Summer 2012

Investigation of k-string energy using the gauge/gravity correspondence

Xiaolong Liu
University of Iowa

INVESTIGATION OF K-STRING ENERGY USING THE GAUGE/GRAVITY CORRESPONDENCE

by

Xiaolong Liu

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Physics in the
Graduate College of The
University of Iowa

July 2012

Thesis Supervisor: Professor Vincent G. J. Rodgers

ABSTRACT

The AdS/CFT correspondence, or in a more general sense, the gauge/gravity correspondence, is a duality between a string theory or gravity defined on one space and a gauge theory living on the conformal boundary of the same space. The AdS/CFT correspondence has drawn a tremendous amount of attention since the late 1990's. After all, it sounds so promising and exciting to connect two drastically different looking theories, gravity and gauge theory, directly.

A lot of effort has been put into the gauge/gravity correspondence related research from the physics community, hoping that connections between string theories and corresponding gauge theories with less supersymmetry would be discovered and more observables of the gauge theory could be calculated in terms of string theory and supergravity. The k-string tension calculation using the gauge/gravity correspondence has been one of many such attempts.

k-string tension is a crucial topic in quark confinement and strong interactions. In this dissertation, we investigate k-strings in $d = 3$ and $d = 4$ spacetime dimensions using the gauge/gravity correspondence. Exploiting the similarities between two supergravity backgrounds, i.e., the Maldacena-Nunez background and the Maldacena-Nastase background, we carry out calculations for k-string energies in $d = 3$ and $d = 4$ spacetime dimensions. The specific calculations investigated are the lowest order tension term for the energy of k-strings and the first order, one loop corrections, i.e., the Lüscher term. The tension term is proportional to L, the length of quark-antiquark pairs and the Lüscher term is the typical $1/L$ Coulombic correction. we reproduced the sine law for the tension term and the Lüscher term which agree with the results from lattice gauge theory.

We also briefly compare the resluts of the Klebanov-Strassler background with the Maldacena Nunez and the Maldacena Nastase backgrounds, with the hope to summarize and generalize the k-string energy calculations in the context of gauge/gravity correspondence using different supergravity backgrounds that have similar geometry.

Abstract Approved: ______________________
Thesis Supervisor

____
Title and Department

____
Date

INVESTIGATION OF K-STRING ENERGY USING THE GAUGE/GRAVITY CORRESPONDENCE

by

Xiaolong Liu

A thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Physics in the Graduate College of The University of Iowa

July 2012

Thesis Supervisor: Professor Vincent G. J. Rodgers

Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Xiaolong Liu

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Physics at the July 2012 graduation.

Thesis Committee: ______________________________
Vincent G. J. Rodgers, Thesis Supervisor

Hao Fang

Yannick L. Meurice

Craig E. Pryor

Leopoldo A. Pando Zayas

To my grandmothers

ACKNOWLEDGMENTS

First, I would like to thank my advisor, Prof. Vincent G. J. Rodgers. The graduate school life in Iowa hasn't always been easy for me. It was his encouragement, support and patience that helped me get through the hard times. He is a very inspiring and enlightening teacher and it wouldn't be possible at all for me to finish this project without his guidance. I am truly grateful for having such a great advisor. Also, I would like to thank the rest of my thesis committee members, Prof. Hao Fang, Prof. Craig Pryor, Prof. Yannick Meurice and Prof. Leopoldo Pando Zayas for their time, their patience and their support. I want to specially thank Prof. Pando Zayas for his guidance throughout the years. I would like to thank Prof. Hallsie Reno for her encouragement and support and I want to thank Prof. Usha Mallik for allowing me to work with her during the early days of my graduate school time. I would also like to express my gratitude and respect to the late Prof. Thomas Branson who had taught me before he left us.

I want to thank all my professors who have taught me and have been very supportive and my colleages in Prof. Rodgers' group. I made a lot of friends at Iowa and I appreciate the friendships that have warmed my heart in cold Iowa winters. I want to thank Dr. Kory Stiffler for his support and help and for being a great friend.

I'd like to thank my partner Julia for her love and support. At the final and difficult stage of my thesis, it was her love that cheered me up.

Finally, I want to thank my parents. They gave me life and love and want nothing back. They always taught me to be a good person. Any word would be too pale to describe how grateful I am. I also want to thank my brother Xiaopan for being the greatest brother.

ABSTRACT

The AdS/CFT correspondence, or in a more general sense, the gauge/gravity correspondence, is a duality between a string theory or gravity defined on one space and a gauge theory living on the conformal boundary of the same space. The AdS/CFT correspondence has drawn a tremendous amount of attention since the late 1990's. After all, it sounds so promising and exciting to connect two drastically different looking theories, gravity and gauge theory, directly.

A lot of effort has been put into the gauge/gravity correspondence related research from the physics community, hoping that connections between string theories and corresponding gauge theories with less supersymmetry would be discovered and more observables of the gauge theory could be calculated in terms of string theory and supergravity. The k-string tension calculation using the gauge/gravity correspondence has been one of many such attempts.

k-string tension is a crucial topic in quark confinement and strong interactions. In this dissertation, we investigate k-strings in $d = 3$ and $d = 4$ spacetime dimensions using the gauge/gravity correspondence. Exploiting the similarities between two supergravity backgrounds, i.e., the Maldacena-Nunez background and the Maldacena-Nastase background, we carry out calculations for k-string energies in $d = 3$ and $d = 4$ spacetime dimensions. The specific calculations investigated are the lowest order tension term for the energy of k-strings and the first order, one loop corrections, i.e., the Lüscher term. The tension term is proportional to L, the length of quark-antiquark pairs and the Lüscher term is the typical $1/L$ Coulombic correction. we reproduced the sine law for the tension term and the Lüscher term which agree with the results from lattice gauge theory.

We also briefly compare the resluts of the Klebanov-Strassler background with

the Maldacena Nunez and the Maldacena Nastase backgrounds, with the hope to summarize and generalize the k-string energy calculations in the context of gauge/gravity correspondence using different supergravity backgrounds that have similar geometry.

TABLE OF CONTENTS

LIST OF TABLES

Table

LIST OF FIGURES

Figure

CHAPTER 1

INTRODUCTION

As complex and multifaceted as the world may seem to be, at the fundamental level, there exist only four different types of interactions in nature, listed in table 1.1: gravitation, electromagnetism, strong interaction and weak interaction. Strong and weak interactions have extremely short ranges. Most forces we directly experience in everyday life are either gravity or composite manifestations of electromagnetism. When we lift weight, walk or even do nothing specific at all, we feel gravity. In fact, gravity is ubiquitous and has become indispensable to our body so that it is a very important research subject how to keep astronauts healthy without experiencing proper bodyweight while in space. Most other forces or interactions of everyday life, to name a few, friction, traction, spring force, basketball shooting and baseball batting, etc. are cumulative manifestitations of electromagnetism and nuclear forces, simply speaking.

The strong interaction is the force that binds protons and neutrons together to form the nucleus. Technically, the strong force between the nucleons is the residuum of the strong interaction that holds quarks together to make up the nucleons. Quarks are known to be confined [3]. In other words, quarks are observed to exist only in groups to form mesons as pairs of quarks or baryons as triples. It may sound naive to think that the reason behind quark confinement is that it just takes too much energy to separate quarks. Actually, that is pretty much the direction physicists are taking to explain quark confinement and this dissertation is mostly about calculating roughly the amount of energy needed to separate quarks, using the gauge/gravity duality. The weak interaction is responsible for the nuclear beta decay, a process in which a neutron decays into a proton, an electron and an anti-neutrino. In general, processes in which neutrinos are involved are mediated by weak interactions.

One of the ultimate goals of physics is unification. Unification usually means the

Fundamental forces	Strength	Theory	Mediator	Range(m)
Strong interaction	10	Chromodynamics	Gluon	10^{-15}
Electromagnetism	10^{-2}	Electrodynamics	Photon	∞
Weak interaction	10^{-13}	Flavordynamics	W and Z bosons	10^{-18}
Gravitation	10^{-42}	General relativity	Graviton	∞

Table 1.1: List of fundamental forces

attempt to describe all fundamental interactions by a single set of mathematical equations, i.e., a single fundamental principle. In a more specific way, unification can mean any action that summarizes and unifies seemingly unrelated and often very different phenomena by a single principle. The motivations of unification in physics lie in different levels: simplicity, elegance and more importantly necessity. History has taught us that unification is an inevitable trend of discovering nature. The more we know about nature, the more we believe that unification is a fundamental feature of nature.

The journey of unification could be considered to have begun as early as in ancient Greece, when philosopher Democritus, along with Leucippus and Epicurus, speculated an atomic theory and claimed the indestructible and indivisible "atoms" to be the smallest constituents of matter. The ancient Greeks perhaps were not alone in terms of having deep thoughts about nature. Roughly around the same time of the above mentioned Greek Greats, in the Far East, ancient Chinese philosophers Zhuangzi and Huizi were arguing what would be left if cutting a stick into two halves with equal length were to be repeated infinite times: where does it end? Similar arguments probably had happened among sages in many early human civilizations in different continents. We'll leave that for philosophers and historians to figure out.

The first major step of unification of modern sense took place in the 19th century when James Clerk Maxwell successfully summarized two drastically different phenomena:

eletricity and magnetism with a consistent set of equations that were later named after him, based on the work of the greatest physicists of his era: Henry Cavendish, Charles Augustin de Coulomb, Hans Christian Oersted, Jean-Baptiste Biot, Felix Savart, Andre-Marie Ampere and Michael Faraday. The beauty and creativity of Maxwell's equations is that not only are they simple, but also, in fact more importantly, they are indispensable. Indeed, separate equations of electricity and magnetism would not be consistent. What's more, the Maxwell's equations resulted in the prediction of electromagnetic waves. The Maxwell's equations in differential form are as follows

$$\begin{aligned}\nabla\cdot \vec{E} &= \frac{\rho}{\epsilon_0}\\ \nabla\cdot \vec{B} &= 0\\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}\\ \nabla \times \vec{B} &= \mu_0 \vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}\end{aligned} \tag{1.0.1}$$

Clearly, one can't write separate equations for $\vec{E}$ or $\vec{B}$. The electric field and the magnetic field simply depend on each other. Of course, now we know that $\vec{E}$ and $\vec{B}$ are just different components of the same physical quantity–the electromagnetic tensor $F_{\mu\nu}$. Actually, we know far more than that.

Another significant breakthrough of unification occured when space and time were finally treated on the same footing, unified as spacetime continuum by Albert Einstein's special theory of relativity in the first decade of the 20th century. In fact, Einstein's starting point was the Maxwell's equations. Space and time had long been thought of as separate notions meaning that the three dimensional flat space is an infinitely large stage and time flows like an eternal river. In other words, there is no interaction between space and time: time is just an independent parameter that measures what happens on the stage of space. In special relativity, the concept of space as a stage has become a spacetime arena. Special relativity was such a revolution that it forever changed the way humans see the world. Technically, everything we had learned in newtonian physics was incorrect or inaccurate. Simple equations like $F = ma$ are just low-speed approximations of what is consistent with the special relativity. The Maxwell's equations 1.0.1, in the covariant formulation, can be written as

$$J = (c\rho, \vec{J})$$
$$\mu_0 J^\mu = \partial_\nu F^{\mu\nu} \tag{1.0.2}$$

where

$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$$
$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$
$$A^\mu = (\varphi/c, \vec{A}) \tag{1.0.3}$$

where $\eta^{\mu\nu}$ is the Minkowski metric that can be used to raise the indices of the tensors in Minkowski space. It shouldn't be surprising that the classical Maxwell's equations 1.0.1 turn out to be consistent with special relativity. After all, one of the two postulates of special relativity, the principle of invariant light speed, is motivated by experimental measurements which agree with the classical Maxwell's equations and the fact that the same speed of light can be derived from the classical Maxwell's equations, regardless of the inertial frame of reference of the observer, which implies the second postulate, the principle of relativity, as well. In other words, special relativity can be seen as a direct consequence of seeking a new spacetime theory that would be consistent with Maxwell's equations. More important than anything else, to qualify as a successful theory, everything has to be consistent with experiments at the end of the day. Special relativity certainly is successful in this regard in its own realm.

Quantum mechanics was another revolutionary discovery in modern physics. The founding fathers of quantum mechanics include the greatest names of the 20th century physicists: Erwin Schrodinger, Werner Heisenberg, and Paul Dirac. Quantum mechanics is the framework for describing microscopic phenomena. In quantum mechanics classical observables are replaced by quantum operators. A physical system is described by a wavefunction Ψ in Schrodinger's formulation or the ket $|\psi\rangle$ in Dirac's. The time-dependent Schrodinger equation takes the following form

$$i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi \tag{1.0.4}$$

which determines how the system evolves with time. In general, we can no longer predict a unique outcome determined by the initial conditions for a system. Instead, we calculate the probability for a given event to occur. Possible outcomes of measuring an observable(operator) are generally discrete quantities, often refered to as the eigenvalues of the operator involved. The correspondence principle between quantum mechanics and classical mechanics states that classical mechanics is just an approximation of quantum mechanics for large systems.

To describe an object of everyday life, as we know, classical mechanics is often accurate enough. When an object travels fast, at speeds comparable to the speed of light, we turn to special relativity. When an object is small, at scales of atomic or subatomic level, we need quantum mechanics. What about the objects that are both small and fast? Well, there came the quantum field theory (QED, QCD) which incorparates quantum principles and relativity. Maxwell's equations, equations of classical electromagnetism, are not quantized. The quantum version of classical electrodynamics, quantum electrodynamics (QED) provides accurate computations for electromagnetic interactions at the microscopic level.

In the late 1960's the Glashow-Weinberg-Salam model of electroweak interactions unified electromagnetism and the weak interaction. Just as the Maxwell's equations were necessary in order to have a consistent model to describe the electric fields or the magnetic fields alone, the electroweak theory wasn't optional either. Instead, it was necessary in order to describe the weak force consistently. Initially, the theory is formulated with four massless particles as the force carriers. The spontaneous symmetry breaking through the Higgs mechanism [4] gives mass to three of these particles: W^+, W^- and Z^0, the carriers of the weak force. The remaining massless particle is the photon, the carrier of the electromagnetic interaction.

The quantum chromodynamics (QCD) is the quantum theory for the strong interactions. The eight massless carriers of the strong force are the gluons. Just like the quarks, the gluons cannot be isolated either. The quarks respond to the gluons for the reason that they carry color. The quarks also carry electric charge and can respond to the weak interaction as well. The electroweak theory and QCD are assembled together to form the standard model. In the standard model, there are twelve force carriers: the eight gluons for the strong force, W^+, W^- and Z^0 for the weak force, the photon for the electromagnetic force. All of the force carriers are bosons. There are also the matter particles which are of two types: leptons and quarks. All of the matter particles are fermions. Fig 1.1 [1] summarizes briefly the present knowledge of elementary particles of the standard model .

On the stage of unification, it seems that the standard model takes care of the strong interaction, the weak interaction and electromagnetism pretty well. What happened to gravity? In 1916, Einstein published his general theory of relativity. The spacetime arena in special relativity gains its own life in general relativity. Gravitational forces arise due to the curvature of this dynamical spacetime manifold. Spacetime is curved by matter and energy in a way determined by the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.0.5}$$

where $R_{\mu\nu}$, $g_{\mu\nu}$, $T_{\mu\nu}$, R and Λ are the Ricci curvature tensor, the spacetime metric tensor, the stress-energy tensor, the Ricci scalar and the cosmological constant respectively. However beautiful the general relativity is, there is an issue: it is not a quantum theory. The natural question is: do we need to quantize gravity and why? The answer is very simple: for the sake of unification, gravity has to be quantized because one can't have a partly classical and partly quantum theory that is still consistent. Also, as a practical matter, gravity is rather negligible compared with the other forces most of the times. However, at the very early stage of the universe near the Big Bang and in black hole

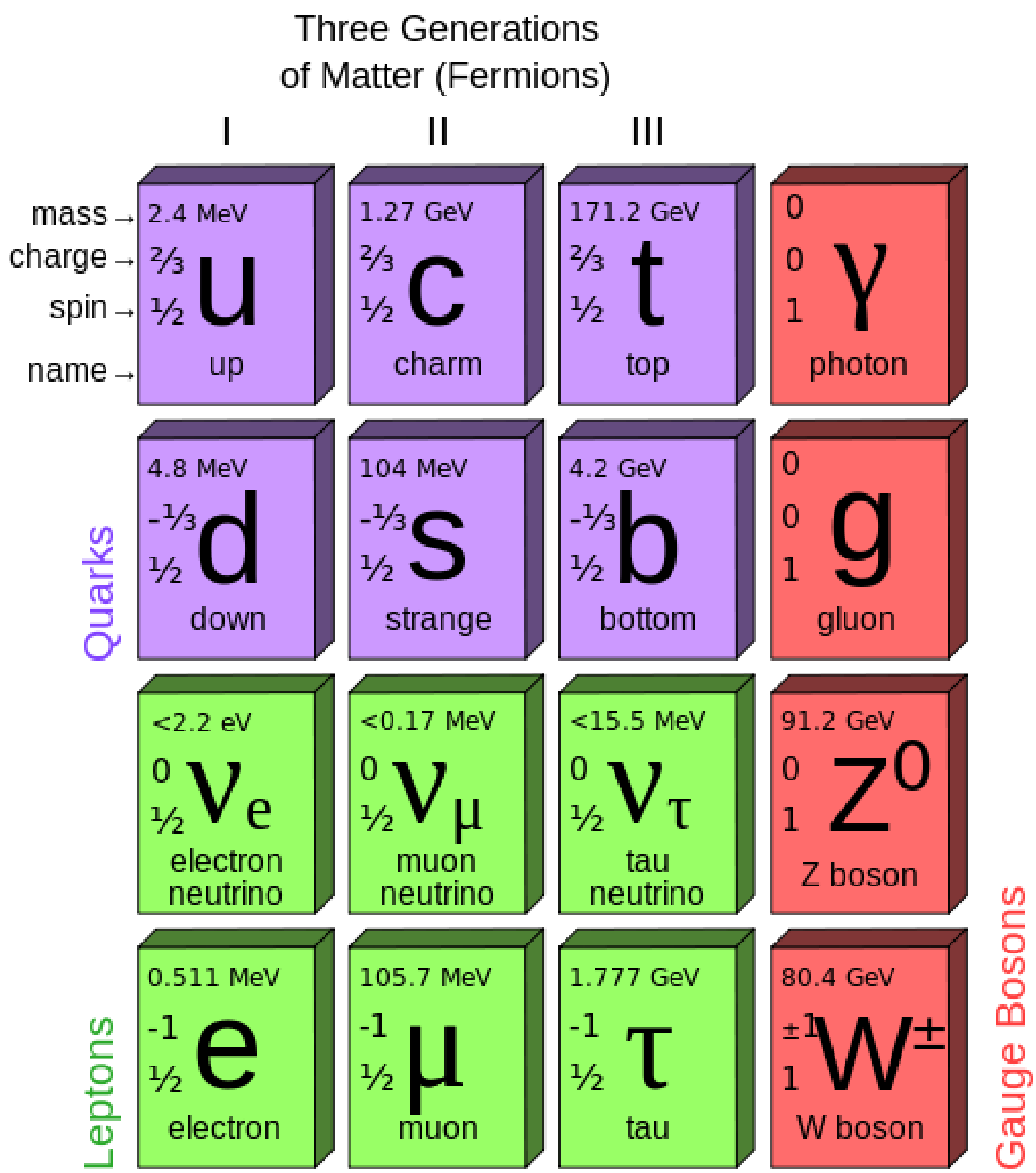

Figure 1.1: Elementary particles of standard model [1]

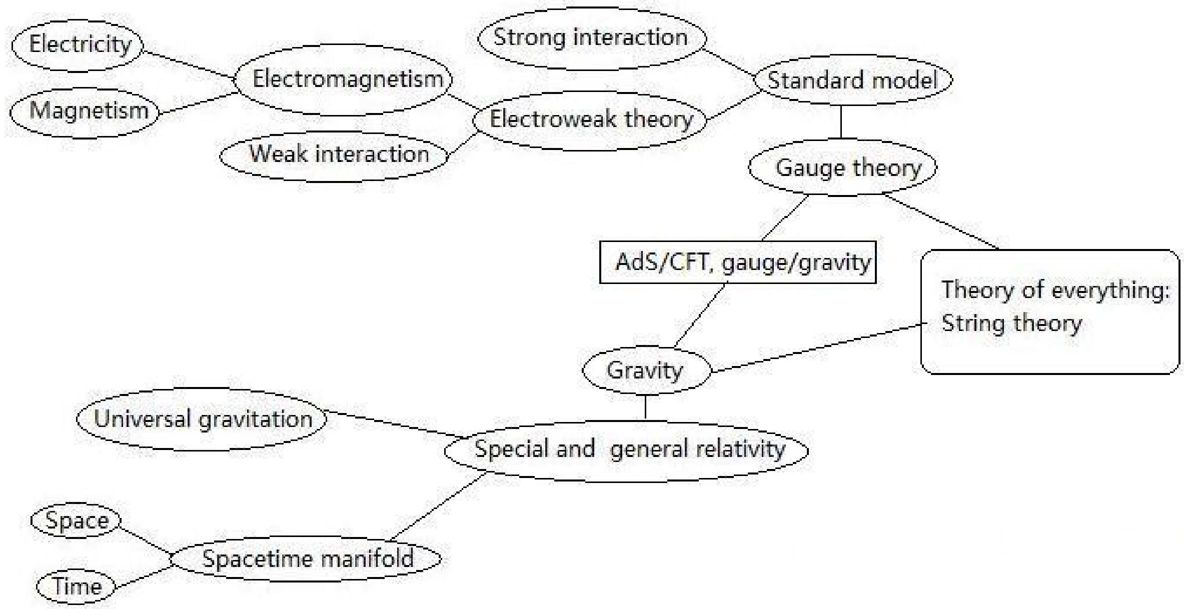

Figure 1.2: A map of unification

studies, the gravitational effect is just too strong to neglect. A quantum theory of gravity is definitely necessary in those situations. It is pretty clear that a quantum theory of gravity may not lead to the ultimate unification but to realize the unification, quantizing gravity is essential.

Since physicists got to the point of identifying the four fundamental interactions of nature, attempts to unify all four interactions have never been stopped. Among all the candiates for unification, string theory is considered very promising by most physicists. Fig 1.2 illustrates a simple map of what physicists have done and what we know about unification.

Widely recognized as a prosperous theory of quantum gravity today, originally string theory was developed with the intent of describing strong interactions in the late 1960's and early 1970's [5, 6, 7] even though QCD ended out being accepted as the correct theory for the strong force. However, what made string theory fail to be an acceptable theory of strong interactions, the existence of the massless spin 2 excitation of closed strings, later makes string theory an even bigger role: a great candidate for unifying all four interactions of nature. The massless spin 2 excitation of closed strings is identified as the graviton.

As a candidate for unification, string theory is considered a very promising theory of quantum gravity and attempts to describe strong interactions through the gauge/gravity correpondence. Although the gauge/gravity correspondence has yet to be proven, a large amount of effort along this line has produced fruitful results thus far. Gravity and gauge theory, two completely different looking theories are related by the so-called holographic principle [8, 9, 10, 11]. Just as the name holography suggests, gravity is dual to a gauge theory which lives on the conformal boundary of the space where gravity lives. Being able to relate gravity and gauge theory through the gauge/gravity correspondence is a huge step foward on the road to unification. The best established example of the gauge/gravity correspondence is the AdS/CFT correspondence [12, 13, 14] which states that a type IIB superstring theory in $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ super Yang-Mills in $4d$ Minkowski space.

In other words, the gauge/gravity correspondence is the generalization of the AdS/CFT correspondence in which more general supergravity backgounds are studied with the hope of either finding something more relevant to nature, e.g., investigating supergravity backgrounds that lead to gauge theories with less supersymmetry or carrying out more realistic calculations for the gauge theory side, e.g., calculating physical spacetime quantities from the string theory perspective that are otherwise hard to calculate in QCD. Attacking k-string physics from the string theory standpoint is one such attempt. In this thesis, we investigate the Maldacena-Nunez background [15] and the Maldacena-Nastase background [16] that are known to be dual to $\mathcal{N} = 1$ super Yang-Mills in the IR and we reproduce the k-string energy in d=3 and d=4 spacetime dimension by calculating the action of a probe D3-brane embedded in each of the above mentioned supergravity backgrounds respectively. The probe D3-brane is conjectured to be dual to a k-string in $3d$ and $4d$ respectively depending on the supergravity background in which the probe D3-brane is embedded.

Among the problems of standard model is quark confinement, which states that individual quarks cannot be isolated. Experimentally, quarks are only observed to exist in groups to form mesons or baryons. k-string physics has been a very important topic in quark confinement and strong interactions. Calculations in both the lattice gauge community [17, 18, 19, 20, 21, 22, 23] and the string theory community [24, 25, 26, 27, 28] have been done to investigate the energy of the k-strings. There are mainly two different formulations, i.e, the Casimir law [29] and the sine law [30] that have been proposed to describe the k-string tension throughout the years.

The k-string energy generally takes the following form

$$E = T_k L \tag{1.0.6}$$

where T_k is the the k-string tension and L is the distance between the quark-antiquark pair.

In the Casimir formulation, the k-string tension reads

$$T_k = k(1 - \frac{k-1}{N-1})T_f \tag{1.0.7}$$

where T_f is the tension for a fundamental string and for large N, the tension T_k can be expanded as

$$T_k = k[1 - \frac{k-1}{N} - \frac{k-1}{N^2} + \mathcal{O}(N^{-3})]T_f \tag{1.0.8}$$

which has both odd and even powers of $1/N$.

While for the sine law, the k-string tension takes the following form

$$T_k = (\sin\frac{\pi}{N})^{-1}\sin(\frac{\pi k}{N})T_f \tag{1.0.9}$$

which, for large N, can be expanded

$$T_k = k[1 - \frac{\pi^2}{6N^2}(k^2-1) + \frac{\pi^4(3k^4 - 10k^2 + 7)}{360N^4} + \mathcal{O}(N^{-6})]T_f \tag{1.0.10}$$

which runs in even powers of $1/N$ only. In this thesis, we show that an exact sine law can be obtained for gauge theories that are dual to the MN and MNa supergravity backgrounds respectively by carrying out calculations in the context of the gauge/gravity correspondence.

The Lüscher term [31] is the Coulombic term that is proportional to 1/L in the k-string energy which includes the quantum fluctuations. A fundamental k string, often refered to as a fundamental string, is a k-string for $k = 1$ and the fundamental string energy including the Lüscher term goes as follows

$$E_f = T_f L + \beta + \frac{\gamma}{L} + \mathcal{O}(L^{-2}) \tag{1.0.11}$$

where $\gamma = -\frac{(d-2)\pi}{24}$ [31].

A general k-string energy with the Lüscher term goes as the following

$$E = T_k L + \frac{\gamma'}{L} + V_c + \mathcal{O}(L^{-2}) \tag{1.0.12}$$

where V_c is a small constant.

We also fluctuate the bosonic fields and the fermionic fields infinitesimally around the ground state of the probe D3-brane and reproduce the Lüscher term for the k-string energy by investigating the quadratic terms of the fluctuated D3-brane action.

One of the motivations of this thesis is the geometrical similarities between the MN, MNa backgounds and the Klebanov-Strassler [32] and CGLP backgounds [33] and the work done in [27, 26] on k-string energy.

CHAPTER 2

THE HOLOGRAPHIC PRINCIPLE AND THE ADS/CFT CORRESPONDENCE

This chapter briefly reviews the development of string theory from a non-technical perspective and introduces the AdS/CFT correspondence as well.

2.1 A Very Brief Review of String Theory History

According to string theory, the fundamental constituents of nature are extended one-dimensional microscopic strings. All elementary particles, including the graviton, are identified as different vibrational modes of the strings. Unlike the standard model, String theory doesn't have adjustable dimensionless parameters. The only dimensionful parameter string theory has is the string length l_s. String theory derives the dimensionality of spacetime, i.e., 26 for bosonic strings or 10 for superstrings, as a consistency requirement of the theory. While in standard model, the number of dimensions is put into it by hand. Therefore, a full flavor of unification is in the gene of string theory.

The original string action, the Nambu-Goto action, describes a relativistic action of an open string embedded in d-dimensional flat spacetime. The string world sheet coordinates are τ and σ.

$$S = -T \int d\tau d\sigma \sqrt{\dot{X}^2 X'^2 - (\dot{X} \cdot X')^2} \tag{2.1.1}$$

where

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}$$
$$X^{\mu\prime} = \frac{\partial X^\mu}{\partial \sigma} \tag{2.1.2}$$

and T is the string tension

$$T = \frac{1}{2\pi\alpha'} \tag{2.1.3}$$

where $\alpha' = l_s^2$ is the Regge slope parameter. Classically, the Nambu-Goto action 2.1.1

is equivalent to the Polyakov action, which tends to make quantization easier

$$S = -\frac{T}{2}\int d^2\xi\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \tag{2.1.4}$$

where γ^{ab} is the induced metric on the string world sheet. Roughly speaking, the Nambu-Goto action and the Polyakov action are the starting point of analyzing string behaviors. Most development of string theory, e.g., deriving equations of motion for the strings and quantizing the strings, are based on this.

In the 1940's, following Wheeler's work earlier, Heisenberg proposed an S-matrix theory to describe the scattering amplitude of the strongly interacting particles which were proposed to be extended objects by himself. The idea was that space and time broke down at microscopic levels and the fundamental quantity was the quantum mechanical amplitude for incoming particles to turn into outgoing ones during a scattering event. This idea was too obscure for any realistic physical calculations to be carried out because of the lack of space-time structure and the lack of any knowledge of the intermediate steps. For this reason, the S-matrix theory was discarded by the physics community after a while.

In 1958, Regge introduced the Regge trajectories [34, 35, 36] to classify the bound states or the resonances in quantum mechanics according to their angular momentum. It was soon realized that Regge trajectories for mesons made straight lines. In other words, there was a linear relationship between the angular momentum and energy-squared for mesons, illustrated in figure 2.1 [2]. Therefore, according to Regge theory, the scattering amplitude of mesons should decrease exponentially at large angles. Out of many attempts by the physics community to find a scattering theory that would match the asymptotic behavior at large angles predicted by the Regge theory, Veneziano used Heisenberg's S-matrix approach to construct the dual resonance model in 1968 [37]. Veneziano discovered that the 4-particle scattering amplitude for particles on Regge trajectories could be described by the Euler Beta-function. This amplitude, known as the Veneziano amplitude, is now interpreted as the scattering amplitude for four open

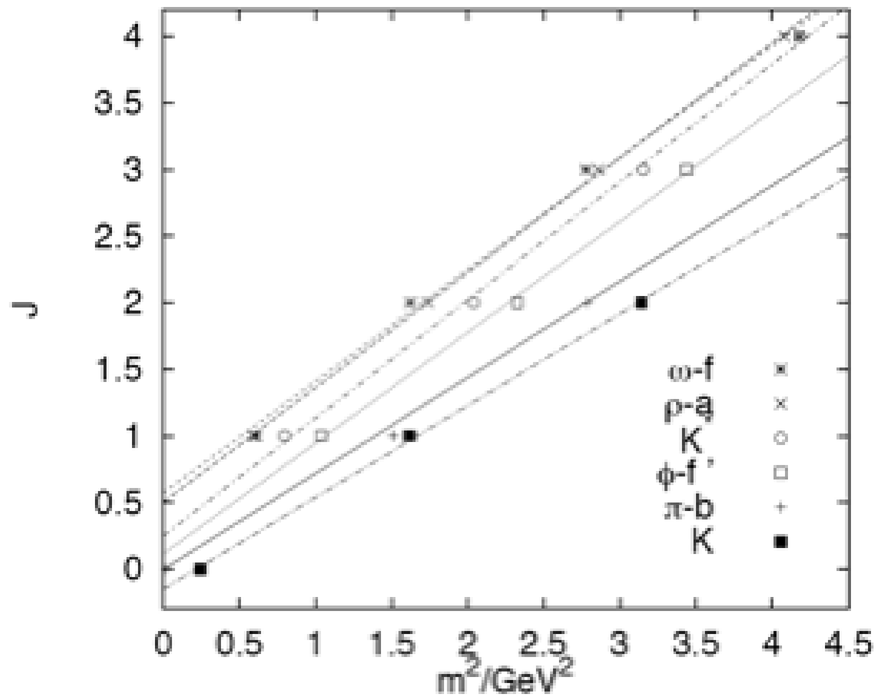

Figure 2.1: Regge trajectories for the low-lying mesons [2]

string tachyons and the linear relationship on Regge trajectories for mesons is now understood to be expected from rotating open strings.

The Veneziano amplitude was soon generalized to the N-particle case. What followed that then was intensive research effort from the physics community revealing that strong interactions [5, 6, 7] of particles represented as one-dimensional strings instead of zero-dimensional particles were described exactly by the Euler Beta-function as well. This was widely recognized as the birth of string theory. According to this newfound string theory, the most fundamental consitutent of nature is a microscopic string and different string vibrational modes produce various observable properties in fundamental particles. However, there was one major problem with string theory: string theory also provided unwanted vibrational patterns, i.e. the spin 2 massless excitation of closed strings, which did not seem to match any known elementary particles. This, among other contradictions between string theory predictions and experiments, caused the physics community to give up string theory as a theory of strong interactions. At the same time, QCD was gradually accepted as the correct theory of strong interactions.

In 1974, however, the massless spin 2 modes were interpreted as gravitons [38, 39], which made a sharp turn for string theory since it marked the beginning of string theory as a theory that quantizes gravity. As supersymmetry [40, 41] was invented, superstring theories [42, 43] which include both bosonic and fermionic vibrational modes were born. Roughly between 1984 and 1986, then came the first superstring revolution during which time string theory was recognized to be a theory that can describe all elementary particles as well as the laws that govern the interactions between them. The revolution was initiated by the discovery of Green-Schwarz mechanism [44] that solved the anomaly of $SO(32)$ in type I string theory in 1984. Other important discoveries made during this period included the heterotic string theory and the recognition that the six extra dimensions need to be compactified on a Calabi-Yau manifold in order to obtain $\mathcal{N} = 1$ supersymmetry [45]. String theory was accepted by the mainstream physics community as an actual candidate of unifying all four interactions of nature since then. There are five known superstring theories: type I, type IIA, type IIB, $SO(32)$ heterotic and $E_8 \times E_8$ heterotic, listed in table 2.1.

Type	I	IIA	IIB	SO(32)	$E_8 \times E_8$	M-theory
Dimension	10	10	10	10	10	11
Chirality	yes	no	yes	yes	yes	no
Gauge group	SO(32)	U(1)	none	SO(32)	$E_8 \times E_8$	none
Heterotic?	no	no	no	yes	yes	no
String type	open(closed)	closed	closed	closed	closed	closed
SUSY generator	N=(1,0)	N=(1,1)	N=(2,0)	N=(1,0)	N=(1,0)	N=1

Table 2.1: Simple comparison of five superstring theories and M-theory

The second superstring revolution took place roughly between 1994 and 1997 as the concept of M-theory emerged [46]. As the string coupling is taken to infinity, the limit of type IIA string theory was shown to yield an 11-dimensional theory, the M-theory. The five different known superstring theories were found to be different limits of the M-theory. There is only one unique theory after all even though a full rigorous proof is still needed. New equivalences that connect different string theories such as S-duality [47], T-duality [48], mirror symmetry [49] were found. D-branes [50], which are higher-dimensional extended objects upon which endpoints of open strings lie with Dirichlet boundary conditions, were also discovered during this period.

The system of N stacked D3-branes is of particaular importance in the duality of string&gravity/gauge theory. The N D3-branes are stacked on top of each other. There are two very different descriptions of this stacked D3-branes system. In the low energy limit, on one hand, the massless open strings propagating on the world voulme of the D3-branes in the weak coupling regime realize the $SU(N)$ super Yang-Mills gauge theory. On the other hand, in the strong coupling regime, the system describes type IIB superstrings in the bulk that is curved by the heavy D3-branes. The investigatation of the relationship between these two descriptions is the core of the AdS/CFT correspondence.

2.2 The AdS/CFT Correspondence

In 1997, Maldacena [12] published a conjecture which proposed that a type IIB superstring theory in $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ super Yang-Mills conformal field theory in $4d$ Minkowski space. Here, AdS_5 stands for 5 dimensional anti de Sitter space which will be reviewed later, S^5 stands for 5-dimensional spheres and CFT stands for conformal field theory. This paper has attracted a tremendous amount of attention and the theoretical physics community responded quickly and has devoted a ton of energy and effort in researching along the line of the duality between gravity and gauge theory which has produced fruitful results. In the next few sections, we'll briefly review a few important theories behind the AdS/CFT correspondence and physics ideas that more or less contributed to the formation of the conjecture. Again, we'll focus on the general ideas of the conjecture instead of technical details. We'll introduce some generalizations and open questions of the AdS/CFT correspondence and how this may be related to our research, the k-string energy investigation.

2.2.1 The original Kaluza-Klein theory

The idea that there is a connection between gravity and gauge theory can be traced back to Kaluza and Klein as early as in the 1920's. In 1921, as an attempt to unify electromagnetism and gravity(strong and weak interactions were not well understood at the time), Kaluza proposed to extend Einstein's general relativity to a 5d spacetime whose fourth spatial dimension would later be proposed by Klein to be curled up as a tiny circle. As a result, electromagnetism can be derived as the gauge theory of the fiber bundle of this circle, i.e., the $U(1)$ gauge and the Einstein equation in 4d can be recovered once electromagnetism and gravity are decoupled in 4d. The 5 dimensional spacetime of the original Kaluza-Klein theory has the following metric

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{00} - \phi A_0 A_0 & g_{01} - \phi A_0 A_1 & g_{02} - \phi A_0 A_2 & g_{03} - \phi A_0 A_3 & -\phi A_0 \\ g_{10} - \phi A_1 A_0 & g_{11} - \phi A_1 A_1 & g_{12} - \phi A_1 A_2 & g_{13} - \phi A_1 A_3 & -\phi A_1 \\ g_{20} - \phi A_2 A_0 & g_{21} - \phi A_2 A_1 & g_{22} - \phi A_2 A_2 & g_{23} - \phi A_2 A_3 & -\phi A_2 \\ g_{30} - \phi A_3 A_0 & g_{31} - \phi A_3 A_1 & g_{32} - \phi A_3 A_2 & g_{33} - \phi A_3 A_3 & -\phi A_3 \\ -\phi A_0 & -\phi A_1 & -\phi A_2 & -\phi A_3 & -\phi \end{pmatrix}$$

and the inverse metric

$$\hat{g}^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g^{\mu\nu} & -A^{\mu} \\ & \\ -A^{\nu} & -1/\phi + A_{\mu}A^{\mu} \end{pmatrix}$$

where hatted indices are for 5 dimensional spacetime and unhatted indices are for 4d: $\hat{\mu} = 0, 1, 2, 3, 4$, $\mu = 0, 1, 2, 3$ and $\hat{x}^{\hat{\mu}} = (x^{\mu}, z)$. A^{μ} will later be identified as the electromagnetic four-potential. The fourth spatial dimension z is of the order of Planck length and is periodic. The 5 dimensional spacetime has the topology of $R^4 \times S^1$. If the compact circle S^1 is to be replaced by more general manifolds whose fiber bundles correspond to general Lie groups, one can expect to acquire Yang-Mills gauge theories and the base space R^4 of Kaluza-Klein theory can be generalized to different topologies such as Riemannian manifolds or supersymmetric manifolds as well, in modern fashions. In the original Kaluza-Klein formulation, the fields $g_{\mu\nu}(x),\ A_{\mu}(x),\ \phi(x)$ are all independent of the fifth dimension z. If one starts with the pure gravity action in the 5 dimensional spacetime whose metric is $\hat{g}_{\hat{\mu}\hat{\nu}}$

$$\hat{S} = \int d^5\hat{x}\sqrt{\hat{g}}\hat{R} \tag{2.2.1}$$

where $\hat{g}$ is the determinant of $\hat{g}_{\hat{\mu}\hat{\nu}}$ which can be shown to be $-g\phi$ and $\hat{R}$ is the Ricci scalar of the 5d spacetime. Using the fact that $\int dz = 1$, giving $\hat{g}_{\hat{\mu}\hat{\nu}}$ a proper conformal rescaling and rescaling the scalar field ϕ properly, one can show that the 5d pure gravity action $\hat{S}$ 2.2.1 can be rewritten as [51]

$$\hat{S} = S = \int d^4x\sqrt{-g}(R - \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - \frac{1}{4}e^{-\sqrt{3}\phi}F_{\mu\nu}F^{\mu\nu}) \tag{2.2.2}$$

where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ which can naturally be identified as the electromagnetic tensor. It's obvious that this action in 4d contains gravity, electromagnetism and a scalar field ϕ which would later be interpreted as a dilaton field. The original Kaluza-Klein theory was not perfect by any means but it is still very inspiring in many different ways. It was truly remarkable that electromagnetism emerged beautifully from pure gravity in higher dimensions. It was pioneering introducing the concept of an extra compact dimension which was long before the extra dimensions and compactification are studied in string theory nowadays.

2.2.2 The holographic principle

As we know from everyday life experience, mass is proportional to volume, assuming the density is a constant. For instance, a 32-oz water bottle can hold twice amount of water as a 16-oz bottle. It would sound crazy if someone said that an object's mass is proportional to the surface area regardless of the shape. However, research in black hole studies has shown that nature might be much richer than we can ever imagine. Bekenstein [52, 53, 54] and Hawking [55] showed that the black hole entropy is proportional to the area of the event horizon

$$S_{BH} = \frac{A}{4} \tag{2.2.3}$$

where A is the area of the event horizon of the black hole. That was an astonishing discovery because one would have naturally expected that the number of possible states, as an exponential of the entropy, would change as a direct function of the volume, not the area. The process of the development of physics has been full of attempts to solve counterintuitive puzzles like this one. Inspired by the fact that the black hole entropy is proportional to the event horizon area instead of the volume of a black hole, 't Hooft proposed [8] the holographic principle stating that physics of a given space can be described by the information on the boundary of that space. In [56], 't Hooft mentioned: "This is what we found out about Natures book keeping system: the data can be written onto a surface, and the pen with which the data are written has a finite size." In some ways, this concept is similar to the hologram, in which a two-dimensonal plate records the interference pattern of light waves so that later when the reconstruction beam shines on the plate, three dimensional image will emerge. 't Hooft's holographic principle was later generalized by Susskind [10] in the context of string theory and it plays a significant role in the AdS/CFT correspondence as one of the two fundamental ideas that directly motivated it. In the AdS/CFT correspondence, relating gravity with conformal field theory in one less dimension is a specific example of the holographic principle.

2.2.3 An example of the holographic principle in 2+1 dimensional gravity

It's worth mentioning that even before the publication of 't Hooft's holographic principle, it had been shown that a correspondence exists between a certain 2+1 dimensional gravity theory and a 2 dimensional conformal field theory. In [57], Brown and Henneaux investigated the central extensions of the asymptotic symmetries of a 2+1 dimensinal gravity theory. They started from a certain asymptotically anti-de Sitter metric, a solution of Einstein field equations in 2+1 dimensions with a negative cosmological constant,

$$dS^2 = -\left(\frac{r^2}{R^2} + \alpha^2\right)dt^2 + 2A\alpha dt d\phi + \left(\frac{r^2 - A^2}{R^2} + \alpha^2\right)^{-1} dr^2 + (r^2 - A^2)d\phi^2 \quad (2.2.4)$$

whose dominant part, when A=0, coincides with the dominant contribution of the a globally anti-de Sitter space

$$dS^2 = -\left(\frac{r^2}{R^2}\right)dt^2 + \left(\frac{R^2}{r^2}\right)dr^2 + r^2 d\phi^2 \quad (2.2.5)$$

In general relativity, asymptotic symmetries are defined to be the gauge transformations that leave the considered field configurations asymptotically invariant. In the Hamiltonian formulation with the gravity theories, after identifying proper boundary conditions for the spacetime metric and converting them into boundary conditions of the canonical variables g_{ab} and π^{ab}, the asymptotic symmetries turn out to be the surface deformation vectors ξ^μ for the space-like hypersurfaces which preserve asymptotic field behaviors at the boundaries. The canonical generators of the transformations on the canonical variables are the global charges $H(\xi)$ whose Poisson bracket algebra is not always isomorphic to the Lie algebra of the infinitesimal asymptotic symmetries in the generic case. Instead,

$$\{H[\xi], H[\eta]\} = H[[\xi, \eta]] + K[\xi, \eta] \tag{2.2.6}$$

where $K[\xi, \eta]$ is the so-called central charges. The Hamiltonian $H[\xi]$ is written as the following

$$H[\xi] = \int d^n x \xi^\mu(x) \mathcal{H}_\mu(x) + J(\xi) \tag{2.2.7}$$

where $\mathcal{H}_\mu(x)$ are the standard constraints for general relativity and the surface term $J(\xi)$, refered to as the charge, is added to make sure that $H[\xi]$ has well defined functional derivatives. Eq 2.2.7 only determines the canonical generators up to the addition of a constant, which leaves the possibility of central charges. Starting from the metric 2.2.4 and specific boundary conditions which can be found in [57], Brown and Henneaux showed that the asymptotic symmetry group is the conformal group in 2 dimensions and the charges $J(\xi)$ form a central extension of the conformal group algebra. They showed that the Dirac bracket algebra of the the charges is a direct sum of two Virasoro algebra and the central charge turn out to be the central charge of the Virasoro algebra. It's hard to tell if Brown and Henneaux's work contributed in motivating Maldacena's original work on the AdS/CFT correspondence, but it can certainly be considered as an example of the holographic principle.

2.2.4 The large N field theory

Another fundamental idea behind the AdS/CFT correspondence, other than the holographic principle, was originated by 't Hooft too. In 1974 [58], 't Hooft proposed to generalize the $SU(3)$ gauge theory to $SU(N)$ in the limit of the color index N approaching infinity while keeping $g_{YM}^2 N$ fixed where g_{YM} is the coupling constant of the $SU(N)$ Yang-Mills gauge theory. 't Hooft showed that, between given initial and final states, the dominant part of the amplitude turns out to be a function of $\lambda \equiv g_{YM}^2 N$, the so called 't Hooft coupling. 't Hooft's large N gauge theory coincides with a low energy limit string theory description. Therefore, we will introduce the large N theory in the context of string theory using string theory notations [59]. A massless open string in string theory can be interpreted as a gauge boson and the open string coupling constant g_o corresponds to the Yang-Mills coupling constant g_{YM}. An open string splitting into two open strings is equivalent to a gauge boson turning into two gauge bosons and two open strings joining as a single one is equivalent to two gauge bosons turning into one.

In the low energy limit, consider a system of N stacked D-branes. One wants to calculate the amplitude of the propagation of an open string $[i, j]$ whose ends lie on the i-th and the j-th D-branes both initially and finally. The amplitude is equal to the sum of all possible intermediate steps, i.e., contributions from all possible diagrams. The simplest nontrivial process is illustrated in fig 2.2, in which an open string splits into two open strings and the two open strings rejoin as one again. The dark dots represent the open string interactions, i.e., splittings and joinings and the letters i, j, k represent the numbering of the branes. Each time an open string splits into two or two open strings join as one, a factor of g_{YM} must be included in the amplitude. The two intermediate open strings $[i, k]$ and $[k, j]$ end on the i-th and the j-th branes respectively and a common k-th brane, which can be any of the N branes. The N possibilities of the shared k-th brane must be included in the amplitude by a factor of N as well. Thus, the diagram 2.2 corresponds to an amplitude A_1

$$A_1 = C_1 g_{YM}^2 N = C_1 \lambda \tag{2.2.8}$$

where C_1 is a constant coefficient.

The intermediate open strings can split too. In fig 2.3, an open string $[i, j]$ splits into two intermediate open strings $[i, k]$ and $[k, j]$. Then an intermediate open string $[i, k]$ splits into $[i, m]$ and $[m, k]$. Later, $[m, k]$ and $[k, j]$ join as $[m, j]$ before $[i, m]$ and $[m, j]$ join as $[i, j]$ again. This process inludes four interaction points which corresponds to a factor of g_{YM}^4 and two times of choosing a common D-brane which corresponds to a factor of N^2 since the values of both k and m can have N possibilities . The amplitude

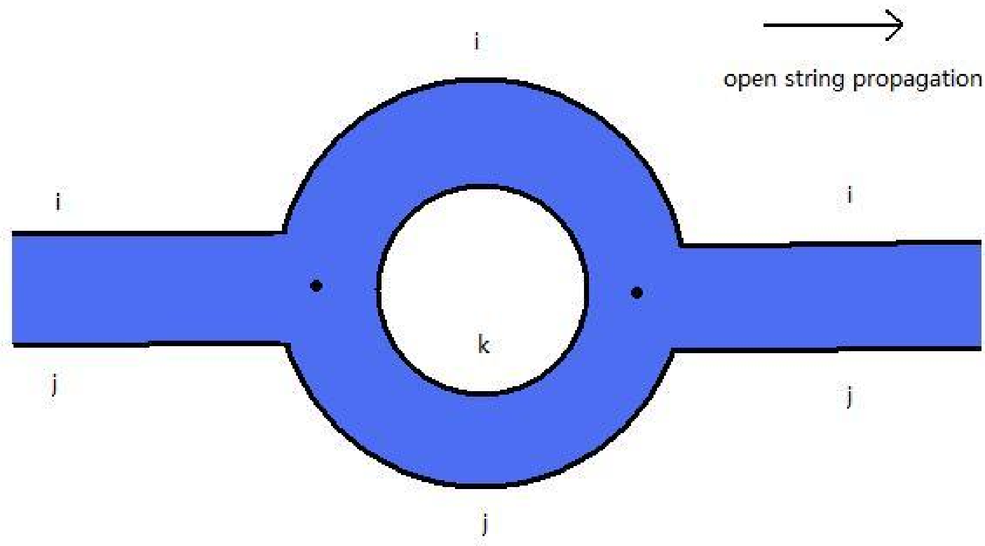

Figure 2.2: An open string ending on the i-th and the j-th branes splits into two open strings ending on the i-th, the j-th and a common k-th branes during propagation and the two open strings rejoin as one open string

that is represented by the diagram 2.3 is given as follows

$$A_2 = C_2 g_{YM}^4 N^2 = C_2 \lambda^2 \tag{2.2.9}$$

It's not hard to see the pattern. Each time an intermediate open string splits into two and later two intermediate open strings join as one, a factor of $\lambda \equiv g_{YM}^2 N$ must be included in the amplitude as long as the splitting and the joining happen in the 'planar' way, which corresponds to the 'planar diagrams' in 't Hooft's formulation [58]. Topologically, in the planar diagrams, each added strip, which adds a new 'loop' to the diagram, contributes two interaction points, which corresponds to g_{YM}^2 and a chance of choosing a brane from N branes, which corresponds to a factor of N. The amplitude that includes contributions from all possible planar diagrams therefore takes the following form

$$A = \sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} C_n \lambda^n \equiv f_0(\lambda) \tag{2.2.10}$$

which includes the trivial term A_0 that corresponds to the open string propagating freely

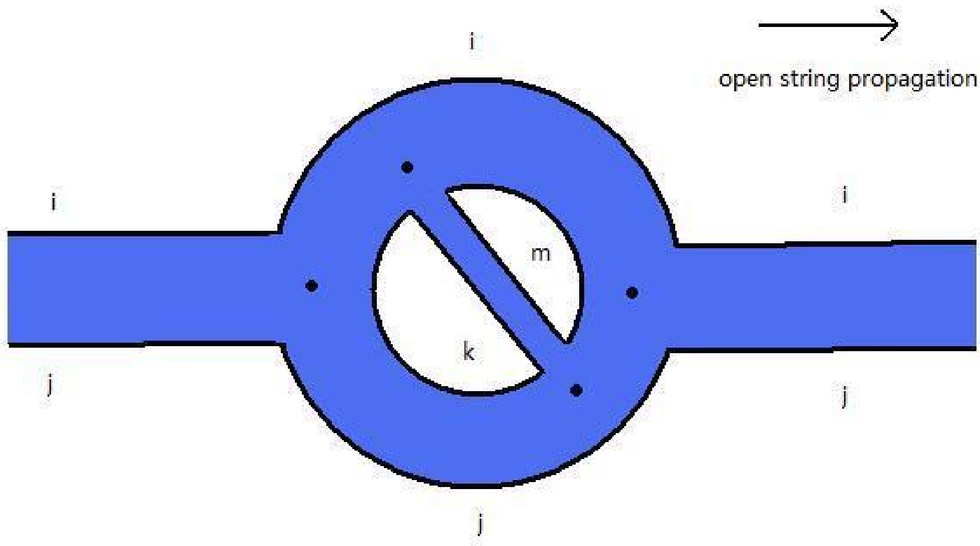

Figure 2.3: One intermediate open string splits during propagation

without any interactions.

The amplitude 2.2.10 is not complete. We've only considered planar diagrams. The simplest non-planar diagram is illustrated in fig 2.4: A new strip doesnt add a new 'loop' to the diagram. $[i, j]$ splits into $[i, k]$ and $[k, j]$. Then $[i, k]$ splits into $[i, k]$ and $[k, k]$, which means an open string that ends on the i-th and the k-th branes splits into another open string that ends on the i-th and the k-th branes and an open string whose both ends lie on the k-th brane. Likewise, the intermediate open string $[k, j]$ splits into $[k, j]$ and $[k, k]$. There is only one free parameter k which can be any of the N branes. Therefore, this diagram has a dependence of $g_{YM}^6 N = \frac{\lambda^3}{N^2}$. The rule for non-planar diagrams can be summarized as the following: each time, if one adds a non-planar strip to a diagram, a factor of $\frac{\lambda}{N^2}$ has to be multiplied to the amplitude. Therefore, everything considered, planar and non-planar, the complete amplitude can be written as follows

$$A_f = f_0(\lambda) + \frac{f_2(\lambda)}{N^2} + \frac{f_4(\lambda)}{N^4} + ... \tag{2.2.11}$$

where, for instance, the second term $\frac{f_2(\lambda)}{N^2}$ represents the contributions from all diagrams

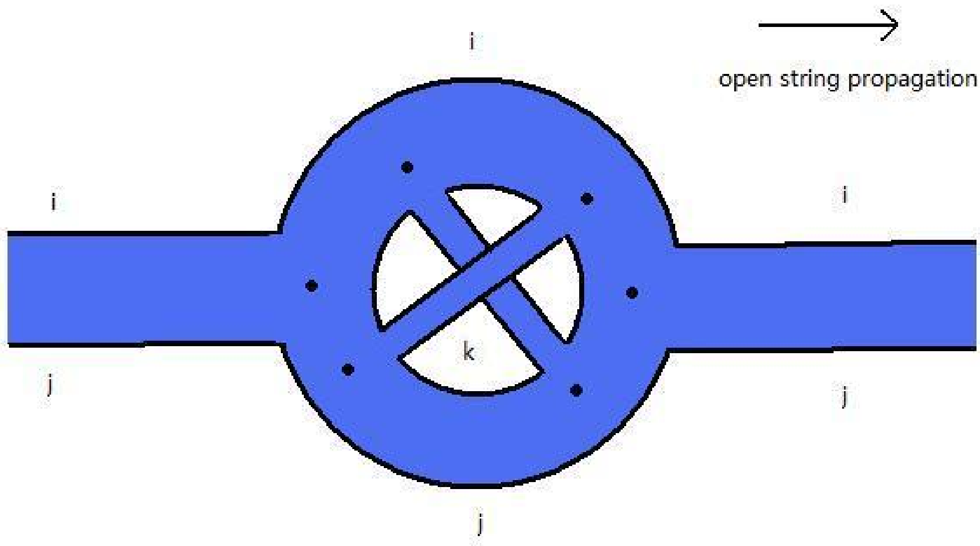

Figure 2.4: A diagram with one non-planar strip, which decreases the number of the 'loops' by one

with only one non-planar strip, and so on. Clearly, in the limit of $N->\infty$ and g_{YM}^2N being finite, only the first term $A = f_0(\lambda)$ plays a significant role and the physically relevant coupling turns out to be the 't Hooft coupling $\lambda \equiv g_{YM}^2N$ instead of g_{YM}. The above results based on the popagation of an open string in a system of N stacked D-branes in the low energy limit apply to the $SU(N)$ gauge theory in the spirit of the AdS/CFT correspondence.

2.2.5 The throat geometry

A real simple and intuitive way of describing the AdS/CFT correspondence is to look at the system of N stacked D3-branes in the low energy limit [59]. The low energy limit means considering energies much lower than the string energy scale, i.e.,

$$E \ll \frac{1}{\sqrt{\alpha'}} \tag{2.2.12}$$

In the low energy limit, one varies the closed string coupling g from $gN \ll 1$ to $gN \gg 1$ smoothly, with N fixed and large, and examines the resulting physical system. If $g = 0$, the system consists of free open strings on the D3-branes and free type IIB closed strings on the 10d flat spacetime because of the lack of interactions. One compares the $gN \ll 1$ regime and the $gN \gg 1$ regime. These two regimes are related by changing g smoothly under the low energy limit. Regardless of the value of g as long as $g \neq 0$, the relevant physical system remains as two decoupled systems: free closed type IIB strings on the 10d flat spacetime and supersymmetric $SU(N)$ Yang-Mills fields when $gN \ll 1$, or type IIB superstrings on the $AdS_5 \times S^5$ when $gN \gg 1$.

Since, the system always remains decoupled as two separate systems and one of the two decoupled systems is free closed strings throughout and the other one changes from the supersymmetric $SU(N)$ Yang-Mills gauge fields to type IIB superstrings on $AdS_5 \times S^5$ as g increases from $gN \ll 1$ to $gN \gg 1$ and vice versa, it would be reasonable to assume that the $SU(N)$ Yang-Mills gauge fields are actually equivalent to the type IIB superstrings on $AdS_5 \times S^5$. Specifically speaking, the same physical system can be described by two equivalent theories. For instance, when $gN \ll 1$, the two equivalent theories are a weakly coupled $SU(N)$ Yang-Mills gauge theory and a complicated intractable superstring theory. When $gN \gg 1$, the two theories are the near horizon type IIB superstring theory on $AdS_5 \times S^5$ and a complicated intractable strongly coupled gauge theory. One has the freedom to choose a theory to work with in order to make calculations manageable. This is exactly the idea of the AdS/CFT correspondence.

Let's examine how the $SU(N)$ Yang-Mills and the type IIB superstrings on $AdS_5 \times S^5$ emerge, respectively. When g is gradually increased from zero to $gN \ll 1$, the gravitational effects are still negligible and so are the interactions between the fields on the D3-branes and the spacetime fields. Thus, the relevant physics of the system becomes a decoupled system of free closed strings on the 10d flat spacetime and the supersymmetric $SU(N)$ Yang-Mills gauge theory on the D3-branes, which was described in the previous section. Since the closed string coupling $g \sim g_{YM}^2$, $gN \ll 1$ means

$$\lambda \equiv g_{YM}^2 N \ll 1 \tag{2.2.13}$$

which makes sure that the planar amplitude A in eq. 2.2.10 and the coefficients of $\frac{1}{N^{2k}}$, i.e., $f_{2k}(\lambda)$ in the full amplitude A_f in eq. 2.2.11 converge.

When g is further increased to $gN \gg 1$, the gravitational effects become strong. Under the low energy limit, however, the D3-branes and the spacetime excitations remain decoupled. For $gN \gg 1$, the geometry of the N stacked D3-branes and the spacetime can be understood with the help of the throat geometry illustrated in fig 2.5: a two-dimensional flat plane with an infinitely deep throat whose radius asymptotically approaches a limit R. On the two-dimensional plane, if one approaches the throat from far away, the circumference of a circle one draws around the throat will get smaller and smaller. Eventually the circumference approaches a limit: $2\pi R$, with R being the asymptotic radius of the throat, when one gets infinitely deep down the throat and the circle becomes the horizon for the throat.

Fig 2.5 certainly is 2-dimensional. But one can try to visualize it as a 6-dimensional image in which the N stacked D3-branes, appearing as a point, are located at the end of the throat, infinitely far away from the 6-dimensional flat surface. The 6 dimensions one can "see" are the dimensions transverse to the D3-branes: 5 dimensions in the 5-dimensional spheres S^5 which are represented by the circles approaching the end of the throat in fig 2.5, and a radial dimension represented by the radial lines extended deep into the throat. In the 6-dimensional world one is able to "see", the D3-branes seem like a point at the end of the throat. The radius of S^5 apporaches a limit R at the horizon. The near horizon spacetime, curved by the energetic D3-branes, is actually a 10-dimensional $AdS_5 \times S^5$ space, i.e., the direct product of a 5-dimensional anti-de Sitter space, AdS_5, and a 5-dimensional sphere space, S^5. The metric is as follows

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 - dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + R^2 d\Omega_5^2 \tag{2.2.14}$$

where $z = \frac{R^2}{r}$. Both the AdS_5 and the S^5 have the same radius of curvature R. The

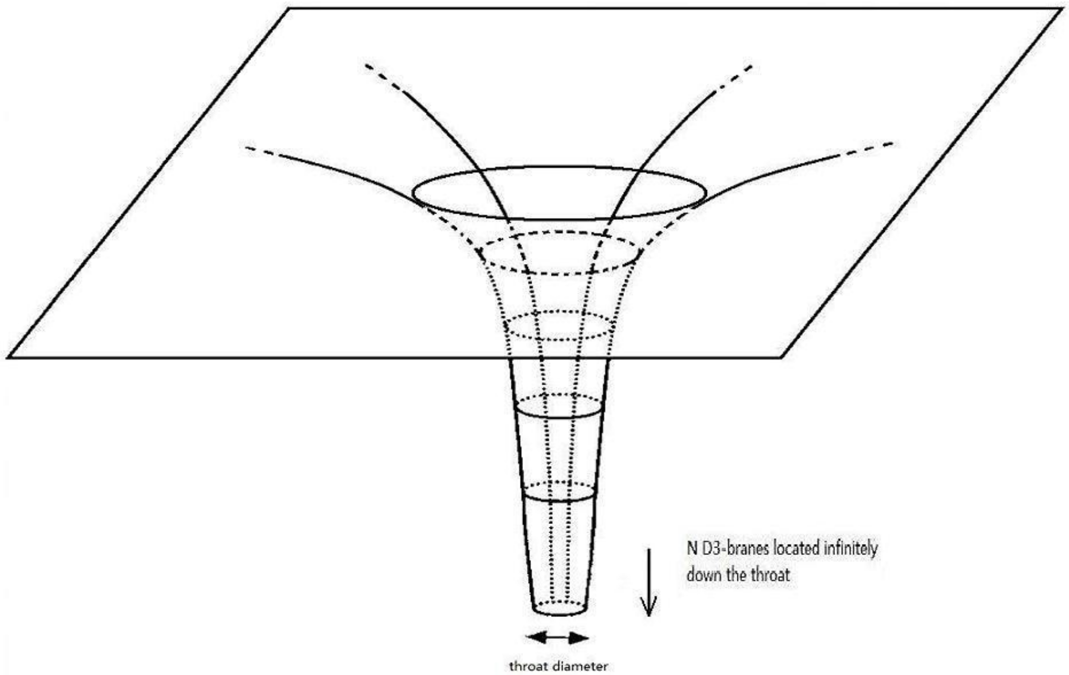

Figure 2.5: A throat geometry of the stacked D3-branes from the transverse dimensions' perspective: the circles can be thought of being 5-dimensional spheres

horizon is located at the limit $z- > \infty$, i.e., $r = 0$. The conformal boundary is 4-dimensional Minkowski space located at $z = 0$. In the low energy limit, the physical system, again, turn outs to be two decoupled systems: free closed strings on the 10d flat spactime far away from the D3-branes and type IIB superstrings on the $AdS_5 \times S^5$ manifold near the horizon.

Thus, we have seen that, for all values of gN, the low energy limit of the system of N stacked D3-branes yields two decoupled systems: free closed strings in 10d flat spacetime and a second system. This second system turns out to be $SU(N)$ super Yang-Mills conformal fields in 4d Minkowski space for $gN \ll 1$ and type IIB superstrings on $AdS_5 \times S^5$ for $gN \gg 1$. Two theories describe the same physical system at different regimes. Therefore, it makes sense to conjecture an equivalence between these two theories, i.e., the AdS/CFT correspondence.

To have a legitimate theory of dualities, the dimensionless parameters of the theories involved have to match up. The $SU(N)$ super Yang-Mills and the type IIB superstrings on $AdS_5 \times S^5$ in the AdS/CFT correspondence have four dimensionless parameters: the string coupling g, the Yang-Mills coupling g_{YM}, the number of D3-branes & the color index of the Yang-Mills theory N and the horizon radius R. It can be shown that the closed string coupling g is related to the Yang-Mills coupling g_{YM} and the 't Hooft coupling λ in the following way

$$g = \frac{g_{YM}^2}{4\pi} \equiv \frac{\lambda}{4\pi N} \tag{2.2.15}$$

The horizon size, i.e., the radius of the S^5 R is related to λ and the string length $\sqrt{\alpha'}$

$$R = \sqrt{\alpha'}\lambda^{\frac{1}{4}} \tag{2.2.16}$$

Eq. 2.2.16 indicates that large R corresponds to large λ and vice versa which brings a dilemma for testing the duality: to make calculations simpler, weak coupling, i.e., small 't Hooft coupling λ or small string coupling g is generally prefered. However, large radius R, which means small curvature, makes calculations easier on the string theory side by approximating the superstring theory as a supergravity theory on a near-flat space. Therefore, it's not realisitic, at least for now, to expect to test the AdS/CFT correspondence directly: calculating a Yang-Mills gauge theory and a type IIB superstring theory simultaneously and comparing the results. After all, large R and small λ cannot co-exist according to eq. 2.2.16. This shouldn't be surprising since, based on the low energy argument we made in the previous section, the $SU(N)$ super Yang-Mills and type IIB superstring theories are two sides of the same theory. They don't coexist, at least, not in the same space. The Yang-Mills fields live on the conformal boundary of the anti-de Sitter space. Nonetheless, there is at least one advantage out of this positively-correlated relationship between R and λ: hard calculations on one side of the AdS/CFT correspondence naturally tend to be easier on the other side based on eq. 2.2.16. In fact, that is one of the beauties of the AdS/CFT correspondence.

Other than the straightforward low energy limit argument of the N stacked D3-branes system, the conjecture of the AdS/CFT correspondence can as well be reinforced by the amazing resemblance between the symmetries that the two theories under consideration possess. The algebra that generates the conformal field theory in 4-dimensional Minkowski space, refered to as the conformal Lie algebra, turns out to be the same algebra that generates the isometries of the AdS_5: the $SO(2,4)$ symmetry. Also, the isometries of S^5 and the R-symmetry, i.e., the rotation of the scalars of the super Yang-Mills turn out to be the same symmetries: the $SU(4) \sim SO(6)$ symmetry.

2.2.6 The anti-de Sitter space

The anti-de Sitter spaces are the most symmetric spacetime with negative curvature and they have one time dimension. There are many different ways of writing the metric of the anti-de Sitter spaces, e.g., the AdS_5 part of eq. 2.2.14. We roughly introduce the strategy of [59] and use the conformal properties of the hyperbolic space H_4 which is a subspace of the AdS_5 to show that the conformal boundary of the AdS_5 leads to the 4-dimensional Minkowski space. We focus on the AdS_5 while the derivation here also applies to anti-de Sitter spaces of other dimensions. The AdS_5 can be defined as a surface embedded in a flat $R^{(2,4)}$ space, which has two time dimensions and four space dimensions, with the metric

$$ds^2 = -du^2 - dv^2 + \sum_{i=1}^{4} \left(dx^i\right)^2 \tag{2.2.17}$$

and the constraint is

$$-u^2 - v^2 + \sum_{i=1}^{4} \left(x^i\right)^2 = -R^2 \tag{2.2.18}$$

Replacing the two time coordinates u, v with new coordinates y, t

$$\begin{aligned} u &= y \cos t \\ v &= y \sin t \end{aligned} \tag{2.2.19}$$

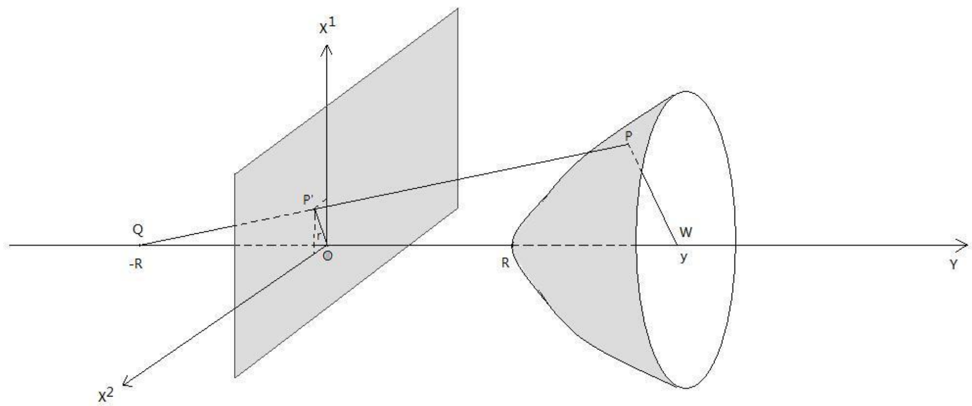

Figure 2.6: In R^3, the projection of a hyperboloid into the $y = 0$ plane by connecting each point on the hyperboloid with $(-R, \vec{0})$. The hyperboloid can be visualized as a hyperbolic space H_4

the ambient metric 2.2.17 and the constraint take the following form

$$ds^2 = -y^2 dt^2 - dy^2 + \sum_{i=1}^{4} (dx^i)^2 \tag{2.2.20}$$

and

$$-y^2 + \sum_{i=1}^{4} (x^i)^2 = -R^2 \tag{2.2.21}$$

Similarly, the 4-dimensional hyperbolic space H_4 is defined to be a hypersurface embedded in the 5-dimensional Minkowski space M_5

$$ds^2 = -dy^2 + \sum_{i=1}^{4} (dx^i)^2 \tag{2.2.22}$$

with a constraint

$$-y^2 + \sum_{i=1}^{4} (x^i)^2 = -R^2 \tag{2.2.23}$$

or

$$y^2 = R^2 + \sum_{i=1}^{4} (x^i)^2 \tag{2.2.24}$$

The constraint has two solutions $y \leq -R$ and $y \geq R$ which correspond to two disconnected leafs and we choose $y \geq R$. As a hypersurface embedded in Minkowski space, the hyperbolic spaces turn out to have no timelike dimension and the full hyperbolic spaces can't be isometrically embedded into the Euclidian spaces. Therefore, to visualize the hyperbolic spaces, we borrow the image of a 2-dimensional hyperboloid embedded in R^3, illustrated in fig 2.6. In order to find the metric of H_4, technically, the ambient metric 2.2.22 and the constraint equation 2.2.23 need to be combined to solve for a metric in 4 dimensions. For the dual purpose of displaying the conformal boundary explicitly and finding the metric of H_4, we follow a trick of projecting the hyperbolic space H_4 into the $y = 0$ plane. In fig 2.6, each point $P(y, \vec{x})$ on H_4 is projected to the $y = 0$ plane by connecting P with the point $Q = (-R, \vec{0})$. The line PQ intersects the $y = 0$ plane at a point P'. Assign to each point P a set of coordinates ζ^i as follows

$$\zeta^i(P) = x^i(P') \tag{2.2.25}$$

This way, the whole hyperbolic space, despite being infinite and boundaryless, can be projected to the interior of the ball B^4 of radius R on the plane of $y = 0$. See fig 2.6 for the details of the projection and the 2-dimensional $y = 0$ plane needs to be visualized as 4-dimensional. One can show that the x^i coordinates and the ζ^i coordinates for any arbitrary point P on H_4 are related as follows

$$x^i = \frac{2R^2}{R^2 - r^2}\zeta^i \tag{2.2.26}$$

and

$$y = \frac{R(R^2 + r^2)}{R^2 - r^2} \tag{2.2.27}$$

where

$$r^2 = \sum_{i=1}^{4} (\zeta^i)^2 \tag{2.2.28}$$

In terms of ζ^i, the metric of H_4 is shown to be

$$ds^2 = \frac{4R^4 \sum_{i=1}^4 (d\zeta^i)^2}{(R^2 - r^2)^2} \tag{2.2.29}$$

Notice the ambient metric 2.2.20 & the constraint 2.2.21 for AdS_5 and the ambient metric 2.2.22 & the constraint 2.2.23 for H_4 only differ by a term $-y^2dt^2$. Thus the projection trick and the metric of the H_4 can be used directly to write the metric of AdS_5

$$ds^2 = -\frac{(R^2 + r^2)^2}{(R^2 - r^2)^2} R^2 dt^2 + \frac{4R^4 \sum_{i=1}^4 (d\zeta^i)^2}{(R^2 - r^2)^2} \tag{2.2.30}$$

which explicitly displays t as the only time dimension.

In the AdS_5 metric 2.2.30, let ζ^i be $R\zeta^i$ and multiply the metric by a conformal factor,

$$ds_c^2 = \frac{(1-r^2)^2}{4R^2}ds^2 = -\frac{(1+r^2)^2}{4}dt^2 + \sum_{i=1}^{4}(d\zeta^i)^2 \tag{2.2.31}$$

which turns into

$$ds_c^2 = -dt^2 + \sum_{i=1}^{4}(d\zeta^i)^2 \tag{2.2.32}$$

at the limit of $r->1$. The metric 2.2.32 has a boundary $R \times S^3$, which, by definition, is the conformal boundary of AdS_5. If the radius of the boundary is taken to ∞, the conformal boundary becomes a 4d Minkowski space M_4

The 4-dimensional hyperbolic space H_4, i.e., the spatial subspace of AdS_5, has infinite volume and has no boundary. However, considering spherical subsets of H_4, one can be show that the ratio between the surface area A_3 of the 3-dmensional boundary surface and the volume V_4 of the 4-dimensional sphere approaches a finite limit when the boundary approaches infinity to cover the full hyperbolic space [59]. Notice that both A_3 and V_4 diverge at this limit.

$$\lim_{Boundary->\infty}\frac{V_4}{A_3} = \frac{R}{3} \tag{2.2.33}$$

which certainly has the flavor of holography since the volume and the surface area turn out to be proportional to each other. The same volume to surface area ratio would diverge for flat spaces as the characteristic size L takes the limit $L->\infty$

$$\lim_{L->\infty}\frac{V_4}{A_3} \sim \lim_{L->\infty}\frac{L^4}{L^3} = L \sim \infty \tag{2.2.34}$$

2.2.7 The generalizations of AdS/CFT correspondence and open questions

In the previous sections, we briefly reviewed some pioneering works that can be considered as the foundation stones of the AdS/CFT correspondence such as the original Kaluza-Klein theory, the holographic principle based on the research on the black hole entropy, and 't Hooft's large N gauge theory. Also, we investigated the geometry of the AdS_5 space and the 4-dimensional hyperbolic space H_4 and explained how the 4-dimensional Minkowski space can emerge as the conformal boundary of the AdS_5. We discussed the low energy limit of the system of N stacked D3-branes and showed how the system becomes two decoupled systems in both $gN \ll 1$ and $gN \gg 1$ regimes. We argued why it makes sense to conjecture the equivalence between an $SU(N)$ super Yang-Mills conformal gauge theory in 4d Minkowski space and a type IIB superstring theory in $AdS_5 \times S^5$, based on the low energy argument of N stacked D3-branes system and symmetry comparisons of the two theories involved in the duality. As one can see, the AdS/CFT correspondence is certainly very exciting as a theory that realizes the holopgraphic principle. Also, since string theory is a quantum theory of gravity in the first place, the AdS/CFT correspondence enables string theory to be a theory capable of describing both gauge theory and gravity as a quantum theory, which makes string theory much more hopeful as a candidate of the ultimate unification of all four fundamental interactions of nature.

The natural question arises: is the AdS/CFT correspondence proven and what are the open questions left? The short answer is: string theory itself, despite being a great and promising candidate for unification, is far from being a complete project. Of course, the AdS/CFT correspondence is not mathematically proven yet in spite of tons of amazing evidence supporting it. To some extent, performing research in string theory is like figuring out a huge puzzle little by little. All one knows is a great amount of independent information and all one can do is to try to put together the known information to hope to figure out the big picture eventually. In the case of the AdS/CFT correspondence, lacking a proof seems certain to lead to new hidden ideas to be discovered [60].

The original Maldacena conjecture proposes the equivalence between $\mathcal{N} = 4$ conformal super Yang-Mills in 4d Minkowski space and type IIB superstrings in 10d $AdS_5 \times S^5$

space. Ever since the publication of the Maldacena's conjecture, the string theory community has been working intensely to generalize the AdS/CFT correspondence. Research in studying gauge theories dual to string theories in a spacetime with deformed $AdS_5 \times S^5$ boundary conditions has been carried out (see [32], for instance). The 5-dimensional sphere S^5 has been replaced by general compact Einstein manifolds X^5 since AdS_5 is recognized as the universal factor, in the string dual description, realizing the $SO(2,4)$ conformal symmetry of large N conformal field theory. Research along the line of replacing AdS_5 with a de Sitter space has been attempted as well [61, 62]. Unfortunately, the de Sitter string vacua seem to produce metastable states only [63, 64, 65, 66]. Also, there have been researches on stacked Dp-branes system with general $p \neq 3$.

Undoubtedly, the AdS/CFT correspondence is a huge step forward in the pursuit of unification. It relates $SU(N)$ super Yang-Mills and type IIB superstrings on $AdS_5 \times S^5$ in an amazing way, as a realization of the holographic principle. However, it would be nicer to find string theories on general supergravity backgrounds other than $AdS_5 \times S^5$ that are dual to gauge theories which are more relevant to nature such as gauge theories with less supersymmetry. Calculating k-string energy [24, 25, 26, 27] from different supergravity backgrounds [15, 16, 32, 33] is one of many attempts along this line.

CHAPTER 3

INTRODUCTION TO K-STRING PHYSICS

Even though the standard model is considered a successful theory of strong interactions, weak interactions and electromagnetism, on the road to unification, it has many obstacles to overcome, one of which is quark confinement which states that quarks cannot be isolated singularly. To speak plainly, let's borrow the image and language of separating two electrically-charged particles to describe the process of separating two quarks. As we all know, when two electrically-charged particles are separated (a positive one and a negative one of course), the electric fields between them diminish quickly in the order of $\frac{1}{r^2}$ so that the two particles become unbound or decoupled. However, as two quarks separate, the gluon fields form narrow tubes of color charge, which produce an attractive force that tends to hold the quarks together. Instead of diminishing quickly like electric fields, the color force between the two quarks remains a constant, regardless of the distance in between, at around ten thousand newtons [67, 68]. At some point during the process of separating two quarks, it is more energetically favorable to spontaneously form a new quark-antiquark pair than to further stretch the tube. As a result, we see new color-neutral mesons or bayrons instead of individual quarks.

3.1 Review of k-strings

Naturally, calculating the tension and energy between the quark pair is very important in quark confinement. k-string physics is a crucial topic in this direction. In $SU(N)$ gauge theories, if a source has k fundamental color indices, a k-string is defined to be the flux tube it generates as illustrated in figure 3.1. The investigation of k and N dependence of the k-string tension is one of the central topics of quark confinement. The simplest case is a fundamental string, which is merely interpreted as a quark antiquark pair at a relatively large distance apart as in figure 3.2. In other words, a fundamental string is the QCD string that connects a quark and an anti-quark. If k fundamental

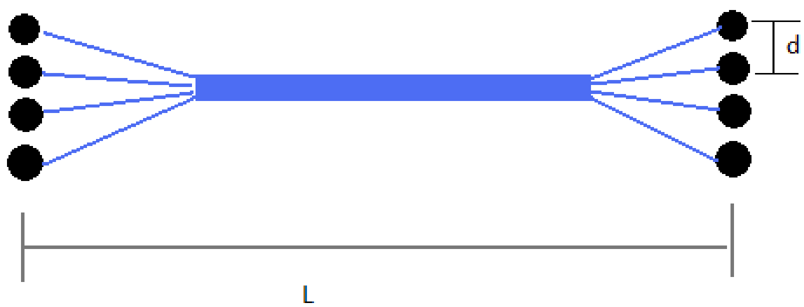

Figure 3.1: A k-string as a bound state of k fundamental strings

strings are brought together as shown in figure 3.3, they will attract each other and form a bound state: a k-string. Therefore, it is natural to assume that the energy of a k-string is less than the sum of the individual fundamental strings, i.e.,

$$E < kE_f \tag{3.1.1}$$

where E_f is the energy of a fundamental string. The difference is defined to be the binding energy.

The k-string tension only depends on its N-ality instead of particular representation of the probe color source due to the effect of gluon cloud screenings [29]. In the large N limit, any quasi-stable strings created will relax to bona fide k-strings (k indices in fully antisymmetric representation) and the relaxtation time is thought to be exponentially large. We investigate only the bona fide k-strings in the context of AdS/CFT correspondence. The previous definiton of a k-string as the bound state of k fundamental strings applies to the $k < N$ case only. For an arbitrary non-negative

Figure 3.2: A fundamental string

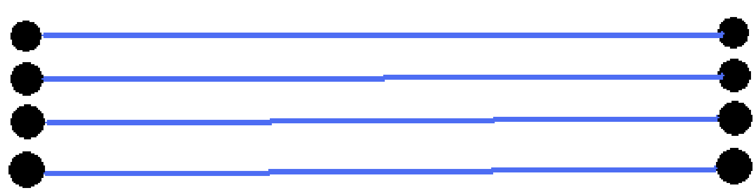

Figure 3.3: Fundamental strings clumped together

integer k, redefine

$$k \equiv k \ mod \ N \tag{3.1.2}$$

so that $0 \leq k < N$. The reason of doing this is, technically, one only needs to consider the $k < N$ situation, since the $k \geq N$ situation is trivial. Consider, for instance, the $k = 5$ and $N = 3$ case. As five fundamental strings are brought together, three quarks and three antiquarks will form two baryons(color-singlets), changing the $k = 5$ string to a $k = 2$ string automatically. Therefore, the only non-trivial cases are $k < N$.

There are a few restrictions due to physical requirements that the formulation of k-string tension has to satisfy. Due to charge conjugation symmetry, the k-string tension needs to be invariant under $k- > N - k$, i.e.,

$$T_k = T_{N-k} \tag{3.1.3}$$

Another restriction is that as N approaches ∞, the interaction between the fundamental strings vanishes so that bringing k fundamental strings together will result in a system of non-interacting fundamental strings instead of a bound state. In other words,

$$T_k = kT_f \tag{3.1.4}$$

for large N.

There is one more important restriction: a k-string has to be the ground state of all possbible states and the k-string energy is the lowest so that a k-string will not decay into separate strings. In other words,

$$T_{k_1+k_2} < T_{k_1} + T_{k_2} \tag{3.1.5}$$

since the k-string energy E equals the tension times string length, i.e., $E = T_k L$.

There have been two major formulations proposed to describe the k-string tension: the Casimir law [29],

$$T_k = k(1 - \frac{k-1}{N-1})T_f \tag{3.1.6}$$

and the Sine law [30]

$$T_k = (\sin \frac{\pi}{N})^{-1} \sin(\frac{\pi k}{N})T_f \tag{3.1.7}$$

Clearly, the tension vanishes when $k = N$ for both formulations and both agree that the

k-string energy approaches infinity if the distance between the quark antiquark pairs gets large. Both formulations also satisfy the restrictions eqs. 3.1.3 and 3.1.4. For instance, as $N->\infty$, the sine law tension turns into

$$T_k->\frac{N}{\pi}\frac{\pi k}{N}T_f=kT_f \tag{3.1.8}$$

for fixed k.

It's very simple to show that

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta<\sin\alpha+\sin\beta \tag{3.1.9}$$

for $0<\alpha,\ \beta<\pi$. Thus, the sine law satisfies $T_{k_1+k_2}<T_{k_1}+T_{k_2}$ obviously. One can show that the Casimir formula satisfies this as well.

For both formulations, $T_k=T_f$ if $k=1$, which is obvious. Therefore, we have two formulations that satisfy all basic requirements. Surprisingly, however, only recently was it recognized that the Casimir formulation is in direct conflict with the 1/N expansion of the Yang-Mills [29]. As shown in eqs. 1.0.8 and 1.0.10, the large-N expansion for Casimir law has both even and odd powers of 1/N while the Sine law large-N expansion runs in even powers of 1/N only, which agrees with the pure Yang-Mills expansion.

There is an argument that the Casimir law should be excluded based on the large N expansion contradiction with the pure Yang-Mills expansion [29]. In the large N limit, consider two fundamental strings being brought together. Near the center of the strings, the interaction should not have any knowledge of the quarks, which are located far away. Hence, the pure Yang-Mills applies there. The expansion of the amplitude of large N pure Yang-Mills has only even powers of $1/N$, as shown in eq. 2.2.11 in the previous chapter. This is certainly not something everyone agrees on and the formulation of the k-string tension remains an open question. The good news for us is that the result we acquire, investigating supergravity backgrounds in the context of gauge/gravity duality, is an exact sine law.

It can also be argued from the binding energy perspective that the Casimir law causes an inconsistency. When fundamental strings are brought together to form a k-string. The fundamental strings interact each other via the exchange of glueballs, which contributes a factor of g^2 where g corresponds to the closed string coupling. In the large N limit, gN is kept fixed, meaning that $g \sim 1/N$ and $g^2 \sim 1/N^2$. Consider the binding energy $kE_f - E$. Again, the Casimir formulation has an expansion that has both even and odd powers of $1/N$ while the sine law contains only even powers with the leading term being

$$kE_f - E = \frac{\pi^2}{6N^2}(k^2 - 1)kT_f L + \mathcal{O}(N^{-4}) \tag{3.1.10}$$

which uses the fact that k-string energy equals the tension multiplying the string length L, i.e.,

$$E = T_k L, \qquad \mathit{and} \quad E_f = T_f L \tag{3.1.11}$$

$E = TL$ only describes the zeroth order of the k-string energy. The first order correction from quantum fluctuations is refered to as the Lüscher term [31]. For a fundamental string, the string energy that includes the Lüscher term, i.e., takes the following form

$$E_f = T_f L + \beta + \frac{\gamma}{L} + \mathcal{O}(L^{-2}) \tag{3.1.12}$$

where the Lüscher term $\frac{\gamma}{L}$ comes from the fluctuations of the transverse dimensions of the string. Similarly, for a k-string, the quantum fluctuations result in a Lüscher term, $\frac{\gamma'}{L}$ as well. The k-string energy including the Lüscher term goes as follows

$$E = T_k L + \frac{\gamma'}{L} + V_c + \mathcal{O}(L^{-2}) \tag{3.1.13}$$

where V_c is a small constant term independent of L.

There is an argument that γ' should be equal to γ based on a similar argument comparing k-string width with fundamental string width [69]: far away from the string, a k-string looks no difference from a fundamental string, which can be described by a Nambu-Goto action. However, the Lüscher term resulting from our calculations is different from the fundamental string Lüscher term.

It should also be noticed that, the large N limit is particularly important for k-string physics: for $N = 3$, the only nontrivial case is $k = 1$($k = 2$ is equivalent to $k = 1$). Of course, one needs to investigate the $N > 3$ situations. The large N field theories have more degrees of freedom and will provide insight into fundamental features of QCD.

3.2 k-string Tension and the AdS/CFT Correspondence

As an important topic in color confinement, the k-string tension calculation has been done by simulations in lattice QCD [17, 18, 19, 20, 21, 22, 23] as well as in string theory [24, 25, 26, 27, 28]. We investigate the Maldacena-Nunez [15] and the Maldacena Nastase [16] backgrounds and are able to reproduce the Sine law in both cases. In the context of AdS/CFT correspondence, a probe D3-brane embedded in the above mentioned supergravity backgrounds is interpreted as a k-string. Investigation of the Dirac-Born-Infeld action of the probe D3-brane leads to the k-string tension.

As mentioned in the previous chapters, since the publication of Maldacena's original AdS/CFT conjecture, which relates $\mathcal{N} = 4$ super Yang-Mills conformal gauge theories in 4d Minkowski space with type IIB superstring theories in $AdS_5 \times S^5$, the physics community has been working on trying to generalize it. It would be nice to find supergravity theories that are dual to Yang-Mills gauge theories with less supersymmetry. Also, it would be nice to be able to use the gauge/gravity duality to carry out some calculations on one side that are otherwise too difficult to perform on the other side. There are supergravity theories, such as the Klebabanov-Strassler background [32], the Maldacena-Nunez background [15] and the Maldacena-Nastase [16] background, which have been proposed to be dual to $\mathcal{N} = 1$ super Yang-Mills in 4d Minkowski space.

The KS background, created by stacking N D3-branes and M wrapped D5-branes on the conifold, is a supergravity solution that is dual to $\mathcal{N} = 1$ $SU(N + M) \times SU(M)$ gauge theory. When N is a multiple of M, the $\mathcal{N} = 1$ $SU(N+M)\times SU(M)$ gauge theory becomes $SU(M)$ gauge theory in the IR. The M D5-branes correspond to M units of R-R 3-form flux.

In [25, 27], the k string tension was calculated in the context of the gauge/gravity duality: a D3-brane is interpreted as a k-string in 4d flat spacetime. A probe D3-brane is embedded in the KS background. One of the 10 dimensions of the background is τ, which controls the energy scale. In the far infrared, i.e., small τ, the background is a warped product of $R^{3,1}$ with $R^3 \times S^3$. The S^3 is threaded by the R-R flux. S-dualize the metric and the M units of R-R flux become M units of NS-NS flux. This S-dual metric, shrinks to $R^{3,1} \times S^3$ at $\tau = 0$. The probe D3-brane wraps over an $S^2 \subset S^3$. The other two dimensions of the probe D3-brane are coincident with a spatial dimension and the time dimension of $R^{3,1}$ so that the D3-brane looks like a one-dimensional string, which is interpreted as a k-string, to an observer in $R^{3,1}$. Calculating the Dirac-Born-Infeld action of the probe D3-brane results in an approximate sine law for the k-string tension.

In this thesis, calculating the k-string tension in the MN background is partially motivated by the similarity beween the geometries of MN and KS backgrounds. In the IR limit, both KS and MN solutions are supergravity duals of $\mathcal{N} = 1$ super Yang-Mills gauge theories. Both solutions have similar geometry in the IR. The geometry of the MN solution is a product of $R^{3,1}$ with $R^3 \times S^3$ with N units of R-R flux, in the IR limit. Just like in [25], a probe D3-brane embedded in the MN background wraps over an $S^2 \subset S^3$ too and the disturbance of the probe D3-brane on the supergravity background is negligible since we are dealing with large N. Again, for an $R^{3,1}$ observer, the D3-brane looks a k-string. It should be noticed that [25] also mentioned that using the very same steps as calculating the action of a probe D3-brane embedded in the KS background to calculate the action of a D3-brane embedded in the MN background will result in an exact sine law as the formula for k-string tension. However, to claculate the k-string tension, we work in the metric with N units of R-R flux, while [25] uses an S-dual metric with NS-NS flux. The string charge k in our model turns out to be a dynamical variable while in [25], the string charge is related to the strength of a magnetic field on the D3-brane.

We proceed with the calculations of the k-string tension simultaneously for both the MNa [16] and the MN background due to the striking resemblance between these two backgrounds. In the IR limit, a probe D3-brane embedded in the MNa background is interpreted as a k-string in 2+1 dimensions instead of 3+1 dimensions for the MN background.

The major part of this thesis is about calculating the one loop correction to the k-string energy, i.e., the Lüscher term in the D3-brane/k-string model investigating the MN and the MNa backgrounds. We fluctuate the Bosonic fileds and Fermionic fields living on the D3-brane and are able to reproduce the Lüscher term from solving the equaions of motion of the fluctuating fields.

CHAPTER 4

THE MALDACENA-NUNEZ BACKGROUND AND THE MALDACENA-NASTASE BACKGROUND

In a recent series of papers [70, 26, 27], k-strings in both $d = 3$ and $d = 4$ spacetime dimensions have been investigated using holographic supergravity dual theories. Investigating this further, we look at the supergravity solutions of Maldacena and Nunez(MN) [15] and Maldacena and Nastase(MNa) [16]. We will show that a probe D3-brane embedded in the Maldacena-Nunez or Maldacena-Nastase background is dual to a confining $SU(N)$ k-string in $d = 4$ or $d = 3$ Minkowski spacetime, respectively. Both Maldacena-Nunez and Maldacena-Nastase backgrounds are known to be dual, in the IR regime, to pure $\mathcal{N} = 1$ super Yang-Mills in 4d and 3d Minkowski space, respectively.

4.1 Introduction to Maldacena-Nunez Background

The AdS/CFT correspondence gives the large N supergravity dual description of $\mathcal{N} = 4$ super Yang-Mills in 4d Minkowski space. Finding similar correspondence for pure Yang-Mills with less supersymmetry would be nice. In [15], Maldacena and Nunez found a supergravity solution created by stacking a large number of NS-5 branes wrapped on a two sphere(after uplifting to 10 dimensions) that was proposed to be dual to pure $\mathcal{N} = 1$ super Yang-Mills in the IR regime. In order to decouple the 4-dimensional Yang-Mills theory, one needs to S-dualize the gravity solution to the low 't Hooft coupling limit and the NS-5 brane description switches to a D5-brane description. The S-dualized metric can be written as

$$ds^2 = e^{\Phi}\Bigg[-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + N\alpha'\Bigg[d\rho^2 + e^{2g(\rho)}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + $$
$$+ \frac{1}{4}\sum_a (w^a - A^a)^2\Bigg]\Bigg] \tag{4.1.1}$$
$$e^{2\Phi} = e^{2\Phi_0}\frac{\sinh 2\rho}{2e^{g(\rho)}} \tag{4.1.2}$$
$$e^{2g} = \rho\coth 2\rho - \frac{\rho^2}{\sinh^2 2\rho} - \frac{1}{4} \tag{4.1.3}$$

where Φ is the dilaton field and the gauge fields A^a can be written explicitly as

$$A^1 = a(\rho)d\theta_1$$
$$A^2 = a(\rho)\sin\theta_1 d\phi_1$$
$$A^3 = \cos\theta_1 d\phi_1 \tag{4.1.4}$$
$$a(\rho) = \frac{2\rho}{\sinh 2\rho} \tag{4.1.5}$$

and w^a, defined as the left invariant one forms of S^3, viewing the sphere as the $SU(2)$ group, can be written as follows:

$$g = e^{\frac{i\psi\sigma^3}{2}}e^{\frac{i\theta_2\sigma^1}{2}}e^{\frac{i\phi_2\sigma^3}{2}}$$
$$\frac{i}{2}w^a\sigma^a = dgg^{-1}$$
$$w^1 + iw^2 = e^{-i\psi}(d\theta_2 + i\sin\theta_2 d\phi_2)$$
$$w^3 = d\psi + \cos\theta_2 d\phi_2 \tag{4.1.6}$$

where σ^a are the pauli matrices and ψ, θ_2, ϕ_2 are the Euler angles of the sphere S^3. w^a read as follows:

$$w^1 = \cos\psi d\theta_2 + \sin\psi\sin\theta_2 d\phi_2$$
$$w^2 = -\sin\psi d\theta_2 + \cos\psi\sin\theta_2 d\phi_2$$
$$w^3 = d\psi + \cos\theta_2 d\phi_2 \tag{4.1.7}$$

The NS-NS 3-form flux H_3 become the R-R 3-form flux F_3:

$$F_3 = N\alpha'\left[-\frac{1}{4}(w^1 - A^1)\wedge(w^2 - A^2)\wedge(w^3 - A^3) + \frac{1}{4}\sum_a F^a\wedge(w^a - A^a)\right] \tag{4.1.8}$$
$$F^a = dA^a + \frac{1}{2}\epsilon^a{}_{bc}A^b\wedge A^c \tag{4.1.9}$$

where the Ramond-Ramond 3-form flux F_3 support the 3-sphere S^3. In component form, F^a can be written as

$$F^a{}_{\mu\nu} = A^a{}_{\nu,\mu} - A^a{}_{\mu,\nu} + \frac{1}{2}\epsilon^a{}_{bc}(A^b{}_\mu A^c{}_\nu - A^b{}_\nu A^c{}_\mu) \tag{4.1.10}$$

The MN solution, in the IR regime(small ρ), has a geometrical structure $R^{(3,1)} \times R^3 \times S^3$, which is similar to the IR limit of the KS gravity solution.

4.2 Maldacena-Nastase Supergravity Background

The Maldacena-Nastase background, corresponding to NS-5 branes wrapped on S^3, is very similar to the Maldacena-Nunez background. In the IR regime, it's proposed to be dual to $\mathcal{N} = 1$ super Yang-Mills in 3d Minkowski space. Again, in order to decouple the 3-dimensional gauge theory, the gravity solution needs to be S-dualized. Thus, the metric, in a D5-brane description, can be written as:

$$ds^2 = e^\Phi \Bigg[- dx_0^2 + dx_1^2 + dx_2^2 + N\alpha' \Bigg[d\rho^2 + R^2(\rho)(d\psi_1^2 + \sin^2\psi_1(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2)) + \frac{1}{4}\sum_a (w^a - A^a)^2 \Bigg]\Bigg] \tag{4.2.1}$$

$$e^\Phi = e^{\Phi_0}(1 + \frac{7}{24}\rho^2) + \mathcal{O}(\rho^3) \tag{4.2.2}$$

$$R^2 = \rho^2 + \mathcal{O}(\rho^3) \tag{4.2.3}$$

where Φ is the dilaton field and w^a and A^a can be written explicitly as

$$A^1 = a(\rho) d\theta_1$$

$$A^2 = a(\rho)\sin\theta_1 d\phi_1$$

$$A^3 = b(\rho)\cos\theta_1 d\phi_1 \tag{4.2.4}$$

$$a(\rho) = 1 - \frac{1}{6}\rho^2 + \mathcal{O}(\rho^4) \tag{4.2.5}$$

$$b(\rho) = 1 - \frac{1}{6}\rho^2 + \mathcal{O}(\rho^4) \tag{4.2.6}$$

and

$$w^1 = \cos\psi d\theta_2 + \sin\psi \sin\theta_2 d\phi_2$$
$$w^2 = -\sin\psi d\theta_2 + \cos\psi \sin\theta_2 d\phi_2$$
$$w^3 = d\psi + \cos\theta_2 d\phi_2 \tag{4.2.7}$$

The R-R flux F_3 are

$$F_3 = N\alpha' \left[-\frac{1}{4}(w^1 - A^1) \wedge (w^2 - A^2) \wedge (w^3 - A^3) + \frac{1}{4}\sum_a F^a \wedge (w^a - A^a) \right] + \\ + \frac{N\alpha'}{16}(\mathcal{O}(\rho^4)) w^1 \wedge w^2 \wedge w^3 \tag{4.2.8}$$

$$F^a = dA^a + \frac{1}{2}\epsilon^a{}_{bc} A^b \wedge A^c \tag{4.2.9}$$

$$F^a{}_{\mu\nu} = A^a{}_{\nu,\mu} - A^a{}_{\mu,\nu} + \frac{1}{2}\epsilon^a{}_{bc}(A^b{}_\mu A^c{}_\nu - A^b{}_\nu A^c{}_\mu) \tag{4.2.10}$$

In the IR regime(small ρ), the geometry of the MNa solution is $R^{(2,1)} \times R^4 \times S^3$ where the S^3 is supported by N units of R-R flux F_3.

4.3 A Succinct Notation for the MN and MNa Background

Due to the similarities between the Maldacena-Nunez and Maldacena-Nastase backgrounds, they can be succinctly written, in the D5-brane description, as

$$ds^2 = e^\Phi \left[dx_d^2 + N\alpha' \left[d\rho^2 + R^2 d\Omega_{6-d}^2 + \frac{1}{4}\sum_a (w^a - A^a)^2 \right] \right] \tag{4.3.1}$$

$$e^\Phi = e^{\Phi_0}(1 + c_1\rho^2) + \mathcal{O}(\rho^3), \quad R^2 = \rho^2 + \mathcal{O}(\rho^3) \tag{4.3.2}$$

$$F_3 = N\alpha' \left[-\frac{1}{4}(w^1 - A^1) \wedge (w^2 - A^2) \wedge (w^3 - A^3) + \frac{1}{4}\sum_a F^a \wedge (w^a - A^a) \right] + \\ + c_2\frac{N\alpha'}{16}(\mathcal{O}(\rho^4)) w^1 \wedge w^2 \wedge w^3 \tag{4.3.3}$$

$$F^a = dA^a + \frac{1}{2}\epsilon^a{}_{bc} A^b \wedge A^c \tag{4.3.4}$$

where

$$\sigma^a A^a = \sigma^1 a(\rho) d\theta + \sigma^2 a(\rho) \sin\theta d\phi + \sigma^3 b(\rho) \cos\theta d\phi,$$

$$w^1 + i w^2 = e^{-i\tilde{\psi}}(d\tilde{\theta} + i \sin\tilde{\theta} d\tilde{\phi}), \qquad w^3 = d\tilde{\psi} + \cos\tilde{\theta} d\tilde{\phi}, \tag{4.3.5}$$

where σ are the Pauli matrices and $d\Omega^2_{6-d}$ is the metric of an S^{6-d} with the untilded coordinates θ, ϕ, etc. What corresponds to the dimension of the k-strings is merely the parameter d which signifies the spacetime dimension of the Minkowski portion of the metric. The probe D3-brane which is the object dual to the k-strings, will be tangent to one space and one time dimension of this Minkowski portion. For MN, $d = 4$ and for MNa, $d = 3$.

The various other functions and parameters which specify each solution, either MN or MNa, up to order ρ^2, are given in Table 4.1.

Supergravity solutions	Maldacena-Nastase solution	Maldacena-Nunez solution
d	3	4
c_1	$\frac{7}{24}$	$\frac{4}{9}$
c_2	1	0
$a(\rho)$	$1 - \frac{1}{6}\rho^2 + \mathcal{O}(\rho^4)$	$\frac{2\rho}{\sinh 2\rho}$
$b(\rho)$	$1 - \frac{1}{6}\rho^2 + \mathcal{O}(\rho^4)$	1

Table 4.1: Different parameters of the MN and MNa solutions

CHAPTER 5

THE CLASSICAL SOLUTION: THE LOWEST ORDER K-STRING ENERGY

In this chapter, we calculate the classical level k-string energy, the $T_k L$ term, simultaneously for both the MN and MNa solutions, thanks to the geometrical similarities. As $\rho -> 0$, the geometry of the MN(MNa) solution shrinks to a structure of $R^{(3,1)} \times S^3$($R^{(2,1)} \times S^3$). The probe D3-brane that wraps over an $S^2 \subset S^3$ and whose two aditional dimensions orient with one spatial dimension and the time dimension of $R^{(3,1)}$ ($R^{(2,1)}$) is interpreted as a k-string in 4d (3d) Minkowski space.

5.1 The Sine Law k-string Tension

To make k-string dual calculations, we consider a probe D3-brane, parametrized by the world volume coordinates

$$\xi^\alpha = (x, \theta, \phi, t) \tag{5.1.1}$$

, electrically charged

$$F = E dt \wedge dx \equiv F_{tx} dt \wedge dx, \tag{5.1.2}$$

and embedded in the MN and MNa backgrounds respectively. This D3-brane action consists of the Dirac-Born-Infeld action [71] and the Chern-Simons term

$$S_{D_3} = -\frac{\mu_3}{g_s} \int d^4\xi \sqrt{-\det(g_{\alpha\beta} + \mathcal{F}_{\alpha\beta})} + \mu_3 \int e^{\mathcal{F}} \wedge \sum_p C_p \tag{5.1.3}$$

where

$$\begin{aligned} g_s &= e^{\Phi_0} \\ \mu_3 &= (2\pi)^{-3} \alpha'^{-2} \end{aligned} \tag{5.1.4}$$

, where α' is the slope parameter [59], which is the proportionality constant between the angular momentum and the square of the energy of a rigidly rotating open string, and

$$g_{\alpha\beta} = \frac{\partial X^a}{\partial \xi^\alpha} \frac{\partial X^b}{\partial \xi^\beta} G_{ab} \tag{5.1.5}$$

which is the pullback on the D3-brane of the 10-dimenional metric G_{ab} in Eq. 4.3.1 and

$$\mathcal{F} = B_2 + 2\pi\alpha' F = 2\pi\alpha' F \tag{5.1.6}$$

since $B_2 = 0$ for the s-dual frame we investigate.

Our ansatz for the classical solution is

$$\begin{aligned} X^a &= (x, X^2 = 0, X^3 = 0, \rho = 0, \theta, \phi, \psi, \theta, \phi, t) \qquad \text{MN} \\ X^a &= (x, X^2 = 0, \rho = 0, \psi, \theta, \phi, \psi, \theta, \phi, t) \qquad\qquad \text{MNa} \end{aligned} \tag{5.1.7}$$

where the D3-brane's world-volume is parametrized by

$$\xi^\alpha = (x, \theta, \phi, t) \tag{5.1.8}$$

and ψ is a constant, yet to be determined.

Plugging this solution into Eqs. 4.3.1 and 4.3.3 and applying the coordinate transformation $\psi \to 4\pi - 2\psi$, we acquire, regardless of background choice MN or MNa:

$$ds^2 = g_{\alpha\beta} d\xi^\alpha d\xi^\beta = e^{\Phi_0}(-dt^2 + dx^2 + N\alpha' \sin^2\psi d\Omega_2^2) \tag{5.1.9}$$

$$F_3 = 2N\alpha' \sin^2\psi \sin\theta d\psi \wedge d\theta \wedge d\phi \tag{5.1.10}$$

Integrating and pulling back the three form, $F_3 = dC_2$, we acquire the pullback of the Ramond-Ramond two form

$$C_2 = N\alpha' \left(\psi - \frac{1}{2}\sin 2\psi\right) \sin\theta d\theta \wedge d\phi \tag{5.1.11}$$

Notice that the gauge fields $F_3 = dC_2$ and $H_3 = dB_2$ are pulled back to the D3 brane in the similar fashion as the pulled back metric in eq 5.1.5.

As all the ingredients for the classical D3-brane action in eq. 5.1.3 are the same for the M-N and M-Na case, they have the same classical action:

$$\begin{aligned} S_{D_3} &= -\frac{\mu_3}{g_s}\int d^4\xi \sqrt{-\det(g_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})} + \mu_3 \int 2\pi\alpha' F \wedge C_2 \\ &= -\frac{\mu_3}{g_s}\int d^4\xi g_s \sqrt{g_s^2 - (2\pi\alpha' E)^2}\alpha' N \sin^2\psi \sin\theta + \\ &\qquad + \mu_3 \int d^4\xi (2\pi\alpha' E) N\alpha' \left(\psi - \frac{1}{2}\sin 2\psi\right)\sin\theta \end{aligned} \tag{5.1.12}$$

Integrating out the angular components $\theta \in (0, \pi), \phi \in (0, 2\pi)$, we acquire

$$S_{D_3} = \int dt dx \mathcal{L}, \quad \mathcal{L} = -\gamma N(\sin^2\psi \sqrt{1 - e^2} - e(\psi - \cos\psi \sin\psi)) \tag{5.1.13}$$

$$\gamma = \mu_3 \alpha' 4\pi g_s, \quad g_s e = 2\pi\alpha' E \tag{5.1.14}$$

Calculating the Hamiltonian with

$$\mathcal{H} = PE - \mathcal{L}, \quad P \equiv \frac{\partial \mathcal{L}}{\partial E} = k, \tag{5.1.15}$$

solving the second equation of 5.1.15 for E in terms of k

$$k = \frac{N\left(2\pi\alpha'(\psi - \sin\psi\cos\psi) + \frac{4\pi^2 \mathrm{E}(\alpha')^2 \sin^2\psi}{\sqrt{e^{2\Phi_0} - 4\pi^2 \mathrm{E}^2(\alpha')^2}}\right)}{2\pi^2\alpha'} \tag{5.1.16}$$

$$E = \frac{e^{\Phi_0}(\pi k - N\psi + N\sin\psi\cos\psi)}{\sqrt{2}\pi\alpha'\sqrt{2\pi^2k^2 - 4\pi k N\psi + 2N\sin 2\psi(\pi k - N\psi) + 2N^2\psi^2 - N^2\cos 2\psi + N^2}} \tag{5.1.17}$$

and plugging this solution 5.1.17 back to the first equation of 5.1.15 results in:

$$\mathcal{H} = \gamma N\sqrt{((\pi k/N) - \psi + \cos\psi\sin\psi)^2 + \sin^4\psi} \tag{5.1.18}$$

The minimization condition for the Hamiltonian with respect to the constant field ψ is

$$\psi = \frac{\pi k}{N} \equiv \psi_0 \tag{5.1.19}$$

which leads to the minimized Hamiltonian

$$\mathcal{H}_{min} = T_k N = \gamma N \sin\frac{k\pi}{N}, \tag{5.1.20}$$

an exact sine law for k-strings in $d = 3$ or $d = 4$ spacetime dimensions from the dual theory of a probe D3-brane in the MNa or MN background, respectively. This should not be surprising since in the IR regime the radius of S^2(MN, S^3 for MNa) vanishes, leading to identical results in the classical level.

CHAPTER 6

QUANTUM FLUCTUATIONS: THE LUSCHER TERM FOR $D = 3$ AND $D = 4$ K-STRINGS

The quantum one-loop correction to the classical solution of the k-string energy, i.e., the Lüscher term is the term proportional to $\frac{1}{L}$ [72, 31] in eq. 3.1.13 where L is interpreted as the separation between the quark antiquark pairs in fig 6.1. We show that fluctuating the classical action by varying the Bosonic and the Fermionic fields and the gauge potentials around the classical solutions will lead to the Lüscher term.

In this chapter, we fluctuate the bosonic fields and solve the resulting equations of motion for the small fluctuations to find the eigenvalues which eventually contribute to

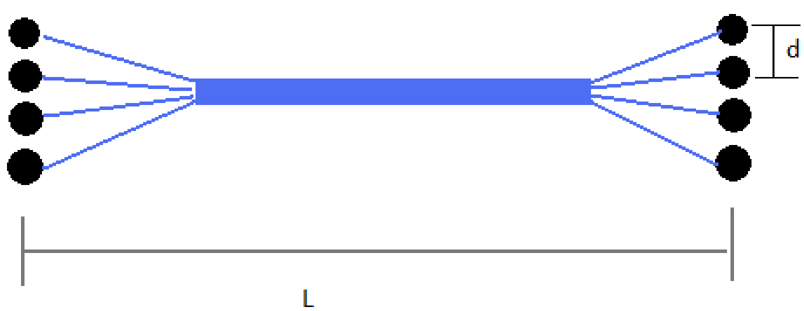

Figure 6.1: A k-string

the Lüscher term.

6.1 Fluctuations of the Bosonic Fields

In general, the D3-brane action [73, 74, 75, 76, 77] includes both the bosonic contributions $S^{(B)}$and the fermionic contributions $S^{(F)}$

$$S = S^{(B)} + S^{(F)} \tag{6.1.1}$$

In the calssical level, the fermionic fields don't contribute, i.e. $\Theta = 0$, which means $S^{(F)} = 0$. For now, let's investigate the fluctuations of bosonic fields first. We will look at the fermionic part in the next chapter.

Starting from the D3-brane action eq. 5.1.3, we vary the bosonic fileds $X^a(x, \theta, \phi, t)$ and the gauge potentials $A^\mu(x, \theta, \phi, t)$ infinitesimally around the classical solutions 5.1.7.

$$\begin{aligned} X^a &= X^a_{(0)} + \delta X^a(\xi) \\ A^\mu &= A^\mu_{(0)} + \delta A^\mu(\xi) \end{aligned} \tag{6.1.2}$$

Using the same strategy that was used in the previous chapter, we proceed simultaneously with MN and MNa backgrounds and indicate the differences only when they appear. Due to the D3-brane reparametrization invariance, there are only six degrees of freedom of fluctuating the 10 Bosonic fields. The ansatz we try are the following:

$$\begin{aligned} & X^0 = t, X^1 = x, X^2 = \delta X^2, \theta_1 = \theta, \phi_1 = \phi, \\ & \theta_2 = \theta + \delta\theta, \phi_2 = \phi + \delta\phi, \psi = \psi_0 + \delta\psi, \rho = \delta\rho \\ & X^3 = \delta X^3, \qquad \text{MN} \\ & \psi_1 = \psi_0 + \delta\psi_1 \qquad \text{MNa} \end{aligned} \tag{6.1.3}$$

$$\begin{aligned} & A^x = Et + \delta A^x \\ & A^t = \delta A^t, A^\theta = \delta A^\theta, A^\phi = \delta A^\phi \end{aligned} \tag{6.1.4}$$

Using the same methods as [70, 26], we find that the 10-dimensional Bosonic metric G_{ab}, the Ramond-Ramond field C_2 and the electromgnatic field $F_{\mu\nu}$ all have up to second order fluctuations:

$$G_{ab} = G^{(0)}_{ab} + G^{(1)}_{ab} + G^{(2)}_{ab}$$

$$C_2 = C^{(0)}_{ab} + C^{(1)}_{ab} + C^{(2)}_{ab}$$

$$F_{\mu\nu} = F^{(0)}_{\mu\nu} + F^{(1)}_{\mu\nu} \tag{6.1.5}$$

where $F_{\mu\nu}$ has only the first order fluctuation since there is no product term of small fluctuations involved.

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu(A_{\nu(0)} + \delta A_\nu) - \partial_\nu(A_{\mu(0)} + \delta A_\mu) \\ &= \partial_\mu A_{\nu(0)} - \partial_\nu A_{\mu(0)} + \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu \\ &= F^{(0)}_{\mu\nu} + \delta F_{\mu\nu} \\ &= F^{(0)}_{\mu\nu} + F^{(1)}_{\mu\nu} \end{aligned} \tag{6.1.6}$$

The induced metric $g_{\alpha\beta}$ therefore also has up to second order fluctuations

$$g_{\alpha\beta} = \frac{\partial X^a}{\partial \xi^\alpha}\frac{\partial X^b}{\partial \xi^\beta} G_{ab} \tag{6.1.7}$$

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + g^{(1)}_{\alpha\beta} + g^{(2)}_{\alpha\beta} \tag{6.1.8}$$

In the bosonic action 5.1.3, the term under the square root $\sqrt{-\det(g_{\alpha\beta} + \mathcal{F}_{\alpha\beta})}$ has to be expanded eventually since both $g_{\alpha\beta}$ and $\mathcal{F}_{\alpha\beta}$ are expanded up to the second order of the fluctuations. It's shown that this term can be eapanded using the following formula

$$\sqrt{|M + \delta M|} = \sqrt{|M|}\Bigg\{1 + \frac{1}{2}Tr(M^{-1}\delta M) + \frac{1}{8}[Tr(M^{-1}\delta M)]^2 - \frac{1}{4}Tr(M^{-1}\delta M M^{-1}\delta M) + \mathcal{O}(\delta M^3)\Bigg\} \tag{6.1.9}$$

where M is an arbitrary square matrix. Clearly, we can identify $g + \mathcal{F}$ as M. Therefore,

$$\begin{aligned} \delta M &= M^{(1)} + M^{(2)} \\ M^{(1)} &= g^{(1)} + \mathcal{F}^{(1)} \\ M^{(2)} &= g^{(2)} + \mathcal{F}^{(2)} \\ M^{(0)} &= g^{(0)} + \mathcal{F}^{(0)} \equiv M_0 \end{aligned} \tag{6.1.10}$$

Plugging eq. 6.1.3 and eq. 6.1.4 into the D3-brane action 5.1.3, collecting all the intermediate results of eq. 6.1.5 and eqs. 6.1.8, 6.1.9, 6.1.10, the bosonic D3-brane action now can be organized in the following way:

$$S = S^{(0)} + S^{(1)} + S^{(2)} \tag{6.1.11}$$

where $S^{(0)}$ is just the calssical actiton 5.1.12 and $S^{(1)}$ is found to vanish up to total derivative terms at the classical solution $\psi_0 = \frac{\pi k}{N}$ which confirms that the ansatz 5.1.7 is indeed the classical solution. $S^{(2)}$ is the second order fluctuation of the action. Solving the Euler-Lagrange equations from $S^{(2)}$ for the small fluctuations will lead to the energy eigenvalues of the fluctuating Hamiltonian which yields the Lüscher term eventually.

6.2 Solving the Equations of Motion for the Bosonic Fluctuations

The second order fluctuations $S^{(2)}$ in eq. 6.1.11 takes the following form:

$$\begin{aligned} S^{(2)} = &-\frac{\mu_3}{g_s}\int d^4\xi\sqrt{-|M_0|}\Bigg\{\frac{1}{2}Tr(M_0^{-1}M^{(2)}) + \\ &+\left[\frac{1}{8}[Tr(M_0^{-1}M^{(1)})]^2 - \frac{1}{4}Tr(M_0^{-1}M^{(1)}M_0^{-1}M^{(1)})\right]\Bigg\} + \\ &+\mu_3\int[\mathcal{F}^{(0)}\wedge C_2^{(2)} + \mathcal{F}^{(1)}\wedge C_2^{(1)} + \mathcal{F}^{(2)}\wedge C_2^{(0)}] \end{aligned} \tag{6.2.1}$$

where the term $\mathcal{F}^{(2)}\wedge C_2^{(0)}$ actually vanishes since $\mathcal{F}^{(2)} = 0$. Recall that

$$\mathcal{F} = B_2 + 2\pi\alpha' F \tag{6.2.2}$$

where the NS-NS field $B_2 = 0$ in the S-dual frame we use and the electromagnetic fields $F_{\mu\nu}$ don't contain second order terms, i.e., $F^{(2)}_{\mu\nu} = 0$.

$S^{(2)}$ is shown to be written in terms of the fluctuating fields as follows

$$\begin{aligned} S^{(2)} = \int d^4\xi\sqrt{g_e}\Bigg[&- c_x \bigtriangledown_\alpha \delta X^i \bigtriangledown^\alpha \delta X^i - c_A\left[\frac{1}{16\pi}\delta F^{\mu\nu}\delta F_{\mu\nu} + \delta A_\mu j^\mu\right] + \\ &- c_\rho\left[\bigtriangledown_\alpha\delta\rho \bigtriangledown^\alpha \delta\rho + m_\rho^2\delta\rho^2\right] + \\ &- c_\Psi\left[\bigtriangledown_a\Psi \bigtriangledown^a \Psi - R\Psi^2\right] + \\ &+ \mathit{total\ derivatives}\Bigg] \end{aligned} \tag{6.2.3}$$

where the constants and parameters are

$$
\begin{aligned}
& i = 2,3 \quad \text{MN} \quad and \quad i = 2 \quad \text{MNa} \\
g_e &\equiv -det(g^{eff}) = \frac{4g_{xx}^2 \sin^2\theta}{R^2}, \;\; g_{xx} = e^{\Phi_0}\sin^2\psi_0 \\
\Psi &\equiv 2\delta\psi - \cos\theta\delta\phi, \;\; c_x = \frac{1}{2}\mu_3 \csc\psi_0 \\
Q_\Psi &= \frac{4\alpha_3 \sin\theta}{\sqrt{g_e}}, \;\; \alpha_3 = \frac{1}{2}\pi\mu_3 N\alpha'^2 \\
c_\rho &= \frac{1}{2}\mu_3 N \csc\psi_0 \alpha'
\end{aligned} \tag{6.2.4}
$$

and

$$
\begin{aligned}
c_A &= \frac{32e^{-\Phi_0}\pi^3\mu_3 \csc\psi_0\alpha'}{g_{xx}RN}, c_\Psi = \frac{e^{\Phi_0}\mu_3 N \sin\psi_0\alpha'}{8g_{xx}} \\
R &= \frac{2e^{-\Phi_0}\csc^2\psi_0}{N\alpha'} \\
m_\rho &= \frac{\sqrt{14}e^{-\Phi_0/2}\csc\psi_0}{3\sqrt{N\alpha'}} \qquad \text{MN} \\
m_\rho &= \frac{\sqrt{2}e^{-\Phi_0/2}\csc\psi_0\sqrt{-\frac{w_2}{2} + \frac{1}{8}(2+3w_2^2) + \sin^2\psi_0}}{\sqrt{N\alpha'}} \qquad \text{MNa} \\
b_2 &= 0 \quad \text{MN}, \;\; b_2 = \frac{-w_2}{2} \quad \text{MNa}, \quad w_2 \in [0,1]
\end{aligned} \tag{6.2.5}
$$

In the second order action 6.2.3, the covariant derivative $\bigtriangledown_\alpha$ is taken with respct to an effective metric g^{eff} on the D3-brane

$$
g^{eff}{}_{\alpha\beta} = \begin{pmatrix} g_{xx} & 0 & 0 & 0 \\ 0 & \frac{2}{R} & 0 & 0 \\ 0 & 0 & \frac{2\sin^2\theta}{R} & 0 \\ 0 & 0 & 0 & -g_{xx} \end{pmatrix}
$$

The $R^{1,1} \times S^2$ topology and the scalar curvature R of this effective metric are the smae as the induced metric 5.1.9. The field $\Psi \equiv 2\delta\psi - \cos\theta\delta\phi$ is a combination of $\delta\psi$ and $\delta\phi$ and contains all second order contributions of $\delta\psi$ and $\delta\phi$. The $U(1)$ gauge current is

$$
j^\mu = (\frac{Q_\Psi \bigtriangledown_t \Psi}{c_A}, \;\; 0, \;\; 0, \;\; \frac{-Q_\Psi \bigtriangledown_x \Psi}{c_A}) \tag{6.2.6}
$$

The Euler-Lagrange equations for the infinitesimal bosonic fluctuations, derived from the second order action 6.2.3, are given by the following

$$\bigtriangledown^2 \delta X^i = 0, \quad i = 2,3 \quad \text{MN}; i = 2 \quad \text{MNa} \tag{6.2.7}$$

$$\bigtriangledown^2 \delta\rho - m_\rho^2 \delta\rho = 0 \tag{6.2.8}$$

$$\bigtriangledown^2 \Psi + R\Psi + \frac{1}{2c_\Psi} Q_\Psi \delta F_{tx} = 0 \tag{6.2.9}$$

$$\bigtriangledown^\mu \delta F_{\mu\nu} - 4\pi j_\nu = 0 \tag{6.2.10}$$

Notice that the field ψ_1 has no dynamics in the second order since the ψ_1 dependence vanishes under $\rho -> 0$ limit. The $\delta\theta$ field has no dynamics either.

With the help of the Riemann curvature tensor, the $U(1)$ gauge equation 6.2.10 can be rewritten as

$$\begin{aligned}
\bigtriangledown^\mu \delta F_{\mu\nu} &= \bigtriangledown^\mu (\bigtriangledown_\mu \delta A_\nu - \bigtriangledown_\nu \delta A_\mu) \\
&= \bigtriangledown^\mu \bigtriangledown_\mu \delta A_\nu - \bigtriangledown_\nu \bigtriangledown_\mu \delta A^\mu - R^\mu{}_{\alpha\mu\nu} \delta A^\alpha \\
&= \bigtriangledown^\mu \bigtriangledown_\mu \delta A_\nu - \bigtriangledown_\nu \bigtriangledown_\mu \delta A^\mu - R_{\alpha\nu} \delta A^\alpha = 4\pi j_\nu
\end{aligned} \tag{6.2.11}$$

where the Ricci tensor $R_{\alpha\nu}$ has only two non-zero components

$$R_{\theta\theta} = 1, \qquad R_{\phi\phi} = sin^2\theta \tag{6.2.12}$$

Using the temporal gauge $\delta A^t = 0$, the $\nu = t$ component of eq. 6.2.11 yeilds the Gauss' law constraint

$$\begin{aligned}
\bigtriangledown_t \bigtriangledown_\mu \delta A^\mu &= -4\pi j_t \\
&= \frac{-4\pi g_{xx} Q_\Psi \bigtriangledown_x \Psi}{c_A}
\end{aligned} \tag{6.2.13}$$

The ansatz we try for the equations of motion 6.2.7 to 6.2.10 are the following

$$\delta A_j = \int dp\, d\omega \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \tilde{A}_j^{(lm)}(p,\omega) e^{i(px-\omega t)} Y_i^{(lm)}(\theta,\phi), \qquad j = x, \theta, \Phi \tag{6.2.14}$$

$$\Psi = \int dp\, d\omega \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \tilde{\Psi}^{(lm)}(p,\omega) e^{i(px-\omega t)} Y^{lm}(\theta,\phi) \tag{6.2.15}$$

$$\delta\rho = \int dp\, d\omega \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \tilde{\rho}^{(lm)}(p,\omega) e^{i(px-\omega t)} Y^{lm}(\theta,\phi) \tag{6.2.16}$$

$$\delta X^j = \int dp\, d\omega \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \tilde{X^j}^{(lm)}(p,\omega) e^{i(px-\omega t)} Y^{lm}(\theta,\phi) \qquad j = 2,3 \;\; \text{MN};\; j = 2 \;\; \text{MNa} \tag{6.2.17}$$

where

$$\begin{aligned} Y_x^{(lm)}(\theta,\phi) &\equiv Y^{lm}(\theta,\phi) \\ Y_\theta^{(lm)}(\theta,\phi) &\equiv \frac{\csc\theta}{\sqrt{l(l+1)}} \partial_\phi Y^{lm}(\theta,\phi) \\ Y_\phi^{(lm)}(\theta,\phi) &\equiv \frac{-\sin\theta}{\sqrt{-l(l+1)}} \partial_\theta Y^{lm}(\theta,\phi) \end{aligned} \tag{6.2.18}$$

and Y^{lm} is the spherical harmonics. $Y_\theta^{(lm)}(\theta,\phi)$ and $Y_\phi^{(lm)}(\theta,\phi)$ satisfy the eigenvalue equation for the vector spherical harmonics

$$\hat{L}^2 Y_i^{(lm)} = [l(l+1)-1] Y_i^{(lm)}, \qquad i = \theta,\ \phi \tag{6.2.19}$$

where

$$\hat{L}^2 = -\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) - \frac{1}{\sin^2\theta}\partial_\phi^2 \tag{6.2.20}$$

which is the angular momentum squared operator.

The eigenvalue equations of the small fluctuations can be acquired by plugging the ansatz 6.2.14 to 6.2.17 to the equations of motion 6.2.7 to 6.2.10. For instance, the massless fields δX^i are found to satisfy the following eigenvalue equation

$$[\omega^2 - p^2 - \frac{g_{xx}}{2} l(l+1)R]\tilde{X}^i = 0, \qquad i = 2,3 \;\; \text{MN};\; i = 2 \;\; \text{MNa} \tag{6.2.21}$$

$\delta\rho$ has the eigenvalue equation

$$[\omega^2 - p^2 - \frac{g_{xx}}{2} l(l+1)R - g_{xx} m_\rho^2]\tilde{\rho} = 0 \tag{6.2.22}$$

To make life easier, try the ansatz $\tilde{A}_\theta = \tilde{A}_\phi$ which makes $\bigtriangledown_\theta \delta A^\theta + \bigtriangledown_\phi \delta A^\phi$ vanish since

$$\bigtriangledown_\theta \delta A^\theta + \bigtriangledown_\phi \delta A^\phi = \partial_\theta \delta A^\theta + \partial_\phi \delta A^\phi + \Gamma^\phi{}_{\theta\phi} \delta A^\theta \tag{6.2.23}$$

and this can easily be shown to vanish, plugging in the ansatz for δA^θ and δA^ϕ and using the features of the vector spherical harmonics $Y_\theta^{(lm)}(\theta, \phi)$ and $Y_\phi^{(lm)}(\theta, \phi)$ and the fact that $\Gamma^\phi{}_{\theta\phi} = \cot\theta$. This further simplifes the Gauss' law constraint 6.2.13

$$\begin{aligned}
\bigtriangledown_x \bigtriangledown_t \delta A_x &= \frac{-4\pi g_{xx}^2 Q_\Psi \bigtriangledown_x \Psi}{c_A} \\
\bigtriangledown_t \delta A_x &= \frac{-4\pi g_{xx}^2 Q_\Psi \Psi}{c_A}
\end{aligned} \tag{6.2.24}$$

Plugging this into the $\nu = x$ component of eq. 6.2.11 and using the fact that $\bigtriangledown_\theta \delta A^\theta + \bigtriangledown_\phi \delta A^\phi = 0$, we acquire

$$\bigtriangledown^\theta \bigtriangledown_\theta \delta A_x + \bigtriangledown^\phi \bigtriangledown_\phi \delta A_x = 0 \tag{6.2.25}$$

which results in the following

$$\hat{L}^2 \tilde{A}_x = 0 \tag{6.2.26}$$

This means $l = 0$ for the coulped fields δA_x and Ψ. Therefore the eigenvalue equation for Ψ goes as follows

$$[\omega^2 - p^2 - g_{xx} m_\Psi^2]\tilde{\Psi} = 0 \tag{6.2.27}$$

where

$$m_\Psi^2 = 2\pi g_{xx}^2 \frac{Q_\Psi^2}{C_\Psi C_A} - R \tag{6.2.28}$$

The equations of motion for δA_θ and δA_ϕ, derived from 6.2.11, taking into account that $\bigtriangledown_\theta \delta A_x = 0$ and $\bigtriangledown_\phi \delta A_x = 0$, can be written in the following simplified form

$$\bigtriangledown^2 \delta A_i - \frac{R}{2} \delta A_i = 0, \qquad i = \theta,\ \phi \tag{6.2.29}$$

which yields the eigenvalue equation for δA_θ and δA_ϕ

$$[\omega^2 - p^2 - \frac{g_{xx}}{2} l(l+1)R]\tilde{A}_i = 0, \quad i = \theta, \phi \tag{6.2.30}$$

Now we organize the eigenvalue equations for all the fields and put everything together for the MN background

$$\mathcal{H}_b^2 \begin{pmatrix} \tilde{\Psi} \\ \tilde{\rho} \\ \tilde{X}^2 \\ \tilde{X}^3 \\ \tilde{A}_\theta \\ \tilde{A}_\phi \end{pmatrix} = \omega^2 \begin{pmatrix} \tilde{\Psi} \\ \tilde{\rho} \\ \tilde{X}^2 \\ \tilde{X}^3 \\ \tilde{A}_\theta \\ \tilde{A}_\phi \end{pmatrix}$$

where $\mathcal{H}_b^2$ is given by the following diagonal matrix

$$\mathcal{H}_b^2 = \begin{pmatrix} \omega_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_2^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_3^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_3^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_3^2 \end{pmatrix}$$

For the MNa background, there's one less field that needs to be considered

$$\mathcal{H}_b^2 \begin{pmatrix} \tilde{\Psi} \\ \\ \tilde{\rho} \\ \\ \tilde{X^2} \\ \\ \tilde{A}_\theta \\ \\ \tilde{A}_\phi \end{pmatrix} = \omega^2 \begin{pmatrix} \tilde{\Psi} \\ \\ \tilde{\rho} \\ \\ \tilde{X^2} \\ \\ \tilde{A}_\theta \\ \\ \tilde{A}_\phi \end{pmatrix}$$

where $\mathcal{H}_b^2$ is given by the following diagonal matrix

$$\mathcal{H}_b^2 = \begin{pmatrix} \omega_1^2 & 0 & 0 & 0 & 0 \\ \\ 0 & \omega_2^2 & 0 & 0 & 0 \\ \\ 0 & 0 & \omega_3^2 & 0 & 0 \\ \\ 0 & 0 & 0 & \omega_3^2 & 0 \\ \\ 0 & 0 & 0 & 0 & \omega_3^2 \end{pmatrix}$$

where

$$\begin{aligned} \omega_1^2 &= p^2 + g_{xx} m_\Psi^2 \\ \omega_2^2 &= \omega_3^2 + g_{xx} m_\rho^2 \\ \omega_3^2 &= p^2 + \frac{g_{xx}}{2} l(l+1)R \end{aligned} \tag{6.2.31}$$

In the next chapter, we'll investigate the fermionic fluctuations and solve the Dirac equation for the fermionic eigenvalues. In the final chapter of this dissertation, we'll show how to acquire the Lüscher term using these eigenvalues.

CHAPTER 7

THE FERMIONIC FLUCTUATIONS

So far, we have only investigated the bosonic portion of the D3 brane action, i.e., $S^{(B)}$ as in eq 6.1.1 since we first started from the classical action 5.1.3 which does not contain fermionic fields. Now that we have investigated the classical action and the quantum fluctuations of the bosonic portion, naturally, the next step would be to look at the fermionic portion of the D3-brane action.

We introduce the fermion fields as

$$\Theta = 0 + \delta\Theta(\xi) \tag{7.0.32}$$

since the fermion fields do not exist at the classical level and investigate the dynamics of the world volume fermion fields $\delta\Theta$. We hope to construct and solve the equations of motion, i.e., the Dirac equations for the fermionic fluctuations $\delta\Theta$ and see if they contribute to the second order term of the k-string energy, i.e., the Lüscher term. Again, as in the previous chapters, we investigate and compare the Maldacena-Nunez and the Maldacena-Nastase supergravity backgrounds and all calculations are to be carried out simultaneously. All steps apply to both backgrands while the differences will be mentioned and evaluated where it matters. We also compare the MN and the MNa results with the results based on Klebanov-Strassler supergravity background [27].

The D3-brane action including both the fermionic and the bosonic fields takes the general form [73, 74, 75, 76, 77]

$$S = S^{(B)} + S^{(F)} \tag{7.0.33}$$

where $S^{(B)}$, shown in equation 5.1.3, and $S^{(F)}$ are the bosonic and the fermionic contributions respectively.

7.1 The Fermionic Action

[78, 79] used the superspace as target space to write a complete superaction which includes both the bosonic fields and the fermionic fields.

$$I = I_{DBI} + I_{WZ} \tag{7.1.1}$$

Notice that the major difference between the superaction 7.1.1 and the bosonic action 5.1.3 is that the bosonic background fields have been replaced by the corresponding superfields in the superaction. Even if compact and elegant, however, the implicit superspace formalism makes explicit calculations involving the world-volume fermions obscure because how the bosonic background fields enter the fermionic terms of the action is hidden. Besides, Calculations starting from the superaction in [78, 79] require a case by case study [76] (see [80, 81] for examples). An important step toward understanding the fermionic terms in D-brane actions in backgrounds with fluxes was made in [73, 74] and a complete quadratic fermionic action for any D-brane in any supergravity background was given in these papers using a Dirac-like form.

The geometrical structure of the fermionic D-brane action in any supergratity background acquired in [73, 74] was further clarified in [76]. Taking into account the non-vanishing world-volume field strength

$$\mathcal{F} = B_2 + 2\pi\alpha' F \tag{7.1.2}$$

which adds new kinetic terms and hence generalizes the Dirac-like operator, the fermionic sector of the type IIB D3-brane action, i.e., $S^{(F)}$, according to [76] can be written as

$$\begin{aligned} S^{(F)} = \frac{\mu_3}{2g_s} \int d^4\xi e^{-\Phi} \sqrt{-det(M_0)}\overline{\Theta}\{ & (M_0^{-1})^{\alpha\beta}\Gamma_\alpha D_\beta^{(0)} - \Delta^{(1)} \\ & - \check{\Gamma}_{D_3}^{-1}(M_0^{-1})^{\alpha\beta}\Gamma_\beta W_\alpha + \check{\Gamma}_{D_3}^{-1}\Delta^{(2)}\}\Theta \end{aligned} \tag{7.1.3}$$

where

$$g_s = e^{\Phi_0}$$

$$\mu_3 = (2\pi)^{-3}\alpha'^{-2} \tag{7.1.4}$$

where α' is the slope parameter. The world volume of the D3-brane is parametrized by

$$\xi^\alpha = (x, \theta, \phi, t) \tag{7.1.5}$$

The fermionic field Θ is a positive chirality doublet Majorana-Weyl spinor. $\mathcal{F}$ is the gauge field living on the brane. Notice that the generalized integration measure $\sqrt{-det(M_0)}$ which depends on the world volume field $\mathcal{F}$ also appears in the bosonic sector 5.1.3. Hence, one can see that the world volume field strength $\mathcal{F}$ deforms the world-volume geometry naturally [76]. The various components in 7.1.3 are written explicitly as follows

$$\begin{aligned} M_0 &= g + \mathcal{F} \\ g_{\alpha\beta} &= \frac{\partial X^a}{\partial \xi^\alpha}\frac{\partial X^b}{\partial \xi^\beta} G_{ab} \\ \mathcal{F} &= B_2 + 2\pi\alpha' F \\ D_\alpha^{(0)} &= \partial_\alpha + \frac{1}{4}\Omega_\alpha{}^{\bar{a}\bar{b}}\Gamma_{\bar{a}\bar{b}} + \frac{1}{4\cdot 2!}H_{\alpha np}\Gamma^{np} \\ W_\alpha &= \frac{1}{8}\left[F_n\Gamma^n + \frac{1}{3!}(F_{npq} + C_0 H_{npq})\Gamma^{npq} + \frac{1}{2\cdot 5!}(F_{npqrs} + H_{[npq}C_{rs]})\Gamma^{npqrs}\right]\Gamma_\alpha \end{aligned} \tag{7.1.6}$$

where $g_{\alpha\beta}$ is the 4-dimensional induced metric of the D3 brane and

$$\begin{aligned} \Delta^{(1)} &= \frac{1}{2}(\Gamma^m\partial_m\Phi + \frac{1}{2\cdot 3!}H_{mnp}\Gamma^{mnp}) \\ \Delta^{(2)} &= -\frac{1}{2}e^\Phi\left[F_m\Gamma^m + \frac{1}{2\cdot 3!}(F_{mnp} + C_0 H_{mnp})\Gamma^{mnp}\right] \end{aligned}$$

$$\check{\Gamma}_{D_p} = (-1)^{\frac{(p-2)(p-3)}{2}}\frac{\epsilon^{\alpha_1\ldots\alpha_{p+1}}\Gamma_{\alpha_1\ldots\alpha_{p+1}}}{(p+1)!\sqrt{-det(M_0)}}\sum_{q=0}^{\frac{p+1}{2}}\frac{\Gamma^{\beta_1\ldots\beta_{2q}}}{q!2^q}\mathcal{F}_{\beta_1\beta_2}\ldots\mathcal{F}_{\beta_{2q-1}\beta_{2q}} \tag{7.1.7}$$

$$\Gamma^{abc\ldots} = \Gamma^a\Gamma^b\Gamma^c\ldots \tag{7.1.8}$$

where the latin indices a,b,c..., are 10-d bosonic indices and the greek indices,α, β, μ, ν..., are the D-brane world volume indices. Moreover, latin indices with an overbar, e.g. $\bar{a}$ are flat space-time indices. The gamma matrices satisfy the Clifford algebra, e.g. the

10-d flat gamma matrices satisfy

$$\{\Gamma^{\bar{a}}, \Gamma^{\bar{b}}\} = 2\eta^{\bar{a}\bar{b}} I_{32} \tag{7.1.9}$$

For the 10-d flat gamma matrices, we use the same representations as [27]:

$$\begin{aligned}
\Gamma^{\bar{0}} &= -i\sigma^1 \otimes \sigma^1 \otimes \sigma^1 \otimes \sigma^1 \otimes \sigma^3 \\
\Gamma^{\bar{1}} &= \sigma^1 \otimes \sigma^1 \otimes \sigma^1 \otimes \sigma^2 \otimes \sigma^0 \\
\Gamma^{\bar{2}} &= \sigma^1 \otimes \sigma^3 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \\
\Gamma^{\bar{3}} &= \sigma^2 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \\
\Gamma^{\bar{4}} &= \sigma^1 \otimes \sigma^1 \otimes \sigma^3 \otimes \sigma^0 \otimes \sigma^0 \\
\Gamma^{\bar{5}} &= \sigma^1 \otimes \sigma^2 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \\
\Gamma^{\bar{6}} &= -\sigma^1 \otimes \sigma^1 \otimes \sigma^1 \otimes \sigma^1 \otimes \sigma^1 \\
\Gamma^{\bar{7}} &= \sigma^1 \otimes \sigma^1 \otimes \sigma^1 \otimes \sigma^1 \otimes \sigma^2 \\
\Gamma^{\bar{8}} &= \sigma^1 \otimes \sigma^1 \otimes \sigma^2 \otimes \sigma^0 \otimes \sigma^0 \\
\Gamma^{\bar{9}} &= \sigma^1 \otimes \sigma^1 \otimes \sigma^1 \otimes \sigma^3 \otimes \sigma^0
\end{aligned} \tag{7.1.10}$$

and

$$\Gamma^{\bar{1}\bar{1}} = \Gamma^{\bar{0}} \cdot \Gamma^{\bar{1}} \cdot \Gamma^{\bar{2}} \cdot \Gamma^{\bar{3}} \cdot \Gamma^{\bar{4}} \cdot \Gamma^{\bar{5}} \cdot \Gamma^{\bar{6}} \cdot \Gamma^{\bar{7}} \cdot \Gamma^{\bar{8}} \cdot \Gamma^{\bar{9}} \tag{7.1.11}$$

where the σ^i are the Pauli matrices:

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Like tensors in flat and curved spaces, the 10-d curved gamma matrices and 10-d flat gamma matrices are related via the viel-biens in the following way

$$\Gamma^a = e_{\bar{a}}{}^a \Gamma^{\bar{a}}$$
$$\Gamma_a = e_a{}^{\bar{a}} \Gamma_{\bar{a}} \quad (7.1.12)$$

where the viel-beins $e_a{}^{\bar{a}}$ are background dependent and have been calculated in previous chapters. Therefore, the 10-d curved gamma matrices satisfy the curved Clifford algebra

$$\{\Gamma^a, \Gamma^b\} = 2G^{ab} I_{32} \quad (7.1.13)$$

where the 10-d curved metric is either the Maldacena-Nunez or the Maldacena-Nastase background

$$G^{ab} = e_{\bar{a}}{}^a e_{\bar{b}}{}^b \eta^{\bar{a}\bar{b}} \quad (7.1.14)$$

The pulled back gamma matrices on D3 brane Γ^α read

$$\Gamma_\alpha = \frac{\partial X^a}{\partial \xi^\alpha} \Gamma_a$$
$$\Gamma^\alpha = g^{\alpha\beta} \Gamma_\beta \quad (7.1.15)$$

The spin connection $\Omega_\alpha{}^{\bar{a}\bar{b}}$ is the pullback of the 10-d spin connection $\Omega_a{}^{\bar{a}\bar{b}}$ to the D3 brane with only the first index being pulled back:

$$\Omega_\alpha{}^{\bar{a}\bar{b}} = \frac{\partial X^a}{\partial \xi^\alpha} \Omega_a{}^{\bar{a}\bar{b}}$$
$$\Omega_a{}^{\bar{a}\bar{b}} = \frac{1}{2} e_a{}^{\bar{c}} (\eta_{\bar{d}\bar{c}} \eta^{\bar{e}\bar{a}} \eta^{\bar{f}\bar{b}} - \delta_{\bar{d}}{}^{\bar{a}} \eta^{\bar{e}\bar{b}} \delta_{\bar{c}}{}^{\bar{f}} - \delta_{\bar{d}}{}^{\bar{b}} \delta_{\bar{c}}{}^{\bar{e}} \eta^{\bar{f}\bar{a}}) C^{\bar{d}}{}_{\bar{e}\bar{f}}$$
$$= \frac{1}{2} e_a{}^{\bar{c}} (C_{\bar{c}}{}^{\bar{a}\bar{b}} - C^{\bar{a}\bar{b}}{}_{\bar{c}} - C^{\bar{b}}{}_{\bar{c}}{}^{\bar{a}})$$
$$C^{\bar{a}}{}_{\bar{b}\bar{c}} = (e_{\bar{b}}{}^a e_{\bar{c}}{}^b - e_{\bar{b}}{}^b e_{\bar{c}}{}^a) \partial_b e_a{}^{\bar{a}}$$
$$C^{\bar{b}}{}_{\bar{c}}{}^{\bar{a}} = \eta^{\bar{a}\bar{d}} C^{\bar{b}}{}_{\bar{c}\bar{d}}, \qquad etc. \quad (7.1.16)$$

where the indices of the holonomy elements $C^{\bar{b}}{}_{\bar{c}}{}^{\bar{a}}$ can be raised or lowered by the flat Minkowski metric $\eta^{\bar{a}\bar{b}}$ as the last eqaution of eq.7.1.16 has shown.

Now, we start to symplify eqs 7.1.6 and 7.1.7. For both the Maldacena-Nunez and Maldacena-Nastase backgrounds, the only non-zero gauge fields are $F_3 = dB_2$. The dilaton fields are given by

$$\Phi = log(e^{\Phi_0}(1 + c_1 \rho^2)) \quad (7.1.17)$$

Clearly,

$$\partial_\rho \Phi = 0 \tag{7.1.18}$$

at $\rho = 0$ which is where the D3 brane is embedded in the 10-d supergravity backgrounds. Therefore, eqs 7.1.6 and 7.1.7 can be symplified as

$$\begin{aligned} M_0 &= g + \mathcal{F} \\ D_\alpha^{(0)} &= \partial_\alpha + \frac{1}{4}\Omega_\alpha{}^{\bar{a}\bar{b}}\Gamma_{\bar{a}\bar{b}} \\ W_\alpha &= \frac{1}{8\cdot 3!}F_{npq}\Gamma^{npq}\Gamma_\alpha \end{aligned} \tag{7.1.19}$$

and

$$\begin{aligned} \Delta^{(1)} &= 0 \\ \Delta^{(2)} &= -\frac{1}{4\cdot 3!}e^{\Phi_0}F_{mnp}\Gamma^{mnp} \\ \check{\Gamma}_{D_3} &= \frac{\epsilon^{\alpha\beta\mu\nu}\Gamma_{\alpha\beta\mu\nu}}{4!\sqrt{-detM_0}}(I_{32} + \frac{1}{2}\Gamma^{\lambda\rho}\mathcal{F}_{\lambda\rho}) \end{aligned} \tag{7.1.20}$$

Then, the symplified D3 brane fermionic action 7.1.3 can now be rewritten as:

$$S^{(F)} = \frac{\mu_3}{2g_s}\int d^4\xi e^{-\Phi_0}\sqrt{-det(M_0)}\overline{\Theta}\{(M_0^{-1})^{\alpha\beta}\Gamma_\alpha\partial_\beta + M_1 + M_2 + M_3\}\Theta \tag{7.1.21}$$

where

$$\begin{aligned} M_1 &= \frac{1}{4}(M_0^{-1})^{\alpha\beta}\Gamma_\alpha\Omega_\beta{}^{\bar{a}\bar{b}}\Gamma_{\bar{a}\bar{b}} \\ M_2 &= -\frac{1}{8\cdot 3!}\check{\Gamma}_{D_3}^{-1}(M_0^{-1})^{\alpha\beta}\Gamma_\beta F_{npq}\Gamma^{npq}\Gamma_\alpha \\ M_3 &= -\frac{1}{4\cdot 3!}e^{\Phi_0}\check{\Gamma}_{D_3}^{-1}F_{mnp}\Gamma^{mnp} \end{aligned} \tag{7.1.22}$$

Combining eqs. 7.1.12 and 7.1.15, the pulled back gamma matrices Γ_α onto the D3 brane can be written as:

$$\begin{aligned} \Gamma_\alpha &= \frac{\partial X^a}{\partial \xi^\alpha}\Gamma_a \\ &= \frac{\partial X^a}{\partial \xi^\alpha}e_a{}^{\bar{a}}\Gamma_{\bar{a}} \end{aligned} \tag{7.1.23}$$

Evaluating the pullback matrices and the frame fields, the pulled back gamma matrices

Γ_α have been found to be the same for both backgrounds:

$$\Gamma_\alpha = \begin{pmatrix} \sqrt{e^{\Phi_0}})\Gamma_{\bar{0}} \\ \\ \sqrt{e^{\Phi_0}}\Gamma_{\bar{1}} \\ \\ \frac{1}{2}\sqrt{e^{\Phi_0}}\sqrt{N\alpha'}\{(cos2\psi_0 - 1)\Gamma_{\bar{6}} + sin2\psi_0\Gamma_{\bar{7}}\} \\ \\ \frac{1}{2}\sqrt{e^{\Phi_0}}\sqrt{N\alpha'}\{-sin2\psi_0 sin\theta\Gamma\bar{6} + (cos2\psi_0 - 1)sin\theta)\Gamma_{\bar{7}}\} \end{pmatrix}$$

where

$$\Gamma_{\bar{0}} = -\Gamma^{\bar{0}}$$
$$\Gamma_{\bar{i}} = \Gamma^{\bar{i}}, \qquad i = 1, 2....9 \tag{7.1.24}$$

Unlike the D3 brane gamma matrices Γ_α which are the same for both backgrounds, the spin connections ${\Omega_\alpha}^{\bar{a}\bar{b}}$ turn out to be different for the MN and MNa backgrounds. Combining eqs 7.1.16 with the frame fields of MN and MNa backgrouds respectively, all nonzero elements of the spin connections have been found to be the following

$$\begin{aligned} MN: \qquad & {\Omega_\theta}^{\bar{4}\bar{9}} = \frac{1}{6}(5 + cos2\psi_0), \quad {\Omega_\theta}^{\bar{5}\bar{9}} = \frac{1}{6}sin2\psi_0 \\ & {\Omega_\theta}^{\bar{6}\bar{8}} = \frac{1}{2}sin2\psi_0, \qquad {\Omega_\theta}^{\bar{7}\bar{8}} = -cos^2\psi_0 \\ & {\Omega_\phi}^{\bar{4}\bar{9}} = -\frac{1}{6}sin2\psi_0 sin\theta, \quad {\Omega_\phi}^{\bar{5}\bar{9}} = \frac{1}{6}(5 + cos2\psi_0)sin\theta \\ & {\Omega_\theta}^{\bar{6}\bar{8}} = cos^2\psi_0 sin\theta, \quad {\Omega_\phi}^{\bar{7}\bar{8}} = \frac{1}{2}sin2\psi_0 sin\theta \\ & {\Omega_\phi}^{\bar{4}\bar{5}} = {\Omega_\phi}^{\bar{6}\bar{7}} = -cos\theta \end{aligned} \tag{7.1.25}$$

and

$$
\begin{aligned}
MNa: \qquad & \Omega_\theta{}^{\bar{4}\bar{9}} = \frac{11}{12} sin\psi_0, \quad \Omega_\theta{}^{\bar{5}\bar{9}} = \frac{1}{12} cos\psi_0 \\
& \Omega_\theta{}^{\bar{6}\bar{8}} = \frac{1}{2} sin2\psi_0, \quad \Omega_\theta{}^{\bar{7}\bar{8}} = -cos^2\psi_0 \\
& \Omega_\theta{}^{\bar{4}\bar{5}} = \frac{1}{24} cot\theta(cot\psi_0 - 12 sin2\psi_0) \\
& \Omega_\phi{}^{\bar{4}\bar{9}} = -\frac{1}{12} cos\psi_0 sin\theta, \quad \Omega_\phi{}^{\bar{5}\bar{9}} = \frac{11}{12} sin\psi_0 sin\theta \\
& \Omega_\phi{}^{\bar{6}\bar{8}} = cos^2\psi_0 sin\theta, \quad \Omega_\phi{}^{\bar{7}\bar{8}} = \frac{1}{2} sin2\psi_0 sin\theta \\
& \Omega_\phi{}^{\bar{6}\bar{7}} = -cos\theta, \quad \Omega_\phi{}^{\bar{4}\bar{5}} = \frac{1}{24} cos\theta(-13 + 12 cos2\psi_0) \qquad (7.1.26)
\end{aligned}
$$

Notice that the spin connection $\Omega_\alpha{}^{\bar{a}\bar{b}}$ is antisymmetric with respect to the top two indices $\bar{a}$ and $\bar{b}$. Therefore, all the elements one gets from switching the top two indices are nonzero as well even though they are not listed.

The inverse of $\check{\Gamma}_{D_3}$ in the last equation of eqs. 7.1.20 is a 32 by 32 diagonal matrix $\check{\Gamma}^{-1}_{D_3}$ with all diagonal elements found to be the following and all off-diagonal elements vanish

$$
\begin{aligned}
\check{\Gamma}^{-1}_{nn} &= \frac{i\sqrt{e^{2\Phi_0} - 4\pi^2\alpha'^2 F_{tx}^2}}{e^{\Phi_0} + 2\pi\alpha' F_{tx}}, \qquad n = 4i+1, \quad i \in [0,7] \\
\check{\Gamma}^{-1}_{nn} &= \frac{i\sqrt{e^{2\Phi_0} - 4\pi^2\alpha'^2 F_{tx}^2}}{e^{\Phi_0} - 2\pi\alpha' F_{tx}}, \qquad n = 4i+2, \quad i \in [0,7] \\
\check{\Gamma}^{-1}_{nn} &= -\frac{i\sqrt{e^{2\Phi_0} - 4\pi^2\alpha'^2 F_{tx}^2}}{e^{\Phi_0} - 2\pi\alpha' F_{tx}}, \qquad n = 4i+3, \quad i \in [0,7] \\
\check{\Gamma}^{-1}_{nn} &= -\frac{i\sqrt{e^{2\Phi_0} - 4\pi^2\alpha'^2 F_{tx}^2}}{e^{\Phi_0} + 2\pi\alpha' F_{tx}}, \qquad n = 4i, \quad i \in [1,8] \qquad (7.1.27)
\end{aligned}
$$

Now with all the ingredients found and put together for the Fermionic action 7.1.21, we can write and start solving the D3 brane Dirac equation:

$$
[(M_0^{-1})^{\alpha\beta}\Gamma_\alpha\partial_\beta + M_1 + M_2 + M_3]\delta\Theta = 0 \qquad (7.1.28)
$$

where $\delta\Theta = \Theta$ since the fermion fields are zero at the classical level.

7.2 Solution to the Dirac Equation

We try the following ansatz for 7.1.28

$$\delta\Theta = \int dp d\omega \sum_{l,m} e^{i(px-\omega t)} \tilde{\tilde{\Theta}}_{l,m}(p,\omega) \cdot \Phi_{l,m}(\theta,\phi) \tag{7.2.1}$$

where $\Phi_{l,m}(\theta,\phi)$, as an arbitrary function of θ and ϕ, is a 32 component complex spinor, and $\tilde{\tilde{\theta}}_{l,m}(p,\omega)$ is a 32 component spinor of Grassman numbers. The operator $\cdot$ is a commutative product operator between two vectors or spinors that acquires the components of the product by multiplying the corresponding components of the two vectors or spinors involved, defined in the following way

$$A \cdot B = \begin{pmatrix} A^1 \\ A^2 \\ . \\ . \\ . \\ A^M \end{pmatrix} \cdot \begin{pmatrix} B^1 \\ B^2 \\ . \\ . \\ . \\ B^M \end{pmatrix} \equiv \begin{pmatrix} A^1 B^1 \\ A^2 B^2 \\ . \\ . \\ . \\ A^M B^M \end{pmatrix}$$

Notice that due to the differences in the spin connections between the Maldacena-Nunez and the Maldacena-Nastase backgrounds as eq. 7.1.25 and eq. 7.1.26 have shown, M1 in the Dirac equation will be different for MN and MNa. Therefore, we will have to solve the Dirac equation separately for M-N and M-Na. First, we will solve the Dirac equation for the M-N case, during which we will show detailed steps. Then the rsults for the M-Na case will be shown briefly since the procedures are going to be the same. There should be no fundamental differences expected. Plugging the ansatz 7.2.1 into eq. 7.1.28, the D3 brane Dirac equation can be simplified and rewritten as

$$\mathcal{H}_f \tilde{\tilde{\Theta}}_{l,m}(p,\omega) = \omega \tilde{\tilde{\Theta}}_{l,m}(p,\omega) \tag{7.2.2}$$

where the $\Phi_{l,m}(\theta,\phi)$ dependence has been absorbed in the operator $\mathcal{H}_f$ which has been found to take the following form

$$\mathcal{H}_f = \begin{pmatrix} \mathcal{H}_1 & 0 & 0 & 0 \\ 0 & \mathcal{H}_2 & 0 & 0 \\ 0 & 0 & \mathcal{H}_3 & 0 \\ 0 & 0 & 0 & \mathcal{H}_4 \end{pmatrix}$$

where each block represents an eight by eight matrix so that $\mathcal{H}_f$ is a 32 by 32 matrix. The symplicity of $\mathcal{H}_f$ is a direct result of the symmetries and the simplicities that the matrices $\Gamma_\alpha, M1, M3, M3$ in the Dirac equation have. It turns out that $\mathcal{H}_1 = \mathcal{H}_3$ and $\mathcal{H}_2 = \mathcal{H}_4$. Hence, Eq. 7.2.2 can be further simplified as two separate eigenvalue equations of $\mathcal{H}^1$ and $\mathcal{H}^2$

$$\begin{aligned} \mathcal{H}_1 \tilde{\tilde{\Theta}}_1(p,\omega) &= \omega \tilde{\tilde{\Theta}}_1(p,\omega) \\ \mathcal{H}_2 \tilde{\tilde{\Theta}}_2(p,\omega) &= \omega \tilde{\tilde{\Theta}}_2(p,\omega) \end{aligned} \tag{7.2.3}$$

The two operators $\mathcal{H}_1$ and $\mathcal{H}_2$ are eight by eight matrices that act on eight-component spinors $\tilde{\tilde{\Theta}}_1$ and $\tilde{\tilde{\Theta}}_2$, respectively. Both $\mathcal{H}_1$ and $\mathcal{H}_2$ have been found. However, they are too large to display. As it was mentioned before, $\mathcal{H}_1$ and $\mathcal{H}_2$ have absorbed $\Phi_i(\theta,\phi)$. Some of their components have θ,ϕ dependence as a result. For the purpose of explaining future steps, either $\mathcal{H}_1$ or $\mathcal{H}_2$ needs to be displayed and we chose to display submatrices of $\mathcal{H}_1$ in the following fashion: the first two columns $\mathcal{H}_{1ij(i=1...8,\ j=1,2)}$, the second two columns $\mathcal{H}_{1ij(i=1...8,\ j=3,4)}$, the third two columns $\mathcal{H}_{1ij(i=1...8,\ j=5,6)}$ and the last two columns $\mathcal{H}_{1ij(i=1...8,\ j=7,8)}$ while $\mathcal{H}_2$, which has a similar structure, is not displayed. The next couple of pages will only dispaly the four submatrices of $\mathcal{H}_1$ in the above mentioned order without repeating the names due to the width restriction.

$$\begin{pmatrix} -p & \frac{\text{c1d}\left(\cot(\theta)\Phi(2)(\theta,\phi)+i\csc(\theta)\Phi(2)^{(0,1)}(\theta,\phi)+\Phi(2)^{(1,0)}(\theta,\phi)\right)}{\text{c2b}\Phi(1)(\theta,\phi)} \\ -\frac{\text{c2c}\left(\csc(\theta)\Phi(1)^{(0,1)}(\theta,\phi)+i\Phi(1)^{(1,0)}(\theta,\phi)\right)}{\text{c1c}\Phi(2)(\theta,\phi)} & p \\ -\frac{\text{c1b}\Phi(1)(\theta,\phi)}{\text{c1c}\Phi(3)(\theta,\phi)} & 0 \\ 0 & \frac{\text{c2a}\Phi(2)(\theta,\phi)}{\text{c2b}\Phi(4)(\theta,\phi)} \\ 0 & 0 \\ \frac{\text{c8b}\Phi(1)(\theta,\phi)}{\text{c1c}\Phi(6)(\theta,\phi)} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\text{c3b}\Phi(3)(\theta,\phi)}{\text{c2b}\Phi(1)(\theta,\phi)} & 0 \\ 0 & \frac{\text{c4a}\Phi(4)(\theta,\phi)}{\text{c1c}\Phi(2)(\theta,\phi)} \\ p & \frac{\text{c1d}\left(\cot(\theta)\Phi(4)(\theta,\phi)+i\csc(\theta)\Phi(4)^{(0,1)}(\theta,\phi)+\Phi(4)^{(1,0)}(\theta,\phi)\right)}{\text{c1c}\Phi(3)(\theta,\phi)} \\ -\frac{\text{c2c}\left(\csc(\theta)\Phi(3)^{(0,1)}(\theta,\phi)+i\Phi(3)^{(1,0)}(\theta,\phi)\right)}{\text{c2b}\Phi(4)(\theta,\phi)} & -p \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{\text{c6b}\Phi(3)(\theta,\phi)}{\text{c2b}\Phi(8)(\theta,\phi)} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\frac{\text{c3a}\Phi(6)(\theta,\phi)}{\text{c2b}\Phi(1)(\theta,\phi)} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -p & \frac{\text{c1d}\left(i\csc(\theta)\Phi(6)^{(0,1)}(\theta,\phi)+\Phi(6)^{(1,0)}(\theta,\phi)\right)}{\text{c2b}\Phi(5)(\theta,\phi)} \\ -\frac{i\text{c2c}\cot(\theta)\Phi(5)(\theta,\phi)+\text{c2c}\csc(\theta)\Phi(5)^{(0,1)}(\theta,\phi)+i\text{c2c}\Phi(5)^{(1,0)}(\theta,\phi)}{\text{c1c}\Phi(6)(\theta,\phi)} & p \\ \frac{\text{c4a}\Phi(5)(\theta,\phi)}{\text{c1c}\Phi(7)(\theta,\phi)} & 0 \\ 0 & -\frac{\text{c3b}\Phi(6)(\theta,\phi)}{\text{c2b}\Phi(8)(\theta,\phi)} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -\frac{\text{c1a}\Phi(8)(\theta,\phi)}{\text{c1c}\Phi(3)(\theta,\phi)} \\ 0 & 0 \\ \frac{\text{c2a}\Phi(7)(\theta,\phi)}{\text{c2b}\Phi(5)(\theta,\phi)} & 0 \\ 0 & -\frac{\text{c1b}\Phi(8)(\theta,\phi)}{\text{c1c}\Phi(6)(\theta,\phi)} \\ p & \frac{\text{c1d}\left(i\csc(\theta)\Phi(8)^{(0,1)}(\theta,\phi)+\Phi(8)^{(1,0)}(\theta,\phi)\right)}{\text{c1c}\Phi(7)(\theta,\phi)} \\ -\frac{i\text{c2c}\cot(\theta)\Phi(7)(\theta,\phi)+\text{c2c}\csc(\theta)\Phi(7)^{(0,1)}(\theta,\phi)+i\text{c2c}\Phi(7)^{(1,0)}(\theta,\phi)}{\text{c2b}\Phi(8)(\theta,\phi)} & -p \end{pmatrix}$$

where $\Phi(1)(\theta,\phi)$, $\Phi(2)(\theta,\phi)$,..., $\Phi(8)(\theta,\phi)$ are the first eight components of the 32 component spinor $\Phi_{l,m}(\theta,\phi)$ in eq. 7.2.1 and $\Phi(i)^{(1,0)} = \partial_\theta \Phi(i)$ and $\Phi(i)^{(0,1)} = \partial_\phi \Phi(i)$ and the coefficient constants are as follows

$$
\begin{aligned}
&c1a = \\
&\frac{e^{-\Phi 0/2}\left(-3i\cot(\psi 0)\sqrt{e^{2\Phi 0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}+2\sqrt{e^{2\Phi 0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}-2i\pi\mathrm{Ftx}\alpha'\right)}{3\sqrt{N}\sqrt{\alpha'}\sqrt{e^{2\Phi 0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}} \\
&+\frac{e^{-\Phi 0/2}\left(ie^{\Phi 0}-i\right)}{3\sqrt{N}\sqrt{\alpha'}\sqrt{e^{2\Phi 0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}} \\
&c1b = \\
&\frac{ie^{-3\Phi 0/2}\left(e^{\Phi 0}\left(3\cot(\psi 0)\sqrt{e^{2\Phi 0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}-e^{\Phi 0}-1\right)+2\pi\mathrm{Ftx}\left(e^{\Phi 0}+2\right)\alpha'\right)}{3\sqrt{N}\sqrt{\alpha'}\sqrt{e^{2\Phi 0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}} \\
&c1c = \frac{e^{\Phi 0/2}}{2\pi\mathrm{Ftx}\alpha'+e^{\Phi 0}} \\
&c1d = \frac{e^{-\Phi 0/2}(1-i\cot(\psi 0))}{\sqrt{N}\sqrt{\alpha'}} \\
&c2a = \\
&\frac{ie^{-3\Phi 0/2}\left(e^{\Phi 0}\left(3\cot(\psi 0)\sqrt{e^{2\Phi 0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}+2e^{\Phi 0}-1\right)+2\pi\mathrm{Ftx}\left(2e^{\Phi 0}+1\right)\alpha'\right)}{3\sqrt{N}\sqrt{\alpha'}\sqrt{e^{2\Phi 0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}} \\
&c2b = \frac{e^{\Phi 0/2}}{e^{\Phi 0}-2\pi\mathrm{Ftx}\alpha'} \\
&c2c = \frac{e^{-\Phi 0/2}(-i+\cot(\psi 0))}{\sqrt{N}\sqrt{\alpha'}}
\end{aligned}
\tag{7.2.4}
$$

$$c3a =$$

$$\frac{ie^{-\Phi0/2}\left(3\cot(\psi0)\sqrt{e^{2\Phi0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}+2i\sqrt{e^{2\Phi0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}\right)}{3\sqrt{N}\sqrt{\alpha'}\sqrt{e^{2\Phi0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}}$$

$$+\frac{ie^{-\Phi0/2}\left(2\pi\mathrm{Ftx}\alpha'+e^{\Phi0}-1\right)}{3\sqrt{N}\sqrt{\alpha'}\sqrt{e^{2\Phi0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}}$$

$$c3b =$$

$$\frac{ie^{-3\Phi0/2}\left(e^{\Phi0}\left(3\cot(\psi0)\sqrt{e^{2\Phi0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}+e^{\Phi0}+1\right)+2\pi\mathrm{Ftx}\left(e^{\Phi0}+2\right)\alpha'\right)}{3\sqrt{N}\sqrt{\alpha'}\sqrt{e^{2\Phi0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}}$$

$$c4a =$$

$$\frac{ie^{-3\Phi0/2}\left(e^{\Phi0}\left(3\cot(\psi0)\sqrt{e^{2\Phi0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}-2e^{\Phi0}+1\right)+2\pi\mathrm{Ftx}\left(2e^{\Phi0}+1\right)\alpha'\right)}{3\sqrt{N}\sqrt{\alpha'}\sqrt{e^{2\Phi0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}} \tag{7.2.5}$$

$$c6b =$$

$$-\frac{ie^{-\Phi0/2}\left(3\cot(\psi0)\sqrt{e^{2\Phi0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}-2i\sqrt{e^{2\Phi0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}\right)}{3\sqrt{N}\sqrt{\alpha'}\sqrt{e^{2\Phi0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}}$$

$$-\frac{ie^{-\Phi0/2}\left(2\pi\mathrm{Ftx}\alpha'+e^{\Phi0}-1\right)}{3\sqrt{N}\sqrt{\alpha'}\sqrt{e^{2\Phi0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}}$$

$$c8b =$$

$$\frac{e^{-\Phi0/2}\left(3i\cot(\psi0)\sqrt{e^{2\Phi0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}+2\sqrt{e^{2\Phi0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}\right)}{3\sqrt{N}\sqrt{\alpha'}\sqrt{e^{2\Phi0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}}$$

$$\frac{e^{-\Phi0/2}\left(2i\pi\mathrm{Ftx}\alpha'-ie^{\Phi0}+i\right)}{3\sqrt{N}\sqrt{\alpha'}\sqrt{e^{2\Phi0}-4\pi^2\mathrm{Ftx}^2\left(\alpha'\right)^2}} \tag{7.2.6}$$

Notice that in eqs. 7.2.3, the RHS $\tilde{\tilde{\Theta}}_1(p,\omega)$ and $\tilde{\tilde{\Theta}}_2(p,\omega)$ do not have θ,ϕ dependence. Therefore, the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ should not have explicit dependence on θ,ϕ either.

Naturally, we should be looking for nontrivial $\Phi(i)(\theta,\phi)$ that make $\mathcal{H}_1$ and $\mathcal{H}_2$ not depend on θ,ϕ. For instance, we can set

$$\begin{aligned}
\mathcal{H}_1(1,2) &= \frac{\text{c1d}\left(\Phi(2)^{(1,0)}(\theta,\phi)+\cot(\theta)\Phi(2)(\theta,\phi)+i\csc(\theta)\Phi(2)^{(0,1)}(\theta,\phi)\right)}{\text{c2b}\Phi(1)(\theta,\phi)} = constant\\
\mathcal{H}_1(2,1) &= -\frac{\text{c2c}\left(\csc(\theta)\Phi(1)^{(0,1)}(\theta,\phi)+i\Phi(1)^{(1,0)}(\theta,\phi)\right)}{\text{c1c}\Phi(2)(\theta,\phi)} = constant\\
\mathcal{H}_1(3,1) &= -\frac{\text{c1b}\Phi(1)(\theta,\phi)}{\text{c1c}\Phi(3)(\theta,\phi)} = constant\\
\mathcal{H}_1(5,6) &= \frac{\text{c1d}\left(\Phi(6)^{(1,0)}(\theta,\phi)+i\csc(\theta)\Phi(6)^{(0,1)}(\theta,\phi)\right)}{\text{c2b}\Phi(5)(\theta,\phi)} = constant, \quad etc.
\end{aligned} \tag{7.2.7}$$

After going through all the nonzero elements of $\mathcal{H}_1$ and setting up equations like 7.2.7, we see a pattern in $\Phi(i)(\theta,\phi)$:

$$\begin{aligned}
&\Phi(1)(\theta,\phi) = \Phi(3)(\theta,\phi) = \Phi(6)(\theta,\phi) = \Phi(8)(\theta,\phi) \equiv Y^{(l,m)}(\theta,\phi)\\
&\Phi(2)(\theta,\phi) = \Phi(4)(\theta,\phi) \equiv Y_-^{(l,m)}(\theta,\phi)\\
&\Phi(5)(\theta,\phi) = \Phi(7)(\theta,\phi) \equiv Y_+^{(l,m)}(\theta,\phi)
\end{aligned} \tag{7.2.8}$$

which satify the following coupled differential equations

$$\begin{aligned}
\mathcal{O}_-^{(1)}Y^{(l,m)} &= \lambda_1 Y_-^{(l,m)}(\theta,\phi)\\
\mathcal{O}_+^{(1)}Y^{(l,m)} &= \lambda_2 Y_+^{(l,m)}(\theta,\phi)\\
\mathcal{O}_-^{(2)}Y_+^{(l,m)} &= \lambda_3 Y^{(l,m)}(\theta,\phi)\\
\mathcal{O}_+^{(2)}Y_-^{(l,m)} &= \lambda_4 Y^{(l,m)}(\theta,\phi)
\end{aligned} \tag{7.2.9}$$

where the operators $\mathcal{O}_\pm^{(i)}$ are defined as

$$\begin{aligned}
\mathcal{O}_\pm^{(1)} &= \partial_\theta \pm icsc\theta\partial_\phi\\
\mathcal{O}_\pm^{(2)} &= \mathcal{O}_\pm^{(1)} + cot\theta
\end{aligned} \tag{7.2.10}$$

Eqs. 7.2.9 remind us the spherical harmonics. In fact

$$\mathcal{O}_-^{(2)}\mathcal{O}_+^{(1)} = \mathcal{O}_+^{(2)}\mathcal{O}_-^{(1)} = \partial_\theta^2 + \frac{1}{sin^2\theta}\partial_\phi^2 + cot\theta\partial_\theta \tag{7.2.11}$$

and the spherical harmonics Y^{lm} are the eigenfunctions of $\mathcal{O}_-^{(2)}\mathcal{O}_+^{(1)}$ and $\mathcal{O}_-^{(2)}\mathcal{O}_+^{(1)}$

$$\begin{aligned}
\mathcal{O}_-^{(2)}\mathcal{O}_+^{(1)}Y^{lm} &= -l(l+1)Y^{lm}\\
\mathcal{O}_+^{(2)}\mathcal{O}_-^{(1)}Y^{lm} &= -l(l+1)Y^{lm}
\end{aligned} \tag{7.2.12}$$

Since

$$\mathcal{O}_+^{(2)}\mathcal{O}_-^{(1)}Y^{(l,m)} = \lambda_1\lambda_4 Y^{(l,m)}$$
$$\mathcal{O}_-^{(2)}\mathcal{O}_+^{(1)}Y^{(l,m)} = \lambda_2\lambda_3 Y^{(l,m)} \tag{7.2.13}$$

it would be very natural and convenient to identify $Y^{(l,m)}$ as the spherical harmonics Y^{lm} and to set $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \lambda_4 = -l(l+1)$. Therefore, eqs. 7.2.9 become

$$Y_-^{(l,m)}(\theta,\phi) = \mathcal{O}_-^{(1)}Y^{(l,m)}$$
$$Y_+^{(l,m)}(\theta,\phi) = \mathcal{O}_+^{(1)}Y^{(l,m)}$$
$$\mathcal{O}_-^{(2)}Y_+^{(l,m)} = -L(L+1)Y^{(l,m)}(\theta,\phi)$$
$$\mathcal{O}_+^{(2)}Y_-^{(l,m)} = -L(L+1)Y^{(l,m)}(\theta,\phi) \tag{7.2.14}$$

From now on, $Y^{(l,m)}$ will be considered to be equivalent to the spherical harmonics Y^{lm} without chanaging notations. Using eqs 7.2.8 and 7.2.14, the operator $\mathcal{H}_1$ will be significantly symplified and be removed of (θ,ϕ) dependence. The operator $\mathcal{H}_2$ has a very similar structure but different elements. For $\mathcal{H}_2$, the identifications are as follows

$$\Phi(10)(\theta,\phi) = \Phi(12)(\theta,\phi) = \Phi(13)(\theta,\phi) = \Phi(15)(\theta,\phi) \equiv Y^{(l,m)}(\theta,\phi)$$
$$\Phi(14)(\theta,\phi) = \Phi(16)(\theta,\phi) \equiv Y_-^{(l,m)}(\theta,\phi)$$
$$\Phi(9)(\theta,\phi) = \Phi(11)(\theta,\phi) \equiv Y_+^{(l,m)}(\theta,\phi) \tag{7.2.15}$$

Now with the help of eqs. 7.2.8, 7.2.15 and 7.2.14, $\mathcal{H}_1$ and $\mathcal{H}_2$ can be simplified as

$$\mathcal{H}_1 = \begin{pmatrix} -p & -\frac{\text{c1d}l(l+1)}{\text{c2b}} & -\frac{\text{c3b}}{\text{c2b}} & 0 & 0 & -\frac{\text{c3a}}{\text{c2b}} & 0 & 0 \\ -\frac{i\text{c2c}}{\text{c1c}} & p & 0 & \frac{\text{c4a}}{\text{c1c}} & 0 & 0 & 0 & 0 \\ -\frac{\text{c1b}}{\text{c1c}} & 0 & p & -\frac{\text{c1d}l(l+1)}{\text{c1c}} & 0 & 0 & 0 & -\frac{\text{c1a}}{\text{c1c}} \\ 0 & \frac{\text{c2a}}{\text{c2b}} & -\frac{i\text{c2c}}{\text{c2b}} & -p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -p & \frac{\text{c1d}}{\text{c2b}} & \frac{\text{c2a}}{\text{c2b}} & 0 \\ \frac{\text{c8b}}{\text{c1c}} & 0 & 0 & 0 & \frac{i\text{c2c}l(l+1)}{\text{c1c}} & p & 0 & -\frac{\text{c1b}}{\text{c1c}} \\ 0 & 0 & 0 & 0 & \frac{\text{c4a}}{\text{c1c}} & 0 & p & \frac{\text{c1d}}{\text{c1c}} \\ 0 & 0 & \frac{\text{c6b}}{\text{c2b}} & 0 & 0 & -\frac{\text{c3b}}{\text{c2b}} & \frac{i\text{c2c}l(l+1)}{\text{c2b}} & -p \end{pmatrix}$$

and

$$\mathcal{H}_2 = \begin{pmatrix} -p & \frac{\text{c1d}}{\text{c2b}} & -\frac{\text{c2a}}{\text{c2b}} & 0 & 0 & 0 & 0 & 0 \\ \frac{i\text{c2c}l(l+1)}{\text{c1c}} & p & 0 & \frac{\text{c1b}}{\text{c1c}} & -\frac{\text{c8b}}{\text{c1c}} & 0 & 0 & 0 \\ -\frac{\text{c4a}}{\text{c1c}} & 0 & p & \frac{\text{c1d}}{\text{c1c}} & 0 & 0 & 0 & 0 \\ 0 & \frac{\text{c3b}}{\text{c2b}} & \frac{i\text{c2c}l(l+1)}{\text{c2b}} & -p & 0 & 0 & -\frac{\text{c6b}}{\text{c2b}} & 0 \\ 0 & \frac{\text{c3a}}{\text{c2b}} & 0 & 0 & -p & -\frac{\text{c1d}l(l+1)}{\text{c2b}} & \frac{\text{c3b}}{\text{c2b}} & 0 \\ 0 & 0 & 0 & 0 & -\frac{i\text{c2c}}{\text{c1c}} & p & 0 & -\frac{\text{c4a}}{\text{c1c}} \\ 0 & 0 & 0 & \frac{\text{c1a}}{\text{c1c}} & \frac{\text{c1b}}{\text{c1c}} & 0 & p & -\frac{\text{c1d}l(l+1)}{\text{c1c}} \\ 0 & 0 & 0 & 0 & 0 & -\frac{\text{c2a}}{\text{c2b}} & -\frac{i\text{c2c}}{\text{c2b}} & -p \end{pmatrix}$$

These two matrices $\mathcal{H}_1$ and $\mathcal{H}_1$ turn out to have exactly the same eight eigenvalues. The eigenvalues are the solutions to the following equation

$$Det(\mathcal{H}_1 - \omega) = \mathrm{c0} + \mathrm{c2}\omega^2 + \mathrm{c4}\omega^4 + \mathrm{c6}\omega^6 + \omega^8 = 0 \tag{7.2.16}$$

which yield the following solutions

$$\omega = \pm\frac{1}{2}\sqrt{-c6(p,l) - \sqrt{e8(p,l)} \pm \sqrt{e9(p,l)}}, \;\; \pm\frac{1}{2}\sqrt{-c6(p,l) + \sqrt{e8(p,l)} \pm \sqrt{e10(p,l)}} \tag{7.2.17}$$

where the constants are

$$\begin{aligned}
c6 =& \frac{\mathrm{c1ac6b}}{\mathrm{c1cc2b}} - \frac{\mathrm{2c1bc3b}}{\mathrm{c1cc2b}} - \frac{4i\mathrm{c1dc2c}l^2}{\mathrm{c1cc2b}} - \frac{4i\mathrm{c1dc2c}l}{\mathrm{c1cc2b}} \\
&- \frac{\mathrm{2c2ac4a}}{\mathrm{c1cc2b}} + \frac{\mathrm{c3ac8b}}{\mathrm{c1cc2b}} - 4p^2 \\
e5 =& \ \mathrm{12c0} - \mathrm{3c2c6} + \mathrm{c4}^2 \\
e6 =& -\mathrm{72c0c4} + \mathrm{27c0c6}^2 + \mathrm{27c2}^2 - \mathrm{9c2c4c6} + \mathrm{2c4}^3 \\
e7 =& \sqrt{\mathrm{e6}^2 - \mathrm{4e5}^3} + \mathrm{e6} \\
e8 =& \ 4\left(-\frac{\mathrm{2c4}}{3} + \frac{\mathrm{c6}^2}{4} + \frac{\sqrt[3]{2}\mathrm{e5}}{3\sqrt[3]{\mathrm{e7}}} + \frac{\sqrt[3]{\mathrm{e7}}}{3\sqrt[3]{2}}\right) \\
e9 =& \ 4\left(-\frac{-\mathrm{8c2} + \mathrm{4c4c6} - \mathrm{c6}^3}{2\sqrt{\mathrm{e8}}} - \frac{\mathrm{4c4}}{3} + \frac{\mathrm{c6}^2}{2} - \frac{\sqrt[3]{2}\mathrm{e5}}{3\sqrt[3]{\mathrm{e7}}} - \frac{\sqrt[3]{\mathrm{e7}}}{3\sqrt[3]{2}}\right) \\
e10 =& \ 4\left(\frac{-\mathrm{8c2} + \mathrm{4c4c6} - \mathrm{c6}^3}{2\sqrt{\mathrm{e8}}} - \frac{\mathrm{4c4}}{3} + \frac{\mathrm{c6}^2}{2} - \frac{\sqrt[3]{2}\mathrm{e5}}{3\sqrt[3]{\mathrm{e7}}} - \frac{\sqrt[3]{\mathrm{e7}}}{3\sqrt[3]{2}}\right) \\
c4 =& -\frac{2i\mathrm{c1ac1dc2cc6b}l^2}{\mathrm{c1c^2c2b^2}} - \frac{2i\mathrm{c1ac1dc2cc6b}l}{\mathrm{c1c^2c2b^2}} - \frac{\mathrm{2c1ac2ac4ac6b}}{\mathrm{c1c^2c2b^2}} + \frac{\mathrm{c1ac3ac6bc8b}}{\mathrm{c1c^2c2b^2}} \\
&+ \frac{\mathrm{c1ac3b^2c8b}}{\mathrm{c1c^2c2b^2}} - \frac{\mathrm{3c1ac6b}p^2}{\mathrm{c1cc2b}} + \frac{\mathrm{c1b^2c3ac6b}}{\mathrm{c1c^2c2b^2}} + \frac{\mathrm{c1b^2c3b^2}}{\mathrm{c1c^2c2b^2}} + \frac{4i\mathrm{c1bc1dc2cc3b}l^2}{\mathrm{c1c^2c2b^2}} \\
&+ \frac{4i\mathrm{c1bc1dc2cc3b}l}{\mathrm{c1c^2c2b^2}} + \frac{2i\mathrm{c1bc1dc2cc4a}l^2}{\mathrm{c1c^2c2b^2}} + \frac{2i\mathrm{c1bc1dc2cc4a}l}{\mathrm{c1c^2c2b^2}} + \frac{\mathrm{4c1bc2ac3bc4a}}{\mathrm{c1c^2c2b^2}} \\
&+ \frac{\mathrm{6c1bc3b}p^2}{\mathrm{c1cc2b}} - \frac{\mathrm{6c1d^2c2c^2}l^4}{\mathrm{c1c^2c2b^2}} - \frac{\mathrm{12c1d^2c2c^2}l^3}{\mathrm{c1c^2c2b^2}} - \frac{\mathrm{6c1d^2c2c^2}l^2}{\mathrm{c1c^2c2b^2}} + \frac{2i\mathrm{c1dc2ac2cc3b}l^2}{\mathrm{c1c^2c2b^2}} \\
&+ \frac{2i\mathrm{c1dc2ac2cc3b}l}{\mathrm{c1c^2c2b^2}} + \frac{4i\mathrm{c1dc2ac2cc4a}l^2}{\mathrm{c1c^2c2b^2}} + \frac{4i\mathrm{c1dc2ac2cc4a}l}{\mathrm{c1c^2c2b^2}} - \frac{2i\mathrm{c1dc2cc3ac8b}l^2}{\mathrm{c1c^2c2b^2}} \\
&- \frac{2i\mathrm{c1dc2cc3ac8b}l}{\mathrm{c1c^2c2b^2}} + \frac{\mathrm{c2a^2c4a^2}}{\mathrm{c1c^2c2b^2}} - \frac{\mathrm{2c2ac3ac4ac8b}}{\mathrm{c1c^2c2b^2}} + \frac{12i\mathrm{c1dc2c}l^2p^2}{\mathrm{c1cc2b}} \\
&+ \frac{12i\mathrm{c1dc2c}lp^2}{\mathrm{c1cc2b}} + \frac{\mathrm{6c2ac4a}p^2}{\mathrm{c1cc2b}} - \frac{\mathrm{3c3ac8b}p^2}{\mathrm{c1cc2b}} + 6p^4
\end{aligned} \tag{7.2.18}$$

where

$c2 =$

$$\frac{4ic1d^3c2c^3l^6}{c1c^3c2b^3} + \frac{12ic1d^3c2c^3l^5}{c1c^3c2b^3} + \frac{12ic1d^3c2c^3l^4}{c1c^3c2b^3} + \frac{12c1d^2c2c^2p^2l^4}{c1c^2c2b^2} + \frac{2c1bc1d^2c2c^2c3bl^4}{c1c^3c2b^3}$$
$$+ \frac{4c1d^2c2ac2c^2c3bl^4}{c1c^3c2b^3} + \frac{4c1bc1d^2c2c^2c4al^4}{c1c^3c2b^3} + \frac{2c1d^2c2ac2c^2c4al^4}{c1c^3c2b^3} - \frac{c1ac1d^2c2c^2c6bl^4}{c1c^3c2b^3}$$
$$- \frac{c1d^2c2c^2c3ac8bl^4}{c1c^3c2b^3} + \frac{4ic1d^3c2c^3l^3}{c1c^3c2b^3} + \frac{24c1d^2c2c^2p^2l^3}{c1c^2c2b^2} + \frac{4c1bc1d^2c2c^2c3bl^3}{c1c^3c2b^3}$$
$$+ \frac{8c1d^2c2ac2c^2c3bl^3}{c1c^3c2b^3} + \frac{8c1bc1d^2c2c^2c4al^3}{c1c^3c2b^3} + \frac{4c1d^2c2ac2c^2c4al^3}{c1c^3c2b^3} - \frac{2c1ac1d^2c2c^2c6bl^3}{c1c^3c2b^3}$$
$$- \frac{2c1d^2c2c^2c3ac8bl^3}{c1c^3c2b^3} - \frac{12ic1dc2cp^4l^2}{c1cc2b} - \frac{2ic1bc1dc2ac2cc3b^2l^2}{c1c^3c2b^3} - \frac{2ic1bc1dc2ac2cc4a^2l^2}{c1c^3c2b^3}$$
$$+ \frac{12c1d^2c2c^2p^2l^2}{c1c^2c2b^2} - \frac{8ic1bc1dc2cc3bp^2l^2}{c1c^2c2b^2} - \frac{4ic1dc2ac2cc3bp^2l^2}{c1c^2c2b^2} - \frac{4ic1bc1dc2cc4ap^2l^2}{c1c^2c2b^2}$$
$$- \frac{8ic1dc2ac2cc4ap^2l^2}{c1c^2c2b^2} + \frac{4ic1ac1dc2cc6bp^2l^2}{c1c^2c2b^2} + \frac{4ic1dc2cc3ac8bp^2l^2}{c1c^2c2b^2} + \frac{2c1bc1d^2c2c^2c3bl^2}{c1c^3c2b^3}$$
$$+ \frac{4c1d^2c2ac2c^2c3bl^2}{c1c^3c2b^3} + \frac{4c1bc1d^2c2c^2c4al^2}{c1c^3c2b^3} + \frac{2c1d^2c2ac2c^2c4al^2}{c1c^3c2b^3} - \frac{2ic1dc2a^2c2cc3bc4al^2}{c1c^3c2b^3}$$
$$- \frac{2ic1b^2c1dc2cc3bc4al^2}{c1c^3c2b^3} - \frac{4ic1bc1dc2ac2cc3bc4al^2}{c1c^3c2b^3} - \frac{c1ac1d^2c2c^2c6bl^2}{c1c^3c2b^3}$$
$$- \frac{2ic1bc1dc2ac2cc3ac6bl^2}{c1c^3c2b^3} + \frac{2ic1ac1dc2ac2cc4ac6bl^2}{c1c^3c2b^3} - \frac{c1d^2c2c^2c3ac8bl^2}{c1c^3c2b^3}$$
$$+ \frac{2ic1dc2ac2cc3ac4ac8bl^2}{c1c^3c2b^3} - \frac{2ic1ac1dc2cc3bc4ac8bl^2}{c1c^3c2b^3} - \frac{2ic1bc1dc2ac2cc3b^2l}{c1c^3c2b^3}$$
$$- \frac{2ic1bc1dc2ac2cc4a^2l}{c1c^3c2b^3} - \frac{8ic1bc1dc2cc3bp^2l}{c1c^2c2b^2} - \frac{4ic1dc2ac2cc3bp^2l}{c1c^2c2b^2} - \frac{4ic1bc1dc2cc4ap^2l}{c1c^2c2b^2}$$
$$- \frac{8ic1dc2ac2cc4ap^2l}{c1c^2c2b^2} + \frac{4ic1ac1dc2cc6bp^2l}{c1c^2c2b^2} + \frac{4ic1dc2cc3ac8bp^2l}{c1c^2c2b^2} - \frac{2ic1dc2a^2c2cc3bc4al}{c1c^3c2b^3}$$
$$- \frac{2ic1b^2c1dc2cc3bc4al}{c1c^3c2b^3} - \frac{4ic1bc1dc2ac2cc3bc4al}{c1c^3c2b^3} - \frac{2ic1bc1dc2ac2cc3ac6bl}{c1c^3c2b^3}$$
$$+ \frac{2ic1ac1dc2ac2cc4ac6bl}{c1c^3c2b^3} + \frac{2ic1dc2ac2cc3ac4ac8bl}{c1c^3c2b^3} - \frac{2ic1ac1dc2cc3bc4ac8bl}{c1c^3c2b^3} - 4p^6$$
$$- \frac{6c1bc3bp^4}{c1cc2b} - \frac{6c2ac4ap^4}{c1cc2b} + \frac{3c1ac6bp^4}{c1cc2b} + \frac{3c3ac8bp^4}{c1cc2b} - \frac{2c1bc2a^2c3bc4a^2}{c1c^3c2b^3}$$
$$- \frac{2c1b^2c3b^2p^2}{c1c^2c2b^2} - \frac{2c2a^2c4a^2p^2}{c1c^2c2b^2} - \frac{8c1bc2ac3bc4ap^2}{c1c^2c2b^2} - \frac{2c1b^2c3ac6bp^2}{c1c^2c2b^2} + \frac{4c1ac2ac4ac6bp^2}{c1c^2c2b^2}$$
$$- \frac{2c1ac3b^2c8bp^2}{c1c^2c2b^2} + \frac{4c2ac3ac4ac8bp^2}{c1c^2c2b^2} - \frac{2c1ac3ac6bc8bp^2}{c1c^2c2b^2} - \frac{2c1b^2c2ac3b^2c4a}{c1c^3c2b^3}$$
$$+ \frac{c1ac2a^2c4a^2c6b}{c1c^3c2b^3} - \frac{2c1b^2c2ac3ac4ac6b}{c1c^3c2b^3} + \frac{c2a^2c3ac4a^2c8b}{c1c^3c2b^3} - \frac{2c1ac2ac3b^2c4ac8b}{c1c^3c2b^3}$$
$$- \frac{2c1ac2ac3ac4ac6bc8b}{c1c^3c2b^3} - \frac{12ic1dc2cp^4l}{c1cc2b} \tag{7.2.19}$$

and

$c0 =$

$$
\begin{aligned}
&\frac{\mathrm{c1d}^4\mathrm{c2c}^4 l^8}{\mathrm{c1c}^4\mathrm{c2b}^4}+\frac{4\mathrm{c1d}^4\mathrm{c2c}^4 l^7}{\mathrm{c1c}^4\mathrm{c2b}^4}+\frac{6\mathrm{c1d}^4\mathrm{c2c}^4 l^6}{\mathrm{c1c}^4\mathrm{c2b}^4}-\frac{4i\mathrm{c1d}^3\mathrm{c2c}^3 p^2 l^6}{\mathrm{c1c}^3\mathrm{c2b}^3}-\frac{2i\mathrm{c1d}^3\mathrm{c2ac2c}^3\mathrm{c3b}l^6}{\mathrm{c1c}^4\mathrm{c2b}^4}\\
&-\frac{2i\mathrm{c1bc1d}^3\mathrm{c2c}^3\mathrm{c4a}l^6}{\mathrm{c1c}^4\mathrm{c2b}^4}+\frac{4\mathrm{c1d}^4\mathrm{c2c}^4 l^5}{\mathrm{c1c}^4\mathrm{c2b}^4}-\frac{12i\mathrm{c1d}^3\mathrm{c2c}^3 p^2 l^5}{\mathrm{c1c}^3\mathrm{c2b}^3}-\frac{6i\mathrm{c1d}^3\mathrm{c2ac2c}^3\mathrm{c3b}l^5}{\mathrm{c1c}^4\mathrm{c2b}^4}\\
&-\frac{6i\mathrm{c1bc1d}^3\mathrm{c2c}^3\mathrm{c4a}l^5}{\mathrm{c1c}^4\mathrm{c2b}^4}+\frac{\mathrm{c1d}^4\mathrm{c2c}^4 l^4}{\mathrm{c1c}^4\mathrm{c2b}^4}-\frac{6\mathrm{c1d}^2\mathrm{c2c}^2 p^4 l^4}{\mathrm{c1c}^2\mathrm{c2b}^2}-\frac{\mathrm{c1d}^2\mathrm{c2a}^2\mathrm{c2c}^2\mathrm{c3b}^2 l^4}{\mathrm{c1c}^4\mathrm{c2b}^4}\\
&-\frac{\mathrm{c1b}^2\mathrm{c1d}^2\mathrm{c2c}^2\mathrm{c4a}^2 l^4}{\mathrm{c1c}^4\mathrm{c2b}^4}-\frac{12i\mathrm{c1d}^3\mathrm{c2c}^3 p^2 l^4}{\mathrm{c1c}^3\mathrm{c2b}^3}-\frac{2\mathrm{c1bc1d}^2\mathrm{c2c}^2\mathrm{c3b}p^2 l^4}{\mathrm{c1c}^3\mathrm{c2b}^3}\\
&-\frac{4\mathrm{c1bc1d}^2\mathrm{c2c}^2\mathrm{c4a}p^2 l^4}{\mathrm{c1c}^3\mathrm{c2b}^3}-\frac{2\mathrm{c1d}^2\mathrm{c2ac2c}^2\mathrm{c4a}p^2 l^4}{\mathrm{c1c}^3\mathrm{c2b}^3}+\frac{\mathrm{c1ac1d}^2\mathrm{c2c}^2\mathrm{c6b}p^2 l^4}{\mathrm{c1c}^3\mathrm{c2b}^3}\\
&-\frac{6i\mathrm{c1d}^3\mathrm{c2ac2c}^3\mathrm{c3b}l^4}{\mathrm{c1c}^4\mathrm{c2b}^4}-\frac{6i\mathrm{c1bc1d}^3\mathrm{c2c}^3\mathrm{c4a}l^4}{\mathrm{c1c}^4\mathrm{c2b}^4}-\frac{4\mathrm{c1bc1d}^2\mathrm{c2ac2c}^2\mathrm{c3bc4a}l^4}{\mathrm{c1c}^4\mathrm{c2b}^4}\\
&-\frac{\mathrm{c1d}^2\mathrm{c2a}^2\mathrm{c2c}^2\mathrm{c3ac6b}l^4}{\mathrm{c1c}^4\mathrm{c2b}^4}-\frac{\mathrm{c1ac1d}^2\mathrm{c2c}^2\mathrm{c4a}^2\mathrm{c8b}l^4}{\mathrm{c1c}^4\mathrm{c2b}^4}-\frac{12\mathrm{c1d}^2\mathrm{c2c}^2 p^4 l^3}{\mathrm{c1c}^2\mathrm{c2b}^2}\\
&-\frac{2\mathrm{c1b}^2\mathrm{c1d}^2\mathrm{c2c}^2\mathrm{c4a}^2 l^3}{\mathrm{c1c}^4\mathrm{c2b}^4}-\frac{4i\mathrm{c1d}^3\mathrm{c2c}^3 p^2 l^3}{\mathrm{c1c}^3\mathrm{c2b}^3}-\frac{4\mathrm{c1bc1d}^2\mathrm{c2c}^2\mathrm{c3b}p^2 l^3}{\mathrm{c1c}^3\mathrm{c2b}^3}\\
&-\frac{8\mathrm{c1bc1d}^2\mathrm{c2c}^2\mathrm{c4a}p^2 l^3}{\mathrm{c1c}^3\mathrm{c2b}^3}-\frac{4\mathrm{c1d}^2\mathrm{c2ac2c}^2\mathrm{c4a}p^2 l^3}{\mathrm{c1c}^3\mathrm{c2b}^3}+\frac{2\mathrm{c1ac1d}^2\mathrm{c2c}^2\mathrm{c6b}p^2 l^3}{\mathrm{c1c}^3\mathrm{c2b}^3}\\
&-\frac{2i\mathrm{c1d}^3\mathrm{c2ac2c}^3\mathrm{c3b}l^3}{\mathrm{c1c}^4\mathrm{c2b}^4}-\frac{2i\mathrm{c1bc1d}^3\mathrm{c2c}^3\mathrm{c4a}l^3}{\mathrm{c1c}^4\mathrm{c2b}^4}-\frac{8\mathrm{c1bc1d}^2\mathrm{c2ac2c}^2\mathrm{c3bc4a}l^3}{\mathrm{c1c}^4\mathrm{c2b}^4}\\
&-\frac{2\mathrm{c1d}^2\mathrm{c2a}^2\mathrm{c2c}^2\mathrm{c3ac6b}l^3}{\mathrm{c1c}^4\mathrm{c2b}^4}-\frac{2\mathrm{c1ac1d}^2\mathrm{c2c}^2\mathrm{c4a}^2\mathrm{c8b}l^3}{\mathrm{c1c}^4\mathrm{c2b}^4}+\frac{4i\mathrm{c1dc2c}p^6 l^2}{\mathrm{c1cc2b}}-\frac{6\mathrm{c1d}^2\mathrm{c2c}^2 p^4 l^2}{\mathrm{c1c}^2\mathrm{c2b}^2}\\
&+\frac{4i\mathrm{c1bc1dc2cc3b}p^4 l^2}{\mathrm{c1c}^2\mathrm{c2b}^2}+\frac{2i\mathrm{c1dc2ac2cc3b}p^4 l^2}{\mathrm{c1c}^2\mathrm{c2b}^2}+\frac{2i\mathrm{c1bc1dc2cc4a}p^4 l^2}{\mathrm{c1c}^2\mathrm{c2b}^2}\\
&-\frac{2i\mathrm{c1ac1dc2cc6b}p^4 l^2}{\mathrm{c1c}^2\mathrm{c2b}^2}-\frac{2i\mathrm{c1dc2cc3ac8b}p^4 l^2}{\mathrm{c1c}^2\mathrm{c2b}^2}-\frac{\mathrm{c1d}^2\mathrm{c2a}^2\mathrm{c2c}^2\mathrm{c3b}^2 l^2}{\mathrm{c1c}^4\mathrm{c2b}^4}\\
&+\frac{2i\mathrm{c1b}^2\mathrm{c1dc2ac2cc3bc4a}^2 l^2}{\mathrm{c1c}^4\mathrm{c2b}^4}+\frac{2i\mathrm{c1bc1dc2ac2cc3b}^2 p^2 l^2}{\mathrm{c1c}^3\mathrm{c2b}^3}+\frac{2i\mathrm{c1bc1dc2ac2cc4a}^2 p^2 l^2}{\mathrm{c1c}^3\mathrm{c2b}^3}\\
&-\frac{2\mathrm{c1bc1d}^2\mathrm{c2c}^2\mathrm{c3b}p^2 l^2}{\mathrm{c1c}^3\mathrm{c2b}^3}-\frac{4\mathrm{c1d}^2\mathrm{c2ac2c}^2\mathrm{c3b}p^2 l^2}{\mathrm{c1c}^3\mathrm{c2b}^3}-\frac{4\mathrm{c1bc1d}^2\mathrm{c2c}^2\mathrm{c4a}p^2 l^2}{\mathrm{c1c}^3\mathrm{c2b}^3}\\
&-\frac{2\mathrm{c1d}^2\mathrm{c2ac2c}^2\mathrm{c4a}p^2 l^2}{\mathrm{c1c}^3\mathrm{c2b}^3}+\frac{2i\mathrm{c1dc2a}^2\mathrm{c2cc3bc4a}p^2 l^2}{\mathrm{c1c}^3\mathrm{c2b}^3}+\frac{2i\mathrm{c1b}^2\mathrm{c1dc2cc3bc4a}p^2 l^2}{\mathrm{c1c}^3\mathrm{c2b}^3}\\
&+\frac{4i\mathrm{c1bc1dc2ac2cc3bc4a}p^2 l^2}{\mathrm{c1c}^3\mathrm{c2b}^3}+\frac{\mathrm{c1ac1d}^2\mathrm{c2c}^2\mathrm{c6b}p^2 l^2}{\mathrm{c1c}^3\mathrm{c2b}^3}+\frac{2i\mathrm{c1bc1dc2ac2cc3ac6b}p^2 l^2}{\mathrm{c1c}^3\mathrm{c2b}^3}\\
&-\frac{2i\mathrm{c1ac1dc2ac2cc4ac6b}p^2 l^2}{\mathrm{c1c}^3\mathrm{c2b}^3}+\frac{\mathrm{c1d}^2\mathrm{c2c}^2\mathrm{c3ac8b}p^2 l^2}{\mathrm{c1c}^3\mathrm{c2b}^3}-\frac{2i\mathrm{c1dc2ac2cc3ac4ac8b}p^2 l^2}{\mathrm{c1c}^3\mathrm{c2b}^3}\\
&+\frac{2i\mathrm{c1ac1dc2cc3bc4ac8b}p^2 l^2}{\mathrm{c1c}^3\mathrm{c2b}^3}+\frac{2i\mathrm{c1bc1dc2a}^2\mathrm{c2cc3b}^2\mathrm{c4a}l^2}{\mathrm{c1c}^4\mathrm{c2b}^4}-\frac{4\mathrm{c1bc1d}^2\mathrm{c2ac2c}^2\mathrm{c3bc4a}l^2}{\mathrm{c1c}^4\mathrm{c2b}^4}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\mathrm{4c1d^2c2ac2c^2c3b}p^2l^4}{\mathrm{c1c^3c2b^3}}+\frac{\mathrm{c1d^2c2c^2c3ac8b}p^2l^4}{\mathrm{c1c^3c2b^3}}-\frac{\mathrm{2c1d^2c2a^2c2c^2c3b^2}l^3}{\mathrm{c1c^4c2b^4}}\\
&-\frac{\mathrm{8c1d^2c2ac2c^2c3b}p^2l^3}{\mathrm{c1c^3c2b^3}}+\frac{\mathrm{2c1d^2c2c^2c3ac8b}p^2l^3}{\mathrm{c1c^3c2b^3}}-\frac{\mathrm{c1b^2c1d^2c2c^2c4a^2}l^2}{\mathrm{c1c^4c2b^4}}\\
&-\frac{\mathrm{c1d^2c2a^2c2c^2c3ac6b}l^2}{\mathrm{c1c^4c2b^4}}+\frac{2i\mathrm{c1bc1dc2a^2c2cc3ac4ac6b}l^2}{\mathrm{c1c^4c2b^4}}-\frac{\mathrm{c1ac1d^2c2c^2c4a^2c8b}l^2}{\mathrm{c1c^4c2b^4}}\\
&+\frac{2i\mathrm{c1ac1dc2ac2cc3bc4a^2c8b}l^2}{\mathrm{c1c^4c2b^4}}+\frac{4i\mathrm{c1dc2c}p^6l}{\mathrm{c1cc2b}}+\frac{4i\mathrm{c1bc1dc2cc3b}p^4l}{\mathrm{c1c^2c2b^2}}\\
&+\frac{2i\mathrm{c1bc1dc2cc4a}p^4l}{\mathrm{c1c^2c2b^2}}+\frac{4i\mathrm{c1dc2ac2cc4a}p^4l}{\mathrm{c1c^2c2b^2}}-\frac{2i\mathrm{c1ac1dc2cc6b}p^4l}{\mathrm{c1c^2c2b^2}}-\frac{2i\mathrm{c1dc2cc3ac8b}p^4l}{\mathrm{c1c^2c2b^2}}\\
&+\frac{2i\mathrm{c1b^2c1dc2ac2cc3bc4a^2}l}{\mathrm{c1c^4c2b^4}}+\frac{2i\mathrm{c1bc1dc2ac2cc3b^2}p^2l}{\mathrm{c1c^3c2b^3}}+\frac{2i\mathrm{c1bc1dc2ac2cc4a^2}p^2l}{\mathrm{c1c^3c2b^3}}\\
&+\frac{2i\mathrm{c1dc2a^2c2cc3bc4a}p^2l}{\mathrm{c1c^3c2b^3}}+\frac{2i\mathrm{c1b^2c1dc2cc3bc4a}p^2l}{\mathrm{c1c^3c2b^3}}+\frac{4i\mathrm{c1bc1dc2ac2cc3bc4a}p^2l}{\mathrm{c1c^3c2b^3}}\\
&+\frac{2i\mathrm{c1bc1dc2ac2cc3ac6b}p^2l}{\mathrm{c1c^3c2b^3}}-\frac{2i\mathrm{c1ac1dc2ac2cc4ac6b}p^2l}{\mathrm{c1c^3c2b^3}}-\frac{2i\mathrm{c1dc2ac2cc3ac4ac8b}p^2l}{\mathrm{c1c^3c2b^3}}\\
&+\frac{2i\mathrm{c1ac1dc2cc3bc4ac8b}p^2l}{\mathrm{c1c^3c2b^3}}+\frac{2i\mathrm{c1bc1dc2a^2c2cc3b^2c4a}l}{\mathrm{c1c^4c2b^4}}+\frac{2i\mathrm{c1bc1dc2a^2c2cc3ac4ac6b}l}{\mathrm{c1c^4c2b^4}}\\
&+\frac{2i\mathrm{c1ac1dc2ac2cc3bc4a^2c8b}l}{\mathrm{c1c^4c2b^4}}+p^8+\frac{\mathrm{2c1bc3b}p^6}{\mathrm{c1cc2b}}+\frac{\mathrm{2c2ac4a}p^6}{\mathrm{c1cc2b}}-\frac{\mathrm{c1ac6b}p^6}{\mathrm{c1cc2b}}-\frac{\mathrm{c3ac8b}p^6}{\mathrm{c1cc2b}}\\
&+\frac{\mathrm{c1b^2c3b^2}p^4}{\mathrm{c1c^2c2b^2}}+\frac{\mathrm{c2a^2c4a^2}p^4}{\mathrm{c1c^2c2b^2}}+\frac{\mathrm{4c1bc2ac3bc4a}p^4}{\mathrm{c1c^2c2b^2}}+\frac{\mathrm{c1b^2c3ac6b}p^4}{\mathrm{c1c^2c2b^2}}-\frac{\mathrm{2c1ac2ac4ac6b}p^4}{\mathrm{c1c^2c2b^2}}\\
&+\frac{\mathrm{c1ac3b^2c8b}p^4}{\mathrm{c1c^2c2b^2}}-\frac{\mathrm{2c2ac3ac4ac8b}p^4}{\mathrm{c1c^2c2b^2}}+\frac{\mathrm{c1ac3ac6bc8b}p^4}{\mathrm{c1c^2c2b^2}}+\frac{\mathrm{c1b^2c2a^2c3b^2c4a^2}}{\mathrm{c1c^4c2b^4}}\\
&+\frac{\mathrm{2c1bc2a^2c3bc4a^2}p^2}{\mathrm{c1c^3c2b^3}}+\frac{\mathrm{2c1b^2c2ac3b^2c4a}p^2}{\mathrm{c1c^3c2b^3}}-\frac{\mathrm{c1ac2a^2c4a^2c6b}p^2}{\mathrm{c1c^3c2b^3}}\\
&-\frac{\mathrm{c2a^2c3ac4a^2c8b}p^2}{\mathrm{c1c^3c2b^3}}+\frac{\mathrm{2c1ac2ac3b^2c4ac8b}p^2}{\mathrm{c1c^3c2b^3}}+\frac{\mathrm{2c1ac2ac3ac4ac6bc8b}p^2}{\mathrm{c1c^3c2b^3}}\\
&+\frac{\mathrm{c1b^2c2a^2c3ac4a^2c6b}}{\mathrm{c1c^4c2b^4}}+\frac{\mathrm{c1ac2a^2c3b^2c4a^2c8b}}{\mathrm{c1c^4c2b^4}}+\frac{\mathrm{c1ac2a^2c3ac4a^2c6bc8b}}{\mathrm{c1c^4c2b^4}}\\
&+\frac{4i\mathrm{c1dc2ac2cc4a}p^4l^2}{\mathrm{c1c^2c2b^2}}+\frac{2i\mathrm{c1dc2ac2cc3b}p^4l}{\mathrm{c1c^2c2b^2}}+\frac{\mathrm{2c1b^2c2ac3ac4ac6b}p^2}{\mathrm{c1c^3c2b^3}}
\end{aligned}
\tag{7.2.20}
$$

7.3 The MNa Fermions

We have solved the Dirac equation and found the frequency eigenvalues for the fermionic fluctuations for the MN solution. As far as the Lüscher term is concerned, the massive fermionic modes have no contribution. As we'll show in the next chapter, in the limit of large L, only massless modes contribute to the Lüscher term since the factor of

e^{-mL} that is associated with the massive fermions vanishes quickly for large L. Based on the similarities between the MN and the MNa solutions and the fact that MN fermions are massive, we don't expect the MNa solution to produce massless fermionic fluctuation modes and we don't expect any fundamental difference between the MN Lüscher term and the MNa Lüscher term, other than the apparent dimensionality difference.

In the next chapter, We'll calculate the Lüscher terms for both the MN and MNa solutions based on the bosonic eigenvalues.

CHAPTER 8

CONCLUSION

We will derive the k-string Lüscher term for both the MN and the MNa solutions and summarize the thesis with a few conclusions.

8.1 The k-string Lüscher Term

The one loop correction to the k-string energy, as in [82, 26, 27], results from the sum of ω of all massless modes of the small fluctuations since the contribution from massive modes dissipates quickly, controlled by a factor of e^{-mL} for large quark antiquark separation L. All fermionic modes are massive so that the fermions don't contribute to the one loop correction. Specifically, the one loop k-string energy correction E_1 goes as follows

$$E_1 = \frac{1}{2} \sum \omega_b \tag{8.1.1}$$

The bosonic fluctuations for the MN and the MNa backgrounds have the same set of eigenvalues

$$\begin{aligned} \omega_1^2 &= p^2 + g_{xx} m_\Psi^2 \\ \omega_2^2 &= \omega_3^2 + g_{xx} m_\rho^2 \\ \omega_3^2 &= p^2 + \frac{g_{xx}}{2} l(l+1) R \end{aligned} \tag{8.1.2}$$

where ω_1 and ω_2 are non-degenerate eigenvalues for both backgrounds. The difference is that ω_3 is a 4-fold degenerate eigenvalue for MN and 3-fold for MNa. Then the one loop correction takes the following form

$$E_1 = \frac{1}{2} \left(\sum_{p,l,m} \omega_1 + \sum_{p,l,m} \omega_2 + 4(\ or\ 3) \sum_{p,l,m} \omega_3 \right) \tag{8.1.3}$$

It turns out only the massless ω_3 modes contribute to the Lüscher term. Using the same trick of [26, 27] to deal with the infinite sum with the application of ζ function regularizations. One can show that the Lüscher term, i.e., the $1/L$ term, turns to be

$$V_{Luscher} = \frac{1}{2} \frac{4\pi}{L} \zeta(-1) = -\frac{\pi}{6L} \tag{8.1.4}$$

for MN and

$$V_{Luscher} = \frac{1}{2}\frac{3\pi}{L}\zeta(-1) = -\frac{\pi}{8L} \tag{8.1.5}$$

for MNa.

Recall that a probe D3-brane embedded in the MN background is interpreted as a k-string in 4d Minkowski space and a probe D3-brane embedded in the MNa background is interpreted as a k-string in 3d Minkowski space. Thus, the results we acquired for the Lüscher term satisfy the empirical formula proposed in [70]

$$V_{Luscher} = -\frac{(d+p-3)\pi}{24} \tag{8.1.6}$$

where d is the dimension of the spacetime where the k-string lives and also the dimension of the Minkowski portion of the supergravity background in which the probe Dp-brane whose spatial dimension is p is embedded. The empirical formula 8.1.6 was confirmed to be true for k-string calculations in KS and CGLP backgrounds [26, 27] as well. Apparently, the Lüscher term for a k-string from our investigation is different from the the Lüscher term for a fundamental string proposed by Lüscher [31]

$$V_f = -\frac{(d-2)\pi}{24L} \tag{8.1.7}$$

There seems to be a discrepancy between the Lüscher term for a fundamental string we acquired and the Lüscher term for a fundamental string proposed in [31]. The Lüscher term we acquired doesn't have k-dependence, which means the Lüscher term for a fundamental string should equal the Lüscher term for a general $k \neq 1$ k-string. Physically, it's simple to show this based on the charge conjugation symmetry, the k-string tensions for k and $N-k$ should be equal, i.e., $T_k = T_{N-k}$ which implies that $E_k = E_{N-k}$. For instance, consider the cases $k = 1$ and $k = 4$ for $SU(5)$ gauge theory. $E_1 = E_4$, in the classical level. If $E_1 = E_4$ still holds when the quantum fluctuations are included, then clearly, the $1/L$ terms in E_1 and E_4 should be equal.

8.2 Conclusions

We reviewed the AdS/CFT correspondence, possible generalizations of this correspondence and some open questions. The ultimate goal of physics is unification, which means finding a single theory that is capable of describing all four known interactions of nature, based one fundamental principle. The discovery of the AdS/CFT correspondence undoubtedly was a great achievement. There have been tremendous amount of effort from the physics community that is devoted to research along the line of finding connections between the Yang-Mills gauge theories with less supersymmetry and supergravity/superstrings, motivated by the AdS/CFT correspondence.

Specifically, we investigated two similar 10d supergravity backgrounds: the Maldacena-Nunez background and the Maldecena-Nastase background, which are both proposed to be dual to $\mathcal{N} = 1$ super Yang-Mills gauge theories in the IR regime. Partially motivated by the similarity between the KS background and the MN, MNa backgrounds and the success of k-string energy investigation using the KS background, we proceeded with similar strategies to calculate the k-string energy using the MN and the MNa backgrounds.

A common key feature of the geometry of all these supergravity backgrounds is an S^3, in the IR regime, supported by N units of $R - R$ flux which correspond to the N stacked D-branes that create the supergravity background under consideration. A probe D3-brane embedded in the MN/MNa background wrapping over an $S^2 \subset S^3$ is interpreted as a one dimensional k-string for an observer in the 4d/3d Minkowski space which is a subspace of the 10d supergravity background. Two of the longitudinal dimensions of the probe D3-brane are oriented with a spatial dimension and the time dimension of the Minkowski portion of the 10d background. In the large N limit, the disturbance of the probe D3-brane on the background geometry created by the stacked D-branes is negligible.

We calculated the Dirac-Born-Infeld action for the probe D3-brane which has an electric field along the Minkowski spatial direction. We found an exact sine law for the k-string tension for both the MN and MNa backgrounds at the classical level. A dynamical variable, the generalized momentum of the electric field is interpreted as the string charge k.

We fluctuated the bosonic fields and the fermionic fields living on the D3-brane. Then the second order of the fluctuated probe D3-brane action led to the equations of motion of the small fluctuations. The equations of motion were solved. The sum of the eigenvalue frequencies of all massless modes of the small fluctuations led to the k-string Lüscher term.

Even though we reproduced the sine law for the k-string tension and the Lüscher term for the k-string energy in the context of gauge/gravity duality, it should be noticed that the supergravity theories we investigated are dual to $\mathcal{N} = 1$ SU(N) gauge theory. To solve the QCD confinement, string theories dual to gauge theories with broken supersymmetry still remain to be discovered.

REFERENCES

[1] MissMJ. Wikipedia.

[2] J. Greensite. *An Introduction to the Confinement Problem*. Springer, 2011.

[3] R. L. Jaffe. Quark confinement. *Nature*, 268:201, 1977.

[4] P. Higgs. Spontaneous symmetry breakdown without massless bosons. *Physical Review*, 145(4):1156–1163, 1966.

[5] Y. Nambu. Quark model and the factorization of the Veneziano amplitude. *Symmetries and Quark models*, page 269, 1970.

[6] H. B. Nielsen. An almost physical interpretation of the integrand of the n-point Veneziano amplitude. *the 15th International Conference on High Energy Physics,Kiev*, 1970.

[7] L. Susskind. Dual-symmetry of hadrons. *Nuovo Cim*, 69A:457, 1970.

[8] G. 't Hooft. Dimensional reduction in quantum gravity. *Utrecht Preprint THU-93/26*, 1993.

[9] C. R. Stephens, G. 't Hooft, and B. F. Whiting. Black hole evaporation without information loss. *Class. Quant. Grav.*, 11:621–648, 1994.

[10] L. Susskind. The world as a hologram. *J. Math. Phys.*, 36:6377–6396, 1995.

[11] R. Bousso. The holographic principle. *Rev. Mod. Phys.*, 74:825–874, 2002.

[12] J. Maldacena. The large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2:231, 1998.

[13] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov. Gauge theory correlators from noncritical string theory . *Phys. Lett.*, B428:105, 1998.

[14] E. Witten. Anti-de Sitter space and holography . *Adv. Theor. Math. Phys.*, 2:253, 1998.

[15] Juan Martin Maldacena and Carlos Nunez. Towards the large N limit of pure N = 1 super Yang Mills. *Phys. Rev. Lett.*, 86:588–591, 2001.

[16] Juan Martin Maldacena and Horatiu Stefan Nastase. The supergravity dual of a theory with dynamical supersymmetry breaking. *JHEP*, 09:024, 2001.

[17] J. Greensite. The Confinement Problem in Lattice Gauge Theory. *Prog. Part. Nucl. Phys*, 51:1, 2003.

[18] L. Del Debbio, H. Panagopoulos, P. Rossi, and E. Vicari. Spectrum of confining strings in SU(N) gauge theories . *JHEP*, 0201:009, 2002.

[19] B. Lucini and M. Teper. Confining strings in SU(N) gauge theories . *Phys. Rev. D*, 64:105019, 2001.

[20] B. Lucini, M. Teper, and U. Wenger. Glueballs and k-strings in SU(N) gauge theories : calculations with improved operators . *JHEP*, 0406:012, 2004.

[21] S. Kratochvila and P. de Forcrand. Observing string breaking with Wilson loops . *Nucl. Phys. B*, 671:103–132, 2003.

[22] L. Del Debbio, H. Panagopoulos, and E. Vicari. Confining strings in representations with common n-ality . *JHEP*, 0309:034, 2003.

[23] L. Del Debbio, H. Panagopoulos, P. Rossi, and E. Vicari. k-string tensions in SU(N) gauge theories. *Phys. Rev. D*, 65:021501, 2002.

[24] C. Herzog. String tensions and three dimensional confining gauge theories . *Phys. Rev. D*, 66:065009, 2002.

[25] C. Herzog and I. Klebanov. On string tensions in supersymmetric SU(M) gauge theory . *Phys. Lett. B*, 526:388–392, 2002.

[26] C. Doran, L. A. Pando Zayas, V. G. J. Rodgers, and K. Stiffler. Tensions and Luscher Terms for (2+1)-dimensional k-strings from Holographic Models. 2009.

[27] Leopoldo A. Pando Zayas, Vincent G. J. Rodgers, and Kory Stiffler. Luscher Term for k-string Potential from Holographic One Loop Corrections. *JHEP*, 12:036, 2008.

[28] J. Ridgway. Confining k-string tensions with D-branes in super Yang-Mills theories. *Phys. Lett. B*, 648:76–83, 2007.

[29] M. Shifman. k strings from various perspectives: QCD, lattices, string theory and toy models. *Acta Phys. Polon.*, B36:3805, 2005.

[30] M.R. Douglas and S.H. Shenker. *Nucl. Phys. B*, 447:271, 1995.

[31] M. Luscher. Symmetry Breaking Aspects of the Roughening Transition in Gauge Theories. *Nucl. Phys. B*, 180:317, 1981.

[32] I. Klebanov and M. Strassler. Supergravity and a confining gauge theory: duality cascades and χ SB-Resolution and naked singularities. *JHEP*, 0008:052, 2000.

[33] M. Cvetic, G. Gibbons, H. Lu, and C. Pope. Supersymmetric non-singular fractional D2-branes and NS-NS 2-branes. *Nucl. Phys. B*, 606:18–44, 2001.

[34] T. Regge. *Nuovo Cimento*, 14:951, 1959.

[35] T. Regge. *Nuovo Cimento*, 18:947, 1960.

[36] P. D. B. Collins. *An Introduction to Regge Theory and High Energy Physics*. Cambridge University Press, 2009.

[37] G. Veneziano. Construction of a crossing-simmetric, Regge-behaved amplitude for linearly rising trajectories. *Nuovo Cimento A*, 57:190–197, 1968.

[38] J. Scherk and Schwarz J. Dual models for non-hadrons. *Nucl. Phys. Rev. B*, 81:118, 1974.

[39] T. Yoneya. Connection of dual models to electrodynamics and gravidynamics. *Prog. Theor. Phys.*, 51(6):1907–1920, 1974.

[40] H. Miyazawa. Baryon number changing currents. *Prog. Theor. Phys.*, 36(6):1266–1276, 1966.

[41] S. Martin. A supersymmetry primer.

[42] A. Neveu and J. Schwarz. Factorizable dual model of pions. *Nucl. Phys. B*, 31:86–112, 1971.

[43] P. Ramond. Dual theory for free fermions. *Phys. Rev. D*, 3:2415–2418, 1971.

[44] J. Schwarz and M. Green. Anomaly cancellations in supersymmetric D=10 gauge theory and superstring theory. *Phys. Lett. B*, 149:117–122, 1984.

[45] B. Greene. String theory on Calabi-Yau manifolds. *TASI96*, 1996.

[46] P. Horava and E. Witten. Heterotic and type I string dynamics from eleven dimensions . *Nucl. Phys. B*, 460:506–524, 1996.

[47] N. Seiberg. Electric-magnetic duality in supersymmetric non-abelian gauge theories. *Nucl. Phys. B*, 435:129–146, 1995.

[48] E. Alvarez, L. Alvarez-Gaume, and Y. Lozano. An introduction to T-duality in string theory. *Nucl. Phys. Proc. Suppl.*, 41:1–20, 1995.

[49] A. Strominger, S. T. Yau, and E. Zaslow. Mirror symmetry is T-duality. *Nucl. Phys. B*, 479:243–259, 1996.

[50] J. Polchinski. Dirichlet-branes and Ramond-Ramond charges. *Phys. Rev. Lett.*, 75:4724–4727, 1995.

[51] M. Duff. Kaluza-Klein theory in perspective.

[52] J. D. Bekenstein. Black holes and the second law. *Nuovo. Cim. Lett.*, 4:737, 1972.

[53] J. D. Bekenstein. Black holes and entropy. *Phys. Rev. D*, 7:2333, 1973.

[54] J. D. Bekenstein. Generalized second law of thermodynamics in black hole physics. *Phys. Rev. D*, 9:3292, 1974.

[55] S. W. Hawking. Black hole explosions. *Nature*, 248:30, 1974.

[56] G. 't Hooft. The holographic principle. Open lecture, 2000.

[57] J. D. Brown and M. Henneaux. Central charges in the canonical realization of asymptotic symmetries: an example from three dimensional gravity. *Comm. Math. Phys.*, 104:207, 1986.

[58] G. 't Hooft. A planar diagram theory for strong interactions. *Nucl. Phys. B*, 72:461–473, 1974.

[59] Barton Zwiebach. *A First Course in String Theory*. Cambridge University Press, 2009.

[60] G. Horowitz and J. Polchinski. Gauge/gravity duality. 2006.

[61] A. Strominger. The dS/CFT correspondence. *JHEP*, 0110:034, 2001.

[62] E. Witten. Quantum gravity in de Sitter space. 2001.

[63] E. Silverstein. (A)dS backgrounds from asymmetric orientifolds. 2001.

[64] S. Kachru, R. Kallosh, A. Linde, and S. Trivedi. de Sitter vacua in string theory. *Phys. Rev. D*, 68:046005, 2003.

[65] S. Giddings. The fate of four dimensions. *Phys. Rev. D*, 68:026006, 2003.

[66] T. Banks. Some thoughts on the quantum theory of stable de Sitter space . 2005.

[67] Andrei Smilga. *Lectures on Quantum Chromodynamics*. World Scientific, 2001.

[68] T. Muta. *Foundations of Quantum Chromodynamics: An Introduction to Perturbative Methods in Gauge Theories*. World Scientific, 2010.

[69] A. Armoni and J. Ridgway. Quantum broadening of k-strings from the AdS/CFT correspondence. *Nucl. Phys. B*, 801:118–127, 2008.

[70] Kory Stiffler. Mesons From String Theory. 2009.

[71] R. G. Leigh. Dirac-Born-Infeld action from Dirichlet sigma model. *Mod. Phys. Lett. B*, A4:2767, 1989.

[72] M. Luscher, K. Symanzik, and P. Weisz. Anomalies of the Free Loop Wave Equation in the WKB Approximation. *Nucl. Phys. B*, 173:365, 1980.

[73] D. Marolf, L. Martucci, and P. J. Silva. Fermions, T-duality and effective actions for D-branes in bosonic backgrounds. *JHEP*, 0304:051, 2003.

[74] D. Marolf, L. Martucci, and P. J. Silva. Actions and fermionic symmetries for D-branes in bosonic backgrounds. *JHEP*, 0307:019, 2003.

[75] D. Marolf, L. Martucci, and P. J. Silva. The explicit form of the effective actions for F1 and D-branes. *Class. Quant. Grav.*, 21:S1385, 2004.

[76] L. Martucci, J. Rosseel, D. Van den Bleeken, and A. Van Proeyen. Dirac actions for D-branes on backgrounds with fluxes. *Class. Quant. Grav.*, 22:2745, 2005.

[77] I. Kirsch. Spectroscopy of fermionic operators in AdS/CFT. *JHEP*, 0609:052, 2006.

[78] M. Cederwall, A. von Gussich, B. E. W. Nilsson, P. Sundell, and A. Westerberg. The Dirichlet super-p-branes in ten-dimensional type IIA and IIB supergravity. *Nucl. Phys. B*, 490:179, 1997.

[79] E. Bergshoeff and P. K. Townsend. Super D-branes. *Nucl. Phys. B*, 490:145, 1997.

[80] M. Grana. D3-brane action in a supergravity background: The fermionic story. *Phys. Rev. D*, 66:045014, 2002.

[81] P. K. Tripathy and S. P. Trivedi. D3 brane action and fermion zero modes in presence of background flux. *JHEP*, 0506:066, 2005.

[82] L. A. Pando Zayas, J. Sonnenschein, and D. Vaman. Regge trajectories revisited in the gauge/string correspondence. *Nucl. Phys. B*, 682:3–44, 2004.