$ blocks. Hence, the Jacobian will have full rank, and the point wB will be smooth. $\square$

Example 4.5. When $h = (2,3,4,4)$ we noted that the cell $BwB/B \cap Hess(X,h)$ was singular for $w = 2143$. This is because $h(w^{-1}(2)) = h(1) = 2 < w^{-1}(4) = 3$ and $h(w^{-1}(1)) = h(2) = 3 < w^{-1}(3) = 4$. However, if $h = (2,4,4,4)$ the cell $BwB/B \cap Hess(X,h)$ for $w = 2143$ is smooth. While $h(w^{-1}(2)) = 2 < w^{-1}(4) = 3$ we have $h(w^{-1}(1)) = 4 \not< w^{-1}(3) = 4$. ⊞

Example 4.6. Consider the matrix patch $\mathcal{M}_{w,h}$ for $w = 2143$ in the Hessenberg variety $Hess(X,h)$ corresponding to $h = (2,3,4,4)$.

$$\left\{ \begin{pmatrix} x_{11} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ x_{31} & x_{32} & x_{33} & 1 \\ x_{41} & x_{42} & 1 & 0 \end{pmatrix} : \begin{array}{c} g_{31} : x_{41} = x_{31}x_{31} + (1 - x_{31}x_{11})x_{32} \\ g_{41} : 0 = x_{31}x_{41} + (1 - x_{31}x_{11})x_{42} \\ g_{32} : x_{42} = x_{32}x_{31} + (-x_{32}x_{11})x_{32} + (-x_{32}x_{41} + x_{32}x_{11}x_{42})x_{33} \end{array} \right\}$$

The Jacobian evaluated at the origin $\mathbf{x} = 0$ is:

	$\partial/\partial x_{32}$	$\partial/\partial x_{42}$	$\partial/\partial x_{31}$	$\partial/\partial x_{41}$
g_{32}	0	-1	0	0
g_{42}	0	0	0	0
g_{31}	1	0	0	-1
g_{41}	0	1	0	0

Note that g_{42} is not a generator of the ideal $I_{w,Hess(X,h)}$. We include it to demonstrate the blocks $-N$ and I of the Jacobian matrix. We see that the row corresponding to $g_{n,w^{-1}(2)}$ cancels with the row corresponding to $g_{n-1,w^{-1}(1)}$. Hence the Jacobian does not have full rank when evaluated at the origin. This proves that

wB is singular in $Hess(X, h)$. ⊞

Example 4.7. Consider patch $\mathcal{M}_{w,h}$ for $w = 321654$ in the Hessenberg variety $Hess(X, h)$ corresponding to $h = (3, 4, 5, 6, 6, 6)$.

	$\partial/\partial x_{43}$	$\partial/\partial x_{53}$	$\partial/\partial x_{42}$	$\partial/\partial x_{52}$	$\partial/\partial x_{62}$	$\partial/\partial x_{41}$	$\partial/\partial x_{51}$	$\partial/\partial x_{61}$
g_{43}	0	-1	0	0	0	0	0	0
g_{53}	0	0	0	0	0	0	0	0
g_{42}	1	0	0	-1	0	0	0	0
g_{52}	0	1	0	0	-1	0	0	0
g_{62}	0	0	0	0	0	0	0	0
g_{41}	0	0	1	0	0	0	-1	0
g_{51}	0	0	0	1	0	0	0	-1
g_{61}	0	0	0	0	1	0	0	0

For $k = 3$, we see that

$$h(w^{-1}(3)) < w^{-1}(6) \qquad h(w^{-1}(2)) < w^{-1}(5) \qquad h(w^{-1}(1)) < w^{-1}(4).$$

We have highlighted the corresponding rows g_{43}, g_{52}, and g_{61}, which fail to have maximal rank. ⊞

Example 4.8 (A combinatorial criterion to test for singularity)**.** This example provides a visual algorithm to check the singularity of a cell $BwB/B \cap Hess(X, h)$.

In an $n \times n$ grid, draw dots in place of the 1's in the permutation matrix of

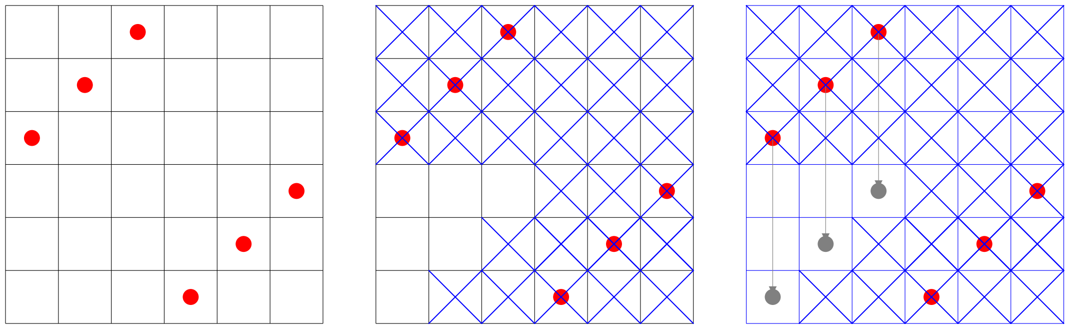


Figure 4.3: An h-patch diagram

w. For $1 \leq i \leq n$ color the boxes corresponding to the matrix coordinates

$$(i, w^{-1}(1)),\ (i, w^{-1}(2)),\ \ldots, (i, w^{-1}(h(i))).$$

We call the resulting grid an *h-patch diagram for w*. If for some k (with $w^{-1}(i) < k$) the first k rows can be dropped by $n-k$ rows and the dots fit into the uncolored part of the h-patch diagram for w, then the cell $BwB/B \cap Hess(X,h)$ is singular.

Figure 4.3 gives the h-patch diagram for $h = (3,4,5,6,6,6)$ and $w = 321654$. For $k = 3$, we see that the first k rows can be dropped $n-k = 3$ rows and the dots fit in the uncolored part of the diagram. Thus the cell $BwB/B \cap Hess(X,h)$ is singular in $Hess(X,h)$.

$\boxplus$

4.6 Equivariant cohomology localizations

Throughout this section let $S = \{(t, t^2, t^3, \ldots, t^n) : t \in \mathbb{C}^*\}$. This is a one-dimensional subtorus of T which preserves the Hessenberg variety. This section describes the localizations of the S-equivariant cohomology classes for the regular nilpotent Hessenberg varieties in the S-equivariant cohomology ring of the full flag variety. It also describes an algorithm for computing these localizations using the *h-patch diagrams* described in the previous section.

As noted earlier, equivariant cohomology can actually be easier to calculate than regular cohomology. One reason for this is that the calculation of equivariant cohomology classes can be reduced to a "local calculation:" Rather than calculate the equivariant cohomology class of the Hessenberg variety as a whole, we calculate the equivariant cohomology class of each patch using techniques from combinatorial commutative algebra. This local calculation is significantly easier than the global one. When a variety is equivariantly formal, it turns out that the equivariant cohomology class of that variety is the direct sum of its localizations. So these local calculations completely determine the equivariant cohomology class of the Hessenberg variety $[Hess(X,h)]_S$ in the equivariant cohomology ring of the flag variety $H^*_S(GL_n(\mathbb{C})/B)$.

We now describe a technique using combinatorial commutative algebra to define the localization of S-equivariant cohomology classes of regular nilpotent Hessenberg varieties. These techniques are described in terms of *multidegrees* of S-graded free resolutions in Chapter 8 of Miller-Sturmfels *Combinatorial Commutative Algebra.* However, at the end of that chapter there is an exercise proving that the *multidegree* of a complete intersection can be calculated without knowing the S-graded free resolution. So we will use Theorem 4.12 as our definition of multidegrees, and then explain why they give equivariant localizations.

A polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ is said to be *multigraded by a group* T if it is equipped with a *degree map* $\deg : \mathbb{C}[x_1, \ldots, x_n] \to T$ [MilStu04, Definition 8.1]. In this paper we take T to be the group $S = \{(t, t^2, \ldots, t^n) : t \in \mathbb{C}^*\}$. The action of S on $Hess(X, h)$ induces an S-multigrading on the polynomial ring $\mathbb{C}[wU^-]$. The degree map of the S-multigrading is defined by $\deg(x_{ij}) = \frac{t^i}{t^{w(j)}} = t^{i-w(j)}$.

From the degree map of the multigrading, we obtain another algebraic quantity, which we call the *S-weight* of x_{ij}. (Miller and Sturmfels simply call this quantity a *degree*, but we use the term S-weight.) We define this quantity by

$$\text{S-weight}(x_{ij}) = (i - w(j))t. \tag{4.6}$$

Lemma 4.11. *As in Section 4.2, let* $\mathbb{C}[\boldsymbol{x}]$ *denote the coordinate ring of a matrix patch* $\mathcal{M}_w$. *The multidegree of a polynomial ring is trivial:*

$$\mathit{multidegree}(\mathbb{C}[\boldsymbol{x}]) = 1.$$

Theorem 4.12 (Miller-Sturmfels, Exercise 8.12)**.** *Let* M *be an* S*-graded* $\mathbb{C}[\boldsymbol{x}]$*-module, and let* $g \in \mathbb{C}[\boldsymbol{x}]$ *be homogeneous of* S*-weight* b. *Assume that* g *is not a zerodivisor in* M. *Then the* S*-multidegree of the quotient module* M/gM *is the multidegree of* M *times the degree of* g*:*

$$\mathit{multidegree}(M/gM) = b \cdot \mathit{multidegree}(M).$$

Knutson-Miller-Shimozono proved that multidegrees are algebraic reformulations of the geometric *equivariant cohomology classes* of varieties in $\mathbb{C}^n$ [KnuMilShi06, Proposition 1.19]. Their result implies that the S-multidegree of the patch is the S-equivariant localization of $[Hess(X,h)]_S$ at the fixed point wB.

Proposition 4.13 (KnuMilShi06, Proposition 1.19)**.** *The multidegree of a graded module Γ over $\mathbb{C}[\boldsymbol{x}]$ is the class $[\Gamma]_T$ in the equivariant cohomology ring $H_T^*(M)$ where $M = \mathrm{Spec}(\mathbb{C}[\boldsymbol{x}])$.*

Here is a brief summary of why multidegrees give equivariant cohomology classes. Geometrically, a multigrading on a polynomial ring comes from the action of an algebraic torus. Multigraded $\mathbb{C}[x_1, \ldots, x_n]$ - modules correspond to *torus-equivariant sheaves* on the vector space $\mathbb{C}^n$. These modules correspond to elements of the Grothendieck group of torus equivariant sheaves over $\mathbb{C}^n$. The Chern character homomorphism is a map from the equivariant K-theory ring to the equivariant cohomology ring. Knutson-Miller-Shimozono proved that the multidegree of an S-multigraded $\mathbb{C}[x_1, \ldots, x_n]$-module is the Chern character of the corresponding equivariant Grothendieck group element [MilStu04, p172]. The coordinate ring $\frac{\mathbb{C}[\mathbf{x}]}{I_{w,Hess(X,h)}}$ is an S-multigraded module over $\mathbb{C}[\mathbf{x}]$. Its multidegree is the localization at w of the S-equivariant Chern class induced by the structure sheaf $\mathcal{O}_{Hess(X,h)}$ in the equivariant cohomology of the flag variety. From this we conclude that the S-equivariant localization of the class $[Hess(X,h)]_S$ at the fixed point wB is the product of the S-weights of the generators of the patch ideal.

Theorem 4.14. *Let wB be a point in $Hess(X,h)$. The localization of the equivariant cohomology class of $[Hess(X,h)]_S$ at wB is*

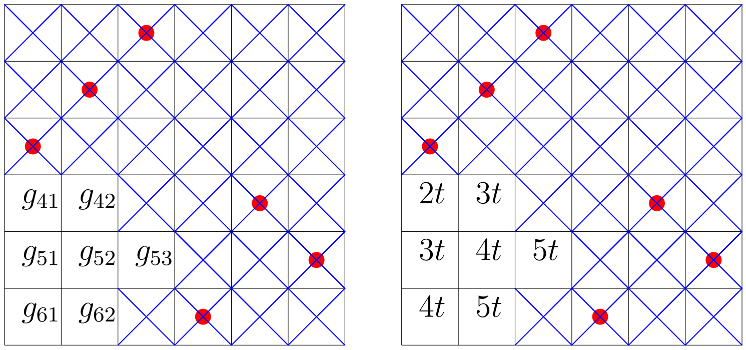
$$\prod_{\{1\le i\le k, 1\le j\le n: h(j)<w^{-1}(i)\}} ((i-w(j)+1)\cdot t)$$

Proof. The action of S on $Hess(X,h)$ restricts to an action on each patch $\mathcal{N}_{w,h}$. Thus we may define an S-multigrading on $\frac{\mathbb{C}[\mathbf{x}]}{I_{w,Hess(X,h)}}$. This S-multigrading assigns weight $(i-w(j))\cdot t$ to the variable x_{ij} in coordinate ring of the patch as noted in 4.6.

The polynomial g_{ij} is homogeneous with respect to the S-multigrading and it contains the term $x_{i+1,j}$ by definition. Hence the S-weight of g_{ij} is $(i+1-w(j))t$.

The patch ideal $I_{w,Hess(X,h)}$ is generated by the g_{ij} with $h(j) < w^{-1}(i)$. The patch ideal is a complete intersection. This means that if we order the generators g_{ij} of $I_{w,Hess(X,h)}$, calling them $g_1, g_2, \ldots, g_k$ then each g_i will not be a zero-divisor in the ring $\frac{\mathbb{C}[\mathbf{x}]}{\langle g_1,\ldots,g_{i-1}\rangle}$. (Otherwise, the codimension of $\frac{\mathbb{C}[\mathbf{x}]}{I_{w,Hess(X,h)}}$ would not equal the number of generators in $I_{w,Hess(X,h)}$.) By Theorem 4.12 and Proposition 4.13 the S-equivariant cohomology localization at wB is the product of the S-weights of the g_{ij} in $I_{w,Hess(X,h)}$. This proves the theorem. □

Example 4.9. Let $h = (3,3,5,6,6,6)$ and $w = 321645$. We will calculate the localization of the equivariant cohomology class $[Hess(X,h)]_S$ at wB. Start with the h-patch diagram for w. Label any uncolored box (i,j) by $(i-w(j)+1)t$. The localization of $[Hess(X,h)]_S$ at wB is the product of the weights in the uncolored portion of the h-patch diagram.



The equivariant cohomology class $[Hess(X,h)]_S$ localized at wB is

$$[Hess(X,h)]_S|_{wB} = (2t)(3t)^2(4t)^2(5t)^2.$$

⊞

Theorem 4.14 allows us to calculate the equivariant cohomology class of each Hessenberg variety $Hess(X,h)$. Since Tymoczko's paving by affines of the regular nilpotent Hessenberg varieties have S-invariant cells, the regular nilpotent Hessenberg varieties are S-equivariantly formal [GorKotMac98, Theorem 14.1]. Thus the S-equivariant class $[Hess(X,h)]_S$ is determined by these localizations. The S-equivariant class $[Hess(X,h)]_S$ is the tuple in $H_S^*(G/B)^S) \cong \oplus_{w\in W}\mathbb{C}[t]$ consisting of the specific localizations at each fixed point wB in $Hess(X,h)$, and zero elsewhere.

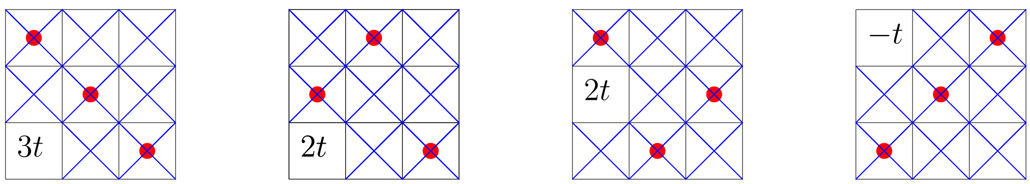
Example 4.10. In this example, we calculate the S-equivariant cohomology class of the Peterson variety $[\mathrm{Pet}_n]_S$ in $GL_3(\mathbb{C})/B$. We then verify that the equivariant cohomology class of the Peterson variety is equal to the following sum of equivariant Schubert basis classes

$$[\mathrm{Pet}_n]_S = [X_{231}]_S + [X_{312}]_S.$$

Recall that this Peterson variety contains four fixed points $321, 213, 132$, and 123. The patch ideals at these points and their S-graded degrees are:

Fixed point	Patch ideal	degree
321	$\langle -x_{21}x_{31} - (x_{31} - x_{21}^2)x_{32} \rangle$	degree $= 3t$
213	$\langle -x_{31}^2 - (1 - x_{31}x_{11})x_{32} \rangle$	degree $= 2t$
132	$\langle x_{31} - x_{21}^2 + x_{21}x_{31}x_{32} \rangle$	degree $= 2t$
123	$\langle x_{21} - x_{12} \rangle$	degree$= -t$.

The localizations can be calculated using the h-diagrams.



$w = 123$ $w = 213$ $w = 132$ $w = 321$

When writing the localizations of an equivariant cohomology class we will order the fixed points of $GL_n(\mathbb{C})/B$ as $123, 213, 132, 231, 312, 321$. The S-equivariant cohomology class of the Peterson variety is

$$[\mathrm{Pet}_n]_S = (3t, 2t, 2t, 0, 0, -t) \in H_S^*(GL_n(\mathbb{C})/B).$$

One can also calculate the following S-equivariant cohomology classes using the h-diagrams:

- $[X_{312}]_S = (2t, 2t, t, 0, t, 0)$

- $[X_{231}]_S = (2t, t, 2t, t, 0, 0)$

- $t[GL_n(\mathbb{C})/B]_S = (t, t, t, t, t, t)$

From these calculations we see that

$$[\mathrm{Pet}_n]_S = (3t, 2t, 2t, 0, 0, -t) = [X_{312}]_S + [X_{231}]_S - t[GL_n(\mathbb{C})/B]_S.$$

Calculating in ordinary cohomology amounts to setting $t = 0$ [GorMac10, Proposition 2.1 (ii)]. When we do this, we see that

$$[\mathrm{Pet}_n] = [X_{312}] + [X_{231}] = [X_{v_\Delta}] + [X_{u_\Delta}]$$

which agrees with our intersection theory calculation in Chapter 3. ⊞

4.7 Hessenberg dimension path diagrams

Tymoczko showed that all Hessenberg varieties are paved by affines and gave a combinatorial formula for the dimension of each cell using fillings of Young tableaux in type A_{n-1} [Tym06b]. In this section, we define Hessenberg dimension path diagrams. We use these diagrams to define another combinatorial criterion for calculting the dimension of the affine cells in Tymoczko's paving and give a criterion for when a fixed point wB is contained a Hessenberg variety.

We define an algorithm for constructing the *Hessenberg diagram of a Hessenberg function h*. The Hessenberg diagram will prove useful in a number of algorithms for computing dimensions of affine cells and defining ideals in the chapters to come.

Algorithm 4.15. *(Constructing the Hessenberg diagram) We construct a Hessenberg diagram for a Hessenberg function h as follows:*

1. *Start with an $n \times n$ grid.*
2. *Index the rows from 1 to n from top to bottom and the columns from 1 to n from left to right.*
3. *In row i cross out the last $n - h(i)$ boxes.*

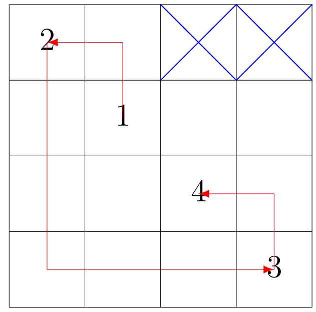
We call the resulting grid, the Hessenberg diagram for the function h.

Algorithm 4.16. *(Constructing the h-dimension path diagam) We now define an algorithm for constructing what we will call a h-dimension path diagram for a permutation w.*

1. *Label the diagonal entry (i, i) of the Hessenberg diagram by $w(i)$, i.e., write the one-line notation along the diagonal.*
2. *Draw an arrow straight up/down from 1 to the row containing 2 staying in the column containing 1. Then draw an arrow straight left/right from that box to the box containing 2 staying in the row containing 2.*
3. *Continue this process from 2 to 3 and so on, until you have reached n.*

We call the resulting diagram, the h-dimension path diagram of w.

Example 4.11. Let $w = 2143$ and $h(1, 2, 3, 4) = (2, 4, 4, 4)$. The h-dimension path diagram for w is shown on the left. The diagram with the Hessenberg diagram on top of it is on the right.



⊞

This h-dimension path diagram of w is a very useful combinatorial object: it can be used to check whether the point wB is contained in the Hessenberg variety $Hess(X, h)$, and to find the dimension of the affine cell $BwB/B \cap Hess(X, h)$, as we shall prove shortly.

Theorem 4.17. *The fixed point wB is contained in $Hess(X, h)$ if and only if the h-dimension path diagram of w fits inside the Hessenberg diagram for h.*

Proof. The fixed point wB is contained inside the Hessenberg variety $Hess(X, h)$ if and only if

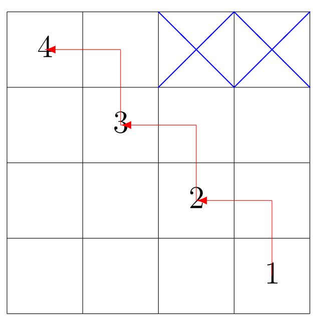
$$h(w^{-1}(i+1)) \leq w^{-1}(i) \text{ for all } 1 \leq i < n.$$

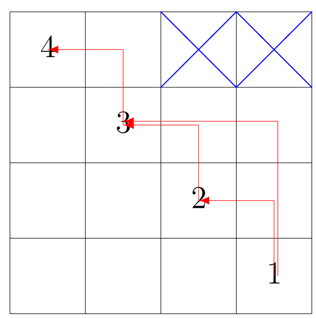
By construction of the Hessenberg diagram, a box in position (j, k) is crossed out if and only if $h(j) < k$. By construction of the h-dimension path diagram, an arrow from i to $i+1$ in the h-dimension path diagram makes a turn at the box (j, k) if and only if $w^{-1}(i+1) = j$ and $w^{-1}(i) = k$. Hence an arrow from i to $i+1$ in the h-dimension path diagram fits inside the Hessenberg diagram if and only $h(w^{-1}(i+1)) \leq w^{-1}(i)$. $\square$

We now construct *inversion paths* for an h-dimension path diagram of w. The number of all such paths in a h-dimension path diagram will tell us the dimension of the cell $BwB/B \cap Hess(X,h)$.

Algorithm 4.18. *Start with the h-dimension path diagram of w. For each pair $i < j$ compare all of the entries (i,i) and (j,j) on the diagonal of the diagram. If $i < j$ and i sits to the right of j in the diagram, then draw an arrow straight up from i to the row containing j. Then continue that arrow over to the box containing j. Any such arrow is an inversion path of the h-dimension path diagram of w. If the arrow does not pass through the crossed out region of the Hessenberg diagram, then it is called an h-allowable inversion path.*

Example 4.12. Let $h(1,2,3,4) = (2,4,4,4)$ and $w = 4321$. The h-dimension path diagram of w is on the left. The collection of all h-allowable inversion paths is drawn on the right.





⊞

We are ready to prove that the dimension $BwB/B \cap Hess(X,h)$ is equal to the number of *h-allowable inversion paths* in the h-path diagram.

Theorem 4.19. *The dimension of the cell $BwB/B \cap Hess(X,h)$ is the number of inversion paths in the h-dimension path diagram of w.*

Proof. If $i < j$ and i sits to the right of j in the h-dimension path diagram then

$w^{-1}(i) > w^{-1}(j)$. So inversion paths correspond to the inversions of w. Furthermore, an inversion path will pass through the crossed out portion of the Hessenberg diagram exactly when the corner of the path is in the crossed-out portions, namely when $h(w^{-1}(j)) < w^{-1}(i)$. Thus the number of inversion paths fitting inside the non-crossed-out portion of the Hessenberg diagram is equal to the cardinality of the set

$$\{(1 \leq i < j \leq n) : w^{-1}(i) > w^{-1}(j) \text{ and } h(w^{-1}(j)) < w^{-1}(i)\}.$$

Tymoczko proved that this is the dimension of the cell $BwB/B \cap Hess(X, h)$ [Tym06b, Corollary 6.3]. $\square$

CHAPTER 5
EQUIVARIANT COHOMOLOGY OF HESSENBERG VARIETIES

The equivariant cohomology $H_T^*(GL_n(\mathbb{C})/B;\mathbb{C})$ of the full flag variety is a well-known and widely studied object. In this chapter we will describe the S-equivariant cohomology of regular nilpotent Hessenberg variety $H_S^*(Hess(X,h);\mathbb{C})$ as a quotient of the ring $H_S^*(GL_n(\mathbb{C})/B;\mathbb{C})$. To do this, we combinatorially construct an ideal I_h and show that

$$H_S^*(Hess(X,h);\mathbb{C}) \cong \frac{H_S^*(GL_n(\mathbb{C})/B;\mathbb{C})}{I_h}.$$

5.1 Goresky-MacPherson calculus

Let X be an equivariantly formal T-space with finitely many T-fixed points, and suppose that the cohomology ring $H^*(X;\mathbb{C})$ is generated by its degree two classes. In 2010 Goresky and MacPherson proved that the spectrum of the equivariant cohomology ring $\mathrm{Spec}(H_T^*(X;\mathbb{C}))$ is a subspace arrangement of the vector space $H_2^T(X;\mathbb{C})$ [GorMac10].

In this chapter we restrict attention to the torus $S = \{(t, t^2, t^3, \ldots, t^n) : t \in \mathbb{C}^*\}$ acting on the flag variety $GL_n(\mathbb{C})/B$ by left multiplication. We use Goresky and MacPherson's techniques to identify the equivariant cohomology ring $H_S^*(Hess(X,h);\mathbb{C})$ with the coordinate ring of a line arrangement in the ambient space

$$H_2^S(Hess(X,h);\mathbb{C}) \cong \mathbb{C}^{d+1}.$$

Here d is the number of degree 2 homology classes in $GL_n(\mathbb{C})/B$. We then give a combinatorial description of this ring using only the Hessenberg function.

Definition 5.1. We call a subspace $X \subset GL_n(\mathbb{C})/B$ a GM-space if:

1. X is equivariantly formal
2. X has finitely many S-fixed points
3. the singular cohomology ring $H^*(X;\mathbb{C})$ is generated by its degree 2 classes.

The following is a simplified version of Theorem 3.1 of Goresky and MacPherson's paper. It says that the $\mathbb{C}^*$-equivariant cohomology of any GM-space is isomorphic to the coordinate ring of a line arrangement with one line for each fixed point $p \in X^{\mathbb{C}^*}$. Each line in this arrangement is isomorphic as a vector space to the Lie algebra $\mathfrak{t}$ of the torus $\mathbb{C}^*$. We will denote the line corresponding a point p by $\mathfrak{t}_p \cong \mathrm{Spec}(\mathbb{C}[t])$.

Theorem 5.2. *[GorMac10, Theorem 3.1] Suppose that a torus S acts on a GM-space X with finitely many fixed points. Then the spectrum of the equivariant cohomology ring*

$$\mathit{Spec}\ (H_S^*(X;\mathbb{C})) \cong \bigcup_{p \in X^{\mathbb{C}^*}} \mathfrak{t}_p$$

is a line arrangement where each $\mathfrak{t}_p$ is a linear subspace identified with the Lie algebra of the torus S. There is one line $\mathfrak{t}_p$ for each fixed point p in X^S.

Remark 5.1. (Intuition behind Theorem 5.2.) There is a canonical isomorphism between the set of characters $\chi^*(S)$ and the second equivariant homology of a point

$H^2(BS;\mathbb{Z}) = H^2_S(pt;\mathbb{Z})$. This isomorphism is defined by sending each character of the torus $\lambda : S \to \mathbb{C}^*$ to the first Chern class of the corresponding line bundle L_λ on BS. This gives the following isomorphisms

$$\mathbb{C}[\mathfrak{t}] \cong H^*_S(BS;\mathbb{C}) = H^*_S(pt;\mathbb{C})$$

and

$$\mathfrak{t} \cong \chi_*(S) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^S_2(pt;\mathbb{C})$$

where $\chi_*(S)$ denotes the set of cocharacters of S. Since X is equivariantly formal, the following sequence of vector spaces is exact [GorMac10, Proposition 2.1]

$$0 \to H_2(X;\mathbb{C}) \to H^S_2(X;\mathbb{C}) \to H^S_2(pt;\mathbb{C}) \cong \mathfrak{t} \to 0.$$

The inclusion of each fixed point $p \hookrightarrow X$ induces a splitting

$$\mu_p : H^S_2(p;\mathbb{C}) \cong \mathfrak{t} \to H^S_2(X;\mathbb{C})$$

of this sequence. The image of this splitting is a line $\mathfrak{t}_p$ in $H^S_2(X;\mathbb{C})$. Let $\mathcal{V} = \bigcup_{p \in X^S} \mathfrak{t}_p$ denote the resulting line arrangement in the vector space $H^T_2(X;\mathbb{C})$.

The maps μ_p also induce surjective maps

$$\phi_p = \mu_p^* : \mathbb{C}[H^S_2(X;\mathbb{C})] \to \mathbb{C}[\mathfrak{t}_p] \cong H^*_S(pt;\mathbb{C}).$$

Let $I(\mathcal{V}) = \bigcap_{p \in X^S} \ker \phi_p$ denote the ideal of functions that vanish on $\mathcal{V}$. The isomorphisms $H^0_S(X;\mathbb{C}) \cong \mathbb{C}$ and $H^2_S(X;\mathbb{C}) \cong \mathrm{Hom}(H^S_2(X;\mathbb{C});C)$ determine a degree-doubling homomorphism of graded $\mathbb{C}[\mathfrak{t}]$-algebras

$$\mathbb{C}[H^S_2(X;\mathbb{C})] \to H^*_S(X;\mathbb{C}).$$

Since $H^*(X;\mathbb{C})$ is generated by its degree two classes, this map is surjective and we obtain the following exact sequence

$$0 \to I(\mathcal{V}) \to \mathbb{C}[H^S_2(X;\mathbb{C})] \to H^*_S(X;\mathbb{C}).$$

Thus the coordinate ring $\mathbb{C}[\mathcal{V}]$ of the line arrangement $\mathcal{V}$ is isomorphic to the equivariant cohomology of $H^*_S(X;\mathbb{C})$ of X. $\boxplus$

We start by applying Goresky and MacPherson's result to the flag variety $GL_n(\mathbb{C})/B$. The flag variety $GL_n(\mathbb{C})/B$ has finitely many S-fixed points, and its cohomology ring is generated by its degree two Schubert classes. Hence, it is a GM-space. Goresky-MacPherson's result states that $\mathrm{Spec}\,(H^*(GL_n(\mathbb{C})/B;\mathbb{C}))$ is a union of lines, one for each fixed point wB in $GL_n(\mathbb{C})/B^S$:

$$\mathrm{Spec}(H^*_S(GL_n(\mathbb{C})/B;\mathbb{C})) \cong \bigcup_{wB \in GL_n(\mathbb{C})/B^{\mathbb{C}^*}} \mathfrak{t}_w \subset H^S_2(GL_n(\mathbb{C})/B;\mathbb{C}) \cong \mathbb{C}^n$$

where $\mathfrak{t} \cong \mathrm{Spec}\mathbb{C}[t]$ is the Lie algebra of $S \cong \mathbb{C}^*$. This line arrangement is isomorphic to Spec $(H_S^*(GL_n(\mathbb{C})/B;\mathbb{C})) \cong \mathrm{Spec}(R)$ where

$$R = \frac{\mathbb{C}[x_1, x_2, \ldots, x_n, t]}{\bigcap w \in W \langle x_1 - w(1)\cdot t, \ldots, x_n - w(n)\cdot t\rangle}. \tag{5.1}$$

The correspondence between a fixed point wB in $GL_n(\mathbb{C})/B$ and a line in this arrangement is given by

$$wB \mapsto \mathrm{Spec}\left(\frac{\mathbb{C}[x_1, \ldots, x_n, t]}{\langle x_1 - w(1)\cdot t, x_2 - w(2)\cdot t, \ldots, x_n - w(n)\cdot t\rangle}\right) \cong \mathrm{Spec}\mathbb{C}[t] \cong \mathfrak{t}_w.$$

Let

- $\mathbf{x}$ denote the set of variables $\{x_1, \ldots, x_n\}$.
- $e_i(\mathbf{x})$ denote the ith elementary symmetric function in $\mathbf{x} = x_1, x_2, \ldots, x_n$.
- $e_i(t)$ denote the ith elementary symmetric function in $t_1, t_2, \ldots, t_n$ after substituting $t_i = i \cdot t$ and
- $E_n(\mathbf{x}, t)$ denote the ideal generated by the polynomials $e_i(\mathbf{x}) - e_i(t)$ for $1 \leq i \leq n$.

Remark 5.2. The following ring presentation of the cohomology of the flag variety is due to Borel [Bor53]

$$H^*(GL_n(\mathbb{C})/B;\mathbb{C}) \cong \frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle e_i(x_1, \ldots, x_n)\rangle_{i=1,\ldots,n}}.$$

The analagous presentation in T-equivariant cohomology is well-known [Ful07, Lecture 9, Proposition 1.1]

$$H_T^*(GL_n(\mathbb{C})/B;\mathbb{C}) \cong \frac{\mathbb{C}[x_1,\ldots,x_n,t_1,\ldots,t_n]}{\langle e_i(x_1,\ldots,x_n) - e_i(t_1,\ldots,t_n)\rangle_{i=1,\ldots,n}}.$$

Substituting $t_i \mapsto t^i$, we obtain the following presentation of the S-equivariant cohomology of the flag variety

$$H_S^*(GL_n(\mathbb{C})/B;\mathbb{C}) \cong \frac{\mathbb{C}[x_1,\ldots,x_n,t]}{\langle e_i(x_1,\ldots,x_n) - e_i(t^1,\ldots,t^n)\rangle_{i=1,\ldots,n}} \tag{5.2}$$

$$= \frac{\mathbb{C}[\mathbf{x},t]}{E_n(\mathbf{x},t)}. \tag{5.3}$$

⊞

Using the embedding identified in Equation (**??**), Goresky-MacPherson's presentation recovers the presentation in Equation (5.2) for equivariant cohomology:

$$\begin{aligned}
\operatorname{Spec}(H_S^*(X)) &= \operatorname{Spec}\left(\frac{\mathbb{C}[x_1,x_2,\ldots,x_n,t]}{\bigcap\limits_{w\in W}\langle x_1 - w(1)\cdot t, x_2 - w(2)\cdot t,\ldots,x_n - w(n)\cdot t\rangle}\right)\\
&= \operatorname{Spec}\left(\frac{\mathbb{C}[x_1,x_2,\ldots,x_n,t]}{\langle e_1(\mathbf{x}) - e_1(t), e_2(\mathbf{x}) - e_2(t),\ldots,e_n(\mathbf{x}) - e_n(t)\rangle}\right)\\
&= \operatorname{Spec}\left(\frac{\mathbb{C}[\mathbf{x},t]}{E_n(\mathbf{x},t)}\right).
\end{aligned}$$

Example 5.1. Recall that we have chosen S to be $S = \{(t,t^2,t^3) : t \in \mathbb{C}^*\}$ on $GL_3(\mathbb{C})/B$. Goresky and MacPherson's result says that the spectrum of the equivariant cohomology ring is a central line arrangement in $H_2^S(GL_n(\mathbb{C})/B;\mathbb{C}) \cong \mathbb{C}^3$ with one line for each fixed point. In other words, the equivariant cohomology ring of the full flag variety is

$$H_S^*(GL_n(\mathbb{C})/B;\mathbb{C}) \cong \frac{\mathbb{C}[x_1,x_2,x_3,t]}{\bigcap_{w\in S_n}\langle x_1 - w(1)\cdot t, x_2 - w(2)\cdot t, x_3 - w(3)\cdot t\rangle}.$$

This presentation agrees with the following presentation

$$H_S^*(GL_n(\mathbb{C})/B;\mathbb{C}) \cong \frac{\mathbb{C}[x_1,x_2,x_3,t]}{\langle x_1+x_2+x_3-6t, x_1x_2+x_1x_3+x_2x_3-11t^2, x_1x_2x_3-6t^3\rangle}.$$

⊞

5.2 Equivariant cohomology for subvarieties of $GL_n(\mathbb{C})/B$

Let V be a GM-subspace of $GL_n(\mathbb{C})/B$ satisfying three conditions:

1. V is equivariantly formal
2. the second homology of V is a subspace of the second homology of $GL_n(\mathbb{C})/B$;
3. the fixed point set V^S is a subset of $GL_n(\mathbb{C})/B^S$;

Then we can identify Spec $(H_S^*(V;\mathbb{C}))$ with a subarrangement of Spec $(H_S^*(GL_n(\mathbb{C})/B))$. The assumption that

$$H_2(V;\mathbb{C}) \subseteq H_2(GL_n(\mathbb{C})/B;\mathbb{C})$$

implies that the second equivariant homology group $H_2^S(V;\mathbb{C})$ is a vector subspace of $H_2^S(GL_n(\mathbb{C})/B;\mathbb{C})$.

The embedding $i : V \hookrightarrow GL_n(\mathbb{C})/B$ induces a map on the second equivariant homology groups

$$i_2 : H_2^S(V, \mathbb{C}) \hookrightarrow H_2^S(GL_n(\mathbb{C})/B, \mathbb{C}).$$

By assumption the embedding i also restricts to an embedding of the fixed points $i : V^S \hookrightarrow GL_n(\mathbb{C})/B^S$. Thus the map i_2 sends the line in $\mathrm{Spec}\,(H_S^*(V;\mathbb{C}))$ corresponding to wB to the line in $\mathrm{Spec}\,(H_S^*(GL_n(\mathbb{C})/B, \mathbb{C}))$ corresponding to wB. Hence we identify the arrangement $\mathrm{Spec}\,(H_S^*(V;\mathbb{C}))$ with the subarrangement of $\mathrm{Spec}\,(H_S^*(GL_n(\mathbb{C})/B;\mathbb{C}))$ which has lines corresponding to the fixed points of V.

Since $\mathrm{Spec}\,(H_S^*(V;\mathbb{C}))$ is a subarrangment of $\mathrm{Spec}\,(H_S^*(GL_n(\mathbb{C})/B;\mathbb{C}))$ we know that the equivariant cohomology of V is a quotient of the equivariant cohomology of $GL_n(\mathbb{C})/B$ by an ideal I:

$$H_S^*(V;\mathbb{C}) \cong \frac{H_S^*(GL_n(\mathbb{C})/B;\mathbb{C})}{I}.$$

The rest of this chapter specifies when a Hessenberg variety is a GM-space and what the ideal I is.

5.3 Which Hessenberg varieties are GM-spaces?

Any regular nilpotent Hessenberg variety $Hess(X, h)$ has only finitely many S-fixed points [Tym06b]. If we assume that the cohomology $H^*(Hess(X, h);\mathbb{C})$ is generated by its degree two cohomology classes, then $Hess(X, h)$ satisfies the conditions of a GM-space.

Lemma 5.3. *Any regular nilpotent Hessenberg variety $Hess(X, h)$ whose cohomology is generated by its degree two classes is a GM-space.*

One way to show that the cohomology of a Hessenberg variety is generated by its degree 2 classes is to prove that $H^*(GL_n(\mathbb{C})/B; \mathbb{C})$ surjects onto $H^*(Hess(X, h); \mathbb{C})$. This has already been done in exactly four cases.

Remark 5.3. The regular nilpotent Hessenberg varieties corresponding to the following Hessenberg functions h are known to have cohomology rings generated by their degree 2 classes:

- $h(i) = i$ for all $1 \leq i \leq n$;
- $h(i) = i + 1$ for all $1 \leq i \leq n$;
- $h(1) = 3$ and $h(i) = i + 1$ for all $2 \leq i \leq n$; and
- $h(i) = n$ for all $1 \leq i \leq n$.

⊞

There are examples of semisimple Hessenberg varieties where $H^*(GL_n(\mathbb{C})/B; \mathbb{C})$ does not surject onto $H^*(Hess(X, h); \mathbb{C})$. However, we conjecture $H^*(Hess(X, h); \mathbb{C})$ is generated by its degree two classes for any nilpotent Hessenberg variety $Hess(X, h)$.

Conjecture 5.4. *It is conjectured that the following statements are true in Type A_{n-1}:*

1. *The cohomology of any regular nilpotent Hessenberg variety is generated by its degree 2 classes.*

2. *The cohomology of any nilpotent Hessenberg variety is generated by its degree* 2 *classes.*

Henceforth, we will assume that all regular nilpotent Hessenberg varieties we work with have cohomology rings generated by their degree 2 cohomology classes.

5.4 Which fixed points wB are in $Hess(X, h)$?

The goal of this chapter is to identify the ideal I_h describing $H^*_S(Hess(X,h);\mathbb{C})$ as a quotient of $H^*_S(GL_n(\mathbb{C})/B;\mathbb{C})$. We know that this ideal I_h should specify which fixed points in $(GL_n(\mathbb{C})/B)^S$ appear in $Hess(X,h)^S$ because the arrangement $\mathrm{Spec}\,(H^*_S(X;\mathbb{C}))$ is a subarrangement of $\mathrm{Spec}\,(H^*_S(GL_n(\mathbb{C})/B;\mathbb{C}))$ containing lines which correspond to the fixed points wB in $Hess(X,h)^S$.

In this section we give several combinatorial criteria for identifying the fixed points $wB \in GL_n(\mathbb{C})/B$ contained in a Hessenberg variety $Hess(X,h)$. These combinatorial statements will be essential to describe the ideal I_h for regular nilpotent Hessenberg varieties. Let h and h' be Hessenberg functions such that h' is bigger than h' in exactly one place. In other words, fix j and let h and h' satisfy

- $h'(i) = h(i)$ for all $1 \leq i \leq n$ with $i \neq j$
- $h'(j) = h(j) + 1$.

We describe which fixed points wB need to be added to $Hess(X,h)^S$ to obtain $Hess(X,h')^S$.

Lemma 5.5. *Let j be the fixed index where $h'(j) = h(j) + 1$. The set $Hess(X,h')^S \setminus Hess(X,h)^S$ contains the fixed points wB satisfying:*

1. $h'(w^{-1}(i+1)) = w^{-1}(i)$ *when* $w^{-1}(i+1) = j$

2. $h(w^{-1}(i+1)) \geq w^{-1}(i)$ *when* $w^{-1}(i+1) \neq j$.

Proof. The fixed points in any Hessenberg variety $Hess(X, h)$ are

$$Hess(X,h)^S = \{wB : w^{-1}(i) \leq h(w^{-1}(i+1)) \text{ for all } 1 \leq i < n\}.$$

Thus the fixed points in $Hess(X, h')$ can be partitioned into disjoint sets

$$\begin{aligned} Hess(X,h')^S &= \left\{wB : w^{-1}(i) \leq h'(w^{-1}(i+1)) \text{ for all } 1 \leq i < n\right\} \\ &= A_1 \cup A_2 \end{aligned}$$

where

$$A_1 = \left\{ wB : \begin{array}{c} w^{-1}(i) \leq h'(w^{-1}(i+1)) \text{ for } i \text{ with } w^{-1}(i+1) \neq j \\ \text{and } w^{-1}(i) < h'(w^{-1}(i+1)) \text{ otherwise} \end{array} \right\}$$

and

$$A_2 = \left\{ wB : \begin{array}{c} w^{-1}(i) \leq h'(w^{-1}(i+1)) \text{ for } i \text{ with } w^{-1}(i+1) \neq j \\ \text{and } w^{-1}(i) = h'(w^{-1}(i+1)) \text{ otherwise} \end{array} \right\}.$$

The fact that $h'(i) = h(i)$ for all $i \neq j$ and $h'(j) = h(j) + 1$ implies that the following sets are equal:

$$\begin{aligned} Hess(X,h)^S &= \{wB : w^{-1}(i) \le h(w^{-1}(i+1)) \text{ for all } 1 \le i < n\} \\ &= \left\{ wB : \begin{array}{c} w^{-1}(i) \le h'(w^{-1}(i+1)) \text{ for } w^{-1}(i+1) \ne j \text{ and} \\ w^{-1}(i) < h'(w^{-1}(i+1)) \text{ when } w^{-1}(i+1) = j \end{array} \right\}. \end{aligned}$$

Hence the fixed points wB in the set $Hess(X,h')^S \setminus Hess(X,h)^S$ are

$$\left\{ wB : \begin{array}{c} w^{-1}(i) \le h'(w^{-1}(i+1)) \text{ for } w^{-1}(i+1) \ne j \text{ and} \\ w^{-1}(i) = h'(w^{-1}(i+1)) \text{ when } w^{-1}(i+1) = j \end{array} \right\}.$$

This proves the lemma. □

Here is an equivalent description which we will use.

Lemma 5.6. *The fixed points wB in the set $Hess(X,h')^S \setminus Hess(X,h)^S$ all satisy the conditions:*

1. $w(j) - w(h(j)) = 1$

2. $h(w^{-1}(i+1)) \le w^{-1}(i)$ *when* $w^{-1}(i+1) \ne j$.

Proof. We start by showing that Condition (1) of Lemma 5.5 is equivalent to Condition (1) of Lemma 5.6, in other words $h'(w^{-1}(i+1)) = w^{-1}(i)$ when $w^{-1}(i+1) = j$

if and only if $h(j) - h(w(j)) = 1$. Let $j = w^{-1}(i+1)$ for some i. Note that

$$\begin{aligned} j = w^{-1}(i+1) &\iff w(j) = i+1 \\ &\iff w(j) - 1 = i \\ &\iff w^{-1}(w(j)-1)) = w^{-1}(i). \end{aligned} \tag{5.4}$$

From Equation (5.4) we obtain the following equivalent statements

$$\begin{aligned} w^{-1}(i) = h(w^{-1}(i+1)) &\iff w^{-1}(w(j)-1) = h(j) \\ &\iff w(j) - 1 = w(h(j)) \\ &\iff w(j) - w(h(j)) = 1. \end{aligned}$$

Hence the two conditions are equivalent. Condition (2) of Lemma 5.5 is the same as Condition (2) of Lemma 5.6. Therefore, the two lemmas are equivalent. □

The h-dimension path diagrams can also be used to describe which fixed points appear in $Hess(X, h')$ and not in $Hess(X, h)$.

Lemma 5.7. *Assume that $h'(i) = h(i)$ for all $i \neq j$ and $h'(j) = h(j) + 1$. The fixed points wB in the set $Hess(X, h')^S \setminus Hess(X, h)^S$ are those whose dimension path diagrams pass through the box $(j, h'(j))$ and stay inside the Hessenberg diagram for h everywhere else.*

Proof. The only difference between the Hessenberg diagram for h' and the Hessenberg diagram for h is that the box $(j, h'(j))$ is not crossed out in the diagram for h'.

Condition (1) of Lemma 5.6 says that the fixed points wB which are contained in $Hess(X, h')$ and not $Hess(X, h)$ are those whose dimension path diagram passes through the box $(j, h'(j))$. Condition (2) of Lemma 5.6 says that the fixed points which are contained in $Hess(X, h')$ and not $Hess(X, h)$ are those whose dimension path diagrams stay inside the Hessenberg diagram for h everywhere except the box $(j, h'(j))$. □

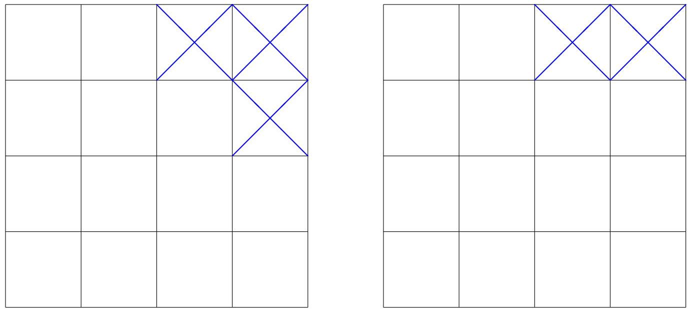
Example 5.2. Let $h(1, 2, 3, 4) = (2, 3, 4, 4)$ and $h'(1, 2, 3, 4) = (2, 4, 4, 4)$. The following table lists the fixed points in $Hess(X, h)$ and $Hess(X, h')$. The last column lists the additional fixed points in $Hess(X, h')$.

$Hess(X,h)$		$Hess(X,h')$		
1234	2134	1234	2134	1423
1243	1324	1243	1324	1342
3214	2143	3214	2143	3241
1432	4321	1432	4321	4312

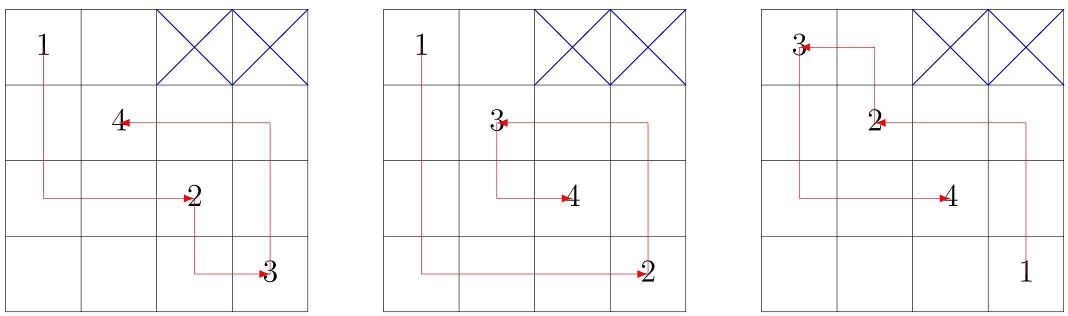
We see that the fixed points that are in $Hess(X, h')$ and not $Hess(X, h)$ are precisely those satisfying

$$w(2) - w(h'(2)) = w(2) - w(4) = 1.$$

Here are the Hessenberg diagrams for h and h' respectively. The only difference between them is that the box $(2, 4)$ is not crossed out in the Hessenberg diagram for h' since $h'(2) = 4$ and $h(2) = 3$.



We have drawn the dimension path diagrams for $w = 1423, 1342$, and 3241. A good exercise is to draw the dimension path diagram of $w = 4312$ and verify that it fits inside the Hessenberg diagram for h'.



$\boxplus$

Thus far, all of the criteria discussed have described the fixed point set $Hess(X, h)^S$ as a set of points satisfying a series of inequalities. Our next criterion describes the set $Hess(X, h)^S$ as the set of points satisfying a series of equations.

We now describe n sets of equations $C_{h(1)}, \ldots, C_{h(n)}$ which describe the fixed point set $Hess(X, h)^S$. Our description uses the h-dimension path diagrams.

Algorithm 5.8. *(To construct the sets $C_{h(1)}, \ldots, C_{h(n)}$ of conditions describing the fixed points of the Hessenberg variety $Hess(X, h)$.)*

1. *Start with a Hessenberg diagram for h.*

2. Label the diagonal entry (i, i) with $w(i) = i$.

3. Label the entry (i, j) by $w(i) - w(j) = 1$.

4. Then the set $C_{h(i)}$ is the union of all unshaded entries on or above the diagonal.

Example 5.3. Let $h(1, 2, 3, 4) = (2, 4, 4, 4)$. This diagram was constructed using the above algorithm.

$w(1) = 1$	$w(1) - w(2) = 1$	$w(1) - w(3) = 1$	$w(1) - w(4) = 1$
	$w(2) = 2$	$w(2) - w(3) = 1$	$w(2) - w(4) = 1$
		$w(3) = 3$	$w(3) - w(4) = 1$
			$w(4) = 4$

The set of conditions $C_{h(i)}$ defining the set of fixed points wB in $Hess(X, h)$ are

- $C_{h(1)} = \{w(1) = 1 \text{ or } w(1) - w(2) = 1\}$
- $C_{h(2)} = \{w(2) = 2 \text{ or } w(1) - w(2) = 1 \text{ or } w(2) - w(3) = 1 \text{ or } w(2) - w(4) = 1\}$
- $C_{h(3)} = \{w(3) = 3 \text{ or } w(2) - w(3) = 1 \text{ or } w(2) - w(4) = 1 \text{ or } w(3) - w(4) = 1\}$
- $C_{h(4)} = \{w(4) = 4 \text{ or } w(2) - w(4) = 1 \text{ or } w(3) - w(4) = 1\}$

$\boxplus$

Lemma 5.9. *The fixed points in $Hess(X, h)$ are precisely the wB for which w satisfies at least one condition from each $C_{h(i)}$. In other words, $Hess(X, h)^S$ is the intersection of the sets defined by $C_{h(1)}, \ldots, C_{h(n)}$.*

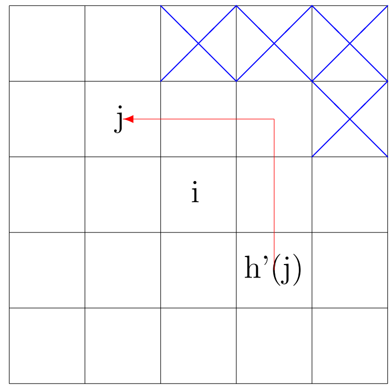
Proof. We proceed by induction on the Hessenberg function. Suppose $h(i) = i$ for all $1 \leq i \leq n$. Then the only fixed point in the Hessenberg variety is $w = id$. It satisfies the conditions $w(i) = i$ for all i. Hence the claim holds.

Fix a Hessenberg function h'. Assume by induction that the claim holds for all h where $h'(j) = h(j) + 1$ for some j and $h'(i) = h(i)$ for $i \neq j$. We will show that the claim holds for h' as well.

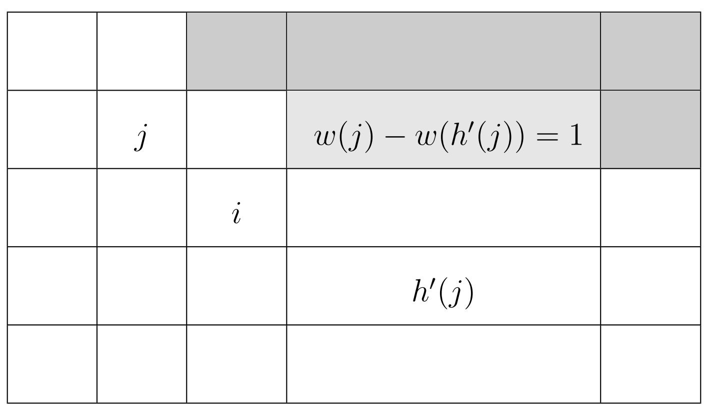
Let h be a Hessenberg function with $h'(i) = h(i)$ for all $i \neq j$ and with $h'(j) = h(j) + 1$. The fixed points that lie in both $Hess(X, h')$ and $Hess(X, h)$ satisfy at least one condition from each $C_{h(i)}$ by our induction hypothesis.

We only need to show that the points which lie in $Hess(X, h')$ and not in $Hess(X, h)$ satisfy a condition in each $C_{h'(i)}$. The fixed points in $Hess(X, h')^S$ and not in $Hess(X, h)^S$ are those having dimension path diagrams passing through the box $(j, h'(j))$ and otherwise staying inside the Hessenberg diagram for h.

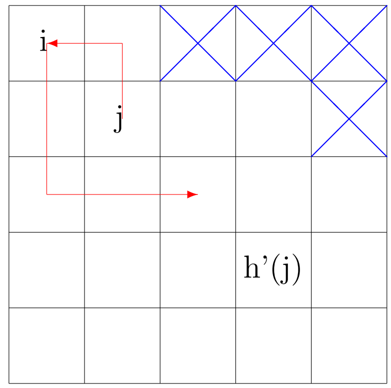
Let wB be one of these points. Then the fixed point wB satisfies the condition $w(j) - w(h'(j)) = 1$. Hence, if $i \in [j, h'(j)]$ then the dimension path diagram of w has an arrow passing through the box $(j, h'(j))$ which sits above and to the right of the diagonal entry (i, i). The diagram below illustrates this scenario. The function h' is represented by the crossed out region, and h contains the one extra box $(j, h'(j))$.



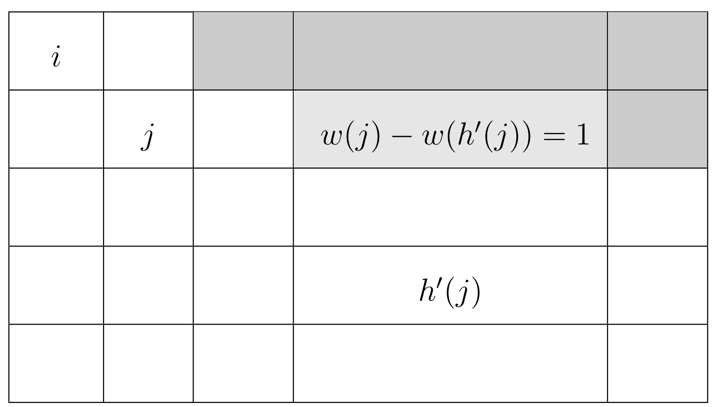
Hence w satisfies the condition $w(j) - w(h'(j)) = 1$. As the diagram below depicts, the new condition $w(j) - w(h'(j)) = 1$ is contained in $C_{h'(i)}$, because the box labeled $w(j) - w(h'(j)) = 1$ sits above and to the right of the box (i, i) on the diagonal.



On the other hand, if $i \notin [j, h'(j)]$ then the arrows passing through (i, i) in the dimension path diagram of w must sit inside the Hessenberg diagram for h.



This means that w satisfies one of the conditions from $C_{h(i)}$. However, if $i \notin [j, h'(j)]$ then $C_{h'(i)} = C_{h(i)}$ because the diagrams of h' and h have the same unshaded boxes above and to the right of the box (i, i).



Hence each fixed point wB in $Hess(X, h')$ satisifies one of the conditions in $C_{h'(i)}$ for $i \in [1, n]$. By induction, we have shown the claim holds for any Hessenberg function. □

Let us look at some examples in $GL_n(\mathbb{C})/B$.

Example 5.4. Consider the six fixed points in $GL_3(\mathbb{C})/B$. The fixed points in $Hess(X, h)^S$ are the wB where w satisfies one condition from each $C_{h(i)}$.

$GL_3(\mathbb{C})/B$		$h=(2,3,3)$		$h=(1,3,3)$	$h=(2,2,3)$	$h=(1,2,3)$
123	213	123	213	123	123	123
132	321	132	321	132	213	
231	312					

When $h=(1,2,3)$ there is only one fixed point wB in the set, and it satisfies

- $C_{h(1)} := \{w(1)=1\}$
- $C_{h(2)} := \{w(2)=2\}$
- $C_{h(3)} := \{w(3)=3\}$.

When $h=(2,2,3)$ there are two fixed points. They satisfy

- $C_{h(1)} := \{w(1)=1 \text{ or } w(1)-w(2)=1\}$
- $C_{h(2)} := \{w(2)=2 \text{ or } w(1)-w(2)=1\}$
- $C_{h(3)} := \{w(3)=3\}$.

When $h=(1,3,3)$ there are also two fixed points. This time they satisfy

- $C_{h(1)} := \{w(1)=1\}$
- $C_{h(2)} := \{w(2)=2 \text{ or } w(2)-w(3)=1\}$
- $C_{h(3)} := \{w(3)=3 \text{ or } w(2)-w(3)=1\}$.

When $h=(2,3,3)$ there are four fixed points. They satisfy the conditions

- $C_{h(1)} := \{w(1) = 1 \text{ or } w(1) - w(2) = 1\}$

- $C_{h(2)} := \{w(2) = 2 \text{ or } w(1) - w(2) = 1 \text{ or } w(2) - w(3) = 1\}$

- $C_{h(3)} := \{w(3) = 3 \text{ or } w(2) - w(3) = 1\}$.

Finally, when $h = (3, 3, 3)$ the six fixed points satisfy

- $C_{h(1)} := \{w(1) = 1 \text{ or } w(1) - w(2) = 1 \text{ or } w(1) - w(3) = 1\}$

- $C_{h(2)} := \{w(2) = 2 \text{ or } w(2) - w(3) = 1 \text{ or } w(1) - w(2) = 1 \text{ or } w(1) - w(3) = 1\}$

- $C_{h(3)} := \{w(3) = 3 \text{ or } w(1) - w(3) = 1 \text{ or } w(2) - w(3) = 1\}$.

$\boxplus$

5.5 The Ideal I_h

In this section, assume $Hess(X, h)$ is a Hessenberg variety whose cohomology is generated by its degree-2 cohomology classes. This implies that $Hess(X, h)$ is a GM-space. The goal of this section is to identify an ideal I_h such that

$$H_S^*(Hess(X, h); \mathbb{C}) = \frac{H_S^*(GL_n(\mathbb{C})/B; \mathbb{C}))}{I_h}.$$

The line arrangement $\mathrm{Spec}\,(H_S^*(Hess(X, h); \mathbb{C})) \subset \mathrm{Spec}\,(H_S^*(GL_n(\mathbb{C})/B; \mathbb{C}))$ is a subarrangement with lines corresponding to the fixed points in $Hess(X, h)$. Thus, the ideal I_h specifies which fixed points from $GL_n(\mathbb{C})/B$ are contained in $Hess(X, h)$. We start this section with an algorithm defining a candidate for the ideal I_h. Then we prove that this ideal I_h describes the equivariant cohomology of $Hess(X, h)$.

Algorithm 5.10. *(Constructing the ideal I_h) The ideal I_h is constructed using essentially the same algorithm for identifying the $C_{h(i)}$ in the previous section.*

1. *Start with the Hessenberg diagram for h.*
2. *Fill in the diagonal with $x_i - it$ in the box (i, i)*
3. *Fill in the box (i, j) with $x_i - x_j - t$.*
4. *Take g_i to be the product of all unshaded entries which are above and to the right of (i, i) in the diagram. In other words*

$$g_i = (x_i - it)\left(\prod_{k<i, j\geq i}(x_k - x_j - t)\right)\left(\prod_{j>i}(x_i - x_j - t)\right).$$

5. *Define I_h to be the ideal $I_h = \langle g_1, g_2, \ldots, g_n\rangle$ generated by the g_i.*

Example 5.5. Let $h(1, 2, 3, 4) = (2, 4, 4, 4)$.

$x_1 - t$	$x_1 - x_2 - t$	$x_1 - x_3 - t$	$x_1 - x_4 - t$
	$x_2 - 2t$	$x_2 - x_3 - t$	$x_2 - x_4 - t$
		$x_3 - 3t$	$x_3 - x_4 - t$
			$x_4 - 4t$

In this case the ideal I_h is $I_h = \langle g_1, g_2, g_3, g_4\rangle$ where

- $g_1 = (x_1 - t)(x_1 - x_2 - t)$
- $g_2 = (x_2 - 2t)(x_1 - x_2 - t)(x_2 - x_3 - t)(x_2 - x_4 - t)$

- $g_3 = (x_3 - 3t)(x_2 - x_3 - t)(x_3 - x_4 - t)(x_2 - x_4 - t)$

- $g_4 = (x_4 - 4t)(x_3 - x_4 - t)(x_2 - x_4 - t)$

$\boxplus$

The ideal I_h was constructed so that it describes exactly which fixed points are in the Hessenberg variety $Hess(X, h)$. We now prove that it does just that.

Theorem 5.11. *The ideals $E_n(\boldsymbol{x}, t) + I_h$ and $\bigcap_{wB \in Hess(X,h)} I(\mathsf{t}_w)$ are equal*

$$E_n(\boldsymbol{x}, t) + I_h = \langle e_1(\mathbf{x}) - e_1(t), e_2(\mathbf{x}) - e_2(t), \ldots, e_n(\mathbf{x}) - e_n(t) \rangle + I_h = \bigcap_{wB \in Hess(X,h)} I(\mathsf{t}_w).$$

In other words, the equivariant cohomology of the regular nilpotent Hessenberg variety $Hess(X, h)$ is

$$H_S^*(Hess(X, h), \mathbb{C}) \cong \frac{\mathbb{C}[\boldsymbol{x}, t]}{\bigcap_{wB \in Hess(X,h)} I(\mathsf{t}_w)} \cong \frac{\mathbb{C}[\boldsymbol{x}, t]}{E_n(\boldsymbol{x}, t) + I_h}.$$

Proof. We proceed by induction on the Hessenberg function h. Our base case is when $h(i) = n$ for all n. In this case the Hessenberg variety is the full flag variety. Since $h(i) = n$ for all $1 \leq i \leq n$ the ideal I_h describes the set of all fixed points in $GL_n(\mathbb{C})/B$. So the subarrangement it describes is the full line arrangement of $H_S^*(GL_n(\mathbb{C})/B; \mathbb{C})$. In other words $E_n(\mathbf{x}, t) + I_h = E_n(\mathbf{x}, t)$. However, we know that

$$E_n(\mathbf{x}, t) \cong \bigcap_{wB \in GL_n(\mathbb{C})/B^S} I(\mathsf{t}_w)$$

because the left hand side is the presentation of S-equivariant cohomology given in Equation (5.2) and the right hand side is Goresky-MacPherson's description of equivariant cohomology. This proves the base case.

Assume by induction that the isomorphism holds for all Hessenberg functions h' with $h'(j) = h(j) + 1$ and $h'(i) = h(i)$ for all $i \neq j$. We will show that the isomorphism holds of the Hessenberg function h as well.

The ideals $I_{h'} + E_n(\mathbf{x}, t) = \bigcap_{wB \in Hess(X,h')} I(\mathbf{t}_w)$ are equal by our induction hypothesis. If $i \notin [j, h'(j)]$ then the generator g_i in I_h is equal to the generator g'_i in $I_{h'}$ because the diagrams of h and h' have the same unshaded boxes above and to the right of the diagonal entry (i, i). (In this diagram the function h is marked by all of the shaded boxes, and h' is the represented by only the darker shaded boxes.)

i				
	j		$x_j - x_{h'(j)} = 1$	
			$h'(j)$	

If $i \in [j, h'(j)]$ the generator $g_i = g'_i/(x_j - x_{h'(j)} - t)$ because the diagram of h' has one more unshaded box above and to the right of the ith box on the diagonal than the diagram for h. Furthermore this unshaded box is labeled with $x_j - x_{h'(j)} - t$.

	j		$x_j - x_{h'(j)} - t$	
		i		
			$h'(j)$	

This means that the line arrangement

$$\operatorname{Spec}\left(\frac{\mathbb{C}[\mathbf{x}]}{E_n(\mathbf{x},t)+I_h}\right)$$

does not include any of the lines $\mathfrak{t}_w$ corresponding to the fixed points wB in $Hess(X,h')^S$ for which the equation $w(j)-w(h'(j))=1$ is satisfied. However, the ideal $E_n(\mathbf{x},t)+I_h$ does include all of the lines $\mathfrak{t}_w$ corresponding to the wB in $Hess(X,h')$ where $w(j)-w(h'(j))\neq 1$. Lemma 5.9 proved that the set of points wB in $Hess(X,h')$ for which $w(j)-w(h'(j))\neq 1$ are exactly the fixed points in $Hess(X,h)^S$. Thus the line arrangement

$$\operatorname{Spec}\left(\frac{\mathbb{C}[\mathbf{x}]}{E_n(\mathbf{x},t)+I_h}\right)$$

contains only the lines corresponding to fixed points in $Hess(X,h)^S$. We conclude that

$$E_n(\mathbf{x},t)+I_h=\bigcap_{wB\in Hess(X,h)} I(\mathfrak{t}_w).$$

By induction the ideals $E_n(\mathbf{x},t)+I_h=\bigcap_{wB\in Hess(X,h)} I(\mathfrak{t}_w)$ are equal for any Hessenberg function h. $\square$

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