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Resource allocation problems in communication and control systems

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RESOURCE ALLOCATION PROBLEMS IN COMMUNICATION AND CONTROL SYSTEMS

by

Manish Goldie Vemulapalli

An Abstract

Of a thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Electrical and Computer Engineering in the Graduate College of The University of Iowa

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Thesis Supervisor: Professor Soura Dasgupta

ABSTRACT

Resource allocation in control and communication systems constitutes the distribution of (finite) system resources in a way that achieves maximum system functionality and or cost effectivesness. Specific resource allocation problems in subband coding, Discrete Multi-tone modulation based systems and autonomous multi-agent control are addressed in this thesis.

In subband coding, the number of bits used (out of a target bit budget) to code a subband signal are allocated in a way that minimizes the coding distortion. In Discrete Multi-tone modulation based systems, high bit rate streams are split into several parallel lower rate streams. These individual data streams are transmitted over different subchannels. Given a target bit rate, the goal of resource allocation is to distribute the bits among the different subchannels such that the total transmitted power is minimized. The last problem is achieving stable control of a fleet of autonomous agents by utilizing the available communication resources (such as transmitted Power and bandwidth) as effectively as possible.

We present an efficient bit loading algorithm that applies to both subband coding and single-user multicarrier communication system. The goal is to effect an optimal distribution of B bits among N subchannels (subbands) to achieve a minimum transmitted power (distortion error variance) for multicarrier (subband coding) systems. All the algorithms in literature, except a few (which provides a suboptimal solution), have run times that increase with B. By contrast, we provide an algorithm that solves the aforementioned problems *exactly* and with a complexity (given by $O(N \log(N))$,) which is dependent *only on* N.

Bit loading in multi-user multicarrier systems not only involves the distribution of bit rates across the subchannels but also the assignment of these subchannels to different users. The motivation for studying suboptimal bit allocation is underscored by implicit and explicit claims made in some of the papers which present suboptimal bit loading algorithms, without a formal proof, that the underlying problem is NP-hard. Consequently, for no other reason than the sake of completeness, we present a proof for NP-hardness of the multiuser multicarrier bit loading problem, thereby formally justifying the search for suboptimal solutions.

There has been a growing interest in the area of cooperative control of networks of mobile autonomous agents. Applications for such a set up include organization of large sensor networks, air traffic control, achieving and maintaining formations of unmanned vehicles operating underwater, air traffic control etc. As in Abel et al, our goal is to devise control laws that, require minimal information exchange between the agents and minimal knowledge on the part of each agent of the overall formation objective, are fault tolerant, scalable, and easily reconfigurable in the face of the loss or arrival of an agent, and the loss of a communication link.

A major drawback of the control law proposed in Abel et al is that it assumes all agents can exchange information at will. This is fine if agents acquire each others state information through straightforward sensing. If however, state information is exchanged through broadcast communication, this assumption is highly unrealistic. By modifying the control law presented in Abel et al, we devise a scheme that allows for a *sharing of the resource*, which is the communication channel, but also achieves the desired formation stably. Accordingly we modify the control law presented in [30] to be compatible with networks constrained by MAC protocols.

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Manish Goldie Vemulapalli

has been approved by the Examining Committee
for the thesis requirement for the Doctor of Philosophy
degree in Electrical and Computer Engineering
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TABLE OF CONTENTS

LIST OF FIGURES

Figure

CHAPTER 1
INTRODUCTION

Resource allocation involves managing the finite system resources among the entities with a goal to achieve optimal system functionality and/or cost effectiveness. These resources include bandwidth, power, buffer size and computational complexity which is indirectly related to power. The primary goal of this thesis is to address certain resource allocation problems in certain signal processing, communication and control applications. The three specific problems we consider are subband coding (SBC), Discrete Multi-Tone (DMT) or multicarrier transmission and autonomous multi-agent control respectively. The common thread in all three problems is the regulation of access to pertinent communication resources, very directly for multi-agent control and the DMT problem, and more tangentially for SBC.

Let us begin with multi-agent control. The specific problem is how to induce a group of agents to organize themselves in a formation, specified by a limited number of relative position information between agents, without excessive reliance on a centralized authority, but rather through limited inter-agent communication. This line of research is relatively new (see [1] and [30] and the references therein). What sets it apart from traditional feedback control problems is that the architecture underlying the information exchange, plays a critical role in control performance. Such communication architectures are only recently being analyzed. In effecting the required information exchange, the agents must intelligently share the communication medium. Specifically, we assume that agents broadcast the information they need neighbors to know. This in turn imposes certain constraints: No two agents can simultaneously broadcast to the same agent, or for that matter only one agent in whose broadcast range an intended receiver resides, can transmit at a given time. Otherwise the broadcasts interfere and smear each other out. This is a classic *Medium Access Control (MAC)* issue, hitherto neglected in the multiagent control literature. Our goal here is to study how to preserve stability despite such MAC constraints.

Medium Access Control has long been considered in all communications systems. In DMT this takes a special form. Specifically, as will be explained at greater length in the sequel, DMT transmits a message using multiple carriers. Different carriers are assigned different number of bits per/symbol. Carriers that experience more adverse channel conditions are assigned fewer bits compared to those that see better *subchannels*. Medium Access Control is partially achieved by *bit loading*, that is defined as the process of allocating the number of bits/symbol to each carrier.

Subband coding on the other hand is a source coding problem. As explained below, it quantizes a signal by first dividing it into multiple subbands, and by then effecting bit loading, by quantizing each subband with varying number of bits/sample, according to the energy in each subband. Subbands with higher energy receive higher number of bits. One can view bit loading in DMT as a channel coding problem. Just as most channel coders have a dual relationship with certain source coders, the bit loading in SBC can be viewed as a dual to that in DMT. As explained in Section 1.1.1, SBC can be viewed as yet another manifestation of MAC, albeit in an indirect sense. We are concerned with effecting optimal bit loading in SBC to minimize the total distortion, and that in DMT, to achieve a given Symbol Error Rate, with the smallest transmission power. While bit loading algorithms abound in the literature, the computational complexity of each such grows with the number of bits to be assigned. Our major, and by now patented, contribution is to provide a bit loading algorithm whose complexity *does not grow with the number of bits to be allocated.*

The thread that ties the three seemingly disparate problems is MAC. In the control problem our emphasis is on stability that is the *sine qua non* of all control laws. In SBC and DMT, our goal is to preserve computational resources that in turn preserve battery power.

1.1 Background and Motivation

The following sections 1.1.1 and 1.1.2 respectively provide some background material on SBC and DMT based systems and fault tolerant multi-agent control.

1.1.1 SBC and DMT based systems

Human ears are generally sensitive over a wide range of frequencies. However, when a lot of audio signal's energy is present over a certain frequency range, the ear fails to pick up lower energy frequencies nearby. Audio signals also typically have a lot of their energy concentrated in the low frequencies. SBC exploits these properties by first dividing the spectrum of the audio signal into a number of subbands and then quantizing these subband signals at a resolution determined by their energy. High energy subbands are encoded using more quantization levels than those with low energy. The problem of bit loading in SBC constitutes the distribution of the number of quantization levels allocated to each subband in a way that keeps the cumulative quantization error to a minimum. SBC is a powerful method of encoding audio signals efficiently. SBC resides at the heart of the popular MP3 format (also known as MPEG 1 audio layer III).

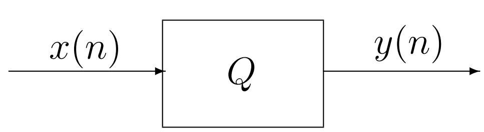

Figure 1.1: A b-bit quantizer

To make things more concrete, consider Figure 1.1, that calls for the discrete time signal $x[n]$ to be quantized by the B-bit quantizer Q. Suppose, the spectrum of $x(n)$ is as in Figure 1.2, the notable feature being that $x(n)$ has dominant low frequency components. Subband coding seeks to exploit the fact that the signal to be quantized is dominantly low pass. It thus employs a 2-channel subband coder depicted in Figure 1.3.

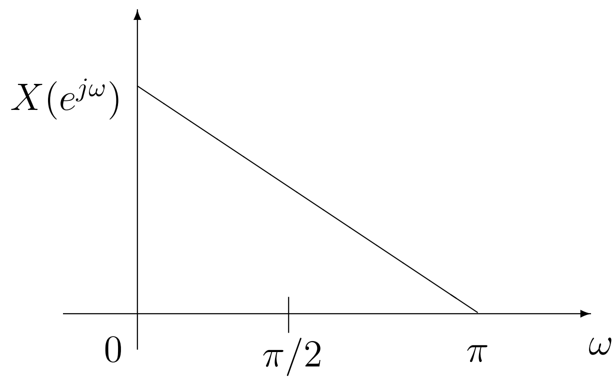

Figure 1.2: Spectrum of the signal to be quantized

In Figure 1.3, the blocks that follow the filters $H_i(z)$ are 2-fold decimators that lower the effective sampling rate by a factor of two by discarding every even indexed sample of its input. In particular should the input to such a device be $v(n)$, then its output is $v(2n)$. The blocks following the blocks labeled Q_i are 2-fold interpolators. These raise the sampling rate by a factor of two by interlacing a zero sample between every consecutive sample of the signals at their input. Thus, should the input to such a device be $v(n)$, then the output at the n-th index is given by:

$$\begin{cases} v\left(\frac{n}{2}\right); & n \mod 2 = 0 \\ 0; & \text{else} \end{cases}$$

Observe $H_0(z)$, with frequency response depicted as in Figure 1.4 is an ideal low pass filter, just as $H_1(z)$, with frequency response depicted as in Figure 1.4 is an ideal high pass filter. Between the two they cover the whole base positive

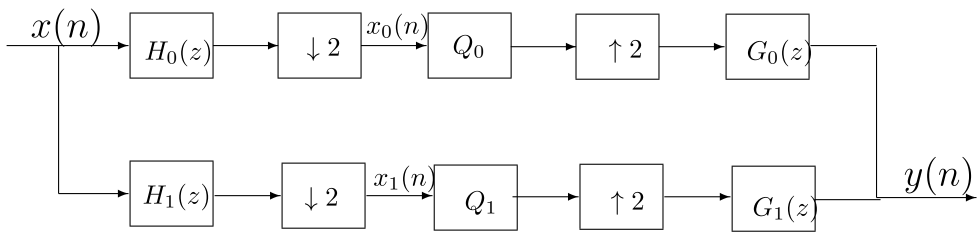

Figure 1.3: A 2-channel subband coder

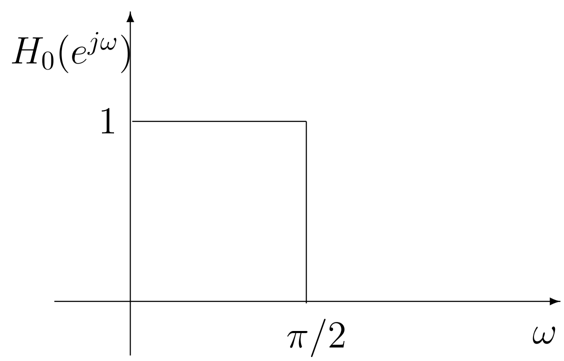

Figure 1.4: Frequency response of $H_0(z)$

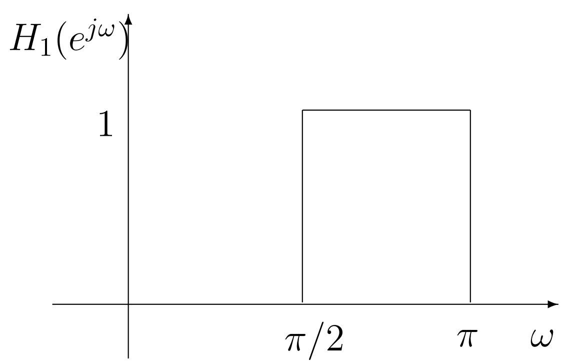

Figure 1.5: Frequency response of $H_1(z)$

frequency range $[0, \pi]$. So how does this 2-channel SBC work? It effectively splits the signal into its low and high frequency components, the former at the output of $H_0(z)$ and the latter at the output of $H_1(z)$. Given that these filtered signals have half the bandwidth of $x(n)$, no information is lost through there being down sampled by a factor of two. Now suppose the Q_i are B_i-bit quantizers. Then the effective coding rate/sample can be kept at the desired value b, should one enforce the equality:

$$B = \frac{B_0 + B_1}{2}. \tag{1.1.1}$$

On the other hand by choosing:

$$B_0 > B_1$$

one allocates more bits to the higher energy component of $x(n)$ than to the lower energy component. One can chose the filters $G_i(z)$ so that the overall output would equal $x(n)$, in the absence of the quantizers Q_i. The net effect is that the output of the SBC has a lower distortion than the output of the arrangement in Figure 1.1, while maintaining the effective desired coding rate of Bb. The bit allocation problem, in the example of such a 2-channel subband coder, is to choose the B_i, so as to minimize this overall distortion subject to (1.1.1). This explains the underlying rudiments of subband coding.

Turn now to DMT. Modern day communication systems must be capable of offering reliable high-speed data transmissions. They are also required to support a plethora of services with vastly different types of data (such as voice, Internet, video, High-Definition voice and video content etc.). Discrete Multi-Tone (DMT) modulation schemes, that generalize their most popular variant, Orthogonal Frequency Division Multiplexed (OFDM) systems, support high capacity communication systems. They are particularly effective over frequency selective communication channels.

Single-Tone modulation systems are more prone to intersymbol interference (ISI). DMT modulation schemes reduce ISI by making the symbol time large enough so that the channel-delay spreads are only a fraction of the symbol duration. A given high bit rate data stream is divided into several parallel lower bit rate streams.

These individual data streams (with potentially different rates) are then transmitted on different subchannels, with different carrier frequencies, and consequently experiencing different channel conditions. The number of bits transmitted over a subchannel is dependent on the channel conditions. The more severe the attenuation experienced in a subchannel the fewer the bits allocated to it. The distribution of the bit rates over the subchannels is to be carried out in a way that minimizes the total transmitted power while simultaneously meeting the symbol error rate (SER) requirements.

Discrete Multitone modulation is a current standard in several wireline applications such as ADSL, VDSL etc, and in the form of Orthogonal Frequency Division Multiplexing (OFDM) is a fixture in wireless standards such as Long Term Evolution (LTE) rel.8 and later, IEEE 802.11x, Mobile Broadband Wireless Access (MBWA) IEEE 802.20 and IEEE 802.16e etc.

The bit loading objective in DMT is as follows. Suppose with N-tones (subchannels) the data rate in each subchannel is reduced by a factor of N relative to the desired data rate. Suppose the desired bit rate is B. Then one must allocate B_i-bits/symbol to the i-th carrier/subchannel/tone to minimize the total transmitted power, to achieve a specified SER subject to bit rate constraint:

$$B = \sum_{i=1}^{N} B_i$$

Though the bit allocation problems in SBC and DMT have different antecedents, and though, as shown in Chapter 2, the functional form of the underlying optimization problems are, on the face of it different, as also shown in Chapter 2, they can be captured within a unifying framework. In particular, the specialization of a common general algorithm solves both. There are several implementation of this general algorithm in the literature. These include those documented in [6], [7], [9], [11], [13]. The two most advanced and recent are [13] and [6]. The complexity of

[13] grows as $O(\max\{N\log(N), B\})$. On the other hand [6] provides a *suboptimal solution* with complexity $O(\max\{N, B^*\})$, where B^* is upper limit on the number of bits allocated to a subchannel.

The assumption of a small B or B^* could prove to be sticky in subband coding, and even in communications settings when certain subchannels experience deep fades (severe attenuation). All the algorithms in literature, except [6] (which provides a suboptimal solution), have run times that increase with B. By contrast, we provide an algorithm, [18, 19], that solves the aforementioned problems *exactly* and with a complexity (given by $O(N\log(N))$,) which is dependent *only on* N.

In a sense the SBC setup can also be viewed as a MAC problem. In particular in Figure 1.3 the arrangement to the left of the quantizers Q_i can be viewed as a transmitter system, and that to the right, as receiver system. Indeed, [21], even DMT has a filter bank interpretation that is a dual to that of Figure 1.3. Specifically, as shown in [21] a DMT system is equivalent to a filter bank in which the arrangement to the right of the quantizers is exchanged with that to the left.

A typical multiuser system has different types of services being offered, for example data, voice, video etc., to different users with varied quality of service requirements. Bit loading in multi-user multicarrier systems not only involves the distribution of bit rates across the subchannels but also the assignment of these subchannels to different users. Algorithms that achieve suboptimal bit loading exist in literature. These include [20], [22], [23], and [24]. The motivation for studying suboptimal bit allocation is underscored by implicit and explicit claims made in some of these papers, without a formal proof, that the underlying problem is NP-hard, i.e. an algorithm whose run time grows polynomially with the number of subchannels N is unlikely to be found.

We note that, in the theory of computational complexity, there exist a class of problems categorized as NP-complete which are known not to have, as yet, algorithms that solve them with a run time that is a polynomial in input size. Thus classifying a problem as NP-hard would mean that the problem is just as "hard" to solve as an NP-complete problem and therefore cannot be solved in polynomial time. Consequently, for no other reason than the sake of completeness, we present a proof, [29], for NP-hardness of the multiuser multicarrier bit loading problem, thereby formally justifying the search for suboptimal solutions.

1.1.2 Multi-Agent Control

Spurred by major advances in computing, wireless communications and networking, and an ever expanding application domain there has been a growing interest in the cooperative control of networks of mobile autonomous agents. Applications for such a set up include organization of large sensor networks, air traffic control, achieving and maintaining formations of unmanned vehicles operating underwater, overland or in the air, organization of communication networks in response to natural or man made disasters, air traffic control, satellite organization, automated highway systems and mobile robots engaged in cooperative tasks.

We are concerned with agents modeled as double integrators in each cartesian coordinate that must organize themselves into formations prescribed by the relative positions between the agents. As in Abel et al [30] our goal is to devise control laws that, require minimal information exchange between the agents and minimal knowledge on the part of each agent of the overall formation objective, are fault tolerant, scalable, and easily reconfigurable in the face of the loss or arrival of an agent, and the loss of a communication link.

A major drawback of [30] is that it assumes all agents can exchange information at will. This is fine if agents acquire each others state information through straightforward sensing. If however, state information is exchanged through broadcast communication, this assumption is highly unrealistic. In particular when agents broadcast their state information they must compete with each other for access to the communication medium and are constrained by Media Access Control (MAC) protocols. Effectively, the access to the communication medium becomes a vital resource. It is therefore imperative to devise a scheme that allows for a *sharing of the resource*, which is the communication channel, but also achieves the desired formation stably. Accordingly we modify the control law in [30] to be compatible with networks constrained by MAC protocols as in [31, 32].

1.2 Organization of the Thesis

In chapter 2 we discuss the problem of bit loading in SBC and single user DMT systems. Sections 2.1 and 2.2 provide brief introductions to these systems. We present bit loading in SBC and single user DMT systems as a convex optimization problem in section 2.3. In section 2.4 we present an algorithm for optimal integer bit loading and a runtime analysis of its complexity along. The section also provides performance analysis of the proposed algorithm and a few of the advanced algorithms that exist in literature.

Chapter 3 introduces the problem of optimal integer bit loading in multiuser DMT systems. We identify the problem to be NP-hard and present a formal proof to this effect as a motivation for seeking suboptimal solutions.

In Chapter 4 we introduce the problem of multi-agent control. A brief review of the previous work in this area is presented before developing the math in section 4.2. In section 4.3 we present a review of [30] which deals with the problem of multi-agent control in the absence of medium access control constraints. Modified control law which respects the medium access constraints is then presented in section 4.4.

In chapter 5 we present the stability analysis for the control law from section 4.4. Section 5.2 presents a proof for the stability of the control law developed in chapter 4. Chapter 6 summarizes the results of the proposed thesis.

CHAPTER 2
RESOURCE ALLOCATION IN SUBBAND CODING AND SINGLE USER DMT SYSTEMS

The problem of optimal integer bit loading in Subband Coding and DMT systems is discussed in this chapter. We start by presenting an overview of SBC and single user DMT systems in sections 2.1 and 2.2. Single user refers to the case where all users supported by the DMT system have the same Quality of Service (QoS) requirements, quantified by overall Symbol Error Rates (SER). The principal contribution of this chapter, a bit loading algorithm for subband coding and single user DMT systems, whose complexity does not grow with the number of bits to be allocated, is presented in section 2.4.

2.1 Subband Coding Systems

Signals must often be communicated over a digital communication channel, such as a modem, or stored in a digital memory, such as a DVD or flash drives. It may however be the case that the original signal is not in an appropriate form for the transmission or storage in the given channel. In this case the original signal has to be transformed or coded into a form suitable for the channel. This general operation is termed as signal coding. Signal coding is usually called signal compression if the bit rate of the digital representation or the bandwidth of the modulated version of the digital representation is less than the rate or bandwidth of the input signal.

When the channels of a filter bank are used for coding, as in Figure 2.1, the resulting scheme is known as subband coding. The symbol $\downarrow M$ denotes an M-fold decimator that only retains those input samples that occur at time instants equal to multiples of M. Thus in Figure 2.1, signals $u_i(k)$ and $v_i(k)$ are related as

$$v_i(k) = u_i(kM)$$

The symbol $\uparrow M$ denotes an M-fold interpolator that inserts $M-1$ zeros between each sample of the input sequence. Hence $y_i(k)$ and $w_i(k)$ in Figure 2.1 are related by

$$y_i(k) = \begin{cases} w_i\left(\frac{k}{M}\right) & \text{if } k \text{ is a multiple of } M \\ 0 & \text{otherwise.} \end{cases}$$

In subband coding the bank of analysis filters, $H_i(z)$, operate on the input signal $x(k)$ to generate a set of narrowband signals each representing a different subband of the input spectrum. The narrow bandwidth of each subband signal $v_i(k)$ allows sub-Nyquist sampling to be performed, reducing the bit rate needed to code each subband. The analysis filters together with the decimators comprise the analysis side of the subband coder. Quantizer Q_i, a lossy compression system, is employed to code each subband signal $v_i(k)$ producing signal $w_i(k)$. On the synthesis side, interpolation is used followed by the synthesis filters $F_i(z)$, resulting in output $\hat{x}(k)$, to synthesize the reproduction of the original signal.

The performance of a subband coding system is generally assessed by some measure of the fidelity of reproduction achieved, or equivalently by the degree of distortion incurred, at a given bit rate. Often the rate is constrained by the

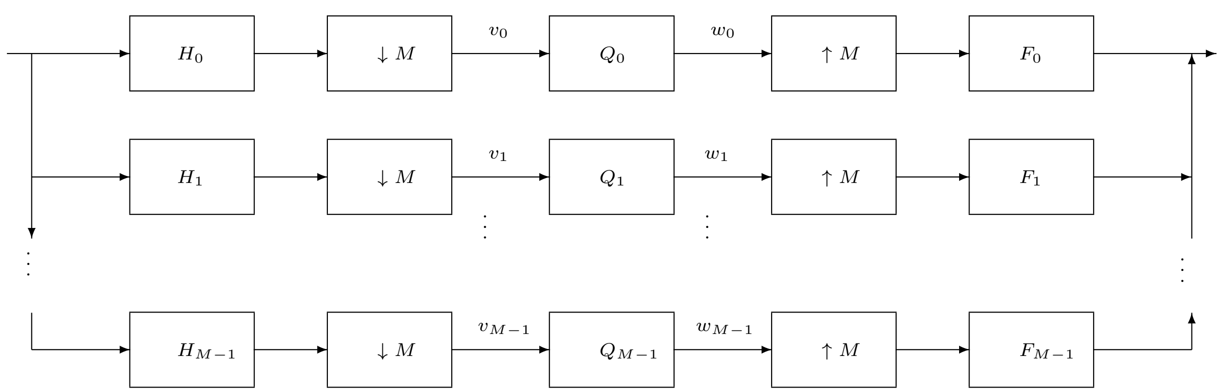

Figure 2.1: An M-channel filter bank as a subband coder.

particular application and the design goal is to achieve some quality objective while maintaining a tolerable complexity. A natural measure of performance in a subband coding system is a quantitative measure of distortion, often measured in the mean-square sense. Hence subband coding systems are considered optimal if they minimize an average distortion measure subject to certain constraints on the coder structure. Minimizing mean-squared distortion is equivalent to maximizing the coding gain of the subband coding system, with coding gain defined as the ratio of distortion incurred using a direct quantization scheme and the distortion using a subband coder.

A major concern of the designer of a subband coding system is bit allocation, the task of distributing a given quota of bits to the various quantizers of the signal coding system to optimize the overall coder performance. Encoding in subbands is advantageous since the number of quantizer levels and hence the reconstruction error variance can be controlled in each band by appropriately allocating the bits to different subbands.

The subband coder is as shown in Figure 2.1, with linear time-invariant filters $H_i(z)$, $F_i(z)$. Quantizing a stationary random sequence with a quantizer can be modeled as adding noise to a sequence. We assume different quantizer functions for the quantizers Q_i with the following model holding:

$$\sigma_{q_i}^2 = c_i 2^{-2b_i} \sigma_{v_i}^2 \tag{2.1.1}$$

where $\sigma_{q_i}^2$,$\sigma_{v_i}^2$ are respectively the quantizer noise and subband signal variances, constants c_i depend on the input signal statistics, and b_i are the number of bits allocated to quantizer Q_i.

The optimal subband coding problem is one of minimizing a weighted overall distortion measure given by

$$\epsilon = \sum_{i=0}^{M-1} w_i \sigma_{q_i}^2 \tag{2.1.2}$$

The weights w_i permit the distortion in different subbands to be assigned varying levels of priority in the overall optimization. This optimization must be performed under certain constraints. The first is the average bit rate constraint:

$$b = \frac{1}{M} \sum_{0}^{M-1} b_i. \tag{2.1.3}$$

The second is a *Perfect Reconstruction* condition. This takes various forms. In all cases the minimal requirement is that absent quantizers, Q_i the output of the filter bank in Figure 2.1 equals its input. Such a filter bank is in general called bi-orthogonal. An additional constraint often imposed, as in e.g. [2], is that of orthogonality. This requires that the total energy of the subband signals v_i equal the input energy, and the total energy of the signals w_i equals the output energy. The variables of optimization are (i) b_i, the bits allocated to the subbands and (ii) the filters $F_i(z)$ and $G_i(z)$. The former of course is the bit loading problem. The latter is the filter selection problem.

On the face of it, this is a highly involved optimization problem. However, both for Orthonormal filter banks and for general biorthogonal filter banks, [2] and [3], respectively, expose what we call here a *separation principle*: This principle, states that under optimal bit loading, regardless of the total number bits, b, that must be allocated, the optimal selection of the analysis and synthesis filters is determined exclusively by the statistics of the input to the filter bank in Figure 2.1. The selection of these filters, and the knowledge of the input statistics, clearly determine c_i and $\sigma_{v_i}^2$ in (2.1.1).

Thus we focus on the bit loading problem, i.e. selecting b_i to minimize:

$$\epsilon = \sum_{i=0}^{M-1} w_i c_i 2^{-2b_i} \sigma_{v_i}^2, \ \ w_i, c_i > 0, \tag{2.1.4}$$

subject to the average bit rate condition (2.1.3), with known w_i, c_i and $\sigma_{v_i}^2$.

2.2 Discrete Multitone Transmission Systems

A central problem in communications concerns reliable and efficient transmission of information carrying signals over distortive channels. One popular approach to achieving high speed data transmissions is Orthogonal Frequency Division Multiplexing (OFDM), which is a special case of Discrete Multitone (DMT) modulation.

To be specific, let us consider OFDM. The basic idea in OFDM is to subdivide the data stream into M streams, each operating at $1/N$-th the original rate with integers $N > M$. In particular $\kappa = N - M$ must exceed the delay spread of the partially equalized channel, i.e. should the finite impulse response (FIR) system representing the partially equalized channel, have degree m, then $\kappa \geq m$. In the sequel denote $C(z)$ to be the transfer function of this FIR partially equalized channel. Each of these data streams is assigned a separate carrier frequency. Successive carrier frequencies are equispaced. These data streams are interlaced and each block of M-symbols is augmented by a cyclic prefix redundancy of length κ. In other words, the M-sized block is converted to an N-sized block by appending the first κ samples to the end of the M-sized block. At the channel output the redundancy is removed prior to demodulation, and subsequent to equalization.

At baseband the overall transmission system takes the form depicted in Figure 2.2. In particular, in OFDM, the x_i represent the baseband data streams prior to modulation. The input transform block G_0 is the Inverse Discrete Fourier Transform (IDFT) matrix. The block labeled P/S is a parallel to serial converter. The block labeled as "insert redundancy" performs cyclic prefix insertion. The inverse transform S_0 is the Discrete Fourier Transform (DFT) matrix. The equalizer is a designer choice and known both to the transmitter and receiver. Interference models co-channel interference.

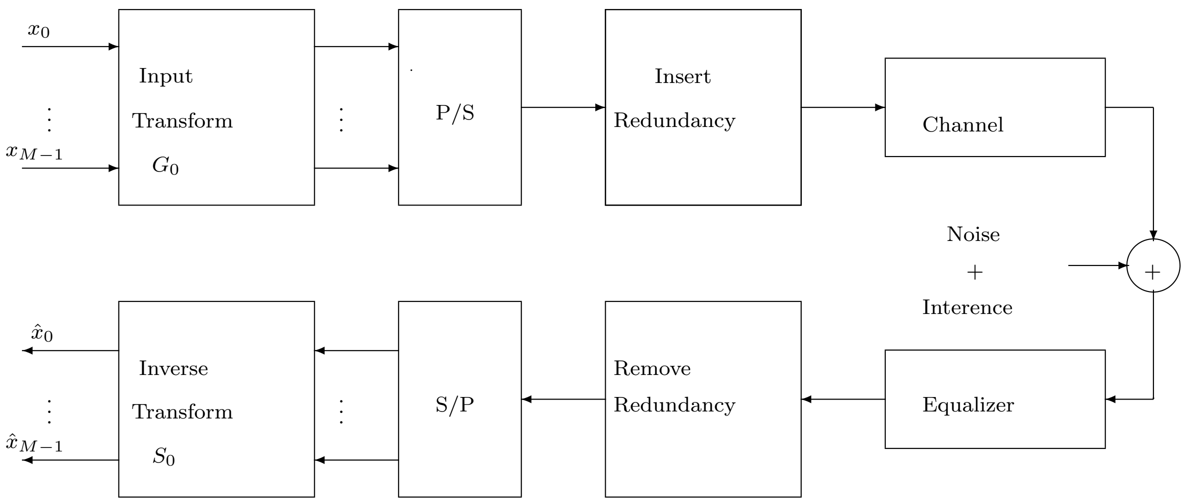

Figure 2.2: A general DMT communications system.

An equivalent representation of this arrangement is depicted in Figure 2.3. One can better appreciate the role of cyclic prefix by observing that in Figure 2.3 there holds:

$$\mathbf{s}(n) = \begin{bmatrix} s_0(n) \\ s_1(n) \\ \vdots \\ s_{N-1}(n) \end{bmatrix} = \begin{bmatrix} y_0(n) \\ y_1(n) \\ \vdots \\ y_{M-1}(n) \\ y_0(n) \\ \vdots \\ y_{\kappa-1}(n) \end{bmatrix}. \tag{2.2.5}$$

In OFDM a perfect reconstruction condition obtains, i.e. absent noise and interference the output stream equals the input stream, as long as κ exceeds the equalized channel delay spread. Several recent papers have considered more generalized settings, [14],[15],[16]. For example, the G_0 and S_0 need not be constrained as IDFT and DFT matrices, respectively. Similarly, redundancy may be more general than cyclic prefix. In fact for a suitably dimensioned matrix Ω, one may replace

(2.2.5) by:

$$\mathbf{s}(n) = \begin{bmatrix} s_0(n) \\ s_1(n) \\ \vdots \\ s_{N-1}(n) \end{bmatrix} = \Omega \begin{bmatrix} y_0(n) \\ y_1(n) \\ \vdots \\ y_{M-1}(n) \\ y_0(n) \end{bmatrix}.$$

As a case in point one can use the so called zero-padding redundancy, where the last κ samples are replaced simply by zero samples, i.e. (2.2.5) becomes:

$$\mathbf{s}(n) = \begin{bmatrix} s_0(n) \\ s_1(n) \\ \vdots \\ s_{N-1}(n) \end{bmatrix} = \begin{bmatrix} y_0(n) \\ y_1(n) \\ \vdots \\ y_{M-1}(n) \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

One constraint common to all of the relevant literature on the design of optimum DMT, e.g. [4], [14], [15] and [16] is that κ be no smaller than the degree of the equalized channel. This ensures that given an arbitrary equalized channel transfer function of the requisite degree, there exist G_0, S_0 and S_1, such that the output in Figure 2.3 equals the input.

As has been shown in [4], [14], [15] and [16], the arrangement in Figure 2.3 has an equivalent filter bank representation depicted in Figure 2.4. The transmitting filters $F_k(z)$, and receiving filters $H_k(z)$ have a precise relationship with S_0 and G_0, respectively. The nature of this relationship is not important for the purposes of this thesis. *What is important is the fact that as asserted in the Introduction, there is a duality between the DMT structure and SBC.* Specifically, if we ignore

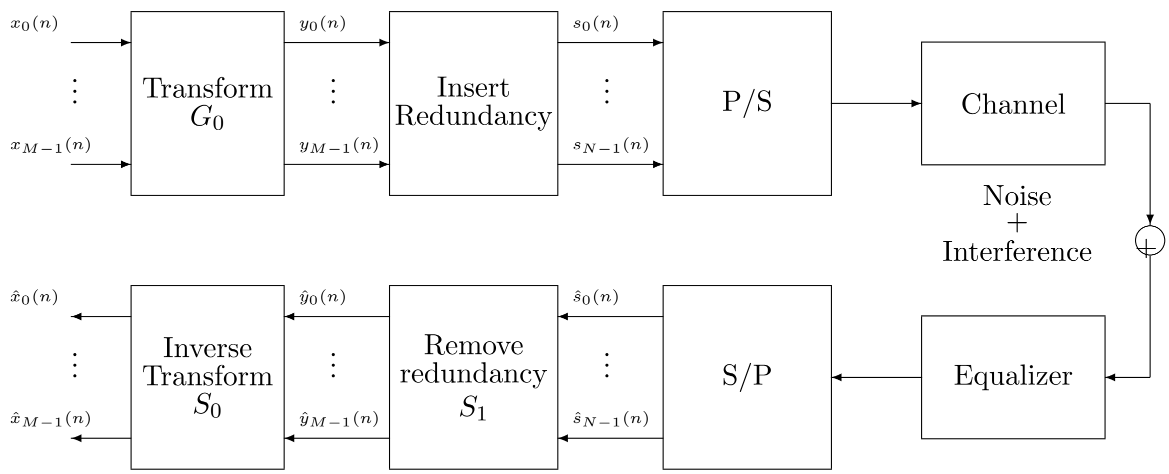

Figure 2.3: DMT communication system.

$C(z)$ then the Filter Bank in Figure 2.4 is obtained by exchanging the synthesis and analysis banks of the filter bank in SBC, albeit with $N > M$, rather than with $N = M$.

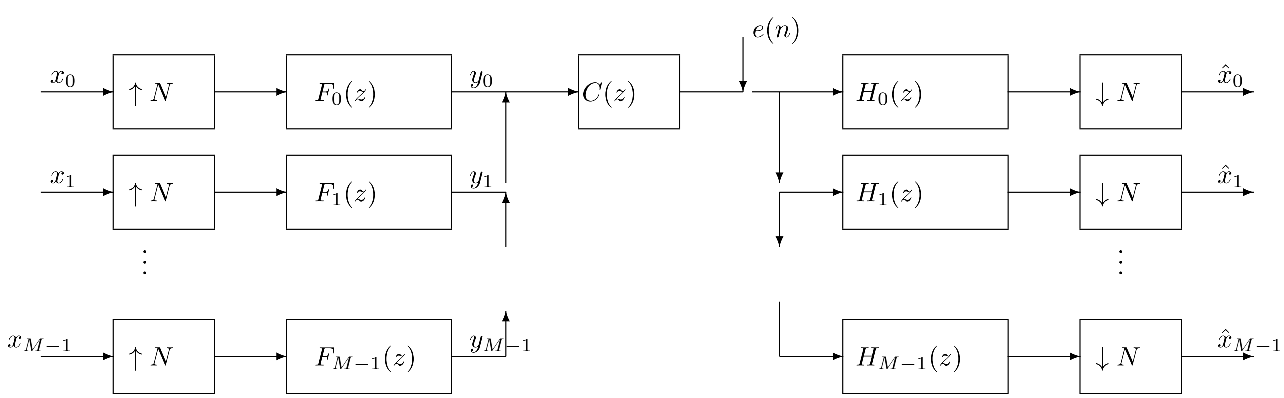

Figure 2.4: Filterbank model of an M-band DMT system.

Now turn to the underlying optimization problem, we wish to address. Each sample of the data streams $x_i(n)$ has b_i-bit symbols. Examples could be Pulse Amplitude Modulation (PAM), Quadrature Amplitude Modulation (QAM) or for that matter other well accepted encoders. A higher b_i is more susceptible to error. Given that each data stream, modulated as it is with a different carrier frequency, experiences a different set of channel conditions, at a macro level, the point of bit-loading in DMT is to assign fewer bits/symbol to $x_i(n)$ if it sees a worse channel condition, while maintaining specified QoS requirements. The QoS it self is quantified in this thesis by the following quantities:

(A) Bit rate.

(B) Symbol error rate (SER), η. Here SER is the probability of mis detecting a symbol at the receiver.

In a single user case, each data stream has an SER no more than η, while the effective bit rate over all the data streams matches the required bit rate. The latter condition in particular can be quantified through the requirement that for a given β there holds:

$$B = \sum_{i=1}^{M} b_i. \tag{2.2.6}$$

The overall optimization problem in DMT is as follows. Given a linear time invariant equalizer, wide sense stationary noise of known spectrum, with κ no smaller than the degree of the equalized channel choose S_i, G_0, and b_i, so that the QoS requirements (A) and (B) are met, with *minimum transmitted power*, subject to the perfect reconstruction requirement that in Figure 2.3, absent noise and interference, the output data steam assuredly matches the input.

The key to the resolution of this problem is the relation between η in (B) above and the number of bits per symbol and the signal to noise ratio (SNR). As shown in [4, 14, 15, 16], all conventional B-bit symbol constellation schemes require an output signal-to-noise ratio (SNR) of $d2^{\zeta B}$, for modestly large B, in order to achieve an SER of no more than η. Here $d > 0$ is determined by the SER, η and the employed modulation scheme, and the constant $\zeta > 0$ depends on the particular modulation scheme used. For example, for a B-bit square QAM, the SER is given by, [14],

$$\eta = 4\left(1 - \frac{1}{\sqrt{2^B}}\right) Q\left(\sqrt{\frac{3\text{SNR}}{(2^B-1)}}\right) \approx 4Q\left(\sqrt{\frac{3\text{SNR}}{2^B}}\right), \tag{2.2.7}$$

whenever 2^{-B} is much smaller than 1, and

$$Q(a) = \int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

Thus for large B, SNR$= d2^{\zeta B}$ with $\zeta = 1$, $d = \frac{1}{3}[Q^{-1}(\frac{\eta}{4})]^2 > 0$. In the case of PAM, $\zeta = 2$, $d = \frac{1}{3}[Q^{-1}(\frac{\eta}{2})]^2 > 0$.

Let the input power in the j-th subchannel be $\sigma^2_{x_j}$, and $\sigma^2_{\nu_j}$ be the output noise variance in this subchannel. Since under perfect reconstruction, the input power equals the output power in that subchannel, the relation between the input signal power and the output noise power are related by

$$\sigma^2_{x_j} = d2^{\zeta b_j}\sigma^2_{e_j}, \tag{2.2.8}$$

where the constant d is determined by the modulation scheme used.

Consequently, the total transmitted power needed to meet the QoS requirements is:

$$P_B = \sum_{i=1}^{M} d2^{\zeta b_i}\sigma^2_{e_i}. \tag{2.2.9}$$

The optimization problem then becomes selecting S_i and G_0, such that under perfect reconstruction and (2.2.6), (2.2.9) is minimized. It has been shown in [4, 14, 15, 16] that as in the SBC problem, a separation principle obtains: Under the optimal selection of the b_i, i.e. optimal bit loading, regardless of ζ and M, the selection of S_i and G_0 is determined solely by the statistics of the overall transmitted signal, noise and interference. Their selection determines ζ and e_i. Consequently, the problem at hand reduces to the following bit loading problem; namely that given B, ζ and $\sigma^2_{e_i}$, select b_i to minimize (2.2.9) under (2.2.6).

2.3 Bit Loading as a Convex Optimization Problem

We will now look at the problem of bit loading in subband coding systems as well as single user DMT based systems. The importance of bit loading in these systems was detailed in the preceding two sections. Specifically, for an N-subchannel (subband) system these problems reduce to a general problem of finding b_k to

$$\text{Minimize: } P(b_1, .., b_N) = \sum_{k=1}^{N} \phi_k(b_k) \tag{2.3.10}$$

$$\text{Subject to : } \sum_{k=1}^{N} b_k = B, b_k \in \{\mathit{0}, \mathit{1}, ...B\}, \tag{2.3.11}$$

where ϕ_k is a convex function, and B is a positive integer. In subband coding

$$\phi_k(b_k) = \alpha_k 2^{-2b_k} \tag{2.3.12}$$

where α_k is determined by the signal variance in the k-th subband, [4] and $P(b_1, .., b_N)$ is the average distortion variance, and b_k is the bits assigned to the k-th subband signal. Further α_k increases with increasing signal variance. In multicarrier systems

$$\phi_k(b_k) = \alpha_k 2^{b_k} \tag{2.3.13}$$

where α_k reflect target performance, and channel and interference conditions experienced in the k-th subchannel, [14], [15] and $P(b_1, .., b_N)$ is the total transmitted power. Higher values of α_k reflects more adverse subchannel conditions and/or stringent performance goals; b_k is the the number of bits assigned to each symbol in the cognizant subchannel.

It is recognized that for general convex functions $\phi_k(\cdot)$, the above constrained minimization grows in complexity with the size of B. Since B can be large, it is important to formulate algorithms for which the complexity bound is *independent of B.* The algorithm we provide here is for a general convex function (2.3.14) and hence could accommodate (2.3.12) (η=-2, a=2) and (2.3.13) (η=1, a=2).

$$\phi_k(b_k) = \alpha_k a^{\eta b_k} \tag{2.3.14}$$

We now present a general result from [17] that solves (2.3.10), (2.3.11) for arbitrary convex $\phi_k(\cdot)$. This result is specialized to the cases of (2.3.12) and (2.3.13) in subsequent section. Denote for $k = 1, ..., N, x = 1, ..., B$,

$$\delta_k(x) = \phi_k(x) - \phi_k(x-1). \tag{2.3.15}$$

The ϕ_k's being convex, it follows that

$$\delta_k(1) < \delta_k(2) < ... < \delta_k(B), \forall k. \tag{2.3.16}$$

Let S denote the set of smallest B elements of

$$\tau = \{\delta_k(x) : k = 1, ..., N, x = 1, ..., B\}$$

The following lemma from [17], gives an optimum solution to (2.3.10), (2.3.11).

Lemma 2.3.1 *The optimal solution* $\mathbf{b}^* = [b_1^*, ..., b_N^*]^T$ *to problem (2.3.10), (2.3.11), is defined as follows*

$$b_k^* = \begin{cases} 0 & : \quad \delta_k(1) \notin S \\ B & : \quad \delta_k(B) \in S \\ y & : \quad \delta_k(y) \in S, \delta_k(y+1) \notin S \end{cases}$$

In essence this lemma provides a conceptual framework for solving (2.3.10), (2.3.11). Specifically, construct S, and for each k, determine the largest integer argument b_k for which $\delta_k(b_k)$ is in S. For general convex functions ϕ_k the complexity of all known solutions grows with B. In the sequel we present an algorithm for the convex functions of the type (2.3.14) whose complexity does not depend on B.

2.4 Bit Loading Algorithm

In the case of (2.3.14), one finds that,

$$\delta_k(x) = \alpha_k \beta^{(x-1)}(\beta - 1).(\text{where } \beta = a^\eta). \tag{2.4.17}$$

The first step of the algorithm requires ordering the α_i, and can be accomplished in $O(N \log N)$ steps. Henceforth assume without sacrificing generality that:

$$\begin{aligned} \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_N \quad \text{if} \quad \beta > 1, \\ \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_N \quad \text{if} \quad \beta < 1. \end{aligned} \tag{2.4.18}$$

Define the sequence:

$$l_i = \lceil \log_\beta(\frac{\alpha_i}{\alpha_1}) \rceil, i = 1, 2, ..., N \tag{2.4.19}$$

with $l_{N+1} = \infty$, where $\lceil a \rceil$ is the smallest integer greater than or equal to a. The significance of the integers l_i is explained by Lemma 2.4.1

Lemma 2.4.1 *With l_i defined in (2.4.19),*

$$\delta_1(l_i) < \delta_i(1) \leq \delta_1(l_i + 1).$$

Proof: From (2.4.19) we have $l_i = \lceil \log_\beta(\frac{\alpha_i}{\alpha_1}) \rceil$. The definition of the ceiling function gives us the following result,

$$l_i - 1 < \log_\beta(\frac{\alpha_i}{\alpha_1}) \leq l_i.$$

Now for $\beta > 1$ the result directly follows from the above equation. When $\beta < 1$ we have the following

$$\alpha_1 \beta^{l_i - 1} > \alpha_i \geq \alpha_1 \beta^{l_i}. \tag{2.4.20}$$

Multiplying throughout by (β-1) we obtain the result (observe that $\beta - 1 < 0$). ■

Then the proposed algorithm for solving (2.3.10), (2.3.11) under (2.3.14) is given below. It assumes that the ordering implicit in (2.4.18), has already occurred, and assigns b_i bits to the i-th subchannel.

Proposed algorithm

Step-1: Find the smallest k such that

$$R_k = \sum_{i=1}^{k-1}(l_k - l_i) \geq B \tag{2.4.21}$$

Then

$$b_i = 0 \quad \forall i \in \{k, k+1, \cdots, N\}. \tag{2.4.22}$$

Step-2: Find

$$\Delta = B - R_{k-1} \tag{2.4.23}$$

$$r = \Delta \bmod (k-1) \tag{2.4.24}$$

$$q = \Delta \mathrm{div}(k-1) \tag{2.4.25}$$

Step-3: Find the r smallest elements of the set

$$\{\delta_1(l_{k-1} - l_1), \delta_2(l_{k-1} - l_2), \cdots, \delta_{k-1}(0)\}. \tag{2.4.26}$$

In particular, with l_{j_i} such that with $l_{j_i} \in \{1, \cdots, k-1\}$,

$$\delta_{j_i}(l_{k-1} - l_{j_i}) \leq \delta_{j_{i+1}}(l_{k-1} - l_{j_{i+1}}), \tag{2.4.27}$$

call

$$J = \{j_1, j_2, ..., j_r\}\,. \tag{2.4.28}$$

If $r = 0$, J is empty.

Step-4: For all $i \in \{1, \cdots, k-1\}$,

$$b_{j_i} = \begin{cases} l_{k-1} - l_i + q + 1 & \text{if} \quad i \in J, \\ l_{k-1} - l_i + q & \text{else.} \end{cases} \tag{2.4.29}$$

2.4.1 Runtime Analysis

Observe that the complexity implicit in achieving (2.4.18) is $O(N \log N)$. Determination of k so that (2.4.21) holds requires at most $2N$ operations, *regardless of* B. Indeed one has, with

$$\rho_1 = 0$$

$$\rho_n = \rho_{n-1} + l_n,$$

$$R_n = (n-1)l_n - \rho_{n-1}.$$

The only impact that B has in the complexity of determining k is that for sufficiently small B, $k < N$ and the number of computations is further reduced to $2(k-1)$. Determining the ranking manifest in (2.4.27) is determined only by r and k, and is

$$O(r \log(k-1)) \leq O((N-1)\log(N-1)).$$

Determination of r requires 2 operations, independent of B. B does affect the precise value of r, which however is no greater than $N-1$. Thus the overall complexity, is bounded by $O(N \log(N))$, with B playing no role in the determination of this bound. The only effect that B has on the overall complexity is to cause fluctuations in the precise number of operations, within a range that is independent of B. To recap, these fluctuations occur when:

- For small B, $k < N$, and finding k requires only $2(k-1)$ operations.

- As B changes r fluctuates between 0 and $N-1$, and the number of operations required to determine the smallest r elements of the set in (2.4.26) changes.

Campello's Algorithm:

To better illustrate the runtime efficiency of the proposed algorithm with regards to the ones existing in literature, we analyze the algorithm (with the best runtime) proposed in [6]. [6] proposes a solution to (2.3.10) and (2.3.11), where

$$\phi_k(x) = \frac{2}{g_n}(2^x - 1). \tag{2.4.30}$$

The discrete bitloading algorithm described in this reference is based on the following theorem.

Theorem 2.4.1 *The discrete bit allocation,* $\boldsymbol{b}(i) = [b_1(i), b_2(i), \ldots, b_N(i)]$*, given by*

$$b_n(i) = [\lfloor \log_2 g_n \rfloor + i]_0^{\bar{b}}, n = 1, 2, \ldots, N. \tag{2.4.31}$$

is efficient for all $i \in \mathbf{Z}$*, where*

$$[x]_a^b = \begin{cases} b & \text{if } b < x \\ x & \text{if } a \leq x \leq b \\ a & \text{if } x < a. \end{cases}$$

The idea is to find an i such that the resulting allocation is as tight as possible. The algorithm could be summarized as follows,

Step 1: Determine

$$i_B = \max\{i \in \mathbf{Z} : |\mathbf{b}(i)| \leq B\}.$$

Step 2: Increment $B' = B - |\mathbf{b}(i_B)|$ subchannels with the smallest incremental energy $\delta_n(i_B + 1)$ in the set

$$I = \{n \in \{1, 2, \ldots, N\} : 0 \leq \lfloor \log_2 g_n + i_B \rfloor \leq \bar{b}$$

We now analyze the steps to find i_B as discussed in [6]. Assuming that g_n's have a dynamic range, i.e., $g_n/g_m < 2^L$ for all $n, m = 1, 2, \ldots, N$, and that they are normalized to be in the set $\left[1, 2^L - 1\right]$, define

$$\mathcal{M}_k = \{n \in \{1, 2, \ldots, N\} : \lfloor \log_2 g_n \rfloor = k\}, k = 0, \ldots, L - 1$$

and let $M_k = |\mathcal{M}_k|$. Now $\mathbf{b}(i)$ can be calculated iteratively for $i = -L + 2, -L + 3, \ldots, \bar{b}$ according to the following equations. We consider the case $\bar{b} \leq L$ for our analysis (the case $\bar{b} > L$ is just as simple to analyze) .

$$S_i = \begin{cases} S_{i-1} + M_{1-i} & \text{if } 1 - L \leq i \leq \bar{b} + 1 - L \\ S_{i-1} + M_{1-i} - M_{\bar{b}+2-i} & \text{if } \bar{b} + 1 - L < i \leq 1 \\ S_{i-1} - M_{\bar{b}-i-1} & \text{if } 1 < i \leq \bar{b} \\ 0 & \text{if } \bar{b} < i. \end{cases}$$

$$|\mathbf{b}(i)| = |\mathbf{b}(i-1)| + S_i$$

where $S_{1-L} = |\mathbf{b}(1 - L)| = 0$. For $1 - L \leq i \leq \bar{b} + 1 - L$ we have,

$$\begin{aligned} |\mathbf{b}(2 - L)| &= M_{L-1}, \\ |\mathbf{b}(3 - L)| &= 2.M_{L-1} + M_{L-2}, \\ |\mathbf{b}(4 - L)| &= 3.M_{L-1} + 2.M_{L-2} + M_{L-3}, \\ &\vdots \\ \left|\mathbf{b}(\bar{b} + 1 - L)\right| &= \bar{b}.M_{L-1} + (\bar{b} - 1).M_{L-2} + \ldots + M_{L-\bar{b}}, \end{aligned}$$

$|\mathbf{b}(i)|$'s for $i > \bar{b} + 1 - L$ take a similar pattern with the added constraint that none of the channels are allocated more than $\bar{b}$ bits. This is guaranteed due to the correction incorporated into computing S_i's for $i > \bar{b}+1-L$. The essence of the above illustration is that determining i_B for this scenario is equivalent to sorting α_i's in ascending or descending order (depending on the application) and finding the largest k such that $R_k \leq B$ in the proposed algorithm (Step-1). The runtime of this step as discussed in [6] is O(L) (O(max$\{N, L\}$) to be precise), where L is the dynamic range of g_n's. This is not a desirable property since the presence of deep fades in the channel could result in a fairly large L.

2.4.2 Proof for Correctness of the Algorithm

We now show that the algorithm in section 2.4 does indeed solve (2.3.10), (2.3.11), under (2.3.13). In view of Lemma 2.3.1 it suffices to show that the set

$$S^* = \{\delta_1(1), \cdots, \delta_1(b_1), \delta_2(1), \cdots, \delta_2(b_2), \ldots, \delta_{k-1}(b_{k-1})\}, \tag{2.4.32}$$

is such that $S^* = S$, defined in section 2.3. This in turn requires the demonstration of the following facts.

(A) $|S^*| = |S| = B$, where $|\cdot|$ represents the cardinality of its argument.

(B) For all $i, j \in \{1, 2, \cdots, N\}$, $\delta_i(b_i + 1) \geq \delta_j(b_j)$.

The first theorem proves (A).

Theorem 2.4.2 *With b_i defined in (2.4.21-2.4.29), $|S^*| = B$.*

Proof: Since $b_i = 0$ for all $i \in \{k, k+1, \cdots, N\}$, we need to show that $\sum_{i=1}^{k-1} b_i = B$. From (2.4.21-2.4.25) we have that

$$\begin{aligned}
\sum_{i=1}^{k-1} b_i &= \sum_{i \in J} b_i + \sum_{i \in \{\{1,\cdots,k-1\}-J\}} b_i \\
&= r(q+1) + (k-1-r)q + \sum_{i=1}^{k-1}(l_{k-1} - l_i) \\
&= \Delta + R_{k-1} \\
&= B.
\end{aligned}$$

■

To prove (B) we need an additional Lemma.

Lemma 2.4.2 *With l_i, k and q as in (2.4.19-2.4.25),*

$$q \begin{cases} \leq l_k - l_{k-1} & \textit{if } r = 0 \\ < l_k - l_{k-1} & \textit{if } r \neq 0 \end{cases}$$

Proof: From (2.4.21-2.4.25)

$$\begin{aligned}
(k-1)q + r &\leq R_k - R_{k-1} \\
&= \sum_{i=1}^{k}(l_k - l_i) - \sum_{i=1}^{k-1}(l_{k-1} - l_i) \\
&= (k-1)(l_k - l_{k-1}).
\end{aligned}$$

Hence the result. ■

We now prove (B) for the case where $r = 0$.

Theorem 2.4.3 *Consider (2.4.19-2.4.29). Suppose $r = 0$. Then (B) above holds.*

Proof:

Consider two cases that exhaust all possibilities.

Case $\beta > 1$: For all $i \in \{2, \cdots, k-1\}$, from Lemma 2.4.1, we have:

$$\delta_i(b_i) = \alpha_i \beta^{l_{k-1}-l_i+q-1}(\beta - 1) \le \alpha_1 \beta^{l_{k-1}+q-1}(\beta - 1) = \delta_1(b_1), \tag{2.4.33}$$

as $l_1 = 0$. Thus $\delta_1(b_1)$ is the largest member of S^* in (2.4.32). From Lemma 2.4.1, for all $i \in \{1, \cdots, k-1\}$,

$$\delta_i(b_i + 1) = \alpha_i \beta^{l_{k-1}-l_i+q}(\beta - 1) > \alpha_1 \beta^{l_{k-1}+q-1}(\beta - 1) = \delta_1(b_1). \tag{2.4.34}$$

Further, as (2.4.22) holds, we have from Lemmas 2.4.1 and 2.4.2 that for all $i \in \{k, k+1, \cdots, N\}$,

$$\delta_1(b_1) = \alpha_1 \beta^{l_{k-1}+q-1}(\beta - 1) \le \alpha_1 \beta^{l_k-1}(\beta - 1) < \alpha_k = \delta_k(1). \tag{2.4.35}$$

In view of (2.4.33), (2.4.34) and (2.4.35), prove the result.

Case $\beta < 1$: From Lemma 2.4.1 and (2.4.20) (multiplying (2.4.20) throughout by $(\beta - 1)$) we have:

$$\delta_i(b_i) = \alpha_i \beta^{l_{k-1}-l_i+q-1}(\beta - 1) \le \alpha_1 \beta^{l_{k-1}+q-1}(\beta - 1) = \delta_1(b_1), \tag{2.4.36}$$

This shows that $\delta_1(b_1)$ is the largest member of S^* in (2.4.32). From Lemma 2.4.1 and (2.4.20), for all $i \in \{1, \cdots, k-1\}$,

$$\delta_i(b_i + 1) = \alpha_i \beta^{l_{k-1}-l_i+q}(\beta - 1) > \alpha_1 \beta^{l_{k-1}+q-1}(\beta - 1) = \delta_1(b_1). \tag{2.4.37}$$

Following the same argument as before from (2.4.22), Lemmas 2.4.1 and 2.4.2 that for all $i \in \{k, k+1, \cdots, N\}$,

$$\delta_1(b_1) = \alpha_1 \beta^{l_{k-1}+q-1}(\beta - 1) \le \alpha_1 \beta^{l_k-1}(\beta - 1) < \alpha_k = \delta_k(1). \tag{2.4.38}$$

Equations (2.4.36), (2.4.37) and (2.4.38) prove the result. ■

Finally we prove (B) for the case where $r \neq 0$.

Theorem 2.4.4 *Consider (2.4.19-2.4.29). Suppose $r \neq 0$. Then (B) above holds.*

Proof: With the indices j_i defined in (2.4.27), we first show that

$$\delta_{j_r} \geq \delta_i(b_i) \quad \forall i \in \{1, \cdots, k-1\}. \tag{2.4.39}$$

In view of (2.4.27) this is clearly true for $i \in J$. Now consider $p \in \{\{1, \cdots, k-1\} - J\}$.

Consider again two cases that exhaust all possibilities.

Case $\beta > 1$*:* As a result of (2.4.29) and Lemma 2.4.1

$$\begin{aligned}
\delta_p(b_p) &= \alpha_p \beta^{l_{k-1}-l_p+q-1}(\beta-1) \\
&\leq \alpha_1 \beta^{l_{k-1}+q-1}(\beta-1) \\
&< \alpha_{j_1} \beta^{l_{k-1}-l_{j_1}+q}(\beta-1) = \delta_{j_1}(b_{j_1}) \\
&\leq \delta_{j_r}(b_{j_r}),
\end{aligned}$$

where the last inequality follows from (2.4.27).

Case $\beta < 1$*:* As a result of (2.4.29), Lemma 2.4.1 and (2.4.20) (multiplying (2.4.20) throughout by $(\beta - 1)$)

$$\begin{aligned}
\delta_p(b_p) &= \alpha_p \beta^{l_{k-1}-l_p+q-1}(\beta-1) \\
&\leq \alpha_1 \beta^{l_{k-1}+q-1}(\beta-1) \\
&< \alpha_{j_1} \beta^{l_{k-1}-l_{j_1}+q}(\beta-1) = \delta_{j_1}(b_{j_1}) \\
&\leq \delta_{j_r}(b_{j_r}),
\end{aligned}$$

where the last inequality once again follows from (2.4.27). For all $i \in \{\{1, \cdots, k-1\} - J\}$, (2.4.27, 2.4.28) demonstrate that

$$\delta_i(b_i + 1) \geq \delta_{j_r}(b_{j_r}). \tag{2.4.40}$$

Further, from Lemma 2.4.1 for all $i \in J$,

$$\begin{aligned}
\delta_i(b_i + 1) &= \alpha_i \beta^{l_{k-1}-l_i+q+1}(\beta-1) \\
&> \alpha_1 \beta^{l_{k-1}+q}(\beta-1) \\
&\geq \delta_{j_r}(b_{j_r})(\beta-1).
\end{aligned}$$

Then the result is proved by observing from Lemma 2.4.2 that

$$\begin{aligned}
\delta_{j_r}(b_{j_r}) &= \alpha_{j_r}\beta^{l_{k-1}-l_{j_r}+q}(\beta-1) \\
&\leq \alpha_{j_r}\beta^{l_{k-1}-l_{j_r}-1}(\beta-1) \\
&\leq \alpha_1\beta^{l_k-1} \\
&< \alpha_k = \delta_k(1).
\end{aligned}$$

■

2.5 Simulations

A comparison of the performance of the algorithms of [13] and [6] and the proposed algorithm with respect to the number of computations required is shown in the figures 2.5 and 2.6, for the cases where $N = 32$ and $N = 64$, respectively. In implementing [6], which is a suboptimal algorithm, the maximum number of bits, B^* that any channel can be assigned is kept at B.

Number of computations needed for each algorithm to converge to the optimal solution was calculated by assuming that addition, subtraction, div, mod, multiplication or division of two numbers would need one computation as would the logical comparisons between two decimal numbers. The results show that the algorithm described in [6] is linear with respect to B while the algorithm in [13] needs large number of computations to converge as B grows. The number of computations needed for the proposed algorithm is independent of the change in B the minor variations seen are attributed to the facts that for small B, k in (2.4.21) is small, reducing the number of computations slightly, and cyclic fluctuations induced by the variation in r (see (2.4.24)) between 0 and $N-1$. The sorting algorithm whose convergence depends on the input vector) and the difference in the run-times becomes very significant for large B. Further the proposed algorithm out performs that of [6], even when B is small, and even in [6] $B = B^*$ will be chosen. This is

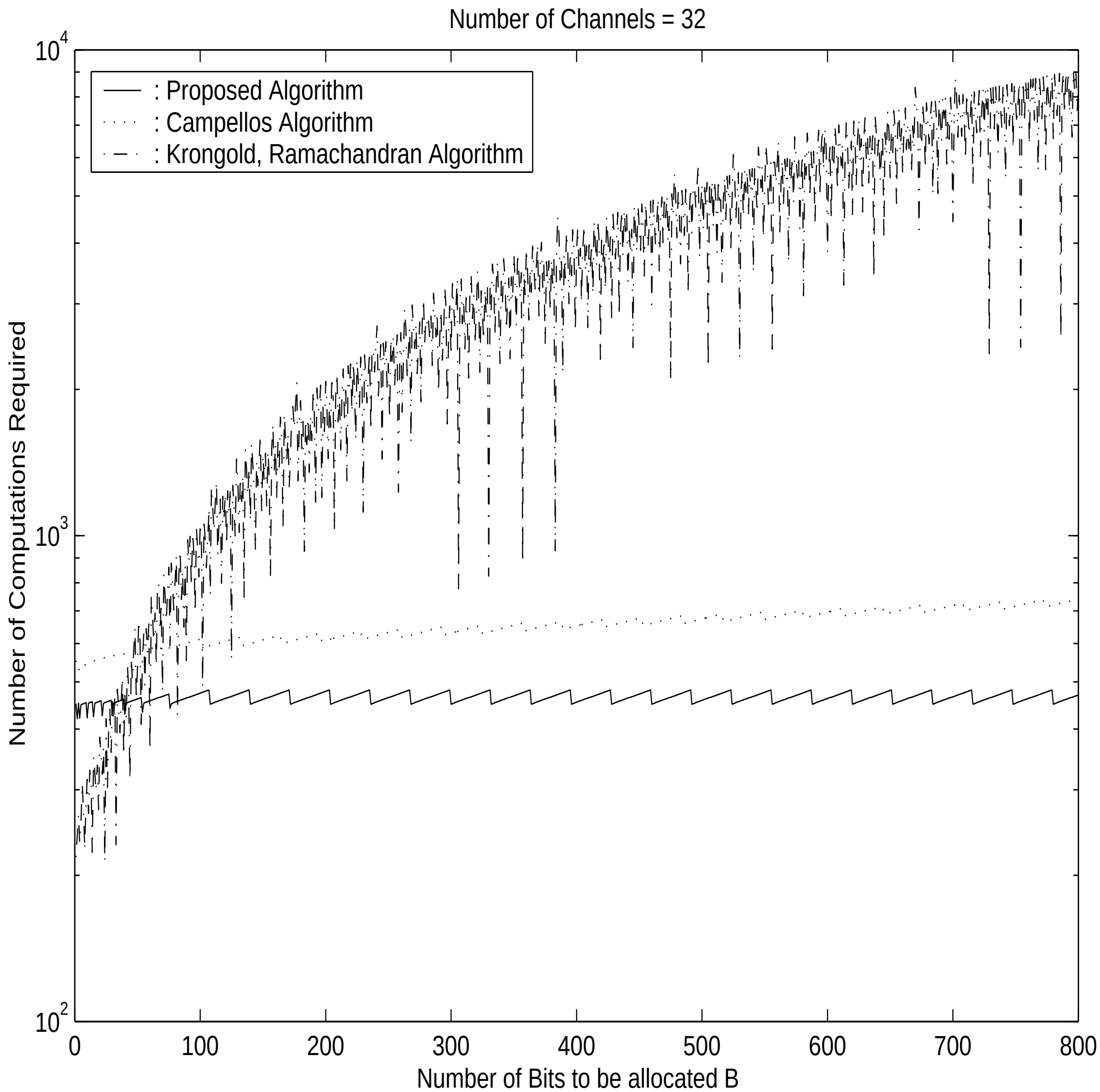

Figure 2.5: Runtime comparisons of the three algorithms for N=32

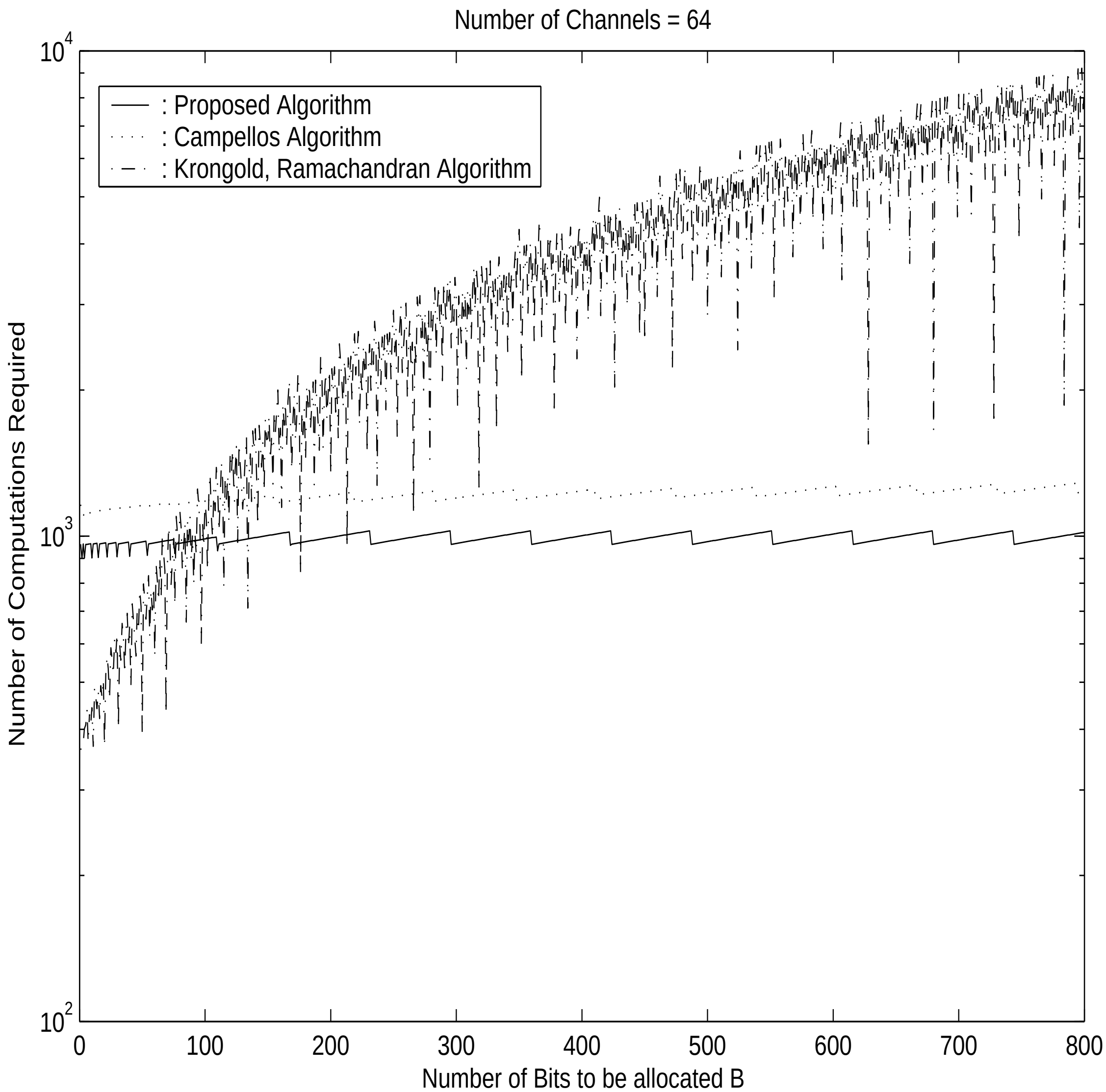

Figure 2.6: Runtime comparisons of the three algorithms for N=64

largely because of the fact that the run time in [6] grows with the dynamic range of α_k.

2.6 Conclusion

In this chapter we have motivated the role of bit loading in subband coding and single user multicarrier communication. The main contribution of this chapter is to formulate a new bit loading algorithm that unlike all others, has complexity that does not grow with the number of bits to be allocated. This has significant practical implications for high fidelity coding and high data rate broadband communications. The next chapter will view the bit loading problem for multiuser DMT, where the same DMT system supports multiple users with differing QoS requirements, quantified by the SERs they require and bit rates they expect. That result however, will be negative, in that we will show the problem to NP-hard.

CHAPTER 3
BIT LOADING IN MULTIUSER MULTICARRIER SYSTEMS

In this chapter, we examine the problem of bit allocation in multiuser multicarrier systems. In particular we examine bit loading within the context of multiuser communications, i.e. when multiple services with different data rate and QoS requirements must share the same multicarrier communications system. The principal contribution is to show that the underlying optimization problem is NP-hard. Section 3.1 describes the general setup, and abstracts a bit loading problem from it. Section 3.2 recounts pertinent facts about Np-completeness and relates it to the problem at hand. Section 3.3 proves NP-hardness.

3.1 The setup

Refer to the general DMT setup of Figure 3.1, note the change of notation from the previous chapter. As in [16] we now consider the problem where the DMT system depicted in Figure 3.1 supports multiple users. To be specific, we assume there are K users and N subchannels. Each user has its own quality of service (QoS) requirement quantified by its bit rate and symbol error rate (SER). More precisely, we assume that the k-th user requires a bit rate of β_k and must not sustain an SER that exceeds η_k. Its overall communication is distributed over several subchannels, as many in fact as needed. The overall optimization goal is to select the input and output block transforms G_0 and S_0, the linear redundancy matrix S_1, the number of bits/symbol assigned to each subchannel, and the subchannel assignment to each service flow, in order to achieve the QoS specifications under a perfect reconstruction condition with *minimum transmitted power.*

Observe the expanded goal of not just to distributing the bit rates across the carriers, but also to assign specific carriers to different services. For an N-subchannel, K-user system this problem reduces to general problem of constrained

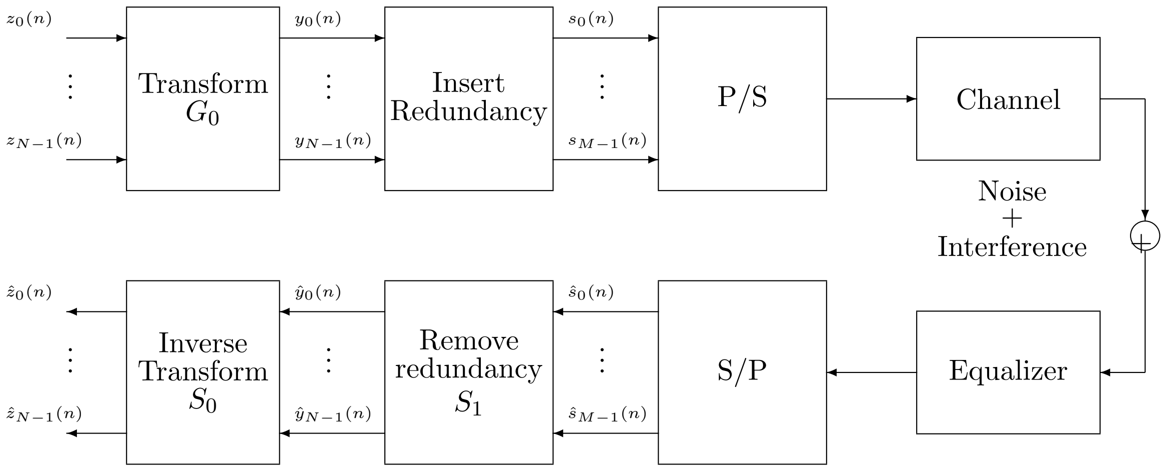

Figure 3.1: Multiuser DMT communication system.

optimization shown below,

$$\min_{b_{n,k},\rho_{n,k}} \sum_{k=1}^{K} \sum_{n=1}^{N} \phi_{nk}(b_{n,k})\rho_{n,k} \tag{3.1.1}$$

$$\text{Subject to : } \begin{cases} \sum\limits_{k=1}^{K} \rho_{n,k} = 1, \text{ for } n = 1, \ldots, N \\ \sum\limits_{k=1}^{N} b_{n,k} = \beta_k,\ k = 1, \ldots, K. \\ \rho_{n,k} = \begin{cases} 1 & \text{if } b_{n,k} \neq 0 \\ 0 & \text{if } b_{n,k} = 0 \end{cases} \\ \hat{z}_i(n) = z_0(n); \quad \forall n \in \mathcal{Z}, i \in \{0, \cdots, N-1\} \quad \text{if no noise and interference.} \end{cases} \tag{3.1.2}$$

Here

$$\phi_{nk}(b_{n.k}) = \alpha_{kn} 2^{b_{n,k}}, \tag{3.1.3}$$

with α_{kn} determined by the SER requirement and the signal variance in the pertinent channel. Further, β_k is the data rate requirement for user-k; $\rho_{n,k}$ takes value 1 if the n-th subchannel is allocated to user k and 0 otherwise. Since the sum of all $\rho_{n,k}$ is 1 for any n, it is assured that none of the subchannels are allocated to more than one user. A key difference from single user case is the multiple bit rate constraints.

The optimization variables are $b_{n,k}$, $\rho_{n,k}$, G_0 and S_i. The selection of $\rho_{n,k}$ is tantamount to the added requirement of subchannel selection, absent in the single user case. Another difference of course is the fact that the SER requirement varies across users. Nonetheless, [16] shows that under optimal bit loading and subchannel selection the matrices G_0, and S_i are determined entirely by the signal, noise and interference statistics. Thus, effectively (3.1.2) can be replaced by the simpler problem below.

$$\text{Subject to :} \begin{cases} \sum_{k=1}^{K} \rho_{n,k} = 1, \text{ for } n = 1, \ldots, N \\ \sum_{k=1}^{N} b_{n,k} = \beta_k, \ k = 1, \ldots, K. \\ \rho_{n,k} = \begin{cases} 1 & \text{if } b_{n,k} \neq 0 \\ 0 & \text{if } b_{n,k} = 0 \end{cases} \end{cases} \tag{3.1.4}$$

This problem is yet to be solved, and its study is the subject of this chapter. Approximation algorithms achieving a suboptimal bit allocation exist in the literature. These include, [20], [22], [23], [24] and [25] the latter considering a variation. Several papers e.g. [22] have stated without proof or reference to the fact that this problem is NP-hard as a motivation for considering suboptimal solutions. The paper [20], does cite a result from [28] to justify the NP-hardness of this problem. However, [28] demonstrates the NP-hardness of the following problem. Given arbitrary convex f_i, positive real a_i, find non-negative integers x_i to:

$$\text{Minimize : } Z(X) = \sum_{i=1}^{n} f_i(\sum_{j=1}^{m} a_{ij}x_{ij}), \tag{3.1.5}$$

$$\text{Subject to} : \sum_{i=1}^{n} x_{ij} = N_j, j = 1, 2, \ldots, m. \tag{3.1.6}$$

This problem is far more general than the problem studied here. Thus its NP-hardness does not prove the NP-hardness of the problem considered. Consequently, if for no other reason than the sake of completeness, we present a proof for NP-hardness of the integer bit loading problem for a Multiuser multicarrier system.

A brief introduction to the theory of NP-completeness is first presented in section 3.2. Section 3.3 presents a formal proof classifying the problem as being NP-hard.

3.2 Theory of NP-Completeness

A problem A is said to be NP-hard if a problem B known to be NP-complete can be transformed to a problem instance of A (in polynomial time) in the sense that any problem instance of B has a solution if and only if the transformed instance A has a solution. (Thus A is not easier than B since any instance of B can be solved by solving the transformed instance of A). Note that the whole set of problem instances of B may be transformed to only a subset of problem instances of A. The following problem $\mathcal{Q}$ is a special case of the constrained optimization problem defined by (3.1.1) and (3.1.4) specialized for the two user case.

Problem $\mathcal{Q}$: Given positive integers β_1 and β_2, positive numbers α_k, $\phi_k(x_k) = \alpha_k 2^{x_k}$, find nonnegative integers x_k and δ_k, $\forall k$ such that (3.2.7) is minimized subject to (3.2.8).

$$\text{Minimize} : P(x_1, .., x_N) = \sum_{k=1}^{N} \phi_k(x_k), \tag{3.2.7}$$

$$\text{Constraint}: \begin{cases} \sum_{k=1}^{N} \delta_k x_k = \beta_1, \text{where} \quad \delta_k = \{0,1\}, \\ \sum_{k=1}^{N} x_k = \beta_1 + \beta_2, \\ x_k \in \{0, 1, ..., \max\{\beta_1, \beta_2\}\}. \end{cases} \tag{3.2.8}$$

To see why this problem is a special case of our original problem in the two user case, observe in the original problem one works with a cost function each of whose summands depends not just on the subchannel but the user. Indeed see that in (3.1.3) the coefficient α_{nk} has two subscripts. It changes from user to user because the different users may have different SER requirements. As stated, in Problem $\mathcal{Q}$ the functional from of the summands only depend on the subchannels, and not on the user. Thus, in effect Problem $\mathcal{Q}$ assumes that *all users have the same SER requirement.* Of course if this problem is NP-Hard then so must be the original problem.

We now present a problem that qualifies as a simpler instance of $\mathcal{Q}$. Thus if this new problem is NP-Hard so is $\mathcal{Q}$.

Problem $\mathcal{A}$: Given a set of positive real numbers $\{\alpha_1, \ldots, \alpha_N\}$, P as defined in Problem $\mathcal{Q}$, and positive integers β_1 and β_2, Find x_k,δ_k, $\forall k$, should they exist, that satisfy the following set of equations.

$$P(x_1, .., x_N) \leq P(b_1, .., b_N), \text{ for a certain } b_1, b_2, \ldots, b_N \tag{3.2.9}$$

$$\text{Constraint}: \begin{cases} \sum_{k=1}^{N} \delta_k x_k = \beta_1, \text{where} \quad \delta_k = \{0,1\}, \\ \sum_{k=1}^{N} x_k = \beta_1 + \beta_2, \\ x_k \in \{0, 1, ..., \max\{\beta_1, \beta_2\}\} \end{cases} \tag{3.2.10}$$

The reason for this is stated as follows. Problem $\mathcal{A}$ is no harder than the corresponding minimization problem $\mathcal{Q}$ because the minimum value $P(x_1^*, x_2^*, \ldots, x_N^*)$ of the minimization problem immediately shows whether $P(x_1, x_2, \ldots, x_N) \leq \mathcal{K}$ for a certain constant $\mathcal{K}$ is possible or not.

We now consider *Subset Cover* problem which is well known to be NP-complete.

Problem $\mathcal{B}$: Given a set of positive integers S $= \{b_1, \ldots, b_N\}$ and a positive integer β_1, decide whether there is a subset $S_1 \subseteq S$ such that elements of S_1 sum to β_1.

Remark 3.2.1 *This is equivalent to determining if $\delta_k \, \forall k$ exist such that*

$$\begin{aligned} &\sum_{k=1}^{N} \delta_k . b_k = \beta_1, \; \delta_k \in \{0, 1\}, \, \forall k. \\ &\sum_{k=1}^{N} b_k = \beta_1 + \beta_2 \end{aligned} \tag{3.2.11}$$

We will now show that given arbitrary positive integers S $= \{b_1, \ldots, b_N\}$ and a positive integer β_1, one can find a set of α_i in polynomial time, such that the solution to **Problem $\mathcal{B}$** is obtained by solving **Problem $\mathcal{A}$** with these α_i. Indeed, without loss of generality arrange:

$$b_i \geq b_{i+1} \forall i \tag{3.2.12}$$

Choose α_i such that,

$$2^{b_1 - b_i - 1} < \alpha_i < 2^{b_1 - b_i}, \; \forall i. \tag{3.2.13}$$

The following additional restrictions are imposed on the choice of α_i's.

$$\begin{aligned} &\alpha_i < \alpha_j, \; \forall i < j, \\ &\alpha_i \neq 2^C \alpha_j, \; \forall i < j, \text{ for any integer } C \end{aligned} \tag{3.2.14}$$

Since α_i's are real numbers, there is always a choice of α_i's satisfying (3.2.13) and (3.2.14). The above transformation could be completed in $O(N)$ and the original sorting of b_i requires $O(N \log N)$ time.

3.3 Proof for NP-Hardness of Bit loading in Multiuser Multicarrier Systems

With the choice of α_i's as explained in Section 3.2 we now show that $\mathcal{Q}$ is NP-hard.

We first state the following Theorem that will be proved through Lemmas that follow it.

Theorem 3.3.1 *Consider positive integers* $S = \{b_1, \ldots, b_N\}$ *and a positive integer* β_1*, under (3.2.12). Suppose* α_i *obey (3.2.13) and (3.2.14). Then the only set* $\{x_1, x_2, \ldots, x_N\}$ *for which* $P(x_1, \ldots, x_N) \leq P(b_1, \ldots, b_N)$ *under the constraint* $\sum_{k=1}^{N} x_k = \beta_1 + \beta_2$*, is* $x_1 = b_1$*,* $x_2 = b_2$*, ...,* $x_N = b_N$*.*

The next four lemmas, (3.3.1-3.3.4), help substantiate the above claim.

The first lemma proves that $\{x_1, \ldots, x_N\} = \{b_1, \ldots, b_N\}$ is an optimal solution for the allocation of $\beta_1 + \beta_2$ bits. It can be seen that the solution space of Problem $\mathcal{A}$ is a subset of the solution space of the constrained optimization problem presented in Theorem 3.3.1 above. Lemma 4 shows that $\{x_1, \ldots, x_N\} = \{b_1, \ldots, b_N\}$ is the only optimal solution under the constraint $\sum_{k=1}^{N} x_k = \beta_1 + \beta_2$. The Lemma uses the algorithm discussed for single user bit allocation problem in the previous chapter.

$$\text{Minimize} : P(x_1, .., x_N) = \sum_{k=1}^{N} \phi_k(x_k), \tag{3.3.15}$$

$$\text{Constraint} : \sum_{k=1}^{N} x_k = B. \tag{3.3.16}$$

Lemma 3.3.1 *Under the conditions of Theorem 3.3.1, the solution to the optimization problem*

$$\begin{aligned} &Minimize: P(x_1,..,x_N) = \sum_{k=1}^{N} \phi_k(x_k), \textit{ where} \quad \phi_k(x_k) = \alpha_k 2^{x_k}, \\ &Constraint: \sum_{k=1}^{N} x_k = \beta_1 + \beta_2, \end{aligned} \tag{3.3.17}$$

is given by $\{x_1, x_2, \ldots, x_N\} = \{b_1, b_2, \ldots, b_N\}$.

Proof: Using the single user discrete bit-loading algorithm presented earlier in the paper, [18], for the choice of α_i's described earlier we have,

$$l_i = \left\lceil log_2(\frac{\alpha_i}{\alpha_1}) \right\rceil = b_1 - b_i.$$

We need to determine the smallest k for which the following holds.

$$R_k = \sum_{i=1}^{k-1}(l_k - l_i) \geq \beta_1 + \beta_2 \tag{3.3.18}$$

It can be seen that

$$\begin{aligned} R_N &= \sum_{i=1}^{N-1}(l_N - l_i) \\ &= \sum_{i=1}^{N-1}(b_i - b_N) \\ &= \beta_1 + \beta_2 - (N \times b_N) \\ &\leq \beta_1 + \beta_2. \end{aligned}$$

Therefore the smallest k for which (3.3.18) holds is $k = N + 1$ (since $l_{N+1} = \infty$).

$$\begin{aligned} &\delta = \beta_1 + \beta_2 - R_N = N \times b_N \\ &r = \delta \operatorname{mod}(N) = 0 \\ &q = \delta \operatorname{div}(N) = b_N \end{aligned}$$

Since $r = 0$,

$$\begin{aligned} x_i &= l_N - l_i + q \\ &= b_i - b_N + b_N \\ &= b_i, \forall i. \end{aligned}$$

■

Having proved that claimed solution claimed in Theorem 3.3.1 is indeed optimal, we will next prove that it is the only optimal solution. To this end we first establish an ordering of any solution to the optimization problem posed in Lemma 3.3.1.

Lemma 3.3.2 *Under the conditions of Theorem 3.3.1, suppose* $\{x_1, x_2, \ldots, x_N\}$ *is an optimal solution to the constrained optimization problem in Lemma 3.3.1. Then,* $x_i \geq x_{i+1}, \forall i.$

Proof: Supposing that $\exists\, i, j$ for which $x_i < x_j$ for some $i < j$.

$$\alpha_i(2^{x_j} - 2^{x_i}) < \alpha_j(2^{x_j} - 2^{x_i}), \text{ because } \alpha_j > \alpha_i, \text{ for } j > i$$

$$\alpha_i.2^{x_j} + \alpha_j.2^{x_i} < \alpha_i.2^{x_i} + \alpha_j.2^{x_j}$$

This means that by swapping the number of bits assigned to channels i and j (while retaining the allocation for the remaining channels) we could minimize the cost. This is a contradiction since the initial assignment was know to be optimal. ■

Next we expose a property of the α_i imposed by (3.2.13) and (3.2.14) under (3.2.12).

Lemma 3.3.3 *With the choice of* α_i*'s explained by (3.2.13) and (3.2.14) the fol-*

lowing inequalities hold,

$$\begin{aligned} \alpha_1.2^{b_1} &< \alpha_i.2^{b_i+1},\ \forall i \\ \alpha_i.2^{b_i} &< \alpha_1.2^{b_1},\ \forall i \end{aligned} \tag{3.3.19}$$

Proof: From equations (3.2.13) and (3.2.14) we have the following.

$$\alpha_1.2^{b_1-b_i-1} < \alpha_i < \alpha_1.2^{b_1-b_i}$$

Using the first half of the above inequality we have,

$$\begin{aligned} \alpha_1.2^{b_1-b_i-1} &< \alpha_i \\ \alpha_1.2^{b_1-1} &< \alpha_i.2^{b_i} \\ \alpha_1.2^{b_1} &< \alpha_i.2^{b_i+1} \end{aligned}$$

Now using the second half of the inequality,

$$\begin{aligned} \alpha_i &< \alpha_1.2^{b_1-b_i} \\ \alpha_i.2^{b_i} &< \alpha_1.2^{b_1} \end{aligned}$$

■

Finally, we show that the solution postulated by Theorem 3.3.1 is the only solution.

Lemma 3.3.4 *Suppose the conditions of Theorem 3.3.1 hold. Then,* $\sum_{k=1}^{N} \alpha_k.2^{x_k} = \sum_{k=1}^{N} \alpha_k.2^{b_k}$ *and* $\sum_{k=1}^{N} x_k = \sum_{k=1}^{N} b_k = \beta_1 + \beta_2$*, imply* $\{x_1, \ldots, x_N\} = \{b_1, \ldots, b_N\}$

Proof: Let us assume that there exists some k such that $x_k \neq b_k$ otherwise we are done.

Case I: $x_1 > b_1$

There exists some k such that $x_k = b_k - C_2$ for some positive constant C_2 while $x_1 = b_1 + C_1$ (for some positive constant C_1). Using Lemma(3.3.3) we have.

$$\alpha_1.2^{b_1+C_1-1} > \alpha_k.2^{b_k-C_2}, \text{ since } C_1, C_2 \geq 1$$
$$\alpha_1.2^{b_1+C_1} + \alpha_k.2^{b_k-C_2} > \alpha_1.2^{b_1+C_1-1} + \alpha_k.2^{b_k-C_2+1}$$
$$\alpha_1.2^{x_1} + \alpha_k.2^{x_k} > \alpha_1.2^{x_1-1} + \alpha_k.2^{x_k+1}$$

The last equation implies that an allocation of $\{x_1-1, x_2, \ldots, x_k+1, \ldots, x_N\}$ yields a smaller cost. This is a contradiction.

Case II: $x_1 < b_1$

There exists some k such that $x_k = b_k + C_2$ for some positive constant C_2 while $x_1 = b_1 - C_1$ (for some positive constant C_1). Using Lemma(3.3.3) we have.

$$\alpha_k.2^{b_k+C_2-1} > \alpha_1.2^{b_1-C_1}, \text{ since } C_1, C_2 \geq 1$$
$$\alpha_1.2^{b_1-C_1} + \alpha_k.2^{b_k+C_2} > \alpha_1.2^{b_1-C_1+1} + \alpha_k.2^{b_k+C_2-1}$$
$$\alpha_1.2^{x_1} + \alpha_k.2^{x_k} > \alpha_1.2^{x_1+1} + \alpha_k.2^{x_k-1}$$

The last equation implies that an allocation of $\{x_1+1, x_2, \ldots, x_k-1, \ldots, x_N\}$ yields a smaller cost. This, again, is a contradiction and therefore $x_1 = b_1$. We now consider this scenario.

Case III: $x_1 = b_1$.

Now $\exists$ k such that $x_k = b_k + C$ for some positive constant C. Using Lemma(3.3.3) we have,

$$\alpha_k.2^{b_k+C} > \alpha_1.2^{b_1}, \text{ since } C \geq 1$$
$$\alpha_1.2^{b_1} + \alpha_k.2^{b_k+C} > \alpha_1.2^{b_1+1} + \alpha_k.2^{b_k+C-1}$$
$$\alpha_1.2^{x_1} + \alpha_k.2^{x_k} > \alpha_1.2^{x_1+1} + \alpha_k.2^{x_k-1}$$

The last equation clearly implies that an allocation of $\{x_1+1, x_2, \ldots, x_k-1, \ldots, x_N\}$ yields a smaller cost. This is a contradiction. Therefore from Cases I, II and III we can conclude that $x_k = b_k, \quad \forall k$

■

Lemma 3.3.1 proves that $\{x_1, \ldots, x_N\} = \{b_1, \ldots, b_N\}$ is an optimal solution for the allocation of $\beta_1+\beta_2$ bits. It can be seen that the solution space of Problem $\mathcal{A}$ is a subset of the solution space of the constrained optimization problem presented in Theorem 3.3.1 above. Lemma 3.3.4 shows that $\{x_1, \ldots, x_N\} = \{b_1, \ldots, b_N\}$ is the only optimal solution under the constraint $\sum_{k=1}^{N} x_k = \beta_1 + \beta_2$. The Lemma uses the algorithm discussed for single user bit allocation problem discussed in the preceding sections.

The solution space of Problem $\mathcal{A}$ is clearly a subset of the solution space of constrained optimization problem described in Lemma 3.3.1. It has been shown using Lemmas 3.3.1 and 3.3.4 that the only possible solution to the constrained optimization problem described in Lemma 1 is $\{x_1, x_2, \ldots, x_N\}$=$\{b_1, b_2, \ldots, b_N\}$. Therefore it is obvious that Problem $\mathcal{A}$ would have a solution if and only if, $\exists\, \delta_k$, such that $\sum_{k=1}^{N} \delta_k b_k = \beta_1$, where, $\delta_k = \{0, 1\}$. It can therefore be concluded that Problem $\mathcal{B}$ has a solution if and only if Problem $\mathcal{A}$ has a solution. Since any instance of Problem $\mathcal{B}$ can be transformed to an instance of $\mathcal{Q}$, $\mathcal{Q}$ is NP-hard.

3.4 Conclusion

We have considered optimal DMT under a multiuser setting, where multiple users with different QoS requirements are supported by the same DMT system. The QoS requirements are quantified by the SERs and bit rates required by each user. We have considered the problem of optimal bit allocation and subchannel allocation. We have shown that when there are only two users present, and both have the same SER requirement, then the underlying optimization is NP-Hard. Consequently, the more general problem is also NP-Hard.

A variant of this problem is worthy of future consideration. In the current instances we have been flexible about the number of subchannels each user is assigned. Rather that implicitly comes out as a variable of optimization. The following alternative formulation is worthy of consideration: That the number of subchannels, each user receives is fixed *a priori*, though which subchannels, each user gets is still a variable of optimization. We conjecture that this too is an NP-hard problem.

CHAPTER 4
MULTI-AGENT CONTROL UNDER MEDIUM ACCESS CONSTRAINTS

In this chapter we introduce the problem of multi-agent control. A brief review of the previous work in this area is presented in section 4.1. We develop the math and introduce the necessary notations and assumptions for the framework in section 4.2. In section 4.3 we present a review of [30] which deals with the problem of multi-agent control in the absence of medium access control constraints. A modified control law which respects the medium access constraints is then presented in 4.4.

4.1 Background and motivation

The sequel extends the work done by Abel et al [30] that concerned the fault tolerant, distributed, scalable control of a group of agents that must move in a formation specified by relative positions between agents and a constant formation velocity. The control law proposed in [30] naturally accommodates various levels of fault tolerance and scalability and requires an amount of inter-agent communication that is commensurate with a designated level of fault tolerance. The control law assumes, however, that this exchange of information occurs simultaneously. In practice communications must occur under Medium Access Control (MAC) constraints. Thus no agent can transmit and receive at the same time, and cannot transmit to another agent who is receiving information from yet another. We modify the control algorithm so that such MAC constraints are respected, and provide a stability analysis of this modified control law.

Networks of mobile autonomous agents, [33]-[62], involve multiple mobile objects that cooperate to achieve any number of objectives. Thus they may achieve a formation, avoid collisions and obstacles, and be robust to malicious and hostile environments. Cooperation between these agents is effected through limited exchange of information between the agents over wireless media with little or no centralized intervention. We are concerned with agents modeled as double integrators in each cartesian dimension that must organize themselves into formations prescribed by the relative positions between the agents. As in [30] the goal is to devise control laws that, require minimal information exchange between the agents and minimal knowledge on the part of each agent of the overall formation objective, are fault tolerant, scalable, and easily reconfigurable in the face of the loss or arrival of an agent, and the loss of a communication link.

A major drawback of [30] is that it assumes that all agents can exchange information at will. This is fine if agents acquire each others state information through straightforward sensing. If however, state information is exchanged through broadcast communication, this assumption is highly unrealistic. In particular when agents broadcast their state information they must compete with each other for access to the communication medium and are constrained by Media Access Control (MAC) protocols. Specifically, if agent A must listen to the broadcast of agent B, then no other agent that has A in its broadcast range can broadcast at that instant. Further in many instances no agent can simultaneously transmit and receive. These requirements limit (often severely, [64]) the number of transmissions that can occur at a given time, and the full schedule of information exchange *can only occur over several time slots.* Consequently information available to a given agent as it executes its control law may not be the most up to date. The principal contribution of the work presented in this chapter is to modify [30] so that MAC protocols are accommodated.

Significant work in this area has been conducted in the robotics community, e.g. [38]-[44], and also in the string stability literature, [36], [52]. [37], force a group of agents to collaboratively lie on a manifold, but require every agent to communicate with all others. The biologically motivated flocking literature, [39]-

[40], seeking to mimic flocks of bird, seeks to organize coherent group movement as opposed to maintaining specified relative positions. To induce a set of agents with same speed to move in the same direction [33] proposes a simple algorithm that is rigorously analyzed in [34]. The rendezvous problem, where agents are induced to converge to a single unspecified location, is studied in [48], [49]. Consensus forming or synchronization are also instructive examples, [53], [60].

We are particularly interested in organizing agents into formations defined by desired relative positions and trajectories, [50] -[51]. These papers contrast with the rigid formations literature where the topology is specified by inter agent distances rather than relative positions and velocities, [45]-[47] and requires a much more extensive communication architecture. The closest approach is in [51] which seeks to organize a network of agents according to specified relative positions, and focuses on the communication topology. Further discussion on this paper and [30] is below.

Papers like [50] and [51] separately propose a desired formation and a state exchange architecture and ask whether the latter suffices to achieve a formation. [30] reverse the question and ask given a desired formation, what state exchange architecture suffices to achieve it. They also focus on control laws that incorporate redundancies that permit the formation to survive the loss of agents and/or communication links.

Since the take off point of the work presented here is [30], we briefly reprise its salient points. [30] recognizes that the same geometry can be described in multiple ways. Thus if the desired geometry is that depicted in fig. 4.1 it can be described by specifying the relative positions between agents joined by arrows. Thus in this figure relative positions and/or relative velocities of the pairs $(1, 2)$, $(1, 4)$, $(2, 3)$ and $(4, 5)$ are specified. One may also specify the same geometry by adding redundant information, as in fig. 4.2, where the additional constraints are

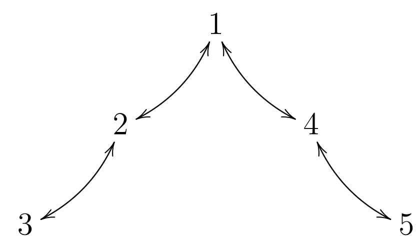

Figure 4.1: Agent formation topology with no redundancy

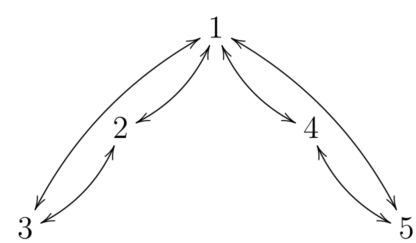

Figure 4.2: Agent formation topology with redundancy

added between the pairs $(1, 3)$ and $(1, 5)$. Such a redundant structure adds fault tolerance to the geometric description. Thus, while the loss of agent 4 in fig. 4.1, implies that 5 is isolated, in fig. 4.2, 5 retains its position relative to agent 1 and the new topology remains viable. Thus additional fault tolerance is achieved in fig. 4.2 by adding redundancies in the geometric configuration such that the loss of an agent still results in an acceptable geometry.

Here on we will call this the *Formation Topology*, as opposed to the *Communication Topology* which defines the state information flow required to implement a cooperative control law. We explore here the relation between these two topologies and argue that issues of fault tolerance, scalability and communication derive from the correct design of the formation topology.

To this end [30] proposes a cost function that incorporates the formation topology. A one step ahead optimal control law obtained on its basis has many features. Foremost among them is the fact that the communication topology required to implement it is *identical* to the underlying formation topology.

The key attractive properties of the approach of [30] are as follows: In the sequel we will call a pair of agent neighbors if they appear in the same geometric constraint. Thus in fig. 4.1 agent 1 has the neighbors 2, and 4, while in fig. 4.2 it has the additional neighbors 3 and 5.

(a) Agent i needs the state information of only its neighbors in the formation topology.

(b) A given agent only *needs to know the constraints imposed on itself* by the formation topology. Thus in figure 4.1, agent 2 needs only to know its desired position/velocity relative to 1 and 3. Should the formation topology explicitly mandate that 2 move with a certain velocity, then of course 2 should be aware of this.

(c) Should the loss of an agent still permit an acceptable topology, e.g. the loss of 4 in fig. 4.2, then only the neighbors of the lost agent need to reconfigure their control law.

(d) Should the loss of a communication channel still permit an acceptable topology, e.g. the loss of the arc joining agents 1 and 5 in fig. 4.2, then only the agent at the end points of the lost arc need to reconfigure their control law.

(e) If a new agent joins the fleet by establishing a geometric position with respect to a subset of the agents, then only these agents need to reconfigure their control law.

(f) Relative position constraints can be augmented by compatible, potentially redundant velocity and/or relative velocity constraints. Thus one may impose a velocity requirement on agent 5 in fig. 4.2, that would automatically specify the direction and movement of the whole formation.

Thus (a) indicates the communication topology highlighted in the foregoing. Item (b) has the added attraction of permitting the control to be implemented by a given agent with only a *local knowledge* of the formation topology. Scalability comes from (e) as a new agent 6 in fig. 4.1 with only 5 as a neighbor would require that only 5 readjust its control law. *Reconfigurability* under the loss of an agent is greatly facilitated.

We propose an alternative control law that retains these attractive properties while respecting MAC requirements. A few points of note are as follows: First [30] have an undirected communication architecture, i.e. if agent i must convey its state to j, then j must convey its state to i. Though over a period of time we also have this requirement, as no agent can simultaneously transmit and receive, in any given sampling interval the architecture here is directional. This contrasts though from the directional control of [65], [66] where if agent i must sense the state of j, then agent j will not know the sense of i at all. Second as will be evident in the sequel, the control law employs a communication architecture that varies from one sampling interval to the next. However, unlike [60] this architecture is periodically varying. Of course [60] is confined to the synchronization problem, as opposed to the harder formation control problem studied here.

4.2 Dynamics and Formation Topology

When considering the problem of an n-agent formation our focus here is on a two dimensional formation topology, even though the ideas trivially extend to three dimensional formations as well. We shall partition the global, $4n \times 1$ state vector x of the network as

$$x = [x_1^T, x_2^T]^T, \tag{4.2.1}$$

where x_1 and x_2 contain the positions and velocities respectively. In particular, denoting $x_{l,j}$ as the j-th element of x_l, we will have

$x_{1,i}$ is the x position of agent i,

$x_{2,i}$ is the x velocity of agent i,

$x_{1,n+i}$ is the y position of agent i, and

$x_{2,n+i}$ is the y velocity of agent i

We shall further assume that each agent has been internally controlled to represent a double integrator with elements u_i and u_{n+i} of the control input vector u representing normalized force variables acting on the i-th agent, in the x and y directions respectively. For notational simplicity we will assume that the sampling interval is 1-second. The ideas trivially extend to non unity sampling intervals. Thus, to within a suitable force normalization the system of agents can be described by:

$$x(k+1) = \Phi x(k) + \Gamma u(k) \tag{4.2.2}$$

where

$$\Phi = \begin{bmatrix} I_{2n} & I_{2n} \\ 0 & I_{2n} \end{bmatrix}, \text{ and } \Gamma = \begin{bmatrix} I_{2n} \\ 2I_{2n} \end{bmatrix}. \tag{4.2.3}$$

To ease notation we will often denote

$$\Phi x[k] = \theta(k). \tag{4.2.4}$$

The formation topology is alternatively characterized graphically and algebraically. In the former case it is described by an *undirected graph* with agents as nodes. An arc exists between two agents if their relative position constraint explicitly appears in the description of the formation topology.

Algebraically, the formation topology will be characterized in the following way. Observe that the relative positions between two agents i and j can be completely specified, for suitable f and g by the pair of equations

$$x_{1,i} - x_{1,j} = f \text{ and } x_{1,n+i} - x_{1,n+j} = g. \tag{4.2.5}$$

Assume that there are L such pairs of constraints. Then with an $L \times n$ matrix A,

$$\mathcal{A} = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0_{2n\times 2n} \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ 0_{2n\times 1} \end{bmatrix} \tag{4.2.6}$$

the topology can be represented by the following equation:

$$\mathcal{A}x = b, \tag{4.2.7}$$

Here x is the target state vector. In all there are $2L$ position constraints and $2m$ velocity constraints. Further A is a a matrix with each row having all but two elements zero and the remaining two being ± 1.

Thus in figure 4.1, L is 4 and in figure 4.2, it is 6. In (4.2.5), for example, f and g are suitable elements of b_1 and b_2 respectively. Also in case of figure 4.1

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Formally, we make the following assumption.

Assumption 4.2.1 *Suppose the formation topology has L arcs. Then the matrix A is $L \times n$. Further if an arc exists between agents i and j then there exists a row of A which has all but the i-th and j-th elements zero and among the remaining* *two one is* 1*, and the other* -1*. Further* b_1, b_2 *are each in the range space of* A*, and*

$$rank[A] = n - 1. \tag{4.2.8}$$

Note that (4.2.8) implies that the graph representing the formation topology is connected, i.e. there is a path joining any two nodes that can be traversed by moving from one nearest neighbor to the next.

Recall that while figures 4.1 and 4.2 describe the same geometry the latter represents a formation topology with redundancies. Observe if the formation topologies in figure 4.1 and figure 4.2 are respectively defined by the pairs $[\mathcal{A}^{(1)}, b^{(1)}]$ and $[\mathcal{A}^{(2)}, b^{(2)}]$, then $[\mathcal{A}^{(1)}, b^{(1)}]$ is a *submatrix* of $[\mathcal{A}^{(2)}, b^{(2)}]$. Moreover, should the loss of an agent result in a topology that remains acceptable, e.g. the loss of 4 in figure 4.2, then this new topology characterized by $[\mathcal{A}^{(3)}, b^{(3)}]$ obtained by removing the rows corresponding to the constraints featuring 4 and columns corresponding to the states of 4, is itself a submatrix of $[\mathcal{A}^{(2)}, b^{(2)}]$. The loss of a communication channel, e.g. that between 1 and 5 would involve the use of a new pair obtained by removing rows characterizing the constraint defining this lost arc. This feature forms a core property to be exploited in fault tolerant design. Scalability is likewise incorporated rather easily. Thus if a new agent 6 appears in figure 4.2 with an arc between it and 5, then the new pair $[\mathcal{A}^{(4)}, b^{(4)}]$ characterizing it has $[\mathcal{A}^{(2)}, b^{(2)}]$ as a submatrix, and involves just the addition of rows and columns, and aumenting rows in $[\mathcal{A}^{(2)}, b^{(2)}]$ that feature in $[\mathcal{A}^{(4)}, b^{(4)}]$ by zero column entries. In other words with $\times$ denoting arbitrary submatrices, one has

$$[\mathcal{A}^{(4)}, b^{(4)}] = \begin{bmatrix} \mathcal{A}^{(2)} & 0 & b^{(2)} \\ \times & \times & \times \end{bmatrix}. \tag{4.2.9}$$

Thus the loss of an agent/communication channel requires working with a submatrix of the original $[A, b]$, and the addition of an agent requires a supermatrix of $[A, b]$.

4.3 Control Law Without MAC Constraints

We first recount the control law of [30], that assumes that communication occurs without access control constraints. It involves a one step ahead optimization law using the cost function

$$J(k) = [\mathcal{A}x(k+1) - b]^T [\mathcal{A}x(k+1) - b] + \\ + u^T(k) Q u(k) \tag{4.3.10}$$

Where $Q = Q^T > 0$ penalizes the input. The key step in achieving the control law with the desired characteristics described in the introduction is to appropriately select Q.

Since $x(k+1)$ is dependent on $u(k)$ we begin by substituting (4.2.2, 4.2.4) into the cost function defined in (4.3.10). Taking the partial derivative of the resultant expression with respect to $u(k)$, we obtain:

$$\left[\Gamma^T \mathcal{A}^T \mathcal{A} \Gamma + Q\right] u(k) = \Gamma^T \mathcal{A}^T [b - \mathcal{A}\theta(k)]$$

Choose, for some $\alpha > 0$,

$$Q = \alpha - \Gamma^T \mathcal{A}^T \mathcal{A} \Gamma > 0.$$

The resulting control law is:

$$u(k) = \alpha^{-1} \Gamma^T \mathcal{A}^T b - \alpha^{-1} \Gamma^T \mathcal{A}^T \mathcal{A} \theta(k) \tag{4.3.11}$$

It has been shown in [30] that stability is guaranteed if Q is positive definite. Thus α must be chosen so that

$$\alpha I - \Gamma^T \mathcal{A}^T \mathcal{A} \Gamma > 0. \tag{4.3.12}$$

Now we reprise the arguments from [30] that show that the communication topology resulting from (4.3.11) is identical to the geometric topology and further that only a local knowledge of the formation is required by each agent. Observe that the control inputs to agent i are u_{2i} and u_{2i-1}. We will show that if i and j do not have an arc between them in the formation topology, then u_{2i} and u_{2i-1} do not depend on $\theta_{2j-1}, \theta_{2j}, \theta_{2j-1+n}$ and θ_{n+2j}. Because of the structure of Φ, this in turn implies that u_{2i} and u_{2i-1} do not depend on $x_{2j-1}, x_{2j}, x_{2j-1+n}$ and x_{n+2j}, establishing the structure of the communication topology. *Furthermore observe that agent i can select λ_i simply by knowing the number of constraints it is involved in.* As theorem 4.3.1 shows and is evident from (4.3.11) λ_i is only needed by agent i to construct its input.

Then we have the following result that establishes the various properties of the communication topology listed in the foregoing.

Theorem 4.3.1 *Consider (4.3.11) under (4.2.1), (4.2.3), and (4.2.6). Then finding $u_{2i-1}(k)$ and $u_{2i}(k)$ requires:*

(A) The states of agent l only if there is an arc between agents l and i in the formation topology.

(B) The l-th row of A only if for some $j \in \{2i-1, 2i, 2i-1+n, 2i+n\}$ $a_{lj} \neq 0$.

(C) The l-th element of $b(k)$ only if for some $j \in \{2i-1, 2i, 2i-1+n, 2i+n\}$ $a_{lj} \neq 0$.

(D) The gain λ_i can be obtained by agent i simply by knowing how many neighbors it has.

Proof: See [67] for a proof. ■

(A) shows that the communication topology is the same as the formation topology. (B) and (C) show that agent i need only know those rows of A and elements of b which define the arcs emanating from it. Thus i must only know its place in the formation topology and a distributed knowledge of the formation topology suffices. This in particular has *security implications* as even if an agent is compromised the global objective is not.

If despite the loss of an agent, e.g. 4 in figure 4.2, the formation topology remains viable, then this modified formation topology is described by a $[\mathcal{A}, b]$ matrix that is a submatrix of its counterpart in the original formation topology, and obtained by removing the rows characterizing the two arcs impacting 4 and the four columns of A corresponding to the states of 4. As the elements of these columns in the rows of the original A matrix defining the arcs of 2 and 3 are zero, the inputs to agents 2 and 3 are unchanged. These agents do not reconfigure their control laws and need not know about the loss. Similarly if communication between 1 and 5 be impaired or lost, then only 1 and 5 must know of this loss and adjust their control law.

4.4 Control Law under Medium Access Constraints

The control law in 4.3 assumes that all agents can communicate at will. In practice when broadcast communication is used MAC constraints must be used to avoid message collisions. At the minimum this requires that when an agent is receiving state information from a neighbor all others in whose broadcast range it resides, must be silent. Nor can an agent receive and broadcast simultaneously.

Further, depending on the circumstance one of the following three situations may hold.

(a) All agents are in each others mutual broadcast range.

(b) Only agents having an arc between them are in each other's mutual broadcast range.

(c) While any two agents that have an arc between them are in each other's mutual broadcast range, other agents without an arc to them may be in their broadcast range.

The control law we propose accommodates all these three settings. At any rate the following assumption will hold.

Assumption 4.4.1 *If an arc exists between i and j in the formation topology, then i and j are in each others broadcast range. Further each agent always knows its position and velocity.*

As is customary in *ad hoc* networks, [63] we assume *a priori* that the agents have settle on a broadcast schedule, that is consistent with the MAC constraints noted above. We note that efficient algorithms for determining such a schedule, that involve only local exchange of information are available in the literature.

This schedule must be implemented over K sampling intervals, in each of which certain agents broadcast in a manner consistent with MAC requirements. Each interval is assumed for simplicity to be one.

This transmission pattern is repeated after every K-samples. We further assume that while every input is updated in every sampling interval, the agent effecting that update does so by modifying (4.3.11), by replacing the instantaneous state information by the latest value it has access to.

Finally, as any two agent that have an arc between them are in each others broadcast range, for broadcast efficiency the following assumption, that also formalizes the other MAC constraints, will apply.

Assumption 4.4.2 *Every agent broadcasts only once in every K sampling intervals, and when it transmits, all agents it has an arc with receive that information. Further no agent can receive while it is broadcasting, and an agent cannot broadcast if an agent it has an arc to is receiving from another source. Moreover, all communication is instantaneous, in that if a broadcast occurs over an interval $[a, b)$, then* *the recepient knows the information at time a.*

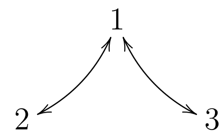

Figure 4.3: Desired formation for a three agent system.

As an example consider the setting of (4.3). Suppose the transmission schedule uses $K = 3$, and is as follows: 1 broadcasts to 2 and 3 at all instants $3k$, 2 transmits to 1 at $3k+1$, and 3 to 1 at $3k+2$. Note that this accords with assumptions, 4.4.1 and 4.4.2, regardless of whether 2 and 3 are in each others broadcast range.

Define

$$D_i = e_i e_i' \tag{4.4.13}$$

where e_i is a $n \times 1$ vector that has 1 in its i-th element, and zeros in all others. Also denote:

$$D_{ij} = I_j \otimes D_i. \tag{4.4.14}$$

Then the control law becomes:

$$
\begin{aligned}
u(3k) &= \frac{\Gamma'}{\alpha}\mathcal{A}'b - \Bigg\{(D_{22}+D_{32})\frac{\Gamma'}{\alpha}\mathcal{A}'\mathcal{A}\Phi D_{14} \\
&+ \sum_{i=1}^{3} D_{i2}\frac{\Gamma'}{\alpha}\mathcal{A}'\mathcal{A}\Phi D_{i4}\Bigg\}\, x(3k) \\
&- D_{12}\frac{\Gamma'}{\alpha}\mathcal{A}'\mathcal{A}\Phi D_{34} x(3k-1) \\
&- D_{12}\frac{\Gamma'}{\alpha}\mathcal{A}'\mathcal{A}\Phi D_{24} x(3k-2) \\
u(3k+1) &= \frac{\Gamma'}{\alpha}\mathcal{A}'b - \Bigg\{D_{12}\frac{\Gamma'}{\alpha}\mathcal{A}'\mathcal{A}\Phi D_{24} \\
&+ \sum_{i=1}^{3} D_{i2}\frac{\Gamma'}{\alpha}\mathcal{A}'\mathcal{A}\Phi D_{i4}\Bigg\}\, x(3k+1) \\
&- (D_{22}+D_{32})\frac{\Gamma'}{\alpha}\mathcal{A}'\mathcal{A}\Phi D_{14} x(3k) \\
&- D_{12}\frac{\Gamma'}{\alpha}\mathcal{A}'\mathcal{A}\Phi D_{34} x(3k-1) \\
u(3k+2) &= \frac{\Gamma'}{\alpha}\mathcal{A}'b - \Bigg\{D_{12}\frac{\Gamma'}{\alpha}\mathcal{A}'\mathcal{A}\Phi D_{34} \\
&+ \sum_{i=1}^{3} D_{i2}\frac{\Gamma'}{\alpha}\mathcal{A}'\mathcal{A}\Phi D_{i4}\Bigg\}\, x(3k+2) \\
&- D_{12}\frac{\Gamma'}{\alpha}\mathcal{A}'\mathcal{A}\Phi D_{24} x(3k+1) \\
&- (D_{22}+D_{32})\frac{\Gamma'}{\alpha}\mathcal{A}'\mathcal{A}\Phi D_{14} x(3k).
\end{aligned}
$$

The term involving

$$\sum_{i=1}^{3} D_{i2}\frac{\Gamma'}{\alpha}\mathcal{A}'\mathcal{A}\Phi D_{i4}$$

recognizes that each agent always has its state information.

To make these ideas more concrete Suppose

$$V = \{1, 2, \cdots, n\}.$$

Define $V_i \subset V$ as the set of all agents that have an arc to agent i in the formation topology. Then the schedule comprises a sequence of sets

$$V(l) \subset V, \quad \forall l \in \{0, 1, \cdots, K-1\},$$

where each agent in $V(l)$ broadcasts in every sampling interval starting with $kK+l$. In keeping with assumption 4.4.2 we obtain the following control law, which we note *retains the attractive properties of (4.3.11)*: For all integer k, and $l \in \{0, 1, \cdots, K-1\}$

$$\begin{aligned} u(kK+l) &= \frac{\Gamma'}{\alpha}\mathcal{A}'b - \left(\sum_{i=1}^{n} D_{i2}\frac{\Gamma'}{\alpha}\mathcal{A}'\mathcal{A}\Phi D_{i4}\right) x(kK+l) \\ &- \sum_{m=0}^{l}\left(\sum_{i\in V(m)}\sum_{j\in V_i} D_{j2}\frac{\Gamma'}{\alpha}\mathcal{A}'\mathcal{A}\Phi D_{i4}\right) x(kK+m) \\ &- \sum_{m=l+1}^{K-1}\left(\sum_{i\in V(m)}\sum_{j\in V_i} D_{j2}\frac{\Gamma'}{\alpha}\mathcal{A}'\mathcal{A}\Phi D_{i4}\right) \\ &\quad x(k(K-1)+m) \end{aligned} \tag{4.4.15}$$

In (4.4.15) the second term captures the fact that all agents have access to their own states at all times. The resulting closed loop system is of course K*-periodic.*

To formalize the underlying rules governing the MAC protocol, that directly impact the stability proof, we make the following assumptions.

Assumption 4.4.3 *The $V(l)$ form a disjoint partition of V, i.e.*

$$V(i) \cap V(j) = \{\phi\}$$

and

$$\cup_{i=0}^{K-1} V(i) = V.$$

This assumption ensures that in every K-cycle each node broadcasts only once, and is consistent with assumption 4.4.2.

Assumption 4.4.4 *If for some $l \in \{0, 1, \cdots, K-1\}$, $i \in V(l)$ then for all $j \in$*

$V(l)$, $i \notin V_j$.

Since every neighbor of $i \in V(l)$ in the formation topology is in receive mode in the pertinent interval, this ensures that no agent receives and transmits simultaneously.

Assumption 4.4.5 *If for some $l \in \{0, 1, \cdots, K-1\}$, $\{i, j\} \subset V(l)$. Then $V(i) \cap V(j) = \{\phi\}$.*

This assumption ensures that no node can receive simultaneously from multiple sources. This is necessitated by the fact that no node can broadcast if a node in its broadcast range is receiving from another node.

4.4.1 Simulation Results

The initial conditions of the fleet are the same for all the simulations. Figure (4.1) illustrates the desired formation topology without any built-in redundancy. The communication protocol for such a configuration is as follows,

- $(kT, kT + h)$: $1 \to 2$, $1 \to 4$
- $(kT + h, kT + 2h)$: $2 \to 1$, $2 \to 3$, $5 \to 4$
- $(kT + 2h, kT + 3h)$: $4 \to 1$, $4 \to 5$, $3 \to 2$

where, $T = 3h$ and the direction of the arrow indicates the direction of the information flow. This assumes that none of the pairs (2,4), (5,1) and (3,5) are in each others broadcast range.

Now consider the topology with redundancy built-in. Figure 4.4 illustrates the desired formation topology with redundancy incorporated via the link between agents 2 and 4. The communication protocol for this configuration is shown below

- $(kT, kT + h)$: $1 \to 2$, $1 \to 4$
- $(kT + h, kT + 2h)$: $2 \to 1$, $2 \to 3$, $2 \to 4$

- $(kT+2h, kT+3h)$: $4 \to 1$, $4 \to 5$, $4 \to 2$
- $(kT+3h, kT+4h)$: $3 \to 2$, $5 \to 4$

where $T = 4h$. This does assume that (2,4) are now in each others broadcast range. Effectively, in going from figure 4.4 to 4.1 the agents have reduced their broadcast range. This is a device that is commonly employed to ensure a more efficient implementation of the broadcast schedule.

The starting points of the fleet are denoted by 'x', the positions at each time step 'T' are denoted by '.'. The final positions of the agents are denoted by 'o'. All the simulations were run until the desired formation was reached. Figures 4.5,4.6 show the trajectories of the individual agents in the fleet for the formations in figures 4.1,4.4 respectively. Figures 4.7,4.8 illustrate the position errors $\|Ax - b\|$ of the fleet for the formations shown in figures 4.1,4.4 respectively. Notice that a redundant formation topology has the added advantage of faster convergence, even without the loss of an agent, i.e. it is more robust from a performance point of view as well.

4.5 Conclusion

We have extended the work of Abel et. al. by imposing a Medium Access Control constraint on formation control. The MAC constraints prohibits simultaneous broadcast by multiple agents that neighbor an agent that is currently receiving.

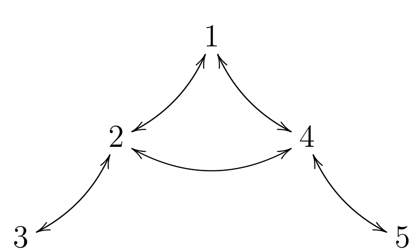

Figure 4.4: Desired formation for a five agent system with redundancy.

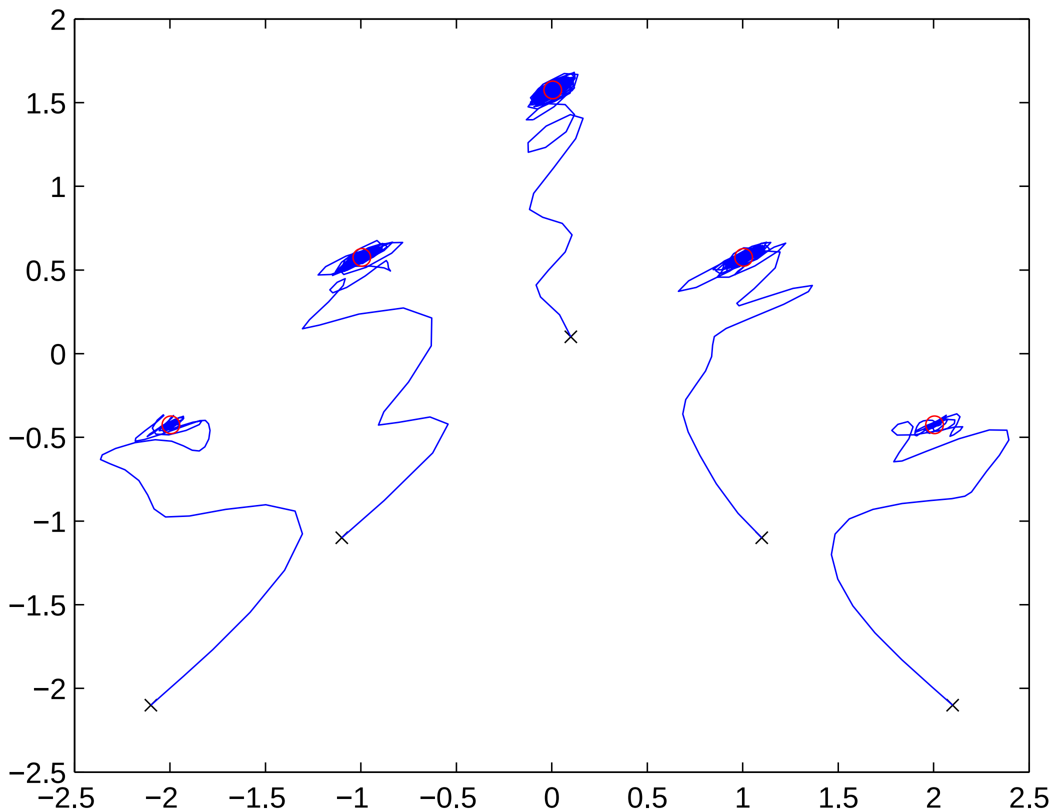

Figure 4.5: Trajectories of the individual agents without redundancy.

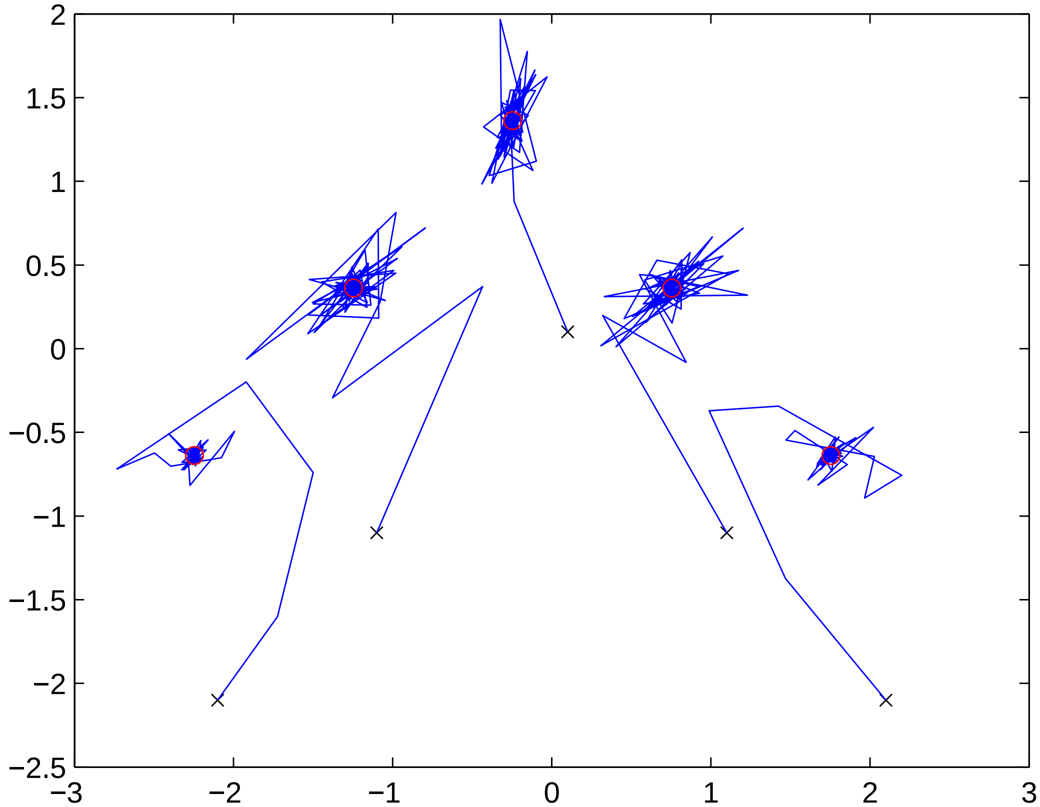

Figure 4.6: Trajectories of the individual agents with redundancy.

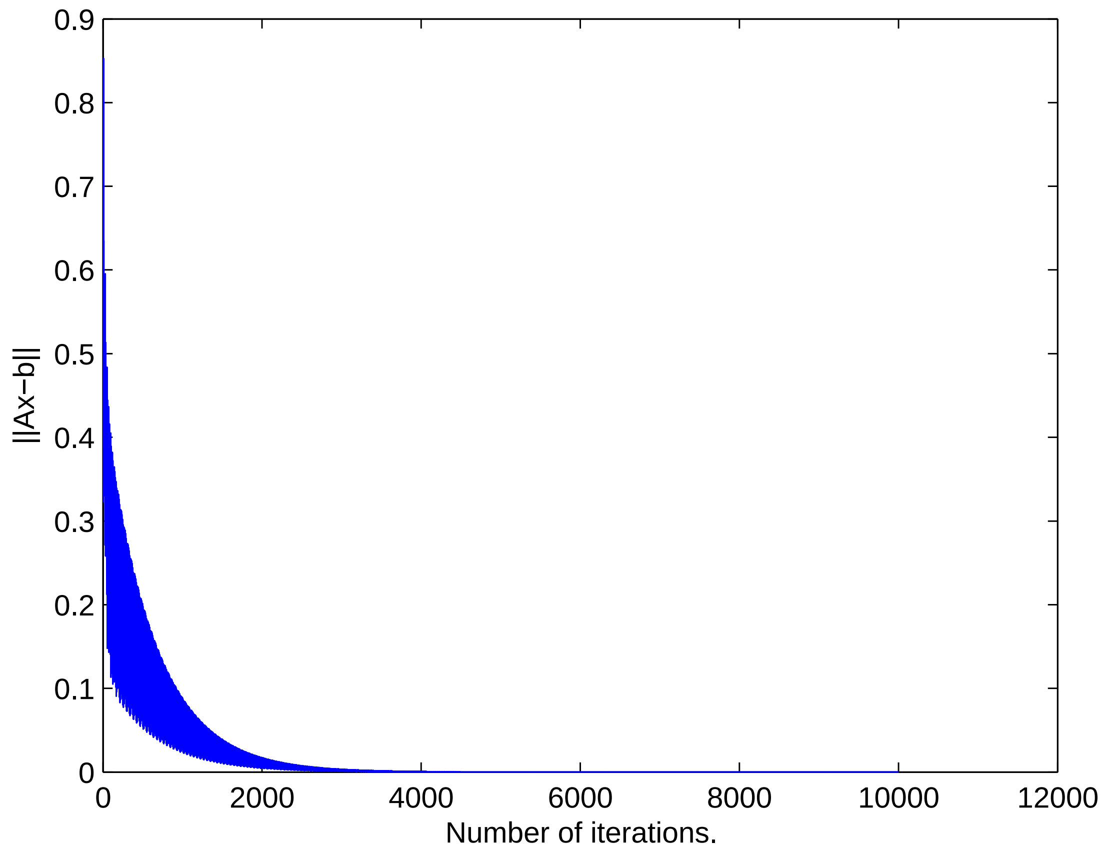

Figure 4.7: The formation error in the case of a non-redundant topology.

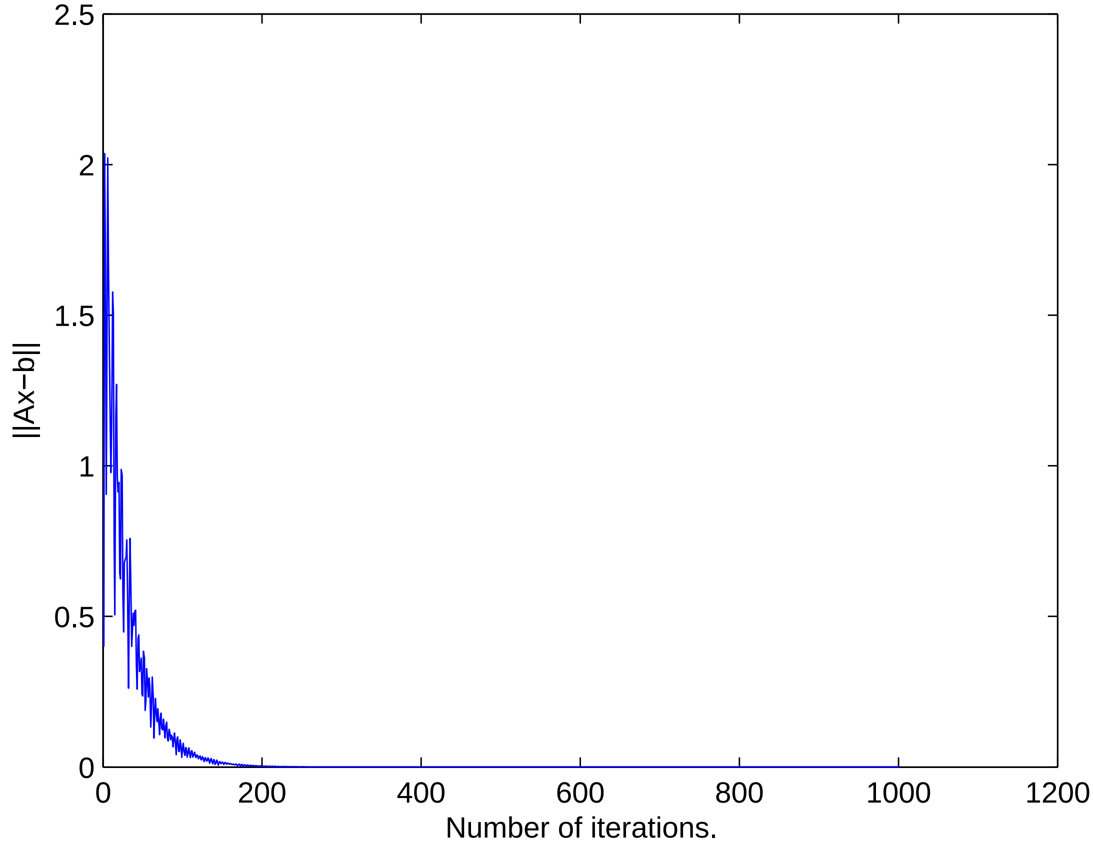

Figure 4.8: The formation error in the case of a redundant topology.

Our algorithm involves a periodic schedule. It is more stringent than is needed in that it permits only one agent to broadcast at a time. Extension to a more relaxed setting is straightforward, but becomes notationally complicated. The next chapter conducts a stability analysis.

CHAPTER 5
STABILITY ANALYSIS WITH MAC CONSTRAINTS

In Chapter 4 we examined the cooperative control of a fleet of autonomous agents. The goal was to devise a control law that would enable the agents to attain a specified formation, specified entirely by the relative position between a subset of agents, while simultaneously adhering to the MAC constraints. In this chapter, we present a proof for the stability of the control law in 5.2. It is noteworthy, that this analysis is quite nontrivial and novel. The MAC constraints can be viewed as engendering a setting with time varying, albeit periodic, links. As only one agent broadcasts at a given time, one can view only its links being active at that time. Such a setting of time varying links has been considered before in the literature, but only under single integrator dynamics. The double integrator dynamics manifest in our setting lack prior definitive analysis.

5.1 Preliminaries

In the sequel we adopt the notation that the direct sum between two matrices H_1 and H_2 is given by:

$$\begin{aligned} H &= H_1 \oplus H_2 \\ &= \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}. \end{aligned}$$

The following fact is self-evident.

Lemma 5.1.1 *Consider invertible $n_i \times n_i$ matrices P_i, and Q_i,*

$$P = \oplus_{i=1}^{K} P_i$$

and

$$Q = \oplus_{i=1}^{K} Q_i.$$

Then the set of eigenvalues of $P^{-1}Q$ are the union of the sets of eigenvalues of $P_i^{-1}Q_i$.

The next fact concerns observability issues.

Lemma 5.1.2 *Consider an $n \times n$ A and $p \times n$, C. Then an eigenvalue of A is unobservable through C iff*

$$C(zI - A)^{-1}$$

does not have this eigenvalue as a pole.

Further the following obtains:

Lemma 5.1.3 *Consider $n_i \times n_i$ A_i and $p_i \times n_i$, C_i. Then some eigenvalues of A_i are unobservable through C_i iff*

$$\left[\oplus_{i=1}^{K} C_i\right]\left(zI - \oplus_{i=1}^{K} A\right)^{-1}$$

does not have these eigenvalues as a poles.

Finally, we note the following result.

Lemma 5.1.4 *Consider $n \times n$ matrices A and B, with A invertible. Then the set of eigenvalues of $A^{-1}B$ are identical to those of $A^{-\top}B^{\top}$.*

Proof: Observe λ is an eigenvalue of $A^{-1}B$ iff

$$\det(\lambda A - B) = 0.$$

Thus, there holds:

$$\det(\lambda A^{\top} - B^{\top}) = 0.$$

Consequently the result follows. ∎

5.2 Stability Analysis

We first examine the approach for proving the stability of the closed loop system defined by (4.2.2) and (4.4.15). Define first for $m \in \{0, \cdots, K-1\}$

$$\mathcal{G}_m = -\Gamma \sum_{i \in V(m)} \sum_{j \in V_i} D_{j2} \frac{\Gamma'}{\alpha}(A'A \oplus A'A \oplus 0)(D_{i2} \oplus 0)\Phi, \tag{5.2.1}$$

and

$$\mathcal{G} = -\Gamma \sum_{i=1}^{n} D_{i2} \frac{\Gamma'}{\alpha}(A'A \oplus A'A \oplus 0)(D_{i2} \oplus 0)\Phi. \tag{5.2.2}$$

Then because of (4.2.6) and (4.2.3), the closed loop becomes: for $l \in \{0, \cdots, M-1\}$,

$$\begin{aligned} x(kK+l+1) &= (\Phi + \mathcal{G})x(kK+l) + \sum_{m=0}^{l} \mathcal{G}_m x(kK+m) \\ &+ \sum_{m=l+1}^{K-1} \mathcal{G}_m x(k(K-1)+m) + \Gamma \frac{\Gamma'}{\alpha} \mathcal{A}' b. \end{aligned}$$

Then it follows that

$$\begin{bmatrix} x(kK+K+1) \\ x(kK+K) \\ \vdots \\ x(kK+1) \end{bmatrix} = \mathcal{F} \begin{bmatrix} x(kK) \\ x(kK-1) \\ \vdots \\ x(kK-K+1) \end{bmatrix} + \hat{\mathcal{G}} \tag{5.2.3}$$

where with $\mathcal{F}_1$ given by

$$\begin{bmatrix} I & -(\Phi + \mathcal{G} + \mathcal{G}_{K-1}) & -\mathcal{G}_{K-2} & \cdots & \cdots & -\mathcal{G}_1 \\ 0 & I & -(\Phi + \mathcal{G} + \mathcal{G}_{K-2}) & -\mathcal{G}_{K-1} & \cdots & -\mathcal{G}_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \tag{5.2.4}$$

and $\mathcal{F}_2$ given by

$$\begin{bmatrix} \mathcal{G}_0 & 0 & \cdots & \cdots & \cdots & 0 \\ \mathcal{G}_0 & \mathcal{G}_{K-1} & 0 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (\Phi + \mathcal{G} + \mathcal{G}_0) & \mathcal{G}_{K-1} & \mathcal{G}_{K-2} & \mathcal{G}_{K-3} & \cdots & \mathcal{G}_1 \end{bmatrix} \tag{5.2.5}$$

$$\mathcal{F} = \mathcal{F}_1^{-1}\mathcal{F}_2 \tag{5.2.6}$$

and

$$\hat{\mathcal{G}} = \mathcal{F}_1^{-1} \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} \Gamma \frac{\Gamma'}{\alpha} \mathcal{A}' b.$$

We first examine the eigenvalues of $\mathcal{F}$. To this end we provide a result that relates its eigenvalues to a lower dimensional matrix. Specifically define:

$$G_m = -\sum_{i \in V(m)} \sum_{j \in V_i} D_j \frac{A'A}{\alpha} D_i, \tag{5.2.7}$$

$$G = -\sum_{i=1}^{n} D_i \frac{A'A}{\alpha} D_i, \tag{5.2.8}$$

F_1 given by

$$\begin{bmatrix} I & -(I + G + G_{K-1}) & -G_{K-2} & \cdots & \cdots & -G_1 \\ 0 & I & -(I + G + G_{K-2}) & -G_{K-1} & \cdots & -G_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}, \tag{5.2.9}$$

$$F_2 = \begin{bmatrix} G_0 & 0 & \cdots & \cdots & \cdots & 0 \\ G_0 & G_{K-1} & 0 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (I+G+G_0) & G_{K-1} & G_{K-2} & G_{K-3} & \cdots & G_1 \end{bmatrix} \tag{5.2.10}$$

and

$$F = F_1^{-1} F_2 \tag{5.2.11}$$

In view of Lemma 5.1.1, (5.2.1)-(2.1.2) and (5.2.8)-(5.2.10) one obtains:

Lemma 5.2.1 *Suppose (4.3.12) and assumptions 4.2.1-4.4.5 hold. Suppose that some eigenvalues of F in (5.2.11) are at 1, and the rest are inside the unit circle. Then the eigenvalues of $\mathcal{F}$ in (5.2.6) are also either 1, or inside the unit circle.*

Similarly, the reduced dimensional matrix F has the following property.

Lemma 5.2.2 *Suppose (4.3.12) and assumptions 4.2.1-4.4.5 hold. Then $(K-1)n$ eigenvalues of F in (5.2.11) are at 0, one eigenvalue is at 1, and the remaining $n-1$ are inside the unit circle.*

Lemmas 5.2.1 and 5.2.2 together show that the eigenvalues of $\mathcal{F}$ are either inside the unit circle or at 1. Our eventual goal is to show that

$$\lim_{k\to\infty} \mathcal{A}x(k) = b. \tag{5.2.12}$$

Observe that this is equivalent to the requirement that

$$\lim_{k\to\infty} (I_K \otimes \mathcal{A}) \begin{bmatrix} x(kK) \\ x(kK-1) \\ \vdots \\ x(kK-K+1) \end{bmatrix} = \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} b. \tag{5.2.13}$$

We next have the following result that assumes that all eigenvalues of F are either at 1 or inside the unit circle.

Lemma 5.2.3 *Suppose all eigenvalues of F are either at 1 or are inside the unit circle. Then all poles of*

$$(I_K \otimes \mathcal{A})\,(zI - \mathcal{F})^{-1}$$

are inside the unit circle.

Proof: From Lemma 5.2.1 the only eigenvalues of $\mathcal{F}$ not inside the unit circle are at 1. Then using Lemma 5.2.2, and a a Kalman like decomposition used in [67] to show that all eigenvalues of $\mathcal{F}$ at 1 are unobservable through $I_K \otimes \mathcal{A}$. Then the result follows from Lemma 5.1.3.

■

Then we have the following main result.

Theorem 5.2.1 *Suppose (4.3.12), assumptions 4.2.1, and 4.4.3-4.4.5 hold. Then under (4.2.2) and (4.4.15), one has*

$$\lim_{k\to\infty} \mathcal{A}x(k) = b.$$

Proof: Because of Lemma 5.2.3 the proof uses the final value theorem. Specifically, it shows that

$$\begin{aligned}
\lim_{z\to 1}(1 - z^{-1}) \quad \mathcal{A} & \left((zI - \mathcal{F})^{-1} \left[\begin{bmatrix} x(0) \\ x(-1) \\ \vdots \\ x(-K+1) \end{bmatrix} + \frac{\mathcal{G}}{1 - z^{-1}} \right] \right. \\
& \left. - \quad \frac{b}{1 - z^{-1}} \right) \\
& = \quad \lim_{z\to 1} \mathcal{A}(zI - \mathcal{F})^{-1}\mathcal{G} - b \\
& = \quad 0.
\end{aligned}$$

■

Thus it suffices to show that F has all eigenvalues either at 1 or inside the unit circle. The sequel establishes this fact. From Lemma 5.1.4 it suffices to show that all eigenvalues of $F_1^{-\top}F_2^{\top}$ either at 1 or inside the unit circle. Suppose λ is such an eigenvalue. Then there holds, for suitable vectors z_i,

$$\begin{bmatrix} G_0' & 0 & \cdots & \cdots & \cdots & 0 \\ G_0' & G_{K-1}' & 0 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (I+G+G_0)' & G_{K-1}' & G_{K-2}' & G_{K-3}' & \cdots & G_1' \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_K \end{bmatrix}$$
$$= \lambda \begin{bmatrix} I & -(I+G+G_{K-1})' & -G_{K-2}' & \cdots & \cdots & -G_1' \\ 0 & I & -(I+G+G_{K-2})' & -G_{K-1}' & \cdots & -G_1' \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_K \end{bmatrix} \tag{5.2.14}$$

Denoting $(I+G)=D$ we have,

$$\begin{bmatrix} (G_0^{\top}-\lambda I) & \lambda(G_{K-1}^{\top}+D) & \cdots & \lambda G_2^{\top} & \lambda G_1^{\top} \\ G_0^{\top} & (G_{K-1}^{\top}-\lambda I) & \cdots & \lambda G_2^{\top} & \lambda G_1^{\top} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (G_0^{\top}+D) & G_{K-1}^{\top} & \cdots & G_2^{\top} & (G_1^{\top}-\lambda I) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_K \end{bmatrix} = 0 \tag{5.2.15}$$

Lemma 5.2.4 *The sums of the rows of the matrix* $G + \sum_{i=0}^{K-1} G_i = \frac{A'A}{\alpha}$ *are each identically equal to zero.*

Proof: The result follows directly from the properties of the incidence matrix A.

■

We need one last preparatory result.

Corollary 5.2.1 *From Lemma (5.2.4), equations (5.2.7) and (5.2.8):*

1. $[G]_{ii} < 0 \ \forall i$

2. $[G_m]_{ij} > 0 \; \forall i, j$

These together with Lemma (5.2.4) we have,

3. The sums of the rows of the matrix $\sum_{i=0}^{K-1} |G_i| - |G| = \frac{A'A}{\alpha}$ *are each identically equal to zero.*

We will now prove the fact that all eigenvalues of the matrix $F_1^{-1}F_2$ are all either at 1 or inside the unit circle.

Lemma 5.2.5 *The eigenvalues of the matrix* $F_1^{-1}F_2$ *are all either at 1 or inside the unit circle.*

Proof:

From Gershgorin's theorem it suffices to show that the coefficient matrix in equation (5.2.15) is diagonally dominant for $|\lambda| \geq 1$. Observe that the main diagonal for $G_i \; \forall i \in \{0, \cdots, K-1\}$ is identically zero. Therefore observe that the diagonal entries in the coefficient matrix in equation (5.2.15) given by the main diagonal of $G_i - \lambda I$ are all identically $-\lambda$. Now consider the first n rows of the coefficient matrix in equation (5.2.15)

$$\Big[(G_0 - \lambda I), \quad \lambda(G_{K-1} + D), \quad \cdots \quad , \lambda G_2, \quad \lambda G_1\Big] \tag{5.2.16}$$

It suffices, therefore, to show that sum of the absolute values of elements in each row of the following matrix

$$\Big[G_0, \quad \lambda(G_{K-1} + I + G), \quad \cdots \quad , \lambda G_2, \quad \lambda G_1\Big] \tag{5.2.17}$$

are each greater than $|\lambda|$ (the absolute value of diagonal entries in the first n rows) for $|\lambda| > 1$. Absolute value of equation (5.2.17) can be written as follows:

$$\left[|G_0|, \quad |\lambda|\,|(G_{K-1} + I + G)|, \quad \cdots \quad, |\lambda|\,|G_2|, \quad |\lambda|\,|G_1|\right] \tag{5.2.18}$$

From equation (5.2.8), it can be noted that G is a diagonal matrix with $[G]_{ii} =$ -$\frac{[A'A]_{ii}}{\alpha}$. From the properties of the incidence matrix, the i-th diagonal entry of $A'A$ is equal to the degree of the i-th node in the graph. Choosing α to be greater than the highest degree of all the nodes in the graph, i.e,

$$\alpha > \max_i\{[A'A]_{ii}\} \tag{5.2.19}$$

equation (5.2.18) can be re-written as follows:

$$\left[|G_0|, \quad |\lambda|\,|G_{K-1}| + |\lambda|\,|I + G|, \quad \cdots \quad, |\lambda|\,|G_2|, \quad |\lambda|\,|G_1|\right]. \tag{5.2.20}$$

Define $|A|$ to be the matrix obtained by applying the absolute value to the matrix A on an element by element basis. Due to the fact that all the diagonal entries in G are negative and $|[G]_{ii}| < 1$ (under 5.2.19), we have

$$\begin{aligned} |I + G| &= I + G \qquad (5.2.21) \\ &= I - |G| \end{aligned}$$

Using (5.2.22) to re-write (5.2.20) we have,

$$\left[|G_0|, \quad |\lambda|\,|G_{K-1}| + |\lambda|\,(I - |G|), \quad \cdots \quad, |\lambda|\,|G_2|, \quad |\lambda|\,|G_1|\right]. \tag{5.2.22}$$

Adding all matrices in (5.2.22) we have,

$$|\lambda|\,(I - |G|) + |\lambda| \left(\sum_{i=1}^{K-1} |G_i|\right) + |G_0|$$

Adding and subtracting $(|\lambda| - 1)|G_0|$ from the above equation,

$$
\begin{aligned}
& |\lambda|(I - |G|) + |\lambda| \left(\sum_{i=0}^{K-1} |G_i| \right) - (|\lambda| - 1)|G_0| \\
= \quad & |\lambda| I - (|\lambda| - 1)|G_0| + |\lambda| \left\{ \left(\sum_{i=0}^{K-1} |G_i| \right) - |G| \right\}
\end{aligned} \tag{5.2.23}
$$

Using Lemma (5.2.4) and Corollary (5.2.1) the sum of the rows in the above equation simplifies to the sum of the rows of the following matrix,

$$
|\lambda| I - (|\lambda| - 1)|G_0| \tag{5.2.24}
$$

Clearly the sums of the rows of the matrix in (5.2.24) are each less than $|\lambda|$ for $|\lambda| > 1$. Similarly, the sum of the absolute values of the non-diagonal elements of rows $jn + 1 : j(n + 1)$ is equal to the sum of the rows of the following matrix,

$$
|\lambda| I - (|\lambda| - 1) \left(|G_0| + \sum_{i=1}^{j} |G_{K-j}| \right)
$$

Once again the sums of the rows of the matrix above are each less than $|\lambda|$ for $|\lambda| > 1$. Therefore, the coefficient matrix in equation (5.2.15) is diagonally dominant for $|\lambda| \geq 1$. ■

This result together with 5.2.3 proves the stability of the control law presented in section 4.4. In this chapter we presented the stability analysis of the control law proposed in chapter 4. This modified control law, presented in section 4.4, respects the medium access constraints.

CHAPTER 6
CONCLUSIONS

In this thesis we considered the resource allocation problems pertaining to Subband Coding Systems, DMT/OFDM based communication systems and Multi-Agent control with medium access constraints.

In Chapter 2 we considered the problem of bit allocation to the quantizers in a subband coding systems that minimizes the average distortion incurred by the system. An algorithm that solves the problem of integer bit loading optimally was presented in section 2.4. The computational complexity of the algorithm was shown to be independent of the bit budget (B).

The problem of Optimum bit loading in DMT/OFDM based communication systems (single user, multicarrier case) was introduced in chapter 2. The goal was to minimize the total transmitted power subject to the user QoS constraints. This minimization was considered through optimum allocation of bits to each subchannel in the communication system. It is observed that the algorithm presented for optimal bit allocation in subband coding systems can be adapted easily to solve the single user multicarrier scenario. The computational complexity, once again, is seen to be independent of the target bit rate (B).

We presented the problem of Optimum bit loading in DMT/OFDM based communication systems (multiuser, multicarrier case) in chapter 3. We then examined the problem of bit allocation in multiuser multicarrier systems. In particular we examined bit loading within the context of multiuser communications, i.e. when multiple services with different data rate and QoS requirements must share the same multicarrier communications system. In this case the goal is not just to distribute the bit rates across the carriers, but also to assign specific carriers to different services. Several papers e.g. [22] have stated without proof or reference to the fact that this problem is NP-hard as a motivation for considering suboptimal solutions. Consequently, if for no other reason than the sake of completeness, we presented a proof for NP-hardness of the integer bit loading problem for a Multiuser multicarrier system.

Multi-Agent Control under Medium Access Constraints was where we examined the cooperative control of a fleet of autonomous units that have to achieve arbitrary relative positions. We proposed a new control strategy that results in distributed control, requiring a communication topology that mirrors exactly the formation topology and respects MAC requirements. A detailed stability analysis of this control law was presented in 5.

Several areas of future work remain open. A variant of the multiuser DMT problem is worthy of future consideration. In the current instances we have been flexible about the number of subchannels each user is assigned. Rather that implicitly comes out as a variable of optimization. The following alternative formulation is worthy of consideration: That the number of subchannels, each user receives is fixed *a priori*, though which subchannels, each user gets is still a variable of optimization. We conjecture that this too is an NP-hard problem. Further, efficient suboptimal algorithms are desirable.

In the mutliagent control problem, we would like to study non-periodic access protocols. Additionally, derivation of stability margins, and performance in the face of temporary link losses need to be studied.

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