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C^1,α regularity for boundaries with prescribed mean curvature

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$C^{1,\alpha}$ REGULARITY FOR BOUNDARIES WITH PRESCRIBED MEAN CURVATURE

by

Stephen William Welch

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

December 2012

Thesis Supervisor: Professor Lihe Wang

ABSTRACT

In this study we provide a new proof of $C^{1,\alpha}$ boundary regularity for finite perimeter sets with flat boundary which are local minimizers of a variational mean curvature formula. Our proof is provided for curvature term $H \in L^\infty(\Omega)$. The proof is a generalization of Caffarelli and Córdoba's method [6], and combines techniques from geometric measure theory and the theory of viscosity solutions which have been developed in the last 50 years. We rely on the interplay between the local nature of sets which are minimizers of a given functional, and the pointwise properties of comparison surfaces which satisfy certain PDE. As a heuristic, in our proof we can consider the curvature as an error term which is estimated and controlled at each point of the calculation.

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Title and Department

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Date

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Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the December 2012 graduation.

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ABSTRACT

In this study we provide a new proof of $C^{1,\alpha}$ boundary regularity for finite perimeter sets with flat boundary which are local minimizers of a variational mean curvature formula. Our proof is provided for curvature term $H \in L^{\infty}(\Omega)$. The proof is a generalization of Caffarelli and Córdoba's method [6], and combines techniques from geometric measure theory and the theory of viscosity solutions which have been developed in the last 50 years. We rely on the interplay between the local nature of sets which are minimizers of a given functional, and the pointwise properties of comparison surfaces which satisfy certain PDE. As a heuristic, in our proof we can consider the curvature as an error term which is estimated and controlled at each point of the calculation.

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CHAPTER 1
INTRODUCTION

This study is intended to provide a new proof of $C^{1,\alpha}$ regularity near non-singular points for sets with a prescribed mean curvature on the boundary. This approach to prescribed mean curvature surfaces has its origin with the work of Ennio De Giorgi, who developed the theory minimal surfaces from the standpoint of sets of finite perimeter. A deep result of his [9] shows that the sets of minimal perimeter, i.e. those with zero mean curvature, enjoy $C^{1,\alpha}$ regularity of the boundary except possibly in a small singular set. Much effort has been made to provide analogous proofs of these results in different measure theoretic settings, as instanced in works by Reifenberg [27], Federer and Fleming [13], Almgren [2], and others. Also, much interest has been placed on the dimension of the set of singular points. In [26] Miranda proved that if the dimension of the surface is this singular set must have $(n-1)-$dimensional Hausdorff measure 0. After stronger results by Almgren [1] and Simons [30], the existence of a singular cone was provided by Bombieri, De Giorgi and Giusi [3] in $\mathbb{R}^8$. Finally, Federer [12] showed that the singular set could have at most $(n-8)-$dimensional Hausdorff measure.

Along a different line of study, Massari extended De Giorgi's proof to boundaries satisfying a variational mean curvature formula with mean curvature term $H \in L^\infty(\Omega)$ in [24] and later with $H \in L^p(\Omega)$ for $p > n$ in [25]. More precisely, he studied the boundary regularity of sets E which are local minimizers in $\Omega \subset \mathbb{R}^n$

of a functional

$$\mathcal{F}_H(E) = \mathcal{P}_\Omega(E) + \int_{E\cap\Omega} H(x)\, dx. \tag{1.1}$$

Here $\mathcal{P}_\Omega(E)$ denotes the perimeter of E in Ω. The motivation for (1.1) is seen by calculating the first variation of $\mathcal{F}_H$ in the case where ∂E is a C^2 surface and H is a continuous function. Then $\mathcal{P}_\Omega(E)$ is the surface area of ∂E, and for $x \in \partial E$, we have that $-H(x)/(n-1)$ is the mean curvature of ∂E. For the details of this calculation, see Appendix A.2.

In 1993 Caffarelli and Córdoba [6] presented a completely new proof of $C^{1,\alpha}$ boundary regularity for sets of minimal perimeter by reformulating the problem in terms of viscosity solutions and using techniques from the theory of fully nonlinear elliptic equations.

In this study we provide a new proof of $C^{1,\alpha}$ boundary regularity for finite perimeter sets with flat boundary having variational mean curvature as in (1.1), where the curvature term is $H \in L^\infty(\Omega)$. The proof is a generalization of Caffarelli and Córdoba's paper, and is strongly influenced by their philosophy and methods. It involves a combination of techniques from geometric measure theory and the theory of viscosity soultions which have been developed over the last 50 years, and thus represents a complete departure from the techniques used in [24, 25]. As such, it relies on the interplay between the local nature of sets which are minimizers of a given functional, and the pointwise properties of comparison surfaces which satisfy certain PDE. As a heuristic, in our proof we can consider the curvature as an error term which is estimated and controlled at each point of the calculation.

CHAPTER 2
DEFINITIONS AND BACKGROUND

2.1 Sets of Finite Perimeter

and the Reduced Boundary

We refer to the standard references [17, 11] for the following definitions and theorems.

Definition 2.1.1 ([17] p. 5, [11] p. 166)**.** *Let E be a Borel set and $\Omega \subset \mathbb{R}^{n+1}$ be an open set. Define the* ***perimeter of*** *E* ***in*** *Ω as*

$$\mathcal{P}_\Omega(E) = \int_\Omega |D\chi_E| = \sup\left\{ \int_E div\, g\, dx \;\middle|\; g \in C_0^1(\Omega; \mathbb{R}^{n+1}),\ |g| \le 1 \right\}.$$

If $\Omega = \mathbb{R}^{n+1}$ then we denote

$$\mathcal{P}(E) = \mathcal{P}_{\mathbb{R}^{n+1}}(E).$$

We say that E ***has finite perimeter in*** *Ω if $\mathcal{P}_\Omega(E) < \infty$. Similarly, we say that E* ***has locally finite perimeter in*** *Ω if $\mathcal{P}_V(E) < \infty$ for each open set $V \subset\subset \Omega$.*

Remark 2.1.2. *Note that if $\partial E \cap \Omega$ is smooth, then from the Gauss-Green theorem, one can see that*

$$\mathcal{P}_\Omega(E) = \mathcal{H}^n(\partial E \cap \Omega).$$

Thus $\mathcal{P}_\Omega(E)$ measures the "surface area" of E in Ω, and the term perimeter is justified. Here $\mathcal{L}^n$ denotes the $n-$dimensional Lebesgue measure.

Remark 2.1.3 ([17], p. 6)**.** *Assume E_1 and E_2 are sets with finite perimeter. Then the following properties follow from Definition 2.1.1:*

1) If $\Omega_1 \subseteq \Omega_2$*, then*

$$\mathcal{P}_{\Omega_1}(E) \leq \mathcal{P}_{\Omega_2}(E)$$

with equality holding when $E \subset\subset \Omega_1$*.*

2) $\mathcal{P}_\Omega(E) = \mathcal{P}_\Omega(\Omega \setminus E)$*.*

3) If $|E| = 0$ *then* $\mathcal{P}(E) = 0$*.*

4) If $|E_1 \Delta E_2| = |(E_1 \setminus E_2) \cup (E_2 \setminus E_1)| = 0$*, then* $\mathcal{P}_\Omega(E_1) = \mathcal{P}_\Omega(E_2)$*.*

Theorem 2.1.4 ([17], p. 172)**.** *If* E_1 *and* E_2 *are sets with finite perimeter, then*

$$\mathcal{P}_\Omega(E_1 \cup E_2) + \mathcal{P}_\Omega(E_1 \cap E_2) \leq \mathcal{P}_\Omega(E_1) + \mathcal{P}_\Omega(E_2),$$

with equality when $dist(E_1, E_2) > 0$*.*

The next theorem shows that for each set E with locally finite perimter in Ω, there exists a measure μ_E that extends the definiton of perimeter to arbitrary $A \subset\subset \Omega$. It can be shown that for open sets U,

$$\mathcal{P}_U(E) = \mu_E(U).$$

Also, there is a "normal vector" to ∂E, denoted ν_E which is defined μ_E *a.e.*. The proof follows from the Reisz Representation theorem.

Theorem 2.1.5 (The Perimeter Measure [11], p. 168)**.** *Let* E *be a set with locally finite perimeter in* Ω*. Then there exists a Radon measure* μ_E *and a* μ_E*-measurable function*

$$\nu_E : \Omega \to \mathbb{R}^{n+1}$$

such that

i) $|\nu_E| = 1 \ \ \mu_E$ *a.e., and*

ii) $\int_E div\, g\, dx = \int_\Omega g \cdot \nu_E \, d\mu_E$ *for all* $g \in C_0^1(\Omega; \mathbb{R}^{n+1})$.

The next two theorems provide the means to prove the existence of minimal perimeter sets.

Theorem 2.1.6 (Lower Semicontinuity of the Perimeter Measure [11], p. 172)**.** *Let* E *and* $\{E_k\}_{k=1}^{\infty}$ *be a sets with locally finite perimeter in* Ω*. If*

$$\chi_{E_k} \to \chi_E \quad in \quad L^1_{loc}(\Omega)$$

then

$$\mu_E(\Omega) \le \liminf_{k\to\infty} \mu_{E_k}(\Omega).$$

Theorem 2.1.7 (Compactness of Sets of Finite Perimeter [11], p. 176)**.** *Let* Ω *be open and bounded with* $\partial\Omega$ *Lipschitz. Suppose* $\{E_k\}_{k=1}^{\infty}$ *is a sequence of sets with finite perimeter in* Ω*. Then there exists a subsequence* $\left\{E_{k_j}\right\}_{j=1}^{\infty}$ *and a set* E *with finite perimeter in* Ω *such that*

$$\chi_{E_k} \to \chi_E \quad in \quad L^1(\Omega).$$

The last topics of this section are the reduced boundary $\partial^* E$ and the generalized Gauss-Green Theorem for the reduced boundary. We will make use of this version of the Gauss-Green theorem several times in the next chapter.

Definition 2.1.8 (The Reduced Boundary [11], p. 194)**.** *Let* $x \in \mathbb{R}^{n+1}$ *We say* $x \in \partial^* E$*, the* ***reduced boundary of*** E *if* *i)* $\mu_E\left(B_r^{n+1}(x)\right) > 0$ *for all* $r > 0$,

ii) the limit $\nu_E(x) = \lim_{r\to 0} \frac{\int_{B_r^{n+1}(x)} \nu_E \, d\mu_E}{\mu_E\left(B_r^{n+1}(x)\right)}$ *exists, and*

iii) $|\nu_E(x)| = 1$.

As we can see from the definition above, the reduced boundary consists of the points of ∂E that have a normal vector defined with respect to μ_E. Furthermore, it can be shown that the perimeter measure coincides with the Hausdorff measure on the reduced boundary:

Theorem 2.1.9 ([11], p. 205)**.** *Assume E has locally finite perimeter in $\mathbb{R}^{n+1}$. Then for any $A \subset \partial^* E$,*

$$\mu_E(A) = \mathcal{H}^n(A).$$

Finally, we can state the Gauss-Green Theorem. This is proved in an equivalent, but slightly different form in [11].

Theorem 2.1.10 ([11], p. 209)**.** *Assume E has locally finite perimeter in $\mathbb{R}^{n+1}$. Then*

i) $\mathcal{H}^n(\partial^* E \cap K) < \infty$ *for each compact* $K \subset \mathbb{R}^{n+1}$, *and*

ii) for $\mathcal{H}^n$ *a.e.* $x \in \partial^* E$, *we have*

$$\int_E div\, g \, dx = \int_{\partial^* E} g \cdot \nu_E \, d\mathcal{H}^n$$

for all $g \in C_0^1(\Omega; \mathbb{R}^{n+1})$.

2.2 Minimizers of Perimeter and Variational Mean Curvature

The space of finite perimeter sets enjoys sufficient compactness to guarantee that a minimizer of $P(E)$ exists, and so it is natural to study minimal surfaces from the perspective of minimizers of the perimeter functional as follows [17]:

Definition 2.2.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be open. Then a set E of finite perimeter has* ***minimal perimeter in*** Ω *if*

$$\mathcal{P}(E) \leq \mathcal{P}(X)$$

for all X such that $E \Delta X \subset\subset \Omega$.

Definition 2.2.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be open, and let $H \in L^1(\Omega)$. We say that a set E of finite perimeter has* ***variational mean curvature*** H ***in*** Ω *if E is a local minimizer of the functional*

$$\mathcal{F}_H(E) = \mathcal{P}_\Omega(E) + \int_{E \cap \Omega} H(x)\, dx. \tag{2.1}$$

That is,

$$\mathcal{F}_H(E) \leq \mathcal{F}_H(X)$$

for all X such that $E \Delta X \subset\subset \Omega$.

When no confusion should arise, we will often drop the adjective 'variational' simply refer to a set E satisfying Definition 2.2.2 as having mean curvature H.

Remark 2.2.3. *The motivation for this definition is that if $\partial E \cap \Omega$ is smooth and H is continuous, then by computing the first variation of $\mathcal{F}_H(E)$, we find that $-\frac{1}{n}H(x)$ is the mean curvature of E at $x \in \partial E \cap \Omega$.*

Remark 2.2.4. *Later, we will make use of the blowup sets*

$$\lambda E = \{ \lambda x \mid x \in E \} .$$

We note that

$$\begin{aligned} \mathcal{F}_H(E) &= \mathcal{P}_\Omega(E) + \int_{E\cap\Omega} H(x)\, dx \\ &= \frac{1}{\lambda^n}\mathcal{P}_\Omega(\lambda E) + \frac{1}{\lambda^{n+1}} \int_{\lambda E\cap\lambda\Omega} H(x/\lambda)\, dx \\ &= \frac{1}{\lambda^n}\left(\mathcal{P}_{\lambda\Omega}(\lambda E) + \frac{1}{\lambda}\int_{\lambda E\cap\lambda\Omega} H(x/\lambda)\, dx\right). \end{aligned} \tag{2.2}$$

Thus we see that if H is a mean curvature for E in Ω, then $\frac{1}{\lambda}H(x/\lambda)$ is a mean curvature for λE in $\lambda\Omega$.

We will now begin to prove volume and density estimates for sets E with mean curvature H in Ω. To do this we will need the following classical theorems from real analysis.

Theorem 2.2.5 (The Isoperimetric Inequalites, [11], pg. 190)**.** *Let E be a bounded set with finite perimeter in $\Omega \subset \mathbb{R}^{n+1}$. Then*

i) $c|E|^{n/n+1} \leq \mathcal{P}(E)$, *and*

ii) Relative Isoperimetric Inequality: For any ball $B_r \subset \Omega$,

$$\min\left\{|B_r \cap E|,\ |B_r \setminus E|\right\}^{n/n+1} \leq C\mathcal{P}_{B_r}(E),$$

for universal constants c and C.

Theorem 2.2.6 (The Vitali Covering Theorem, [11], pg. 27)**.** *Let $\mathcal{G}$ be a collection of nondegenerate closed balls in $\mathbb{R}^{n+1}$ with*

$$\sup\{\, diam\, B \mid B \in \mathcal{G}\,\} < \infty.$$

Then there exists a countable family $\mathcal{A}$ of disjoint balls in $\mathcal{G}$ such that

$$\bigcup_{B\in\mathcal{G}} B \subset \bigcup_{B\in\mathcal{A}} \hat{B},$$

where $\hat{B}$ denotes the concentric ball with radius 5 times the radius of B.

It will be useful to have the following notation in the sequel:

Definition 2.2.7. *We denote the volume which is $\delta-$flat in $B_r^{n+1}(0)$ as*

$$S_{r,\delta} = B_r^{n+1}(0) \cap \{|x_{n+1}| \leq \delta r\}.$$

Lemma 2.2.8 (The Boundary Density Estimate for Flat Surfaces)**.** *Let E be a set with finite perimeter in $\Omega \subset \mathbb{R}^{n+1}$, and let E have mean curvature $H \in L^{n+1}(\Omega)$. Also, assume that $0 \in \partial E$ and ∂E satisfies*

$$\partial E \cap B_r^{n+1}(0) \subset \{|x_{n+1}| \leq \delta r\}.$$

Then

$$\mathcal{P}_{B_r^{n+1}}(E) \leq \omega_n \left(1 + \delta n + \delta^{n/n+1} \|H\|_{L^{n+1}\left(B_r^{n+1}\right)}\right) r^n.$$

Proof. By comparison of E with $E \setminus S_{r,\delta}$ we see that

$$\mathcal{P}_{B_r^{n+1}}(E) + \int_{E\cap S_{r,\delta}} H(x)\, dx \leq \int_{\partial S_{r,\delta}} \chi_E(x)\, d\mathcal{H}^n. \tag{2.3}$$

Similarly, by comparison of E with $E \cup S_{r,\delta}$ we have

$$\mathcal{P}_{B_r^{n+1}}(E) + \int_{E \cap S_{r,\delta}} H(x)\, dx \leq \int_{\partial S_{r,\delta}} 1 - \chi_E(x)\, d\mathcal{H}^n + \int_{S_{r,\delta}} H(x)\, dx. \tag{2.4}$$

Adding (2.3) and (2.4) and bounding the surface area of $S_{r,\delta}$ by the surface area of a cylinder with radius r and height δr, we find

$$\begin{aligned} 2\left(\mathcal{P}_{B_r^{n+1}}(E) + \int_{E \cap S_{r,\delta}} H(x)\, dx\right) &\leq \int_{\partial S_{r,\delta}} d\mathcal{H}^n + \int_{S_{r,\delta}} H(x)\, dx. \\ &= 2\omega_n r^n + 2\delta n \omega_n r^n + \int_{S_{r,\delta}} H(x)\, dx. \end{aligned} \tag{2.5}$$

Thus from (2.5) and Hölder's inequality we have

$$\begin{aligned} \mathcal{P}_{B_r^{n+1}}(E) &\leq \omega_n r^n + \delta n \omega_n r^n + \|H\|_{L^{n+1}\left(B_r^{n+1}\right)} (\delta \omega_n r^{n+1})^{n/n+1} \\ &= \left(\omega_n(1+\delta n) + \|H\|_{L^{n+1}\left(B_r^{n+1}\right)} (\delta\omega_n)^{n/n+1}\right) r^n. \end{aligned}$$

□

Lemma 2.2.9 (The Volume Density Estimate in Balls, [19]). *Let E be a set with finite perimeter in $\Omega \subset \mathbb{R}^{n+1}$, and let E have mean curvature $H \in L^{n+1}(\Omega)$. Assume that $z \in \partial E$ and, and suppose that for some r_0 with $B_{r_0}^{n+1}(z) \subset \Omega$,*

$$\|H\|_{L^{n+1}\left(B_{r_0}^{n+1}(z)\right)} \leq c/2,$$

where c is the isoperimetric constant from lemma 2.2.5. Then there exist a universal constant $c_0 > 0$ such that

$$c_0 r^{n+1} \leq \left|E \cap B_r^{n+1}(z)\right| \leq (\omega_{n+1} - c_0)\, r^{n+1}$$

when

$$0 < r < r_0.$$

Proof. Without loss of generality, we assume that $z = 0$. By comparison of E with $E \setminus B_r^{n+1}(0)$ we see that

$$\mathcal{P}_{B_r^{n+1}(0)}(E) + \int_{E \cap B_r^{n+1}(0)} H(x)\, dx \leq \int_{\partial B_r^{n+1}(0)} \chi_E(x)\, d\mathcal{H}^n. \tag{2.6}$$

Then by Hölder's inequality we have

$$\mathcal{P}_{B_r^{n+1}(0)}(E) - \|H\|_{L^{n+1}\left(B_r^{n+1}(0)\right)} \left|E \cap B_r^{n+1}(0)\right|^{n/n+1} \leq \int_{\partial B_r^{n+1}(0)} \chi_E(x)\, d\mathcal{H}^n. \tag{2.7}$$

The isoperimetric inequality (lemma 2.2.5) states that

$$c\left|E \cap B_r^{n+1}(0)\right|^{n/n+1} \leq \mathcal{P}\left(E \cap B_r^{n+1}(0)\right). \tag{2.8}$$

But

$$\mathcal{P}\left(E \cap B_r^{n+1}(0)\right) = \mathcal{P}_{B_r^{n+1}(0)}(E) + \int_{\partial B_r^{n+1}(0)} \chi_E(x)\, d\mathcal{H}^n. \tag{2.9}$$

Thus combining (2.7), (2.8) and (2.9) we find that

$$\left(c - \|H\|_{L^{n+1}\left(B_r^{n+1}(0)\right)}\right) \left|E \cap B_r^{n+1}(0)\right|^{n/n+1} \leq 2 \int_{\partial B_r^{n+1}(0)} \chi_E(x)\, d\mathcal{H}^n,$$

and the lemma's assumption on $\|H\|_{L^{n+1}\left(B_{r_0}^{n+1}(0)\right)}$ gives

$$c\left|E \cap B_r^{n+1}(0)\right|^{n/n+1} \leq 4 \int_{\partial B_r^{n+1}(0)} \chi_E(x)\, d\mathcal{H}^n, \tag{2.10}$$

for $r < r_0$. Now define

$$g(r) := \left|E \cap B_r^{n+1}(0)\right|.$$

We can see that for almost every $r < r_0$,

$$g'(r) = \int_{\partial B_r^{n+1}(0)} \chi_E(x)\, d\mathcal{H}^n,$$

so that (2.10) can be written as

$$\frac{c}{4} \leq g(r)^{-n/n+1} g'(r) = (n+1)\left(g(r)^{1/n+1}\right)'.$$

Then by integrating from 0 to r for any $r < r_0$ we have

$$\frac{cr}{4(n+1)} \leq \left|E \cap B_r^{n+1}(0)\right|^{1/n+1},$$

which completes one side of the inequality in the lemma.

To complete the other side, we claim that E^c is a minimizer of

$$\mathcal{G}(F) := \mathcal{P}_{B_r^{n+1}(0)}(F) - \int_{F \cap B_r^{n+1}(0)} H(x)\, dx. \tag{2.11}$$

To see this, note that by definition of E, for any set A,

$$\mathcal{P}_{B_r^{n+1}(0)}(E) + \int_{E \cap B_r^{n+1}(0)} H(x)\, dx \leq \mathcal{P}_{B_r^{n+1}(0)}(A^c) + \int_{A^c \cap B_r^{n+1}(0)} H(x)\, dx. \tag{2.12}$$

But

$$\mathcal{P}_{B_r^{n+1}(0)}(E) = \mathcal{P}_{B_r^{n+1}(0)}(E^c) \text{ and } \mathcal{P}_{B_r^{n+1}(0)}(A) = \mathcal{P}_{B_r^{n+1}(0)}(A^c),$$

so from (2.12) we see that

$$\begin{aligned}\mathcal{P}_{B_r^{n+1}(0)}(E^c) - \int_{E^c \cap B_r^{n+1}(0)} H(x)\, dx &\leq \mathcal{P}_{B_r^{n+1}(0)}(A^c) + \int_{A^c \cap B_r^{n+1}(0)} H(x)\, dx - \int_{B_r^{n+1}(0)} H(x)\, dx \\ &\leq \mathcal{P}_{B_r^{n+1}(0)}(A) - \int_{A \cap B_r^{n+1}(0)} H(x)\, dx,\end{aligned}$$

which proves (2.11). It follows from a similar argument to above that

$$\left|E^c \cap B_r^{n+1}(0)\right| \geq c_0 r^{n+1}$$

for some universal c_0 when

$$0 < r < r_0,$$

which implies

$$\left|E \cap B_r^{n+1}(0)\right| \leq (\omega_{n+1} - c_0)\, r^{n+1}$$

in the same regime. □

Corollary 2.2.10 (The Lower Boundary Density Estimate in Balls)**.** *Let E be a set with finite perimeter in $\Omega \subset \mathbb{R}^{n+1}$, and let E have mean curvature $H \in L^{n+1}(\Omega)$. Assume that $z \in \partial E$ and, and suppose that for some r_0 with $B_{r_0}^{n+1}(z) \subset \Omega$,*

$$\|H\|_{L^{n+1}\left(B_{r_0}^{n+1}(z)\right)} \leq c/2,$$

where c is the isoperimetric constant from lemma 2.2.5. Then

$$\mathcal{P}_{B_r^{n+1}(z)}(E) \geq c_0 r^n$$

for some universal c_0 when

$$0 < r < r_0.$$

Proof. Without loss of generality, we assume that $z = 0$. The proof follows from an application of the relative isoperimetric inequality (Lemma 2.2.5) and Lemma 2.2.9. Indeed, given the conditions described in the lemma statement, we have

$$cr^n \leq \min\left\{|B_r^{n+1}(0) \cap E|,\ |B_r^{n+1}(0) \setminus E|\right\}^{n/n+1} \leq C\mathcal{P}_{B_r^{n+1}(0)}(E),$$

for some universal constants c and C. □

The following proof unfortunately did not end up being used in this thesis, but could be useful for later research.

Lemma 2.2.11 (The Clean Ball Lemma)**.** *Let E be a set with finite perimeter in $\Omega \subset \mathbb{R}^{n+1}$, and let E have mean curvature $H \in L^{n+1}(\Omega)$. Assume that $0 \in \partial E$ and, and suppose that for some r with $B_r^{n+1}(0) \subset \Omega$,*

$$\|H\|_{L^{n+1}\left(B_r^{n+1}(0)\right)} \leq c/2,$$

where c is the isoperimetric constant from lemma 2.2.5. Then there exists a ball

$$B_{cr}^{n+1}(y) \subset\subset E \cap B_r^{n+1}(0),$$

for some small universal $c > 0$.

Proof. Consider an open covering of $\partial E \cap B_r^{n+1}(0)$ by balls $B_{\gamma r}^{n+1}(x)$, where $x \in \partial E \cap B_r^{n+1}(0)$ and

$$\gamma << 1.$$

Let $\mathcal{A}_\gamma$ be the disjoint Vitali subfamily of this cover, given by Lemma 2.2.6. We denote by

$$\mathcal{A}_{5\gamma} := \left\{ B_{5\gamma r}^{n+1}(x) \;\middle|\; B_{\gamma r}^{n+1}(x) \in \mathcal{A}_\gamma \right\}$$

The Vitali cover of $\partial E \cap B_r^{n+1}(0)$. Let N_γ be the number of balls in $\mathcal{A}_\gamma$. Our first goal is to show that

$$N_\gamma \leq \frac{C}{\gamma^n}. \tag{2.13}$$

This follows from the disjointess of $\mathcal{A}_\gamma$ and an application of the lower boundary density estimate, Lemma 2.2.10:

$$\begin{aligned} cN_\gamma(\gamma r)^n &\leq \sum_{B^{n+1} \in \mathcal{A}_\gamma} \mathcal{P}_{B^{n+1}}(E) \\ &\leq \mathcal{P}_{B_r^{n+1}(0)}(E). \end{aligned} \tag{2.14}$$

And by comparison of E with $E \setminus B_r^{n+1}(0)$ we have

$$\begin{aligned} \mathcal{P}_{B_r^{n+1}(0)}(E) &\leq \int_{\partial B_r^{n+1}(0)} \chi_E(x)\, d\mathcal{H}^n - \int_{E \cap B_r^{n+1}(0)} H(x)\, dx \\ &\leq Cr^n + C\|H\|_{L^{n+1}\left(B_r^{n+1}(0)\right)} r^n \\ &\leq Cr^n. \end{aligned} \tag{2.15}$$

Thus (2.13) follows from (2.14) and (2.15).

Now consider a finitely overlapping cover $\mathcal{B}_{5\gamma}$ of $E \cap B_{r/2}^{n+1}(0)$ by balls of the form $B_{5\gamma r}^{n+1}(y)$, where $y \in E \cap B_{r/2}^{n+1}(0)$. Our next goal is to find an upper bound on the number M_γ of balls $B^{n+1} \in \mathcal{B}_{5\gamma}$ such that

$$B^{n+1} \cap \partial E \neq \emptyset.$$

To this end, note that for each $B_{5\gamma r}^{n+1} \in \mathcal{A}_{5\gamma}$, by finite overlapping there exists a constant C such that $B_{5\gamma r}^{n+1}$ intersects at most C balls from $\mathcal{B}_{5\gamma}$. Since $\mathcal{A}_{5\gamma}$ covers $\partial E \cap B_{r/2}^{n+1}(0)$, and

$$|\mathcal{A}_{5\gamma}| = |\mathcal{A}_\gamma|\,,$$

from (2.13) we see that

$$M_\gamma \leq CN_\gamma \leq \frac{C}{\gamma^n}. \tag{2.16}$$

Finally we wish to find a lower bound on K_γ, then number of balls in $\mathcal{B}_{5\gamma}$. This follows from the volume density estimate, Lemma 2.2.9:

$$CK_\gamma(\gamma r)^{n+1} = \sum_{B^{n+1} \in \mathcal{B}_{5\gamma}} |B^{n+1}| \geq \left| \bigcup_{B^{n+1} \in \mathcal{B}_{5\gamma}} B^{n+1} \right| \geq \left| E \cap B_{r/2}^{n+1}(0) \right| \geq cr^{n+1}. \tag{2.17}$$

From (2.16) and (2.17) we can see that the number of balls in $\mathcal{B}_{5\gamma}$ that do not intersect ∂E is given by

$$K_\gamma - M_\gamma \geq C\left(\frac{1}{\gamma^{n+1}} - \frac{1}{\gamma^n}\right). \tag{2.18}$$

Clearly we can choose γ universally so that the right hand side of (2.18) is greater than 1. Thus there will exist at least one ball in $\mathcal{B}_{5\gamma}$ that does not intersect ∂E. This ball satisfies the conclusion of the lemma. $\square$

CHAPTER 3
BOUNDARY REGULARITY

3.1 The Flatness Improvement Lemma

Lemma 3.1.1. *Suppose* $g \in C^{1,1}(\Omega)$ *for some domain* Ω*. Then* $D^2 g(x)$ *exists a.e. and*

$$D^2 g(x) = 0$$

almost everywhere in the set

$$\{\, x \mid g(x) = 0 \,\}\,.$$

Proof. Because $g \in C^{1,1}(\Omega)$, g is semi-convex. Thus Aleksandrov's Theorem ([11], p. 242) states that g is twice differentiable $\mathcal{L}^n$ *a.e* in Ω.

Let A denote the set of density points of $\{\, x \mid g(x) = 0 \,\}$. Recall that

$$|\{\, x \mid g(x) = 0 \,\} \setminus A| = 0.$$

We first show that at any $x_0 \in A$,

$$g(x) = o(|x - x_0|),$$

which implies that

$$Dg(x_0) = 0.$$

Suppose that this is not the case. Then we can find a sequence of points x_i such that $x_i \to x_0$ as $i \to \infty$ and

$$\frac{g(x_i)}{|x_i - x_0|} \geq c > 0. \tag{3.1}$$

Because g is Lipschitz, there exists a constant K such that

$$g(x_i) - K|x - x_i| \leq g(x) \tag{3.2}$$

for each i. Let

$$r_i = \frac{c}{2K}|x_i - x_0|.$$

Then from (3.1) and (3.2), we see that,

$$g(x) \geq \frac{c}{2}|x_i - x_0| > 0 \quad \text{in} \quad B_{r_i}(x_i).$$

Since

$$|x_i - x_0| + r_i = 2\left(\frac{K + c}{c}\right) r_i,$$

we can clearly choose a constant C such that

$$B_{r_i}(x_i) \subset B_{Cr_i}(x_0)$$

for all i. But then because

$$A \cap B_{r_i}(x_i) = \emptyset,$$

we see that for all i

$$\frac{|B_{Cr_i}(x_0) \cap A|}{|B_{Cr_i}(x_0)|} \leq \frac{|B_{Cr_i}(x_0) \setminus B_{r_i}(x_i)|}{|B_{Cr_i}(x_0)|} \leq \tilde{c} < 1,$$

contradicting the fact that x_0 is a density point of A.

The argument above shows that $Dg(x) = 0$ at every point in A. Using the Lipschitz continuity of $Dg(x)$ and an argument nearly identical to the one given above, we see that that $D^2 g(x) = 0$ at every point in A. Thus $D^2 g(x) = 0$ almost everywhere in $\{ x \mid g(x) = 0 \}$. $\square$

Lemma 3.1.2. *Let $f \in L^p(B_1)$ for $p > n/2$, and $g \in C(\partial B_1)$ with*

$$g \geq 0.$$

If $u \in W^{2,p}_{loc}(B_1) \cap C(\overline{B_1})$ is a strong solution of

$$\begin{cases} \Delta u = f & \text{in} \quad B_1 \\ u = g & \text{on} \quad \partial B_1, \end{cases} \tag{3.3}$$

then

$$\|u\|_{L^\infty(B_{1/2})} \leq C \left(\fint_{\partial B_1} g \, dS + \|f\|_{L^p(B_1)} \right).$$

Proof. Let h be a solution of

$$\begin{cases} \Delta h = 0 & \text{in} \quad B_1 \\ h = g & \text{on} \quad \partial B_1. \end{cases}$$

Then h is a nonnegative classical solution and we can use the mean value property and harnack inequality for harmonic functions to derive

$$\|h\|_{L^\infty(B_{1/2})} \leq C \fint_{\partial B_1} h \, dS.$$

Now let w be a strong solution of

$$\begin{cases} \Delta w = f & \text{in} \quad B_1 \\ w = 0 & \text{on} \quad \partial B_1. \end{cases} \tag{3.4}$$

By Corollary 9.18 of [15], a solution $w \in W^{2,p}_{\text{loc}}(B_R) \cap C(\overline{B_R})$ of (3.4) exists and is unique. Then from $W^{2,p}$ estimates,

$$\|w\|_{W^{2,p}(B_{1/2})} \leq C\|f\|_{L^p(B_1)}.$$

By the Sobolev embedding theorem, since $p > n/2$,

$$\|w\|_{L^\infty(B_{1/2})} \leq C\|f\|_{L^p(B_1)}.$$

We see that $u = h + w$ is the unique strong solution of (3.3), and the conclusion follows. $\square$

The following lemma is a slight modification of the lemma from [6]. It provides a means to improve the "flatness" of a surface when restricted to a smaller ball. For our purposes, f_1 and f_2 will be scaled upper and lower envelopes of ∂E. The idea of the lemma is to approximate f_1 and f_2 by solutions $u^\pm$ of a Poisson equation with f_1 and f_2 as boundary data. Then the goal will be to use the linear part of u^+ to estimate f_1 and f_2 in the L^∞-norm.

Lemma 3.1.3 (The Flatness Improvement Lemma, [6]). *Let f_1 and f_2 be functions in $B_1(0)$ satisfying for $i = 1, 2$,*

$$f_i \in C^{1,1}(B_{1-5\delta^{1/2}}(0)), \quad -1 \leq f_1 \leq f_2 \leq 1.$$

$$\Delta f_1 \leq K \quad \textit{and} \quad -K \leq \Delta f_2 \quad \textit{in} \quad B_{1-8\delta^{1/2}}(0),$$

$$f_1(0) \leq 0 \leq f_2(0),$$

and

$$\mathcal{L}^n \left(B_{1-8\delta^{1/2}}(0) \cap \{ f_1 \neq f_2 \}\right) \leq T\delta^{1/2},$$

where K, T and δ are constants with δ small. Then there exists a linear function l satisfying

$$l(0) = 0$$

such that

$$\|f_i - l\|_{L^\infty(B_r)} \le C\left(\delta^{\frac{1}{n+2}} + r^{1+\alpha}\right),$$

where

$$r \le 1/4, \qquad C = C(n, \alpha, K, T)$$

and α *is any positive number satisfying* $0 < \alpha < 1$. *In particular* $\|\nabla l\| \le \tilde{C}$.

Proof. Let

$$R < 1 - 8\delta^{1/2}.$$

Define $g^{\pm}$ in $B_R(0)$ by

$$g^{\pm}(x) = \begin{cases} \Delta f_2 & \text{in} \quad \{\, f_1 = f_2 \,\} \\ \mp K & \text{in} \quad \{\, f_1 < f_2 \,\}. \end{cases}$$

Note that if $x \in B_R$ is a density point of $\{f_1 = f_2\}$, then by Lemma 3.1.1,

$$\Delta f_1(x) = \Delta f_2(x).$$

Thus, for almost every $x \in B_R$,

$$-K \le g^{\pm}(x) \le K. \tag{3.5}$$

In particular, $g^{\pm} \in L^p\,(B_R)$ for all $p > 1$.

We approximate f_1 and f_2 in by solutions u^+ and u^- of the following equations:

$$\begin{cases} \Delta u^{\pm} = & g^{\pm} \quad \text{in} \quad B_R \\ u^{+} = & f_2 \quad \text{on} \quad \partial B_R \\ u^{-} = & f_1 \quad \text{on} \quad \partial B_R. \end{cases} \tag{3.6}$$

By Corollary 9.18 of [15], unique strong solutions $u^{\pm}$ of (3.6) exist, and $u^{\pm} \in W^{2,p}_{\mathrm{loc}}(B_R) \cap C(\overline{B_R})$ for every $p > n/2$. Also, we see that in the strong sense,

$$\begin{cases} -2K\chi_{\{f_1<f_2\}} \leq \Delta(u^+ - u^-) & \leq 0 \qquad \text{in} \quad B_R \\ \qquad\qquad u^+ - u^- & = f_2 - f_1 \quad \text{on} \quad \partial B_R. \end{cases} \tag{3.7}$$

Then by Lemma 3.1.2 rescaled and applied to $u^+ - u^-$ we have

$$\begin{aligned} \|u^+ - u^-\|_{L^\infty(B_{R/2})} &\leq C \fint_{\partial B_R} (f_2 - f_1)\, dS + CR^2 \left(\fint_{B_R} |2K\chi_{\{f_1<f_2\}}|^p \, dx \right)^{1/p} \\ &\leq C \fint_{\partial B_R} (f_2 - f_1)\, dS + C(K,T) R^{2-n/p} \delta^{1/2p}, \end{aligned} \tag{3.8}$$

provided $p > n/2$.

Because

$$\mathcal{L}^n \left(B_{1-3\delta^{1/2}} \cap \{ f_1 \neq f_2 \} \right) \leq T\delta^{1/2},$$

we find that

$$\int_{1/2}^{1-8\delta^{1/2}} \fint_{\partial B_s} (f_2 - f_1)\, dS\, s^{n-1} ds = C \int_{B_{1-8\delta^{1/2}} \setminus B_{1/2}} (f_2 - f_1)\, dx \leq CT\delta^{1/2}.$$

Then using the mean value inequality for integrals, it follows that there exists a t such that

$$1/2 < t < 1 - 8\delta^{1/2}$$

and

$$\fint_{\partial B_t} (f_2 - f_1)\, dS \leq CT\delta^{1/2}. \tag{3.9}$$

If we set

$$R = t$$

and fix

$$p = n/2 + 1,$$

then from (3.8),

$$\|u^+ - u^-\|_{L^\infty(B_{1/4})} \le C(T,K)\left(\delta^{\frac{1}{2}} + \delta^{\frac{1}{2p}}\right)$$
$$\le C(T,K)\delta^{\frac{1}{n+2}}.$$

Now in $B_R(0)$

$$\Delta u^- \ge \Delta f_1 \quad \text{and} \quad \Delta u^+ \le \Delta f_2,$$

so by the weak maximum principle for strong solutions ([15] Theorem 9.1),

$$u^- \le f_1 \le f_2 \le u^+$$

in $B_R(0)$. Since

$$u^-(0) \le f_1(0) \le 0 \le f_2(0) \le u^+(0),$$

we see that in fact

$$u^- - u^+ \le f_1 - (u^+ - u^+(0)) \le f_2 - (u^+ - u^+(0)) \le u^+(0) - u^-(0)$$

in $B_R(0)$. Therefore, for $x \in B_R(0)$, we have

$$|f_i(x) - (u^+(x) - u^+(0))| \le \|u^+ - u^-\|_{L^\infty(B_R)}.$$

From (3.5) and Theorem 9.9 of [15] , we know that $u^\pm \in W^{2,p}(B_{1/4}(0))$ for every $p > 1$. Thus from the Sobolev embedding theorem $u^\pm \in C^{1,\alpha}(B_{1/4}(0))$ for every $0 < \alpha < 1$. Consider the linear function

$$l(x) = \langle \nabla u^+(0), x\rangle.$$

Then

$$|(u^+(x) - u^+(0)) - l(x)| \le C|x|^{1+\alpha} \text{ for } x \in B_{1/4},$$

and we conclude

$$|f_i(x) - l(x)| \leq C(\delta^{\frac{1}{n+2}} + |x|^{1+\alpha}) \text{ for } x \in B_{1/4}, \tag{3.10}$$

where

$$C = C(n, \alpha, K, T).$$

Lastly, note that by (3.10),

$$\|\nabla l\| = \sup_{\partial B_{1/4}} \frac{l(x)}{|x|} \leq 4(1 + C)$$

. □

3.2 Envelope Definition and Regularity

For the remainder of the text, we assume that ∂E satisfies

$$\partial E \cap B_1^{n+1}(0) \subset \{|x_{n+1}| \leq \delta\},$$

where

$$\delta \leq \delta(n)$$

is to be determined. We begin by defining the upper and lower envelopes of ∂E by paraboloids of opening ε. Then we demonstrate that these envelopes are $C^{1,1}$ graphs with estimates.

Definition 3.2.1. *We say that $P : \mathbb{R}^n \to \mathbb{R}$ is a paraboloid of opening ε whenever*

$$P(x) = \pm\frac{\varepsilon}{2}|x|^2 + l(x) + b,$$

where ε is a postitive constant, b is a constant, l is a linear function and

$$x = (x_1, \ldots, x_n).$$

Definition 3.2.2. *We define the surface S^- (resp. S^+) as the lower (resp. upper) envelope of the family of paraboloids of opening -1 (resp. 1). That is, the envelope of all paraboloids of the form*

$$x_{n+1} = -\frac{1}{2}|x-c|^2 + b \qquad \left(\text{resp. } x_{n+1} = \frac{1}{2}|x-c|^2 + b \right)$$

which are tangent to $\partial E \cap B_1$ from below (resp. from above).

Note that in the discussion and proofs that follow, we often focus on the lower envelope S^-; the details for the upper envelope S^+ are similar.

Proposition 3.2.3. *Assume that ∂E satisfies*

$$\partial E \cap B_1^{n+1}(0) \subset \{|x_{n+1}| \leq \delta\}, \tag{3.11}$$

for some $\delta \leq \delta(n)$ small. Then $S^\pm \cap B_1^{n+1}(0)$ are the graphs of continuous functions $\varphi^\pm$, which are Lipschitz on a subset of $B_1(0)$, satisfying the estimate

$$\|D\varphi^\pm\|_{L^\infty\left(B_{1-3\delta^{1/2}}(0)\right)} \leq 2\delta^{1/2}.$$

Proof. The continuity of $\varphi^\pm$ is clear from the definition. We focus on the second assertion of the proposition. Let $(\tilde{x}, \tilde{x}_{n+1}) \in S^-$ satisfy

$$\tilde{x} \in B_{1-3\delta^{1/2}}(0).$$

Then there exist constants $c_{\tilde{x}}$ and $b_{\tilde{x}}$, and a paraboloid

$$P(x) = -\frac{1}{2}|x - c_{\tilde{x}}|^2 + b_{\tilde{x}},$$

such that

$$\tilde{x}_{n+1} = P(\tilde{x}).$$

A simple geometric argument using condition (3.11) shows that

$$|\tilde{x} - c_{\tilde{x}}| \leq 2\delta^{1/2}.$$

It follows that

$$\begin{aligned} |DP(\tilde{x})| &\leq |\tilde{x} - c_{\tilde{x}}| \\ &\leq 2\delta^{1/2}. \end{aligned}$$

Since $(\tilde{x}, \tilde{x}_{n+1})$ is arbitrary, we can consider S^- to be the finite supremum of a family of lipschitz functions with a fixed lipschitz constant $2\delta^{1/2}$. Hence we can write S^- as the graph of a lipschitz function φ^- with the same constant. □

We would like to show that, after a certain scaling, the envelopes $\varphi^\pm$ satisfy the conditions of Lemma 3.1.3, so that we can obtain the decay of flatness of ∂E in dyadic balls. We do this in two steps. The first step is to show that $\varphi^\pm$ are $C^{1,1}(B_{1-\varepsilon}(0))$. This is accomplished in Proposition 3.2.7. It will be useful to have the following definition and lemma in the proof of Lemma 3.2.7.

Definition 3.2.4. *We denote* ***the contact set of the graph of*** φ^- ***with*** ∂E ***in*** $B_{1-3\delta^{1/2}}(0)$ *by*

$$\mathbb{C}^- = B_{1-3\delta^{1/2}}(0) \cap \left\{ x \;\middle|\; (x, \varphi^-(x)) \in \partial E \right\}.$$

Similarly, ***the contact set of the graph of*** φ^+ ***with*** ∂E ***in*** $B_{1-3\delta^{1/2}}(0)$ *will be denoted by*

$$\mathbb{C}^+ = B_{1-3\delta^{1/2}}(0) \cap \left\{ x \;\middle|\; (x, \varphi^+(x)) \in \partial E \right\}.$$

Definition 3.2.5 (see [32]). *For sets $A, B \subset \mathbb{R}^{n+1}$ define*

$$A \oplus B = \{ (x, y) \mid y = y_1 + y_2 \ \textit{for some} \ (x, y_1) \in A \ \textit{and} \ (x, y_2) \in B \},$$

*which we refer to as **addition in the** e_{n+1} **direction**.*

Now consider

$$P_1 = \left\{ (x, x_{n+1}) \,\middle|\, x_{n+1} = \frac{1}{2}|x|^2 \right\},$$

and let

$$\Gamma(x) := \Gamma\left((\partial E \cap B_1) \oplus P_1\right)(x)$$

denote the convex envelope of $(\partial E \cap B_1) \oplus P_1$. We claim that

$$\Gamma(x) = \varphi^-(x) + \frac{1}{2}|x|^2.$$

Indeed, it can be shown that this is the case using the following three facts:

1) $\varphi^-(x)$ is the pointwise supremum of paraboloids of opening -1.

2) The convex envelope is the pointwise supremum of affine functions.

3) Addition of P_1 in the e_{n+1} maps paraboloids of opening -1 to affine functions.

Then we can see clearly that there is an equality of contact sets

$$\mathbb{C}_\Gamma^- = \{ x \mid (x, \Gamma(x)) \in (\partial E \cap B_1) \oplus P_1 \} = \mathbb{C}^-.$$

This is of importance to us, because of the following heuristic: we find estimates for the second derivatives of $\varphi^\pm(x)$ only at points in $\mathbb{C}^-$. We then see that similar estimates hold for the function $\Gamma(x) = \varphi^-(x) + \frac{1}{2}|x|^2$. Using convexity, we will be able infer the same estimates for the second derivatives of $\Gamma(x)$ in all of $B_{1-5\delta^{1/2}}(0)$ (in a manner described below). Thus, because the second derivatives of $\frac{1}{2}|x|^2$ are easily computable, we have second derivative estimates for $\varphi^{\pm}(x)$ in all of $B_{1-5\delta^{1/2}}(0)$.

To clarify the comments above, we note some facts about convex sets and functions. It follows from Caratheodory's theorem and the definition of the convex envelope that any point $x_0 \in B_{1-5\delta^{1/2}}(0) \setminus \mathbb{C}^-$ belongs to a simplex S with at most $n+1$ vertices $x_1, \ldots, x_{n+1}$ (i.e. S is the convex hull of $\{ x_1, \ldots, x_{n+1} \}$), where each x_i belongs in the "contact set", or set of extreme points of $(\partial E \cap B_1) \oplus P_1$:

$$x_i \in \{ x \mid (x, \Gamma(x)) \in (\partial E \cap B_1) \oplus P_1 \}$$

for each i. Thus there exist $\lambda_1, \ldots \lambda_{n+1}$ such that

$$\sum_{i=1}^{n+1} \lambda_i = 1 \quad \text{and} \quad 0 \le \lambda_j \le 1$$
$$x_0 = \sum_{i=1}^{n+1} \lambda_i x_i \quad \text{and} \quad \Gamma(x_0) = \sum_{i=1}^{n+1} \lambda_i \Gamma(x_i).$$

See [5], p. 25, for a proofs of the above facts. By an elementary geometric argument using the flatness δ of ∂E and the opening 1 of the touching paraboloids, we can furthermore ensure that each x_i above satisfies

$$x_i \in B_{2\delta^{1/2}}(x_0),$$

so that for each i,

$$x_i \in B_{1-3\delta^{1/2}}(0).$$

Now consider the second order differential quotient

$$\Delta_{he}^2 \Gamma(x_0) = \frac{1}{h^2} \left[\Gamma(x_0 + he) + \Gamma(x_0 - he) - 2\Gamma(x_0) \right],$$

where $x_0 \in B_{1-5\delta^{1/2}}(0) \setminus \mathbb{C}^-$. We can see that for x_i as described above,

$$\Delta_{he}^2 \Gamma(x_0) \leq \sum_{i=1}^{n+1} \lambda_i \frac{1}{h^2} \left[\Gamma(x_i + he) + \Gamma(x_i - he) - 2\Gamma(x_i) \right].$$

Hence, the second order differential quotients of $\varphi^-(x) + \frac{1}{2}|x|^2$ at any point are a convex combination of the differential quotients at points in $\mathbb{C}^-$. So it suffices to prove the next two propositions at points in $\mathbb{C}^-$. We first need the following lemma.

Lemma 3.2.6. *For a function $\psi(x) : \mathbb{R}^n \to \mathbb{R}$, let*

$$\mathcal{M}(D^2\psi, D\psi) = \frac{(1 + |D\psi|^2)\Delta\psi - (D\psi)^t D^2\psi(D\psi)}{(1 + |D\psi|^2)^{3/2}}$$

denote the mean curvature of $\psi(x)$. If

$$\Delta\psi(x) \geq 0$$

then

$$\mathcal{M}(D^2\psi, D\psi) \geq \Delta\psi - C|D\psi|^2|D^2\psi|$$

for a universal constant C.

Proof. We can use the inequality

$$\frac{1}{(1 + |D\psi|^2)^{1/2}} = 1 - \frac{(1 + |D\psi|^2)^{1/2} - 1}{(1 + |D\psi|^2)^{1/2}} \geq 1 - \frac{|D\psi|^2}{(1 + |D\psi|^2)^{1/2}} \geq 1 - |D\psi|^2$$

to find the lower bound for the mean curvature of ψ as follows:

$$\begin{aligned}\mathcal{M}(D^2\psi, D\psi) &= \frac{\Delta\psi}{(1+|D\psi|^2)^{1/2}} - \frac{(D\psi)^t D^2\psi(D\psi)}{(1+|D\psi|^2)^{3/2}} \\ &\geq \Delta\psi - |D\psi|^2\Delta\psi - |D\psi|^2|D^2\psi| \\ &\geq \Delta\psi - C|D\psi|^2|D^2\psi|.\end{aligned}$$

□

Proposition 3.2.7. *Let E be a set with finite perimeter in $\Omega \subset \mathbb{R}^{n+1}$, let E have mean curvature H satisfying*

$$\|H\|_{L^\infty\left(B_1^{n+1}(0)\right)} \leq \frac{n}{2},$$

and let $\varphi^\pm$ be the envelopes of ∂E described above. Assume $B_1^{n+1}(0) \subset \Omega$, and $0 \in \partial E$. Then there exists $\delta(n)$ such that if ∂E satisfies

$$\partial E \cap B_1^{n+1}(0) \subset \{|x_{n+1}| \leq \delta\},$$

for $\delta \leq \delta(n)$, then $\varphi^\pm \in C^{1,1}\left(B_{1-5\delta^{1/2}}(0)\right)$ with the estimate

$$\|\varphi^\pm\|_{C^{1,1}\left(B_{1-5\delta^{1/2}}(0)\right)} \leq C$$

for a universal constant C.

Proof. We focus on the lower envelope φ^-, as the calculations for φ^+ are similar. Because φ^- is an envelope of tangent paraboloids of opening -1 touching ∂E from below, an elementary calculation shows that the second order differential quotient $\Delta^2_{he}\varphi^-(x)$ satisfies

$$\Delta^2_{he}\varphi^-(x) = \frac{1}{h^2}\left[\varphi^-(x+he) + \varphi^-(x-he) - 2\varphi^-(x)\right] \geq -1$$

for any $x \in B_1(0)$ and unit vector e, so we easily have a lower bound *a.e.* on the second order derivatives. Note that bcause

$$\Gamma(x) = \varphi^-(x) + \frac{1}{2}|x|^2,$$

is convex, Aleksandrov's Theorem ([11], p. 242) states that Γ is second order differentiable $\mathcal{L}^n$ *a.e* in $B_1(0)$, and hence φ^- is also second order differentiable $\mathcal{L}^n$ *a.e* in $B_1(0)$.

From the discussion above, it suffices to prove the proposition at contact points. So let $x \in \mathbb{C}^-$. We choose coordinates so that

$$x = 0 \in \mathbb{C}^- \text{ and } \Gamma(0) = 0,$$

with the supporting hyperplane to Γ at 0 as

$$l = 0.$$

We want to show that

$$\alpha(\rho) = \sup_{B_\rho(0)} \Gamma(x) \leq C\rho^2$$

for some universal $C > 0$ when ρ is small enough. Without loss of generality, assume that

$$\alpha(\rho) = \Gamma(\rho e_n).$$

Then the supporting plane to Γ at re_n has the form $ax_n + b$. So by the convexity of Γ, for any $x' = (x_1, \ldots, x_{n-1})$,

$$\Gamma(x', \rho) \geq \alpha(\rho).$$

Now consider the auxiliary function

$$P(x) = \frac{n(n+2)}{2}\left((x_n+\rho)^2 - \frac{1}{n}|x'|^2\right) - \frac{1}{2}|x|^2$$

in the strip

$$R = \left\{\, x \;\middle|\; |x_n| \le \rho,\, |x'| \le 2\sqrt{n}\,\rho \,\right\}.$$

We check that the following two items are true in R:

1) $P(x) \le \varphi^-(x)$ on $\partial R \setminus \{\, x_n = \rho \,\}$.

2) $P(0) > 0$.

To see that 1) is true, note that when

$$x_n = -\rho,$$

we have

$$P(x) + \frac{1}{2}|x|^2 \le 0 \le \Gamma,$$

which implies

$$P(x) \le \varphi^-(x).$$

On the other hand, when

$$|x'| = 2\sqrt{n}\,\rho,$$

we also have

$$P(x) + \frac{1}{2}|x|^2 \le 0 \le \Gamma.$$

In order to compare the auxiliary function $P(x)$ with $\varphi^-(x)$ we first show that the mean curvature of $P(x)$ is positive. To this end we we calculate the following equalities and inequalities in R for reference:

$$
\begin{aligned}
DP(x) &= (-(n+3)\,x_1, \ldots, -(n+3)\,x_{n-1},\ (n(n+2)-1)\,x_n + n(n+2)\rho)\,. \\
|DP(x)|^2 &= (n+3)^2\,|x'|^2 + [(n(n+2)-1)\,x_n + n(n+2)\rho]^2 \\
&\le 4n\,(n+3)^2\,\rho^2 + 4n^2(n+2)^2\rho^2 \\
&< C\rho^2. \\
D^2P(x) &= \begin{pmatrix} -(n+3) & 0 & \ldots & 0 \\ 0 & -(n+3) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & n(n+2)-1 \end{pmatrix}.
\end{aligned}
$$

and

$$
\begin{aligned}
\left|D^2P(x)\right| &= \left[(n-1)\,(n+3)^2 + (n(n+2)-1)^2\right]^{1/2} \\
&\le C.
\end{aligned}
$$

Let

$$
\mathcal{M}(D^2P, DP) = \frac{(1+|DP|^2)\Delta P - (DP)^t D^2P(DP)}{(1+|DP|^2)^{3/2}}
$$

denote the mean curvature of P. Since

$$
\Delta P(x) = -(n-1)(n+3) + (n(n+2)-1) = 2 > 0,
$$

we apply Lemma 3.2.6 to estimate the mean curvature of P in the strip R as follows:

$$
\begin{aligned}
\mathcal{M}(D^2P, DP) &\ge \Delta P - C|DP|^2|D^2P| \\
&> 2 - C\rho^2.
\end{aligned}
$$

We wish the last expression above to be larger than 1. To achieve this we require ρ

to be small enough that

$$C\rho^2 < 1. \tag{3.12}$$

Now suppose

$$P(\rho e_n) + \frac{1}{2}|\rho e_n|^2 < \alpha(\rho)$$

for some ρ satisfying (3.12). We claim that this contradicts E being a set of variational mean curvature H.

By the convexity of Γ, we see that

$$P(x) + \frac{1}{2}|x|^2 \leq \Gamma(x) \text{ on } \{ x_n = \rho \}.$$

It follows from the work above that then

$$P(x) \leq \varphi^-(x) \text{ on } \partial R.$$

Now we denote the epigraph of P by:

$$G = \left\{ (x, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; x_{n+1} > P(x) \right\}$$

Also denote

$$U = E \setminus \overline{G} = E \cap \left\{ (x, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; x_{n+1} < P(x) \right\}$$

See figure 3.1 below for the configuration.

By translating $P(x)$ downwards in the x_{n+1} direction, we can assume that U is small enough that

$$\text{Proj}_{e_{n+1}} U \subset\subset R,$$

where $\text{Proj}_{e_{n+1}}$ is the projection of $\mathbb{R}^{n+1}$ onto $e_{n+1}^{\perp}$. Furthermore, if we let $d_{\partial G}(x)$ be the signed distance function defined in $(A.1)$, then by translating $P(x)$ downward enough, we can ensure that

$$\sum_{i=1}^{n} \frac{\kappa_i}{1-\kappa_i d_{\partial G}(x)} \geq \frac{n}{2}\mathcal{M}(y) > \frac{n}{2},$$

where $y \in \partial G$ such that

$$d_{\partial G}(x) = |x-y|,$$

$\kappa_1, \ldots, \kappa_n$ are the principal curvatures of ∂G at y, and $\mathcal{M}(y)$ is the mean curvature of of ∂G at y. In other words, we can ensure that U is small enough that

$$\frac{1}{1-\kappa_i d_{\partial G}(x)} \geq \frac{1}{2}$$

for each i and $x \in U$. See Appendix A for the details on the relationship between $\mathcal{M}$ and $d_{\partial G}$.

Then letting $L \subset \Omega$ such that $U \subset\subset L$, letting N denote the *outward* unit normal to $\partial^* U$, and using Lemma A.1.2 and the Gauss Green Theorem (2.1.10), we have

$$\begin{aligned} |U|\frac{n}{2} &< \int_U \sum_{i=1}^{n} \frac{\kappa_i}{1-\kappa_i d_{\partial G}(x)}\, dx \\ &= -\int_U \Delta d_{\partial G}(x)\, dx \\ &= -\int_{\partial^* U} D d_{\partial G}(x) \cdot N \, d\mathcal{H}^n \\ &= -\int_{\partial^* E \cap \overline{U}} D d_{\partial G}(x) \cdot N \, d\mathcal{H}^n - \int_{\partial G \cap \overline{U}} D d_{\partial G}(x) \cdot N \, d\mathcal{H}^n \\ &< \mathcal{P}_{\overline{G}^c}(E) - \mathcal{P}_{\overline{U}}(G) \\ &= \mathcal{P}_L(E) - \mathcal{P}_L(G). \end{aligned} \tag{3.13}$$

Note that by minimality,

$$\mathcal{P}_L(E) - \mathcal{P}_L(G) \leq \|H\|_{L^\infty(B_1)}|U|. \tag{3.14}$$

Combining (3.13) and (3.14), we see that

$$\frac{n}{2} < \|H\|_{L^\infty(B_1)},$$

contradicting the assumptions on $\|H\|_{L^\infty(B_1)}$. It follows that

$$\alpha(\rho) \leq P(\rho e_n) + \frac{1}{2}|\rho e_n|^2 \leq C\rho^2,$$

and in general,

$$\sup_{B_\rho(0)} \Gamma(x) \leq C\rho^2 \tag{3.15}$$

for small enough ρ.

From (3.15) and the definiton of Γ, we have

$$-\frac{1}{2}|x|^2 \leq \varphi^-(x) \leq C|x|^2. \tag{3.16}$$

This implies that there exists a universal r_0 such that $\varphi^-(x)$ has tangent balls of radius r_0 from above (and below) at 0. Since 0 is arbitrary, and the tangent balls do not depend on the choice of coordinates, we see that $\varphi^-(x)$ has tangent balls of radius r_0 from above at every $x \in \mathbb{C}^-$. We can then choose $\delta(n)$, depending only on r_0, so that the tangent balls contact the graph of $\varphi^-(x)$ only in a small cap. Finally, we see that there exists a universal constant $\tilde{C}$ such that every point in the cap of each tangent ball has a tangent paraboloid of the form

$$p(x) = \frac{\tilde{C}}{2}|x-c|^2 + b$$

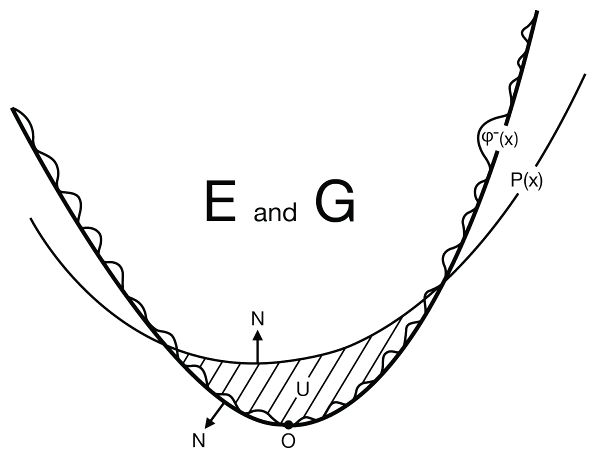

Figure 3.1: Schematic diagram of barrier function and set E

for constants c and b. It follows from the discussion before the lemma that for every $x \in B_{1-5\delta^{1/2}}(0)$, the second order difference quotient satisfies

$$\Delta^2_{he}\varphi^-(x) \leq \tilde{C}.$$

□

Now we improve the result of the lemma above to obtain a slightly stronger upper bound on $\Delta\varphi^-(x)$ when restricted to the a subset of the contact set. Eventually we will show that this upper bound holds in all of $B_{1-8\delta^{1/2}}(0)$. This estimate is proved in the context of integral averages; we recall that for a second order differentiable function f, there exists a universal constant C such that

$$\lim_{\rho\to 0}\frac{1}{\rho^2}\left(⨍_{\partial B_\rho(x)} f(y)\, dS(y) - f(x)\right) = C\Delta f(x).$$

Proposition 3.2.8. *Let E be a set with finite perimeter in $\Omega \subset \mathbb{R}^{n+1}$ with mean curvature H. Let $\varphi^\pm$ be the envelopes of ∂E described above. Assume $B_1^{n+1}(0) \subset \Omega$,* *and* $0 \in \partial E$*. Then there exists* $\delta(n)$ *such that if* ∂E *satisfies*

$$\partial E \cap B_1^{n+1}(0) \subset \{|x_{n+1}| \leq \delta\},$$

for $\delta \leq \delta(n)$*, and*

$$\|H\|_{L^\infty\left(B_1^{n+1}(0)\right)} \leq \delta,$$

then for $x \in \mathbb{C}^- \cap B_{1-6\delta^{1/2}}(0)$

$$\frac{1}{\rho^2}\left(\fint_{\partial B_\rho(x)} \varphi^-(y)\, dS(y) - \varphi^-(x)\right) \leq C\delta,$$

where $\rho \leq \rho(n)$ *and* C *is a universal constant.*

Proof. Fix $x_0 \in \mathbb{C}^- \cap B_{1-6\delta^{1/2}}(0)$. Without loss of generality, we assume

$$x_0 = 0,$$

and

$$\varphi^-(0) = 0 \tag{3.17}$$

For a given ρ, consider the solution u of

$$\begin{cases} \Delta u = 0 & \text{in } B_\rho(0) \\ u = \varphi^- & \text{on } \partial B_\rho(0). \end{cases}$$

We will approximate the average integral of $\varphi^-(x)$ by the average of a small pertubation of u, so we first collect some derivative estimates for $\varphi^-(x)$ and u. From (3.17) and Propositions 3.2.7 and 3.2.3, the following holds for φ:

$$\|\varphi^-\|_{L^\infty(B_\rho)} \leq 2\delta^{1/2}\rho + C\rho^2 \tag{3.18}$$

Thus if

$$\rho \leq \delta^{1/2},$$

we have the following interior estimates for u:

$$\begin{aligned} \|Du\|_{L^\infty\left(B_{\rho/2}\right)} &\leq C(\delta^{1/2}+\rho) \leq C\delta^{1/2} \\ \|D^2u\|_{L^\infty\left(B_{\rho/2}\right)} &\leq C. \end{aligned} \tag{3.19}$$

The first estimate for u follows from interior schauder estimates for harmonic functions, the maximum principle and (3.18). The second estimate follows by applying interior schauder estimates to the function

$$w(x) = u(x) - l(x),$$

where $l(x) = \langle D\varphi^-(0), x\rangle$, because then

$$\Delta w = 0 \ \text{ in } B_\rho(0)$$

and

$$\|w\|_{L^\infty(B_\rho)} \leq \|\varphi^-(x) - l(x)\|_{L^\infty(\partial B_\rho)} \leq C\rho^2.$$

Define v in $B_\rho(0)$ by

$$v(x) = u(x) + M\delta(|x|^2 - \rho^2).$$

Then v is a solution of

$$\begin{cases} \Delta v = 2nM\delta & \text{in } B_\rho(0) \\ v = \varphi^- & \text{on } \partial B_\rho(0). \end{cases}$$

We claim that we can choose δ and $M > 0$ universally such that

$$v(0) \leq \varphi^-(0) \tag{3.20}$$

for all

$$\rho < \rho(n).$$

To demonstrate this, assume by way of contradiction that

$$v(0) > \varphi^{-}(0) \tag{3.21}$$

Then from (3.19) we have the following estimates for v:

$$\begin{aligned} \|Dv\|_{L^\infty(B_{\rho/2})} &\leq C(\delta^{1/2} + M\delta\rho) \\ \|D^2 v\|_{L^\infty(B_{\rho/2})} &\leq C(1 + M\delta). \end{aligned} \tag{3.22}$$

Now let $\mathcal{M}(D^2v, Dv)$ denote the mean curvature operator. By Lemma 3.2.6 and (3.22) we see that in $B_{\rho/2}$

$$\begin{aligned} \mathcal{M}(D^2v, Dv) &\geq \Delta v - C|Dv|^2|D^2v| \\ &\geq 2nM\delta - C(\delta^{1/2} + M\delta\rho)^2(1 + M\delta) \\ &\geq 2nM\delta - C(\delta + M^2\delta^2\rho^2)(1 + M\delta). \end{aligned} \tag{3.23}$$

Now we employ an argument similar to Proposition 3.2.7 to get an upper bound for

$$\varepsilon := \mathcal{M}(D^2v, Dv)(0).$$

Define

$$G_\eta = E \cap \left\{ (x, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; x_{n+1} > v(x) - \eta \right\},$$

(the supergraph in E of a translate of v) and denote

$$U_\eta = E \setminus \overline{G_\eta} = E \cap \left\{ (x, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; x_{n+1} < v(x) - \eta \right\}.$$

By (3.21) we can choose

$$0 \le \eta < v(0) - u(0)$$

so that the following two conditions hold

1) $\mathrm{Proj}_{e_{n+1}} U_\eta \subset\subset B_{r/2}(0)$,

2) $\mathcal{M}(D^2 v, Dv)(x) \ge \frac{\varepsilon}{2}$ for $x \in \mathrm{Proj}_{e_{n+1}} U_\eta$,

where $\mathrm{Proj}_{e_{n+1}}$ denotes projection in the e_{n+1} direction. Then just as in Proposition 3.2.7, letting $L \subset \Omega$ such that $U \subset\subset L$, letting N denote the outward unit normal to $\partial^* U$, and using Lemma A.1.2, we have

$$\begin{aligned} |U_\eta| \frac{\varepsilon}{4} &< \int_U \sum_{i=1}^n \frac{\kappa_i}{1 - \kappa_i d_{\partial G}(x)}\, dx \\ &= -\int_{U_\eta} \Delta d_{\partial G}(x)\, dx \\ &= -\int_{\partial^* U_\eta} D d_{\partial G}(x) \cdot N \, d\mathcal{H}^n \\ &= -\int_{\partial^* E \cap \overline{U_\eta}} D d_{\partial G}(x) \cdot N \, d\mathcal{H}^n - \int_{\partial G \cap \overline{U_\eta}} D d_{\partial G}(x) \cdot N \, d\mathcal{H}^n \\ &< \mathcal{P}_L(E) - \mathcal{P}_L(G) \le \|H\|_{L^\infty(B_1)} |U_\eta|. \end{aligned}$$

By the assumptions on $\|H\|_{L^\infty(B_1)}$, we see that

$$\varepsilon \le 2\delta. \tag{3.24}$$

Thus from (3.23) and (3.24),

$$2nM\delta \le 2\delta + C(\delta + M^2\delta^2\rho^2)(1 + M\delta),$$

or

$$2nM \le 2 + C(1 + M^2\delta\rho^2)(1 + M\delta). \tag{3.25}$$

Then if we fix

$$M = \frac{2(2+C)}{n},$$

and assume

$$\delta \leq \frac{1}{M},$$

we find from (3.25)

$$4(2+C) \leq 2C + \frac{4C(2+C)}{n}\rho^2. \tag{3.26}$$

Clearly we can find $\rho(n)$ such that (3.26) does not hold for $\rho \leq \rho(n)$, a contradiction.

So assuming we have chosen ρ and δ so that (3.20) holds, we have

$$\frac{1}{\rho^2}\left(⨍_{\partial B_\rho(0)} \varphi^-(x)\, dS(x)\right) \leq \frac{1}{\rho^2}\left(⨍_{\partial B_\rho(0)} \varphi^-(x)\, dS(x) - v(0)\right). \tag{3.27}$$

We find an upper bound for the right hand side of (3.27) using the representation formula for the solution to the dirichlet problem (3.2) (see for example [23], p. 13). We have

$$v(0) = \int_{\partial B_\rho(0)} \varphi^-(y)\frac{\partial G}{\partial \nu}(0,y)\, dS(y) + 2nM\delta \int_{B_\rho(0)} G(0,y)\, dy, \tag{3.28}$$

where $G(x,y)$ is the Green's function for the ball, given at $G(0,y)$ by

$$G(0,y) = \frac{1}{n(2-n)\omega_n}\left(\frac{1}{|y|^{n-2}} - \frac{1}{\rho^{n-2}}\right).$$

It can be calculated that

$$\frac{\partial G}{\partial \nu}(0,y) = \frac{\rho}{n\omega_n}\frac{1}{|y|^n},$$

so that (3.28) reads

$$⨍_{\partial B_\rho(0)} \varphi^-(y)\, dS(y) - v(0) = -\frac{2nM\delta}{n(2-n)\omega_n}\int_{B_\rho(0)}\left(\frac{1}{|y|^{n-2}} - \frac{1}{\rho^{n-2}}\right)\, dy.$$

But then a simple calculation gives that

$$\fint_{\partial B_\rho(0)} \varphi^-(y)\, dS(y) - v(0) = \frac{M\delta}{(2-n)}\rho^2. \tag{3.29}$$

Combining (3.27) and (3.29) we have the desired estimate. □

We are finally able to prove the upper bound on $\Delta\varphi^-(x)$ in all of $B_{1-8\delta^{1/2}}(0)$. This is accomplished in the next proposition.

Proposition 3.2.9. *Let E be a set with finite perimeter in $\Omega \subset \mathbb{R}^{n+1}$ with mean curvature H. Let $\varphi^\pm$ be the envelopes of ∂E described above. Assume $B_1^{n+1}(0) \subset \Omega$, and $0 \in \partial E$. Then there exists $\delta(n)$ such that if ∂E satisfies*

$$\partial E \cap B_1^{n+1}(0) \subset \{|x_{n+1}| \leq \delta\},$$

for $\delta \leq \delta(n)$, and

$$\|H\|_{L^\infty\left(B_1^{n+1}(0)\right)} \leq \delta,$$

then

$$\Delta\varphi^-(x) \leq C\delta \quad a.e. \quad x \in B_{1-8\delta^{1/2}}(0)$$

for a universal constant C.

Proof. Recall from the previous proposition that for δ and ρ sufficiently small,

$$\frac{1}{\rho^2}\left(\fint_{\partial B_\rho(x)} \varphi^-(y)\, dS(y) - \varphi^-(x)\right) \leq C\delta$$

for every $x \in \mathbb{C}^- \cap B_{1-6\delta^{1/2}}(0)$. So fix $x_0 \in B_{1-8\delta^{1/2}}(0) \setminus \mathbb{C}^-$. Also by the discussion preceding Proposition 3.2.7, we can find at most $n+1$ points $x_1, \ldots, x_{n+1} \in \mathbb{C}^- \cap$

$B_{1-6\delta^{1/2}}(0)$ and constants $\lambda_1, \ldots, \lambda_{n+1}$ such that

$$\sum_{i=1}^{n+1} \lambda_i = 1 \quad \text{and} \quad 0 \leq \lambda_j \leq 1$$
$$x_0 = \sum_{i=1}^{n+1} \lambda_i x_i \quad \text{and} \quad \Gamma(x_0) = \sum_{i=1}^{n+1} \lambda_i \Gamma(x_i).$$

Then by the convexity of Γ and the discrete Jensen's Inequality,

$$\begin{aligned}
\frac{1}{\rho^2}\left(\fint_{\partial B_\rho(x_0)} \Gamma(y)\, dS(y) - \Gamma(x_0)\right) &= \frac{1}{\rho^2}\left(\fint_{\partial B_\rho(0)} \Gamma(x_0 + y)\, dS(y) - \Gamma(x_0)\right) \\
&= \frac{1}{\rho^2}\left(\fint_{\partial B_\rho(0)} \Gamma(\sum_{i=1}^{n+1} \lambda_i (x_i + y))\, dS(y) - \sum_{i=1}^{n+1} \lambda_i \Gamma(x_i)\right) \\
&\leq \sum_{i=1}^{n+1} \frac{\lambda_i}{\rho^2}\left(\fint_{\partial B_\rho(0)} \Gamma(x_i + y)\, dS(y) - \Gamma(x_i)\right) \\
&= \sum_{i=1}^{n+1} \frac{\lambda_i}{\rho^2}\left(\fint_{\partial B_\rho(x_i)} \Gamma(y)\, dS(y) - \Gamma(x_i)\right) \\
&\leq C\delta + \sum_{i=1}^{n+1} \frac{\lambda_i}{2\rho^2}\left(\fint_{\partial B_\rho(x_i)} |y^2\, dS(y) - |x_i|^2\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{\rho^2}\left(\fint_{\partial B_\rho(x_0)} \varphi^-(y)\, dS - \varphi^-(x_0)\right) \\
&\qquad \leq C\delta + \sum_{i=1}^{n+1} \frac{\lambda_i}{2\rho^2}\left(\fint_{\partial B_\rho(x_i)} |y|^2\, dS - |x_i|^2\right) \qquad (3.30) \\
&\qquad - \frac{1}{2\rho^2}\left(\fint_{\partial B_\rho(x_0)} |y|^2\, dS - |x_0|^2\right).
\end{aligned}$$

If φ^- is second order differentiable at x_0, then the limit as $\rho \to 0$ of the left side of (3.30) exists and equals $\Delta\varphi^-(x_0)$. On the other hand, since $\Delta\left(\frac{1}{2}|x|^2\right)$ is constant, after taking the limit as $\rho \to 0$, the integrals on the right side of (3.30) cancel. Thus

$$\Delta\varphi^-(x_0) \leq C\delta.$$

□

3.3 The Measure of the Contact Set

Our goal for this section will be to obtain an estimate on the measure of the contact set $\mathbb{C}$ of the envelopes $\varphi^{\pm}$ with ∂E. To do this we will use a variant of the Alexandrov Bakelman Pucci estimate, and the envelope regularity estimates from the previous section.

Lemma 3.3.1 (The Measure of the Contact Set for Each Side)**.** *Let E be a set with finite perimeter in $\Omega \subset \mathbb{R}^{n+1}$ with mean curvature H. Let $\varphi^{\pm}$ be the envelopes of ∂E described in the previous section. Assume $B_1^{n+1}(0) \subset \Omega$, and $0 \in \partial E$. Then there exists $\delta(n)$ such that if*

$$\partial E \cap B_1^{n+1}(0) \subset \{|x_{n+1}| \leq \delta\},$$

for $\delta \leq \delta(n)$, and

$$\|H\|_{L^\infty\left(B_1^{n+1}(0)\right)} \leq \delta,$$

then the contact set $\mathbb{C}^{\pm}$ satisfies

$$(1 - C\delta)\,|B_{1-8\delta^{1/2}}(0)| \leq |\mathbb{C}^-|.$$

for a universal constant C.

Proof. Assume δ is at least as small as the $\delta(n)$ specified in Proposition (3.2.9). We define the function

$$y(z) = z + D\varphi^-(z),$$

which maps contact points z of $\varphi^-(z)$ to the center $y(z)$ of the corresponding paraboloid. Then the differential of y is

$$D_z y = I + D^2\varphi^-(z).$$

Because φ^- has a touching paraboloid of opening -1 from below,

$$D^2\varphi^-(z) \geq -I,$$

so that

$$D_z y \geq 0.$$

Note that y is surjective from $\mathbb{C}^-$ onto $B_{1-8\delta^{1/2}}(0)$. To see this, take a paraboloid with center

$$\tilde{y} \in |B_{1-8\delta^{1/2}}(0)|$$

and opening -1, and then lift the paraboloid from $-\infty$ until it touches ∂E from below. If $(\tilde{z}, \tilde{z}_{n+1}) \in \partial E$ is the point of contact of the paraboloid with ∂E, then by the defintion of φ^-, we must have

$$\varphi^-(\tilde{z}) = \tilde{z}_{n+1}.$$

A simple geometric argument using the opening of the paraboloid and the flatness of ∂E shows that $\tilde{z} \in B_{1-6\delta^{1/2}}(0)$. Thus $\tilde{z} \in \mathbb{C}^-$ and

$$\tilde{y} = y(\tilde{z}).$$

Since φ^- is $C^{1,1}$, $y(z)$ is certainly Lipschitz. So by the area formula ([11], p. 96)

$$\int_{\mathbb{C}^-} |\text{det} D_z y| \, dz = \int_{R^n} \text{card}\left(\mathbb{C}^- \cap y^{-1}(x)\right) \, dx \geq |B_{1-8\delta^{1/2}}(0)| \, .$$

Then by the arithmetic-geometric mean inequality, and Lemma 3.2.9, we have

$$\begin{aligned} |B_{1-8\delta^{1/2}}(0)| &\leq \int_{\mathbb{C}^-} |\mathrm{det} D_z y| \; dz \\ &\leq \int_{\mathbb{C}^-} \left(\frac{\mathrm{tr} D_z y}{n} \right)^n \; dz \\ &= \int_{\mathbb{C}^-} \left(1 + \frac{1}{n} \Delta \varphi^- \right)^n \; dz \\ &\leq \int_{\mathbb{C}^-} (1 + C\delta)^n \; dz \\ &\leq (1 + C\delta) |\mathbb{C}^-|. \end{aligned}$$

Thus

$$\begin{aligned} |\mathbb{C}^-| &\geq |B_{1-8\delta^{1/2}}(0)| \, / (1 + C\delta) \\ &\geq (1 - C\delta) \, |B_{1-8\delta^{1/2}}(0)| \, , \end{aligned}$$

for $\delta < 1/C$. □

Using the previous lemma to we are now in a position to provide an upper bound on the quantity

$$\mathcal{H}^n \left(B_{1-8\delta^{1/2}}(0) \cap \left\{ \, x \mid \varphi^-(x) \neq \varphi^+(x) \, \right\} \right),$$

which is accomplished in the next proposition.

Proposition 3.3.2 (The Measure of the Contact Set)**.** *Let E be a set with finite perimeter in $\Omega \subset \mathbb{R}^{n+1}$ with mean curvature H. Let $\varphi^{\pm}$ be the envelopes of ∂E described in the previous section. Assume $B_1^{n+1}(0) \subset \Omega$, and $0 \in \partial E$. Then there exists $\delta(n)$ such that if*

$$\partial E \cap B_1^{n+1}(0) \subset \{ |x_{n+1}| \leq \delta \} \, ,$$

for $\delta \le \delta(n)$ *and*

$$\|H\|_{L^\infty\left(B_1^{n+1}(0)\right)} \le \delta,$$

then

$$\mathcal{H}^n\left(B_{1-8\delta^{1/2}}(0) \cap \left\{\, x \;\middle|\; \varphi^-(x) \neq \varphi^+(x) \,\right\}\right) \le C\delta^{1/2}.$$

Proof. We assume δ is at least small enough to satisfy Lemma 3.3.1. For simplicty of notation, we set

$$r_1 = 1 - 6\delta^{1/2},$$

and

$$r_2 = 1 - 8\delta^{1/2}.$$

Note that

$$\begin{aligned} \mathcal{H}^n\left(B_{1-8\delta^{1/2}}(0) \cap \left\{\, x \;\middle|\; \varphi^-(x) \neq \varphi^+(x) \,\right\}\right) &= \mathcal{H}^n\left(B_{1-8\delta^{1/2}}(0) \setminus \left(\mathbb{C}^- \cap \mathbb{C}^+\right)\right) \\ &\le \omega_n r_2^n - \mathcal{H}^n(\mathbb{C}^- \cap \mathbb{C}^+). \end{aligned} \tag{3.31}$$

Following (3.31), and using Lemma 3.3.1 and the obvious inequality

$$\mathcal{H}^n(\mathbb{C}^- \cup \mathbb{C}^+) \le \mathcal{P}_{B_{r_2}^{n+1}}(E),$$

we calculate

$$\begin{aligned} \omega_n r_2^n - \mathcal{H}^n(\mathbb{C}^- \cap \mathbb{C}^+) &\le \omega_n r_2^n - \mathcal{H}^n(\mathbb{C}^-) - \mathcal{H}^n(\mathbb{C}^+) + \mathcal{H}^n(\mathbb{C}^- \cup \mathbb{C}^+) \\ &\le \omega_n r_2^n - 2\omega_n r_1^n(1 - C\delta) + \mathcal{H}^n(\mathbb{C}^- \cup \mathbb{C}^+) \\ &\le \omega_n r_2^n - 2\omega_n r_1^n(1 - C\delta) + \mathcal{P}_{B_{r_2}^{n+1}}(E). \end{aligned} \tag{3.32}$$

But by the density estimate, Lemma 2.2.8, for δ small enough depending on ω_{n+1},

$$\begin{aligned} \mathcal{P}_{B_{r_2}^{n+1}}(E) &\leq \omega_n \left(1 + \delta n + \delta^{n/n+1} \|H\|_{L^{n+1}\left(B_{r_2}^{n+1}\right)}\right) r_2^n \\ &= \omega_n \left(1 + \delta n + \omega_{n+1}^{1/n+1} \delta^{2n+1/n+1} r_2\right) r_2^n \qquad (3.33) \\ &\leq \omega_n \left(1 + \delta n + \delta r_2\right) r_2^n. \end{aligned}$$

Combining (3.32) and (3.33), we see that

$$\begin{aligned} \omega_n r_2^n - \mathcal{H}^n(\mathbb{C}^- \cap \mathbb{C}^+) &\leq \omega_n r_2^n - 2\omega_n r_1^n (1 - C\delta) + \omega_n \left(1 + \delta n + \delta r_2\right) r_2^n \\ &= 2\omega_n (r_2^n - r_1^n) + 2C\delta r_1^n + \delta n \omega_n r_2^n + \delta \omega_n r_2^{n+1} \\ &\leq 2\omega_n (r_2^n - r_1^n) + C\delta. \end{aligned}$$

It only remains to show that

$$r_2^n - r_1^n \leq C\delta^{1/2}$$

for some universal C. But this follows easily, since

$$\begin{aligned} r_2^n - r_1^n &= (r_2 - r_1)(r_2^{n-1} + r_2^{n-2} r_1 + \cdots + r_2 r_1^{n-2} + r_1^{n-1}) \\ &\leq C\delta^{1/2}. \end{aligned}$$

□

3.4 Flatness Implies $C^{1,\alpha}$

Theorem 3.4.1. *Let E be a set with finite perimeter in $\Omega \subset \mathbb{R}^{n+1}$ with mean curvature H. Assume $B_1^{n+1}(0) \subset \Omega$, and $0 \in \partial E$. Then there exists $\delta(n)$ small such that if*

$$\partial E \cap B_1^{n+1}(0) \subset \left\{|x_{n+1}| \leq \delta\right\},$$

for $\delta \leq \delta(n)$, and

$$\|H\|_{L^\infty\left(B_1^{n+1}(0)\right)} \leq \delta,$$

then ∂E *is* $C^{1,\alpha}$ *at* 0.

Proof. Fix

$$0 < \alpha < 1,$$

and let

$$f_1(x) = \frac{\varphi^-(x)}{\delta} \quad \text{and} \quad f_2(x) = \frac{\varphi^+(x)}{\delta}.$$

Then by the flatness of ∂E we have

$$-1 \leq f_1 \leq f_2 \leq 1,$$

and because $0 \in \partial E$, we see that

$$f_1(0) \leq 0 \leq f_2(0).$$

From Proposition 3.2.8, it follows that

$$\Delta f_1(x) \leq K \quad \text{and} \quad -\Delta f_2(x) \geq K$$

for some universal K. Applying Proposition 3.3.2 to get a universal T such that

$$\mathcal{H}^n \left(B_{1-8\delta^{1/2}}(0) \cap \left\{ x \mid \varphi^-(x) \neq \varphi^+(x) \right\} \right) \leq T\delta^{1/2},$$

we see that we are in the setting of Lemma 3.1.3. Thus there exists a linear function l_1 satisfying

$$l(0) = 0$$

such that for any $0 \leq \beta \leq 1$,

$$\|f_i - l_1\|_{L^\infty(B_r)} \leq C \left(\delta^{\frac{1}{n+2}} + r^{1+\beta} \right)$$

for

$$r \leq 1/2$$

where

$$C = C(n, \beta).$$

Note that C does not explicitly depend on T and K, because they are universal in this context. We find

$$\begin{aligned} \|\varphi^{\pm} - \delta l_1\|_{L^\infty(B_{r_0})} &\leq C\left(\delta^{\frac{n+3}{n+2}} + \delta r_0^{1+\beta}\right) \\ &\leq \delta r_0^{1+\alpha}, \end{aligned}$$

where we have chosen r_0 and $\alpha \leq \beta$ such that

$$C r_0^{1+\beta} < \frac{1}{2} r_0^{1+\alpha},$$

and

$$\delta = \delta(n, \alpha)$$

small enough so that

$$C\delta^{\frac{1}{n+2}} \leq \frac{1}{2} r_0^{1+\alpha}.$$

Since ∂E lies between the graphs of $\varphi^{\pm}$, we see that if ν_1 is the normal to δl_1, then

$$\partial E \cap B_{r_0}^{n+1}(0) \subset \left\{|x \cdot \nu_1| \leq \delta r_0^{1+\alpha}\right\}. \tag{3.34}$$

Now we rotate coordinates so that δl_1 is the horizon in the new coordinate system. In general, we will define the blowup sets

$$E_k = \frac{1}{r_0^k} E,$$

so that in this notation, after the rotation (3.34) becomes

$$\partial E_1 \cap B_1^{n+1}(0) \subset \{|x_n| \leq \delta r_0^{\alpha}\}.$$

Recall from (2.2) that E_1 is a set with prescribed mean curvature given in $B_1^{n+1}(0)$ by

$$H_{r_0}(x) = r_0 H(r_0 x).$$

We note that

$$\|H_{r_0}\|_{L^\infty\left(B_1^{n+1}(0)\right)} \leq r_0 \delta.$$

Taking $\varphi^\pm$ as the envelopes of the rescaled surface ∂E_k, we can repeat the argument above in general to find a linear function l_k with normal ν_k to $\delta r_0^{(k-1)\alpha} l_k$ such that

$$\partial E_k \cap B_1^{n+1}(0) \subset \left\{|x \cdot \nu_k| \leq \delta r_0^{k\alpha}\right\}. \tag{3.35}$$

By the definition of ν_k, we see that

$$\begin{aligned} |\nu_{k+1} - \nu_k| &\leq \left|\left(\delta r_0^{k\alpha} D l_{k+1}, 1\right) - \left(\delta r_0^{(k-1)\alpha} D l_k, 1\right)\right| \\ &\leq C \delta r_0^{(k-1)\alpha}, \end{aligned} \tag{3.36}$$

where Dl_k denotes the gradient of l_k in the first coordinate system. Thus

$$\nu_k \to \nu_\infty \quad \text{as} \quad k \to \infty.$$

Also from (3.36), we see that

$$|\nu_k - \nu_\infty| \leq C\delta \sum_{n=k}^{\infty} r_0^{(n-1)\alpha} \leq \frac{C\delta r_0^{\alpha(k-1)}}{1 - r_0^{\alpha}} \leq C(\alpha, r_0)\delta r_0^{k\alpha}. \tag{3.37}$$

Note that for $x \in B_1^{n+1}(0)$ we have from (3.37)

$$\begin{aligned} |x \cdot \nu_k| &\geq |x \cdot \nu_\infty| - |x \cdot \nu_\infty - x \cdot \nu_k| \\ &\geq |x \cdot \nu_\infty| - |x||\nu_\infty - \nu_k| \\ &\geq |x \cdot \nu_\infty| - C(\alpha, r_0)\delta r_0^{k\alpha} \end{aligned} \tag{3.38}$$

Combining (3.35) and (3.38), we find

$$\begin{aligned} \partial E_k \cap B_1^{n+1}(0) &\subset \left\{|x \cdot \nu_k| \leq \delta r_0^{k\alpha}\right\} \\ &\subset \left\{|x \cdot \nu_\infty| \leq C(\alpha, r_0)\delta r_0^{k\alpha}\right\}. \end{aligned} \tag{3.39}$$

Rescaling, we find

$$\partial E \cap B_{r_0^k}^{n+1}(0) \subset \left\{|x \cdot \nu_\infty| \leq C(\alpha, r_0)\delta r_0^{k(1+\alpha)}\right\},$$

which implies ∂E is $C^{1,\alpha}$ at 0 with normal ν_∞. □

APPENDIX A
THE SIGNED DISTANCE FUNCTION AND FIRST VARIATION FORMULA

A.1 The Signed Distance Function

In this appendix, we collect some facts regarding the signed distance function, following the exposition of [17] and [15].

For a bounded open set $E \subset \mathbb{R}^{n+1}$ define the signed distance function

$$d_{\partial E}(x) = \begin{cases} dist(x, \partial E) & \text{if } x \in E \\ -dist(x, \partial E) & \text{if } x \notin E. \end{cases} \tag{A.1}$$

Also, define the set

$$\Gamma_\mu = \left\{ x \in R^{n+1} \;\middle|\; |d_{\partial E}(x)| < \mu \right\},$$

which is the "tubular neigborhood" around ∂E of radius μ.

Lemma A.1.1. *Assume that $E \subset \mathbb{R}^{n+1}$ is bounded with C^k boundary for some $k \geq 2$. Then there exists an $R_0 > 0$, depending on E, such that $d_{\partial E}(x)$ is C^k in Γ_{R_0}.*

Proof. Because E has a C^k boundary, at each point $y \in \partial E$ there exists a ball $B \subset E$ with

$$\overline{B} \cap \partial E = \{ y \}. \tag{A.2}$$

Let $R(y)$ denote the radius of the largest ball satisfying $(A.2)$. Because ∂E is compact,

$$R_0 = \inf_{y \in \partial E} R(y) > 0.$$

We see that R_0^{-1} gives an upper bound for the principal curvatures of ∂E.

Now let $y_0 \in \partial E$ and let $T(y_0)$ denote the tangent hyperplane to ∂E at y_0. By rotating coordinates, we assume without loss of generality that

$$y_0 = 0 \text{ and } T(y_0) = \{\, x_{n+1} = 0 \,\}\,.$$

Then in a neighborhood U of $0 \in \mathbb{R}^n$, we can represent ∂E as the graph of a function

$$x_{n+1} = f(x_1, \ldots, x_n),$$

with

$$Df(0) = 0.$$

By further rotation of E around the x_{n+1} direction we can assume that $D^2 f(0)$ is diagonal. If we assume that E lies above the graph of f in the x_{n+1} direction, then

$$D^2 f(0) = diag(\kappa_1, \ldots, \kappa_n),$$

where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of ∂E at 0. Here, the signs of the principal curvatures are chosen to assign positive curvatures to convex (upwards) surfaces.

In general, let

$$y = (y', y_{n+1})$$

denote a point of R^{n+1}. Given $y' \in U$, if

$$y = (y', f(y')),$$

then the inner normal $\nu(y)$ to ∂E at y will be given by

$$\nu_i(y) = \begin{cases} \dfrac{-D_i f(y')}{\sqrt{1+|Df(y')|^2}} & \text{for } \ i = 1, \ldots, n \\ \\ \dfrac{1}{\sqrt{1+|Df(y')|^2}} & \text{for } \ i = n+1. \end{cases} \tag{A.3}$$

Define the map $g : U \times \mathbb{R} \to E$ by

$$g(y', d) = y + \nu(y)d. \tag{A.4}$$

Then $g \in C^{k-1}(U \times \mathbb{R})$. From $(A.3)$ we calculate that for $j = 1, \ldots, n+1$,

$$D_j\nu_i(0) = \begin{cases} -\delta_{ij}\kappa_i & \text{for } \ i = 1, \ldots, n \\ \\ 0 & \text{for } \ i = n+1. \end{cases} \tag{A.5}$$

So then

$$Dg(0, d) = diag(1 - \kappa_1 d, \ldots, 1 - \kappa_n d, 1). \tag{A.6}$$

Now, if

$$-R_0 < d < R_0, \tag{A.7}$$

then

$$\det(Dg(0, d)) = \prod_{i=1}^{n}(1 - \kappa_i d) > 0.$$

From the inverse function theorem, we see that in a neighborhood V of

$$x_d = (0, d) \in \mathbb{R}^{n+1},$$

y' and d can be written as C^{k-1} functions of x. Writing $(A.4)$ as

$$x - y(x) = \nu(y(x))d_{\partial E}(x),$$

it is geometrically clear that

$$Dd_{\partial E}(x) = \nu(y(x)) = \nu(y'(x)). \tag{A.8}$$

Since $\nu(y'(x))$ is a C^{k-1} function, we conclude that $d_{\partial E}(x)$ is a C^k function, provided $(A.7)$ is satisfied. □

Lemma A.1.2. *Assume that $E \subset \mathbb{R}^{n+1}$ is bounded with C^k boundary for some $k \geq 2$. Let $R_0 > 0$ be the constant from Lemma A.1.1. If $x \in \Gamma_{R_0}$ and $y \in \partial E$ such that*

$$d_{\partial E}(x) = |x - y|,$$

then

$$\Delta d_{\partial E}(x) = -\sum_{i=1}^{n} \frac{\kappa_i}{1 - \kappa_i d_{\partial E}(x)}, \tag{A.9}$$

where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of ∂E at y.

Proof. From $(A.8)$,

$$D_i D_i d_{\partial E}(x) = D_i(\nu_i(y(x))) = \sum_{k=1}^{n} D_k \nu_i(y(x)) D_i y_k(x) \quad \text{for } i = 1, \ldots, n.$$

Without loss of generality, assume

$$y(x) = 0 \quad \text{and} \quad T(y(x)) = \{\, x_{n+1} = 0 \,\}.$$

Using $(A.5)$ implies

$$\Delta d_{\partial E}(x) = -\sum_{i=1}^{n} \kappa_i D_i y_i(x).$$

By the inverse function theorem and $(A.6)$,

$$D_i y_i(x) = \frac{1}{1 - \kappa_i d_{\partial E}(x)} \quad \text{for } i = 1, \ldots, n,$$

and $(A.9)$ follows. □

Corollary A.1.3. *Assume that $E \subset \mathbb{R}^{n+1}$ is bounded with C^k boundary for some $k \geq 2$. Let $R_0 > 0$ be the constant from Lemma A.1.1. Let $x \in \Gamma_{R_0}$ and $y \in \partial E$ such that*

$$d_{\partial E}(x) = |x - y|.$$

If $\kappa_1, \ldots, \kappa_n$ are principal curvatures of ∂E at y, and

$$\mathcal{M}(y) = \frac{1}{n} \sum_{i=1}^{n} \kappa_i$$

denotes the mean curvature at y, then

$$\begin{cases} \Delta d_{\partial E}(x) \leq -n\mathcal{M}(y) & \text{for } x \in E \\ \\ \Delta d_{\partial E}(x) \geq -n\mathcal{M}(y) & \text{for } x \notin E. \end{cases} \tag{A.10}$$

Proof. If $x \in E$, then

$$d_{\partial E}(x) > 0,$$

so

$$\frac{-\kappa_i}{1 - \kappa_i d_{\partial E}(x)} \leq -\kappa_i \text{ for } i = 1, \ldots, n,$$

regardless of whether κ_i is positive or negative.

On the other hand if $x \notin E$, then

$$d_{\partial E}(x) \leq 0.$$

But then

$$\frac{-\kappa_i}{1 - \kappa_i d_{\partial E}(x)} \geq -\kappa_i \text{ for } i = 1, \ldots, n,$$

and $(A.10)$ follows. □

A.2 First Variation of Perimeter and Variational Mean Curvature

In this section of the appendix, we prove the assertion of Remark (2.1.2).

Lemma A.2.1 ([17], p. 115)**.** *Assume* $\Omega \subset \mathbb{R}^{n+1}$ *and* $f \in BV_{loc}(\Omega)$*. Let* $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ *be a diffeomorphism. Then for every compact* $K \subset \Omega$*,*

$$\int_{F(K)} \left| D\left(f(F^{-1}(x))\right) \right| dx = \int_K |G_F(x)\nu(x)||Df(x)|\, dx$$

where

$$G_F(x) = |\det DF(x)| DF(x)^{-1},$$

and

$$\nu(x) = \frac{Df(x)}{|Df(x)|} \quad a.e.,$$

the Radon-Nikodym derivative of $Df(x)$ *with respect to* $|Df(x)|$*.*

Lemma A.2.2 (First Variation of Perimeter for Regular Sets)**.** *Assume that* $E \subset \mathbb{R}^{n+1}$ *has* C^k *boundary for some* $k \geq 2$*. For* R_0 *as in Lemma A.1.1, let*

$$K \subset\subset \Gamma_{R_0/2}$$

and $g \in C^1_0(K)$*. Define a normal variation of* E *by*

$$E_t = \{\, x + tg(x)Dd_{\partial E}(x) \mid x \in E \,\},$$

where we require

$$tg(x) < R_0/2$$

for $x \in E$*. Then*

$$\left.\frac{d\,P_K(E_t)}{dt}\right|_{t=0} = -\int_{\partial E\cap K} n\mathcal{M}(x)g(x)\,dH^n(x),$$

where $\mathcal{M}(x)$ *denotes the mean curvature of* ∂E *at* x*.*

Proof. We define $F_t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$F_t(x) = x + tg(x)Dd_{\partial E}(x).$$

Note that

$$F(K) = K \quad\text{and}\quad \chi_{E_t}(x) = \chi_E(F^{-1}(x)).$$

Then from Lemma $A.2.1$,

$$\begin{aligned}\mathcal{P}_K(E_t) &= \int_K |D\chi_{E_t}(x)|\,dx = \int_K \left|D\left(\chi_E(F_t^{-1}(x))\right)\right| dx\\ &= \int_K \left|G_{F_t}(x)\frac{D\chi_E(x)}{|D\chi_E(x)|}\right| |D\chi_E(x)|\,dx\\ &= \int_{K\cap\partial E} |G_{F_t}(x)\nu(x)|\,d\mathcal{H}^n(x),\end{aligned}$$

since

$$\nu(x) = \frac{D\chi_E(x)}{|D\chi_E(x)|} \quad\text{for a.e.}\quad x \in \partial E.$$

Taking derivatives of both sides gives

$$\left.\frac{d\,\mathcal{P}_K(E_t)}{dt}\right|_{t=0} = \int_{K\cap\partial E} \left.\frac{d\,|G_{F_t}(x)\nu(x)|}{dt}\right|_{t=0} d\mathcal{H}^n(x). \tag{A.11}$$

Because

$$G_{F_t}(x) = I \quad\text{when } t = 0,$$

it follows that

$$\left.\frac{d\,|G_{F_t}\nu|}{dt}\right|_{t=0} = \left\langle \left.\frac{d\,G_{F_t}}{dt}\right|_{t=0}\nu,\ \nu\right\rangle. \tag{A.12}$$

Now we examine $(A.12)$. Because

$$DF_t = I + tD(g\nu),$$

we see that for small enough t,

$$(DF_t)^{-1} = I - tD(g\nu) + O(t^2).$$

Also, for t sufficiently small,

$$\det(DF_t) = 1 + t\,Tr(D(g\nu)) + O(t^2) > 0. \tag{A.13}$$

Hence

$$\begin{aligned}\left.\frac{d\,G_{F_t}}{dt}\right|_{t=0} &= \left.\frac{d\,\det(DF_t)}{dt}(DF_t)^{-1}\right|_{t=0} + \left.\frac{d\,(DF_t)^{-1}}{dt}\det(DF_t)\right|_{t=0} \\ &= Tr(D(g\nu)) - D(g\nu).\end{aligned}$$

So from $(A.12)$ we find

$$\left.\frac{d\,|G_{F_t}\nu|}{dt}\right|_{t=0} = Tr(D(g\nu)) - \sum_{i,j=1}^{n+1} \nu_i\nu_j D_i(g\nu_j). \tag{A.14}$$

The last expression is the *tangential divergence* of $g\nu$. To clarify the expression above, consider the *tangential gradient*, defined for a function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ as follows:

$$D_S f = Df - \nu\,\langle Df, \nu\rangle\,.$$

We can see that D_S is the projection of the gradient D onto the hyperplane perpendicular to ν. Hence,

$$\nu \cdot D_S f = 0.$$

We use the following notation for the tangential derivatives, which are components of the tangential gradient:

$$D_S f = \left(\underline{D}_1 f, \ldots, \underline{D}_{n+1} f\right).$$

Then the tangential divergence of $g\nu$ may be written as

$$\mathrm{div}_S(g\nu) := Tr(D(g\nu)) - \sum_{i,j=1}^{n+1} \nu_i \nu_j D_i(g\nu_j) = \sum_{j=1}^{n+1} \underline{D}_j(g\nu_j).$$

Note that the tangential derivatives satisfy the usual Leibniz rule. Thus we have

$$\begin{aligned}\mathrm{div}_S(g\nu) &= \sum_{j=1}^{n+1} \underline{D}_j(g\nu_j) \\ &= \sum_{j=1}^{n+1} \nu_j \underline{D}_j(g) + g\underline{D}_j(\nu_j) \\ &= \nu \cdot D_S f + g\,\mathrm{div}_S(\nu) \\ &= g\,\mathrm{div}_S(\nu).\end{aligned} \tag{A.15}$$

Finally, recalling the signed distance function $(A.1)$, we note that

$$\nu = Dd_{\partial E}(x)$$

By differentiating

$$1 = |\nu|^2 = \sum_{i=1}^{n+1} (D_i d_{\partial E})^2$$

we obtain

$$0 = \frac{1}{2} D_j |\nu|^2 = \sum_{i=1}^{n+1} D_i d_{\partial E}\, D_{ji} d_{\partial E}.$$

Using $(A.5)$, it follows that for $j = 1, \dots, n$

$$\begin{aligned} \underline{D}_j(\nu_j) &= D_j\nu_j - \nu_j \sum_{i=1}^{n+1} \nu_i D_i \nu_j \\ &= D_{jj}d_{\partial E} - D_j d_{\partial E} \sum_{i=1}^{n+1} D_i d_{\partial E} D_{ij} d_{\partial E} \\ &= D_{jj}d_{\partial E} - D_j d_{\partial E} \sum_{i=1}^{n+1} D_i d_{\partial E} D_{ji} d_{\partial E} \\ &= D_{jj}d_{\partial E} = D_j\nu_j = -\kappa_j, \end{aligned} \tag{A.16}$$

and for $j = n+1$,

$$\underline{D}_j(\nu_j) = 0. \tag{A.17}$$

Combining $(A.11)$, $(A.14)$, $(A.15)$ $(A.17)$and $(A.16)$ gives

$$\begin{aligned} \left.\frac{d\mathcal{P}_K(E_t)}{dt}\right|_{t=0} &= -\int_{K\cap\partial E} g(x) \sum_{j=1}^{n} \kappa_j \; d\mathcal{H}^n(x) \\ &= -\int_{K\cap\partial E} n\mathcal{M}(x) g(x) \; d\mathcal{H}^n(x) \end{aligned} \tag{A.18}$$

□

Proposition A.2.3 (The Equivalence of Classical and Variational Curvatures)**.** *Assume that $E \subset \mathbb{R}^{n+1}$ has variational mean curvature H in Ω, where $H \in L^1(\Omega) \cap C(\Omega)$. Suppose that E has C^k boundary for some $k \geq 2$. Then for $x \in \partial E$,*

$$-\frac{1}{n} H(x) = \mathcal{M}(x),$$

where $\mathcal{M}(x)$ is the mean curvature of ∂E at x.

Proof. We first prove the proposition assuming $H \in C^1(\Omega)$. For any open $K \subset\subset \Omega$ and $g \in C_0^1(K)$ we can define $F_t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$F_t(x) = x + tg(x)\nu(x),$$

and

$$E_t = \{\, x + tg(x)\nu(x) \mid x \in E \,\} = F_t(E),$$

as in Lemma $A.2.2$. Note that

$$F_t(E \cap K) = E_t \cap K.$$

Then since

$$\mathcal{F}_H(E_t) = \mathcal{P}_K(E_t) + \int_{E_t \cap K} H(x)\, dx$$

has a minimum at $t = 0$, we see that

$$\left.\frac{d\,\mathcal{F}_H(E_t)}{dt}\right|_{t=0} = \left.\frac{d\,\mathcal{P}_K(E_t)}{dt}\right|_{t=0} + \frac{d}{dt}\left\{\int_{E_t \cap K} H(x)\, dx\right\}_{t=0} = 0. \tag{A.19}$$

Changing variables, we see that

$$\begin{aligned}\frac{d}{dt}\left\{\int_{E_t \cap K} H(x)\, dx\right\}_{t=0} &= \frac{d}{dt}\left\{\int_{F_t(E \cap K)} H(x)\, dx\right\}_{t=0} \\ &= \frac{d}{dt}\left\{\int_{E \cap K} H(F_t(x))|\det DF_t(x)|\, dx\right\}_{t=0} \\ &= \int_{E \cap K} \left\{|\det DF_t(x)| \sum_{i=1}^{n+1} \frac{dH}{dy_i}(F_t(x)) \frac{d(F_t(x))_i}{dt}\right. \\ &\left. + H(F_t(x)) \frac{d|\det DF_t(x)|}{dt}\right\}_{t=0} dx.\end{aligned}$$

We observe

$$F_0(x) = x,$$

$$|\det DF_t(x)|_{t=0} = 1$$

$$\left.\frac{d(F_t(x))_i}{dt}\right|_{t=0} = g(x)\nu_i(x)$$

and using $(A.13)$, $\left.\frac{d|\det DF_t(x)|}{dt}\right|_{t=0} = Tr(D(g\nu))$.

so the expression above simplifies to

$$\begin{aligned}\int_{E\cap K} g(x)DH(x)\cdot\nu(x) + H(x)Tr(D(g\nu)(x))\;dx &= \int_{E\cap K}\operatorname{div}(g(x)H(x)\nu(x))\;dx\\ &= \int_{\partial(E\cap K)} gH\nu\cdot N\;d\mathcal{H}^n\\ &= -\int_{\partial E\cap K} gH\;d\mathcal{H}^n,\end{aligned}$$

where N is the outward pointing normal of $E\cap K$. By Lemma $A.2.2$ and $(A.19)$,

$$-\int_{\partial E\cap K} ng\mathcal{M}\;dH^n = \int_{\partial E\cap K} gH\;d\mathcal{H}^n.$$

Since g is arbitrary, the result follows.

For $H\in L^1(\Omega)\cap C(\Omega)$, the result can be demonstrated by approximation. □

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