Contributions to geomagnetic theory

Joseph Emil Kasper
State University of Iowa

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CONTRIBUTIONS TO GEOMAGNETIC THEORY

by

Joseph Emil Kasper

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, in the Department of Physics in the Graduate College of the State University of Iowa

August, 1958

Chairman: Assistant Professor Ernest C. Ray
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I Introduction and Summary

This paper will describe two investigations into the theory of the effects of the earth's magnetic field on the cosmic radiation.

The first of these, described in Part II, concerned itself with the theory of the earth's simple shadow effect. This work was undertaken because of the unsatisfactory state of the subject previously. A detailed examination of the problem is given, adequate proofs are given for the chief theorems involved (desiderata previously lacking), and unproved assumptions are exhibited. Since, as is explained in the body of the paper, the computed simple shadow cones published earlier and used for decades by experimental workers in cosmic radiation are grossly in error, it was also considered very useful to recompute those cones. The newly computed cones are presented with some discussion of their significance and with an account of the manner in which they were calculated.

In the second investigation, described in Part III, there was constructed a simple new geomagnetic theory for the novel case in which the point of observation is above the earth's surface, and in fact may be at very great altitudes. Motivation for carrying out this work was supplied by the fact that we are currently in an era during which observations of the cosmic radiation are actually being made at such altitudes. An explanation of the theory is given; this has been made very brief in its parts dealing with the Störmer and main cones under the new conditions, but detailed discussions of the newly defined principal
shadow cones and penumbral region are given. Further, in order to render 
this theory practically usable, there are exhibited and described the 
results of very extensive computations of principal shadow cones and of 
other data.

The magnitude of an undertaking, the object of which is the pro-
duction of nearly complete numerical results for the case of observation 
points at general altitudes, is prohibitive and it is only by much 
judicious compromise that the numerical parts of this project were made 
manageable. Nevertheless the computed results are believed to be quite 
sufficiently complete and precise, and it is the author's hope that 
they will be found to be of utility in investigations of the natures of 
the earth's field and of the cosmic radiation.
II Contributions to the Theory of the Earth's Simple Shadow Effect

1. Introduction

There will be described here a new investigation of the earth's simple shadow effect on the cosmic radiation. A re-examination of this problem has long been called for by several considerations. One of these is that experimentalists have for many years believed the previously published simple shadow cones to be grossly in error because of their inherent unreasonableness. Another is that it can be shown easily without reference to any computed trajectories that at least parts of some of those cones are indeed incorrect quantitatively (vide infra). Further, it has recently been shown by numerical integration of the equations of motion that a trajectory which should be in an earth's simple shadow region according to the published cones, is not in such a region.  


Despite such considerations, the urgency of the need for new computations of the cones has not been great enough to have justified undertaking the large program of numerical work required for the integration of the equations of motion hundreds of times over and for the reduction of the integrations to actual simple shadow cones. However, in pursuing other and more elaborate goals, the author recently computed
some 2000 trajectories of charged particles in the field of a magnetic dipole with the use of an IBM Type 704 computer, some 500 of these trajectories having direct applicability to the problem of the shadow cones. This availability of the required trajectories, together with an awareness of the need for a re-computation of the cones, prompted performance of the work discussed in this paper.

There are given here a definition of the concept of the simple shadow cone, a discussion of some theoretical aspects of the subject, a set of newly computed cones, and a brief discussion of some of the significance of these cones for interpretation of cosmic ray observations.

Since even a partially self-contained paper on this subject would necessarily contain excessively lengthy preliminary explanations which are generally and readily available, there will be assumed familiarity with the older work on aspects of geomagnetic theory. Familiarity with the manner of treatment, and with the principal results of that treatment, of the problem of the reduced motion of a charged particle in the meridian plane of a magnetic dipole, as carried out by Lemaitre and Vallarta\(^2\), and by Schremp\(^3,4\) will be assumed. It will be assumed in

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particular that the contents of the second paper of 1938 by Schremp\(^4\) are known.
The mathematical symbolism used by the writers referred to will be employed without explanation where it is uniform among them; otherwise explicit new conventions will be adopted. All symbols not defined here are defined in the papers of the writers named $^2,^3,^4$. The symbol $\mathcal{V}$ will be used instead of the symbol $\mathcal{V}$, used in much of the older literature.

It will be assumed that the earth is spherical and its magnetic field will be taken to be that of a magnetic dipole located at the geometrical center of the earth. The dipole strength is incorporated in the Störmer unit of length and need not be specified. Latitudes ($\lambda$) referred to are geomagnetic latitudes defined with respect to the magnetic dipole axis, and are to be understood to be northern latitudes ($\lambda \geq 0^\circ$) throughout this paper, except where this is explicitly denied.

2. Definition of the Simple Shadow Cones

A. Pencils of Trajectories with Properties A

We begin by claiming that for at least some values $x_o$ and $\lambda_o$ of the variables $x$ and $\lambda$, and for some values of the parameter $\mathcal{V}$, the pencil of trajectories in the $x,\lambda$ plane issuing from the point $x_o, \lambda_o$ will have the following properties: (1) If the initial value $\eta_o$ of the angle $\eta$ at the point $x_o, \lambda_o$ is such that $90^\circ < \eta_o < 270^\circ$, the associated trajectory immediately upon issuance from the point lies in a region where $x < x_o$. (2) As $\eta_o$ decreases monotonically from $90^\circ$ a continuum of trajectories associated with that continuum of values of $\eta_o$ occurs, all
of which immediately upon issuance from the point $x_o$, $\lambda$ lie in a region where $x > x_o$ and all of which return after traversal of such a region to a region where $x < x_o$ with no points of inflection or extrema in $x$ or $\lambda$ in the region where $x > x_o$ except for a relative maximum in $x$ followed by a relative maximum in $\lambda$, or except for a relative maximum in $x$ only.

(3) The continuum of trajectories referred to in (2), above, is terminated when $\eta_o$ reaches a value $\eta_{SR}$ for which the associated trajectory issues into a region where $x > x_o$ and is a self-reversing trajectory with no points of inflection or extrema in $x$ or $\lambda$ between $x_o, \lambda$ and the self-reversal point at the contour $P = 0$ (upper branch). (4) As $\eta$ decreases from $\eta_{SR}$ monotonically, a continuum of associated trajectories occurs, all of which immediately upon issuance from $x_o, \lambda_o$ lie in a region where $x > x_o$ and which return after traversing such a region to a region where $x < x_o$ with no points of inflection or extrema in $x$ or $\lambda$ except for a relative maximum in $\lambda$ followed by a relative maximum in $x$. (5) The continuum of trajectories referred to in (4), above, is terminated when $\eta_o$ reaches a value $\eta_{NT}$ for which the associated trajectory differs from those in (4) only in that on returning to the value $x_o$ for $x$, it there has a point of tangency with the line $x = x_o$. Such a trajectory is of the second kind in the terminology of Schremp, as are certain others presently to be specified. The properties (1) through (5) will be referred to as properties A in the balance of this paper.

The existence of points with the properties A has not been shown with analytic rigor, nor can such a pencil with the required infinitely
large population be produced by computation. However, the claim can be regarded as a postulate, and arguments adduced to lend it a high degree of plausibility. Section II 3D of this paper may be consulted in this connection; in that section is proved a theorem very closely related to the question of the existence of points with properties A. In Figure 1 are shown some members of a pencil of trajectories for \( x_o = -0.08, \lambda_o = 10^\circ \), and \( \gamma = 0.92 \) produced by numerical integration of the equations of motion. Although neither the self-reversing trajectory referred to in (3) above nor the trajectory of the second kind referred to in (5) above were computed, it is apparent that those two special trajectories are approximately like those sketched in with dotted lines, and that the entire pencil has the properties A. One sees this by considering the form of the contour map over which the motions take place, the manner in which the x-minima for the trajectories move uniformly to the right as \( \gamma_o \) decreases from \( 90^\circ \), and the role played by the self-reversing trajectories in separating families which turn in a clockwise sense near the curve \( P = 0 \) from those which turn in a counterclockwise sense near there.

When such a point exists, then all trajectories issuing from the point \( x_o, \lambda_o \) with \( 270^\circ > \gamma_o > \eta^N \) are said to lie in the earth's simple shadow. The line \( x = x_o \) can be interpreted as representing the earth's surface in these coordinates, and the point \( x_o, \lambda_o \) as an observation point on the earth, while the reversibility of the motions in the \( x, \lambda \) plane with respect to the independent variable \( \sigma \) makes the trajectories open to interpretation as trajectories arriving at the point \( x_o, \lambda_o \), so
that those which lie in the earth's simple shadow correspond to paths for charged particles in the earth's magnetic dipole field which have to traverse part of the impenetrable earth in order to arrive at the observation point.

B. Pencils of Trajectories with Properties B

The further claim is made that at least for some values $x_o$, $\lambda_o$ of $x$, $\lambda$ and for some values of $\eta$, the pencil issuing from the point $x_o$, $\lambda_o$ will have the following characteristics: (1) If $\eta_o$ is such that $90^\circ < \eta_o < 270^\circ$ the associated trajectory immediately upon issuance from the point $x_o$, $\lambda_o$ lies in a region where $x < x_o$. (2) As $\eta_o$ increases monotonically from $270^\circ$ a continuum of trajectories associated with that continuum of values of $\eta_o$ occurs, all of which immediately upon issuance from the point $x_o$, $\lambda_o$ lie in a region where $x > x_o$ and all of which return after traversal of such a region to a region where $x < x_o$ with no points of inflection or extrema in $x$ or $\lambda$ in the region where $x > x_o$ except for a relative maximum in $x$. (3) The continuum of trajectories just referred to is terminated when $\eta_o$ reaches a value $\eta^S_T$ for which the associated trajectory differs from those in (2) only in that on returning to $x = x_o$, it there has a point of tangency with that line. Such a trajectory will also be called a trajectory of the second kind. Again, all trajectories from such a point with $\eta_o$ such that $90^\circ < \eta_o < \eta^S_T$ will be said to lie in the earth's simple shadow. The properties (1) through (3) will be called properties B.
The validity of the claim is supported by examination of computed trajectories, together with considerations of the effects of the contours $P = \text{constant}$ on the forms of the trajectories as $\eta_0$ varies, as was the validity of the first claim. An example of such a point is shown in Figure 2 for $x_o = -0.08, \lambda_o = 31.7^\circ, \gamma = 0.92$.

A special kind of point of the first type described above exists when the self-reversing trajectory referred to in (3) itself has a point of tangency with the line $x = x_o$ at the initial point $x_o, \lambda_o$ and is thus identical with the trajectory referred to in (5), so that $\eta_{SR} = \eta^N_T$ and the continuum referred to in (4) is empty of members. Another special kind of point exists when families with properties A and B exist simultaneously for one point. Examples will be omitted.

C. Relations of Pencils with Properties A and B to Features of Cone Diagrams

It is to be noticed that for any point for which $\eta^N_T$ exists, the corresponding values of $x_o, \lambda_o, \gamma$ determine a value of the quantity $\sin \theta$, the horizontal coordinate in a cone diagram (as a $\sin \theta, \cos \theta$ plot will be called) and that the value of $\eta^N_T$ there, together with the value of $\theta$, fixes a quantity $\cos \theta \times \sin \gamma$, the vertical coordinate in the cone diagram. In a similar way, $x_o, \lambda_o, \gamma$ and $\eta^S_T$, if the latter quantity exists at the point in question, determine a point on a cone diagram. All points in a cone diagram on a line $\sin \theta = \text{constant}$ above a point corresponding to $\eta^N_T$ or below a point corresponding
to $\eta_t^S$, or both if the two points exist simultaneously, correspond to trajectories lying in the earth's simple shadow. This claim is proved in section 3C below. The points themselves will also be said to be in the earth's simple shadow for simplicity of expression. Thus the line \( \sin \theta = \text{constant} \) may be divided into segments, some of which represent directions of arrival at the observation point on the earth which are not physically possible.

Figure 3 is a schematic representation of typical cases. In the figure, lines of constant \( \sin \theta \) are shown; these are the same as lines of constant \( \eta \). For \( \eta'(1) \) all points on the segment ab above a, for \( \eta'(2) \) all points on cd below c, and for \( \eta'(3) \) all on ef above e and on gh below g are in the earth's simple shadow, as are all those in the lower hemisphere \( (90^\circ < \eta_o < 270^\circ) \) for all values of \( \eta' \).

It should be remarked somewhat parenthetically here that it is not claimed that the trajectories corresponding to the points a, c, e, or g or to points below a, above c, or between e and g escape to infinite values of x in the \( x, \lambda \) plane without ever intersecting the line \( x = x_o \) after having issued from the point \( x_o, \lambda_o \). Thus a point such as a, for example, may have points neighboring it on both sides along the line \( \eta' = \eta'(1) \) which correspond to trajectories which arrive at \( x_o, \lambda_o \) only after traversing regions where \( x < x_o \).

Suppose that for some set of values of \( x_o, \lambda_o \), and \( \eta' \) a pencil having properties A exists. By virtue of the definition \( e^x = 2\pi \eta' \) the family can equivalently be said to exist for certain values of
Now let it be supposed that while \( r \) and \( \lambda \) are held constant at the values \( r_0 \) and \( \lambda_0 \), \( \gamma \) is allowed to vary from its initial value, which will now be called \( \gamma_0 \). It is apparent that one consequence of a minute variation of \( \gamma \) from the value \( \gamma_0 \) will be a minute variation of the topography of the \( x, \lambda \) plane — that is, the contour map formed by the lines \( P = \text{constant} \) will change its structure slightly. A second consequence will be that a new point \( x, \lambda \) will correspond to the new value of \( \gamma \) and the fixed values of \( r \) and \( \lambda \), and this new point will lie embedded in the new contour map in changed relation to its features.

At least for some \( x_0, \lambda_0, \gamma_0 \) there will exist pencils with the properties \( A \) for the new point and for all the points generated during the variation of \( \gamma \). Also, the value of \( N_T \) associated with the pencils will vary continuously during the variation of \( \gamma \). An adequate proof for these claims is the following: the trajectories in question do not pass through singular points, and therefore they will be continuous functions of the initial points and slopes and of the parameter \( \gamma \).

Entirely similar claims are made for the continuous variation of the angle \( N_T^S \) during variation of \( \gamma \) when at least some points are considered for which pencils having properties \( B \) exist, and at least for suitably small ranges of variation of \( \gamma \).

When variations of the kinds just discussed occur, there will be generated at least short continuous arcs on the faces of corresponding
cone diagrams due to the continuous variations of $\beta$ and of $\eta_T^N$ and $\eta_T^S$. Such arcs are indicated very schematically in Figure 3.

It is in fact an empirical result that there will occur the continuous generation of an entire single locus in any cone diagram as $\lambda$ varies continuously through its values across the face of the diagram, for given $r$ and $\lambda$, which begins at some point on the upper edge of the cone diagram, moves continuously in a clockwise sense around the diagram, and terminates at some point on the lower edge of the diagram. The loci for various values of $r$ and $\lambda$ constitute the simple shadow cones contemplated by Schremp. The generation of such loci will be discussed in a special way in the sequel. The approach to be adopted exhibits, it is believed, the manner of generation of the cones, the significance of special points on them (such as the upper initial point, the lower terminal point, and the maximum in the coordinate $\sin \beta$), and the limitations to be placed on the allowable values of $r$, $\lambda$, and $\beta$ with clarity, and makes a meaningful definition of the simple shadow cones possible.

D. Families of Trajectories with Properties C and Their Relation to Pencils with Properties A and B

We begin by discussing the properties of certain kinds of families of related trajectories. Let the segment of the line $x = x_0$ (for some value of $\lambda'$) be considered which lies in a region where the values of $P$ are allowed values for the motions. If, at each point on such a segment, a trajectory is initiated with $\eta_0 = 90^\circ$ then the family resulting
contains among its members all the trajectories of the second kind that can exist for those values of x and \( \dot{y} \). Computed members of two such families are shown in Figures 4 and 5 for widely different values of x and \( \dot{y} \). By reliable interpolations and extrapolations, one sees that in these families there occur in order with increasing \( \lambda \): (1) A continuum of trajectories all of which begin at points below the point at which there appears a self-reversing trajectory with a point of tangency to the line \( x = x_0 \). This self-reversing trajectory has no points of inflection or extrema in x or \( \lambda \) between the initial point and the self-reversal point. (2) The self-reversing trajectory just referred to. (3) A continuum of trajectories all of which are trajectories of the second kind. (4) A limiting trajectory for the preceding continuum which occurs at the point at which the line \( x = x_0 \) intersects the curve \( \frac{d^2x}{d\sigma^2} = 0 \), and which immediately upon issuing from its initial point turns to the left. (5) Above this last point a continuum of trajectories, all of which immediately turn left as did the one referred to in (4). (The region above the curve \( \frac{d^2x}{d\sigma^2} = 0 \) is a region in which the x-component of the acceleration is negative.) A family with these characteristics will be said to have properties C.

For any value of \( \dot{y} \) between 0 and 1 the curve \( \frac{d^2x}{d\sigma^2} = 0 \) is given analytically, and it is also known that there exists a continuous locus (which for values of \( \dot{y} \) near 1 may be of more than one branch) of points of first minima for self-reversing trajectories. The latter loci have been given by Schremp; any such locus will hereafter be referred to as an LMSR (for locus of minima of self-reversers). A
curve of the first kind will be called $F_x = 0$.

If a line $x = x_0$ is so located for a given $\gamma$ that it intersects both an LMSR and $F_x = 0$, the intersection with the LMSR falls at a smaller value of $\lambda$ than does the intersection with $F_x = 0$ because the LMSR must fall in a region where $F_x > 0$. It may of course happen for a given $\gamma$ that a line $x = x_0$ intersects only $F_x = 0$, or may intersect neither $F_x = 0$ nor the LMSR.

Now the claim is made that under the conditions to be specified immediately after this sentence the family of trajectories along $x = x_0$ with $\gamma_0 = 90^\circ$ will have properties C. The conditions are: (1) $0.1 \leq \gamma \leq 1.0$, (2) $x = x_0$ intersects both $F_x = 0$ and the LMSR (or $F_x = 0$ and the branch of the LMSR which lies nearest the dipole if the value of $\gamma$ is such that the LMSR has two branches), and (3) $x = x_0$ does not intersect the upper branch of $P = 0$ to the right of the $\lambda$-minimum of that branch of $P = 0$ in the $x, \lambda$ plane. (The latter restriction is made in order to avoid the supposed possibility that if the restriction is violated self-reversers may leave their self-reversal points on $P = 0$ and reach tangencies at $x = x_0$, but fail to have the simple forms required here.) The author has in his possession some 60 such families for values of $\gamma$ from 0.1 to 1.0 and for widely distributed values of $x_0$ (from $-0.0$ to $+0.1$). Examination of these families together with consideration of the contour maps for the regions of the $x, \lambda$ planes involved show that the truth of the claim is overwhelmingly likely. Since such evidence cannot be adequately presented here, the claim is simply postulated.
Some properties of the trajectories in the families being discussed, which have been deduced from inspection of computed families, will be of importance later, and will be enumerated here. (1) Along the segment of the line $x = x_o$ between the LMSR and $F_x = 0$, there will be at each point one and only one trajectory of the second kind starting there with $\eta_o \neq 90^\circ$. (2) The segment between $F_x = 0$ and $P = 0$ can be divided into two parts, the lower of which is such that for each point on it there will be two and only two trajectories of the second kind, and the upper of which is such that for points on it there will be no trajectories of the second kind at all. (3) There is in each family a trajectory of the second kind which begins along the line $x = x_o$ at a point at which the value of $\lambda$ is greater than it is for any other point along $x = x_o$ at which a trajectory of the second kind exists. This happens, of course, at the limiting point of the segment containing points, each of which has two trajectories of the second kind issuing from it. (4) In the neighborhood below such a point along the line $x = x_o$ the two trajectories of the second kind issuing jointly from any point in the neighborhood have values of $\eta_o$ not greatly different, and as the points approach the northernmost point approach equality and attain it at that northernmost point.

Let it be understood now that when a trajectory of the second kind is mentioned, it is meant to be taken as issuing from the northernmost of its two points for which $x = x_o$. This will always mean that the trajectory issues from the earth with $\eta_o \neq 90^\circ$ except in the case of a trajectory of the second kind which is also a self-reversing trajectory;
in such a special case $\eta_o = 90^\circ$. It is now claimed that whenever a family of trajectories having properties C exists each trajectory of the second kind in the family terminates a pencil of trajectories which intersect the earth in the manner of one of those described in the fifth of the properties A or the third of the properties B, as may be appropriate. That is, for each such trajectory $\eta_o = \eta^N_T$ and all trajectories in its pencil with $270^\circ > \eta_o > \eta^N_T$ lie in the earth's simple shadow, or else $\eta_o = \eta^S_T$ and all trajectories in its pencil with $90^\circ < \eta_o < \eta^S_T$ lie in the earth's simple shadow. This theorem will be proved in section 3 of this paper.

E. Generation of Simple Shadow Cones

Now we begin an examination of the generation of arcs in a cone diagram as $\gamma$ varies across the face of the diagram for fixed values of $r$ and $\lambda$. This will be done at the outset for a more or less specific case; the mode of generation to be described is illustrated in Figure 6, which is somewhat schematic but nevertheless represents fairly accurately the case for any value of $r$ from about 0.1 to 0.3 and for $\lambda \approx 0.6$ radians.

Since at the left edge of a cone diagram $\sin \theta = -1$ and since $\sin \theta$ is an analytically given function of $r$, $\lambda$, and $\gamma$, the value of $\gamma$ at the left edge is fixed. As $\gamma$ increases from this value, the lines of $\sin \theta = \text{constant}$ are generated across the face of the diagram until that value of $\gamma$ is encountered for which $\sin \theta = 1$, at the right edge.
of the diagram. For each value of $\gamma$, if $r$ and $\lambda$ are held fixed, there will be a corresponding point in an $x, \lambda$ plane whose position with respect to the contours $P = \text{constant}$ will depend on the value of $\gamma$, as will the structure of the contour map itself.

Reference should now be made to Figure 6. At the left edge of the cone diagram the point in question lies on the inner branch of the curve $P = 0$, as in (a). As $\gamma$ increases the point moves along the line $\lambda = \text{constant}$, and for every point $x, \lambda$ generated a family with properties $C$ exists along the vertical line through $x$. The point first traverses a region such that the point is always below an LMSR as in (b), where no trajectories of the second kind exist, and then moves in a region such that the point is always between an LMSR and a curve $F_x = 0$, as in (c), having somewhere between crossed an LMSR. At that crossing occurs the first encounter by the point with a trajectory of the second kind and there the value of $\frac{N}{T}$ is $90^\circ$. In (c) there is one trajectory of the second kind per point and a family with the properties $A$ with which that trajectory is associated as a limiting trajectory. The assumptions of continuity previously discussed are applied here and with what has just been said show that an arc starting on the upper edge of the diagram is generated continuously in this region. Next the point crosses the curve $F_x = 0$ and then traverses a region in which two trajectories of the second kind exist per point, as in (d). In this region there is generated a continuation of the arc already begun and a second arc in addition. This second arc begins at the
lower edge of the cone diagram when the point $x, \lambda$ crosses $F_x = 0$. This is true because for points after $F_x = 0$, but infinitesimally near that curve, the newly acquired second trajectories of the second kind have \( S_T \) infinitesimally near 270°, and these angles increase as $F_x = 0$ is left farther behind. As the point progresses through the region in which two arcs are being generated, the arcs approach each other and join smoothly together. After this a region is encountered in which no trajectories of the second exist at the points $x, \lambda$ as in (e). (The reasons for the joining of the arcs and the subsequent disappearance of trajectories of the second kind will be clear because of properties of families of trajectories of the second kind previously stated.) Finally the other branch of the curve $F_x = 0$ is met as in (f), and the cone diagram has then been swept over completely.

On the basis of what has preceded this analysis it is clear that the trajectories corresponding to all points above the northern arc of the curve generated on the cone diagram are in the earth's simple shadow as are all those corresponding to points below the southern arc of the curve. The points to the right of the maximum reach of the curve in $\sin \theta$ do not correspond to physically allowable trajectories. That this is so can be proven: the proof is given in section 36. In crude language, the $x, \lambda$ points corresponding to such cone diagram points are nestled so closely against the upper branch of $P = 0$ where its slope is negative that all trajectories leaving those points to the right reflect back to the left sufficiently to intersect the earth.
F. Definition of a Simple Shadow Cone

It will be noticed that in the case illustrated and described two salient features are present: (1) The generated points \( x, \lambda \) traverse in order (as \( \gamma \) increases) the inner \( P = 0 \), a region below an LMSR, an LMSR, a region between an LMSR and a curve \( F_x = 0 \), a curve \( F_x = 0 \), a region between a curve \( F_x = 0 \) and the upper \( P = 0 \), and finally the upper \( P = 0 \). (2) Every generated point \( x, \lambda \) is such that a family with properties C exists along the vertical line through it. (As a consequence of a claim made in section D, above, every trajectory of the second kind which exists at any generated point \( x, \lambda \) is a limiting trajectory for a pencil with properties A, or with properties B.) Whenever the values of \( r \) and \( \lambda \) are such these two features are properties of the generated \( x, \) points as \( \gamma \) varies across the face of the corresponding cone diagram the curve developed in the cone diagram will be called a simple shadow cone.

For some values of \( r \) and \( \lambda \) it may happen that the generation of a curve in a cone diagram begins in the manner in which it begins when the curve is a simple shadow cone as just defined, but that the value 1 for the parameter \( \gamma \) occurs before generation of a complete curve is accomplished. In any such case the segment (or segments) up to the line \( \gamma = 1 \) will also be called a simple shadow cone. All trajectories corresponding to points above the northern reach of such a cone or below the southern reach, if the latter exists, will be in the earth's simple shadow. Trajectories corresponding to points on or to the right of the
line \( y = 1 \) will be said to be Störmer dark. For some values of \( r \) and \( \lambda \) generation of a cone is not begun before \( y \) reaches the value 1, and for others the value of \( y \) at the left edge of the cone diagram is equal to or greater than 1. In either of these cases it will be considered that no simple shadow cones exist, and it will be said that they are prevented from existing by the Störmer condition. Any trajectory for which \( y \geq 1 \) and which has its initial point inside the closed region of the \( x, \lambda \) plane where motions can occur is not (when reversed) a physically allowable trajectory for cosmic ray particles coming from the outer, open region, as is well known.

G. Restrictions on the Variables for Validity

of the Definition

The values of \( r \) and \( \lambda \) for which a simple shadow cone can exist must yet be specified. To decide whether it is possible for a simple shadow cone to exist for a given \( r \) and \( \lambda \), one first calculates for those values of \( r \) and \( \lambda \) the range through which \( y \) must vary in moving across the face of the cone diagram, and traces the course of the gener-ates \( x, \lambda \) points. One then observes whether the sequence of traversals of special curves and regions required by the definition occur or not, and whether families with properties C exist for each generated \( x \).

The author has thus determined that for all latitudes such that \( 30^\circ \leq \lambda \leq 60^\circ \) for all values of \( r \) not greater than 0.6, and for all latitudes such that \( 60^\circ < \lambda \leq 80^\circ \) for all values of \( r \) not greater than
simple shadow cones exist, provided they are not prevented from existing by the Størmer condition.

These ranges of \( r \) and \( \lambda \) can be extended, but no practical gain is to be effected by their extension because of the extremely small sizes of cones for larger values of \( r \), and the negligible practical importance of the cones for smaller values of \( \lambda \) (because the penumbral region tends toward complete darkness for low latitudes); the computed cones given later in this paper may be consulted in this connection. However, it is of importance to observe that while those ranges can be extended, they cannot be extended very widely. One limitation effective for a given value of \( \lambda \) is that when \( r^2 > \cos \lambda \) there can be no simple shadow cone for such \( r \) at that \( \lambda \), as has been pointed out by Schremp. A second limitation that must be mentioned is the following: As long as \( r \) is small enough or \( \lambda \) large enough, the points \( x, \lambda \) generated during variation of \( \psi \) will lie well up in the "neck" between the branches of the curve \( P = 0 \) where the vertical line through \( x \) must intersect both branches of \( P = 0 \), an LMSR, and \( F_x = 0 \), but for larger values of \( r \) and for latitudes near the equator this is no longer so because for some locations of the points \( x, \lambda \) the vertical lines through such points may fail to intersect an LMSR, and even may lie entirely in a region where \( F_x < 0 \).

The above consideration leads directly to the claim that some of the cones shown by Schremp are either not simple shadow cones (generated by trajectories of the second kind), or else are numerically incorrect.
simple shadow cones. It is easy to show by elementary calculations that at a value of roughly 0.86 for $\gamma$ the $F_x < 0$ region of the $x, \lambda$ plane touches the equator at a small positive value of $x$. For larger values of $\gamma$ (but values less than 1) such regions cover extended segments of the equator, so that for any value of $x$ along such segments, vertical lines lie entirely in regions where $F_x < 0$, between branches of the curve $P = 0$. Whenever a set of values for $r, \lambda$ and $\gamma$ is considered which corresponds to a point on such a line in such a region it is obvious that no trajectories of the second kind exist along that line. Now measurements on Schremp's figures and easy computations show that in some cases all points on parts of his cones are such that the values of $r, \lambda$ and $\gamma$ associated with those parts require that the corresponding points in the $x, \lambda$ planes fall in regions of the kinds just mentioned. This is in contradiction with the claim that the points correspond to trajectories of the second kind. One example of such a case is the most easterly arc of Schremp's cone for $r = 0.8, \lambda = 10^0$, as can be readily verified by the reader.

3. Proofs for Theorems Used

A. Mathematical Preliminaries

The central theorem of the theory of the simple shadow cones can be formulated as follows: In a cone diagram, if a single point on any line $\gamma = \text{constant}$ corresponds to a trajectory of the second kind, all trajectories corresponding to points on that line above the point just
referred to are forbidden trajectories; if two points on any line
\( \gamma = \text{constant} \) correspond to trajectories of the second kind, all tra-
jectories corresponding to points on that line above the northernmost
of the two points and below the southernmost, are forbidden trajectories.

A proof will now be given for this theorem. The method used was
suggested by that employed by Ray in proving the central theorem of the

6. E. C. Ray, private communication. (To be published.)

theory of the main cone. It does not proceed entirely from first prin-
ciples, but requires the granting of the existence of families with
properties C.

The statement of the proof, while itself concise, requires the
prior discussion of some geometrical matters. In an attempt to illumи-
nate this lengthy discussion, Figure 13 is supplied; this is meant to
show the surfaces, points and curves which play roles in the development
of the argument. Figures 4 and 5 may also be helpful.

The equations of motion in the meridian plane can be written in
the form

\[
\frac{dx'}{d\sigma} = \frac{e^{2\lambda}}{16 \sigma^4} - e^{-\lambda} + e^{-2\lambda} \cos^2 \lambda
\]

(1)

\[
\frac{dx}{d\sigma} = \chi'
\]

(2)

\[
\frac{d\lambda}{d\sigma} = e^{-2\lambda} \sin \lambda \cos \lambda - \sin \lambda \cos^{-3} \lambda
\]

(3)

\[
\frac{d\lambda'}{d\sigma} = \chi'
\]

(4)
and by use of the energy integral, these may be reduced to the set

\[
\frac{dx'}{d\sigma} = \pm \left[ \frac{e^{2x}}{16y^2} + 2e^{-x} - e^{-2x} \cos^2 \lambda - \cos^{-2} \lambda - (\lambda')^2 \right]^{\frac{1}{2}} \tag{5}
\]

\[
\frac{d\lambda'}{d\sigma} = e^{-2x} \sin \lambda \cos \lambda - \sin \lambda \cos^{-2} \lambda \tag{6}
\]

\[
\frac{d\lambda}{d\sigma} = \lambda'. \tag{7}
\]

In the space \((x, \lambda, x', \lambda')\) for a given value of \(\lambda\) there passes through each point one and only one solution for the equations (1) through (4) since at all points the right members of the equations have bounded partial derivatives, so that at all points a Lipschitz condition holds, except that this is not true at \(\lambda = \pi/2\) and \(x = -\infty\).

7. Ince, loc. cit.

The energy integral can be written

\[
(x')^2 = \frac{e^{2x}}{16y^4} + 2e^{-x} - e^{-2x} \cos^2 \lambda - \cos^{-2} \lambda - (\lambda')^2 \tag{8}
\]

and the following relation must be satisfied, obviously

\[
\frac{e^{2x}}{16y^4} + 2e^{-x} - e^{-2x} \cos^2 \lambda - \cos^{-2} \lambda - (\lambda')^2 \geq 0. \tag{9}
\]

The surface obtained by using the equality in (9) will be called the motion boundary.

One sees that for any point \(x, \lambda, \lambda'\) which lies on a trajectory there are determined by (8) two values of \(x'\) which differ only in sign. The motion described by equations (5), (6), and (7) takes place in the space \((x, \lambda, \lambda')\), and through each point in this space there pass
exactly two trajectories, which differ only in the sign of $x'$. In this space the motion boundary is a "cylinder" with the line $\lambda = \lambda' = 0$ as axis, and a trajectory can have $x' = 0$ at a point on itself only if that point is a point at which the trajectory is tangent to the motion boundary.

**B. Geometrical Description**

Consider a family of trajectories in the $x, \lambda$ plane with properties C. Each trajectory of the second kind in this family is to be considered as starting from the northernmost of the two points on it for which $x = x_0$; the self-reversing one is to start with $\lambda' > 0$. These trajectories of the second kind form a surface in the space $(x, \lambda, \lambda')$ because of the continuous dependence of the solutions of the equations on the initial conditions.

The surface just mentioned will be described now. That the description is a true one follows from the nature of the properties of the trajectories. The surface, henceforth to be called the shadow surface for brevity, is completely bounded by a continuous curve in the $(x, \lambda, \lambda')$ space. This curve, to be called simply the shadow surface bounding curve, consists of two segments, one of which lies in the plane $x = x_0$ and the other of which lies entirely in the region where $x > x_0$. The segment in the plane $x = x_0$ begins on the motion boundary at a point $P_1$ where $\lambda$ has the special value $\lambda_{p_1}$ and where $\lambda' > 0$. $\lambda_{p_1}$ is the value of $\lambda$ at the point $x, \lambda$ in the meridian plane at which
the self-reversing trajectory of the second kind has a point of tangency with the earth. The segment being discussed terminates on the motion boundary at the point where \( \lambda \) again has the same value \( \lambda_{P_1} \) and \( \lambda' \) differs from its value at \( P_1 \) in sign only. This point corresponds to the return of the self-reversing trajectory to the earth, and will be called \( P_2 \).

Between \( P_1 \) and \( P_2 \) this segment of the shadow surface bounding curve is at first the locus of the points \( x, \lambda, \lambda' \) \((x = x_0)\) given by the trajectories of the second kind in the family in the \( x, \lambda \) plane at the points at which they leave the earth, up to a point \( P_3 \) between \( P_1 \) and \( P_2 \), at which point the locus touches the motion boundary. From \( P_3 \) to \( P_2 \) the segment is the intersection of the motion boundary with the plane \( x = x_0 \). \( P_3 \) occurs at that value of \( \lambda \) at which the curve \( F_{x} = 0 \) intersects the line \( x = x_0 \) in the \( x, \lambda \) plane. The locus proceeds from its beginning at \( P_1 \) to a point where \( \lambda \) has a maximum (at which point \( \lambda' < 0 \)). For values of \( \lambda \) between this maximum and that at the point \( P_3 \) the locus is double-valued with respect to \( \lambda' \); this corresponds to the occurrence of two trajectories of the second kind for points in a certain range of \( \lambda \) along the line \( x = x_0 \) in the \( x, \lambda \) plane. The part of the curve in the plane \( x = x_0 \) which lies entirely in the motion boundary (between \( P_3 \) and \( P_2 \)) is the locus of the points of tangency with the earth of the trajectories of the second kind in the family.

The other segment of the shadow surface bounding curve is itself the self-reversing trajectory of the second kind in the family, as this
trajectory is realized in the space \((x, \lambda, \lambda')\). It proceeds from \(P_1\) into the region where \(x > x_0\) with \(\lambda' > 0\) until it reaches a point, to be called \(P_4\), on the motion boundary at \(\lambda' = 0\). This point corresponds to the attainment of the value zero by the function \(P\) on the self-reversing trajectory of the second kind. From \(P_4\) to \(P_2\) this segment has \(x > x_0\) and \(\lambda' < 0\) for all points on the segment.

The shadow surface nowhere intersects itself, nor does it touch the motion boundary except at the point \(P_1\) and along its bounding curve between \(P_3\) and \(P_2\), and also except along a certain continuous curve reaching from \(P_3\) to \(P_4\). This latter curve from \(P_3\) to \(P_4\) runs in the motion boundary with \(\lambda' < 0\) everywhere on it, and corresponds to the locus of the maxima in \(x\) of the trajectories of the second kind in the family. It should be particularly noticed that \(x' \geq 0\) on the part of the shadow surface bounded by the segments of the shadow surface bounding curve between \(P_3\) and \(P_2\), \(P_1\) and \(P_4\), and by the curve from \(P_3\) to \(P_4\) just mentioned. On the other part of the shadow surface, \(x' \leq 0\).

The part of the plane \(x = x_0\) which will be of interest is bounded by the intersection of this plane with the motion boundary and by the line \(\lambda = \lambda_{P_1}\) in the plane. This limited part of the plane \(x = x_0\) will be called \(R\) for convenience.

\(R\) is divided into two regions by the part of the shadow surface bounding curve which lies in the plane \(x = x_0\). The regions, to be called \(I\) and \(II\), are distinguishable by the specification that \(II\) is that one of the two which is bounded in part by the line \(\lambda = \lambda_{P_1}\) in
the plane \( x = x_0 \).

Now let an auxiliary surface be generated by the motion of a line which is orthogonal to the plane \( \lambda^* = 0 \), the motion to take place so that the surface generated contains the self-reversing trajectory of the second kind as it appears in the three-space. The part of this surface which is bounded by its intersection with the motion boundary and its intersection with the plane \( x = x_0 \) (which intersection is the line \( \lambda = \lambda_{P_1} \)) will be called \( S \).

\( S \) is divided into three regions. The first, to be called \( A \), is that one which is bounded by the intersection of \( S \) with the motion boundary and that part of the shadow surface bounding curve which lies between \( P_1 \) and \( P_4 \). The second, to be called \( B \), is bounded by the intersection of \( S \) with the motion boundary and the part of the shadow surface bounding curve which lies between \( P_4 \) and \( P_2 \). The third, to be called \( C \), is bounded by the part of the shadow surface bounding curve which runs from \( P_1 \) to \( P_4 \) to \( P_2 \), and by the line \( \lambda = \lambda_{P_2} \) in the plane \( x = x_0 \). Clearly \( A \), \( B \), and \( C \) exhaust \( S \).

\( C. \) Proof of the Central Theorem of the Theory of the Simple Shadow Cone and of Another Theorem

The theorem to be proved can be restated in the new terminology as follows: No trajectory with its initial point in \( I \) and with its initial value of \( x' \) greater than zero can cross \( S \) unless it first crosses the plane \( x = x_0 \). The proof follows.
It will first be shown that a trajectory starting in I with \( x' > 0 \) cannot cross the surface \( S \) in region B without first crossing \( S \) in A or C, or the plane \( x = x_0 \). In order to do so, it would obviously first have to cross the part of the shadow surface on which \( x' > 0 \), and this would require that it first change its sign in \( x' \). After this latter crossing it would lie in the volume between the shadow surface, the plane \( x = x_0 \), and region C of surface \( S \). It would then have \( x' < 0 \) and it would then be required to cross the shadow surface where \( x' < 0 \). But this is impossible because the trajectory would first have to change its sign in \( x' \) again, and no allowed points of contact with the motion boundary exist for it except in region I, which is no help.

Next, a trajectory starting in I with \( x' > 0 \) also cannot cross the surface \( S \) in region C without first having crossed \( S \) in A, or the plane \( x = x_0 \). To require it to do so is to require it first to cross the shadow surface in its part where \( x' > 0 \), and as seen above, the trajectory would then find itself in the volume described above, with negative \( x' \). At the point of crossing of C by the trajectory at some point \( x, \lambda, \lambda' \) the value of \( \lambda' \) would be intermediate between those of \( \lambda' \) on the self-reversing trajectory at the points where \( x \) and \( \lambda \) have the same values as at the point of crossing. However, a trajectory crossing \( S \) with \( x' < 0 \) must in fact have \( \lambda' \) less than the negative value of \( \lambda' \) on the self-reversing trajectory at the point with the same values of \( x \) and \( \lambda \).
The only possibility left is that the first crossing of S by the trajectory starting in I with \( x' > 0 \) occurs in A, but at such a crossing the trajectory would have a positive value of \( \lambda' \) greater than that on the self-reversing trajectory at the point at which \( x \) and \( \lambda \) have the same values as at the crossing point, whereas in fact any trajectory crossing S must at the crossing point have its value of \( \lambda' \) less than the value on the self-reverser at the point just referred to. This completes the proof.

Earlier in this paper a proof was promised for the following theorem: In a cone diagram trajectories corresponding to points lying in the diagram to the east of the extreme reach of a simple shadow cone in that diagram (but not to the east of a Störmer cut-off) are physically forbidden trajectories. It is obvious that the proof for this theorem has been incorporated into the above proof, since such trajectories are among those starting in region I with \( x' > 0 \).

D. Proof of a Secondary Theorem

The general method just made use of makes possible simple treatment of another theorem of interest earlier in this paper (in section II 2A). Again the properties of a family of trajectories will be assumed, the surface in the space \((x, \lambda, \lambda')\) generated by those trajectories will be described, and the proof will then follow directly.

The theorem can be stated as follows: Consider a self-reversing trajectory in the \( x, \lambda \) plane with self-reversal point on the northern-
most branch of the curve \( P = 0 \) where the slope of that curve is negative. For each such trajectory there will be a segment of interest lying between the self-reversal point and the first minimum in \( x \) occurring on the trajectory after the self-reversal. Consider any point on such a segment of such a self-reversing trajectory, and let it be understood that at that point \( \lambda' > 0 \) — that is, the segment is to be traversed northward. Now if the value of \( \eta \) for the self-reversing trajectory at that point be allowed to increase or decrease, the trajectories corresponding to the values of \( \eta \) generated will be such that initially and for some range of \( \eta \) when this angle increases the trajectories will turn to the left away from the self-reverser, and initially and for some range of \( \eta \) when this angle decreases the trajectories will turn to the right away from the self-reverser. The phrase "turn to the left" is here equivalent to the phrase "attain a maximum in \( x \) and then a maximum in \( \lambda \)" and the phrase "turn to the right" to the phrase "attain a maximum in \( \lambda \) before attaining an extremum in \( x \)".

The family of trajectories required for proof of the theorem is the family of a self-reversing trajectories with self-reversal points along the northernmost branch of the curve \( P = 0 \) where the slope of that curve is negative. Each of these trajectories is of the following form: After leaving its point of self-reversal, each travels with \( \lambda' < 0 \) and \( x' < 0 \) and with constant sign of curvature until a minimum in \( x \) occurs along it.

The minima in \( x \) of the members of the family form a continuous
locus because the solutions of the equations of motion depend continuously on the initial conditions. It is known from the work of Schremp

8. E. J. Schremp, Phys. Rev. 54, 158 (1938), Fig. 4.

that the locus of the minima of the self-reversers in the region of the $x, \lambda$ plane being considered here is a curve of simple form running down the arm of the $P>0$ region very roughly parallel to the curve $\theta = 0$ in that region.

The segments of the trajectories in the family between the LMSR and the curve $P = 0$ are now to be considered as being traversed from their points on the LMSR to $P = 0$ and then back to the LMSR. These segments in their totality form in the $(x, \lambda, \lambda')$ space a surface, which is a sheet of simple form. This surface touches the motion boundary along the intersection of the plane $\lambda' = 0$ with the motion boundary and along the curve on the motion boundary which is in this space the realization of the LMSR, and touches the motion boundary nowhere else. For every point on it $\lambda' > 0$ (except at its intersection with the motion boundary).

Now if in the $x, \lambda$ plane a trajectory be started on a self-reverser belonging to the family, but with $\lambda'$ greater than the value of $\lambda'$ on the self-reverser at the point in question, then in the $(x, \lambda, \lambda')$ space this trajectory starts in the volume bounded by the motion boundary and the surface just described above. There $\lambda' > 0$. It also has by assumption $x' > 0$ initially. If it is to turn to the
right in the sense in which that phrase is to be understood now, it must pass through a point at which it has \( \lambda' = 0 \) before it attains an extremum in \( x \). The trajectory can reach \( \lambda' = 0 \) only by crossing the surface, and for this to happen, the trajectory must first change its sign of \( x' \). The trajectory thus cannot have an extremum in \( \lambda \) before having one in \( x \).

If a trajectory be started as was the last one considered, with \( x' > 0 \), but with \( \lambda' \) less than the value of \( \lambda' \) on the self-reverser, then clearly this trajectory can turn right. It remains to show that it cannot turn left. Initially the trajectory is in the volume bounded by the surface formed by the self-reversers and the part of the motion boundary other than the one which bounded the volume referred to in the last paragraph. It there has \( x' > 0 \) by assumption; it also has \( \lambda' > 0 \). If this trajectory is to turn left, as that phrase is understood here, it must attain a maximum in \( x \) before a maximum in \( \lambda \). It cannot cross the surface formed by the self-reversers before changing sign in \( x' \), nor can it first change its sign in \( x' \) on the motion boundary where \( \lambda' < 0 \), for this would require occurrence of an extremum in \( \lambda \) before one in \( x \). Thus the required change of sign in \( x' \) which is to occur before an extremum in \( \lambda \) can occur only on the motion boundary where \( \lambda' > 0 \), and where \( \lambda \) is less than the value of \( \lambda \) at the initial point for the trajectory. To reach the boundary there, the trajectory would first have to cross the plane \( \lambda' = 0 \), which is to say, turn right.
A. Computation of Trajectories

As Schremp has pointed out, the calculation of simple shadow cones can be carried out if one computes sufficiently extensively families of trajectories of the second kind like those shown in Figures 4 and 5. The method used by the author depends on integration of the equations of motion to produce such families, but differs from the method of Schremp in by-passing the $\lambda, \eta$ interpolations mentioned by him as having been used to reduce data taken from his trajectories to points on cone diagrams.

All trajectories used were computed with the use of IBM Type 704 computers. Of the approximately 550 trajectories of the second kind computed, about 500 were computed personally by the author with a program written by him and used on the computer at the General Motors Technical Center, Detroit, Michigan. The rest were computed for him by personnel of Martin Aircraft Company, Baltimore, Maryland, at the request of RIAS, Inc. of that city, with an independently prepared program. In addition, some 100 other trajectories were used in the associated investigations.

The method used by the author for the numerical integrations was Gill's modification of the Kutta-Runge method. The precision of the trajectories was checked by several investigations: (1) Machine produced
results were compared with known trajectories, such as those given by Störmer\textsuperscript{10}. (2) Trajectories of great lengths in \( \mathcal{F} \) and of convoluted

\begin{quote}

forms were run forwards and backwards and the results compared; this constitutes a severe test. (3) Trajectories were computed independently by the author and the Martin group and compared. (i) Trajectories were computed several times over, with different step sizes in the independent variable \( \mathcal{F} \). It was concluded that the machine produced values of \( x, \lambda \) and \( \eta \) along the trajectories of the second kind, all of which were short and had simple forms, could be considered to be precise to at least five digits. This precision is very adequate for application of the trajectories to the computation of the cones.

Each family of trajectories of the second kind was computed by picking values of \( r \) and \( \mathcal{F} \) (and hence of \( x \)) and starting the integrations of the members of the family at points distributed between the IMSR and the curve \( F_x = 0 \) along the vertical line through \( x \). The values of \( r \) used were 0.1, 0.2, 0.3, 0.4, and 0.6. For each value of \( r \) the values of \( \mathcal{F} \) used were from 0.1 to 1.0 in steps of 0.1, except that the value of 1.0 was not used for \( r = 0.6 \). An average of about 10 members per family was computed.

B. Interpolations Performed on Trajectories

The trajectories calculated with the General Motors computer were
produced by a program written with an object in mind other than the compu-
tation of simple shadow cones, and no provision was made in the pro-
gram for printing out values of $x$, $\lambda$ and $\gamma$ precisely at the points on
the trajectories at which they return to the earth after starting out
with $\eta_0 = 90^\circ$. Instead, values of those variables were printed out
after each increment in the independent variable $\sigma$. The increment in
$\sigma$ was constant during each trajectory computation; for about 40 of the
trajectories it was $0.0031666666$, for about 120 it was $0.0063333333$,
and for the rest, 0.05. (The smaller step sizes were used when large
negative values of $x$ were involved.) As a result, print-outs occurred
on each trajectory extremely near the line $x = x_0$ when the trajectories
returned to the earth, and linear interpolations were done between
printed points bracketing the line $x = x_0$ in order to find the values
of $\lambda$ and $\gamma$ at $x = x_0$. The points at which $x$, $\lambda$ and $\gamma$ were printed
out near the earth fell at $x$-values differing from $x = x_0$ by roughly
0.00005 or less in about half the cases, by about 0.00005 to 0.0005 in
about two thirds of the remaining cases, and by larger amounts up to a
maximum of about 0.001 in the rest of the cases. Thus the interpo-
lations were done over small arcs of the trajectories. The interpolative
procedure was subjected to an analysis in which estimates were made with
liberality of the maximum possible errors that could be incurred in the
angles $\lambda$ and $\gamma$. These estimates were checked in two ways: (1) Tra-
jectories were computed with $\Delta\sigma = 0.05$ and with much smaller step
sizes, and the interpolations made on the various runs of single
trajectories were compared, and (2) some trajectories computed by the
author were recomputed by the Martin group, whose program provided for
print-out of values of the angles at the earth. It was concluded that
the maximum possible errors in \( \lambda \) (in radians) were less than 0.0001 in
nearly all cases. The maximum possible errors in \( \eta \) were less than 0.01
in all but about 20 cases. In those 20 cases (in which the trajectories
were turning rapidly near the earth and so varying rapidly in \( \eta \)) the
data were not used when they seemed inconsistent with the other data.

The question of importance in connection with the possible errors
in the interpolated angles \( \lambda \) and \( \eta \) is that of their effects on the
coordinates \( \sin \theta \) and \( \cos \theta \times \sin \eta \). The effects depend on the sizes
of the angles as well as on the sizes of the errors because of the trigonometric functions involved, and because the errors in \( \eta \) were smallest
at large latitudes. The conclusions are summarized by the assertion
that the values of the coordinates for all points computed are believed
to be correct to within \( \pm 0.01 \).

C. Reduction of Trajectory Data to Simple Shadow Cones

One way to utilize the \( r, \lambda, \eta \) data taken from the trajectories
is that described by Schrempl, which involves interpolations over the
\( \lambda, \eta \) values for each family of trajectories of the second kind. This
method was tried with exercise of care and it was found by making fair
and reasonable estimates of the errors introduced that the procedure
was essentially unproductive of simple shadow cones. Errors in the
coordinates of points on the resulting cones as large as 0.1 were estimated to be possible.

For this reason a different method was used for much more direct determination of the cones. It was observed that a consequence of the choices of the values of r and \( \gamma \) made was that points \( \sin \theta, \cos \theta \cdot \sin \eta \) (computed by use of the values of \( r, \lambda, \eta, \gamma \) taken from the trajectories) resulted which generally could be divided into subgroups for each group pertaining to one value of \( r \), such that the points in each subgroup pertained to latitudes nearly equal. This made it easy to draw in segments of simple shadow cones for any latitude included in subgroups with sufficient precision by plotting the \( \sin \theta, \cos \theta \cdot \sin \eta \) points on cone diagrams and letting the pencil be guided by them.

For a given \( r \) this process fixed segments of cones of various lengths lying along different parts of the rims of the cone diagrams. When all those segments were plotted together in one diagram, the interrelations between them made it possible often to extend the lengths of segments with good reliability, and also made a consistency check possible.

When all the simple shadow cones for one value of \( r \) were thus determined, the process was repeated for the other values of \( r \). Then for each value of \( r \), interpolations between the cones (which fell at unpreselected latitudes) to find the cones for any special desired latitudes (such as 30°, 40°, 50°, and so on) could be readily done because the number of cones was large and they were not spread very
widely over the cone diagrams.

The terminal points of the cones on the lower and upper edges of the cone diagrams were not given by this procedure. The lower terminal points were computed from the relation $F_x = 0$ and the cones extended down to such points. The upper terminal points depend on locating accurately the self-reversing trajectory for each family of trajectories of the second kind, or on knowing accurately the LMSR for each of the values of $\nu$ involved. In this work, such self-reversers were not computed, and it is not known whether the loci of minima of self-reversers given by Schremp are accurate. However, the cones are so small that there exists no practical need for carrying out the rather extensive investigations necessary for finding those terminals.

In Figures 7 through 12 are shown simple shadow cones produced as described in this section for latitudes at 10° intervals from 30° to 80°. When a section of a cone approaches the rim of a diagram very closely, such a section is not shown, as in the cases $r = 0.2, 0.3, 0.4$ at $\lambda = 60°$, and when entire cones are very near the edge, they are not shown, as in the cases $r = 0.2$ and $0.3$ at $\lambda = 80°$. This has been done to avoid obscuring the drawings.

There are also shown in the figures some of the cones of Schremp in dotted curves for comparison. These were drawn with fair precision by expanding the published cones with an optical projector and tracing the images. The southern terminal points for the cones of Schremp apply to the new cones.
5. Discussion of Results

One striking feature of the new simple shadow cones is that they differ greatly from those of Schremp. It is, of course, necessarily the author's contention that the old cones were badly in error. Some evidence for the incorrectness of the old cones has already been mentioned (the work of Schwartz, and the evident faults in the old cones for large r values and low latitudes). More evidence could be given -- for example, visual inspection of families of trajectories of the second kind shows that most simple shadow cones must be considerably "fatter" in their northern reaches than in their southern, whereas Schremp's cones tend to near-symmetry about their sin θ axes in their easterly parts -- however, it is not desired to belabor this matter.

An associated question is that of the origin of the errors in the old cones. It is not intended to produce here a list of possible origins, since that would at least partially involve conjecture. However, it ought to be pointed out that at least one certain and strong criticism can be made. The values of γ needed for the production of the earlier published cones can be readily computed and compared with the values used by Schremp, which were from 0.62 to 0.98 in steps of 0.01. One finds that the cones for r = 0.1 and latitudes greater than roughly 70°, r = 0.2 and latitudes greater than roughly 65°, r = 0.3 and latitudes greater than about 60°, r = 0.4 and latitudes greater than about 50°, and other cones, require values of γ between 0 and 0.6. How such cones were obtained is an open question. (Schremp does not mention
extrapolations, which would in any event have been extremely dubious over such a wide range of missing \( y \) values.)

A brief commentary on the newly computed simple shadow cones will be appended here. It is clear that for experimentalists these cones will ordinarily be considered as negligible because the fractions of the sky made dark by them are small, and because they fall at large zenith angles whose investigation of azimuthal effects is very difficult.

At high latitudes the penumbral region is known to be nearly completely a region of allowed directions of arrival at the observation point. In such cases the simple shadow cones are so small that the total allowed cones will probably be viewed as being given by the Störmer cones alone.

At intermediate latitudes the simple shadow cones are again negligible for practical purposes and the entire region outside any main cone will probably be considered a penumbral region.

No simple shadow cones are given for latitudes below 30°. However, the omission of these cones is of no real importance. The cones of Schremp for these latitudes are doubtless too large, but even they only slightly affect the main cones, and it is known that with decreasing latitude the penumbral region tends toward a condition of complete darkness. Thus at low latitudes the total allowed cones reduce essentially to the main cones.

A penultimate remark is in order concerning a recent result
produced by Vallarta, Gall, and Lifshitz. This work consisted of the computation of three simple shadow cones, with the earth's field being taken as that of an eccentric magnetic dipole plus magnetic quadrupole.

The authors compare their cones with the appropriate ones of Schremp, find considerable discrepancies, and attribute the differences mainly to the influence of the quadrupole moment, which is then said by them thus to have been shown to create a marked effect in the sizes of simple shadow cones.

The cones given by Vallarta et al agree well with the comparable ones computed by the author of the present paper with use of only a centered dipole field. If the cones given by Schremp are incorrect and if those given in this paper are substantially correct, then the conclusion to be drawn from the work of Vallarta et al is that the inclusion of the quadrupole magnetic field has negligibly small consequences for the simple shadow effect. This is in direct contradiction with the conclusions drawn by the authors referred to.

Finally, some new independent evidence for the correctness of the author's simple shadow cones, and hence for the incorrectness of those of Schremp, can be mentioned here. The principal shadow cones to be defined and discussed presently are of such natures that they are bounds of regions of forbidden trajectories for observation points above
the surface of the earth, and in particular as the altitude of observation decreases toward zero, these cones approach the appropriate simple shadow cones as limits. Now in many of the figures given later in this paper, which exhibit principal shadow cones, the approach of those cones to the simple shadow cones is clearly to be seen, and the figures show convincingly that the approach is certainly not to the simple shadow cones of Schremp, but is to those of the present author. The principal shadow cones were computed by operations totally separate from those described in this part of this paper, and derived from totally different computed trajectories, as will be seen. Therefore the evidence in the principal shadow cones for the great incorrectness of the cones of Schremp and the correctness of the new ones can properly be called independent.
III. Geomagnetic Theory for Observation Points Above the Earth

1. Introduction

This paper will now address itself to the problem of geomagnetic theory in the case in which the point of observation is at non-zero altitude above the earth. By the problem of geomagnetic theory is meant here what it has come to mean in the past — the theoretical description and eventual computation of regions of darkness and light in cone diagrams for ranges of values of particle energy and position of the observer. This work is done with the end in mind of making possible the combination of such results with assumptions about the nature of the cosmic ray beam in external space for the computation of expected particle intensities as functions of position and of properties of the particles and earth's field.

There are only two general approaches to the solution of this problem known to the author. The first would, if carried out sufficiently extensively, make possible a fairly complete numerical solution. With this method, a very great many points in the $x, \lambda$ plane for many values of the parameter $\lambda$ would be chosen, and very many trajectories in families would be computed numerically, each family consisting of trajectories radiating out from one of the points $x, \lambda$. The earth's surface, represented by a vertical line in the meridian plane when the coordinates $x, \lambda$ are used, could be inserted at various locations appropriate for various altitudes of observation. Visual inspection would make it possible to delineate in each case the bounds of dark
ranges of values of the angle $\eta$, and the transition to points in cone diagrams could be made in an obvious way.

However, rough estimates of the number of trajectories required for carrying out this project in even a minimal fashion show the task to be one of prodigious size even in this age of very high-speed digital computers.

The other method of solution depends on a theoretical classification of trajectories out of a general point $x$, $\lambda$, the classification being required to separate such trajectories into groups of those which do, and groups of those which do not, succeed in going from the point to infinity without intersecting the impenetrable earth. Even in the old geomagnetic theory, which has been much worked on, this problem has been only partly solved, and in the present case no complete solution has been found. However, a partial solution is adequate in a practical sense if the light and dark regions dealt with are guaranteed to be those of major importance, so that the computed regions in cone diagrams exhaust most of the areas of those diagrams.

The work to be described in the sequel has been done with this latter criterion in mind, and also has been governed by the requirement that the entire undertaking reduce to manageable proportions when suitable compromises are made in choices of ranges of values of the variables and parameters involved.
2. Preliminary Technical Matters

The field of the earth will be taken to be that of a magnetic dipole lying at the geometric center of a spherical earth. Use of a dipole field is justified by the facts that analysis based on field measurements made on the surface of the earth have shown the magnitudes of contributions from multipole moments of higher order than that of the dipole moment to be small, and that no work yet done has shown the dipole approximation not to be adequate in general geomagnetic theory. (Part I of the present paper contains some evidence in this connection.) It is taken as obvious that the eccentricity of the dipole and the oblateness of the earth will be of even much less importance in the present context than in the theory for points on the earth's surface, and can safely be neglected here.

In an \( x, \lambda \) plane for any value of \( \lambda' \), the surface of the earth is a vertical line. Hereafter a value of \( x \) which is meant to correspond in this way to the surface of the earth will be labelled with the subscript \( e \). An observation point which is not on the earth will be represented by coordinates with the subscript \( s \), as \( x_s, \lambda_s \). (The letter \( s \) is chosen to stand for "satellite".) Other quantities, such as lengths \( r \) measured in störners, may also be labelled with subscripts \( e \) or \( s \), according to whether they pertain to points on the earth or not.

The observation point will hereafter be taken to be above the surface of the earth, except when otherwise specified, and the relative positions of the observation point and the surface of the earth will be
referred to in various ways. It will sometimes be given by the quantity \((x_s - x_e)\); sometimes by the ratio \(r_s/r_e\); sometimes by the ratio \(\rho_s/\rho_e\) where \(\rho_s\) and \(\rho_e\) are the same as \(r_s\) and \(r_e\), respectively, but measured in ordinary units of length; and sometimes by the quantity \(h\), which is the distance in ordinary units of length of the observation point above the earth's surface. The following relations hold, on the assumption that one value of each of the quantities \(\gamma\), \(M\) (the value of the magnetic dipole moment of the earth), and \(R\) (the particle rigidity), is being considered:

\[
\frac{\rho_s}{\rho_e} = \frac{r_s}{r_e} = e^{x_s-x_e} ; \quad h = \rho_s - \rho_e = \rho_e \left( \frac{r_s}{r_e} - 1 \right).
\]

When particles of a given rigidity are being considered, the radius of the earth fixes a value of \(r_e\) for that rigidity; alternatively, the particle rigidity can be specified by stating the value of \(r_e\). However, when an observation point above the surface of the earth is being considered, its radial distance \(\rho_s\) from the center of the earth, together with values of \(M\) and \(R\), determine a different value of \(r\), called \(r_s\), even for particles of the same rigidity \(R\) as those referred to by \(r_e\). Therefore, it is the observation point which fixes the cone diagram, regarded as a coordinate system, even though the features of the dark regions in the cone diagram are affected by the location of the earth. That is, in the expression

\[
\sin \theta = \frac{2y}{r \cos \lambda} - \frac{\cos \lambda}{r^2}
\]
for the horizontal coordinate in a cone diagram, it is \( r_s \) and not \( r_e \) which must be inserted along with values of \( \delta \) and \( \lambda_s \).

3. Reduction of the Generality of the Problem

Some of the main features of a perfectly general theory, in which the ranges of values of the particle rigidity and altitude above the earth of the point of observation are unlimited, are easily discoverable. It is obvious that such a theory would have many ramifications and be of considerable complexity; it would furthermore be difficult to implement with computed results, for this would require computation of very large numbers of trajectories of many very different kinds.

However, such a general theory is not needed. It is sufficient for practical purposes to treat particles of rigidities bounded by some maximum value, the number-rigidity cosmic ray spectrum evidently being such that by far the majority of the particles in the primary beam are particles with rigidities of the general order of \( 0.1 \) or \( 0.2 \), in störmer units. Also it is unnecessary to deal with a completely unrestricted range of altitudes, obviously.

Fortuitously, when suitable restrictions are placed on the values of the variables and parameters of the problem, it becomes possible to construct a theory which is simple in comparison, while practical needs are met. For this reason, and more especially to make possible computation of usable results, such restrictions will be imposed.

In connection with the discussion of the restrictions to follow,
reference may be had to Figures 14 through 23. Of these, Figures 14 through 18 show the values of the parameter $\gamma$ which occur at the left and right edges of cone diagrams, as functions of latitude $\lambda$ and of the variable $r$, now to be interpreted as $r_s$. Since the $\sin \theta$ axis in a cone diagram corresponds to a linear scale of values of $\gamma$, the figures also show the $\gamma$-axes across the faces of cone diagrams. The places of appearance of special values of $\gamma$, such as zero or one, can be read off immediately in semi-quantitative fashion. These figures will be of frequent utility in the sequel. They were computed by hand in an obvious manner.

Figures 19 through 23 show the curves $P = 0$, which bound allowed regions for the motions in the meridian plane, $F_x = 0$, which separate regions where the $x$-component of the force is positive from regions where it is negative, and $\theta = 0$, all for various values of $\gamma$. These curves are of great usefulness, and are not elsewhere available in the literature. The curves were computed by use of an IBM 704 computer.

The first restriction imposed is that no value of $r_s$ greater than 0.8 will be considered. This value, which corresponds to an energy of about 37 BeV for protons, is sufficiently high so that the range of energy to be dealt with is easily the range that includes all the primary spectrum except a very small high-energy tail. For particles of higher energies the geometric shadow of the earth at the observation point can, as will be seen, be used in good approximation for the computation of the principle dark regions. The advantage of the restriction in the
theory is that the line \( x = x_e \) representing the surface of the earth will now always lie across the arms of the \( P > 0 \) region of the meridian plane or else just right of the arms, in the flat plane which exists there, for all values of \( \delta \) between 0.1 and 1.0, except that for values of \( r_e \) near 0.8 and values of \( \delta \) near 1, \( x = x_e \) may lie rather far out in that plane, near the outer principal periodic orbit. Much of the simplicity of the method of computation of the principal shadow regions will be seen to depend on having such configurations in the meridian plane.

Secondly, no negative values of \( \beta \) will be considered. The effect of this restriction can be gauged by consulting Figures 14 through 18. It can be seen that neglect of negative values of \( \beta \) corresponds to neglect of small westerly portions of cone diagrams for the largest values of \( r_s \), if altitudes up to about 3, measured by the ratio \( r_s/r_e \), are taken into account — that is, if altitudes from 0 miles to about 8,000 miles are to be considered. The neglect of those portions would not be important in any event, but they are at least nearly completely light regions for low latitudes, and their neglect can be said to be of nearly no consequence under those conditions.

Finally, these will apply a lower bound to the size of the range of altitudes of the observation points, by which all altitudes from zero to at least several thousands of miles must be considered. This requirement is imposed in order that the theory and computed results may be of practical utility.
4. The Störmer Cone Under the New Conditions

We begin with a very succinct statement of the old theory of the Störmer cone for observation points on the earth such that for them $r < 1$. Motions allowed in the $x, \lambda$ plane are restricted to regions where $|\sin \theta| \leq 1$, i.e. $P \geq 0$. For any positive value of $\gamma$ between zero and one, this region is a simply connected region, but for values greater than one, it is separated into two regions by a region of the plane where $P < 0$. (For $\gamma = 1$, the two regions allowed for the motions touch at the equator at the point corresponding to $r = 1$.) It can be verified that $r < 1$ for the inner of the two allowed regions when $\gamma > 1$, while for the outer of the two regions $r > 1$; therefore when $\gamma > 1$ no trajectories from infinity can arrive at any observation point for which $r < 1$.

According to a restriction just imposed, $r_e \leq 0.8$. Now if observation points above the earth are considered, and if the ratio $r_s/r_e$ is allowed to assume values as large as 3, then for some values of $r_e$ and of the ratio $r_s/r_e$, $r_s$ may be greater than one, while for other it may be less than one while $r_e < 1$ always. In either event $\gamma$ may or may not exceed unity. A sufficiently good guide to the various cases is to be found in Figures 14 through 18. From these it may be seen that for values of $r_s$ up to about $0.4$, cone diagrams will be completely dark ("Störmer dark"), even for latitudes up to $50^\circ$ or $60^\circ$ in some cases. As $r_s$ increases toward unity there will be Störmer dark regions in the easterly parts of the cone diagrams for small latitudes.

When $r_s$ exceeds unity there will again come into play at the
lower latitudes values of $r'$ greater than one, but now the significance of the Störmer cone is different, for it does not now mark off a region of complete darkness, but of complete light, as will be proved.

In every case in question, the values of $r_e$ and $r_s$ are such that the surface of the earth lies in the inner allowed region while the observation point lies in the outer allowed region. Therefore, not the solid earth but only the boundary of the allowed region (the outer branch of the curve $P = 0$) has significance as an obstacle to trajectories seeking to reach the observation point from infinity. It can be verified that the $x$-component of the force is, in this outer region, everywhere directed to the right. Therefore each trajectory leaving any point in the outer allowed region either leaves to the right and proceeds to infinity (being then unable to achieve a maximum in $x$) or else proceeds to the left, attains an absolute minimum in $x$ before or upon reaching the curve $P = 0$, and then proceeds to the right to infinity.

Thus in the new context, the Störmer cone may have either of two different kinds of significance, depending on the values of $r_e$ and $r_s$ jointly.

5. The Main Cone Under the New Conditions

A full discussion of the theory of the main cone may be found in the Toronto paper of Vallarta.  

The existence of a main cone for any observation point depends on the existence of certain trajectories asymptotic to the outer principal periodic orbit, and is independent of the presence of the impenetrable earth. If any trajectory on or inside the main cone somewhere intersects the surface of the earth, that fact is taken account of separately in the theory of the simple shadow cone in the old geomagnetic theory.

Clearly an analogous scheme can be employed in the new context. For any observation point above the earth the main cone will have the same significance as in the theory for points on the earth's surface, and trajectories intersecting the earth will be dealt with in other parts of the theory. Indeed, the main cones computed and published by Lemaitre and Vallarta\(^\text{13}\) can be used directly subject only to the condition that each value of \(r\) referred to by them must now be taken to be a value of \(r_s\).

Since Lemaitre and Vallarta and all others who have contributed to the theory and computations of main cones have been concerned with the case in which the observation point is on the earth, no main cones have been published for the lower hemisphere of directions at an observation point.

Not only is an extension of the published main cones into the lower hemisphere of directions of arrival lacking, but also the computation of many more cones for the upper hemisphere would be needed since

the cones now extant do not cover all the range of values of $r_s$ and $\lambda_s$
of interest. Computation of main cones in quantity must be considered a
task of some magnitude, and the author has chosen to consider that such
an undertaking should not be entered upon until it has been shown that
a genuine need exists.

6. The Principal Shadow Cone

A. Considerations Preceding Definition of Principal Shadow Cones

The object in view now is the discovery of a mode of specifying
the nature of special trajectories bounding dark ranges of values of the
angle $\eta$ for given points $x_s, \lambda_s$, when $x_s > x_e$.

At this point Figures 24, 25 and 26 may be consulted. These show
some of the computed members of pencils of trajectories radiating from
two quite differently located points in the $x, \lambda$ plane, and different
values of $\delta$. Each pencil has been broken into parts to avoid obfusca-
tion. Examination of the trajectories shows the existence of often
large dark ranges of the angle $\eta$ at the observation points which are
bounded by values of $\eta$ corresponding to trajectories with points of
tangency at the earth. It is apparent that the dark ranges which exist
in general may well be defined by use of such "grazing" trajectories.
However, it is evident that sometimes such "grazing" trajectories do not
have forms of a degree of simplicity comparable with those bounding
simple shadow regions. For example, highly oscillating ones may be
seen in the figures. Also, some grazing trajectories clearly cut through
the surface of the earth shortly after grazing the earth, and are dis­

tinct from others which after grazing the earth proceed to the right and
perform more convoluted motions before arriving again at the earth, or
pass to infinity.

After these heuristic considerations, we proceed to a classifica­
tion of trajectories, establishment of nomenclature, and the statement
of some postulates for subsequent use.

Consider any trajectory which issues from a point \( x_s, \lambda s \) and
reaches a point of tangency with the earth at \( x_e < x_s \) without having
intersected the earth in the interval. The segment of such a trajectory
between the initial point and the point of tangency with the earth will
be called a grazing trajectory segment.

A separation of grazing trajectory segments into classes will now
be made for the purpose of eliminating from consideration those in one
of the classes, which will not be of concern hereafter. Let it first be
noticed that the point of tangency at the earth of a grazing trajectory
segment lies above, below, or on a locus of the minima of self-reversing
trajectories (the locus called an LMSR in part II of this paper) if the
line \( x = x_e \) intersects an LMSR, or else the line \( x = x_e \) does not inter­
sect an LMSR. In each case, \( \lambda' \) at the point of tangency may be less
than, or greater than, or equal to, zero. Let it be noticed that now,
and in some of the discussions to follow immediately, the terminology
used has been adapted to cases in which the points of tangency lie in
the northern hemisphere of latitudes; modification of the terminology to
make it apply to points of tangency in either hemisphere obviously can be done.

If the point of tangency of a grazing trajectory segment lies above an LMSR, and if the trajectory has $\lambda' > 0$ at that point, then the continuation of the segment past the point of tangency clearly is a trajectory of the second kind — that is, after the tangency, the continued segment turns in a counterclockwise sense and cuts through the surface of the earth. It is evidently appropriate to call the minimum in $x$ of the grazing trajectory segment at the point of tangency a relative minimum in $x$.

In all other of the possible cases it can be shown that a grazing trajectory segment, if continued past its point of tangency at the earth, must in its continuation not intersect the earth again at all, or else must cross the equator at least twice before intersecting the earth in the same hemisphere as that in which the first point of tangency lies. This at least two-fold crossing of the equator implies wanderings in the $x, \lambda$ plane generally of some length in $\sigma$ and often paths of some high degree of complexity of form. It seems appropriate to refer to the minimum in $x$ at its point of tangency with the earth of a grazing trajectory segment as an absolute minimum in $x$, if the trajectory does not again intersect the earth or else wanders as just described.

Because of the large expenditure of words which would be required to carry out the demonstration which was claimed possible in the preceding paragraph for all possible cases, the author takes the liberty of
giving the demonstration in one case only, and of abbreviating it in that case. The case considered will be that in which the grazing trajectory segment has $\lambda' < 0$ at its point of tangency at the earth, which point furthermore lies below an IMSR. In this case the continuation of the segment past the point of tangency is one of the trajectories below the self-reversing member of a family with properties C (as this was defined in part II) — that is, the continued segment proceeds from the point of tangency by turning in a clockwise sense away from the earth, and passes through a maximum in $\lambda$ before reaching a maximum in $x$. Now use can be made of a theorem found and proved by Størmer\(^{11}\), according to which a trajectory between two consecutive points of intersection with the equator does not have more than one point at which the function

$$V = h \beta^2 \sin \lambda / \exp 2x$$

has a maximum or a minimum; if the point is above the equator $V$ has a maximum, and if below, a minimum. It is easy to see that for the continued segment being considered here, because of the forms of the curves $V = \text{constant}$ in the $x, \lambda$ plane, the function $V$ must attain a maximum during the initial turning away from the earth of the continued segment, and in fact that the point at which this happens will usually occur on the continued segment before $\lambda$ reaches its first maximum. Therefore in this case the grazing trajectory segment has an absolute $x$-minimum at the point of tangency at the earth, in the sense in which that phrase is now
to be understood.

It is sufficiently obvious that a grazing trajectory segment with a relative \( x \)-minimum at the point of tangency is neighbored on both sides by trajectories which intersect the earth, as it does itself, while a grazing trajectory segment with an absolute \( x \)-minimum at the point of tangency will separate a continuum of trajectories which intersect the earth from a continuum of those which do not, or which at worst intersect the earth only after at least one crossing of the equator. Grazing trajectory segments with absolute \( x \)-minima at the earth bound between them continua of trajectories which intersect the earth, and among these latter trajectories will be grazing trajectory segments (continued past their points of tangency) with relative \( x \)-minima at the earth. Therefore the grazing trajectory segments with relative \( x \)-minima at the earth ought to be excluded from consideration as bounds of dark regions.

Henceforth, for simplicity of terminology, and by extension of already established nomenclature in the older geomagnetic theory, grazing trajectory segments with absolute \( x \)-minima at the earth will be called trajectories of the fourth kind.

As was done in part II, let \( \eta^T_N \) signify the value of \( \eta \) at a point which corresponds to a trajectory of the second kind bounding a simple shadow region in its northern reach, and \( \eta^T_S \) the similar quantity for the southern reach of a simple shadow region. If at a point to be considered there is no southern reach of a simple shadow region,
we will take $\eta_S^T = 270^\circ$, and if there is no northern reach, $\eta_N^T = 90^\circ$.

Now it is claimed that for all points $x_s, \lambda_s$ and for all values of $\eta'$, the following is true: Let $\eta$ at the point begin at the value $\eta_N^T$ and increase until the value $\eta_S^T$ is reached. Then all the trajectories corresponding to the values taken on by $\eta$ are trajectories of the fourth kind. (Some of these trajectories will have relative $x$-minima at their first points of tangency with the earth; for them the claim is that they later have absolute $x$-minima. This is only a matter of the terminology used here.)

That the claim is true is well-nigh obvious. If $90^\circ < \eta < 270^\circ$ every trajectory generated by the variation of $\eta$ proceeds initially to the left from the point $x_s, \lambda_s$ and must somewhere in that region have an absolute minimum, with the provision that a trajectory through the dipole must be counted as having an absolute minimum in $x$. If $\eta_N^T \leq \eta < \eta_N^S$ but not $90^\circ < \eta < 270^\circ$, the generated trajectories initially proceed to the right of the initial point, but then turn and travel in the region left of the initial point by application of the theory of the simple shadow cones.

Now it is further claimed that for at least some points $x_s, \lambda_s$ and some values of $\eta'$, the locations of the absolute $x$-minima of the trajectories of the fourth kind generated by the variation of $\eta$ from $\eta_N^T$ to $\eta_S^T$ recede monotonically to the left in $x$ as $\eta$ increases from the value $\eta_N^T$ until for some value of $\eta$ the corresponding trajectory of the fourth kind has its absolute $x$-minimum farther to the left in $x$ than
does any other trajectory of the fourth kind generated during the entire variation of the angle $\eta$, after which the locations of the absolute x-minima of the trajectories of the fourth kind generated during the rest of the variation of $\eta$ move monotonically to the right. A point $x_s, \lambda_s$ of this character will be called a point of kind 1, for short.

The validity of the claim will not be proved here, but evidence will be given, and the claim regarded as a postulate. The evidence is contained in Figures 24, 25, and 26, which show pencils of trajectories of the fourth kind for two different initial points and values of $\eta$. In the cases of the members of the pencils shown, the movement of the absolute x-minima claimed can be convincingly traced, and the intuition suggests that the infinity of missing members must be of such nature as to conform to the claim.

It can be shown by use of the first order constant of the motion for charged particles in a magnetic field found by Alfven in the realm of discourse fixed by use of asymptotic series expansions for solutions of the equations of motion, that the qualitative behavior of the x-minima is as described in the claim. However, the approximation is good only under conditions of little interest here ($r << 0.1$ or $\lambda' \sim 2$) and even then the development shows some unrealistic features, so that it will not be discussed in detail here.

It is interesting to observe that the claim was early taken as

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true by Størmer, though of course in a different context; he discusses
the movement of the $x$-minima and gives some fragmentary pictorial evi-
dence.


17. Ibid., Fig. 132, p. 250.

Now the additional claim is made that for some points $x_s, \lambda_s$ and
some values of $\delta$, as $\eta$ varies from $\eta_N^T$ to $\eta_S^T$, the movement of the
absolute $x$-minima of the trajectories of the fourth kind generated dur-
ing the variation of $\eta$ through its range consists first of a monotonic
recession to the left in $x$, followed by a monotonic movement to the
right, followed by a second monotonic recession to the left, with a
final monotonic movement to the right ensuing until $\eta$ reaches the value
$\eta_S^T$. Such a point will be called a point of kind 2, for short.

The reader can convince himself of this validity of this claim
by referring to Figures 24 and 25 and imagining the initial point for
the pencil shown to be moved to a latitude nearer the equator.

Points of the kinds 1 or 2 are not isolated in $\delta$ — that is,
if $x_s, \lambda_s$ is for some value of $\delta$ a point of kind 1 or 2 then if $\delta'$ is
allowed to vary while $\lambda_s$ is held constant and $r_s$ is held at the value
fixed by the initial values of $x_s$ and $\delta$, the points thus generated will
be also points of kind 1 or 2, at least over some range of $\delta'$. That this
is true can be seen by considering some point which is obviously of kind
1 or 2, and by examining the nature of the deformation of the P-contours which occurs when \( \xi \) varies.

Also, points of kind 1 or 2 are not isolated points of those kinds with respect to \( x \) for a given value of \( \xi \). In particular all points well to the left in the \( x, \lambda \) plane, in the arms of the \( P > 0 \) region are without doubt always of kind 1, and all points near the equator just to the right of the inner branch of the curve \( P = 0 \) are doubtless of kind 2. If any latitude be chosen, and a point of either kind on it well to the left, then if the point be moved to the right along that latitude, the points occurring will be of kind 1 or 2 at least over some range of \( x \).

However, it is not the case that the range of \( x \) just referred to is unlimited. It is possible to find examples of trajectories which show that when the initial point moves far enough to the right in the \( x, \lambda \) plane the trajectories, which in the broad flat plane in the region right of the inner branch of \( P = 0 \) can perform cyclic motions crossing the equator many times, do not depend for the location of their absolute \( x \)-minima on the initial value of \( \xi \) at the point in such a simple way as is true for points of kind 1 or 2.

B. Definition of a Principal Shadow Cone

If any given point \( x_s, \lambda_s \) is a point of kind 1, then if the earth be inserted with its surface at \( x = x_e, x_e \) lying anywhere at will left of \( x_s \), there will be either (1) no trajectories of the fourth kind to the earth at \( x = x_e \) and issuing from the point with values of \( \xi \) between
the earth lying so far left that none of the absolute x-minima of the pencil in question can reach it, or (2) just one such trajectory of the fourth kind out of the point to the earth, that trajectory having its absolute x-minimum farther left than for any other in the pencil, or (3) exactly two such trajectories of the fourth kind to the earth out of the point.

In the last case, it follows immediately that all trajectories leaving the point between the two distinguished trajectories of the fourth kind cut through the surface of the earth. All such trajectories will be said to constitute a principal shadow region for that point, location of the earth, and value of $\gamma$. Sometimes for simplicity of expression the range of values of $\eta$ between those for the bounding trajectories of the fourth kind will be called the principal shadow region for that point.

If the point is of kind 2, then there may be 0, 1, 2, 3 or 4 trajectories of the fourth kind to the earth out of the point, with initial values of $\eta$ at the point between the values $\eta^T_N$ and $\eta^T_S$. The reader will readily see how these various cases can arise. He will also see that there may now be none, or one, or two continua of trajectories which cut the earth and are bounded by the trajectories of the fourth kind with initial values of $\eta$ in the range specified. In any such case the forbidden trajectories will be said to constitute a principal shadow region at that point, location of the earth, and value of $\gamma$. Again, the corresponding range of values of $\eta$ may be called the principal shadow region.
The relation of the preceding discussions and cone diagrams will now be considered. For any point of kind 1 or of kind 2, the value of $\gamma$ and the value of $x_e$ will fix a value of $r_e$, and hence of particle rigidity. Also, the values of $x_e$ and $x_s$ will fix a value for the altitude of the point $x_s$, $\lambda_s$ above the surface of the earth. For the values of $r_s$ and $\lambda_s$ in question, there will correspond a cone diagram, regarded as a coordinate system, which must also be labelled with the value of $r_e$ (or an equivalent quantity or quantities). The particular value of $\gamma$ in question will fix a vertical line in the cone diagram, and the principal shadow region for the given point and particle rigidity will be represented by a segment or segments of that vertical line, all points on such segments corresponding to the forbidden trajectories in the principal shadow region.

Figure 27 (which is highly schematic) is presented to facilitate understanding of this matter. Fixed values of $r_e$, $r_s/r_e$, and $\lambda_s$ are implied. In (a) $\gamma_1$ and $\gamma_5$ are such that no principal shadow region exists, $\gamma_2$ and $\gamma_4$ are such that principal shadow regions consisting of single points exist, and $\gamma_3$ is such that a principal shadow region exists. In (b) values of $\gamma$ occur such that there may exist the most complicated cases, which the reader will readily relate to the discussions given previously.

Suppose that for some value of $\gamma$, the surface of the earth lies at $x = x_e$, and let a value of $\lambda_s$ be taken, all such that the point $x_e$, $\lambda_s$ and points at the same latitude but right of $x_e$ are points of
kinds 1 or 2 at least for some range of $x$. Then if the observation point begins at the earth and rises continuously above it, the principal shadow regions for the points will vary continuously.

Now a prime consequence of the mode of definition used here can be seen. It is guaranteed, if values of the parameters involved are suitably chosen, that principal shadow regions for one set of $r_e, \lambda_s, \gamma$ will begin for $r_s = r_e$ by being identical with the simple shadow region when the observation point is at the earth, and will vary continuously from this initial state as the altitude increases. Since it is to be expected that penumbral dark regions will shrink with increasing altitude and are often of relatively little importance even at the surface of the earth, it is to be expected that the principal shadow regions will generally be the dark regions of chief importance, justifying the use of the modifier "principal". (Störmer dark regions may, of course, continue to be of greater importance than any others.)

Let the values of $r_e, r_s$ and $\lambda_s$ in a particular case be held constant, and a value of $\gamma$ be considered which is such that the point $x_s, \lambda_s$ is a point of kind 1 or 2. If now $\gamma$ be allowed to vary, there will be generated by the corresponding movement and gradual deformation of the vertical line segments representing principal shadow regions, whole areas of darkness in the cone diagram for the given $r_e, r_s, \lambda_s$. Such areas in cone diagrams will also be called principal shadow regions, and the curves bounding them above and below will be called principal shadow cones.
It has already been explained that in the computational work to be described later no value of $r_e$ greater than 0.8 was considered. The actual values of $r_e$ for which trajectories were computed were 0.1, 0.2, 0.3, 0.4, 0.6, and 0.8, this selection having been made to reduce the total number of values to a small value, and to cover well the range of rigidities of practical interest. In order to reduce the total number of computed trajectories to a manageable value, it was decided to use the values 0.1, 0.2, 0.3...1.0 for $\theta$ for each value of $r_e$, except that the values 0.9 and 1.0 were not used for $r_e = 0.8$ nor the value 1.0 for $r_e = 0.6$.

The author is satisfied that for each $r_e$ value used and for $\theta = 0.1$, points on the earth and above the earth to altitudes at least as high as 500 miles are of kind 1 or 2. This can be reliably judged by inspection of the P-contours and computed trajectories; sufficient data are given among the figures included with this paper to make it possible for the reader to satisfy himself that the claim is true. Also for the smaller $r_e$ values and all values of $\theta$ used, points above the earth are of kind 1 or 2 out to altitudes in excess of 8000 miles, which is the maximum altitude dealt with in the computations. However, for the larger values of $r_e$, large used values of $\theta$, and large altitudes, it cannot be guaranteed so certainly that points representing observation points above the earth are points of kind 1 or 2. This matter will be discussed next; it will be seen that it is not necessary to attempt to state numerical values of the parameters involved, such that observation points may
cease being points of kind 1 or 2.

Let values of $r_s, \lambda_s$ and $\gamma$ be selected such that the point $x_s, \lambda_s$ corresponding to these values is safely of kind 1 or 2. Then if the earth at $x = x_e$ be inserted left of this point, there will in general exist a principal shadow region for the chosen configuration. Now if $\gamma$ be allowed to increase, or the altitude to increase, or both, then at least over some ranges of increase of these quantities the continuous changes of those quantities will be accompanied by continuous changes in the principal shadow regions. For the fixed $r_e$ and $\lambda_s$ and for any values of $r_s$ and $\gamma$ encountered during the variation, there will be certain trajectories of the fourth kind out of the point $x_s, \lambda_s$ which bound the principal shadow region, and now if the altitude and the parameter $\delta$ be allowed to continue to vary there will be generated in continuous fashion a sequence of trajectories of the fourth kind which begin with the certain ones referred to, and always bound continua of forbidden trajectories, which continua are themselves generated in continuous fashion beginning with the certain one referred to. Since this is true, then if any observation point which arises for consideration cannot be guaranteed to be a point of kind 1 or 2, there can yet in principle be selected from among the trajectories of the fourth kind emanating from the point those which bound continua of forbidden trajectories such that the forbidden continua are natural extensions of the principal shadow regions which existed in the case in question at lower altitudes or $\delta$-values, or both. This can also be done in practice if for a given
r-value enough values of $^y$ were used in computing trajectories, and if the number of computed trajectories per pair of values of $r_e$ and $^y$ is large enough. This is clearly a claim, the truth of which the reader can satisfy himself only by examining extensive collections of computed trajectories. When members of a computed family of trajectories of the fourth kind for a given $r_e$ are examined, it is usually apparent immediately when one arises which at a given observation point does not bound a continuation of a principal shadow region.

The significance of the foregoing discussion will be reviewed. A particular mode of definition of a principal shadow region was at first used which was an arbitrary choice made among alternative modes. The choice was made because with it the existence and meaning of the principal dark regions is especially easy to understand and requires no lengthy discussion involving legalistic verbalism which seems to be a necessary concomitant of use of other modes of definition. Then since in the actual computations it may be true in extreme cases which are desired to be dealt with that the observation points in question are not of kinds 1 or 2, a way of extending the definition to such points is used. Specifically, a principal shadow region for any point $x_s, \lambda_s$ and values of $r_e$ and $^y$ will now be taken to be a region as previously defined if the point $x_s, \lambda_s$ is a point of kind 1 or 2, or otherwise will be that continuum of forbidden trajectories which is a member of the continuous sequence of continua of forbidden trajectories which can be traced as the altitude or value of $^y$ or both are reduced until a continuum of
forbidden trajectories is achieved which is a principal shadow region as
previously defined. For given $r_o$, $r_s$ and $\lambda_s$ such principal shadow
regions, in the now extended sense, will generate dark areas in cone
diagrams as $\beta$ varies, and such areas will now also be called principal
shadow regions, bounded by curves to be called principal shadow cones.

7. The Penumbral Dark Regions

For observation points above the surface of the earth, some tra­
jectories other than those inside principal shadow regions may be physi­
cally unallowed trajectories for the reason that they cannot be traversed
by particles from infinity to the observation point without intersecting
the impenetrable earth in the interval. That there are such trajectories
can be verified by locating some in the figures given.

These trajectories will display great variety of form. While some
will be re-entrant trajectories like the penumbral trajectories of the
older theory, some will not. Nevertheless it seems quite appropriate
to refer to any forbidden trajectory which is outside a principal shadow
region for points off the earth as a penumbral trajectory, and to call
any dark region in a cone diagram which is outside a principal shadow
region, and is not a region of Størmer darkness, a penumbral dark region.
The whole of a cone diagram outside the principal shadow region, left
of a Størmer cut-off and right of a main cone, will be called the
penumbral region in the cone diagram.

Penumbral dark regions now arise in much the same way as in the
old theory: When the earth is inserted in the $x, \lambda$ plane to the left of the point $x_s, \lambda_s$, some trajectories out of that point but not in principal shadow regions may be cut by the surface of the earth. Because of the continuous deformation of the trajectories with change of the initial conditions, the trajectories affected lie in bands, bounded by penumbral grazing trajectories.

One can see by inspection of collections of computed trajectories that it can be expected that in most cases penumbral trajectories will have re-entrant sections. It may be recalled that it has been shown by Schremp that trajectories with re-entrant sections can exist only for values of $\delta'$ between 1 and a certain other value, found by calculation to be 0.788511. Thus most penumbral dark regions vanish for values of $\delta'$ outside that range, in the present context, whereas all the penumbral dark regions in the old theory vanished for such values.

8. Computation of Principal Shadow Cones

A. Description of Method

An expeditious systematic method for the computation of principal shadow cones is suggested by the method used for the computation of simple shadow cones, since in the present case too there are needed...
families of trajectories with points of tangency at the earth. However, the trajectories needed now must have absolute rather than relative minima in $x$ at their points of tangency, and sufficiently long segments of them are needed so that they will pass through observation points at great altitudes above the earth. Also, the quantity of data required to be dealt with is large. Each trajectory of the second kind yielded at the earth one value of $\eta$ and one of $\lambda$ as usable data, but each trajectory of the fourth kind yields values of $x$, $\lambda$, and $\eta$ at every point on it as usable data. Also the number of trajectories required for the computation of principal shadow cones for wide ranges of rigidity, latitude, and altitude will clearly be much larger than the number required for production of simple shadow cones. In fact the task is one of such magnitude that it was only by judicious selection of values of parameters and by making compromises in the matter of precision that the project could be carried out without delimiting overly much the ranges of rigidity, latitude and altitude treated.

Pairs of values of the quantities $r_e$ and $\delta'$ were first selected. As already explained, the values selected were $0.1$, $0.2$, $0.3$, $0.4$, $0.6$, and $0.8$ for $r_e$, and $0.1$ through $1.0$ in steps of $0.1$ for $\delta'$. The value $\delta' = 1.0$ was not used for $r_e = 0.6$, and $\delta' = 0.9$ and $1.0$ were not used for $r_e = 0.8$. These selections were made to keep the size of the undertaking within bounds while covering the required ranges well.

The decision concerning the values of $\delta'$ used is capable of affecting critically the extent and precision of the computed cones, and
its discussion is worth while here. The number of trajectories to be computed and subjected to lengthy data reduction processes is given by \( abc \), where \( a \) is the number of values of \( r_e \) to be used, \( b \) is the average number of values of \( \gamma \) to be used per \( r_e \) value, and \( c \) is the average number of trajectories per pair \( r_e, \gamma \). The first factor was fixed at 6, the last factor could be expected to be from about 20 to about 30, and the circumstances surrounding the project were such that it appeared that the second factor could not exceed about 10. It is difficult to find one set of 10 values of \( \gamma \) which lead to adequate coverage of all cone diagrams for all latitudes and \( r_e \) values. The meaning of this can be seen by referring to Figures 11 through 18; there it will be seen that the ranges of values of \( \gamma \) across the faces of cone diagrams vary widely with the values of \( r_e \) and \( \lambda_s \). If suitable values of \( \gamma \) are picked to deal adequately with each \( r_e, \lambda_s \) case separately, the total number of values becomes excessive. The selection of values actually made turned out to give reasonably good distributions of points in cone diagrams in most cases, but for the smaller values of \( r_e \) the high latitudes become difficult, with only a few of the selected \( \gamma \) values appearing in any cone diagram. However, this situation is almost completely alleviated even for very high altitudes by the fact that in such cases the Störmer cone is very important. Thus, for example, all cone diagrams are completely dark for \( r_e = 0.1 \) and 0.2, even up to considerable latitudes and very high altitudes.

Then along each line \( x = x_e \) there were selected values of \( \lambda \) in
the allowed ranges (where $P > 0$) for beginning the trajectories. At the outset it was not at all known how to make these selections, but some trial computations served as guides. These indicated that for given $x_0, y$ some 10 to 20 trajectories starting out with $\eta_0 = 90^\circ$ at latitudes more or less evenly spaced over the allowed range of latitudes, and some 6 to 10 starting out with $\eta_0 = 270^\circ$ and generally starting out closer to the lower than to the upper branch of the curve $P = 0$, should give optimal results. Later work with the computed trajectories showed that this was a reasonably good decision.

Only trajectories beginning at latitudes north of the equator were computed, because any such trajectory which crosses the equator and so yields principal shadow cone data for latitudes south of the equator can be reflected in the equator and the resulting data used for the corresponding points north of the equator.

The next step was to compute the trajectories. It was early decided to use a high speed digital computer only for this purpose, with all the subsequent reduction of data to be done by hand. The reasons for making this decision are complicated and of little interest, and will not be discussed here. The computer used was the General Motors IBM 704 computer, except that the Martin Aircraft IBM 704 computer was used for some of the pilot calculations and subsidiary computations. The integrations method used, checks on precision of the computations, and similar matters were discussed in part II of this paper.

Figures 28 and 29 exhibit trajectories of the kinds computed. A
few members of the families for $r = 0.0, \beta = 0.4,$ and $r = 0.6, \beta = 0.6$
are shown; these illustrate the natures of the trajectories. Each trajectory
of the fourth kind gives at each crossing of a particular latitude values of the altitude and the angle $\eta$ there, and so a point on a
cone diagram which lies on a principal shadow cone, if the trajectory of
the fourth kind at the crossing bounds a principal shadow region. It
can be seen that getting crossings of a particular latitude with those
values of $\eta$ intermediate between those used, for the trajectories which were made, the distribution was generally
adequate, and it is in fact felt that if many more trajectories were to
be computed it would contribute more to the cone computations to calcu-
late trajectories for values of $\eta$ intermediate between those used,
than to proliferate the trajectories for those already used. After computation of the trajectories, there was on hand a great
collection of printed $\eta, \lambda, \gamma$ data. For each trajectory these data
were printed out by the computer at preselected intervals in the inde-
pendent variable $r$. Now, principal shadow cones were being sought not
at all latitudes (an impossible task) but only at certain selected lati-
tudes, the specific ones being from $0^\circ$ northward in steps of $10^\circ$. From-
ever, almost never would a machine-produced point on a trajectory fall
at one of the preselected latitudes. Thus an elaborate point-by-point
reading of all trajectories and performance of linear interpolations
for the trajectories which were made, the distribution was generally
prior knowledge of the proper initial conditions or computation of very
large numbers of trajectories. With the selections of initial conditions
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late trajectories for values of $\eta$ intermediate between those used,
along them to produce \( x_s, \lambda_s, \eta, \gamma \) data at points at the preselected latitudes were done. The interpolations were done in the general fashion of those done for the computations of the simple shadow cones described in part II.

Next, principal shadow cones at preselected altitudes were desired, and of course the \( x_s \) values gotten as just outlined would almost never happen to fall at appropriate altitudes. Therefore a further treatment of all the data gotten from the first treatment of the trajectories had to be gone through to get tables of sets of numbers \( x_s, \lambda_s, \eta \), now with the latitudes being the preselected ones, with the \( x_s \) being such as to correspond to the preselected altitudes, and the whole collection of data being arranged in sets, each pertaining to one value of \( r_\circ \), or particle rigidity, and to one value of the parameter \( \gamma \). This stage of the work was carried out by plotting, for each set of data gotten from the latitude-interpolations performed directly on the trajectories, the values of \( \eta \) against those of \( x_s \), one plot being drawn for each set of pairs of values of \( x_s \) and \( \eta \) which pertained to one value of \( \gamma \) and to one latitude. After passing smooth curves through the plotted points, values of \( \eta \) for the preselected altitudes could be read off directly with about three-digit accuracy (in the reading alone).

Samples of the kinds of \( \eta, x_s \) curves obtained are shown in Figures 30 and 31. One of these is a rather favorable case and one a rather unfavorable case, having rather few and poorly distributed points in one segment. Often drawing of such curves was much and very reliably
helped by prior locating of special points or features pertaining to them. For example, if a line of latitude of interest in the $x, \lambda$ plane is seen to intersect the outer region of that plane where $P < 0$, then the $\eta, x_s$ curve for that case obviously must have closed off and terminated before the value of $x_s$ corresponding to the intersection just referred to is reached, and furthermore drawing the envelope to the family of trajectories of the fourth kind at their northern "peaks" fixes by its intersection with the latitude of interest often to very good accuracy the point in $x_s$ at which the $\eta, x_s$ curve terminates, which it does with a value of $\eta$ which often could be found quite closely. Use of this kind of device, and of others, and exercise of great care in verifying the points used and in the graphical work rendered the interpolations in altitude sufficiently precise, it is believed.

At this stage there was on hand essentially the collection of cone diagram points wanted, in the form of values of $x_e, x_s, \lambda_s, \eta$, and $\gamma$. Each such set had to be used to produce corresponding values of the cone diagram coordinates $\sin \theta$ and $\cos \theta \sin \gamma$. Fortunately it was found possible to by-pass the insertion of each of long lists of numbers into the formula for $\sin \theta$ by carrying out preliminary computations for the production of curves showing $\sin \theta$ as a function of $x_s$, of $\lambda_s$, and of $\gamma$. The total amount of computation for this job was small, and from the curves produced by it values of $\sin \theta$ could be read off with accuracy to about three decimal digits, or with as much accuracy as the coordinate could be plotted in a cone diagram.
This left only the very tedious work of listing values of \( \cos \theta \) and \( \sin \theta \), and of carrying out the multiplications to produce the coordinates \( \cos \theta \sin \eta \), and, finally, the graphical work of plotting the computed points in cone diagrams and of connecting them by smooth curves. The total number of cone diagram points actually computed was set largely by the decision concerning the specific altitudes which were to be dealt with, and amounted to about 6,000 in all.

B. Analysis of Errors

In the numerical part of the work described in part II of this paper, as much precision as could be obtained was important, since part of the purpose of the work was the correction of previous erroneous work. In the computation of the principal shadow cones, however, it was found to be very advantageous to relax the requirements for precision previously in force, when doing so would simultaneously leave the final precision in the cones acceptably high and reduce the size of the entire task.

To begin with, it is certain that errors in the machine-computed trajectories are so much smaller than errors introduced during the various later stages of the calculations, that they can be completely neglected.

Then the first introduction of errors occurred during the linear interpolations between machine-produced points to get data at preselected latitudes. These interpolations were done much as were the interpolations involved in the computation of simple shadow cones. Gross errors
and even merely unusually large ones were always, it is believed, quickly detected when the results were plotted for drawing the \( \eta, x_s \) graphs. Examination of the operations involved and testing of the results led to the conclusion that the interpolations over latitudes were such that the values of \( \eta \) produced were in the vast majority of cases reliable to within at least \( \pm 0.03 \) radians.

It is convenient to regard all the errors arising in the interpolations over latitudes and over altitudes as being attached only to the values of \( \eta \) produced. This is desirable because it makes possible a simple kind of estimate of errors in the final cone diagram points, and it is possible because of two considerations. The first of these is that one can always take the value of \( x_s \) produced by the interpolation on a trajectory to find the crossing by that trajectory of a given line of latitude as being absolutely correct, with the understanding that the \( x, \lambda \) point fixed by that value of \( x_s \) and the value of \( \lambda_s \) in question really then lies on a trajectory neighboring the one being dealt with. Then using the interpolated value of \( \eta \) amounts to using a value which may be somewhat incorrect for that neighboring trajectory. The other consideration is that the values of \( x_s \) produced by these first interpolations are not of interest in themselves, but only as fixing (with their associated values of \( \eta \)) curves showing \( \eta \) as functions of \( x_s \). In the next series of interpolations for collecting data at preselected altitudes, values of \( x_s \) are chosen arbitrarily, and so are absolutely correct. Values of \( \eta \) to be associated with these latter values of \( x_s \)
are read off the $\eta$, $x_s$ curves, and in the process it is obvious that all the accompanying error can be attached to the values of $\eta$.

In the interpolations performed by use of the $\eta$, $x_s$ plots, the errors introduced depended on errors already present in the graphs and on errors in reading off values from those graphs. Values of $\eta$ could be read from the graphs to about three decimal digits accuracy, and the graphs were rarely themselves so precise; therefore the errors in the $\eta$-values used in computing cone diagram points were nearly entirely those inherent in the $\eta$, $x_s$ graphs. This assumes that pure misreadings were detected and corrected when cone diagram points were computed; it is believed that this was always done.

It is estimated that at least nearly every value of $\eta$ abstracted from the graphs was reliable to within $\pm 0.05$ or $0.06$ radians, for a single overlarge figure. The majority of the cone diagram points are determined to within much better accuracy than this nominal figure would indicate, but the quantity of data dealt with was such that attachment of error estimates to individual $\eta$-values was not attempted, and it was believed quite adequate to use one estimate to apply to all.

In order that the significance of such errors in $\eta$ for actual cone diagrams may be judged, Figure 32 is presented. This shows the intervals in the vertical coordinates in a cone diagram at various locations in a quadrant, each interval being that corresponding to a range in $\eta$ of 0.1 radian.

Errors in cone diagram points due to errors in $\sin \theta$ or $\cos \theta$ are
considered as negligible because each of those quantities was obtained with as much precision as that with which points could be plotted in cone diagrams, and because the errors are small compared to the nominal overall error estimate in $\sin \eta$.

Finally, errors in the principal shadow cones, as opposed to errors in the points lying on them, were introduced in the process of drawing the cones through the points. Usually this could be done very well, but often the number of points determining a cone was somewhat small and sometimes the points were not distributed in optimal manner over the cones. It is largely left to the reader to gauge the reliability of the curves drawn. Segments are shown in solid curves when they are considered well established. (It should be borne in mind that often such segments are supported by many computed points lying on neighboring cones but not shown in the figures to avoid an excessive profusion of points.) Segments are shown in dashed curves otherwise.

Segments of cones near the edges of cone diagrams are considered to be reliable to within at least amounts comparable to those indicated by Figure 32. Elsewhere, and especially when long interpolations between computed points must be made, arcs of cones may, as the reader can judge for himself, be in doubt by perhaps 0.05 in cone diagram units in the worst cases. However such doubt is not generally to be felt, the circumspectory eye indicating that considerably better faith can usually be had in them.

It should be taken into account that the most significant question
that can be asked concerning the overall accuracy of the cones is that having to do with the accuracy with which the fraction of the sky rendered dark by any given cone can be determined with the data shown. Inspection of the figures and some consideration will show that the accuracy in such fractions will in most cases easily be much better than 10%, large cone uncertainties generally accompanying large cone areas and small cone errors generally accompanying small cone areas, and such precision is held by the author to be more than sufficient for the uses to which the data will be put.

Where individual cones have peculiar features such as small projecting lobes or indentations, those features can be well understood and will be discussed later. The fineness with which such features have been determined nearly always depends almost entirely on the precision in the computed points and is considered very acceptable. Again, the question of the accompanying precision in the total fraction of the sky rendered dark by the cones is a more significant question.

C. Explanations of the Figures Showing Principal Shadow Cones

Each of the Figures 33 through 63 exhibits a collection of principal shadow cones for one value of particle rigidity ($r_e$) and one latitude of observation ($\lambda_s$) over a range of altitudes which usually extends to some 7,000 or 8,000 miles. The cones shown in each figure are a selection from the totality computed, an effort having been made to avoid an unnecessarily excessive number, while giving cones spaced
closely enough in altitude to make interpolations for cones at inter-
mediate altitudes easy.

Complete identification of the values of the parameters pertaining
to any given cone is to be made by referring to the values of \( r_s \) and \( \lambda_s \) attached to the figure containing the cone, and to the number labelling
the cone itself. This latter number is a value of the quantity \((x_s - x_0)\)
multiplied by 100. That is, the altitude in miles (or other units of
length) for any cone is obtained by first moving to the left by two
places an imagined decimal point originally at the right of the labelling
number and then by referring to a table of the exponential function (or
of natural logarithms) to make use of the relation

\[ e^{x_s - x_0} = \frac{r_s}{r_0}. \]

The coordinate system in each figure is a rectangular system with
origin at the center, the axes corresponding to \( \sin \theta \) and \( \cos \theta \sin \eta \) in
the usual way. Labelling of axes and numbering of the principal coordi-
nate lines has not been done to avoid unnecessary markings in the figures.

Special coding devices used are few and simple. Since each cone
diagram is now a projection on a plane of a sphere of directions, it is
necessary to distinguish points in the observer's upper hemisphere of
directions from those in the lower hemisphere. Segments of cones lying
in the upper hemisphere are usually easily distinguishable since they
pass through circled points; they have also been drawn with finely dashed
curves whereas segments lying in the lower hemisphere are drawn in solid
curves when they are considered quite well determined and in dashed
D. Discussion of the Principal Shadow Cones

The principal shadow cones shown in the figures are, it is safe to say, surprising when seen for the first time. Some of them are of odd forms unlikely to be expected beforehand and the variation in size exhibited by them with the change in the values of the parameters sometimes is most unlikely to have been anticipated. For example, one may note the decrease in size for a given altitude at latitudes near the equator that accompanies a decrease in \( r_e \).

Such features are, of course, understandable qualitatively either on physical grounds or in terms of peculiarities of the \( x, \lambda \) plane and of motions in that plane. The discussion to follow will have as its purpose the explanation of the salient features of the cones.

In this connection it is very useful to be able to gauge the effect of the dipole magnetic field of the earth in creating the principal shadow regions, as opposed to the effect which an impenetrable earth would have if it possessed no magnetic field. This can be very easily done in a way now to be explained. Draw in a plane representing a meridian plane of the earth a circle with radius \( \rho_s \) representing the surface of the earth, and take a point outside that circle at radial distance \( \rho_s \) from the center of the earth. In the meridian plane \( \cos \theta = 1 \). The half-angle subtended by the earth at the point is \( \eta \) and one reads off from the figure the relation \( \sin \eta = \rho_e / \rho_s \). This means that the
purely geometrical shadow of the earth at the observation point is given on a cone diagram by a circle in that diagram with center at the center of the diagram and with radius $\frac{\varphi_e}{\varphi_s}$, or the reciprocal of the quantity $\frac{\varphi_s}{\varphi_e}$ applying to any principal shadow cone as a measure of the altitude.

There is a characteristic shift to the north and east for most cones as compared with the geometric shadow. This is the same feature that marks the simple shadow cones at the earth's surface and is understandable in the following way: Westerly directions are distinguished as generally comparatively favored directions of arrival because the sense in which paths of positively charged particles in the earth's field in ordinary three-dimensional space and as given by the Lorentz force is clockwise as seen from a point over the earth's geographic north pole. Northerly directions tend to be more difficult as directions of arrival than southerly ones because particles coming to an observation point in the northern hemisphere from the south tend to follow the lines of force of the earth's field more than do those from the north, the lines of force bending over in the north and obliging the particles to cross them.

There is also a shift to the north at low latitudes as the latitudes increase from the equator, this being the loss of symmetry to be expected as the equator is left behind. There is, when high latitudes begin to be attained, a strong tendency for the cones to move in toward the centers of the cone diagrams and to become circular, this obviously being an approach to the kind of symmetry appropriate for the polar
regions.

For the high rigidities and at low latitudes there is, as can be verified, a general agreement between the sizes of the cones and the sizes of corresponding geometrical shadows. At higher latitudes there can be observed a tendency for the principal shadow cones to be smaller by small amounts than the corresponding geometrical shadows. This is attributed to a kind of magnetic lens effect of the earth's field; reference to three-dimensional wire models of particle trajectories, which are readily to be found pictured in the literature, support the existence of this effect. For the lower rigidities and low latitudes it is impossible to make direct comparison of the dark regions and geometrical shadows, but for such rigidities and higher latitudes a general correspondence can be seen again, as can some tendency for the cones to be smaller than the geometrical shadows.

The nesting of the cones which predominates in most of the diagrams, by which is meant the tendency for cones not to intersect each other, is apparently not associated with any interesting physical traits of the particle motion, and in fact this effect does not always occur, contrary examples being easy to find. The nesting which was earlier noticed for the simple shadow cones corresponds to the distinct nesting which usually is present for low altitudes, for the principal shadow cones.

Most often the cones are of simple forms which do not seem to ask for qualitative explanation, but in some cases there are curious indenta-
tions in them, and in some equatorial or near-equatorial cases there is to be seen a rounded capital sigma-form, both of which effects do ask to be accounted for. Whenever the observation point in the $x, \lambda$ plane lies right of the inner branch of the curve $P = 0$ and on the equator, and at the same time the earth intersects that branch, there will be for the value of $\psi'$ in question two principal shadow regions in the angle $\psi$ at that observation point, one of which reaches up to the surface of the earth in the northern hemisphere and the other of which is symmetrical with respect to the first and reaches down to the surface of the earth in the southern hemisphere. Between these two is a penumbral region. Now with an increase in the value of $\psi'$ when the latitude and altitude of the observation point are held fixed, the surface of the earth as well as the observation point move right in the $s, \lambda$ plane, and when the surface of the earth has moved so far right as no longer to intersect the inner branch of the curve $P = 0$, then only one principal shadow region for the observation point will exist. This effect, together with the symmetry which exists for observation points on the equator, accounts for the sigma-forms sometimes found for the cones. A very similar effect can occur when the observation point in the $x, \lambda$ plane is not on the equator, but now the north-south symmetry will have been lost.

Observation points well up in the northern arm of the $P > 0$ region will have the trajectories of the fourth kind out of that point which bound a principal shadow region directed, generally speaking, to the left and somewhat upward. If the value of $\psi'$ involved now increases,
while the latitude and altitude are held constant, the earth and observation point move to the right, and for some values of the parameters involved the earth begins to move over the steep rounding-off of the inner branch of the curve $P = 0$. As it does so, the trajectories of the fourth kind bounding principal shadow regions can point more and more in a southerly direction, and when the steep round-off of $P = 0$ is passed over sufficiently the transition to southern-pointing trajectories of the fourth kind occurs rapidly, so that there can occur in the case of a segment of a principal shadow cone in a cone diagram first an arc over which the vertical coordinate remains roughly constant, followed by an arc bending more or less abruptly downward; such cases can be seen among the cones.

The variation in the cones with decreasing rigidity $r_e$ but fixed altitude and latitude will now be discussed. Near the equator, notably, this decrease is accompanied by a decrease in the size of the principal shadow cones, whereas the intuition might seem to suggest that there should be an accompanying increase in the sizes of the cones. However, what is properly to be expected with decrease in rigidity is an increase in the total amount of darkness in the cone diagrams and not necessarily an increase in the amount of darkness contained in the principal shadow regions. An examination of the events in the $x, \lambda$ plane with decreasing $r_e$ which can easily be made but which requires too lengthy description for inclusion here, shows that an increase in the total amount of darkness with decreasing $r_e$ does take place, especially for values of $\lambda$ in
the vicinity of \( l \), precisely where the penumbral regions become important. An increase in the total amount of darkness with a decrease in \( r_e \) which does not appear in the change of size of the principal shadow region is accounted for by the concomitant change in the amount of penumbral darkness and Störmer darkness. This effect is much more marked at lower altitudes than at higher ones; in the latter cases the topographical nature of the contour map of the function \( P \) in the neighborhood of the observation point and surface of the earth is much more independent of the value of \( r_e \).

Some concluding remarks will be appended here about work in progress or soon to be instituted to utilize the principal shadow cones. For application of the results contained in this paper to cosmic ray observational data obtained with use of omnidirectional detectors, there is needed for each cone a value for the fraction of the sky rendered dark by that cone, along with a reliable estimate of contributions to the dark fraction by the penumbral dark regions. The work of computing such data has been started at the State University of Iowa and will be carried on in a continuing way. For application to observational data obtained with detectors with directional properties (such as have already been proposed for inclusion in earth satellites) curves showing cut-off rigidities as functions of latitude, altitude and direction of observation are more useful, and it is proposed to extract such data from the cones. There will be carried out an investigation of the effect on the interpretations of cosmic ray data obtained during balloon or rocket
flights of failure heretofore to apply principal shadow cones appropriate for the altitudes in question instead of the simple shadow cones.
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Books


Conferences and Symposia

Figure 2

\[ \gamma_1 = 0.92 \]
\[ \chi_0 = -0.08 \]
\[ \lambda_0 = 31.7^\circ \]
FIGURE 3
\[ \lambda = 30^\circ \]
\lambda = 60^\circ
FIGURE 13
FIGURE 16
FIGURE 19
FIGURE 20
FIGURE 22
FIGURE 23
FIGURE 27
FIGURE 28
FIGURE 29
Figure 30

\[ r_e = 0.6 \]
\[ \lambda_c = 40 \circ \]
\[ \gamma' = 0.8 \]
FIGURE 31
FIGURE 32
\[ r_0 = 0.8 \]
\[ \lambda_s = 10^\circ \]

FIGURE 3h
$r_e = 0.8$

$\lambda_s = 30^\circ$

FIGURE 36
\( \eta = 0.8 \)
\( \lambda = 50^\circ \)

FIGURE 38
\[ r_e = 0.8 \]
\[ \lambda_1 = 60^\circ \]

FIGURE 39
\( G = 0.8 \)
\( \lambda_s = 70° \)

**Figure 40**
FIGURE 59
FIGURE 63