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Essays on dynamic contracts

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ESSAYS ON DYNAMIC CONTRACTS

by

Yaping Shan

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
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Thesis Supervisor: Professor Srihari Govindan

ABSTRACT

This dissertation analyzes the contracting problem between a firm and the research employees in its R&D department. The dissertation consists of two chapters. The first chapter addresses a simplified problem in which the R&D unit has only one agent. The second chapter studies a scenario in which the R&D unit consists of a team.

In the first chapter, I look at problem in which a principal hires an agent to do a multi-stage R&D project. The transition from one stage to the next is modeled by a Poisson-type process, whose arrival rate depends on the agents choice of effort. I assume that effort choice is binary and unobservable by the principal. To overcome the repeated moral-hazard problem, the principal offers the agent a long-term contract which specifies a flow of payments based on his observation of the outcome of the project. The optimal contract combines rewards and punishments: the payment to the agent decrease over time in case of failure and jumps up to a higher level after each success. I also show that the optimal contract can be implemented by using a risky security that has some of the features of the stocks of these firms, thereby providing a theoretical justification for the wide-spread use of stock-based compensation in firms that rely on R&D.

In the second chapter, I look at a scenario in which the R&D unit consists of a team, which I assume, for simplicity, comprises two risk-averse agents. Now, the Poisson arrival rate is jointly determined by the actions of both agents with the action of each remaining unobservable by both the principal and the other agent. I assume that when success in a phase occurs the principal can identify the agent who was responsible for it. In this model, incentive compatibility means that each agent is willing to exert effort conditional on his coworker putting in effort, and thus exerting effort continuously is a Nash-equilibrium strategy played by the agents. In this multi-agent problem, each agents payment depends not only on his own performance, but is affected by the other agents performance as well. Similar to the single-agent case, an agent is rewarded when he succeeds, and his payment decreases over time when both agents fail. Regarding how an agents payment relates to his coworkers performance, I find that the optimal incentive regime is a function of the way in which agents efforts interact with one another: relative-performance evaluation is used when their efforts are substitutes whereas joint-performance evaluation is used when their efforts are complements. This result sheds new light on the notion of optimal incentive regimes, an issue that has been widely discussed in multi-agent incentive problems.

Abstract Approved: ________________________________
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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Economics at the December 2012 graduation.

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To my wife Pei and my incoming first baby.

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This dissertation analyzes the contracting problem between a firm and the research employees in its R&D department. The dissertation consists of two chapters. The first chapter addresses a simplified problem in which the R&D unit has only one agent. The second chapter studies a scenario in which the R&D unit consists of a team.

In the first chapter, I look at problem in which a principal hires an agent to do a multi-stage R&D project. The transition from one stage to the next is modeled by a Poisson-type process, whose arrival rate depends on the agents choice of effort. I assume that effort choice is binary and unobservable by the principal. To overcome the repeated moral-hazard problem, the principal offers the agent a long-term contract which specifies a flow of payments based on his observation of the outcome of the project. The optimal contract combines rewards and punishments: the payment to the agent decrease over time in case of failure and jumps up to a higher level after each success. I also show that the optimal contract can be implemented by using a risky security that has some of the features of the stocks of these firms, thereby providing a theoretical justification for the wide-spread use of stock-based compensation in firms that rely on R&D.

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action of each remaining unobservable by both the principal and the other agent. I assume that when success in a phase occurs the principal can identify the agent who was responsible for it. In this model, incentive compatibility means that each agent is willing to exert effort conditional on his coworker putting in effort, and thus exerting effort continuously is a Nash-equilibrium strategy played by the agents. In this multi-agent problem, each agents payment depends not only on his own performance, but is affected by the other agents performance as well. Similar to the single-agent case, an agent is rewarded when he succeeds, and his payment decreases over time when both agents fail. Regarding how an agents payment relates to his coworkers performance, I find that the optimal incentive regime is a function of the way in which agents efforts interact with one another: relative-performance evaluation is used when their efforts are substitutes whereas joint-performance evaluation is used when their efforts are complements. This result sheds new light on the notion of optimal incentive regimes, an issue that has been widely discussed in multi-agent incentive problems.

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1.1 Introduction

Over the last decade, the industries of information and communication technologies have become the engines of the U.S. economy: during this period, they have had an average share of 4.6 percent of gross GDP and have accounted for one-fourth of GDP-growth. A distinct feature of these so-called "new-economy" industries is a substantial investment in R&D, which constituted a third of business-sector expenditure on R&D in 2007 according to a recent report of the Bureau of Economic Analysis. Clearly, the success of firms in these industries depends crucially on the performance of the employees in their R&D units, and compensation schemes for these researchers become a major decision for these firms. This decision-problem shares some features with the standard problem of providing incentives to workers, but it also has some unique features. Like its standard counterpart, a moral-hazard phenomenon arises in this specific agency relationship. The outcome of research is uncertain, i.e. effort put into research today will not necessarily lead to a discovery tomorrow. However, the stochastic process governing the outcomes is influenced by how much effort is put into research: higher levels of effort increase the chance of making a discovery. Owing to task-complexity, effort exerted by researchers is difficult to monitor. Now, if the effort level is unobservable, then the imperfect monitoring of effort combined with the stochastic feature of innovation maps into a moral-hazard problem. Furthermore, since most R&D projects last a long period of time, the moral-hazard problem is dynamic in nature.

The point of departure from standard agency problems is the feature that some R&D projects progress through different phases, with research in each phase depending on the outcomes of previous phases. In these new-economy industries, this feature is particularly prominent. For example, in the software industry, Microsoft has released a sequence of Windows operating systems since 1985, from Windows 3.0, to Windows XP, and then to the most recent version, Windows 7. In each upgrade, Microsoft introduced a number of new features which make the use of computers easier and more convenient. In the hardware industry, the development of Intel's CPU is an example of multi-stage R&D. From its earlier 8086 and 8088 processors to the advanced Intel Core processor family, besides the fast-growing initial clock speed (from 2MHz to 3GHz), Intel has also added new instructions to each new generation, which are specially optimized for the demand of new applications.

The agency problem faced by these new-economy firms combines the two features described above, namely an imperfect correlation between outcome and effort, and the multistage nature of the innovative process. Firms try to overcome this agency problem by adopting stock-based grants, especially employee stock-options, which have become a primary component of compensation for employees in R&D departments in the past two decades. Since the researchers' actions have a great impact on the performance of the firms, which in turn affects the return of their stocks, stock-based compensation reduces the agency problem by providing a direct link between company performance and researchers' wealth, thereby providing incentive for researchers to put in effort in research. Moreover, since the employee stock-options have a vesting period during which they cannot be exercised, they are widely used by firms to provide long-term incentives. The question then is whether these schemes are optimal.

We approach the problem by first studying the contracting problem in the abstract, deriving the optimal contract and demonstrating an implementation of the optimal contract. Finally we relate our implementation result with the observed business practices. Our finding is that the optimal contract can be implemented by using a risky security, which shares features of the stock of these firms, thereby providing a theoretical justification for the wide-spread use of stock-based compensation in firms that rely on R&D.

Briefly, the setup of the paper is as follows. At any point in time, the agent can choose whether to put in effort or shirk. Conditional on putting in effort, the transition from one stage to the next is a Poisson-type process with a constant arrival rate. If the agent chooses to shirk, the Poisson arrival rate is zero. The principal cannot observe the agent's action. However, the whole history of the innovation process is publicly observable, and the principal will use precisely this information to provide incentives optimally. To overcome the repeated moral-hazard problem, the principal offers the agent a long-term contract which specifies a flow of payments based on his observation of the outcome of the project.

We use recursive techniques to characterize the optimal dynamic contract.

First, we start with a problem in which the R&D project has only one stage. After characterizing the optimal contract in this problem, we use the results for this case to analyze the multi-stage problem by backward induction. We find that in the optimal contract, the principal uses a compensation scheme that combines punishments with rewards. If the agent fails to make a discovery, his continuation utility and payment decrease over time until a discovery is made. If the agent completes a stage, the principal rewards the agent by a discrete increase in the continuation utility.

We also provide a way to implement the optimal contract, in which a primary component of the agent's compensation is a state-contingent security whose return in case of success is higher than that in case of failure. We assume that investing in this security is the only saving-technology for the agent to smooth consumption overtime. At any point in time, besides the effort-choice, the agent also chooses how much to consume and how much to invest in the security, subject to a minimum-holding requirement. Different from the optimal contract, in which the principal controls the agent's consumption directly, the agent chooses the consumption process by himself in this implementation, which nonetheless generates the same effort and consumption process as the optimal contract. This implementation overcomes the problem pointed out by Rogerson (1985) which is that, if the agent is allowed access to credit, he would choose to save some of his wages, if he could, because of a wedge between the agent's Euler equation and the inverse Euler equation implied by the principal's problem. In our implementation, however, the return on savings is state contingent. When we choose the state-dependent rates of return appropriately, the agent's Euler equation mimics the inverse Euler equation; put differently, the wedge between the Euler equation and the inverse Euler equation disappears.

This implementation is similar to the stock-based compensation scheme used in the real-world in two aspects. First, the return of the state contingent security and the stock price have a similar trend, with an notable increase after each breakthrough in R&D. Second, in the implementation, the agent is required to hold a certain amount of the state-contingent security until he completes the entire project. Similarly, stock options have a vesting period during which they cannot be exercised. Capturing these two main features, our implementation provides a theoretical explanation for the compensation scheme used in reality.

This paper is related to three strands of literature: memoryless patent races, management compensation and dynamic contracts. In the current paper, the innovation process is modeled by a memoryless process—the probability of making a discovery at a point of time depends only on the agent's current action. This way of modeling the stochastic innovation process is commonly used in the patent-race literature, for example, Dasgupta and Stiglitz (1980), Lee and Wilde (1980).

In the management-compensation literature, there is extensive research on stock-based grants for CEO compensation. For researchers' compensation, Anderson, Banker, and Ravindran (2000), Ittner, Lambert, and Larcker (2003), and Murphy (2003) have documented that executives and employees in new-economy firms receive more stock-based compensation than do their counterparts in old-economy firms. Sesil, Kroumova, Blasi, and Kruse (2002) compares the performance of 229

'New Economy' firms offering broad-based stock options to that of their non-stock option counterparts, and shows that the former have higher shareholder returns. Our implementation contributes to this literature by giving a rationale for the use of stock-based compensation in new economy firms, from a theoretical point of view.

In terms of methodology, this paper relates to a rich and growing literature on dynamic contracts. Starting with Green (1987) and Spear and Srivastava (1987), using recursive techniques to characterize optimal dynamic contracts has become a standard approach in dynamic-contract theory. Finally, our use of a Poisson process is similar to Biais, Mariotti, Rochet, and Villeneuve (2010) and Myerson (2008). In these two papers, bad events happen with higher Poisson arrival rate when agents do not put enough effort to prevent such events. This current paper differs from these two papers mainly in the assumption of the agent's preference, which leads to different dynamics of the agent's payment. Both Biais, Mariotti, Rochet, and Villeneuve (2010) and Myerson (2008) assume that the agent is risk neutral, and hence he does not receive any payment until the continuation utility reaches a payment threshold. In our model the agent is risk-averse, and his payment decreases over time if he fails to make a discovery.

The rest of the paper is organized as follows. Section 1.2 describes the model. In the first part of section 1.3, a single stage innovation problem is studied as a benchmark. The results of the benchmark model are used in the second part of section 1.3 to analyze the finite-stage problem. In this section, we also discuss the infinite-stage problem. In section 1.4, we provide an implementation of the optimal dynamic contract. Section 1.5 concludes.

1.2 The Model

We consider a dynamic principal-agent model in continuous time. At time 0, a principal hires an agent to do an R&D project. This project has N stages, which must be completed sequentially, i.e. to develop the stage n $(0 < n \leq N)$ innovation, the agent must have finished the innovation of stage $n-1$.

We model the transition from one stage to the next by a Poisson-type process, which is affected by the agent's choice of effort. For simplicity, we assume that the agent has only two choices of effort: he can either put in effort or shirk. Conditional on putting in effort, the probability that during a period of length $\triangle t$ the agent has not made a discovery is $e^{-\lambda \triangle t}$, where λ is the Poisson arrival rate. If the agent chooses to shirk, the Poisson arrival rate is equal to zero.

Whether the agent puts in effort or shirks cannot be monitored by the principal. However, the principal can observe exactly when each stage of the R&D project is completed. Thus, at any point of time, the principal knows the current stage and the length of time it took the agent to finish each previous stage. Let H_t denote the stage at time t. The stage-level process $H = \{H_t, 0 \leq t < \infty\}$ is stochastic and depends on the agent's choice of effort. The history of H, denoted as $H^t = \{H_s, 0 \leq s \leq t\}$, is the realization of the stage-level process till time t. By assumption, H^t is publicly observable, which is the only information that the principal can use to provide incentives to the agent.

At time 0, the principal offers the agent a contract that specifies a flow of consumption $c_t(H^t)$ based on the principal's observation of the stage-level process. Let T denote the stochastic stopping time when the agent finishes the last stage innovation. Note that the history of the stage-level process will not get updated after the agent finishes the last stage of the project. Thus, the principal can equivalently give the agent a lump-sum consumption transfer at T.

The agent's utility is determined by his consumption flow and the effort put in research. The utility function is assumed to have a separable form $U(c)-L(a)$, where $U(c)$ is the utility from consumption, and $L(a)$ is the disutility of doing research. We assume that $U:[0,+\infty)\to[0,+\infty)$ is an increasing, concave and C^2 function with the property that $U'(c)\to+\infty$ as $c\to 0$. The agent's choice of effort is binary, indicated by $a\in\{0,1\}$. $a=1$ means that the agent chooses to put in effort, and $a=0$ means that the agent chooses to shirk. Moreover, we assume that the disutility of putting in effort equals some $l>0$ and the disutility of shirking equals zero, i.e. $L(1)=l$ and $L(0)=0$.

Given the contract, at any time t, the agent makes the effort choice based on the observation of H^t. Denote the effort process as $a=\{a_t(H^t), 0\leq t<\infty\}$. The agent's objective is to choose the effort process a to maximize the total expected utility. Thus, the agent's problem is

$$\max_{\{a_t, 0\leq t<+\infty\}} E\left[\int_0^T re^{-rt}(U(c_t)-L(a_t))dt+e^{-rT}U(c_T)\right],$$

where r is the discount rate. Moveover, the agent has a reservation-utility v_0. If the maximum expected utility he can get from the contract is less than v_0, then the agent will reject the principal's offer.

For simplicity, we assume that the agent and the principal have the same discount rate. Hence, the principal's expected cost is given by

$$E\left[\int_0^T re^{-rt}c_t dt + e^{-rT}c_T\right].$$

The principal's objective is to minimize the expected cost by choosing an incentive-compatible payment scheme subject to delivering the agent the requisite initial value of expected utility v_0. Therefore, the principal's problem is

$$\min_{\{c_t, 0\leq t<+\infty\}} E\left[\int_0^T re^{-rt}c_t dt + e^{-rT}c_T\right]$$

s.t.

$$E\left[\int_0^T re^{-rt}(U(c_t) - l)dt + e^{-rT}U(c_T)\right] \geq v_0.$$

Finally, to simplify the analysis, we could recast the problem as one where the principal directly transfers utility to the agent instead of consumption. In the transformed problem, the principal chooses a stream of utility transfers $u_t(H^t)$ $(0 \leq t < +\infty)$ to minimize the expected cost of implementing positive effort. Then, the principal's problem becomes

$$\min_{\{u_t, 0\leq t<+\infty\}} E\left[\int_0^T re^{-rt}S(u_t)dt + e^{-rT}S(u_T)\right]$$

s.t.

$$E\left[\int_0^T re^{-rt}(u_t - l)dt + e^{-rT}u_T\right] \geq v_0,$$

where $S(u) = U^{-1}(u)$, which is the principal's cost of providing the agent with utility u. It can be shown that $S(u)$ is a decreasing and strictly convex function. Moreover, $S(0) = 0$ and $S'(0) = 0$.

1.3 The Optimal Dynamic Contract

In this section, we derive the optimal dynamic contract and discuss its properties. In doing so, we follow the standard approach in the contracting literature: the optimal contract is written in terms of the agent's continuation-utility v_t, which is the total utility that the principal expects the agent to derive at any time t. At any moment of time, given the continuation utility, the contract specifies the agent's utility flow, the continuation utility if the agent makes a discovery, and the law of motion of the continuation utility if the agent fails to make a discovery.

1.3.1 Single-stage Problem

Before analyzing the multi-stage case, we first look at a simple case where the R&D project has only one stage.

The continuous-time model can be interpreted as the limit of discrete-time models in which each period lasts Δt. When Δt is small, conditional on putting in effort, the probability that the agent successfully finishes the innovation during Δt is approximately $\lambda \Delta t$. For any given continuation-utility v, the principal needs to decide a triplet $(u, \underline{v}, \bar{v})$ in each period, where

- u is the transferred-utility flow in the current period.

- $\underline{v}$ is the next-period continuation utility if the agent fails to make a discovery during this period of time.

- $\bar{v}$ is the next-period continuation utility if the agent completes the innovation during this period of time.

If the agent chooses to exert effort, his expected utility in the current period is

$$r(u-l)\Delta t + e^{-r\Delta t}((1-\Delta t\lambda)\underline{v} + \Delta t\lambda\bar{v}),$$

where the first term is the current-period utility flow and the second term is the discounted expected continuation utility.

If the agent chooses to shirk, he does not incur any utility cost and will fail to make a discovery with probability 1. Thus, his expected utility in the current period is

$$ru\Delta t + e^{-r\Delta t}\underline{v}.$$

The triplet $(u, \underline{v}, \bar{v})$ should satisfy two conditions. First, this policy should indeed guarantee that the agent gets the promised-utility v. That is

$$r(u-l)\Delta t + e^{-r\Delta t}((1-\Delta t\lambda)\underline{v} + \Delta t\lambda\bar{v}) = v.$$

Second, the policy should implement positive effort, i.e. the expected utility of putting in effort should be higher than the expected utility of shirking. Thus,

$$r(u-l)\Delta t + e^{-r\Delta t}((1-\Delta t\lambda)\underline{v} + \Delta t\lambda\bar{v}) \geq ru\Delta t + e^{-r\Delta t}\underline{v}.$$

Let $C(v)$ be the principal's minimum expected cost of providing the agent with continuation-utility v. Then, the Bellman equation is

$$C(v) = \min_{u,\underline{v},\bar{v}} r(S(u))\Delta t + e^{-r\Delta t}((1-\lambda\Delta t)C(\underline{v}) + \lambda\Delta t S(\bar{v}))$$

s.t.

$$r(u-l)\Delta t + e^{-r\Delta t}((1-\Delta t\lambda)\underline{v} + \Delta t\lambda\bar{v}) = v, \tag{1.1}$$

$$r(u-l)\Delta t + e^{-r\Delta t}((1-\Delta t\lambda)\underline{v} + \Delta t\lambda\bar{v}) \geq ru\Delta t + e^{-r\Delta t}\underline{v}, \tag{1.2}$$

where $S(u)$ is the principal's cost given the transferred-utility u and $S(\bar{v})$ is the principal's cost of providing the agent with the lump-sum utility-transfer $\bar{v}$ when the R&D project is completed. Equation (1.1) is the promise-keeping condition and equation (1.2) is the incentive-compatibility condition.

Multiplying both sides of the Bellman equation and the promise-keeping condition (1.1) by $(1 + r\Delta t)/\Delta t$ and letting Δt converge to 0, we derive the following Hamilton-Jacobi-Bellman (HJB) equation in continuous time[1]

$$rC(v) = \min_{u,\bar{v}} rS(u) + C'(v)\dot{v} + \lambda(S(\bar{v}) - C(v))$$

s.t.

$$\dot{v} = rv - r(u-l) - \lambda(\bar{v} - v), \tag{1.3}$$

$$\bar{v} \geq v + \frac{rl}{\lambda}. \tag{1.4}$$

[1]In this paper, we derive the HJB equation, evolution of continuation utility, and the incentive-compatibility condition in continuous time by considering the limit of a discrete-time approximation. We can also derive these formally using stochastic-calculus techniques (see Biais et al. (2010)). The reason we choose this method is because it is more intuitive and generates the same result.

The promise-keeping condition (1.1) becomes the evolution of the agent's continuation utility in case of failure (1.3). In the discrete-time case, after choosing u and $\bar{v}$, $\underline{v}$ is given by the promise-keeping condition. When Δt converges to 0, $\underline{v}$ converges to v. Hence, in continuous time, the continuation utility in the case of failure changes smoothly and its rate of change is determined by u and $\bar{v}$. The continuation utility can be explained as the value that the principal owes the agent. It grows at the discount-rate r and falls due to the flow of repayment $r(u-l)$ plus the expected repayment $\lambda(\bar{v}-v)$ if the agent completes the innovation.

The incentive-compatibility constraint becomes a very simple expression (1.4). To get the agent to put in positive effort, the continuation utility should jump up by at least $\frac{rl}{\lambda}$ in case of success. The term $\frac{rl}{\lambda}$ is the minimum reward that the principal should give the agent when he completes the project. It is determined by three parameters: r, l, and λ, which have the following interpretations: (1) r is discount rate. The agent discounts the future utility at higher rate when r is bigger. (2) l measures the cost of doing research. When l is big, the cost of doing research is high. (3) λ measures the difficulty of the R&D project. Small λ implies a small chance of success. Thus, a big reward is associated with a high discount-rate, or a high cost of doing research, or a low chance of success.

Note that the continuation utility cannot be less than 0, because the agent can guarantee a utility level of 0 by not putting in any effort. Therefore, a negative continuation utility is not implementable.

To characterize the solution of the HJB equation, we do a diagrammatic analysis in the v-$C'(v)$ plane. The dynamics of v and $C'(v)$ are determined by the sign of dv/dt and $dC'(v)/dt$. The expression of dv/dt is given by the evolution of the continuation utility, which is known. However, the expression of $dC'(v)/dt$ depends on whether the incentive-compatibility condition is binding or not. The following lemma gives the condition under which this condition binds.

Lemma 1.3.1. *The incentive-compatibility condition binds if and only if* $C'(v) \leq S'(v + \frac{rl}{\lambda})$.

The next two lemmas determine the sign of $dC'(v)/dt$ and dv/dt under these two different conditions.

Lemma 1.3.2. *If* $C'(v) < S'(v + \frac{rl}{\lambda})$*, then* $\frac{dC'(v)}{dt} < 0$ *and*

$$\frac{dv}{dt}\begin{cases} < 0, & \text{if } C'(v) > S'(v); \\ = 0, & \text{if } C'(v) = S'(v); \\ > 0, & \text{if } C'(v) < S'(v). \end{cases}$$

Lemma 1.3.3. *If* $C'(v) \geq S'(v + \frac{rl}{\lambda})$*, then* $\frac{dC'(v)}{dt} = 0$ *and* $\frac{dv}{dt} < 0$.

The proof of these lemmas can be found in appendix A.

Lemmas 1.3.1-1.3.3 characterize the dynamics of v and $C'(v)$ in the v-$C'(v)$ plane. The $S'(v) = C'(v)$ locus determines the dynamics of v: v is decreasing over time above it and increasing over time below it. The $S'(v + \frac{rl}{\lambda}) = C'(v)$ locus determines the dynamics of $C'(v)$: $C'(v)$ is constant over time above it and decreasing over time below it. The dynamics are summarized in Figure 1.1.

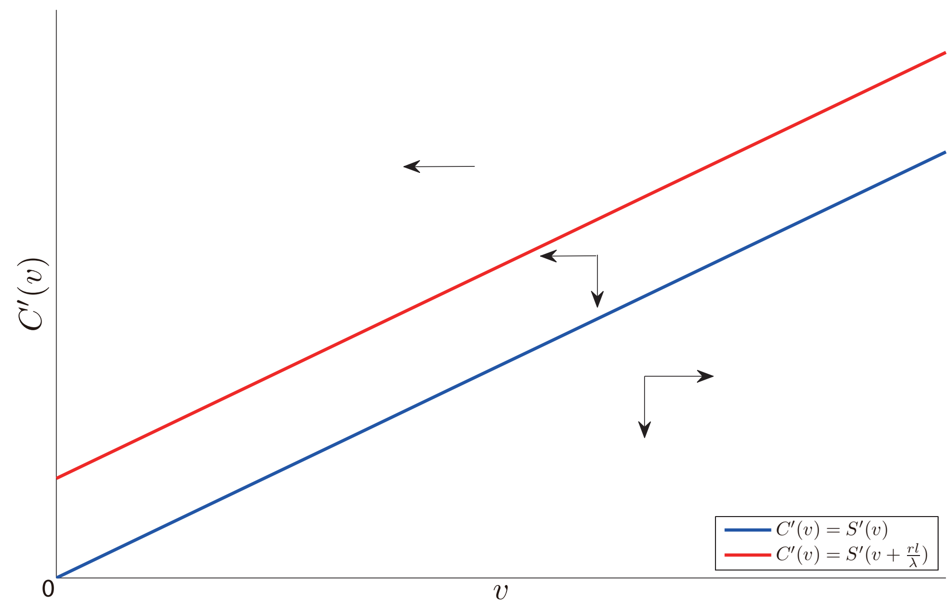

Figure 1.1: Phase Diagram

The next step is to find the optimal path in the phase diagram. From the theorem regarding the existence of a solution to a differential equation, there is an unique path from the line $v = v_0$ to the origin (Path 1 in Figure 1.2). First, any path on which the state variable v diverges to infinity could be ruled out (such as Path 2). This contains the area below Path 1. In the area above Path 1, the continuation-utility v is decreasing over time. When v hits the lower bound 0, it cannot decrease any further. Thus, we must have $dv/dt \geq 0$ at $v = 0$. This condition rules out any path above Path 1 (such as Path 3) because $dv/dt < 0$ when v reaches 0 for any path in this area. Then, Path 1 is the only candidate path left in the phase diagram, and hence it is the optimal path that we are looking for. The final step is to pin down the boundary condition at $v = 0$. At this point, we have $u = 0$ and $\bar{v} = \frac{rl}{\lambda}$. Thus, when v reaches 0, the agent's continuation utility and transferred-utility flow remain at 0 until he makes a discovery. To force the agent to put in positive effort, the principal needs to offer a lump-sum utility transfer of $\frac{rl}{\lambda}$ when the agent completes the single stage R&D project. We can pin down the boundary condition at $v = 0$

$$C(0) = \int_{t=0}^{\infty} e^{-rt} e^{-\lambda t} \lambda S(\frac{rl}{\lambda}) dt = (r+\lambda)^{-1} \lambda S(\frac{rl}{\lambda}).$$

To summarize, starting at the initial point $(v_0, C'(v_0))$, the optimal path locates between the $S'(v) = C'(v)$ locus and the $S'(v + \frac{rl}{\lambda}) = C'(v)$ locus and reaches the lower bound of the continuation utility at the origin (Figure 1.2).

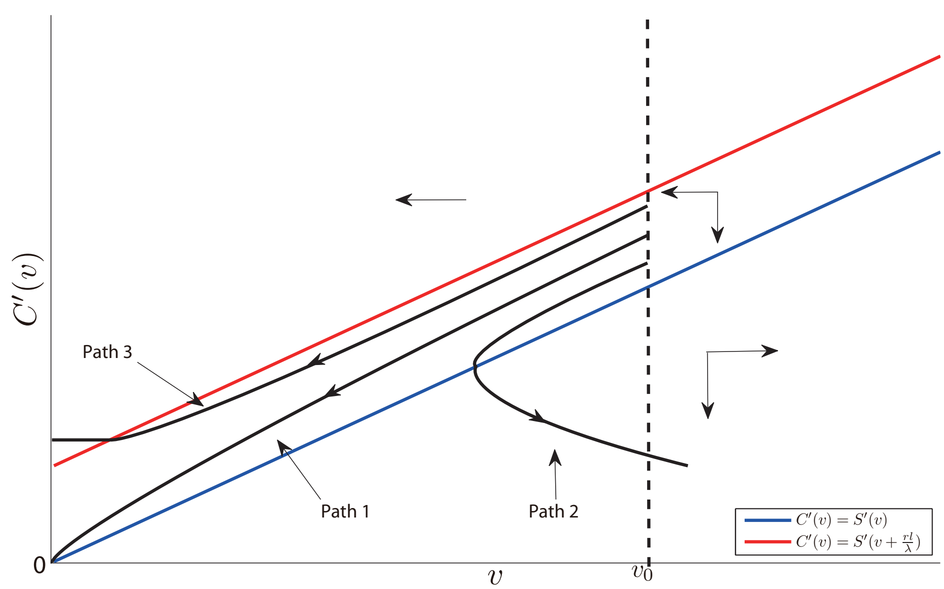

Figure 1.2: Optimal Path

The optimal path and the boundary condition together determine the solution of the HJB equation. The properties of the optimal dynamic contract are summarized in Proposition 1.3.4.

Proposition 1.3.4. *The contract that minimizes the principal's cost takes the following form:*

1. *The principal's expected cost at any point is given by an increasing and convex function $C(v)$, which satisfies*

$$rC(v) = rS(u) + C'(v)(r(v-u)) + \lambda(S(\bar{v}) - C(v)),$$

and boundary condition $C(0) = \frac{\lambda S(\frac{rl}{\lambda})}{r+\lambda}$.

2. *The transferred-utility u satisfies $S'(u) = C'(v)$.*

3. *When the agent completes the innovation, he receives a lump-sum transfer of $\bar{v}$, which satisfies $\bar{v} = v + \frac{rl}{\lambda}$.*

4. *In case of failure to complete the innovation, the continuation-utility v evolves according to $\dot{v} = r(v-u)$, which is decreasing over time and asymptotically goes to 0.*

5. *u and $\bar{v}$ have the same dynamics as v.*

Proof. For part 1, it has been shown that $C(v)$ is determined by the HJB equation and the boundary condition. On the optimal path, $C'(v)$ is strictly increasing in v, which implies that $C(v)$ is strictly convex. In addition, $C'(0) = S'(0) = 0$. Thus $C'(v) > 0$ for all v. Consequently, $C(v)$ is an increasing function.

Part 2 is due to the fact that the transferred-utility flow is determined by the first-order condition $S'(u) = C'(v)$.

For part 3, note that the optimal path locates in the area where the incentive-compatibility constraint binds. Hence, $\bar{v} = v + \frac{rl}{\lambda}$.

For part 4, note that on the optimal path v is decreasing over time and asymptotically converges to 0.

Finally, from part 2, $S'(u) = C'(v)$. Because $S(u)$ and $C(v)$ are both convex, u and v are positively related. From part 3, $\bar{v} = v + \frac{rl}{\lambda}$. Thus, u and $\bar{v}$ have the same dynamics as v, which proves part 5. □

1.3.2 Multi-stage Problem

When the innovation process has multiple but finite number of stages, the optimal dynamic contract can be derived by backward induction. When the project is at stage n ($0 < n \leq N$), we mean that the agents have finished the $(n-1)$-th innovation and are working on the n-th innovation. As in the last subsection, let u be the transferred-utility flow, $S(u)$ be the principal's cost flow given the agent's utility flow u, and $C_n(v)$ be the principal's minimum expected cost of providing the agent with continuation-utility v in stage n. In each stage n, given continuation-utility v, the contract specifies the agent's current utility flow u, the continuation-utility $\bar{v}$ when the agent successfully completes the innovation of stage $(n+1)$, and the evolution of the continuation utility in case of failure.

The backward induction starts from the last stage. After the agent completes the last-stage innovation, no further research work needs to be done and the agent receives a lump-sum utility transfer of $\bar{v}$. Therefore, $C_{N+1}(\bar{v}) = S(\bar{v})$, which is known. Then, the principal's problem in the last stage is

$$rC_N(v) = \min_{u,\bar{v}} rS(u) + C'_N(v)\dot{v} + \lambda(S(\bar{v}) - C_N(v))$$

s.t.

$$\begin{aligned} \dot{v} &= rv - r(u - l) - \lambda(\bar{v} - v), \\ \bar{v} &\geq v + \frac{rl}{\lambda}. \end{aligned}$$

This problem is the same as the single-stage problem. Thus, we have the following proposition.

Proposition 1.3.5. *The contract in the last stage takes the following form:*

1. *The principal's expected cost at any point is given by an increasing and convex function $C_N(v)$, which satisfies*

$$rC_N(v) = rS(u) + C'_N(v)(r(v - u)) + \lambda(S(\bar{v}) - C_N(v)),$$

   and boundary condition $C_N(0) = \frac{\lambda S(\frac{rl}{\lambda})}{r+\lambda}$.

2. *The transferred-utility u satisfies $S'(u) = C'_N(v)$.*

3. *When the agent completes the last stage innovation, he receives a lump-sum utility transfer of $\bar{v}$, which satisfies $\bar{v} = v + \frac{rl}{\lambda}$.*

4. *In case of failure to complete the innovation, the continuation-utility v evolves according to $\dot{v} = r(v - u)$, which is decreasing over time and asymptotically goes to 0.*

5. *u and $\bar{v}$ have the same dynamics as v.*

The proof is similar to that of the proof of Proposition 1.3.4 and is therefore omitted.

From the last-stage problem, we have figured out the principal's minimum expected cost $C_N(v)$ given the agent's continuation-utility v in stage N. Now, given $C_{n+1}(v)$, the principal's problem in stage n is

$$rC_n(v) = \min_{u,\bar{v}} rS(u) + C_n'(v)\dot{v} + \lambda(C_{n+1}(\bar{v}) - C_n(v))$$

s.t.

$$\begin{aligned}\dot{v} &= rv - r(u-l) - \lambda(\bar{v} - v), \\ \bar{v} &\geq v + \frac{rl}{\lambda}.\end{aligned}$$

Similar to the single-stage problem discussed in section 1.3.1, the dynamics are determined by the $C_n'(v) = C_{n+1}'(v + \frac{rl}{\lambda})$ locus and the $C_n'(v) = S'(v)$ locus in the phase diagram. It can be shown that the $C_n'(v) = C_{n+1}'(v + \frac{rl}{\lambda})$ locus is always above the $C_n'(v) = S'(v)$ locus. By doing a similar phase-diagram analysis, we get the following proposition.

Proposition 1.3.6. *The optimal contract in an intermediate stage takes the following form:*

1. *The principal's expected cost at any point is given by an increasing and convex function $C_n(v)$, which satisfies*

$$rC_n(v) = rS(u) + C_n'(v)(r(v-u)) + \lambda(C_{n+1}(\bar{v}) - C_n(v)),$$

and boundary condition $C_n(0) = \frac{\lambda C_{n+1}(\frac{rl}{\lambda})}{r+\lambda}$.

2. *The transferred-utility* u *satisfies* $S'(u) = C_n'(v)$.

3. *When the agent completes stage* $n+1$ *innovation, he enters stage* $n+1$ *and starts with continuation-utility* $\bar{v}$*, which satisfies* $\bar{v} = v + \frac{rl}{\lambda}$.

4. *In case of failure to complete the innovation, the continuation-utility* v *evolves according to* $\dot{v} = r(v-u)$*, which is decreasing over time and asymptotically goes to 0.*

5. u *and* $\bar{v}$ *have the same dynamics as* v.

6. $C_n(v) > C_{n+1}(v)$ *for all* v*;* $C_n'(v) > C_{n+1}'(v)$ *for all* $v > 0$.

Part 6 of Proposition 1.3.6 shows that given the same continuation v, the cost of delivering continuation-utility v is higher at an earlier stage than the cost at a later stage. Moreover, the corresponding-transferred utility at an earlier stage is also higher than the transferred utility at a later stage. When the project is at an earlier stage, there are more stages left, and hence there are more uncertainties in the future. Therefore, the cost of delivering the same level of continuation utility is higher at an earlier stage than the cost at a later stage. Due to the same reason, at an earlier stage, the principal chooses higher transferred utility, because delivering utility in the future is costlier.

The difference between the last-stage problem and any of the intermediate-stage problems is that in the last-stage problem the agent receives a lump-sum utility transfer when he finishes the innovation of the last stage; while in an intermediate stage n, the agent enters stage $(n+1)$ and starts with a higher continuation-utility after finishing the innovation of stage n. Figure 1.3 is a sample path of the continuation utility for a 3-stage R&D project.

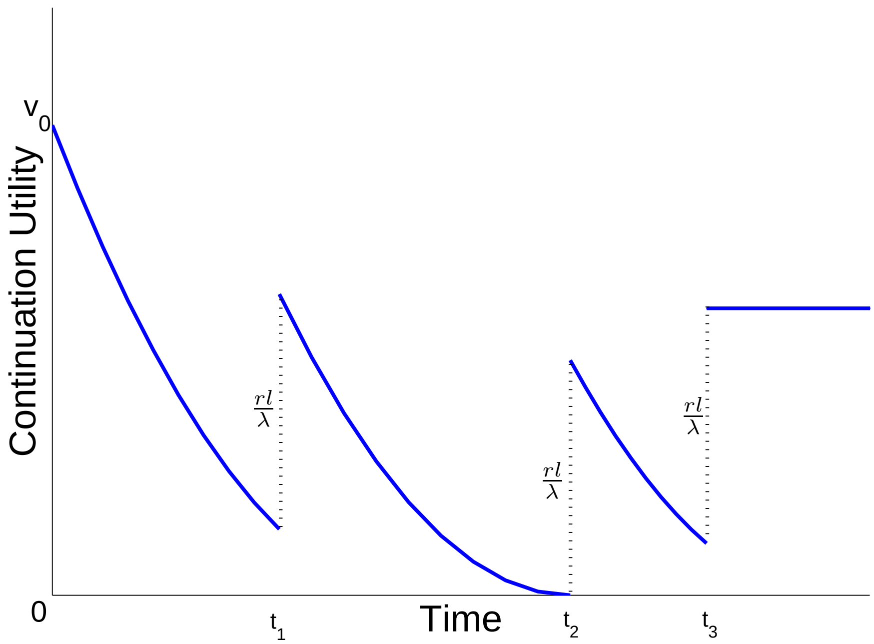

Figure 1.3: Multi-stage

1.3.3 Infinite Stages

In this subsection, we consider the case in which the R&D project has an infinite number of stages. Now, the principal needs to solve the same problem after each success. Let $C(v)$ be the principal's minimum cost of providing continuation-utility v. Then, the optimal contract is characterized by the following HJB equation

$$rC(v) = \min_{u,\bar{v}} rS(u) + C'(v)\dot{v} + \lambda(C(\bar{v}) - C(v))$$

s.t.

$$\begin{aligned}\dot{v} &= rv - r(u-l) - \lambda(\bar{v} - v), \\ \bar{v} &\geq v + \frac{rl}{\lambda},\end{aligned}$$

where u is transferred-utility flow and $\bar{v}$ is the agent's continuation utility if he completes one innovation. Note that this differential equation is a delay differential equation. The derivative of the cost function at a point v depends on the value of the cost function at another point $\bar{v}$. Moreover, u is implicitly determined by $C'(v)$ by first-order condition, which makes the problem even more complicated[2]. Shan (2010b) provides a proof of the existence of a solution to this HJB equation under the assumption that the derivative of S is bounded. Unfortunately, we cannot prove the existence in more general case. A natural conjecture of the property of the cost function is that the cost function is twice-differentiable, strictly convex, and increasing. Suppose there exists a solution to the HJB equation that satisfies these properties. Due to strict convexity of the cost function, the incentive-compatibility condition binds, which implies $\bar{v} = v + \frac{rl}{\lambda}$ and $\dot{v} = rv - r(u-l) - \lambda(\bar{v} - v) = r(v-u)$.

[2]Biais, Mariotti, Rochet, and Villeneuve (2010) and Myerson (2008) analyze a similar version of this kind of problem. Under the assumption of risk-neutrality, the agent does not receive any payment until the continuation utility reaches a payment threshold at which he receives constant payment per unit time, such that his continuation utility remains constant. Then the HJB equation becomes a tractable delay differential equation.

Substituting these into the HJB equation, we get

$$rC(v) = \min_{u,\bar{v}} rS(u) + C'(v)r(v-u) + \lambda(C(v+\frac{rl}{\lambda}) - C(v)).$$

Using envelop theorem,

$$rC'(v) = C''(v)r(v-u) + rC'(v) + \lambda(C'(v+\frac{rl}{\lambda}) - C'(v)).$$

Then,

$$\dot{v} = r(v-u) = \frac{-\lambda(C'(v+\frac{rl}{\lambda}) - C'(v))}{C''(v)} < 0.$$

Thus, if there exists a strictly convex and increasing function $C(v)$ that solves the HJB equation, the optimal contract is similar to the optimal contact in finite-stage case. In the optimal contract, the agent's continuation utility decreases over time in case of failure; after each success, it jumps up by $\frac{rl}{\lambda}$.

1.3.3.1 An Example

If the agent's utility from consumption takes the logarithmic form $U(c) = \log(c)$, then we can provide a closed form solution to the HJB equation[3]. For logarithmic utility, the cost of providing transferred utility flow u is $S(u) = e^u$. Let X be the set of all differentiable functions. Define an operator $G : X \to X$ by

$$(GC)(v) = \min_{u,\bar{v}} \frac{rS(u) + C'(v)\dot{v} + \lambda(C(\bar{v}) - C(v))}{r}$$

[3] Note that the logarithmic utility function is unbounded from below. Hence there is no lower bound the continuation utility.

s.t.

$$\begin{aligned} \dot{v} &= rv - r(u-l) - \lambda(\bar{v} - v), \\ \bar{v} &\geq v + \frac{rl}{\lambda}. \end{aligned}$$

Then, the solution to the HJB equation is a fixed point of this operator.

Consider a cost function in the form of $C(v) = qe^v$, where q is a constant. Apply the operator G to this cost function. Since $C(v)$ is strictly convex, $\bar{v} = v + \frac{rl}{\lambda}$ and $\dot{v} = rv - r(u-l) - \lambda(\bar{v} - v) = r(v-u)$. u is determined by the first order condition $S'(u) = C'(v)$. Thus,

$$e^u = qe^v \Rightarrow u = v + \log q,$$

and

$$\dot{v} = r(v-u) = -r \log q.$$

Then we have

$$\begin{aligned} (GC)(v) &= \frac{rS(u) + C'(v)\dot{v} + \lambda(C(\bar{v}) - C(v))}{r} \\ &= \frac{rqe^v + qe^v(-r\log q) + \lambda(qe^{v+\frac{rl}{\lambda}} - qe^v)}{r} \\ &= \frac{rq - rq\log q + \lambda q e^{\frac{rl}{\lambda}} - \lambda q}{r} e^v. \end{aligned}$$

This result shows that if operator G is applied to C of the form qe^v, $G(C)$ takes the same form as C—a constant times e^v. Thus, the solution to the HJB equation has the form $C(v) = q^* e^v$ where q^* solves

$$q^* = \frac{rq^* - rq^* \log q^* + \lambda q^* e^{\frac{rl}{\lambda}} - \lambda q^*}{r}.$$

Solving the equation, we get

$$q^* = e^{\frac{\lambda}{r}(e^{\frac{rl}{\lambda}}-1)}.$$

1.4 Implementation

The optimal contract derived in the previous sections is written in terms of continuation utility, which is highly abstract. Moreover, the principal controls the agent's consumption directly, i.e. the agent consumes all the payments from the principal at any point in time. In this section, we provide an implementation of the optimal contract, in which a primary component of the agent's compensation is a state-contingent security. In this implementation, besides the decision of exerting effort or shirking, the agent also chooses consumption by himself. Yet, the implementation generates the same allocation as the original optimal contract. Finally, we briefly discuss how this implementation relates to the compensation scheme used in reality.

To introduce the design of the state-contingent security, we first look at a discrete-time approximation of the continuous-time setting. The security lasts for one period. When the project is at stage n, y shares of this security bought in period t pays y in period $t+1$ if the agent fails to make a discovery. If the agent succeeds, the payoff is $Y_{n+1}(y)$, where $Y_{n+1}(y)$ is a function of y, which is stage specific. The price of the security is determined by fair-price rule, i.e. the price of the security equals the present value of this security. Let $P_n(y)$ denote the price of y shares of the security when the project is at stage n. Then,

$$P_n(y) = e^{-r\Delta t}((1 - \lambda\Delta t)y + \lambda\Delta t Y_{n+1}(y)).$$

Remark: In general, the pricing function P_n is non-linear. But, if the utility function is logarithmic, then $Y_{n+1}(y)$ is a linear function of y, and hence the pricing function becomes linear $P_n(y) = p_n y$, where p_n is the price for each share of the security and is stage specific.

To implement the optimal contract, before the project starts, the principal provides the agent with initial-wealth y_0, and $\underline{y}_0$ ($\underline{y}_0 \leq y_0$) of the initial wealth is paid in terms of this security. When the project proceeds, in each period, the agent is required to hold a minimum amount of this security until the whole project is completed. The minimum amount requirement, denoted by $\underline{y}_n$, is also stage specific. We assume that investing in this security is the only saving technology for the agent to smooth consumption overtime. Hence, in each period, besides effort choice, the agent also decides how much to consume and how much to invest in the security. Let y_t denote the agent's wealth in period t. Then, his budget constraint is

$$rc_t\Delta t + e^{-r\Delta t}((1 - \lambda\Delta t)y_{t+1} + \lambda\Delta t Y_{n+1}(y_{t+1})) \leq y_t,$$

where the first term on the left-hand side is his consumption in the current period, and the second term is his investment in the security. Note that y_{t+1} is the number of shares of the security that the agent purchases in period t, which is also his wealth in period $t + 1$ is he fails to make a discovery. Let Δt converses to 0, we can derive the evolution of the agent's wealth in case of failure, which satisfies

$$\dot{y} = ry - rc - \lambda(Y_{n+1}(y) - y).$$

When the project is in stage n, the agent's wealth in case of failure grows at rate r, and decreases due to the spending on consumption c and the loss of the investment in the security $\lambda(Y_{n+1}(y) - y)$. If the agent succeeds, his wealth jumps to $Y_{n+1}(y)$.

The agent's problem is to choose an effort process and a consumption process to maximize his discounted expected utility. Let $V_n(y)$ be the maximum expected utility that the agent can get in stage i, given income y. Then, in recursive form, the agent's problem in stage n is to solve the following HJB equation

$$rV_n(y) =$$

$$\max\{\max_c r(U(c) - l) + V_n'(y)\dot{y} + \lambda(V_{n+1}(Y_{n+1}(y)) - V_n(y)), \max_c rU(c) + V_n'(y)\dot{y}\}$$

s.t.

$$\begin{aligned} \dot{y} &= ry - rc - \lambda(Y_{n+1}(y) - y), \\ y &\geq \underline{y}_n. \end{aligned}$$

The next proposition shows that under certain conditions this implementation generates the same allocation as the original optimal contract. The proof is in the appendix.

Proposition 1.4.1. *Suppose the principal provides the agent with initial wealth* y_0

$$y_0 = C_0(v_0),$$

and in stage n

$$Y_{n+1}(y) = C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda}),$$

$$\underline{y}_n = C_n(0).$$

Then, given income y, the highest discounted expected utility the agent can get is

$$V_n(y) = C_n^{-1}(y),$$

and he chooses consumption flow c that satisfies

$$S'(U(c)) = C_n'(V_n(y)).$$

In addition, the agent always exerts effort until he completes the last-stage innovation.

In stage n, given income y, the highest expected utility that the agent can get is $V_n(y)$, and he chooses consumption flow c which satisfies $S'(U(c)) = C_n'(V_n(y))$. In the optimal contract, the agent's continuation utility is equal to $V_n(y)$ at this point of time. Given this continuation utility, the transferred-utility flow satisfies $S'(u) = C_n'(V_n(y))$. This implies that $U(c) = u$, or the consumption flow chosen by the agent in this implementation attains the same utility flow as what is chosen by the principal in the optimal contract for all possible histories. Hence, this implementation generates the same consumption allocation as the optimal contract.

The idea of this implementation comes from the fact that the agent's utility maximization problem is the dual problem of the principal's cost minimization problem in section 1.3. Given continuation-utility v, $C_n(v)$ is the minimum expected-cost to finance the incentive-compatible compensation scheme. From the dual perspective, given expected wealth $y = C_n(v)$, the maximum expected utility that the agent can reach should equal v. Furthermore, the consumption allocation should be the same.

In this implementation, the state-contingent security plays a key role in providing incentives. The gap between the payoff in case of success and that in case of failure guarantees that the agent is willing to exert effort. In fact, the required minimum amount is the lowest level that can provide an incentive for exerting effort. When the agent's wealth drops to this level, the highest expected utility he can from the contract is zero, which is the lower bound of the continuation utility.

However, in the financial market, there does not exist such an exotic asset that has the exact same payoff structure as the state-contingent security used in this implementation. However, the stock of a company is a reasonable proxy for this security. Since these firms rely intensely on R&D, the performance of the employees in the R&D units have a great impact on these firms' performance outcomes, which bring a close relationship between employees' performance and the return of firms' stocks. In particular, after each breakthrough in R&D, it always follows a notable increase in the firm's stock price. When there is no arrival of such good news for a period time, its stock price tends to decline. Thus, among all available assets, the company's stock has the closest payoff-pattern to that of the state-contingent security. Another feature of our implementation is the minimum amount holding requirement that the agent has to meet until he completes the project. In the real-world, this feature is mimicked by using employee stock-options, which has vesting period during which the options cannot be exercised. The time restriction provides long-term incentives to overcome the repeated moral-hazard problem.

In the past two decades, stock-based grants, especially stock-options, have become the most popular compensation scheme used by new-economy firms. The similarities between this compensation scheme and our implementation of the optimal contract suggest that firms are getting as close to optimality as is allowed by the market structure. In other worlds, our implementation gives a justification for the wide-spread use of stock-based compensation in firms that rely on R&D from a theoretical point of view.

1.5 Conclusion

This paper constructed an optimal dynamic contract to solve the repeated moral-hazard problem when a principal hires an agent to do a multi-stage R&D project. The R&D process is modeled by a jump process (Poisson). In the optimal contract, incentive is provided in two ways: (1) the agent's continuation utility jumps up to a higher value when he successfully completes an innovation (reward); (2) If the agent fails to make a discovery, his continuation utility decreases continuously over time (punishment). The evolution of the continuation utility depends on the entire history of the innovation process up to time t, i.e. it is based on how many innovations have been made before time t and how long it takes the agent to complete each innovation.

We also show that the optimal contract could be implemented by a risky security, whose return depends on the outcome of the project. The agent is required to hold a minimum amount of this security until he completes the whole project. In this implementation, instead of the principal directly controlling the agent's consumption as in the optimal contract, the agent chooses consumption level by himself. By a duality argument, we show that this implementation yields the same allocation as the optimal contract. This implementation provides a theoretical justification for the stock-based compensation used in reality.

CHAPTER 2
DYNAMIC CONTRACTS FOR A CLASS OF MULTI-AGENT R&D MODELS

2.1 Introduction

This paper studies the agency problem between a firm and its in-house R&D unit. The agency problem in this specific relationship differs from the standard principal-agent problem studied in literature in two aspects. First, R&D projects are nowadays typically undertaken by groups of researchers. Unlike the era when Edison invented the light bulb, and Bell telephone, R&D projects are now so complicated that great technological breakthroughs are seldom obtained by individual effort. Large efficiencies can be achieved when multiple researchers target the same hurdle in technological development. Hence, the most innovative companies in the world, like Apple, Google, Microsoft, IBM, and Sony, have adopted innovation-teams, which enable them to launch innovations faster. The wide spread use of team in R&D projects suggests that a multilateral environment is the appropriate setting to think about the agency problem between a firm and its in-house R&D unit.

The second feature that distinguishes the agency problem in this relationship from the standard principal-agent problem is that in some industries R&D projects are carried out in distinct phases. In each phase, researchers are required to achieve some specific goals, and hence the success or failure of each phase is verifiable. For example, in the software industry, Microsoft has released a sequence of Windows operating systems since 1985, from Windows 3.0, to Windows XP, and then to the most recent version, Windows 7. In each upgrade, software engineers are expected to realize a number of new features. During the multi-stage R&D process, the firm and its researchers interact repeated over time. Therefore, the agency problem is a dynamic problem in nature.

These two features are captured here in an abstract principal-agent model in continuous time, and we use recursive techniques to characterize the optimal contract that solves the resulting repeated, multi-agent, moral-hazard problem. We show that the optimal incentive regime is a function of how an agent's effort interacts with those of other agents: relative-performance evaluation is used when their efforts are substitutes whereas joint-performance evaluation is used when their efforts are complements. This result sheds new light on the notion of optimal incentive regimes, an issue that has been widely discussed in multi-agent incentive problems.

Briefly, setup of the paper is as follows. A principal hires two risk-averse agents to perform an R&D project. At any point in time, the agents can either choose to devote effort to work or shirk; and their actions cannot be monitored by the principal, which creates a moral-hazard problem. The R&D project has multiple stages. The transition from one stage to the next is modeled by a Poisson-type process, and the arrival rate is jointly determined by the effort choice of both agents. Hence, the principal cannot treat each agent separately. To overcome the moral-hazard problem, the principal offers a long-term contact to each agent that specifies a history-contingent payment-scheme based on the information that the principal can observe. In the body of the paper, we consider a situation in which the principal can observe each individual's performance, i.e. when an innovation is made, the principal can identify the agent who makes the discovery. In another scenario, the principal can only observe the joint performance of the agents. The analysis of the optimal contract for joint-performance case is a direct extension of the single-agent model analyzed in Shan (2010a), which is included in appendix B.

The optimal compensation-scheme combines reward and punishment. In the optimal contract, the agents' payments decrease continuously over time if both of them fail to make a discovery; and the agent who makes the discovery is rewarded by an upward jump in payment.

Since the principal can observe each agent's performance, an agent's compensation depends not only on his own performance, but may also be tied to the other agent's performance as well. This feature of the optimal compensation scheme in our set-up provides new a viewpoint on optimal incentive regimes used in multi-agent contracting problems. Broadly speaking, there are two types of incentive regimes commonly considered in the literature. The first one is called relative-performance evaluation, which punishes an employee when his coworkers perform well. The second one is called joint-performance evaluation, which rewards an employee when his peers perform well. In a static setting, Lazear and Rosen (1981), Holmstrom (1982), and Green and Stokey (1983) give a rationale for relative-performance evaluation when the performance measures of workers have a common noise component. Che and Yoo (2001) argues that joint-performance evaluation could be used in a repeated setting because a shirking agent is punished by the subsequent shirking of his partner, which provides stronger incentive for working. However, the current paper shows that the type of compensation scheme that the principal should use depends crucially on how the agents' efforts interact. When their efforts are substitutes, an agent's action has a negative externality on the performance of his coworker, and hence relative-performance evaluation is used in which the principal penalizes him when his coworker succeeds. When there is complementarity between agents' efforts, the principal uses joint-performance evaluation, in which an agent also receives a reward when his coworker succeeds, but the reward is lower than the reward when he makes the discovery.

This paper is related to two strands of literature: multi-agent incentive problem and dynamic contracts. There is a large literature about incentive for multiple agents in static setting that lasts for just one transaction (see Lazear and Rosen (1981), Holmstrom (1982), etc.). This paper is the first paper that uses dynamic contracts to analyze repeated multi-agent moral-hazard problem. Che and Yoo (2001) and Rayo (2007) also study moral hazard in teams in a repeated setting but they do so using relational contracts. Both of these two papers assume risk-neutrality for the agents. Risk-neutrality implies that immediate payments and continuation payments have equivalent effects in providing incentive. This property allows them to focus on stationary contracts. Besides using a different type of contracts, the current paper differs from these two papers by assuming that agents are risk averse. Risk aversion gives rise to a trade-off in the contracting problem. On the one hand, to introduce incentives, the principal needs to change agent's payments discontinuously after each success. On the other hand, risk-aversion suggests gains from consumption smoothing. This paper describes the precise dynamic pattern of the optimal contract in which the payment is history contingent and varies over time.

In terms of methodology, this paper follows the rich and growing literature on dynamic moral hazard that uses recursive techniques to characterize optimal dynamic contracts (e.g. Green (1987), Spear and Srivastava (1987), and more recently Sannikov (2008)). Biais, Mariotti, Rochet, and Villeneuve (2010), Myerson (2008), and Shan (2010a) consider the dynamic moral-hazard problem in a similar continuous time and Poisson framework. The current paper contributes this literature by looking at the dynamic-contracting problem in a multi-agent setup instead of single-agent environment.

The rest of the paper is organized as follows. Section 2.2 describes the model. Section 2.3 analyzes the optimal contract. We provide an example in which there is a closed-form solution in Section 2.4. Section 2.5 concludes. A discussion of joint-performance is included in the appendix.

2.2 The Model

Time is continuous. At time 0, a principal hires two agents to perform an R&D project. The project has N stages, which must be completed sequentially. When the project is at stage n $(0 < n \leq N)$, we mean that the agents have finished the $(n-1)$-th innovation and are working on the n-th innovation.

At any point in time, each agent, indexed by i $(i = 1, 2)$, faces a binary-choice

Table 2.1: Arrival Rates

		Agent 2	
		Work	Shirk
Agent 1	Work	λ_1, λ_2	$\hat{\lambda}_1, 0$
	Shirk	$0, \hat{\lambda}_2$	$0, 0$

problem of taking an action $a_i \in A_i = \{0,1\}$. $a_i = 1$ means that agent i chooses to put in effort, and $a_i = 0$ means that he chooses to shirk. Let $A = A_1 \times A_2$ and denote a typical profile of A by $a = (a_1, a_2)$. The completion of each stage of the project is modeled by a Poisson-type process. The agents' actions jointly determine the Poisson arrival-rate in the following way. Each agent's arrival rate of making a discovery is determined by a function $\lambda_i(a) : A \to \mathbb{R}_+$. Then, the total arrival rate of completion of each stage is $\lambda(a) = \lambda_1(a) + \lambda_2(a)$. For simplicity, we assume that if agent i shirks he fails with probability 1, i.e. $\lambda_i(a) = 0$ when $a_i = 0$. The following table describes all the possible actions and the arrival rates for each action taken by the agents: In the above table, λ_i is agent i's arrival rate when both agents exert effort, and $\hat{\lambda}_i$ is his arrival rate when he exerts effort and the other agent shirks. We assume that the probability of success increases when both agents put in effort, i.e. $\lambda = \lambda_1 + \lambda_2 > \max\{\hat{\lambda}_1, \hat{\lambda}_2\}$. To simplify notation, we use λ_{-i} and $\hat{\lambda}_{-i}$ to indicate agent i's coworker's corresponding arrival rates from now on.

Effort-choice is private information, and thus cannot be observed by the principal or the other agent. However, the principal can observe exactly when each stage of the R&D project was completed. Moreover, he can also identify the agent who made the discovery. Let H^t summarize all the public information up to time t. Then, H^t includes information about how many innovations were made before time t, the exact time when each innovation was made, and the identity of the agent who completed that innovation.

We assume that the completion of the project is sufficiently valuable to the principal that he always wants to induce both agents to work. Hence, the principal's problem is to minimize the cost of providing incentives. At time 0, the principal offers each agent a contract that specifies a flow of consumption $\{c_i^t(H^t), 0 \leq t < +\infty\}$ $(i = 1, 2)$, based on the principal's observation of their performance. Let T denote the stochastic stopping time when the last stage of the project is completed, which is endogenously determined by the agents' actions. Note that the history of H^t will not get updated after the project is completed, which implies that agents' payment-flow is constant after the completion of the project. Therefore, the principal can equivalently give the agents a lump-sum consumption transfer at T.

Each agent's utility is determined by his consumption flow and his effort. For simplicity, we assume that the two agents have the same utility function, which is further assumed to have a separable form $U(c_i) - L(a_i)$, where $U(c_i)$ is the utility from consumption and $L(a_i)$ is the disutility of exerting effort. We assume that $U : [0, +\infty) \to [0, +\infty)$ is an increasing, concave, and C^2 function with the property that $U'(c) \to +\infty$ as $c \to 0$. We also assume that the disutility of putting in effort equals some $l > 0$ and the disutility of shirking equals zero, i.e. $L(1) = l$ and $L(0) = 0$.

Given a contract, agent i's objective is to choose an effort process $\{a_i^t(H^t), 0 \leq t < \infty\}$ to maximize his total expected utility. Thus, agent i's problem is

$$\max_{\{a_i^t, 0 \leq t < \infty\}} E\left[\int_0^T re^{-rt}(U(c_i^t) - L(a_i^t))dt + e^{-rT}U(c_i^T)\right],$$

where r is the discount rate. Since agent i's consumption-flow is constant after time T when the project is completed, his total discounted utility at time T equals $U(c_i^T)$. Moveover, the agents have a reservation-utility v_0. If the maximum expected utility they can get from the contract is less than v_0, then they will reject the principal's offer.

For simplicity, we assume that the agents and the principal have the same discount rate. Hence, the principal's expected cost is given by

$$E\left[\int_0^T re^{-rt}(c_1^t + c_2^t)dt + e^{-rT}(c_1^T + c_2^T)\right].$$

The principal's objective is to minimize the expected cost by choosing an incentive-compatible payment scheme subject to delivering the agents the requisite initial value of expected utility v_0. Therefore, the principal's problem is

$$\min_{\{c_1^t, c_2^t, 0 \leq t < +\infty\}} E\left[\int_0^T re^{-rt}(c_1^t + c_2^t)dt + e^{-rT}(c_1^T + c_2^T)\right]$$

s.t.

$$E\left[\int_0^T re^{-rt}(U(c_i^t) - l)dt + e^{-rT}U(c_i^T)\right] \geq v_0$$

for $i = 1, 2$. We assume that the agents play a noncooperative game. Therefore, incentive compatibility in this context means that, at any point in time, each agent is willing to exert effort conditional on the other agent is putting in effort, and that it is Nash equilibrium to put in effort.

Finally, to simplify the analysis, we could recast the problem as one where the principal directly transfers utility to the agents instead of consumption. In the transformed problem, the principal chooses a stream of utility transfers $\{u_i^t(H^t), 0 \leq t < +\infty\}$ to minimize the expected cost of implementing positive effort. Then, the principal's problem becomes

$$\min_{\{u_1^t, u_2^t, 0 \leq t < +\infty\}} E\left[\int_0^T re^{-rt}(S(u_1^t) + S(u_2^t))dt + e^{-rT}(S(u_1^T) + S(u_2^T))\right]$$

s.t.

$$E\left[\int_0^T re^{-rt}(u_i^t - l)dt + e^{-rT}u_i^T\right] \geq v_0,$$

where $S(u) = U^{-1}(u)$, which is the principal's cost of providing the agent with utility u. It can be shown that S is a decreasing and strictly convex function. Moreover, $S(0) = 0$ and $S'(0) = 0$.

2.3 Optimal Contract

In this section, we derive the optimal contract of each agent i ($i = 1, 2$). The contract is written in terms of his continuation-utility v_i, which is the total utility that the principal expects the agent to derive at any time t. Given v_i, agent i's contract specifies his utility-flow u_i, his continuation-utility $\bar{v}_{i,i}$ if he makes a discovery, his continuation-utility $\bar{v}_{i,-i}$ if his coworker makes a discovery, and the law of motion of his continuation utility if both agents fail.

Although our model is a continuous-time model, it can be interpreted as the limit of discrete-time models in which each period lasts Δt. In the discrete-time approximation, for any given continuation-utility v_i, the principal needs to decide $\{u_i, \underline{v}_i, \bar{v}_{i,i}, \bar{v}_{i,-i}\}$ in each period:

- u_i is agent i's transferred-utility flow in the current period.
- $\underline{v}_i$ is his next-period continuation utility if both agents fails.
- $\bar{v}_{i,i}$ is his next-period continuation utility if he successes.
- $\bar{v}_{i,-i}$ is his next-period continuation utility if his coworker successes.

First of all, the contract should indeed provide agent i with continuation-utility v_i. Hence, the contract should satisfy the following promise-keeping condition:

$$r(u_i - l)\Delta t + e^{-r\Delta t}((1-\lambda\Delta t)\underline{v}_i + \lambda_i \Delta t \bar{v}_{i,i} + \lambda_{-i}\Delta t \bar{v}_{i,-i}) = v_i.$$

The left-hand side of the promise-keeping condition is agent i's expected utility from putting in effort when the other agent exerts effort. The expected utility should equal the promised level v_i. Moreover, the contract should provide an incentive to agent i to exert effort conditional on the other agent putting in effort. Hence, the contract should also satisfy the following Nash-Incentive-Compatibility (NIC) condition:

$$\begin{aligned} r(u_i - l)\Delta t + e^{-r\Delta t}((1-\lambda\Delta t)\underline{v}_i + \lambda_i \Delta t \bar{v}_{i,i} + \lambda_{-i}\Delta t \bar{v}_{i,-i}) \\ \geq r u_i \Delta t + e^{-r\Delta t}((1-\hat{\lambda}_{-i}\Delta t)\underline{v}_i + \hat{\lambda}_{-i}\Delta t \bar{v}_{i,-i}). \end{aligned}$$

where the right-hand side agent i's expected utility from shirking when the other agent exerts effort.

Let $W_n(v_1, v_2)$ be the principal's minimum cost of delivering the continuation-utility pair (v_1, v_2) when the project is at stage n. In recursive form, W_n satisfies the following Bellman equation

$$W_n(v_1, v_2) = \min_{u_1,u_2;\underline{v}_1,\underline{v}_2;\bar{v}_{1,1},\bar{v}_{1,2};\bar{v}_{2,1},\bar{v}_{2,2}} r(S(u_1) + S(u_2))\Delta t$$

$$+ e^{-r\Delta t}((1 - \lambda\Delta t)W_n(\underline{v}_1, \underline{v}_2) + \lambda_1 \Delta t W_{n+1}(\bar{v}_{1,1}, \bar{v}_{2,1}) + \lambda_2 \Delta t W_{n+1}(\bar{v}_{1,2}, \bar{v}_{2,2}))$$

s.t.

$$r(u_i - l)\Delta t + e^{-r\Delta t}((1 - \lambda\Delta t)\underline{v}_i + \lambda_i \Delta t \bar{v}_{i,i} + \lambda_{-i}\Delta t \bar{v}_{i,-i}) = v_i,$$

$$r(u_i - l)\Delta t + e^{-r\Delta t}((1 - \lambda\Delta t)\underline{v}_i + \lambda_i \Delta t \bar{v}_{i,i} + \lambda_{-i}\Delta t \bar{v}_{i,-i})$$

$$\geq r u_i \Delta t + e^{-r\Delta t}((1 - \hat{\lambda}_{-i}\Delta t)\underline{v}_i + \hat{\lambda}_{-i}\Delta t \bar{v}_{i,-i}),$$

for $i = 1, 2$.

When the last-stage innovation is completed, both agents receive a lump-sum payment, and the principal's cost of providing these payments is given by $W_{N+1}(v_1, v_2) = S(v_1) + S(v_2)$. Note that agent i's promise-keeping condition and NIC condition only involve $(u_i, \underline{v}_i, \bar{v}_{i,i}, \bar{v}_{i,-i})$ and do not depend on the other agent's policy variables. This property implies that the cost function of the last stage is separable: $W_N(v_1, v_2) = C_{1,N}(v_1) + C_{2,N}(v_2)$, where $C_{i,N}$ is the principal's cost function of providing agent i with continuation utility v_i when the project is at stage N. Then, using the argument again, the result that W_N is separable implies that W_{N-1}

is also separable. Finally, we could conclude that the cost function of every stage is separable: $W_n(v_1, v_2) = C_{1,n}(v_1) + C_{2,n}(v_2)$, where $C_{i,n}$ satisfies the following Bellman equation

$$C_{i,n}(v_i) =$$

$$\min_{u_i,\underline{v}_i,\bar{v}_{i,i},\bar{v}_{i,-i}} rS(u_i)\Delta t + e^{-r\Delta t}((1-\lambda\Delta t)C_{i,n}(\underline{v}_i) + \lambda_i \Delta t C_{i,n+1}(\bar{v}_{i,i}) + \lambda_{-i}\Delta t C_{i,n+1}(\bar{v}_{i,-i}))$$

s.t.

$$r(u_i - l)\Delta t + e^{-r\Delta t}((1-\lambda\Delta t)\underline{v}_i + \lambda_i \Delta t \bar{v}_{i,i} + \lambda_{-i}\Delta t \bar{v}_{i,-i}) = v_i,$$

$$r(u_i - l)\Delta t + e^{-r\Delta t}((1-\lambda\Delta t)\underline{v}_i + \lambda_i \Delta t \bar{v}_{i,i} + \lambda_{-i}\Delta t \bar{v}_{i,-i})$$
$$\geq r u_i \Delta t + e^{-r\Delta t}((1-\hat{\lambda}_{-i}\Delta t)\underline{v}_i + \hat{\lambda}_{-i}\Delta t \bar{v}_{i,-i}),$$

for $i = 1, 2$. Hence, we could decentralize the two-agent problem and focus on the contract for each agent i.

Since the continuous-time model can be interpreted as the limit of the discrete-time approximation, multiplying both sides of the Bellman equation and the promise-keeping condition by $(1+r\Delta t)/\Delta t$ and letting Δt converge to 0, we derive the following Hamilton-Jacobi-Bellman (HJB) equation in continuous time [1]

$$rC_{i,n}(v_i) = \min_{u_i,\bar{v}_{i,i},\bar{v}_{i,-i}} rS(u_i) + C'_{i,n}(v_i)\dot{v}_i - \lambda C_{i,n}(v_i) + \lambda_i C_{i,n+1}(\bar{v}_{i,i}) + \lambda_{-i} C_{i,n+1}(\bar{v}_{i,-i})$$

[1] In this paper, we derive the HJB equation, evolution of continuation utility, and the NIC condition in continuous time by considering the limit of a discrete-time approximation. We can also derive these formally using stochastic-calculus techniques (see Biais et al. (2010)). The reason we choose this method is because it is more intuitive and generates the same result.

s.t.

$$\dot{v}_i = rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i),$$

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) \geq rl \text{ (NIC)}.$$

In continuous time, the promise-keeping condition becomes the evolution of agent i's continuation utility. In the discrete-time model, after choosing $(u_i, \bar{v}_{i,i}, \bar{v}_{i,-i})$, $\underline{v}_i$ is given by the promise-keeping condition. When Δt converges to 0, $\underline{v}_i$ converges to v_i. Hence, the continuation utility in the case of failure changes smoothly, and its rate of change is determined by $(u_i, \bar{v}_{i,i}, \bar{v}_{i,-i})$. The continuation utility can be explained as the value that the principal owes the agent: it grows at the discount-rate r; and it falls due to the flow of repayment $r(u_i - l)$, the expected repayment $\lambda_i(\bar{v}_{i,i} - v_i)$ if agent i completes the innovation, and the expected repayment $\lambda_{-i}(\bar{v}_{i,-i} - v_i)$ if his coworkers completes the innovation.

By putting in effort, agent i increase his arrival rate of success from 0 to λ_i and changes his coworker's arrival rate from $\hat{\lambda}_{-i}$ to λ_{-i}. Therefore, the left-hand side of the NIC condition is his benefit of putting in effort. The right-hand side is his cost of putting in effort. The NIC condition indicates that, to induce agent i to work, the contract should offer him a higher benefit than the cost of working.

The sign of $(\lambda_{-i} - \hat{\lambda}_{-i})$ in the NIC condition has very important implications on the optimal contract. Recall that λ_{-i} is the arrival rate of the event that agent i's coworker makes a discovery when both agents put in effort, and $\hat{\lambda}_{-i}$ is the arrival rate of the event that agent i's coworker makes a discovery when agent i shirks but his coworker exerts effort. When $\lambda_{-i} = \hat{\lambda}_{-i}$, the efforts of agent i and the efforts of his coworker are independent because agent i's action does not affect his coworker's performance. When $\lambda_{-i} < \hat{\lambda}_{-i}$, their efforts are substitutes. When agent i chooses to exert effort, this action decreases his coworker's arrival rate from $\hat{\lambda}_{-i}$ to λ_{-i}. Thus, agent i's effort has negative externality on his coworker's performance. When $\lambda_{-i} > \hat{\lambda}_{-i}$, their efforts are complements. When agent i chooses to exert effort, this action increases his coworker's arrival rate from $\hat{\lambda}_{-i}$ to λ_{-i}. Thus, agent i's effort has positive externality on his coworker's performance.

Note that the continuation utility cannot be less than 0, because the agents can guarantee a utility level of 0 by not putting in any effort. Therefore, a negative continuation utility is not implementable.

In the HJB equation, to solve the stage-n problem, we need to know the functional form of $C_{i,n+1}$. Observe that when the last-stage innovation is completed, the cost of providing continuation-utility v_i is given by $S(v_i)$, which is known. Hence, we solve the principal's problem by backward induction. Our plan is the following. First, we assume that $C_{i,n+1}$ satisfies the following assumption

Assumption A: *$C_{i,n+1}$ is a C^2 function. Its derivative, $C'_{i,n+1}$, is a continuous and strictly increasing function. Moreover, $C'_{i,n+1}$ satisfies:*

1. *If $\lambda_{-i} \leq \hat{\lambda}_{-i}$, then $C'_{i,n+1}(v_i) \geq S'(v_i)$ for all $v_i > 0$, and $C'_{i,n+1}(0) = S'(0) = 0$.*

2. *If $\lambda_{-i} > \hat{\lambda}_{-i}$, then $C'_{i,n+1}(v_i) > S'(v_i - \frac{\hat{\lambda}_{-i}l}{\lambda - \hat{\lambda}_{-i}})$ for all $v_i \geq \frac{\hat{\lambda}_{-i}l}{\lambda - \hat{\lambda}_{-i}}$.*

Then, we derive $C_{i,n}$ from the HJB equation given $C_{i,n+1}$, and show that $C_{i,n}$ also satisfies Assumption A. It is straightforward to check that S satisfies Assumption A.

This result allows us to keep doing the backward-induction exercise until we solve the entire multi-stage problem.

To characterize the solution of the HJB equation, we do a diagrammatic analysis in the v_i-$C'_{i,n}(v_i)$ plane. Given a point $(v_i, C'_{i,n}(v_i))$ in this plane, $(u_i, \bar{v}_{i,i}, \bar{v}_{i,-i})$ are determined by the following Kuhn-Tucker conditions:

$$S'(u_i) - C'_{i,n}(v_i) + \eta_1 = 0, \tag{2.1}$$

$$\lambda_i C'_{i,n+1}(\bar{v}_{i,i}) - \lambda_i C'_{i,n}(v_i) + \gamma\lambda_i + \eta_2 = 0, \tag{2.2}$$

$$\lambda_{-i} C'_{i,n+1}(\bar{v}_{i,-i}) - \lambda_{-i} C'_{i,n}(v_i) + \gamma(\lambda_{-i} - \hat{\lambda}_{-i}) + \eta_3 = 0, \tag{2.3}$$

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) - rl \geq 0, \tag{2.4}$$

$$u_i \geq 0, \tag{2.5}$$

$$\bar{v}_{i,i} \geq 0, \tag{2.6}$$

$$\bar{v}_{i,-i} \geq 0, \tag{2.7}$$

$$\gamma(\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) - rl) = 0, \tag{2.8}$$

$$\eta_1 u_i = 0, \tag{2.9}$$

$$\eta_2 \bar{v}_{i,i} = 0, \tag{2.10}$$

$$\eta_3 \bar{v}_{i,-i} = 0, \tag{2.11}$$

where γ, η_1 , η_2 and η_3 are Lagrangian multipliers and $\gamma, \eta_1, \eta_2, \eta_3 \leq 0$. Equation (2.1)-(2.3) are first-order conditions, (2.4) is the NIC condition, and inequality (2.5)-(2.7) imply that utility flow and continuation utility should be nonnegative.

To do the phase-diagram analysis, we need to determine the dynamics of v_i and $C'_{i,n}(v_i)$ at any point in the v_i-$C'_{i,n}(v_i)$ plane, which are determined by the sign of $dC'_{i,n}(v_i)/dt$ and dv_i/dt. The dynamics of v_i is given by

$$\frac{dv_i}{dt} = rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i).$$

Using the envelope theorem, we can derive the expression for $dC'_{i,n}(v_i)/dt$ from the HJB equation, which is

$$\frac{dC'_{i,n}(v_i)}{dt} = \gamma(\lambda - \hat{\lambda}_{-i}).$$

Therefore, given a point in the v_i-$C'_{i,n}(v_i)$ plane, the values of $(u_i, \bar{v}_{i,i}, \bar{v}_{i,-i}, \gamma)$ can be derived from Kuhn-Tucker conditions, which in turn determine the values of dv_i/dt and $dC'_{i,n}(v_i)/dt$. Finally, the sign of dv_i/dt and $dC'_{i,n}(v_i)/dt$ determine the dynamics at this point. Moreover, fixing v_i, the value of dv_i/dt could be treated as a function of $C'_{i,n}(v_i)$ and define it by $f(C'_{i,n}(v_i))$. The following six lemmas analyze the dynamics.

Lemma 2.3.1. *If $C'_{i,n}(v_i) \geq C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$, then the NIC condition is non-binding. The dynamics of $C'_{i,n}(v_i)$ and v_i satisfy*

$$\begin{aligned} \frac{dC'_{i,n}(v_i)}{dt} &= 0, \\ \frac{dv_i}{dt} &< 0. \end{aligned}$$

Lemma 2.3.2. *If $C'_{i,n}(v_i) < C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$, then the NIC condition is binding. The dynamics $C'_{i,n}(v_i)$ satisfies*

$$\frac{dC'_{i,n}(v_i)}{dt} < 0.$$

Fixing v_i, $f(C'_{i,n}(v_i))$ is a continuous function of $C'_{i,n}(v_i)$, which is decreasing in $C'_n(v_i)$.

For the case in which $\lambda_{-i} \leq \hat{\lambda}_{-i}$, we have

Lemma 2.3.3. *If $C'_{i,n}(v_i) = S'(v_i)$, then $dv_i/dt \geq 0$ when $v_i > 0$, and $dv_i/dt = 0$ when $v_i = 0$.*

Lemma 2.3.4. *The $dv_i/dt = 0$ locus is a continuous curve that locates below the $C'_{i,n}(v_i) = C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ locus and above the $C'_{i,n}(v_i) = S'(v_i)$ locus and intersects the $C'_{i,n}(v_i) = S'(v_i)$ locus at the origin.*

For the case in which $\lambda_{-i} > \hat{\lambda}_{-i}$, we have

Lemma 2.3.5. *If $C'_{i,n}(v_i) = S'(v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}})$, then $dv_i/dt > 0$.*

Lemma 2.3.6. *The $dv_i/dt = 0$ locus is a continuous curve that locates below the $C'_{i,n}(v_i) = C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ locus and above the $C'_{i,n}(v_i) = S'(v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}})$ locus.*

These lemmas characterize the dynamics of v_i and $C'_{i,n}(v_i)$ in the v_i-$C'_{i,n}(v_i)$ plane. The $dv_i/dt = 0$ locus determines the dynamics of v_i: v_i is decreasing over time above it and increasing over time below it. The $C'_{i,n}(v_i) = C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ locus determines the dynamics of $C'_{i,n}(v_i)$: $C'_{i,n}(v_i)$ is constant over time above it and decreasing over time below it (Figure 2.1 and Figure 2.2).

The next step is to find the optimal path in these phase diagrams. First consider the phase diagram for the case in which $\lambda_{-i} \leq \hat{\lambda}_{-i}$ (Figure 2.1). From the theorem regarding the existence of a solution to a differential equation, there is an unique path from any $v_i > 0$ to the origin (Path 1 in Figure 2.1). First, any path on which the state variable v_i diverges to infinity could be ruled out. This contains the

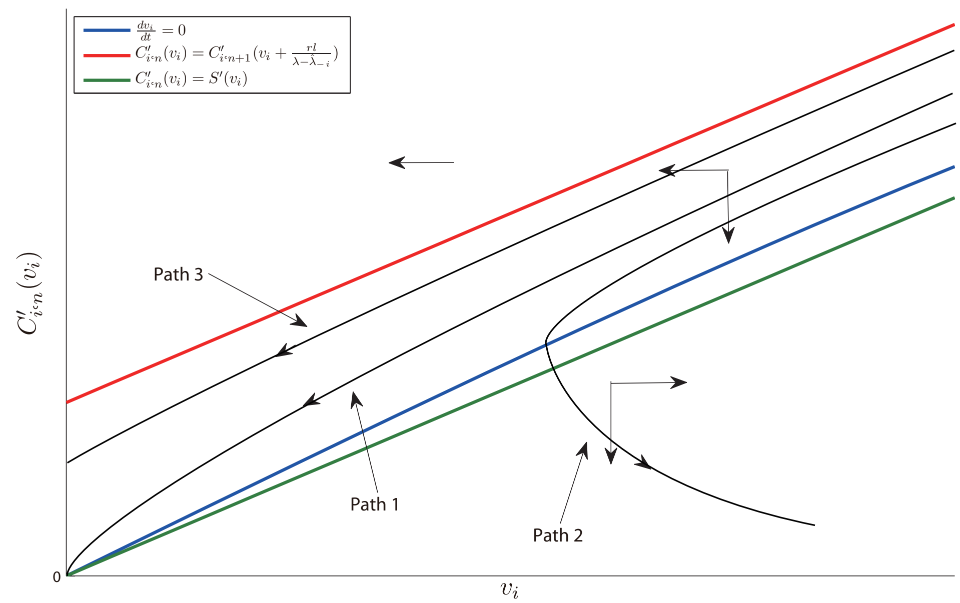

Figure 2.1: Phase Diagram ($\lambda_{-i} \leq \hat{\lambda}_{-i}$)

area below Path 1. In the area above Path 1, the continuation-utility v_i is decreasing over time. When v_i hits the lower bound 0, it cannot decrease any further. Thus, we must have $dv_i/dt \geq 0$ at $v_i = 0$. This condition rules out any path above Path 1, because $dv_i/dt < 0$ at $v_i = 0$ on these paths. Then, Path 1 is the only candidate path left in the phase diagram, and hence it is the optimal path that we are looking for. The final step is to pin down the boundary condition at $v_i = 0$. At the lower-bound, we have $u_i = \bar{v}_{i,-i} = 0$ and $\bar{v}_{i,i} = \frac{rl}{\lambda_i}$. Then,

$$\frac{dv_i}{dt} = rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i) = 0.$$

Therefore, when agent i's continuation utility reaches 0, his continuation utility and transferred-utility flow remain at 0 until he makes a discovery. To force agent i to put in positive effort, the principal rewards him by increasing his continuation utility to $\frac{rl}{\lambda_i}$ when he makes a discovery. We also can pin down the following boundary condition at $v_i = 0$ from the HJB equation

$$C_{i,n}(0) = \frac{\lambda_i C_{i,n+1}(\frac{rl}{\lambda_i}) + \lambda_{-i} C_{i,n+1}(0)}{r + \lambda}.$$

The optimal path and the boundary condition together determine the solution of the HJB equation. The phase-diagram analysis for the case in which $\lambda_{-i} > \hat{\lambda}_{-i}$ is similar (Figure 2.2).

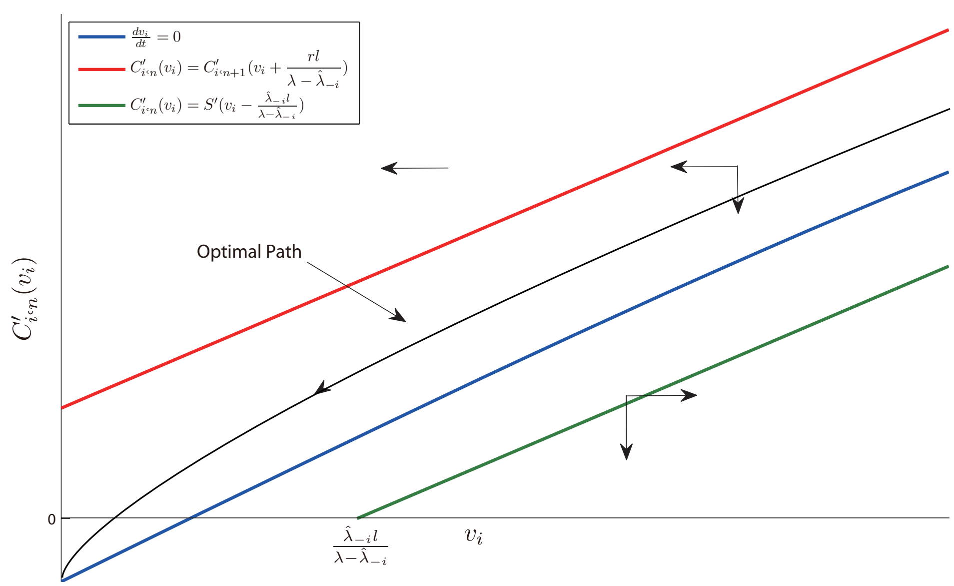

Figure 2.2: Phase Diagram ($\lambda_{-i} > \hat{\lambda}_{-i}$)

Finally, from the phase-diagram, when $\lambda_{-i} \leq \hat{\lambda}_{-i}$, the optimal path is located above the $C'_{i,n}(v_i) = S'(v_i)$ locus and intersects the $C'_{i,n}(v_i) = S'(v_i)$ locus at the origin; when $\lambda_{-i} > \hat{\lambda}_{-i}$, the optimal path is located above the $C'_{i,n}(v_i) = S'(v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}})$ locus. Therefore, we have the following Lemma.

Lemma 2.3.7. *1. If $\lambda_{-i} \leq \hat{\lambda}_{-i}$, then $C'_{i,n}(v_i) \geq S'(v_i)$ for all $v_i > 0$, and $C'_{i,n}(0) = S'(0) = 0$.*

2. If $\lambda_{-i} > \hat{\lambda}_{-i}$, then $C'_{i,n}(v_i) > S'(v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}})$ for all $v_i \geq \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}$.

Lemma 2.3.7 indicates that, if $C_{i,n+1}$ satisfies Assumption A, $C_{i,n}$ also satisfies Assumption A. This result completes the final step of the backward-induction argument.

The following proposition summarizes the properties of the optimal dynamic contract for agent i.

Proposition 2.3.8. *The contract that minimizes the principal's cost takes the following form:*

1. *The principal's expected cost of delivering continuation utility v_i at stage n is given by a convex function $C_{i,n}(v_i)$ that solves the HJB equation and satisfies the boundary condition*

$$C_{i,n}(0) = \frac{\lambda_i C_{i,n+1}(\frac{rl}{\lambda_i}) + \lambda_{-i} C_{i,n+1}(0)}{r + \lambda}.$$

2. *If agent i completes the innovation, then his utility flow jumps up.*

3. *If agent i's coworker completes the innovation, then: 1) his utility flow does not*

change if $\lambda_{-i} = \hat{\lambda}_{-i}$; 2) his utility flow drops down if $\lambda_{-i} < \hat{\lambda}_{-i}$; 3) his utility flow jumps up if $\lambda_{-i} > \hat{\lambda}_{-i}$.

4. *If both agents fail to complete the project, agent i's continuation-utility v_i and transferred utility u_i are decreasing over time and v_i asymptotically goes to* 0.

In the optimal contract, the principal rewards agent i when he makes an innovation by increasing his utility flow. In our setup, agent i has a chance to make a discovery only when he puts in effort. Thus, a discovery by him indicates that he is exerting effort, and therefore he should be rewarded.

Part 3 of Proposition 2.3.8 is the main result of this paper, which demonstrates the way in which the optimal incentive regime is a function of how agents' efforts interact with one another. When $\lambda_{-i} < \hat{\lambda}_{-i}$, the principal uses relative-performance evaluation in which he punishes agent i by decreasing his payment-flow when his coworker makes a discovery. The reason for using relative-performance evaluation is the following. In this case, agent i's effort has a negative externality on his coworker's performance. Thus, when his coworker makes a discovery, this event provides suggestive information that agent i is shirking. Therefore, the principal should punish agent i for not putting in effort.

On the contrary, when $\lambda_{-i} > \hat{\lambda}_{-i}$, agent i's effort has a positive externality on his coworker's performance. The event that his coworker achieves a success gives the principal a hint that agent i is also exerting effort. Therefore, the principal uses joint-performance evaluation in which he rewards agent i by an upward jump in his payment-flow when his coworker makes a discovery.

When $\lambda_{-i} = \hat{\lambda}_{-i}$, since agent i's action does not affect his coworker's performance, the event that his coworker makes a discovery does not offer any useful information about whether agent i puts in effort or not. Hence, agent i's utility flow does not change when his coworker makes a discovery.

2.4 Example

In this section, we provide an example for which we obtain a closed-form solution. In fact, in this example, we can handle the case in which the project has infinitely many stages. When the project has infinite stages, principal's cost function $W(v_1, v_2)$ no longer depends the stage level. Similarly, since agent i's NIC condition and evolution of continuation utility does not relate to the other agent's policy variables, the cost function has a separated form: $W(v_1, v_2) = C_1(v_1) + C_2(v_2)$, where $C_i(v_i)$ is the principals minimum cost of providing agent i with continuation-utility v_i. The optimal contract for agent i is characterized by the following HJB equation

$$rC_i(v_i) = \min_{u_i, \bar{v}_{i,i}, \bar{v}_{i,-i}} rS(u_i) + C_i'(v_i)\dot{v}_i - \lambda C_i(v_i) + \lambda_i C_i(\bar{v}_{i,i}) + \lambda_{-i} C_i(\bar{v}_{i,-i})$$

s.t.

$$\dot{v}_i = rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i),$$

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) \geq rl.$$

We assume that the utility function is logarithmic $U(c_i) = \log c_i$. We solve this HJB equation by guess-and-verify. First, note that, for logarithmic utility function, the cost of providing transferred utility-flow u_i is $S(u_i) = e^{u_i}$. Inspired by this functional form, we make a guess that the cost function takes the form qe^{v_i} $(q > 0)$— a constant times e^{v_i}. Then, using this guess, we solve the minimization problem on the right-hand side of the HJB equation. If the right-hand side also takes the form of a constant times e^{v_i}, then this guess is verified, and we can pin down the constant q from the HJB equation.

Taking $C_i(v_i) = qe^{v_i}$ into the right-hand side of the HJB equation, we have

$$RHS = \min_{u_i, \bar{v}_{i,i}, \bar{v}_{i,-i}} re^{u_i} + qe^{v_i}\dot{v}_i - \lambda qe^{v_i} + \lambda_i qe^{\bar{v}_{i,i}} + \lambda_{-i} qe^{\bar{v}_{i,-i}}$$

s.t.

$$\dot{v}_i = rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i),$$

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) \geq rl.$$

Utility-flow u_i satisfies the first-order condition $S'(u_i) = C_i'(v_i)$. Therefore,

$$e^{u_i} = qe^{v_i},$$

which implies $u_i = v_i + \log q$.

The NIC condition must be blinding, otherwise first-order conditions imply that $\bar{v}_{i,i} = \bar{v}_{i,-i} = v_i$, which violates the NIC condition. Thus, $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ are determined by the following system

$$\lambda_i qe^{\bar{v}_{i,i}} - \lambda_i qe^{v_i} + \gamma\lambda_i = 0 \tag{2.12}$$

$$\lambda_{-i} qe^{\bar{v}_{i,-i}} - \lambda_{-i} qe^{v_i} + \gamma(\lambda_{-i} - \hat{\lambda}_{-i}) = 0 \tag{2.13}$$

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) - rl = 0 \tag{2.14}$$

where (2.12) and (2.13) are first-order conditions, and (2.14) is the NIC condition. γ, the Lagrangian multiplier, satisfies $\gamma < 0$.

If $\lambda_{-i} = \hat{\lambda}_{-i}$, then it follows from (2.13) and (2.14) that

$$\bar{v}_{i,i} = v_i + \frac{rl}{\lambda_i},$$

$$\bar{v}_{i,-i} = v_i.$$

If $\lambda_{-i} \neq \hat{\lambda}_{-i}$, define $\Delta v_{i,i} = \bar{v}_{i,i} - v_i$ and $\Delta v_{i,-i} = \bar{v}_{i,-i} - v_i$. Combining (2.12) and (2.13), we could get

$$\frac{e^{\Delta v_{i,i}} - 1}{e^{\Delta v_{i,-i}} - 1} = \frac{\lambda_{-i}}{\lambda_{-i} - \hat{\lambda}_{-i}}. \tag{2.15}$$

Equation (2.14) could be rewritten as

$$\lambda_i \Delta v_{i,i} + (\lambda_{-i} - \hat{\lambda}_{-i})\Delta v_{i,-i} - rl = 0. \tag{2.16}$$

Then, we can pin down $\Delta v_{i,i}$ and $\Delta v_{i,-i}$ by solving (2.15) and (2.16). Note that neither (2.15) nor (2.16) contains v_i, which implies that both $\Delta v_{i,i}$ and $\Delta v_{i,-i}$ depend only on the parameters of the model and are independent of the state-variable v_i.

Consequently, in both cases, we have $\bar{v}_{i,i} = v_i + \Delta v_{i,i}$ and $\bar{v}_{i,i} = v_i + \Delta v_{i,-i}$, where both $\Delta v_{i,i}$ and $\Delta v_{i,-i}$ are independent of v_i.

Taking the solution for u_i, $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ into the right-hand side of the HJB equation, it becomes

$$\begin{aligned} RHS &= re^{v_i + \log q} + qe^{v_i}(-r\log q - \hat{\lambda}_{-i}\Delta v_{i,-i}) - \lambda q e^{v_i} + \lambda_i q e^{v_i + \Delta v_{i,i}} + \lambda_{-i} q e^{v_i + \Delta v_{i,-i}} \\ &= (rq + q(-r\log q - \hat{\lambda}_{-i}\Delta v_{i,-i}) - \lambda q + \lambda_i q e^{\Delta v_{i,i}} + \lambda_{-i} q e^{\Delta v_{i,-i}})e^{v_i} \end{aligned}$$

This result verifies that the right-hand side also takes the form of a constant times e^{v_i}. Finally, letting the left-hand side of the HJB equation equal to the right-hand side, we have

$$rq = rq + q(-r \log q - \hat{\lambda}_{-i} \Delta v_{i,-i}) - \lambda q + \lambda_i q e^{\Delta v_{i,i}} + \lambda_{-i} q e^{\Delta v_{i,-i}}.$$

Solving q, we get

$$q = \exp\left(\frac{\lambda_i e^{\Delta v_{i,i}} + \lambda_{-i} e^{\Delta v_{i,-i}} - \lambda - \hat{\lambda}_{-i} \Delta v_{i,-i}}{r} \right).$$

The above computation provides the solution to the HJB equation. Next, we derive some properties of the optimal contract implied by this solution. First, it follows from from (2.12) that $\bar{v}_{i,i} > v_i$, which means that the principal rewards agent i by an upward jump in continuation utility when he makes a discovery. From (2.13), we have

$$\bar{v}_{i,-i} \begin{cases} < v_i, & \text{if } \lambda_{-i} < \hat{\lambda}_{-i}; \\ = v_i, & \text{if } \lambda_{-i} = \hat{\lambda}_{-i}; \\ > v_i, & \text{if } \lambda_{-i} > \hat{\lambda}_{-i}. \end{cases}$$

Thus, when $\lambda_{-i} = \hat{\lambda}_{-i}$, agent i's continuation utility does not depends on the other agent's performance; when $\lambda_{-i} < \hat{\lambda}_{-i}$, agent i is punished by a downward jump in continuation utility when his coworker succeeds (relative-performance evaluation); and when $\lambda_{-i} > \hat{\lambda}_{-i}$ agent i is rewarded by an upward jump in continuation utility when his coworker succeeds (joint-performance evaluation). Finally, the evolution of continuation utility in case of failure follows

$$\begin{aligned}
\dot{v}_i &= rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i) \\
&= r(v_i - u_i) - \hat{\lambda}_{-i}(\bar{v}_{i,-i} - v_i) \\
&= \lambda - \lambda_i e^{\Delta v_{i,i}} - \lambda_{-i} e^{\Delta v_{i,-i}} \\
&= \lambda_i(1 - e^{\Delta v_{i,i}}) + \lambda_{-i}(1 - e^{\Delta v_{i,-i}}) \\
&= (\lambda - \hat{\lambda}_{-i})(1 - e^{\Delta v_{i,i}}) \\
&< 0,
\end{aligned}$$

which implies that agent i's continuation utility decreases over time when both agents fail. Since $u_i = v_i + \log q$, the utility-flow u_i has the same dynamics as the continuation-utility v_i. The properties of the optimal contract are summarized in the following proposition

Proposition 2.4.1. *The optimal contract has the following properties:*

1. *If agent i completes the innovation, then his utility flow jumps up.*

2. *If agent i's coworker completes the innovation, then: 1) agent i's utility flow does not change if $\lambda_{-i} = \hat{\lambda}_{-i}$; 2) his utility flow drops down if $\lambda_{-i} < \hat{\lambda}_{-i}$; 3) his utility flow jumps up if $\lambda_{-i} > \hat{\lambda}_{-i}$.*

3. *If both agents fail to complete the project, agent i's continuation-utility v_i and transferred utility u_i decrease over time.*

Observe that all the properties are consistent with those of the optimal contract discussed in Section 2.3, and the intuition behind these results are also the same.

2.5 Conclusion

This paper studies the agency problem between a firm and its in-house R&D unit. The problem is analyzed in a set-up that captures two distinct aspects, namely the multilateral feature and the multi-stage feature, in this specific agency relationship. We use recursive techniques to characterize the dynamic contract that overcome the multi-agent repeated moral-hazard problem. In the optimal contract, the principal provides incentive combining punishment and reward. He decreases every agent's payment if no discovery is made and rewards an agent by an upward jump in payment when he makes a discovery. Moreover, agents' payments not only depend on their own performances, but also may be tied to their peers' performances as well. Relative-performance evaluation is used if their efforts are substitutes whereas joint-performance evaluation is used if their efforts are complements.

APPENDIX A
APPENDIX FOR CHAPTER 1

A.1 Proofs

Proof of Lemma 1.3.1

Proof. In the HJB equation, the principal chooses $\bar{v}$ to minimize $-C'(v)\bar{v} + S(\bar{v})$ subject to the incentive-compatibility constraint $\bar{v} \geq v + \frac{rl}{\lambda}$. By assumption, $-C'(v)\bar{v} + S(\bar{v})$ is a convex and twice-differentiable function of $\bar{v}$. The unconstraint minimum is reached at $\bar{v}'$ that satisfies the first order condition $C'(v) = S'(\bar{v}')$. If $C'(v) > S'(v + \frac{rl}{\lambda})$, then $S(\bar{v}') > S(v + \frac{rl}{\lambda})$, which implies that $\bar{v}' > v + \frac{rl}{\lambda}$. Thus, the optimal choice of $\bar{v}$ is $\bar{v}'$ and the incentive-compatibility constraint is not binding. $C'(v) \leq S'(v + \frac{rl}{\lambda})$ implies that $\bar{v}' \leq v + \frac{rl}{\lambda}$. In this case, the optimal choice of $\bar{v}$ is $v + \frac{rl}{\lambda}$ and the incentive-compatibility constraint binds. □

Proof of Lemma 1.3.2

Proof. By Lemma 1.3.1, the incentive-compatibility constraint binds in this case. Using the equation $\bar{v} = v + \frac{rl}{\lambda}$ in (1.3), the rate of change of v becomes $\frac{dv}{dt} = r(v - u)$. Therefore, the HJB equation is

$$rC(v) = \min_u rS(u) + C'(v)(r(v-u)) + \lambda(S(v + \frac{rl}{\lambda}) - C(v)).$$

From the envelope theorem,

$$(r + \lambda)C'(v) = rC'(v) + \lambda S'(v + \frac{rl}{\lambda}) + C''(v)\frac{dv}{dt}.$$

Thus,

$$\frac{dC'(v)}{dt} = \lambda(C'(v) - S'(v + \frac{rl}{\lambda})).$$

Since $C'(v) < S'(v + \frac{rl}{\lambda})$, it follows that $\frac{dC'(v)}{dt} < 0$.

As $\frac{dv}{dt} = r(v - u)$, the sign of $\frac{dv}{dt}$ is determined by the values of v and u. Note that u is chosen to minimize $S(u) - C'(v)u$, which is a strictly convex function of u. The first order condition implies that $C'(v) = S'(u)$. If $C'(v) = S'(v)$, then $S'(v) = S'(u)$. Since $S(v)$ is strictly convex, we have $v = u$ and $\frac{dv}{dt} = r(v - u) = 0$. Similarly, $C'(v) > S'(v)$ implies that $\frac{dv}{dt} < 0$ and $C'(v) < S'(v)$ implies that $\frac{dv}{dt} > 0$ □

Proof of Lemma 1.3.3

Proof. In this case, $\bar{v}$ is unconstrained optimal and $\bar{v} \geq v + \frac{rl}{\lambda}$. Taking (1.3) into the HJB equation,

$$rC(v) = \min_{u,\bar{v}} rS(u) + C'(v)(rv - r(u - l) - \lambda(\bar{v} - v)) + \lambda(S(\bar{v}) - C(v)).$$

From envelope theorem

$$(r + \lambda)C'(v) = (r + \lambda)C'(v) + C''(v)\frac{dv}{dt}.$$

Therefore,

$$\frac{dC'(v)}{dt} = 0.$$

For the dynamics of v, note that in this case

$$\begin{aligned} \frac{dv}{dt} &= rv - r(u - l) - \lambda(\bar{v} - v) \\ &= r(v - u) + (rl + \lambda v - \lambda\bar{v}). \end{aligned}$$

Since $\bar{v} \geq v + \frac{rl}{\lambda}$, the second term is non-positive. For the first term, u is determined by the first order condition $C'(v) = S'(u)$. Since $C'(v) \geq S'(v + \frac{rl}{\lambda})$, we have $S'(u) \geq S'(v + \frac{rl}{\lambda})$, which implies that $u \geq v + \frac{rl}{\lambda} > v$. Thus, the first term is strictly negative. It follows that $\frac{dv}{dt} < 0$. □

Proof of Proposition 1.3.6

Proof. The proofs for part 1 to part 5 are similar to the proofs for Proposition 1.3.4. Hence, we only prove part 6 of Proposition 1.3.6. We prove this by backward induction. From Proposition 1.3.4, we have that $C_N(v) > C_{N+1}(v) = S(v)$ for all v, and $C'_N(v) > C'_{N+1}(v) = S'(v)$ for all $v > 0$. Suppose these two inequalities hold for stage $n + 1$. We want to show that they also hold for stage n.

Note that on the optimal path, we have

$$\frac{dC'_n(v)}{dt} = \lambda(C'_n(v) - C'_{n+1}(v + \frac{rl}{\lambda})),$$

and

$$\frac{dv}{dt} = r(v - u).$$

Hence, we have

$$\frac{dC'_n(v)}{dv} = \frac{\lambda(C'_n(v) - C'_{n+1}(v + \frac{rl}{\lambda}))}{r(v - u)},$$

where u satisfies $C'_n(v) = S'(u)$.

Suppose $C'_n(v^*) = C'_{n+1}(v^*)$ for some $v^* \geq 0$. Then, we have

$$\frac{dC'_n(v^*)}{dv^*} = \frac{\lambda(C'_n(v^*) - C'_{n+1}(v^* + \frac{rl}{\lambda}))}{r(v^* - u^*)} > \frac{\lambda(C'_{n+1}(v^*) - C'_{n+2}(v^* + \frac{rl}{\lambda}))}{r(v^* - u^*)} = \frac{dC'_{n+1}(v^*)}{dv^*},$$

where $S'(u^*) = C'_n(v^*) = C'_{n+1}(v^*)$ and the inequality follows from $C'_{n+1}(v^* + \frac{rl}{\lambda}) > C'_{n+2}(v^* + \frac{rl}{\lambda})$.

This result implies that there exists at most one v^* such that $C'_n(v^*) = C'_{n+1}(v^*)$. Moreover, $C'_n(v) < C'_{n+1}(v)$ for all $v < v^*$, and $C'_n(v) > C'_{n+1}(v)$ for all $v > v^*$. Since $C'_n(0) = C'_{n+1}(0) = 0$, it implies that $C'_n(v) > C'_{n+1}(v)$ for all $v > 0$.

Finally, we have

$$C_n(0) = \frac{\lambda C_{n+1}(\frac{rl}{\lambda})}{r+\lambda} > \frac{\lambda C_{n+2}(\frac{rl}{\lambda})}{r+\lambda} = C_{n+1}(0).$$

Then, $C'_n(v) > C'_{n+1}(v)$ for all $v > 0$ implies that $C_n(v) > C_{n+1}(v)$ for all $v > 0$. By backward induction, we can show that, at any stage n, $C_n(v) > C_{n+1}(v)$ for all $v \geq 0$, and $C'_n(v) > C'_{n+1}(v)$ for all $v > 0$. □

Proof of Proposition 1.4.1

Proof. We first verify that $V_n(y) = C_n^{-1}(y)$ solves the HJB equation under the conditions in Proposition 1.4.1. Then,we show that this implementation generates the same consumption allocation as the optimal contract. First note that

$$\begin{aligned} rl - \lambda(V_{n+1}(Y_{n+1}(y)) - V_n(y)) &= rl - \lambda(V_{n+1}(C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})) - V_n(y)) \\ &= rl - \lambda(C_n^{-1}(y) + \frac{rl}{\lambda} - C_n^{-1}(y)) \\ &= 0. \end{aligned}$$

This result implies that for any consumption flow c, the agent is indifferent between exerting effort and shirking. Thus, we have

$$\begin{aligned} RHS &= rU(c) + V_n'(y)(ry - rc - \lambda(C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda}) - y)) \\ &= rU(c) + \frac{(r+\lambda)y - rc - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})}{C_n'(C_n^{-1}(y))}, \end{aligned}$$

where c is determined by the first-order condition $U'(c) = \frac{1}{C_n'(C_n^{-1}(y))}$.

Since $C_n(v)$ satisfies the following differential equation

$$(r+\lambda)C_n(v) = rS(u) + C_n'(v)(r(v-u)) + \lambda C_{n+1}(v + \frac{rl}{\lambda}),$$

then

$$\frac{1}{C_n'(v)} = \frac{r(v-u)}{(r+\lambda)C_n(v) - rS(u) - \lambda C_{n+1}(v + \frac{rl}{\lambda})},$$

where u satisfies $S'(u) = C_n'(v)$. Taking $v = C_n^{-1}(y)$ into the equation above, we get

$$\begin{aligned} \frac{1}{C_n'(C_n^{-1}(y))} &= \frac{r(C_n^{-1}(y) - u)}{(r+\lambda)C_n(C_n^{-1}(y)) - rS(u) - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})} \\ &= \frac{r(C_n^{-1}(y) - u)}{(r+\lambda)y - rS(u) - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})}, \end{aligned}$$

where $S'(u) = C_n'(C_n^{-1}(y))$. Since $S(u) = U^{-1}(u)$, it follows that $\frac{1}{U'(S(u))} = C_n(C_n^{-1}(y))$. Hence, $S(u) = c$ and $u = U(c)$ since c satisfies $U'(c) = \frac{1}{C_n'(C_n^{-1}(y))}$. Therefore,

$$\frac{1}{C_n'(C_n^{-1}(y))} = \frac{r(C_n^{-1}(y) - U(c))}{(r+\lambda)y - rc - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})}.$$

Taking this expression for $\frac{1}{C_n'(C_n^{-1}(y))}$ into the right-hand side of the HJB equation, we have

$$\begin{aligned} RHS &= rU(c) + \frac{(r+\lambda)y - rc - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})}{C_n'(C_n^{-1}(y))} \\ &= rU(c) + r(C_n^{-1}(y) - U(c)) \\ &= rC_n^{-1}(y) \\ &= rV_n(y) \\ &= LHS. \end{aligned}$$

Thus, $V_n(y) = C_n^{-1}(y)$ solves the following HJB equation.

The first order condition implies that $S'(U(c)) = C_n'(C_n^{-1}(y)) = C_n'(V_n(y))$. Moreover, since the agent is indifferent between exerting effort and shirking, he is always willing to put in effort. □

APPENDIX B
APPENDIX FOR CHAPTER 2

B.1 Joint Performance

In this appendix, we derive the optimal contract for the case in which the principal can only observe joint performance of the team. The optimal contract for agent i is written in terms of his continuation-utility v_i. At any moment of time, given v_i, the contract specifies agent i's utility-flow u_i, the continuation-utility $\bar{v}_i$ if the one of the agents makes a discovery, and the law of motion of the continuation utility if both agents fail.

When the other agent exerts effort, by putting in effort instead of shirking, agent i increases the team's arrival rate from $\hat{\lambda}_{-i}$ to λ. After success, his continuation utility jumps from v_i to $\bar{v}_i$. Thus, his benefit of exerting effort is $(\lambda - \hat{\lambda}_{-i})(\bar{v}_i - v_i)$. His cost of putting in effort is rl. To induce agent i to work, the contract should offer him a higher benefit than cost of working, which leads to the following NIC condition

$$(\lambda - \hat{\lambda}_{-i})(\bar{v}_i - v_i) \geq rl.$$

When agent i exerts effort, his continuation utility grows at the discount-rate r and falls due to the flow of repayment $r(u_i - l)$ plus the expected repayment $\lambda(\bar{v}_i - v_i)$ if the team completes the innovation. Thus, his continuation utility in case of failure evolves according to

$$\dot{v}_i = rv_i - r(u_i - l) - \lambda(\bar{v}_i - v_i).$$

Let $W_n(v_1, v_2)$ be the principal's minimum cost of delivering continuation utility (v_1, v_2) when the project is at stage n. Note that agent i's NIC condition and evolution of continuation utility only depend on his own policy variables. This property implies that the cost function $W_n(v_1, v_2)$ has a separated form : $W_n(v_1, v_2) = C_{1,n}(v_1) + C_{1,n}(v_2)$, where $C_{i,n}$ is the principal's cost function of providing agent i with continuation utility v_i when the project is at stage n. $C_{i,n}$ is determined by the following HJB equation

$$rC_{i,n}(v_i) = \min_{u,\bar{v}} rS(u_i) + C'_{i,n}(v_i)\dot{v}_i + \lambda(C_{i,n+1}(\bar{v}_i) - C_{i,n}(v_i))$$

s.t.

$$\begin{aligned} \dot{v}_i &= rv_i - r(u_i - l) - \lambda(\bar{v}_i - v_i), \\ (\lambda - \hat{\lambda}_{-i})(\bar{v}_i - v_i) &\geq rl. \end{aligned}$$

Similar to the individual-perforce case, we could use a diagrammatic analysis to characterize the solution to the HJB equation. The property of the optimal contract is summarized in the following proposition

Proposition A: *At stage n $(0 < n \leq N)$, the contract for agent i that minimizes the principal's cost takes the following form:*

1. *The principal's expected cost at any point is given by an increasing and convex function $C_{i,n}(v_i)$ that satisfies the HJB equation and the boundary condition*
$$C_{i,n}\left(\frac{\hat{\lambda}_{-i}l}{\lambda - \hat{\lambda}_{-i}}\right) = \frac{\lambda C_{i,n+1}(\frac{(r+\hat{\lambda}_{-i})l}{\lambda - \hat{\lambda}_{-i}})}{r + \lambda}.$$

2. *When the the team completes the project, agent i's continuation utility jumps to $\bar{v}_i$, which satisfies $\bar{v}_i = v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}}$.*

3. *In case of failure to complete the innovation, the continuation-utility* v_i *is decreasing over time and asymptotically goes to* $\frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}$.

4. *The transferred-utility flow* u_i *has the same dynamics as continuation-utility* v_i.

Different from the individual-performance case, the lower bound on implementable continuation utility is a positive level: $\frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}$. The positive lower-bound is due to a free-rider problem that arises when only joint performance is observable. To provide incentive, the principal should reward every agent when the team completes an innovation. Thus, even if an agent shirks, he still has a chance to get the reward when the other agent succeeds. Therefore, the principal cannot punish the agents too severely. Otherwise, an agent will choose to shirk and free ride on the other agent's success if he cannot expect to get enough payments from his contract.

B.2 Proofs

Proof of Lemma 2.3.1

Proof. Suppose that the NIC condition is not binding and both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ are strictly positive. It follows that all the Lagrangian multipliers γ, η_2 and η_3 equal to 0. Then, first-order conditions (2.2) and (2.3) imply that $C'_{i,n+1}(\bar{v}_{i,i}) = C'_{i,n+1}(\bar{v}_{i,-i}) = C'_{i,n}(v_i)$. Since $C'_{i,n}(v_i) \geq C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ and $C'_{i,n+1}(v_i)$ is strictly increasing, it follows that $\bar{v}_{i,i} = \bar{v}_{i,-i} \geq v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}}$. Hence,

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) \geq (\lambda - \hat{\lambda}_{-i}) \frac{rl}{\lambda - \hat{\lambda}_{-i}} = rl.$$

Thus, the NIC condition is non-binding, and both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ are strictly positive, which verifies our guess. Then, the dynamics of $C'_{i,n}(v_i)$ satisfies

$$\frac{dC'_{i,n}(v_i)}{dt} = \gamma(\lambda - \hat{\lambda}_{-i}) = 0.$$

Next, we analyze the dynamics of v_i under two cases.

(i) $\lambda_{-i} \leq \hat{\lambda}_{-i}$

Since $C'_{i,n+1}(v_i) \geq S'(v_i)$ for all v_i by Assumption A, it follows that

$$C'_{i,n}(v_i) \geq C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}}) \geq S'(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}}) > S'(v_i).$$

When $C'_{i,n}(v_i) \geq C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}}) > 0$, utility-flow u_i is determined by first-order condition $S'(u_i) = C'_{i,n}(v_i)$. Consequently, $S'(u_i) > S'(v_i)$, which implies that $u_i > v_i$. Finally, we have

$$\begin{aligned} \frac{dv_i}{dt} &= rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v) - \lambda_{-i}(\bar{v}_{i,-i} - v) \\ &\leq r(v_i - u_i) - \frac{rl\hat{\lambda}_{-i}}{\lambda - \hat{\lambda}_{-i}} \\ &< 0, \end{aligned}$$

where the first inequality follows from $\bar{v}_{i,i} = \bar{v}_{i,-i} \geq v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}}$, and the second inequality follows from $u_i > v_i$.

(ii) $\lambda_{-i} > \hat{\lambda}_{-i}$

When $v_i \geq \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}$, we have

$$S'(u_i) = C'_{i,n}(v_i) \geq C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}}) > C'_{i,n+1}(v_i) > S'(v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}),$$

where the last inequality follows from Assumption A. Hence, $u_i > v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}$. When $v_i < \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}$, we have $u_i \geq 0 > v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}$. Thus, we always have $u_i > v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}$.

Therefore,

$$\begin{aligned}\frac{dv_i}{dt} &= rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i) \\ &\leq r(v_i - u_i) - \frac{r\hat{\lambda}_{-i}l}{\lambda - \hat{\lambda}_{-i}} \\ &< \frac{r\hat{\lambda}_{-i}l}{\lambda - \hat{\lambda}_{-i}} - \frac{r\hat{\lambda}_{-i}l}{\lambda - \hat{\lambda}_{-i}} \\ &= 0.\end{aligned}$$

□

Proof of Lemma 2.3.2

Proof. On the contrary, suppose the NIC condition is non-binding and hence $\gamma = 0$. The first-order conditions (2.2) and (2.3) imply that

$$\bar{v}_{i,i} = \bar{v}_{i,-i} < v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}},$$

where the last inequality follows from the condition that $C'_{i,n}(v_i) < C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$. Then,

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) < (\lambda - \hat{\lambda}_{-i})\frac{rl}{\lambda - \hat{\lambda}_{-i}} = rl.$$

The NIC condition is violated, which is a contradiction. Therefore, we must have the NIC condition binds and $\gamma < 0$. Then, the dynamics $C'_{i,n}(v_i)$ satisfies

$$\frac{dC'_{i,n}(v_i)}{dt} = \gamma(\lambda - \hat{\lambda}_{-i}) < 0.$$

To analyze the dynamics of v_i, we fix the value of v_i and variate the value of $C'_{i,n}(v_i)$. When we do this, the value of dv_i/dt is a function of the value of $C'_{i,n}(v_i)$.

Denote this function by $f(C'_{i,n}(v_i))$, the sign of which determined the dynamics of v_i. When the NIC condition is binding, we have

$$f(C'_{i,n}(v_i)) = r(v_i - u_i) - \hat{\lambda}_{-i}(\bar{v}_{i,-i} - v_i).$$

Given $C'_{i,n}(v_i)$, $\{u_i, \bar{v}_{i,i}, \bar{v}_{i,-i}\}$ are determined by the system of Kuhn-Tucker conditions. Moreover, since both S' and $C'_{i,n+1}$ are continuous functions, it follows that $f(C'_{i,n}(v_i))$ is continuous in $C'_{i,n}(v_i)$. Next, we derive the sign of $f(C'_{i,n}(v_i))$ under three cases.

Case 1: $\lambda_{-i} = \hat{\lambda}_{-i}$

When $C'_{i,n}(v_i) \geq 0$, u_i and $\bar{v}_{i,-i}$ are determined by the following first-order conditions

$$\begin{aligned} S'(u_i) - C'_{i,n}(v_i) &= 0, \\ C'_{i,n+1}(\bar{v}_{i,-i}) - C'_{i,n}(v_i) &= 0. \end{aligned}$$

Since both S' and $C'_{i,n+1}$ are strictly increasing functions, when we decrease the value of $C'_{i,n}(v_i)$, both u_i and $\bar{v}_{i,-i}$ decrease, and hence $f(C'_{i,n}(v_i))$ increases. Therefore, $f(C'_{i,n}(v_i))$ is a strictly decreasing function of $C'_{i,n}(v_i)$ when $C'_{i,n}(v_i) \geq 0$. If $C'_{i,n}(v_i) < 0$, then both u_i and $\bar{v}_{i,-i}$ equal 0. Hence, $f(C'_{i,n}(v_i))$ is a constant function of $C'_{i,n}(v_i)$ when $C'_{i,n}(v_i) < 0$.

Case 2: $\lambda_{-i} < \hat{\lambda}_{-i}$

When $C'_{i,n}(v_i) = C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$, from the proof of Lemma 2.3.1, both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ equal to $v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}} > 0$. Hence, both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ are positive when $C'_{i,n}(v_i)$ is very close to $C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$. When both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ are positive, they are determined by the following system of equations

$$\lambda_i C'_{i,n+1}(\bar{v}_{i,i}) - \lambda_i C'_{i,n}(v_i) + \gamma\lambda_i = 0, \tag{B.1}$$

$$\lambda_{-i} C'_{i,n+1}(\bar{v}_{i,-i}) - \lambda_{-i} C'_{i,n}(v_i) + \gamma(\lambda_{-i} - \hat{\lambda}_{-i}) = 0, \tag{B.2}$$

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) = rl. \tag{B.3}$$

Since $\lambda_i > 0$, $\lambda_{-i} - \hat{\lambda}_{-i} < 0$, and $\gamma < 0$, (B.1) and (B.2) imply that

$$C'_{i,n+1}(\bar{v}_{i,i}) > C'_{i,n}(v_i) > C'_{i,n+1}(\bar{v}_{i,-i}).$$

Combining (B.1) and (B.2), we have

$$(\lambda_{-i} - \hat{\lambda}_{-i})C'_{i,n+1}(\bar{v}_{i,i}) - \lambda_i C'_{i,n+1}(\bar{v}_{i,-i}) = -\hat{\lambda}_{-i} C'_{i,n}(v_i). \tag{B.4}$$

When we decrease $C'_{i,n}(v_i)$ starting from $C'_{i,n+1}(v + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$, the right-hand side of (B.4) increases. It follows from (B.3) that $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ variate in the same direction. Suppose both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ go up. Then, the left-hand side of (B.4) drops off, and the equality fails to hold. Thus, to let (B.4) hold, we must have both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ decrease when $C'_{i,n}(v_i)$ decreases. Moreover, we know that u_i also decreases when $C'_{i,n}(v_i)$ goes dowm. Hence, $f(C'_{i,n}(v_i))$ increases when $C'_{i,n}(v_i)$ decreases, as long as both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ are positive.

When we keep decreasing $C'_{i,n}(v_i)$, $\bar{v}_{i,-i}$ may hit the lower bound 0. Then, if we decrease $C'_{i,n}(v_i)$ further, $\bar{v}_{i,-i}$ remains at 0 and u_i continues to decrease as long as $C'_{i,n}(v) \geq 0$. Thus, $f(C'_{i,n}(v_i))$ still increases as $C'_{i,n}(v_i)$ decreases until $C'_{i,n}(v)$ reaches 0. Finally, when $C'_{i,n}(v_i) < 0$, both u_i and $\bar{v}_{i,-i}$ equal 0, and $f(C'_{i,n}(v_i))$ becomes constant function of $C'_{i,n}(v_i)$.

Case 3: $\lambda_{-i} > \hat{\lambda}_{-i}$

Similar to the previous case, we decrease $C'_{i,n}(v_i)$ starting from $C'_{i,n+1}(v + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$. When both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ are positive, they are determined by the following system of equations (B.1)-(B.3).

Since $\lambda_i > 0$, $\lambda_{-i} - \hat{\lambda}_{-i} > 0$ and $\gamma < 0$, (B.1) and (B.2) imply that

$$C'_{i,n+1}(\bar{v}_{i,i}) > C'_{i,n+1}(\bar{v}_{i,-i}) > C'_{i,n}(v_i).$$

When $C'_{i,n}(v_i)$ decreases, the right-hand side of (B.4) goes up. Different from the previous case, (B.3) implies that $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ variate in different direction. Suppose $\bar{v}_{i,i}$ decreases and $\bar{v}_{i,-i}$ increases, then the left-hand side of (B.4) goes down, and the equality fails to hold. Thus, to let (B.4) hold, we must have $\bar{v}_{i,i}$ increases and $\bar{v}_{i,-i}$ decreases when $C'_{i,n}(v)$ decreases. Moreover, u_i decreases when $C'_{i,n}(v_i)$ decreases. Hence, $f(C'_{i,n}(v_i))$ increases as we decrease $C'_{i,n}(v_i)$ and fix v_i, as long as both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ are positive.

When $C'_{i,n}(v_i)$ becomes non-positive, utility flow u_i remains at 0. But $\bar{v}_{i,-i}$ keeps going down as we continue to decrease $C'_{i,n}(v_i)$. Thus, $f(C'_{i,n}(v_i))$ keeps increasing as we decrease $C'_{i,n}(v_i)$ and fix v_i. Finally, $\bar{v}_{i,-i}$ hits the lower bound 0. Denote the value of $C'_{i,n}(v_i)$ at which $\bar{v}_{i,-i}$ reaches 0 at the first time by $\tilde{C}'_{i,n}(v_i)$ (we will use it in the proof of Lemma 2.3.6). From then on, both $\bar{v}_{i,-i}$ and u_i remains at 0 as we keep decreasing the value of $C'_{i,n}(v_i)$, and therefore $f(C'_{i,n}(v_i))$ becomes a constant function of $C'_{i,n}(v_i)$.

To summarize, in all of the above three cases, $f(C'_{i,n}(v_i))$ is a continuous and decreasing function of $C'_{i,n}(v_i)$. $\square$

Proof of Lemma 2.3.3

Proof. Since $C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}}) > C'_{i,n+1}(v_i) \geq S'(v_i)$ by Assumption A, if $C'_{i,n}(v_i) = S'(v_i)$, then $C'_{i,n}(v_i) < C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$. By Lemma 2.3.2, the NIC condition is binding, and therefore

$$\frac{dv_i}{dt} = r(v_i - u_i) - \hat{\lambda}_{-i}(\bar{v}_{i,-i} - v_i).$$

Utility flow u_i satisfies the first-order condition that $S'(u_i) = C'_{i,n}(v_i)$. Hence, $S'(u_i) = C'_{i,n}(v_i) = S'(v_i)$, which implies that $u_i = v_i$. From the proof of Lemma 2.3.2, we have $C'_{i,n+1}(\bar{v}_{i,-i}) \leq C'_{i,n}(v_i)$. Consequently,

$$S'(\bar{v}_{i,-i}) \leq C'_{i,n+1}(\bar{v}_{i,-i}) \leq C'_{i,n}(v_i) = S'(v_i),$$

which implies that $\bar{v}_{i,-i} \leq v_i$. Finally, combining $u_i = v_i$ and $\bar{v}_{i,-i} \leq v_i$, we get $dv_i/dt \geq 0$.

When $v_i = 0$, $C'_{i,n}(0) = S'(0) = 0$, which implies that $u_i = \bar{v}_{i,-i} = 0$. Thus, $dv_i/dt = 0$ when $v_i = 0$. □

Proof of Lemma 2.3.4

Proof. By Lemma 2.3.1, $dv_i/dt < 0$ on the $C'_{i,n}(v_i) = C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ locus. By Lemma 2.3.3, $dv_i/dt \geq 0$ on the $C'_{i,n}(v_i) = S'(v_i)$ locus, with strict inequality when $v_i > 0$. Moreover, Lemma 2.3.2 implies that, given v_i, the value of dv_i/dt is a continuous and strictly decreasing function of $C'_{i,n}(v_i)$ when $C'_{i,n}(v_i) \geq 0$. Therefore, for any $v_i \geq 0$, there exists an unique value of $C'_{i,n}(v_i)$ between $S'(v_i)$ and $C'_{i,n+1}(v_i +$

$\frac{rl}{\lambda-\hat{\lambda}_{-i}}$) such that $dv_i/dt = 0$. Moreover, the $dv_i/dt = 0$ locus is determined by the system of Kuhn-Tucker conditions and the following condition

$$rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i) = 0,$$

and both S', $C'_{i,n+1}$ are continuous functions. Therefore, the $dv_i/dt = 0$ locus is a continuous curve that locates below the $C'_{i,n}(v_i) = C'_{i,n+1}(v_i + \frac{rl}{\lambda-\hat{\lambda}_{-i}})$ locus and above the $C'_{i,n}(v_i) = S'(v_i)$ locus. Finally, $dv_i/dt = 0$ at $C'_{i,n}(0) = S'(0) = 0$ by Lemma 2.3.3. Thus, the $dv_i/dt = 0$ locus intersects the $C'_{i,n}(v_i) = S'(v_i)$ locus at the origin. □

Proof of Lemma 2.3.5

Proof. Since $C'_{i,n}(v_i) = S'(v_i - \frac{\hat{\lambda}_{-i}l}{\lambda-\hat{\lambda}_{-i}}) < C'_{i,n+1}(v_i + \frac{rl}{\lambda-\hat{\lambda}_{-i}})$, Lemma 2.3.1 implies that the NIC condition is binding. Thus,

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) - rl = 0. \tag{B.5}$$

From the proof of Lemma 2.3.2, we have $C'_{i,n+1}(\bar{v}_{i,i}) > C'_{i,n+1}(\bar{v}_{i,-i})$, and hence $\bar{v}_{i,i} > \bar{v}_{i,-i}$. Then it follows from (B.5) that

$$\bar{v}_{i,i} > v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}} > \bar{v}_{i,-i}.$$

Utility flow u_i is determined by the first-order condition $S'(u_i) = C'_{i,n}(v_i)$, which implies $u_i = v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}$. It follows that

$$\begin{aligned}
\frac{dv_i}{dt} &= r(v_i - u_i) - \hat{\lambda}_{-i}(\bar{v}_{i,-i} - v_i) \\
&= \frac{r\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}} - \hat{\lambda}_{-i}(\bar{v}_{i,-i} - v_i) \\
&> \frac{r\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}} - \frac{r\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}} \\
&= 0.
\end{aligned}$$

□

Proof of Lemma 2.3.6

Proof. When $v_i \geq \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}$, $dv_i/dt < 0$ on the $C'_{i,n}(v_i) = C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ locus by Lemma 2.3.1, and $dv_i/dt > 0$ on the $C'_{i,n}(v_i) = S'(v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}})$ locus by Lemma 2.3.5. Moreover, By Lemma 2.3.2, fixing v_i, the value of dv_i/dt is a continuous and strictly decreasing function of $C'_{i,n}(v_i)$ when $C'_{i,n}(v_i) \geq 0$. Therefore, there exists an unique value of $C'_{i,n}(v_i)$, which is between $C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ and $S'(v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}})$, such that $dv_i/dt = 0$.

Next, we consider the case when $0 \leq v_i < \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}$. From the proof of Lemma 2.3.2, when $C'_{i,n}(v_i)$ equals $\tilde{C}'_{i,n}(v_i)$, we have $u_i = \bar{v}_{i,-i} = 0$ and hence $dv_i/dt = r(v_i - u_i) - \hat{\lambda}_{-i}(\bar{v}_{i,-i} - v_i) \geq 0$. Moreover, the value of dv_i/dt is a continuous and strictly decreasing function of $C'_{i,n}(v_i)$ when $C'_{i,n}(v_i) \geq \tilde{C}'_{i,n}(v_i)$. Therefore, there exists an unique value of $C'_{i,n}(v_i)$, which is between $C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ and $\tilde{C}'_{i,n}(v_i)$, such that $dv_i/dt = 0$.

Thus, the $dv_i/dt = 0$ locus is a continuous curve that locates below the $C'_{i,n}(v_i) = C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ locus and the $C'_{i,n}(v_i) = S'(v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}})$ locus. □

Proof of Proposition 2.3.8

Proof. For part 1, it has been shown that $C_{i,n}(v_i)$ is determined by the HJB equation and the boundary condition. From the phase diagram, $C'_{i,n+1}(v_i)$ is a continuous and strictly increasing function of v_i. It follows that $C_{i,n}(v_i)$ is a convex function.

To describe the dynamics of transferred utility flow, let u_i, $\bar{u}_{i,i}$ and $\bar{u}_{i,-i}$ be the corresponding utility flow when the continuation utility are v_i, $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$.

When all of $C'_{i,n}(v_i)$, $C'_{i,n+1}(\bar{v}_{i,i})$ and $C'_{i,n+1}(\bar{v}_{i,-i})$ are positive, $(u_i, \bar{u}_{i,i}, \bar{u}_{i,-i})$ are determined by the following first-order condition

$$\begin{aligned} S'(u_i) &= C'_{i,n}(v_i), \\ S'(\bar{u}_{i,i}) &= C'_{i,n+1}(\bar{v}_{i,i}), \\ S'(\bar{u}_{i,i}) &= C'_{i,n+1}(\bar{v}_{i,-i}). \end{aligned}$$

If $\lambda_{-i} = \hat{\lambda}_{-i}$, then on the optimal path we have

$$C'_{i,n+1}(\bar{v}_{i,i}) > C'_{i,n}(v_i) = C'_{i,n+1}(\bar{v}_{i,-i}) \geq 0,$$

which implies $\bar{u}_{i,i} > u_i = \bar{u}_{i,-i}$.

If $\lambda_{-i} < \hat{\lambda}_{-i}$, then on the optimal path we have

$$C'_{i,n+1}(\bar{v}_{i,i}) > C'_{i,n}(v_i) \geq C'_{i,n+1}(\bar{v}_{i,-i}) \geq 0,$$

where the second inequity is strict when $v_i > 0$. Hence, $\bar{u}_{i,i} > u_i \geq \bar{u}_{i,-i}$, with strict inequality when $v_i > 0$.

If $\lambda_{-i} > \hat{\lambda}_{-i}$, then on the optimal path we have

$$C'_{i,n+1}(\bar{v}_{i,i}) > C'_{i,n+1}(\bar{v}_{i,-i}) > C'_{i,n}(v_i).$$

Therefore, when $C'_{i,n}(v_i) \geq 0$, we have $\bar{u}_{i,i} > \bar{u}_{i,-i} > u_i$. However, derivative of the cost function could be negative when $\lambda_{-i} > \hat{\lambda}_{-i}$. In this case, the utility flow equal 0. Therefore,

- if $C'_{i,n+1}(\bar{v}_{i,i}) > C'_{i,n+1}(\bar{v}_{i,-i}) > 0 \geq C'_{i,n}(v_i)$, we have $\bar{u}_{i,i} > \bar{u}_{i,-i} > u_i = 0$.
- If $C'_{i,n+1}(\bar{v}_{i,i}) > 0 \geq C'_{i,n+1}(\bar{v}_{i,-i})) > C'_{i,n}(v_i)$, we have $\bar{u}_{i,i} > 0 = \bar{u}_{i,-i} = u_i$.
- If $0 \geq C'_{i,n+1}(\bar{v}_{i,i}) > C'_{i,n+1}(\bar{v}_{i,-i}) > C'_{i,n}(v_i)$, we have $\bar{u}_{i,i} = \bar{u}_{i,-i} = u_i = 0$.

To summarize, if agent i completes the innovation, the principal rewards him by an upward jump in utility flow. If his coworker completes the innovation, then: 1) his utility flow does not change if $\lambda_{-i} = \hat{\lambda}_{-i}$; 2) his utility flow drops down if $\lambda_{-i} < \hat{\lambda}_{-i}$; 3) his utility flow jumps up if $\lambda_{-i} > \hat{\lambda}_{-i}$. These results prove part 2 and part 3.

Finally, for part 4, note that on the optimal path v_i is decreasing over time and asymptotically converges to 0. Moreover, the transferred utility satisfies $S'(u_i) = C'_{i,n}(v_i)$ and both S and $C_{i,n}$ are convex functions. Therefore, transferred utility u has the same dynamics as continuation utility in case of failure. □

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