Fall 2012

Some representation theory of the group Sl*(2,A) where A=M(2,O/p^2) and * equals transpose

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SOME REPRESENTATION THEORY OF THE GROUP $Sl_*(2, A)$ WHERE

$A = M_2(\mathcal{O}/\mathfrak{p}^2)$ AND $*$ EQUALS TRANSPOSE

by

Carmen Maria Wright

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

December 2012

Thesis Supervisor: Professor Philip Kutzko

ABSTRACT

Let A be a ring with involution $*$. The group $Sl_*(2, A)$, defined by Pantoja and Soto-Andrade, is a non-commutative version of $Sl(2, F)$ where F is a field. In the case of A being artinian, they determined when $Sl_*(2, A)$ admitted a Bruhat presentation, and with Gutiérrez, constructed a representation for $Sl_*(2, A)$ from its generators. In particular, if $A = M_n(F)$ and $*$ is transposition, then $Sl_*(2, A) = Sp(2n, F)$. In this paper, we are interested in the representation theory of $G = Sp_4(\mathcal{O}/\mathfrak{p}^2)$ where $A = M_2(\mathcal{O}/\mathfrak{p}^2)$ and $\mathcal{O}$ is a local ring with prime ideal $\mathfrak{p}$. It has a normal, abelian subgroup K, and by Clifford's theorem we can find distinct irreducible representations of G starting with one-dimensional representations of K. The outline of our strategy will be demonstrated in the example of finding irreducible representations of $SL_2(\mathcal{O}/\mathfrak{p}^2)$.

Abstract Approved: ______________________________

Thesis Supervisor

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Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Carmen Maria Wright

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the December 2012 graduation.

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ACKNOWLEDGEMENTS

I have received a lot of help and support from some programs and individuals. It would not have been possible to write this doctoral thesis without them. As I finish my thesis, I would sincerely like to give my thanks to them.

Foremost, I would like to express my sincere gratitude to my advisor, Prof. Philip Kutzko. Throughout my thesis-writing period, he provided encouragement, sound advice, good teaching, good company, and a wealth of information. Most of all, I thank him for his patience. I could not have imagined having a better advisor and mentor for my Ph.D study.

I would like to thank the many people who have taught me mathematics: my high school math teachers (especially Delmas Campbell), my undergraduate teachers at Oakwood University (especially Kathleen Dobbins, Shushannah Smith, and Albert Osei), and my graduate teachers (especially Frauke Bleher and Muthu Krishnamurthy).

A heartfelt thanks goes to the QEM Network Internship Program and the EDGE Program, which provided numerous benefits for me in terms of networking and opportunities to get me to where I am today. A special thanks goes to Sylvia Bozeman, Rhonda Hughes, Ami Radunskaya, and Ulrica Wilson of the EDGE program for their personal support in addition to the support I received from the EDGE Program.

I would like to thank my church family and student colleagues, with whom I have shared this journey. I am especially grateful to Jeannine Abiva, Shannon

Talbott, Kamuela Yong, Candice Price, and Syvillia Averett.

I wish to thank the Department of Mathematics at The University of Iowa for providing a supportive environment in which to pursue my studies.

Lastly, and most importantly, I wish to thank my family. My parents, Douglas and Margaret Wright, my sisters, Phaidra Boyd and Shaina Farrington, and my extended family have all given me their unconditional love and constant emotional and spiritual support.

ABSTRACT

Let A be a ring with involution $*$. The group $Sl_*(2, A)$, defined by Pantoja and Soto-Andrade, is a non-commutative version of $Sl(2, F)$ where F is a field. In the case of A being artinian, they determined when $Sl_*(2, A)$ admitted a Bruhat presentation, and with Gutiérrez, constructed a representation for $Sl_*(2, A)$ from its generators. In particular, if $A = M_n(F)$ and $*$ is transposition, then $Sl_*(2, A) = Sp(2n, F)$. In this paper, we are interested in the representation theory of $G = Sp_4(\mathcal{O}/\mathfrak{p}^2)$ where $A = M_2(\mathcal{O}/\mathfrak{p}^2)$ and $\mathcal{O}$ is a local ring with prime ideal $\mathfrak{p}$. It has a normal, abelian subgroup K, and by Clifford's theorem we can find distinct irreducible representations of G starting with one-dimensional representations of K. The outline of our strategy will be demonstrated in the example of finding irreducible representations of $SL_2(\mathcal{O}/\mathfrak{p}^2)$.

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CHAPTER 1
INTRODUCTION

1.1 Introduction

Much work has been done for the representation theory of $SL(2, F)$ where F is a field. By a suggestion of Cartier, Pantoja and Soto-Andrade introduced the group $Sl_*(2, A)$, where the field is replaced by a ring A with involution $*$. For example, taking $A = M_n(F)$ with the transpose operation as the involution, $Sl_*(2, A)$ is in fact $Sp(2n, F)$. Previously, Weil found a representation of the symplectic group when F is finite. Representations for $Sp(2n, F)$ can also be found using its presentation, and that method can be generalized to $Sl_*(2, A)$. If a Bruhat presentation may be found for $Sl_*(2, A)$, a generalized Weil representation may be constructed.

This paper delves into the representation theory of $Sp_4(\mathcal{O}/\mathfrak{p}^2)$, where $\mathcal{O}$ is a local ring with prime ideal $\mathfrak{p}$ and finite residue class field $\mathcal{O}/\mathfrak{p}$. (Note that $Sp_4(\mathcal{O}/\mathfrak{p}^2) = Sl_*(2, A)$ where $A = M_2(\mathcal{O}/\mathfrak{p}^2)$ and $*$ is transposition.) Let $G = Sp_4(\mathcal{O}/\mathfrak{p}^2)$ and $\overline{G} = Sp_4(\mathcal{O}/\mathfrak{p})$. We show that there exists a surjective homomorphism φ from G to $\overline{G}$ and we have the sequence

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\varphi} \bar{G} \longrightarrow 1.$$

where $K = \ker \varphi$.

Now suppose σ is an irreducible representation of G. If K is in the kernel of σ, then σ may be inflated from a representation of $\overline{G} = Sp_4(\mathcal{O}/\mathfrak{p})$; irreducible representations of $Sp_4(\mathcal{O}/\mathfrak{p})$ are known by the work of Soto-Andrade ([5]) while irreducible characters of $Sp_4(\mathcal{O}/\mathfrak{p})$ are known by the work of Srinivasan ([6]). (More generally, if χ is the character of a representation α of a group G, it is defined on the group elements by $\chi(g) = \mathrm{tr}\ (\alpha(g))$.) However, if K is not in the kernel, the restriction of σ to K, $\sigma|_K$, contains some non-trivial irreducible representation of K. We deal with the latter case in this paper.

We show that K is a normal, abelian subgroup of the form $K = 1 + \mathfrak{p}\mathfrak{k}$ for a certain (additive) abelian group $\mathfrak{k}$. Characters on K can be defined for each element in $\mathfrak{k}$. By the conjugacy action of G on $\mathfrak{k}$, we can find class representatives for the orbits, i.e. the conjugacy classes. The inertia subgroups for the characters on K are the stabilizers of the action. It suffices to consider only the inertia subgroups pertaining to class representatives. Clifford's theorem helps us find distinct, irreducible representations of G starting with our defined characters on K.

1.2 Organization of thesis

In chapter 2, we state the necessary definitions and mathematical preliminaries.

In chapter 3, we reference Pantoja and Soto-Andrade's paper to define $Sl_*(2, A)$ *when* A *is a ring with involution. We also discuss when* $Sl_*(2, A)$ *admits a presentation.*

In chapter 4, we use Clifford theory to explicitly find irreducible representations for $SL_2(\mathcal{O}/\mathfrak{p}^2)$ *by first defining a normal subgroup* K*, then defining characters on* K*, one for each element in* $\mathfrak{k}$*. The characters of* K *defined for the class representatives* *of* $\mathfrak{k}$ *are extended to their respective inertia groups. Finally, the extended characters are induced to* $SL_2(\mathcal{O}/\mathfrak{p}^2)$.

In chapter 5, we begin our work of finding irreducible representations of $Sp_4(\mathcal{O}/\mathfrak{p}^2)$, *using the outline of chapter 4, modifying where necessary.*

CHAPTER 2
MATHEMATICAL PRELIMINARIES

In this chapter, I state definitions and mathematical preliminaries which I used throughout this thesis.

2.1 Representation Theory

We assume the following: G is a group and V is a complex vector space.

- A *linear representation* of G is a pair (π, V) where $\pi : G \to GL(V)$ is a homomorphism. The degree of the representation is the dimension of V.

- Given two representations of G, say (π, V) and (σ, W), a linear map $T : V \to W$ is said to be a *G-homomorphism* if the following property holds for all $g \in G$: $T \circ \pi(g) = \sigma(g) \circ T$. The representations π and σ are said to be equivalent, denoted $\pi \cong \sigma$, if T is an isomorphism.

- A subspace $W \subseteq V$ is said to be *G-stable* if for all $w \in W$, then $\pi(g)w \in W$ for any $g \in G$.

- One says that (π, V) is *irreducible* if $V \neq 0$ and V has no nontrivial G-stable subspace U, $U \neq V$.

- Given a representation (π, V), one can define a *subrepresentation* (π_W, W) on a subspace $W \subseteq V$ by restriction, where $\pi_W : G \to GL(W)$.

- For abelian groups, irreducible representations are often called *characters*; they are one-dimensional representations.

- If (σ, W) is a representation on a subgroup H of G, then the induced representation from H to G, denoted $(\mathrm{ind}_H^G\sigma, \mathrm{ind}_H^G W)$, is defined by

$$\begin{aligned}\mathrm{ind}_H^G W &= \{f : G \to W \mid f(hx) = \sigma(h)f(x), \quad h \in H,\ x \in G\},\\ \mathrm{ind}_H^G\sigma(x)f(y) &= f(yx), \qquad x, y \in G.\end{aligned}$$

- Let (σ, W) be a representation of a subgroup H of G, and g an element of G. We define the *conjugate representation* (σ^g, W) on gHg^{-1} by $\sigma^g(x) = \sigma(g^{-1}xg)$, where $x \in gHg^{-1}$. In particular, if H is normal, both σ and σ^g are representations on H.

- Let σ be a irreducible representation of a normal subgroup H of G. Then the *inertia subgroup* is defined as

$$T(\sigma) = \{g \in G \mid \sigma^g \cong \sigma\}.$$

- *Clifford's Theorem*: Let N be a normal subgroup of G and σ an irreducible representation on N. Suppose $\mathrm{ind}_N^{T(\sigma)}\sigma$ decomposes into a sum of irreducible representations of $T(\sigma)$. That is, $\mathrm{ind}_N^{T(\sigma)}\sigma = \oplus n_i\tau_i$, where $\{\tau_i\}$ runs through the irreducible representations of $T(\sigma)$ and n_i is the number of copies of τ_i in the sum. Then $\mathrm{ind}_N^G\sigma = \oplus n_i \mathrm{ind}_{T(\sigma)}^G\tau_i$, where the $\mathrm{ind}_{T(\sigma)}^G\tau_i$ are irreducible and distinct representations of G.

CHAPTER 3
BACKGROUND

3.1 The group $Sl_*(2, A)$

The following information about $Sl_*(2, A)$ is found in Pantoja and Soto-Andrade's article [3].

3.1.1 Definition of the group $Sl_*(2, A)$

Let A be a ring with identity and an involution $*$, so it is an anti-automorphism of order two. The involution on A can be extended to an involution on the set of matrices $M(2, A)$. If T is such a matrix, T^* is defined by $(T^*)_{ij} = (T_{ji})^*$.

Let

$$M_*(2, A) = \{g \in M(2, A) \mid g^* J g J^{-1} \in Z(A) I_2\},$$

where $Z(A)$ is the center of A and $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. For $g \in M_*(2, A)$, we set $\delta(g) I_2 = g^* J g J^{-1}$. So $g^* J g = \delta(g) J$ and $\delta(g) \in Z(A)$.

Let $Gl_*(2, A)$ be the set of invertible elements in $M_*(2, A)$.

Properties of δ ([3], Lemma 1.1 and Corollary 1.2):

1. $(\delta(g))^* = \delta(g),\ g \in M_*(2, A)$.

2. $\delta(gh) = \delta(g)\delta(h)$ for $g, h \in M(2, A)$; $M_*(2, A)$ is closed under multiplication.

3. For $g \in Gl_*(2, A),\ \ gJg^* = g^* J g = \delta(g),\ \ \delta(g) = \delta(g^*)$, and $\delta(g^{-1}) = \delta(g)^{-1}$.

Remark. Alternative definition for $Gl_*(2, A)$:

$$Gl_*(2,A) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_*(2,A) \mid a^*c = c^*a,\ b^*d = d^*b,\ ab^* = ba^*,\ cd^* = dc^*, \right.$$

$$\left. ad^* - bc^* = a^*d - c^*b \in Z_s(A)^\times \right\},$$

where $Z_s(A)^\times$ denotes the group of the symmetric, central, invertible elements of A.

The $*$-determinant is the function $\det_*$ on $M_*(2, A)$ given by

$$\det_*(g) = ad^* - bc^* \quad \text{for } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We call $Sl_*(2, A)$ the subset of $M_*(2, A)$ of all g such that $\det_*(g) = 1$.

Remark. $\delta = \det_* : Gl_*(2, A) \to Z_s(A)^\times$ is an epimorphism. Furthermore, $Sl_*(2, A) = \ker \det_*$ ([3], Lemma 1.5).

We also have an alternative definition for $Sl_*(2, A)$:

$$Sl_*(2,A) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_*(2,A) \mid a^*c = c^*a,\ b^*d = d^*b,\ ab^* = ba^*,\ cd^* = dc^*, \right.$$

$$\left. ad^* - bc^* = a^*d - c^*b = 1 \right\}. \tag{3.1}$$

In the following section, we define Bruhat generators for $Sl_*(2, A)$ and discuss what it means for the group to have a Bruhat decomposition, which in certain cases leads to a Bruhat presentation.

3.1.2 Bruhat generators for $Sl_*(2,A)$

Let

$$h_t = \begin{bmatrix} t & 0 \\ 0 & t^{*^{-1}} \end{bmatrix}, \qquad w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \text{and } u_r = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}, \ (r = r^*).$$

Definition 3.1.1. The set of matrices $\{h_t, w, u_r \ : \ t \in A^\times, r \in A_s\}$ is called the set of *Bruhat generators* for $Sl_*(2,A)$.

One can check that these elements satisfy the relations:

- $h_t h_{t'} = h_{tt'}$,
- $w^2 = h_{-1}$,
- $u_r u_s = u_{r+s}$,
- $w h_t = h_{t^{*-1}} w$,
- $h_t u_r = u_{trt^*} h_t$,
- $u_t w u_{t^{-1}} w u_t = w h_{-t^{-1}}$.

We call these relations the Bruhat relations.

Let

$$B = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sl_*(2,A) : \ c = 0 \right\},$$

$$D = \{h_t \in B : \ t \in A^\times\},$$

$$N = \{u_r \in B : \ r \in A_s\}.$$

Then B, D and N are subgroups of $Sl_*(2,A)$ such that $B = DN$.

Definition 3.1.2. Let $SSl_*(2,A)$ be the subgroup of $Sl_*(2,A)$ given by

$$SSl_*(2,A) = \bigcup_{j=0}^{\infty} (BwB)^j, \qquad \text{where } (BwB)^0 = B,$$

and $BwB = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sl_*(2,A) : \ c \in A^\times \right\}$ ([3], Lemma 2.5).

Definition 3.1.3. Let $\widetilde{SSl_*(2,A)}$ be the group generated by the set $\{\tilde{h}_t, \tilde{u}_r, \tilde{w} : \ t \in A^\times, r \in A_s\}$ subject only to the relations

- $\tilde{h}_t \tilde{h}_{t'} = \tilde{h}_{tt'}$,
- $\tilde{u}_r \tilde{u}_s = \tilde{u}_{r+s}$,
- $\tilde{h}_t \tilde{u}_r = \tilde{u}_{trt^*} \tilde{h}_t$,
- $\tilde{w}^2 = \tilde{h}_{-1}$,
- $\tilde{w} \tilde{h}_t = \tilde{h}_{t^{*-1}} \tilde{w}$,
- $\tilde{u}_t \tilde{w} \tilde{u}_{t^{-1}} \tilde{w} \tilde{u}_t = \tilde{w} \tilde{h}_{-t^{-1}}$.

That is, $\widetilde{SSl_*(2,A)}$ is a group that has a presentation whose generators are Bruhat generators and whose relations are Bruhat relations.

$$\widetilde{SSl_*(2,A)} = \langle \tilde{h}_t, \tilde{u}_r, \tilde{w} : \ t \in A^\times, r \in A_s, \ \tilde{h}_t \tilde{h}_{t'} = \tilde{h}_{tt'}, \ \tilde{u}_r \tilde{u}_s = \tilde{u}_{r+s},$$
$$\tilde{h}_t \tilde{u}_r = \tilde{u}_{trt^*} \tilde{h}_t, \ \tilde{w}^2 = \tilde{h}_{-1}, \ \tilde{w} \tilde{h}_t = \tilde{h}_{t^{*-1}} \tilde{w}, \ \tilde{u}_t \tilde{w} \tilde{u}_{t^{-1}} \tilde{w} \tilde{u}_t = \tilde{w} \tilde{h}_{-t^{-1}} \rangle.$$

Since the group $SSl_*(2,A)$ is generated by elements that satisfy the Bruhat relations, we obtain a surjective map from $\widetilde{SSl_*(2,A)}$ to $SSl_*(2,A)$. We note that this map need not be injective. Note below.

Definition 3.1.4. We say that $Sl_*(2,A)$ has a *Bruhat decomposition* if $SSl_*(2,A) = Sl_*(2,A)$. If the above union is finite, then the minimal n such that $Sl_*(2,A) = \cup_{j=0}^{n} (BwB)^j$ is called the *Bruhat length* of G.

Lemma 3.1.5. *$Sl_*(2,A)$ is generated by the Bruhat generators if and only if $Sl_*(2,A)$ has a Bruhat decomposition ([3], Lemma 2.5).*

It is important to note that the generators of $Sl_*(2, A)$ and their relations as given may not define a presentation for $Sl_*(2, A)$.

Definition 3.1.6. We say that $Sl_*(2, A)$ (with a Bruhat decomposition) has a *Bruhat presentation* if $\widetilde{SSl_*(2, A)} \simeq Sl_*(2, A)$, where the Bruhat generators of $\widetilde{SSl_*(2, A)}$ appropriately map to the Bruhat generators of $Sl_*(2, A)$. That is, the kernel of the surjective map $\widetilde{SSl_*(2, A)} \to SSl_*(2, A)$ is trivial.

3.2 The Weil representation

Originally, Weil constructed representations of a symplectic group over locally compact fields via the Heisenberg group ([2]). These representations are, in fact, projective representations, but when the field is finite, they are ordinary representations. Collectively, they are known as the Weil representation.

Cartier conjectured that all the irreducible representations of $Sp_{2n}(F)$, F locally compact, could be constructed by decomposing the Weil representation. By looking at the classical case of $Sp_{2n}(F)$, since we know a priori that it has a Weil representation, it must have a Bruhat presentation with Bruhat generators and Bruhat relations. This also allows us to conveniently construct a representation from its presentation, mapping each generator to an appropriate linear operator on a complex vector space where the operators satisfy the same Bruhat relations.

Further work in this area seeks to generalize this method of constructing a Weil representation by generators and relations for the group $Sl_*(2, A)$ where A is any involutive ring. The process in the classical case can be imitated for $Sl_*(2, A)$

and its generators, and we can "guess" what the operators are and verify that they satisfy the same relations. What needs to be checked in each case is that $Sl_*(2, A)$ admits a presentation. If so, it suffices to check directly that the operators satisfy the Bruhat relations. In [1], Gutierrez, Pantoja, and Soto-Andrade explicitly constructed a generalized Weil representation for $G = Sl_*(2, A)$ for certain involutive rings A, and the symplectic group is recovered as an example.

This paper deals with the cases of $Sp_2(\mathcal{O}/\mathfrak{p}^2) = Sl_*(2, \mathcal{O}/\mathfrak{p}^2)$, where $*$ is the identity, and $Sp_4(\mathcal{O}/\mathfrak{p}^2) = Sl_*(2, M_2(\mathcal{O}/\mathfrak{p}^2))$, where $*$ is transposition, and $\mathcal{O}$ is a local ring with prime ideal $\mathfrak{p}$.

Remark. We need to make a general statement that applies to both cases.

Preliminaries: First, we define a surjective ring homomorphism f from $\mathcal{O}/\mathfrak{p}^2$ to $\mathcal{O}/\mathfrak{p}$ by $f(r + \mathfrak{p}^2) = r + \mathfrak{p},\ r \in \mathcal{O}$; the kernel of f is $\mathfrak{p}/\mathfrak{p}^2$. For ease of notation, we will write this map as $r \mapsto \overline{r}$, where $r \in \mathcal{O}/\mathfrak{p}^2$ and $\overline{r} \in \mathcal{O}/\mathfrak{p}$. It extends to another ring homomorphism $f_n : M_n(\mathcal{O}/\mathfrak{p}^2) \to M_n(\mathcal{O}/\mathfrak{p})$ where $R \mapsto \overline{R}$ is defined entry-wise by

$$(\overline{R})_{ij} = \overline{(R_{ij})}. \tag{3.2}$$

That is, f is applied to each entry of $R \in M_n(\mathcal{O}/\mathfrak{p}^2)$.

Now let $G = Sp_{2n}(\mathcal{O}/\mathfrak{p}^2)$ and $\overline{G} = Sp_{2n}(\mathcal{O}/\mathfrak{p})$. We can define a map φ from G to $\overline{G}$, or equivalently, from $Sl_*(2, M_n(\mathcal{O}/\mathfrak{p}^2))$ to $Sl_*(2, M_n(\mathcal{O}/\mathfrak{p}))$.

Let $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$, where $a, b, c, d \in M_n(\mathcal{O}/\mathfrak{p}^2)$. Then

$$\varphi(x) = \begin{bmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{bmatrix};$$

f_n is applied to each entry of x as by Equation 3.2. Then φ is also a homomorphism.

Lemma 3.2.1. *The map φ is surjective.*

Proof. Let $\overline{x} = \begin{bmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{bmatrix} \in \overline{G}$.

Suppose $\overline{c} = \overline{0}$. Then $\det_* \overline{x} = \overline{ad^t} = \overline{1}$, and there exists $y \in M_n(\mathcal{O})$ such that $ad^t - 1 = \varpi y$. Let ℓ be an element of $M_n(\mathcal{O})$ satisfying $\ell \equiv -ya \bmod \mathfrak{p}$. Then

$$\varphi\left(\begin{bmatrix} a + \varpi\ell & b \\ 0 & d \end{bmatrix}\right) = \overline{x}.$$

Let G contain the subgroup B and element w as in §3.1.2. It is already known that $\overline{G}$ has a length 2 Bruhat decomposition ([3]), say

$$\overline{G} = \overline{B} \cup \overline{B}\,\overline{w}\,\overline{B} \cup \overline{B}\,\overline{w}\,\overline{B}\,\overline{w}\,\overline{B},$$

where $\overline{w} = \varphi(w)$.

We have already shown that if $\overline{x} \in \overline{B}$, then there exists $x \in B$ such that $\varphi(x) = \overline{x}$. Since φ is a homomorphism, the result follows:

- If $\overline{x} \in \overline{B}\,\overline{w}\,\overline{B}$, then $\overline{x} = \overline{b_1}\,\overline{w}\,\overline{b_2}$ for some $\overline{b_1}, \overline{b_2} \in \overline{B}$. Let $x = b_1 w b_2$ where $b_i \in B$ and $\varphi(b_i) = \overline{b_i}$, $i = 1, 2,$.

- If $\overline{x} \in \overline{B}\overline{w}\overline{B}\overline{w}\overline{B}$, then $\overline{x} = \overline{b_1}\overline{w}\overline{b_2}\overline{w}\overline{b_3}$ for some $\overline{b_i} \in \overline{B}$, $i = 1, 2, 3$. Let $x = b_1 w b_2 w b_3$, where $b_i \in B$ and $\varphi(b_i) = \overline{b_i}$, $i = 1, 2, 3$.

In either case, $\varphi(x) = \overline{x}$.

$\square$

CHAPTER 4
REPRESENTATION THEORY OF $SL_2(\mathcal{O}/\mathfrak{p}^2)$

We first turn our attention to the group $Sp_2(\mathcal{O}/\mathfrak{p}^2) = SL_2(\mathcal{O}/\mathfrak{p}^2)$ where $\mathcal{O}$ is a local ring with unique maximal ideal $\mathfrak{p}$ generated by the element ϖ. Here, $A = \mathcal{O}/\mathfrak{p}^2$ and $*$ is the identity. The methodology used in finding irreducible representations of $SL_2(\mathcal{O}/\mathfrak{p}^2)$ will be used as a strategy to find irreducible representations of $Sp_4(\mathcal{O}/\mathfrak{p}^2)$. These results appear in [4] using a different method. Shalika found representations for $SL_2(\mathcal{O}/\mathfrak{p}^n)$ in general using the Weil representation.

4.1 The groups K and $\mathfrak{k}$

Let $G = SL_2(\mathcal{O}/\mathfrak{p}^2)$ and $\overline{G} = SL_2(\mathcal{O}/\mathfrak{p})$.

Remark. Throughout this paper, computations done "mod $\mathfrak{p}^2$" will generally use the $=$ sign, while computations done "mod $\mathfrak{p}$" will generally use the $\equiv$ sign.

Let $\varphi : G \to \overline{G}$ be the $n = 1$ case of the surjective map referenced in Lemma 3.2.1, p. 12.

Lemma 4.1.1. *The map φ is surjective.*

Lemma 4.1.2. *The kernel of φ is $K = (1 + \mathfrak{p}M_2(\mathcal{O}/\mathfrak{p}^2)) \cap G$, thus it is normal in G.*

Proof.

$$
\begin{aligned}
\ker\varphi &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G \ : \ \begin{bmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{bmatrix} = \begin{bmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{1} \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G \ : \ \begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix} \in M_2(\mathfrak{p}/\mathfrak{p}^2) \right\} \\
&= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G \ : \ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in 1 + \mathfrak{p}M_2(\mathcal{O}/\mathfrak{p}^2) \right\} \\
&= K
\end{aligned}
$$

□

We obtain the sequence

$$1 \longrightarrow K \hookrightarrow G \longrightarrow \overline{G} \longrightarrow 1$$

and rewrite K as $K = 1 + \mathfrak{p}\mathfrak{k}$, where

$$\mathfrak{k} = \{k \in M_2(\mathcal{O}/\mathfrak{p}^2) \mid 1 + \mathfrak{p}k \in G\}.$$

The set $\mathfrak{k}$ can also be seen as a subset of $M_2(\mathcal{O}/\mathfrak{p})$. By definition, if $x_1 = 1+\varpi k_1$ and $x_2 = 1 + \varpi k_2$ in K are equivalent, then

$$1 + \varpi k_1 = 1 + \varpi k_2$$

$$\text{implies} \quad k_1 \equiv k_2 \bmod \mathfrak{p}.$$

Now, let $x = 1 + \varpi k$ be in K, where $k = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathcal{O}/\mathfrak{p})$. Then

$$x = \begin{bmatrix} 1+\varpi a & \varpi b \\ \varpi c & 1+\varpi d \end{bmatrix}, \quad \text{and}$$

$$\det x = (\varpi a + 1)(\varpi d + 1) - \varpi^2 bc = 1 + \varpi(a+d)$$

implies $a + d \in \mathfrak{p}$ since the determinant must be 1. Hence

$$\mathfrak{k} = \{k \in M_2(\mathcal{O}/\mathfrak{p}) \mid \text{tr }(k) \equiv 0 \bmod \mathfrak{p}\} \tag{4.1}$$

It is clear that $(\mathfrak{k}, +)$ is an abelian group.

Suppose char $(\mathcal{O}/\mathfrak{p}) \neq 2$. Define a pairing $\mathfrak{k} \times \mathfrak{k} \to \mathcal{O}/\mathfrak{p}$ by

$$\langle k_1 \mid k_2 \rangle = \text{tr }(k_1 k_2).$$

Proposition 4.1.3. *The form $\langle \cdot \mid \cdot \rangle$ on $\mathfrak{k} \times \mathfrak{k}$ defined above is non-degenerate.*

Proof. Let $k_1 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ be a non-zero element of $\mathfrak{k}$. Then at least one of its entries is non-zero. We must show there exists a non-zero k_2 in $\mathfrak{k}$ such that $\langle k_1 \mid k_2 \rangle \neq 0$.

- If $a \neq 0$, set $k_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $\langle k_1 \mid k_2 \rangle = 2a \neq 0$.
- If $b \neq 0$, set $k_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $\langle k_1 \mid k_2 \rangle = b \neq 0$.
- If $c \neq 0$, set $k_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $\langle k_1 \mid k_2 \rangle = c \neq 0$.

□

Lemma 4.1.4. *The map $\tau : \mathfrak{k} \to K$ defined by $k \mapsto 1 + \varpi k$ is an isomorphism.*

Proof. It is a homomorphism: for any $k_1, k_2 \in \mathfrak{k}$,

$$\tau(k_1 + k_2) = 1 + \varpi(k_1 + k_2) = (1 + \varpi k_1)(1 + \varpi k_2) = \tau(k_1)\tau(k_2).$$

Since $|K| = |\mathfrak{k}|$, proving injectivity is sufficient:

$$\begin{aligned} \ker \tau &= \{k \in \mathfrak{k} \mid 1 + \varpi k = 1\} \\ &= \{k \in \mathfrak{k} \mid k \equiv 0 \bmod \mathfrak{p}\} \\ &= M_2(\mathfrak{p}). \end{aligned}$$

□

It follows easily then that K is also abelian.

Fix a non-trivial (additive) character ψ of $\mathcal{O}/\mathfrak{p}$. For $k \in \mathfrak{k}$, define a character ψ_k on K by

$$\psi_k(1 + \varpi k_1) = \psi(\mathrm{tr}\ k k_1), \quad 1 + \varpi k_1 \in K. \tag{4.2}$$

Then $\psi_k \in \widehat{K}$, the set of one-dimensional representations of K.

Proposition 4.1.5. *The map $\Phi : \mathfrak{k} \to \widehat{K}$ defined by $k \mapsto \psi_k$ is an isomorphism.*

First, we need to prove a few statements. Note that $\mathcal{O}/\mathfrak{p} = F$ is a field with q elements, where q is some prime power, i.e. $q = p^n$. We take our character ψ on F to be the composition of a character $\bar{\psi}$ on $\mathbb{F}_p$ and the field trace map $\mathrm{Tr}_{F/\mathbb{F}_p}$ on F:

$$\psi = \bar{\psi} \circ \mathrm{Tr}_{F/\mathbb{F}_p}. \tag{4.3}$$

Lemma 4.1.6. *Let $\alpha \in F^\times$. Then there exists a non-zero $\beta \in F$ such that $Tr_{F/\mathbb{F}_p}(\beta\alpha) \neq 0$.*

Proof. Let $\gamma \neq 0$ be in F such that $\mathrm{Tr}_{F/\mathbb{F}_p}(\gamma) \neq 0$. (Note: We know such γ exists because $F/\mathbb{F}_p$ is a separable extension; moreover, it is cyclic.) Set $\beta = \gamma\alpha^{-1}$. Then

$$\mathrm{Tr}_{F/\mathbb{F}_p}(\beta\alpha) = \mathrm{Tr}_{F/\mathbb{F}_p}((\gamma\alpha^{-1})\alpha) = \mathrm{Tr}_{F/\mathbb{F}_p}(\gamma) \neq 0.$$

□

Lemma 4.1.7. *For $k \in \mathfrak{k}$, $\psi(tr\ kk_1) = 1$ for all k_1 in $\mathfrak{k}$ if and only if $k = 0$.*

Proof. If $k = 0$, the result is clear. Conversely, suppose $k \in \mathfrak{k}$ and $\psi(\mathrm{tr}\ kk_1) = 1$ for all $k_1 \in \mathfrak{k}$. The subset $\mathfrak{k} \subset M_2(\mathcal{O}/\mathfrak{p})$ may be viewed as a vector space over $\mathcal{O}/\mathfrak{p}$. If $k_1 \in \mathfrak{k}$ and $\beta \in \mathcal{O}/\mathfrak{p}$, then βk_1 is also in $\mathfrak{k}$. Fixing $\beta \in (\mathcal{O}/\mathfrak{p})^\times$, for all $k_1 \in \mathfrak{k}$, we have

$$\psi(\mathrm{tr}\ k\beta k_1) = \psi(\beta \cdot \mathrm{tr}\ kk_1) = 1.$$

Then $\psi = \bar{\psi} \circ \mathrm{Tr}_{F/\mathbb{F}_p}$ implies

$$\bar{\psi}(\mathrm{Tr}_{F/\mathbb{F}_p}(\beta \cdot \mathrm{tr}\ kk_1)) = 1,$$

and since $\bar{\psi}$ is faithful, we must have

$$\mathrm{Tr}_{F/\mathbb{F}_p}(\beta \cdot \mathrm{tr}\ kk_1) = 0,$$

implying $\mathrm{tr}\ kk_1 = 0$ for all $k_1 \in \mathfrak{k}$ (Lemma 4.1.6). Therefore $k = 0$ by Proposition 4.1.3. □

Now we prove Proposition 4.1.5.

Proof. First we show that $\Phi : k \mapsto \psi_k$ is a homomorphism, i.e. $\psi_{k_1+k_2} = \psi_{k_1}\psi_{k_2}$ for any $k_1, k_2 \in \mathfrak{k}$.

For any $1 + \varpi k \in K$, $k \in \mathfrak{k}$,

$$\begin{aligned}
\psi_{k_1+k_2}(1+\varpi k) &= \psi(\text{tr } (k_1+k_2)k) \\
&= \psi(\text{tr } k_1 k + \text{tr } k_2 k) \\
&= \psi(\text{tr } k_1 k)\psi(\text{tr } k_2 k) \\
&= \psi_{k_1}(1+\varpi k)\psi_{k_2}(1+\varpi k) \\
&= \psi_{k_1}\psi_{k_2}(1+\varpi k)
\end{aligned}$$

Since $|\widehat{K}| = |K| = |\mathfrak{k}|$, we are done if the map is injective. Here we use *triv* to denote the trivial representation, which acts as the identity map on the vector space. Then

$$\begin{aligned}
\ker \Phi &= \{k \in \mathfrak{k} \mid \psi_k = triv\} \\
&= \{k \in \mathfrak{k} \mid \psi_k(1+\varpi k_1) = 1 \quad \text{for all } k_1 \in \mathfrak{k}\} \\
&= \{k \in \mathfrak{k} \mid \psi(\text{tr } k k_1) = 1 \quad \text{for all } k_1 \in \mathfrak{k}\} \\
&= \{0\}
\end{aligned}$$

by Proposition 4.1.7.

$\square$

4.1.1 Clifford Theory

We have proved that K is a normal abelian subgroup of G. We have also shown that the characters ψ_k for all non-zero $k \in \mathfrak{k}$ are non-trivial irreducible representations of K. We are now in a position to start applying Clifford's theorem.

Define an action of G on $\mathfrak{k}$ for $g \in G,\ k \in \mathfrak{k}$ by $g \cdot k = gkg^{-1}$. Recall that the matrices in $\mathfrak{k}$ were defined by their trace. Note that tr $(k) \equiv 0 \bmod \mathfrak{p}$ implies tr $(gkg^{-1}) \equiv 0 \bmod p$. Of course, this also defines an action of $\overline{G}$ on $\mathfrak{k}$.

Let $1 + \varpi k_1,\ k_1 \in \mathfrak{k}$, be an element of K. For any $g \in G$,

$$\begin{aligned} \psi_k^g(1+\varpi k_1) &= \psi_k(1+\varpi(g^{-1}k_1 g)) \\ &= \psi(\text{tr } (kg^{-1}k_1)g) \\ &= \psi(\text{tr } (gkg^{-1})k_1) \\ &= \psi_{g\cdot k}(1+\varpi k_1). \end{aligned}$$

This says that the action is G-equivariant for the map $\mathfrak{k} \to \widehat{K}$, where $k \mapsto \psi_k$. Therefore, $\psi_k^g = \psi_{g\cdot k}$, and the inertia group for ψ_k is

$$T(\psi_k) = \{g \in G \mid \psi_{g\cdot k} = \psi_k\} = \{g \in G \mid gkg^{-1} \equiv k \bmod \mathfrak{p}\}.$$

With the action defined above, it is clear that $T(\psi_k)$ is the stabilizer of k. Note that for all $x \in K = 1 + \mathfrak{p}\mathfrak{k}$, we have $xkx^{-1} = k$ since $x \bmod \mathfrak{p} = 1$. Therefore K is a subgroup of $T(\psi_k)$.

In the next section, we determine class representatives for the orbits.

4.2 Determining conjugacy classes

We say that two matrices k_1, k_2 in $\mathfrak{k}$ are equivalent, denoted by $k_1 \sim k_2$, if they lie in the same orbit. Let $[\mathfrak{k}]$ be a complete set of class representatives. That is, for each $k \in \mathfrak{k}$, there exists a $g \in G$ such that $k \sim k'$ for one of the k' in $[\mathfrak{k}]$. In fact, it is enough to find a $g \in \overline{G}$.

Let $k = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, and set $\Delta = \det k$. Our class representatives for the orbits should have the same trace and determinant. We will show that the representatives are of the form

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad k_\Delta = \begin{bmatrix} 0 & 1 \\ -\Delta & 0 \end{bmatrix} \quad \text{and} \quad k_\varepsilon = \begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \end{bmatrix},$$

where $\Delta \neq 0$ and $\varepsilon = 1$ or non-square. Of course, when k is the zero matrix it is its own representative.

Suppose $k \sim k_\Delta$. Then there exists a matrix $g = \begin{bmatrix} t & u \\ v & w \end{bmatrix} \in \overline{G}$ satisfying $gkg^{-1} = k_\Delta$. By matrix multiplication we get the equations

$$v = ta + uc$$

$$w = tb - ua$$

Substituting these equations for v and w in the determinant of g yields

$$bt^2 - 2atu - cu^2 = 1 \tag{4.4}$$

Hence the existence of g depends on the existence of appropriate t and u in $\mathcal{O}/\mathfrak{p}$.

<u>When $\Delta \neq 0$</u>

Case 1: $b \neq 0$.

Applying the completing-the-square method to Equation 4.4 yields

$$\left(t - \frac{a}{b}u\right)^2 - \left(\frac{a^2 + bc}{b^2}\right)u^2 = \frac{1}{b} \tag{4.5}$$

Using the substitutions $\theta = t - \frac{a}{b}u$ and $-\Delta = a^2 + bc$, we have

$$\theta^2 - \left(\frac{-\Delta}{b^2}\right)u^2 = \frac{1}{b}. \tag{4.6}$$

Our approach to this equation depends on whether $-\Delta$ is a square or not.

(1a) $-\Delta$ *is a square*:

Let δ be an element in $\mathcal{O}/\mathfrak{p}$ that satisfies $-\Delta = \delta^2$. By Equation 4.6,

$$\theta^2 - \left(\frac{\delta}{b}\right)^2 u^2 = \frac{1}{b}$$
$$\left(\theta - \frac{\delta}{b}u\right)\left(\theta + \frac{\delta}{b}u\right) = \frac{1}{b}$$

One way to obtain a solution is to set $\theta - \frac{\delta}{b}u = 1$ and $\theta + \frac{\delta}{b}u = \frac{1}{b}$. Since these two equations are linearly independent, we solve for u and θ, by which we find $t = \theta + \frac{a}{b}u$:

$$u = \frac{1-b}{2\delta}, \qquad t = \frac{\delta(b+1) + a(1-b)}{2b\delta}.$$

(1b) $-\Delta$ *is a nonsquare*

As before, we let $F = \mathcal{O}/\mathfrak{p}$ and $\gamma = \sqrt{-\Delta}$ so that $F[\gamma]$ is a quadratic extension of F. Combining the definition of the field norm with Equation 4.6, we have

$$N_{F[\gamma]/F}\left(\theta + \frac{u}{b}\gamma\right) = b^{-1}.$$

Recall that the norm map is surjective for finite fields. This means that for $b^{-1} \in F$, we know there exists such $\theta,\ u$, which give us t.

Case 2: $b = 0$.

Since $\Delta \neq 0$, we also have $a \neq 0$. From Equation 4.4, we have $-2atu - cu^2 = 1$. Fix some nonzero u in $\mathcal{O}/\mathfrak{p}$ and let $t = \dfrac{-1 - cu^2}{2au}$.

When $\Delta = 0$

We show $k = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ is equivalent to either $k_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ or to $k_\varepsilon = \begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \end{bmatrix}$, where ε is non-square.

Case 3: $b \neq 0$

(3a) Suppose b is a square, i.e. $b = \beta^2$ for some $\beta \in \mathcal{O}/\mathfrak{p}$. Then $gkg^{-1} = k_0$ for $g = \begin{bmatrix} \beta^{-1} & 0 \\ \beta^{-1}a & \beta \end{bmatrix}$.

(3b) Suppose b is not a square. If $k \sim k_\varepsilon$, there exists $g = \begin{bmatrix} t & u \\ v & w \end{bmatrix} \in \overline{G}$ such that $gkg^{-1} = k_\varepsilon$. By matrix multiplication,

$$v = \varepsilon^{-1}(ta + uc)$$

$$w = \varepsilon^{-1}(tb - ua)$$

Substituting these equations for v and w in the determinant of g yields

$$bt^2 - 2atu - cu^2 = \varepsilon. \tag{4.7}$$

Hence the existence of g depends on the existence of appropriate t and u in $\mathcal{O}/\mathfrak{p}$.

Since $b \neq 0$, we can solve for t:

$$t = \frac{au \pm \sqrt{b\varepsilon}}{2b}.$$

After choosing u, this is a solution if $b\varepsilon$ is a square. The product of two non-squares in a finite field is a square, so $\sqrt{b\varepsilon} \in \mathcal{O}/\mathfrak{p}$.

Case 4: $b = 0$

It follows easily from $\Delta = 0$ that $a = 0$.

(4a) Suppose $c \neq 0$. Then $\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & -c \\ 0 & 0 \end{bmatrix}$ for $g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. See Case 3.

(4b) If $c = 0$ then k is the zero matrix, the only element in its conjugacy class.

We have proved the following:

Theorem 4.2.1. *Let $k \in \mathfrak{k}$ and $\Delta = \det k$. Then the conjugacy classes are determined by Δ. If $\Delta \neq 0$, we denote each unique class representative by $k_\Delta = \begin{bmatrix} 0 & 1 \\ -\Delta & 0 \end{bmatrix}$. If $\Delta = 0$ and k is non-zero, then the class representative is $k_\varepsilon = \begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \end{bmatrix}$ where $\varepsilon = 1$ or is some fixed non-square. Also, the zero matrix is its own representative.*

$\det k$	$k = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$	class representative
$\Delta \neq 0$	$-a^2 - bc \neq 0$	$k_\Delta = \begin{bmatrix} 0 & 1 \\ -\Delta & 0 \end{bmatrix}$
$\Delta = 0$	$b \neq 0$, b a square	$k_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
	$b \neq 0$, b non-square	$k_\varepsilon = \begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \end{bmatrix}$
	$a = b = 0$, $c \neq 0$	k_0 if $-c$ is a square, k_ε if $-c$ non-square
	zero matrix	zero matrix

4.3 Extending K by conjugacy classes

4.3.1 The inertia group

We would like to extend the non-trivial characters on K to characters on their inertia groups. It is sufficient to carry out this process for the non-zero class representatives. Set $\psi_\Delta = \psi_{k_\Delta}$ and similarly, set $\psi_\varepsilon = \psi_{k_\varepsilon}$. Thus, the only inertia groups that we need to consider are

$$T(\psi_\Delta) = \{x \in G \mid x k_\Delta x^{-1} \equiv k_\Delta \bmod \mathfrak{p}\},$$

$$T(\psi_\varepsilon) = \{x \in G \mid x k_\varepsilon x^{-1} \equiv k_\varepsilon \bmod \mathfrak{p}\}.$$

For $k \in \mathfrak{k}$, define the subgroup

$$S(k) = \{x \in G \mid x k x^{-1} = k\}.$$

For all $k \in [\mathfrak{k}]$, these are abelian subgroups of G. Let $S_0 = S(k_0) = S(k_\varepsilon)$ and $S_\Delta = S(k_\Delta)$. Note that this notation is consistent with the case $\Delta = 0$. We have

$$\begin{aligned} S_\Delta &= \{x \in G \mid xk_\Delta x^{-1} = k_\Delta\} = \left\{ \begin{bmatrix} t & u \\ -\Delta u & t \end{bmatrix} \in G \ : \ t^2 + \Delta u^2 = 1 \right\}, \\ S_0 &= \{x \in G \mid xk_\varepsilon x^{-1} = k_\varepsilon\} = \left\{ \begin{bmatrix} t & u \\ 0 & t \end{bmatrix} \in G \ : \ t = \pm 1 \right\}. \end{aligned}$$

Proposition 4.3.1. *For $x \in G$, $xk_\Delta x^{-1} \equiv k_\Delta$ mod $\mathfrak{p}$ if and only if $x \in S_\Delta K$. Similarly, $xk_\varepsilon x^{-1} \equiv k_\varepsilon$ mod $\mathfrak{p}$ if and only if $x \in S_0 K$. That is,*

$$T(\psi_\Delta) = S_\Delta K,$$

$$T(\psi_\varepsilon) = S_0 K.$$

Proof. Suppose $x \in S_\Delta K$, $x = sy$ for $s \in S_\Delta$, $y \in K$. Recall that $yk_\Delta y^{-1} = k_\Delta$ and it follows easily from the definition of S_Δ that $sk_\Delta s^{-1} \equiv k_\Delta$ mod $\mathfrak{p}$. These two statements together imply that

$$(sy)k_\Delta(sy)^{-1} \equiv k_\Delta \text{ mod } \mathfrak{p}.$$

Thus $sy = x \in T(\psi_\Delta)$. The same argument holds for k_ε, S_0, and $T(\psi_\varepsilon)$.

Case (a) Conversely, let $x \in T(\psi_\Delta)$,

$$T(\psi_\Delta) = \left\{ \begin{bmatrix} t & u \\ -\Delta u + \varpi v & t + \varpi w \end{bmatrix} \in G \ : \ t^2 - (-\Delta)u^2 + \varpi(tw - uv) = 1 \right\}.$$

Then

$$xk_\Delta x^{-1} = k_\Delta + \varpi \begin{bmatrix} -(tv + \Delta uw) & -(tw - uv) \\ -\Delta(tw - uv) & (tv + \Delta uw) \end{bmatrix}.$$

We will show that $x \in KS_\Delta = S_\Delta K$ by showing there exists $1+z \in K$ with $z \in \mathfrak{pk}$ such that $(1+z)x \in S_\Delta$. Let $z = \varpi \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \mathfrak{pk}$, where

$$c + \Delta b \equiv -(tv + \Delta uw) \bmod \mathfrak{p}, \qquad a \equiv \frac{tw - uv}{2} \bmod \mathfrak{p}.$$

Then

$$(1+z)^{-1} k_\Delta (1+z) = x k_\Delta x^{-1}.$$

But this implies

$$(1+z) x k_\Delta x^{-1} (1+z)^{-1} = k_\Delta.$$

Therefore $(1+z)x \in S_\Delta$.

Case (b) We follow the steps in Case (a) using the data $(k_\varepsilon, S_0, T(\psi_\varepsilon))$ in place of $(k_\Delta, S_\Delta, T(\psi_\Delta))$. Let $x \in T(\psi_\varepsilon)$,

$$T(\psi_\varepsilon) = \left\{ \begin{bmatrix} t & u \\ \varpi\varepsilon^{-1} v & t + \varpi w \end{bmatrix} \in G \ : \ t^2 + \varpi(tw - \varepsilon^{-1}uv) = 1 \right\}.$$

Then

$$x k_\varepsilon x^{-1} = k_\varepsilon + \varpi \begin{bmatrix} -tv & -(\varepsilon tw - uv) \\ 0 & \varpi tv \end{bmatrix}.$$

Let $z = \varpi \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \mathfrak{pk}$, where

$$c \equiv -\varepsilon^{-1} tv \bmod \mathfrak{p}, \qquad a \equiv \frac{tw - \varepsilon^{-1}uv}{2} \bmod \mathfrak{p}.$$

Then

$$(1+z)^{-1} k_\varepsilon (1+z) = x k_\varepsilon x^{-1}.$$

But this implies

$$(1+z)xk_\varepsilon x^{-1}(1+z)^{-1} = k_\varepsilon.$$

Therefore $(1+z)x \in S_0$.

$\square$

4.4 Extending to the inertia group

Let θ be a one-dimensional representation of the abelian subgroup S (which represents either S_Δ or S_0) such that $\theta|_{S\cap K} = \psi_k|_{S\cap K}$, k non-zero in $[\mathfrak{k}]$.

$$\begin{tikzcd}[column sep=small]
 & \widetilde{\psi}_\theta,\ SK \arrow[dl, dash] \arrow[dr, dash] & \\
\theta,\ S \arrow[dr, dash] & & K,\ \psi_k \arrow[dl, dash] \\
 & S\cap K &
\end{tikzcd}$$

We extend ψ_k to a map $\widetilde{\psi}_\theta : SK \to \mathbb{C}$, defined for $sx \in SK$ by

$$\widetilde{\psi}_\theta(sx) = \theta(s)\psi_k(x). \tag{4.8}$$

Note that for any $s \in S,\ x \in K$, we have that $\psi_k(sxs^{-1}) = \psi_k(x)$ by definition of the inertia group.

Claim. *The map $\widetilde{\psi}_\theta$ is well-defined.*

Proof. Let $sx = ty$ where $s, t \in S,\ x, y \in K$. Then

$$\widetilde{\psi}_\theta(sx) = \theta(s)\psi_k(x) = \theta(s)\psi_k(s^{-1}txt^{-1}s) = \widetilde{\psi}_\theta(txt^{-1}s).$$

But $sx = ty$ implies $xt^{-1} = s^{-1}tyt^{-1}$. Continuing, we get

$$\widetilde{\psi}_\theta(ts^{-1}tyt^{-1}s) = \theta(t)\psi_k(s^{-1}tyt^{-1}s) = \theta(t)\psi_k(y) = \widetilde{\psi}_\theta(ty).$$

□

Claim. *The map $\widetilde{\psi}_\theta$ is a character.*

Proof. It is enough to show $\widetilde{\psi}_\theta$ is a homomorphism. Let $sx, ty \in SK$ for $s, t \in S$, $x, y \in K$. Note that since S is in the normalizer of K, $SK = KS$, and SK is a subgroup. Then we may write $xt = t_1x_1$ for some $t_1 \in S, x_1 \in K$.

$$\begin{aligned}
\widetilde{\psi}_\theta(sx \cdot ty) &= \widetilde{\psi}_\theta(st_1x_1y) \\
&= \theta(st_1)\psi_k(x_1y) \\
&= \theta(st_1)\psi_k(t_1^{-1}tx_1yt^{-1}t_1) \\
&= \widetilde{\psi}_\theta(stx_1yt^{-1}t_1) \\
&= \widetilde{\psi}_\theta(stx_1yx_1^{-1}t_1^{-1}xt_1) \qquad \text{using the substitution } t^{-1} = x_1^{-1}t_1^{-1}x \\
&= \theta(st)\psi_k(x_1yx_1^{-1})\psi_k(t_1^{-1}xt_1) \\
&= \theta(s)\psi_k(x)\theta(t)\psi_k(y) \\
&= \widetilde{\psi}_\theta(sx) \cdot \widetilde{\psi}_\theta(ty).
\end{aligned}$$

□

Let $\widetilde{\psi_k}$ be an extension of ψ_k on K to SK such that $\widetilde{\psi_k}|_S = \theta$. Then for $sx \in SK$,

$$\widetilde{\psi_k}(sx) = \widetilde{\psi_k}(s)\widetilde{\psi_k}(x) = \theta(s)\psi_k(x) = \widetilde{\psi}_\theta(sx).$$

Thus, as θ varies over all the one-dimensional representations of S, $\widetilde{\psi}_\theta$ gives us all the extensions of ψ_k to SK.

Specifically, denote by $\widetilde{\psi}_{\theta,\Delta}$ a character of $S_\Delta K$, and by $\widetilde{\psi}_{\theta',\varepsilon}$ a character of S_0K. Here, θ is a character of S_Δ and θ' is a character of S_0. Moreover, for $x = 1 + \varpi k \in K$, where $k = (k)_{ij} \in \mathfrak{k}$, then

$$\widetilde{\psi}_{\theta,\Delta}(sx) = \theta(s)\psi_{k_\Delta}(x) = \theta(s)\psi(\mathrm{tr}\ k_\Delta k) = \theta(s)\psi(k_{21} - \Delta k_{12}), \tag{4.9}$$

$$\widetilde{\psi}_{\theta',\varepsilon}(sx) = \theta'(s)\psi_{k_\varepsilon}(x) = \theta'(s)\psi(\mathrm{tr}\ k_\varepsilon k) = \theta'(s)\psi(\varepsilon k_{21}). \tag{4.10}$$

4.5 Induction to G

At this time, we induce the representations $(\widetilde{\psi}_{\theta',\varepsilon}, \mathbb{C})$ on S_0K and the representations $(\widetilde{\psi}_{\theta,\Delta}, \mathbb{C})$ on $S_\Delta K$ to obtain representations on G. By the definition of induction using $\widetilde{\psi}_{\theta',\varepsilon}$ and $\widetilde{\psi}_{\theta,\Delta}$ from S_0K and $S_\Delta K$, respectively, we have the induced spaces

$$\begin{aligned}
\mathrm{ind}_{S_\Delta K}^G \mathbb{C} =& \{f : G \to \mathbb{C} \mid f((sx)y) = \widetilde{\psi}_{\theta,\Delta}(sx) f(y), \quad s \in S_\Delta,\ x \in K,\ y \in G\},\\
\mathrm{ind}_{S_0 K}^G \mathbb{C} =& \{f : G \to \mathbb{C} \mid f((sx)y) = \widetilde{\psi}_{\theta',\varepsilon}(sx) f(y), \quad s \in S_0,\ x \in K,\ y \in G\}.
\end{aligned}$$

Or more specifically, by Equations 4.9 and 4.10,

$$\begin{aligned}
\mathrm{ind}_{S_\Delta K}^G \mathbb{C} =& \{f : G \to \mathbb{C} \mid f((sx)y) = \theta(s)\psi(k_{21} - \Delta k_{12}) f(y), \quad s \in S_\Delta,\ k \in \mathfrak{k},\ y \in G\},\\
\mathrm{ind}_{S_0 K}^G \mathbb{C} =& \{f : G \to \mathbb{C} \mid f((sx)y) = \theta'(s)\psi(\varepsilon k_{21}) f(y), \quad s \in S_0,\ k \in \mathfrak{k},\ y \in G\}.
\end{aligned}$$

As Δ and ε vary, we obtain irreducible, distinct representations of G by Clifford's theorem.

4.6 The degree of the induced representation

We recall an interesting fact in representation theory concerning the degree of an induced representation: If (σ, W) is a representation on the subgroup H of G, then the degree of the induced representation $(\mathrm{ind}_H^G \sigma, \mathrm{ind}_H^G W)$ is

$$\dim \mathrm{ind}_H^G W = [G : H] \dim W.$$

In our case, the degree of the induced representation $\mathrm{ind}_{SK}^G \widetilde{\psi}_\theta$ is

$$\dim \mathrm{ind}_{SK}^G \mathbb{C} = [G : SK] \dim \mathbb{C} = \frac{|G|}{|SK|}.$$

Again, this S represents S_0 or S_Δ.

The cardinality of G is known since $|G| = |K||SL_2(\mathcal{O}/\mathfrak{p})| = q^3 \cdot q(q^2-1) = q^4(q^2-1)$. What remains is to calculate the orders of S_0K and $S_\Delta K$.

Lemma 4.6.1. *We have $|S_0| = 2q^4$ and $|S_0 \cap K| = q$. Also, for $\Delta \neq 0$, we have $|S_\Delta \cap K| = q$.*

Proof. Recall that elements of S_0 are of the form $x = \begin{bmatrix} t & u \\ 0 & t \end{bmatrix}$, $t = \pm 1$. Then $|S_0| = 2q^2$. If x is also in K, it must be of the form $1 + \varpi k$ for some $k = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ in $\mathfrak{k}$:

$$\begin{bmatrix} t & u \\ 0 & t \end{bmatrix} = \begin{bmatrix} 1+\varpi a & \varpi b \\ \varpi c & 1-\varpi a \end{bmatrix}.$$

This implies elements in $S_0 \cap K$ are of the form

$$\begin{bmatrix} 1 & \varpi b \\ 0 & 1 \end{bmatrix}, \quad b \in \mathcal{O}/\mathfrak{p}.$$

Therefore $|S_0 \cap K| = q$.

The elements of S_Δ are of the form $x = \begin{bmatrix} t & u \\ -\Delta u & t \end{bmatrix}$ where $t^2 + \Delta u^2 = 1$. If x is also in K, it must be of the form $1 + \varpi k$ for some $k = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ in $\mathfrak{k}$:

$$\begin{bmatrix} t & u \\ -\Delta u & t \end{bmatrix} = \begin{bmatrix} 1+\varpi a & \varpi b \\ \varpi c & 1-\varpi a \end{bmatrix}.$$

This implies $\varpi c = -\varpi b \Delta$, so elements in $S_\Delta \cap K$ are of the form

$$\begin{bmatrix} 1 & \varpi b \\ -\varpi b \Delta & 1 \end{bmatrix}, \quad b \in \mathcal{O}/\mathfrak{p}.$$

We obtain easily that $|S_\Delta \cap K| = q$. $\square$

To calculate the order of S_Δ, first we define $\overline{S_\Delta} = \{x \bmod \mathfrak{p} \mid x \in S_\Delta\}$; it is a subgroup of $\overline{G} = SL_2(\mathcal{O}/\mathfrak{p})$. If $1 \longrightarrow S_\Delta \cap K \longrightarrow S_\Delta \longrightarrow \overline{S_\Delta} \longrightarrow 1$ is a short exact sequence, we can use $|\overline{S_\Delta}|$ to calculate $|S_\Delta|$.

1. The "mod $\mathfrak{p}$" map $\alpha : S_\Delta \to \overline{S_\Delta}$ is surjective by definition of $\overline{S_\Delta}$.

2. Also, $\ker \alpha = S_\Delta \cap K$. The argument is the same as in the proof of Lemma 4.1.2 (p. 14).

Lemma 4.6.2.

$$|\overline{S_\Delta}| = \begin{cases} q-1, & -\Delta \textit{ is a square} \\ q+1, & -\Delta \textit{ is not a square} \end{cases}$$

Proof. Elements of the form $\begin{bmatrix} t & u \\ -\Delta u & t \end{bmatrix}$ mod $\mathfrak{p}$ in $\overline{S_\Delta}$ satisfy the equation $t^2 - (-\Delta)u^2 \equiv 1 \bmod \mathfrak{p}$. (Even though t, u, and Δ are elements of $\mathcal{O}/\mathfrak{p}^2$, for ease of notation in this proof we will view them as their "mod $\mathfrak{p}$" images in $\mathcal{O}/\mathfrak{p}$.)

Suppose first that $-\Delta$ is a square, i.e. $-\Delta = r^2$ for some r in $\mathcal{O}/\mathfrak{p}$. Then $t^2 - r^2u^2 = (t+ru)(t-ru) = 1$. Making the substitutions $y = t + ru$ and $x = t - ru$ we obtain $yx = 1$. Note that y can be any nonzero element of $\mathcal{O}/\mathfrak{p}$, so there are $q-1$ choices for y. Since $x = 2t - y$, x is dependent on our choice of y.

Now suppose $-\Delta$ is not a square. As before, we adjoin $\gamma = \sqrt{-\Delta}$ to the field $F = \mathcal{O}/\mathfrak{p}$ to get a quadratic field extension $F[\gamma]$ of F. Recall the norm is a surjective homomorphism between $F[\gamma]^\times$ and $F^\times$:

$$N_{F[\gamma]/F}\,(t + u\gamma) = t^2 + \Delta u^2.$$

Thus the number of elements in the kernel of the norm is the same number of elements in $\overline{S_\Delta}$. Since $F[\gamma]^\times / \ker(N_{F[\gamma]/F}) \cong F^\times$, we must have

$$\ker N_{F[\gamma]/F} = \frac{q^2-1}{q-1} = q + 1.$$

$\square$

Finally, we can calculate the order of S_Δ:

$$|S_\Delta| = |S_\Delta \cap K||\overline{S_\Delta}| = \begin{cases} q(q-1), & -\Delta \text{ is a square} \\ q(q+1), & -\Delta \text{ is not a square} \end{cases}.$$

Since S_0K and $S_\Delta K$ are subgroups of G, using Lemma 4.6.1, we obtain the following:

$$|S_0K| = \frac{|S_0||K|}{|S_0 \cap K|} = 2q^4,$$

$$|S_\Delta K| = \frac{|S_\Delta||K|}{|S_\Delta \cap K|} = \begin{cases} q^3(q-1), & -\Delta \text{ is a square} \\ q^3(q+1), & -\Delta \text{ is not a square} \end{cases}.$$

Therefore, recalling $|G| = q^4(q^2-1)$, the degrees of the induced representations are

$$\dim \operatorname{ind}_{S_0K}^G \mathbb{C} = \frac{q^2-1}{2},$$

$$\dim \operatorname{ind}_{S_\Delta K}^G \mathbb{C} = \begin{cases} q(q+1), & -\Delta \text{ is a square} \\ q(q-1), & -\Delta \text{ is not a square} \end{cases}.$$

CHAPTER 5
REPRESENTATION THEORY OF $Sp_4(\mathcal{O}/\mathfrak{p}^2)$

Now that we have found irreducible representations for $Sl_*(2, \mathcal{O}/\mathfrak{p}^2) = SL_2(\mathcal{O}/\mathfrak{p}^2)$, we would like to do the same for $Sl_*(2, M_2(\mathcal{O}/\mathfrak{p}^2))$. Although the increase in matrix dimension raises the complexity of the problem, we know we can still use Clifford's theorem, so we use the same strategy to find irreducible representations of $Sp_4(\mathcal{O}/\mathfrak{p}^2)$.

1. Define non-trivial characters on a normal, abelian subgroup K of $Sp_4(\mathcal{O}/\mathfrak{p}^2)$.

2. Extend certain characters to characters on their respective inertia groups.

3. Induce the extended characters to $Sp_4(\mathcal{O}/\mathfrak{p}^2)$; they are irreducible and distinct.

Remark. If the proof of a statement for $SL_2(\mathcal{O}/\mathfrak{p}^2)$ was dependent only on the general properties of the groups $Sl_*(2, A)$, K and $\mathfrak{k}$, then the result extends to $Sp_4(\mathcal{O}/\mathfrak{p}^2)$ and we will not include its proof in this chapter. Neither included are full proofs for statements on $Sp_4(\mathcal{O}/\mathfrak{p}^2)$ where the argument is the same as the $SL_2(\mathcal{O}/\mathfrak{p}^2)$ case, but the details are only minimally modified, which will be indicated.

5.1 The groups K and $\mathfrak{k}$

In this chapter, we will use the $Sl_*(2, A)$ structure of $G = Sp_4(\mathcal{O}/\mathfrak{p}^2)$ where $A = M_2(\mathcal{O}/\mathfrak{p}^2)$ and $*$ is the transpose to find certain irreducible representations of G. Let $\overline{G} = Sp_4(\mathcal{O}/\mathfrak{p})$ and $\mathfrak{p} = \langle \varpi \rangle$.

Let $\varphi : G \to \overline{G}$ be the $n = 2$ case of the surjective map referenced in Lemma 3.2.1, p. 12.

Lemma 5.1.1. *The map φ is surjective.*

Lemma 5.1.2. *The kernel is $K = (1+\mathfrak{p}M_2(A)) \cap G$, and thus K is a normal subgroup of G.*

Proof. Similar argument as given in the proof of Lemma 4.1.2 on p. 14. Modification: An element in G has entries in $M_2(\mathcal{O}/\mathfrak{p}^2)$, so the kernel is now $(1 + \mathfrak{p}M_2(M_2(\mathcal{O}/\mathfrak{p}^2))) \cap G$. $\square$

Again, we write our normal subgroup K as $K = 1 + \mathfrak{p}\mathfrak{k}$, where

$$\mathfrak{k} = \{k \in M_2(A) \mid 1 + \mathfrak{p}k \in G\}.$$

The set $\mathfrak{k}$ can also be seen as a subset of $M_2(M_2(\mathcal{O}/\mathfrak{p}))$. As shown previously, two elements $1 + \varpi k_1$ and $1 + \varpi k_2$ of K are equivalent when $k_1 \equiv k_2 \mod \mathfrak{p}$.

Now let $k = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(M_2(\mathcal{O}/\mathfrak{p}))$. Then

$$x = 1 + \varpi k = \begin{bmatrix} 1+\varpi a & \varpi b \\ \varpi c & 1+\varpi d \end{bmatrix},$$

and since $x \in G$, $\det_* x = 1$. Then

$$\det_* x = (\varpi a + 1)(\varpi d + 1)^t - \varpi^2 bc^t = 1 + \varpi(a + d^t)$$

implies $a + d^t \in \mathfrak{p}$. The other properties of $Sl_*(2, A)$ imply that $b \equiv b^t \mod \mathfrak{p}$, $c \equiv c^t \mod \mathfrak{p}$ (see Equation 3.1, p. 7). Hence

$$\mathfrak{k} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(M_2(\mathcal{O}/\mathfrak{p})) \mid a + d^t = 0,\ b, c \in M_2(\mathcal{O}/\mathfrak{p})_s \right\} \tag{5.1}$$

Then $d \equiv -a^t \bmod \mathfrak{p}$. Also, note that $(\mathfrak{k}, +)$ is an abelian group.

Suppose char $\mathcal{O}/\mathfrak{p} \neq 2$. Define a pairing $\mathfrak{k} \times \mathfrak{k} \to \mathcal{O}/\mathfrak{p}$ by

$$\langle k_1 \mid k_2 \rangle = \text{tr }(k_1 k_2).$$

Proposition 5.1.3. *The form $\langle \cdot | \cdot \rangle$ defined on $\mathfrak{k} \times \mathfrak{k}$ on is non-degenerate.*

Proof. Let $k_1 = \begin{bmatrix} a & b \\ c & -a^t \end{bmatrix}$ be a non-zero element in $\mathfrak{k}$. Since $k_1 \neq 0$, at least one entry is non-zero.

Suppose $a \neq 0$. Let $r \in M_2(\mathcal{O}/\mathfrak{p})$ such that tr $(ar) \neq 0$. Set $k_2 = \begin{bmatrix} r & 0 \\ 0 & -r^t \end{bmatrix}$. Then $\langle k_1 \mid k_2 \rangle = 2\text{tr }(ar) \neq 0$.

Suppose $b = \begin{bmatrix} b_1 & b_2 \\ b_2 & b_4 \end{bmatrix}$ is non-zero. At least one entry is non-zero.

- If $b_1 \neq 0$, let $t = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $k_2 = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix}$. Since t is symmetric, $k_2 \in \mathfrak{k}$, and $\langle k_1 \mid k_2 \rangle = \text{tr }(bt) = b_1 \neq 0$.
- If $b_2 \neq 0$, let $t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so t is symmetric. Then $k_2 = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} \in \mathfrak{k}$ and $\langle k_1 \mid k_2 \rangle = 2b_2 \neq 0$.
- If $b_4 \neq 0$, take $t = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in M_2(\mathcal{O}/\mathfrak{p})_s$. Set $k_2 = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} \in \mathfrak{k}$. Then $\langle k_1 \mid k_2 \rangle = b_4 \neq 0$.

Suppose $c \neq 0$. This is similar to when $b \neq 0$, but now we find a symmetric

$s \in M_2(\mathcal{O}/\mathfrak{p})$ such that tr $(cs) \neq 0$. Then for $k_2 = \begin{bmatrix} 0 & s \\ 0 & 0 \end{bmatrix}$, $\langle k_1 \mid k_2 \rangle = \text{tr } (cs) \neq 0$. □

Lemma 5.1.4. *The map $\tau : \mathfrak{k} \to K$ defined by $k \mapsto 1 + \varpi k$ is an isomorphism.*

Proof. The proof of Lemma 4.1.4 (p. 17) holds here since it only used the properties of K and $\mathfrak{k}$. □

Then K is abelian. To apply Clifford's theorem, we need to define characters on K.

Once again, we fix a non-trivial character ψ of $\mathcal{O}/\mathfrak{p}$. For $k \in \mathfrak{k}$, define a character ψ_k on K by $\psi_k(1 + \varpi k_1) = \psi(\text{tr } kk_1)$, $k_1 \in \mathfrak{k}$.

Proposition 5.1.5. *The map $\Phi : \mathfrak{k} \to \widehat{K}$ defined by $k \mapsto \psi_k$ is an isomorphism.*

Proof. See proof of Proposition 4.1.5 on p. 17, making two modifications. First, $\mathfrak{k}$ is now a subset of $M_2(M_2(\mathcal{O}/\mathfrak{p}))$, but it is still a vector space over $\mathcal{O}/\mathfrak{p}$. Second, for the non-degeneracy of the $\mathfrak{k} \times \mathfrak{k} \to \mathcal{O}/\mathfrak{p}$ form, we must use Proposition 5.1.3 instead of Proposition 4.1.3. □

5.1.1 Clifford Theory

Up to this point, we have proved that K is a normal abelian subgroup of $G = Sp_4(\mathcal{O}/\mathfrak{p}^2)$ and have also given characters ψ_k of K for all $k \in \mathfrak{k}$. Recall the inertia group for ψ_k is

$$T(\psi_k) = \{g \in G \mid gkg^{-1} \equiv \psi_k \text{ mod } \mathfrak{p}\},$$

and K is a subgroup of $T(\psi_k)$. As before, the inertia groups are the stabilizers of the action of G on $\mathfrak{k}$, where $g \cdot k = gkg^{-1},\ g \in G,\ k \in \mathfrak{k}$. Note: For $k \in \mathfrak{k}$,

$$g(1 + \varpi k)g^{-1} = 1 + \varpi(gxg^{-1}) \in K \text{ if and only if } gxg^{-1} \in \mathfrak{k}.$$

What we would like to find next are representatives of each orbit.

We would like to proceed as before, but now the entries of $k \in \mathfrak{k}$ are elements in $M_2(\mathcal{O}/\mathfrak{p})$, a non-commutative ring instead of the commutative field $\mathcal{O}/\mathfrak{p}$. Hence, we will need to employ different techniques.

We would also like to find a way to extend characters on K to the inertia group. Recall the subgroup

$$S(k) = \{x \in G \mid xkx^{-1} = k\},$$

which in general is not abelian, but we still need to know its representations. We hope to show $T(\psi_k) = S(k)K$ for the class representatives so that representations of the inertia group are defined in terms of the representations of $S(k)$ and K, and finally use Clifford's theorem to induce up from the inertia groups to get our distinct, irreducible representations of $Sp_4(\mathcal{O}/\mathfrak{p}^2)$.

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