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Some topics in abstract factorization

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SOME TOPICS IN ABSTRACT FACTORIZATION

by

Jason Robert Juett

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
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Thesis Supervisor: Professor Daniel Anderson

ABSTRACT

Anderson and Frazier [6] defined a generalization of factorization in integral domains called τ-factorization. If D is an integral domain and τ is a symmetric relation on the nonzero nonunits of D, then a *τ-factorization* of a nonzero nonunit $a \in D$ is an expression $a = \lambda a_1 \cdots a_n$, where λ is a unit in D, each a_i is a nonzero nonunit in D, and $a_i \tau a_j$ for $i \neq j$. If $\tau = D^{\#} \times D^{\#}$, where $D^{\#}$ denotes the nonzero nonunits of D, then the τ-factorizations are just the usual factorizations, and with other choices of τ we get interesting variants on standard factorization. For example, if we define $a \tau_d b \Leftrightarrow (a, b) = D$, then the τ_d-factorizations are the *comaximal factorizations* introduced by McAdam and Swan [19]. Anderson and Frazier defined τ-factorization analogues of many different factorization concepts and properties, and proved a number of theorems either generalizing standard factorization results or the comaximal factorization results of McAdam and Swan. Some of these concepts include τ-UFD's, τ-atomic domains, the τ-ACCP property, τ-BFD's, τ-FFD's, and τ-HFD's. They showed the implications between these concepts and showed how each of the standard variations implied their τ-factorization counterparts (sometimes assuming certain natural constraints on τ). Later, Ortiz-Albino [21] introduced a new concept called Γ-factorization that generalized τ-factorization. We will summarize the known theory of τ-factorization and Γ-factorization as well as introduce several new or improved results.

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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ABSTRACT

Anderson and Frazier [6] defined a generalization of factorization in integral domains called τ-factorization. If D is an integral domain and τ is a symmetric relation on the nonzero nonunits of D, then a *τ-factorization* of a nonzero nonunit $a \in D$ is an expression $a = \lambda a_1 \cdots a_n$, where λ is a unit in D, each a_i is a nonzero nonunit in D, and $a_i \tau a_j$ for $i \neq j$. If $\tau = D^{\#} \times D^{\#}$, where $D^{\#}$ denotes the nonzero nonunits of D, then the τ-factorizations are just the usual factorizations, and with other choices of τ we get interesting variants on standard factorization. For example, if we define $a \tau_d b \Leftrightarrow (a, b) = D$, then the τ_d-factorizations are the *comaximal factorizations* introduced by McAdam and Swan [19]. Anderson and Frazier defined τ-factorization analogues of many different factorization concepts and properties, and proved a number of theorems either generalizing standard factorization results or the comaximal factorization results of McAdam and Swan. Some of these concepts include τ-UFD's, τ-atomic domains, the τ-ACCP property, τ-BFD's, τ-FFD's, and τ-HFD's. They showed the implications between these concepts and showed how each of the standard variations implied their τ-factorization counterparts (sometimes assuming certain natural constraints on τ). Later, Ortiz-Albino [21] introduced a new concept called Γ-factorization that generalized τ-factorization. We will summarize the known theory of τ-factorization and Γ-factorization as well as introduce several new or improved results.

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CHAPTER 1
INTRODUCTION

In [6], the authors define and study a generalization of factorization in integral domains called τ-factorization. If D is an integral domain and τ is a symmetric relation on the nonzero nonunits of D, then a *τ-factorization* of a nonzero nonunit $a \in D$ is an expression $a = \lambda a_1 \cdots a_n$, where λ is a unit in D, each a_i is a nonzero nonunit in D, and $a_i \tau a_j$ for $i \neq j$. If $\tau = D^{\#} \times D^{\#}$, where $D^{\#}$ denotes the nonzero nonunits of D, then the τ-factorizations are just the usual factorizations, and with other choices of τ we get interesting variants on standard factorization. For example, if we define $a \tau_d b \Leftrightarrow (a, b) = D$, then the τ_d-factorizations are the *comaximal factorizations* introduced in [19]. In [6], the authors defined τ-factorization analogues of many different factorization concepts and properties, and proved a number of theorems either generalizing standard factorization results or the comaximal factorization results of [19]. Some of these concepts include τ-UFD's, τ-atomic domains, the τ-ACCP property, τ-BFD's, τ-FFD's, and τ-HFD's. (See [3] for a study of the analogous standard factorization concepts.) They showed the implications between these concepts and showed how each of the standard variations implied their τ-factorization counterparts (sometimes assuming certain natural constraints on τ). Later, in [21], a new concept called Γ-factorization was introduced that generalized τ-factorization. We will summarize the known theory of τ-factorization and Γ-factorization as well as introduce several new or improved results.

While one usually has an integral domain in mind when thinking of factorization, we will find it useful to work in full generality as much as possible, not assuming the presence of any additive structure or cancellative property, especially when formulating definitions. A careful reading of the previous work on "abstract factorization" theories such as τ-factorization or Γ-factorization reveals that virtually none of the general theory is lost by removing the additive structure, and large portions of the theory do not require the cancellative property. We will see several interesting examples of factorization occurring with no additive structure, thus showing that our extra generality is far from frivolous and allows some interesting special cases to be subsumed by the theory. One such special case of particular interest is the comaximal factorization of ring ideals studied in [19][Section 5] and later generalized in [17]. Some aspects of the theory of abstract factorization are considerably simplified by restricting to the cancellative monoid setup, while others are developed in absolutely identical fashion whether one assumes the cancellative property or not. This thesis will have a cancellative focus in the sense that the cancellative property will always be assumed in the first situation, while in the second situation we will often (but not always) state results and definitions in their full generality. We mention the upcoming thesis [20] as an important resource for the non-cancellative focus.

1.1 Basic Definitions

Throughout this thesis, a *monoid* will refer to a commutative multiplicative semigroup with $1 \neq 0$ unless stated otherwise. Similarly, all rings will be commutative with $1 \neq 0$.

Let H be a monoid. We will use use $H^\times$ to denote the group of units of H, and set $H^* = H \setminus \{0\}$, $H^\# = H^* \setminus H^\times$, and $H_0^\# = H \setminus H^\times$. Elements $a, b \in H$ are called:

(1) *associates*, denoted $a \sim b$, if $a \mid b$ and $b \mid a$, or, equivalently, if $aH = bH$;

(2) *strong associates*, denoted $a \approx b$, if $a = \lambda b$ for some $\lambda \in H^\times$; and

(3) *very strong associates*, denoted $a \cong b$, if $a \sim b$ (equivalently, $a \approx b$) and either $a = b = 0$ or $a \neq 0$ and $a = rb \Rightarrow r \in H^\times$.

The relation $\cong$ is symmetric and transitive, the relations $\sim$ and $\approx$ are congruence relations, and $\cong \leq \approx \leq \sim$. If $\approx = \sim$, then H is called a *strongly associate monoid.* We say H is *présimplifiable* if one of the following equivalent statements holds: (1) $a = ab$ implies $a = 0$ or $b \in H^\times$, (2) $\cong = \sim$, (3) $\cong$ is reflexive, or (4) $\cong$ is a congruence relation. The monoid H is called *cancellative* if $ac = bc$ implies $a = b$ or $c = 0$. It is easily seen that cancellative $\Rightarrow$ présimplifiable $\Rightarrow$ strongly associate. The papers [9] and [4] are our primary references on associate relations and the présimplifiable property. Those papers work in the context of rings, but the results from them that we will use generalize to monoids with essentially the same proofs.

The most rigorous way to define factorizations was first laid out in [7]. That paper worked in the context of integral domains, but this and many other definitions lead to obvious extensions to the full generality of monoids. A *(reduced) factorization* in a monoid H is a formal word $(\lambda, a_1, \ldots, a_n, 1, 1, \ldots)$, where $\lambda \in H^\times$ ($\lambda = 1$), $n \geq 1$, and each $a_i \in H^\#$. We will also regard $(\lambda, 0, 1, 1, \ldots)$ as a factorization for each $\lambda \in H^\times$, and call it *reduced* if $\lambda = 1$. For a nonunit a, the set of (reduced) factorizations $(\lambda, a_1, \ldots, a_n, 1, 1, \ldots)$ with $a = \lambda a_1 \cdots a_n$ are the *(reduced) factorizations of* a. For the sake of convenience, we will take the usual approach of identifying a factorization $(\lambda, a_1, \ldots, a_n, 1, 1, \ldots)$ with the expression $\lambda a_1 \cdots a_n$, and in the case $\lambda = 1$ we will write simply $a_1 \cdots a_n$. In a factorization $\lambda a_1 \cdots a_n$, the *leading unit* is λ, the *factors* are the a_i's, and the *length* of the factorization is n. A *trivial factorization* is a factorization of length 1. The set of factorizations (resp., trivial factorizations, reduced factorizations, trivial reduced factorizations) will be denoted by $\mathrm{fact}(H)$ (resp., $\mathrm{tfact}(H)$, $\mathrm{rfact}(H)$, $\mathrm{trfact}(H)$), and the set of factorizations (resp., trivial factorizations, reduced factorizations, trivial reduced factorizations) of a nonunit a by $\mathrm{fact}(a)$ (resp., $\mathrm{tfact}(a)$, $\mathrm{rfact}(a)$, $\mathrm{trfact}(a)$).

Historically, the vast majority of work done with factorization has been in the integral domain context, where the factorizations of 0 are hopelessly boring, so in such a setup it would make good sense to not allow 0's to appear in factorizations. However, in the presence of zero divisors it makes sense to consider the factorizations of 0, and to formulate definitions for when 0 is irreducible, prime, and so on. It is beneficial to translate definitions such as these so that they are wholly in terms of factorizations, so that they can be readily generalized to the abstract factorization systems we will introduce shortly. For example, a nonunit p is *prime* if whenever it divides a factorization $\lambda a_1 \cdots a_n$, it divides some a_i, and the notion of *divides* itself is easily translated as being a factor in a factorization. In order to be able to extend this approach to 0 in a way consistent with how it is done with all the other nonunits, there is no way around allowing 0 to appear in some factorizations, which is our motivation for allowing the factorizations $0 = \lambda \cdot 0$. However, allowing 0 to appear in nontrivial factorizations causes inconveniences in formulating some definitions, which has led us to restrict it to being a factor in these trivial factorizations. For example, we want the definitions to work out so that H has no zero divisors if and only if 0 has no nontrivial factorizations, and clearly this desired theorem holds if and only if 0 is not allowed as a factor in any nontrivial factorizations.

We call a subset Γ of $\mathrm{fact}(H)$ a *factorization system* on H. Often we will shorten "factorization system on H" to "factorization system" or simply "system" if what we mean is clear from context. For a nonunit a, we define $\Gamma(a) = \Gamma \cap \mathrm{fact}(a)$. A Γ-*factorization (of a nonunit a)* is an element of Γ ($\Gamma(a)$). We call a nonunit a Γ-*expressible* if $\Gamma(a) \neq \emptyset$. If $a = \lambda a_1 \cdots a_n$ is a Γ-factorization, then we call a a Γ-*product* of $a_1, \ldots, a_n$. We call the factors of the Γ-factorizations of a nonunit b the Γ-*factors* or Γ-*divisors* of b; if a is a Γ-factor of b, then we say a Γ-*divides* b and write $a \mid_\Gamma b$. The set of reduced (resp., trivial, trivial reduced) Γ-factorizations is denoted by Γ_r (resp., Γ_t, Γ_{tr}).

We introduce a new concept of ψ-factorization lying somewhere between τ-factorization and Γ-factorization. For a relation ψ on H, we define a factorization system Γ_ψ that consists of the factorizations $\lambda a_1 \cdots a_n$ with $a_i \psi a_j$ for $i < j$ and each $\lambda \psi a_i$. (Note that $a_1 \cdots a_n \in \Gamma_\psi \Leftrightarrow a_i \psi a_j$ for $i < j$ and each $1 \psi a_i$.) A ψ-*factorization* is a Γ_ψ-factorization. We similarly abbreviate all other "Γ_ψ" phrases where the abbreviation will not cause confusion, even abbreviating "Γ_ψ" itself with "ψ" when it does not cause difficulties. (One thing to avoid is writing a factorization system of the form $\bigcup_{i\in I}\psi_i$, as the two possible interpretations $\bigcup_{i\in I}\Gamma_{\psi_i}$ and $\Gamma_{\bigcup_{i\in I}\psi_i}$ are in general not the same thing. However, writing factorization systems in the form $\bigcap_{i\in I}\psi_i$ is all right, since $\bigcap_{i\in I}\Gamma_{\psi_i} = \Gamma_{\bigcap_{i\in I}\psi_i}$.) We will write ψ_H or τ_H for the relation $H \times H$ and $\psi_\emptyset$ for the empty relation on H. Note that the factorization system $\psi_\emptyset$ is simply the empty factorization system, which we will sometimes prefer to denote with $\Gamma_\emptyset$.

Finally, we are ready to define the τ-factorizations. For a relation τ on $H^\#$, we define a factorization system Γ_τ by defining $\lambda a_1 \cdots a_n \in \Gamma_\tau \Leftrightarrow a_i \tau a_j$ for $i < j$. A *τ-factorization* is an element of Γ_τ, and we abbreviate all the Γ_τ phrases like we did for ψ-factorization. If we think of τ as a relation on H by defining $\lambda\tau a$ for every $\lambda \in H^\times$ and $a \in H_0^\#$, then this definition is consistent with the one above for ψ-factorizations. We will write $\tau_\emptyset$ for the empty relation on $H^\#$; expanding $\tau_\emptyset$ to a relation on H in the above way gives $\tau_\emptyset = H^\times \times H_0^\#$. Note that the factorization system $\tau_\emptyset$ is not $\Gamma_\emptyset$, but rather $\mathrm{tfact}(H)$. If τ is symmetric, then we can equivalently replace the "$i < j$" in the definition with "$i \neq j$". Most of the literature only considers the case where τ is symmetric and would use the phrase *ordered τ-factorizations* to refer to τ-factorizations with τ possibly non-symmetric, but we will carry things out for non-symmetric relations as far as possible. Historically, the progression of ideas went: τ-factorization, reduced τ-factorization, Γ-factorization, and then the ψ-factorization introduced here. However, in the following chapter, we will find it most efficient and illuminating to develop the theory going from most general to most specific: Γ-factorization, ψ-factorization, and then τ-factorization.

For a thorough introduction to τ-factorization along with the original proofs of several key results, see [6]; for a brief summary, see the introductory sections of [18]. For a survey of the three major theories (τ-factorization, reduced τ-factorization, and Γ-factorization) and the relationship between them, see [7]. Those papers work within the context of integral domains, but a fair portion of it carries over with essentially no change to rings with zero divisors or even non-cancellative monoids. For those more subtle parts of the theory that do not trivially carry over, one should consult [20].

We end this introductory chapter with a few examples of interesting factorization systems that motivate our study of abstract factorization.

Example 1.1.1.

(1) For any monoid H, the τ_H-factorizations are simply the usual factorizations.

(2) Let R be a ring. In analogy with [6], we define a symmetric relation τ_d on $R^\#$ by $a\tau_d b \Leftrightarrow (a,b) = R$. The τ_d-factorizations are the *comaximal factorizations* studied in [19]. More generally, if $\star$ is a weak ideal system on a monoid H, then we have a symmetric relation $\tau_\star$ on $H^\#$ given by $a\tau_\star b \Leftrightarrow \{a,b\}_\star = H$, which give rise to the $\star$-*comaximal factorizations* introduced in [6]. (A very thorough reference for weak ideal systems is [15], while the reader may refer to [17][Section 3] for a quicker survey. The comaximal factorizations correspond to the special case where $\star$ is the d-operation that takes a subset of a ring to the ideal that it generates.)

(3) Similarly, we can study comaximal factorizations of ideals, by letting $\mathcal{I}(R)$ be

the monoid of ideals of a ring R and defining $I\tau_d J \Leftrightarrow I+J=R$. More generally, if $\star$ is a weak ideal system on a monoid H, then its $\star$-ideals form a monoid $\mathcal{I}_\star(H)$ under the $\star$-multiplication $I \star J = (IJ)_\star = \{ab \mid a \in I, b \in J\}_\star$, and we define $I\tau_\star J \Leftrightarrow (I \cup J)_\star = H$, giving rise to the $\star$-comaximal factorizations of ideals that are the subject of [17].

(4) Let H be a monoid. Analogously to [6], we define a symmetric relation $\tau_{[]}$ on $H^\#$ by $a\tau_{[]}b \Leftrightarrow a$ and b are relatively prime. Then $\tau_{[]} \leq \tau_d$. This relation and the factorizations associated with it are further studied in [22] and [8].

(5) Let X be a set. Then $(\mathcal{P}(X), \cup)$ is a monoid. Define a relation $\tau_\sqcup$ on $\mathcal{P}(X)^\# = \mathcal{P}(X) \setminus \{\emptyset\}$ by $Y\tau_\sqcup Z \Leftrightarrow Y \cap Z = \emptyset$. So the $\tau_\sqcup$-factorizations of a subset of X are the different ways of writing it as a disjoint union of nonempty subsets. If X is a topological space and we replace $\mathcal{P}(X)$ with the closed subspaces of X, then the irreducible closed subspaces of X are precisely those with no nontrivial $\tau_\sqcup$-factorization.

(6) Let R be a ring and J be an ideal of R. Define a relation τ_J on $R^\#$ by $a\tau_J b \Leftrightarrow a - b \in J$. An important special case is the $\tau_{(n)}$-relation on $\mathbb{Z}^\#$, where $n \in \mathbb{Z}$. The (reduced) $\tau_{(n)}$-factorizations have been extensively studied in [6], [14], and [16].

(7) While a lot of the most natural examples of factorization systems correspond to some sort of τ-factorization, there are many that cannot be written this way. For instance, the factorization system on $\mathbb{Z}$ consisting of factorizations with at most two even factors is one such system.

CHAPTER 2
FROM Γ-FACTORIZATION TO τ-FACTORIZATION

The purpose of this chapter is to redevelop the elementary theory of τ-factorization starting from the more general theory of Γ-factorization. As we do this, we will get an appreciation for what results about τ-factorization are actually true in a much greater generality, and which things about τ-factorization are truly special.

2.1 Γ-factorization

Before introducing the basic Γ-factorization properties, we need to give a precise definition of "Γ-refinement". Let Γ, Γ_1, and Γ_2 be factorization systems on a monoid H. A Γ_1-Γ_2-*refinement* of a factorization $\lambda a_1 \cdots a_n$ is a Γ_2-factorization of the form $\lambda b_{1,1} \cdots b_{1,m_1} \cdots b_{n,1} \cdots b_{n,m_n}$, where each $a_i = b_{i,1} \cdots b_{i,m_i}$ is a $((\Gamma_1)_r \cup \text{trfact}(H))$-factorization; in this case we call the first factorization a Γ_1-Γ_2-*partition* of the second, and, if some $m_i > 1$, then we call the Γ_1-Γ_2-refinement and Γ_1-Γ_2-partition *proper*. A Γ-*refinement* is a Γ-Γ-refinement and a Γ-*partition* is a Γ-Γ-partition. A *refinement* is a $\text{fact}(H)$-refinement and a *partition* is a $\text{fact}(H)$-partition. Some care should be taken with this terminology, since a partition of a factorization $\lambda b_1 \cdots b_m$ only corresponds to a specific kind of partition of the set $\{1, \ldots, m\}$.

We are now ready to introduce the main Γ-factorization properties that we will study. Let Γ and Γ' be factorization systems on a monoid H, and let ρ be a relation on $H^\#$. We call Γ:

(1) *symmetric* if permuting the factors of a Γ-factorization results in a Γ-factorization;

(2) *reflexive* if $0 = 0$ is a Γ-factorization and $\underbrace{a \cdots a}_{n \text{ times}}$ is a Γ-factorization for each $a \in H^{\#}$ and $n \geq 1$;

(3) *transitive* if for every two Γ-factorizations $\lambda a_1 \cdots a_m$ and $\mu a_m \cdots a_n$, the factorization $\lambda a_1 \cdots a_n$ is a Γ-factorization;

(4) *(weakly) pseudo-transitive* if whenever $m \geq 0$ (and $n = m+2$) and $\lambda a_1 \cdots a_m a_{m+1}$, $\lambda a_1 \cdots a_m a_{m+2}$, $\ldots$, $\lambda a_1 \cdots a_m a_n$, and $\mu a_{m+1} \cdots a_n$ are Γ-factorizations, then $\lambda a_1 \cdots a_n$ is a Γ-factorization;

(5) *unital* if changing the leading unit of a Γ-factorization results in a Γ-factorization;

(6) *ρ-preserving* if whenever $\lambda a_1 \cdots a_n$ is a Γ-factorization and some $b \rho a_i$, then $\lambda a_1 \cdots a_{i-1} b a_{i+1} \cdots a_n$ is a Γ-factorization;

(7) *associate-preserving* (resp., *strong associate-preserving,* Γ'*-divisive, divisive*) if it is $\sim$-preserving (resp., $\approx$-preserving, $|_{\Gamma'}$-preserving, $|$-preserving);

(8) Γ'*-refinable* if any Γ'-fact(H)-refinement of a Γ-factorization is a Γ-factorization,

(9) *refinable* if it is Γ-refinable;

(10) *combinable* if every partition of a Γ-factorization is a Γ-factorization;

(11) *(weakly) multiplicative* if whenever $\lambda a_1 \cdots a_m$ and $\lambda b_1 \cdots b_n$ are Γ-factorizations with $m \leq n$ ($m = n$) and $a_i = b_i$ for each $i \leq m$ except possibly $i = j$, then $\lambda a_1 \cdots a_{j-1}(a_j b_j) a_{j+1} \cdots a_m$ is a Γ-factorization;

(12) *(weakly) divisible* if for any Γ-factorization $\lambda a_1 \cdots a_n$, $k \leq n$ ($k \leq 2$), and $1 \leq i_1 < \cdots < i_k \leq n$, the factorization $\lambda a_{i_1} \cdots a_{i_k}$ is a Γ-factorization;

(13) *(weakly) reduced divisible* if for any Γ-factorization $\lambda a_1 \cdots a_n$, $k \leq n$ ($k \leq 2$), and $1 \leq i_1 < \cdots < i_k \leq n$, the factorization $a_{i_1} \cdots a_{i_k}$ is a Γ-factorization;

(14) *(reduced) expressive* if every nonunit has a (reduced) Γ-factorization; and

(15) *(reduced) normal* if it contains every (reduced) trivial factorization.

Note that for "refinable" we broke with our standard of dropping the "Γ" from names if $\Gamma = \text{fact}(H)$. It made the most sense to use "refinable" to mean "Γ-refinable" because the property of being fact(H)-refinable is too strong to be of any use in practice. When making some general discussion of the ρ-preserving property, nothing is lost by assuming ρ to be reflexive, since the ρ-preserving property is equivalent to the $(\rho \cup =)$-preserving property.

Most of the above definitions are from [7] and [21], where we give the newer [7] preference in the event of a conflict of definitions. Some of the definitions above are generalizations of or slight variants on the corresponding ones in those papers. The properties (reduced) expressive and (weakly) pseudo-transitive are stated for the first time here. One conflict of definitions is "multiplicative", which is defined to mean something entirely different in [21] (it is not included in the newer [7]), but we have modified it so that it is consistent with the prior definition of "multiplicative" for τ-factorization given in [6]. A more subtle conflict of definitions is that "(strong) associate-preserving" is defined to include the unital property in [7]. We have chosen not to go in this direction in order to be more consistent with prior work on reduced τ-factorization, and also because several theorems require only our weaker version, and we can obtain more powerful results in this way. (Perhaps the most important application is the aforementioned reduced τ-factorization.)

We pause to illustrate some of the above properties by returning to our examples from Chapter 1.

Example 2.1.1.

(1) The system consisting of all factorizations satisfies all of the above properties.

(2) We will later see that a factorization system Γ is of the form $\Gamma = \Gamma_\psi$ for some relation ψ on H if and only if it is (weakly) divisible and (weakly) pseudo-transitive. Similarly, it turns out Γ is of the form $\Gamma = \Gamma_\tau$ for some relation τ on $H^\#$ if and only if it is (weakly) divisible, (weakly) pseudo-transitive, and normal. The latter systems also possess the unital property. We will see that it is easy to characterize when the system τ is symmetric, reflexive, transitive, ρ-preserving, combinable, or multiplicative in terms of the relation τ. (A similar comment applies for the system ψ, though some things work out slightly less cleanly.) On the other hand, a variety of interesting facts previously shown for τ-factorization are still true for Γ-factorization with these assumptions weakened or even removed entirely, and it is for this reason that the generalization of Γ-factorization has value.

(3) The system τ_d (or more generally $\tau_\star$) is symmetric, divisive, and multiplicative. (This is true whether we are talking about the factorization of elements or ideals.) In the case of τ-factorization, divisive implies refinable, and multiplicative implies combinable. So this system has quite a few nice properties.

(4) The system $\tau_{[\,]}$ is symmetric and divisive. It is multiplicative in a GCD domain,

but in general it need not even be combinable. (For an example of a domain where $\tau_{[]}$ is not combinable, see [18, Example 2.3] or [6, Example 2.1(6)].)

(5) The system $\tau_{\sqcup}$ is symmetric, divisive, and multiplicative.

(6) The relation τ_J extends to a congruence relation in an obvious way, so the system τ_J is clearly symmetric, transitive, and reflexive. However, it is very rarely associate-preserving, refinable, or combinable. In the usual case where we are working in a domain, conditions for when τ_J is associate-preserving, divisive, or multiplicative are known – see [16, Theorem 3.13]. Examining the proof reveals that the multiplicative and combinable properties are equivalent for τ_J when working in a domain.

(7) The factorization system on $\mathbb{Z}$ consisting of factorizations with at most two even factors is symmetric, unital, normal, divisive, divisible, combinable, and multiplicative, but not refinable.

Often, when we state some result with a strong associate-preserving assumption, all we really need is the weakly strong associate-preserving property, where H is *weakly strong associate-preserving* if for any Γ-factorization $\lambda a_1 \cdots a_n$ and $\mu \in H^{\times}$, some $\lambda a_1 \cdots a_{i-1}(\mu a_i)a_{i+1} \cdots a_n$ is a Γ-factorization. Similarly, when we state some result with a (reduced) divisible assumption, it can very often be weakened to assuming the (reduced) truncatable property, where Γ is *(reduced) truncatable* if for any Γ-factorization $\lambda a_1 \cdots a_n$ and $1 \leq i \leq j \leq a_n$, the factorization $\lambda a_i \cdots a_j$ $(a_i \cdots a_j)$ is a Γ-factorization. We leave it to the reader who feels so inclined to examine the proofs and determine when these slight improvements of results can be achieved. Of course, these weaker properties are equivalent to their stronger counterparts in the usual case where Γ is symmetric.

Let H be a monoid. If P is one of the above properties other than expressive, reduced expressive, and weakly strong associate-preserving, and if $\{\Gamma_i\}_{i\in I}$ is any family of factorization systems satisfying property P, then $\bigcap_{i\in I}\Gamma_i$ satisfies property P. Because $\mathrm{fact}(H)$ satisfies all of the above properties, this enables us to define for any such P the *P closure* of a factorization system Γ to be the unique smallest factorization system containing Γ and satisfying property P. We will use Γ_u, Γ_{ap}, Γ_{sap}, Γ_s, Γ_{nl}, Γ_{rnl}, and Γ_c to denote the unital, associate-preserving, strong associate-preserving, symmetric, normal, reduced normal, and combinable closures of Γ, respectively. More explicitly,

$$\Gamma_u = \{\lambda a_1 \cdots a_n \mid \lambda \in H^\times, \mu a_1 \cdots a_n \in \Gamma\},$$

$$\Gamma_{ap} = \{\lambda a_1' \cdots a_n' \mid \lambda a_1 \cdots a_n \in \Gamma, a_i' \sim a_i\},$$

$$\Gamma_{sap} = \{\lambda(\mu_1 a_1)\cdots(\mu_n a_n) \mid \mu_1, \ldots, \mu_n \in H^\times, \lambda a_1 \cdots a_n \in \Gamma\},$$

$$\Gamma_s = \{\lambda a_{\sigma(1)} \cdots a_{\sigma(n)} \mid \lambda a_1 \cdots a_n \in \Gamma, \sigma \in S_n\},$$

$$\Gamma_{nl} = \Gamma \cup \mathrm{tfact}(H),$$

$$\Gamma_{rnl} = \Gamma \cup \mathrm{trfact}(H),$$

and

$$\Gamma_c = \{\lambda(a_{1,1}\cdots a_{1,m_1})\cdots(a_{n,1}\cdots a_{n,m_n}) \mid \lambda a_{1,1}\cdots a_{n,m_n} \in \Gamma\}.$$

Similarly, if $P_1, \ldots, P_n$ are some of these properties, then we can define the *P_1, …, P_{n-1}, and P_n closure* of a factorization system Γ to be the smallest factorization system containing Γ that satisfies $P_1, \ldots, P_n$. If these are some of the above properties whose closure operations have been assigned symbols, then we form the symbol for this combination by separating the symbols with commas. For example, the unital, associate-preserving, and symmetric closure of a factorization system Γ is denoted $\Gamma_{u,ap,s}$. The form of the (strong) associate-preserving closure can be generalized to the ρ-preserving property, with ρ any reflexive relation on $H^{\#}$. We set $\Gamma_{\rho,0} = \Gamma$ and recursively define $\Gamma_{\rho,k+1} = \{\lambda a_1' \cdots a_n' \mid \lambda a_1 \cdots a_n \in \Gamma_{\rho,k}, a_i' \rho a_i\}$ for $k \geq 0$, and then the ρ-preserving closure is given by $\Gamma_\rho = \bigcup_{k=0}^{\infty} \Gamma_{\rho,k}$. When ρ is transitive, as it is in the (strong) associate-preserving and divisive closures, then we have simply $\Gamma_\rho = \Gamma_{\rho,1}$. Closures will prove an important tool later on, as proving that certain properties are preserved by taking certain closures will allow us to know when certain simplifying assumptions are harmless. For instance, it is clear that $|_\Gamma = |_{\Gamma_s}$, and that the Γ-expressible and Γ_s-expressible nonunits coincide, so in proving any theorem about Γ-factorization that is only dependent on the $|_\Gamma$ relation and the Γ-expressible nonunits, it suffices to consider the case where Γ is symmetric.

We will now prove some equivalent ways to define the combinable, (weakly) multiplicative, refinable, ρ-preserving, and divisible properties. It is useful to have all the different possible definitions at our disposal. For the most part, we will use whichever is most convenient without an explicit reference to the following lemma.

Lemma 2.1.2. *Let Γ be a factorization system on a monoid H.*

(1) *The following are equivalent.*

  (a) *The system Γ is combinable.*

(b) *For any* Γ*-factorization* $\lambda a_1 \cdots a_n$ *and* $1 \leq i \leq n-1$*, the factorization* $\lambda a_1 \cdots a_{i-1}(a_i a_{i+1})a_{i+2} \cdots a_n$ *is a* Γ*-factorization.*

(2) *Consider the following statements.*

(a) *The system* Γ *is multiplicative.*

(b) *Whenever* $\lambda a_{1,1} \cdots a_{1,n_1}, \ldots, \lambda a_{m,1} \cdots a_{m,n_m}$ *are* Γ*-factorizations with* $n_1 \leq \cdots \leq n_m$ *and* $a_{1,i} = \cdots = a_{m,i}$ *for each* $i \leq n_1$ *except possibly* $i = j$*, then* $\lambda a_{1,1} \cdots a_{1,j-1}(a_{1,j} \cdots a_{m,j})a_{1,j+1} \cdots a_{1,n_1}$ *is a* Γ*-factorization.*

(c) *The system* Γ *is weakly multiplicative.*

(d) *Whenever* $\lambda a_{1,1} \cdots a_{1,n}, \ldots, \lambda a_{m,1} \cdots a_{m,n}$ *are* Γ*-factorizations with each* $a_{1,i} = \cdots = a_{m,i}$ *except possibly for* $i = j$*, then* $\lambda a_{1,1} \cdots a_{1,j-1}(a_{1,j} \cdots a_{m,j})a_{1,j+1} \cdots a_{1,n}$ *is a* Γ*-factorization.*

Then (a) $\Leftrightarrow$ *(b)* $\Rightarrow$ *(c)* $\Leftrightarrow$ *(d). If* Γ *is divisible, then all four statements are equivalent.*

(3) *The following are equivalent.*

(a) *The system* Γ *is (reduced) divisible.*

(b) *For any nontrivial* Γ*-factorization* $\lambda a_1 \cdots a_n$ *and* $1 \leq i \leq n$*, the factorization* $\lambda a_1 \cdots \widehat{a_i} \cdots a_n$ *(*$a_1 \cdots \widehat{a_i} \cdots a_n$*) is a* Γ*-factorization.*

(4) *The following are equivalent.*

(a) *The system* Γ *is refinable.*

(b) *Whenever* $\lambda a_1 \cdots a_n$ *and some* $a_i = b_1 \cdots b_m$ *are* Γ*-factorizations, then so is* $\lambda a_1 \cdots a_{i-1} b_1 \cdots b_m a_{i+1} \cdots a_n$*.*

(5) *The following are equivalent for a reflexive relation* ρ *on* $H^{\#}$.

(a) *The system* Γ *is* ρ*-preserving.*

(b) *Whenever* $\lambda a_1 \cdots a_n$ *is a* Γ*-factorization and each* $a_i' \rho a_i$*, the factorization* $\lambda a_1' \cdots a_n'$ *is a* Γ*-factorization.*

Proof. We prove only (1), as (2) – (5) are very simple. (a) $\Rightarrow$ (b): Clear. (b) $\Rightarrow$ (a): Assume (b) and let $\lambda b_1 \cdots b_m$ be a Γ-factorization with a partition $\lambda a_1 \cdots a_n$. We need to show that $\lambda a_1 \cdots a_n$ is a Γ-factorization. For each $i = 1, \ldots, n$ write $a_i = b_{k_i} \cdots b_{k_{i+1}-1}$, where $1 = k_1 < \cdots < k_{n+1} = n + 1$. If each $k_{i+1} = k_i + 1$, then each $b_i = a_i$, so let us assume that $m \geq 2$ and some $k_{i+1} \geq k_i + 2$. By assumption, $\lambda b_1 \cdots b_{k_i - 1}(b_{k_i} b_{k_i+1}) b_{k_i+2} \cdots b_m$ is a Γ-factorization, and the fact that $k_{i+1} \geq k_i + 2$ ensures that $\lambda a_1 \cdots a_n$ is a partition of it. Thus $\lambda a_1 \cdots a_n$ is a Γ-factorization by induction. $\square$

Now we are ready to prove several implications between the Γ-factorization properties. Admittedly, a great many of these implications are obvious from the definitions, and there are several additional trivial implications that we could have added to the list but opted not to. Additionally, some of the results in this theorem and others can be strengthened if we define yet more properties of factorization systems, but the process has to end somewhere. The conditions for when $\Gamma = \text{fact}(H)$ were done for the τ-factorization case in [21, Lemma 4.3].

Theorem 2.1.3. *Let* Γ *be a factorization system on a monoid* H.

(1) *All of the above properties are satisfied by* $\text{fact}(H)$.

(2) *Reduced normal and unital* $\Rightarrow$ *normal* $\Rightarrow$ *reduced normal* $\Rightarrow$ *reduced expressive* $\Rightarrow$ *expressive.*

(3) *Transitive* $\Rightarrow$ *pseudo-transitive* $\Rightarrow$ *weakly pseudo-transitive.*

(4) *Divisive* $\Rightarrow$ *associate-preserving* $\Rightarrow$ *strong associate-preserving.*

(5) *The system* Γ_r *is reflexive and divisive if and only if* $\Gamma_r = \mathrm{rfact}(H)$. *Hence* $\Gamma = \mathrm{fact}(H)$ *if and only if* Γ *is unital, reflexive, and divisive.*

(6) *Divisible and multiplicative* $\Rightarrow$ *combinable.*

(7) *If* Γ_r *is refinable and weakly multiplicative, then it is pseudo-transitive. Hence, if* Γ *is unital, refinable, and weakly multiplicative, then it is pseudo-transitive.*

(8) *If* Γ *is divisible and refinable, then it is* Γ_r*-divisive.*

(9) *Pseudo-transitive and normal* $\Rightarrow$ *unital.*

(10) *Combinable and divisive* $\Rightarrow$ *divisible.*

Proof. Parts (1)-(4) are clear. We now prove the remainder.

(5) ($\Rightarrow$): Assume Γ_r is reflexive and divisive. Then $0 = 0$ is a Γ_r-factorization. Now let $a_1 \cdots a_n$ be any other reduced factorization. By reflexivity, the factorization $(a_1 \cdots a_n)^n$ is a Γ_r-factorization, and so is $a_1 \cdots a_n$ by divisiveness. ($\Leftarrow$): Clear.

(6) Assume Γ is multiplicative and divisible. Then for any Γ-factorization $\lambda a_1 \cdots a_n$ and $1 \le i \le n-1$, by divisibility $\lambda a_1 \cdots \widehat{a_{i+1}} \cdots a_n$ and $\lambda a_1 \cdots \widehat{a_i} \cdots a_n$ are Γ-factorizations, and by the weak multiplicative property $\lambda a_1 \cdots a_{i-1}(a_i a_{i+1}) a_{i+2} \cdots a_n$ is a Γ-factorization. Therefore Γ is combinable.

(7) Assume Γ_r is refinable and weakly multiplicative. Let $a_1 \cdots a_m a_{m+1}$,

$a_1 \cdots a_m a_{m+2}$, $\ldots$, $\lambda a_1 \cdots a_m a_n$, and $a_{m+1} \cdots a_n$ be Γ_r-factorizations. By the weak multiplicative property, we know $a_1 \cdots a_m(a_{m+1} \cdots a_n)$ is a Γ_r-factorization, and by refinability $a_1 \cdots a_n$ is a Γ_r-factorization.

(8) Assume Γ is divisible and refinable. Let $\lambda a_1 \cdots a_n$ be any Γ-factorization and a_i' be a Γ_r-divisor of some a_i, say $a_i = b_1 \cdots b_k a_i' b_{k+1} \cdots b_m$ is a reduced Γ-factorization. Because Γ is refinable, we know $\lambda a_1 \cdots a_{i-1} b_1 \cdots b_k a_i' b_{k+1} \cdots b_m a_{i+1} \cdots a_n$ is a Γ-factorization, and hence so is $\lambda a_1 \cdots a_{i-1} a_i' a_{i+1} \cdots a_n$ by divisibility.

(9) Assume Γ is pseudo-transitive and normal. Take any $\mu \in H^\times$ and Γ-factorization $\lambda a_1 \cdots a_n$. By normality, each μa_i is a Γ-factorization. Therefore by pseudo-transitivity $\mu a_1 \cdots a_n$ is a Γ-factorization.

(10) Assume Γ is combinable and divisive. Let $\lambda a_1 \cdots a_n$ be any nontrivial Γ-factorization. Then for any $1 \leq i \leq n-1$, we know $\lambda a_1 \cdots a_{n-2}(a_{n-1}a_n)$ and $\lambda a_1 \cdots a_{i-1}(a_i a_{i+1}) a_{i+2} \cdots a_n$ are Γ-factorizations by the combinable property, so $\lambda a_1 \cdots a_{n-1}$ and $\lambda a_1 \cdots \widehat{a_i} \cdots a_n$ are Γ-factorizations by divisiveness.

□

The following proposition lists some simple observations about the $|_\Gamma$ and $|_{\Gamma_r}$ operators.

Proposition 2.1.4. *Let Γ be a factorization system on a monoid H.*

(1) $|_{\Gamma_r} \leq |_\Gamma \leq |$.

(2) *If H is cancellative, then Γ is normal if and only if* $\approx \leq |_\Gamma$.

(3) *If H is cancellative, then Γ is reduced normal if and only if* $= \leq |_{\Gamma}$.

(4) *If* $\mathrm{tfact}(H) \cup \mathrm{rfact}(H) \subseteq \Gamma$, *then the notions of* $|_{\Gamma}$, $|_{\Gamma_r}$, *and* $|$ *coincide.*

For any monoid H, we have $|_{\mathrm{fact}(H)} = |$ as relations on $H^{\#}$, but there is a subtle difference when the zero element is involved. More specifically, for a nonzero nonunit a, we always have $a \mid 0$, but $a \,|_{\mathrm{fact}(H)}\, 0$ if and only if a is a zero divisor.

Let Γ_1 and Γ_2 be factorization systems on a monoid H. We call a finite sequence of factorizations a Γ_1-Γ_2-*sequence* if each is a Γ_1-Γ_2-refinement of the last; the number of terms in such a sequence is its *length*. A term in a Γ_1-Γ_2-sequence is called a *sequential* Γ_1-Γ_2-*refinement* of the first term in the sequence. In the case $\Gamma_1 = \Gamma_2 = \Gamma$, we write simply "$\Gamma$" in place of "$\Gamma_1$-$\Gamma_2$" in the above. When speaking of Γ-sequences, a natural question comes up: Is a sequential Γ-refinement a Γ-refinement? Unfortunately, in general the answer is no.

Example 2.1.5. An example of a length 3 Γ-sequence whose last term is not a Γ-refinement of the first. Let $H = \mathbb{Z}$ and obtain a factorization system Γ by taking the normal, associate-preserving, unital, and symmetric closure of the set of factorizations of the forms $(2^{n_1})(2^{n_2})$, and $(3^m)(2^{n_1})\cdots(2^{n_k})$, where $m \geq 1$, $k \geq 0$, and each $n_i \geq 1$. We now form a length 3 Γ-sequence $3 \cdot 8$, $3 \cdot 4 \cdot 2$, $3 \cdot 2 \cdot 2 \cdot 2$, and we note that the last term is not a Γ-refinement of the first since $2 \cdot 2 \cdot 2$ is not a Γ-factorization.

It turns out that we avoid such pathologies if Γ is either refinable or reduced divisible.

Theorem 2.1.6. *Let Γ be a factorization system on a monoid H. Assume refining* *the factorization* $\lambda \prod_i a_i$ *twice yields a factorization* $\lambda \prod_i c_i$*. Assume further that one of these conditions is met:*

(1) *The refinements are* Γ-fact(H)*-refinements, the system* Γ *is refinable, and* $\lambda \prod_i a_i$ *is a* Γ*-factorization.*

(2) *The system* Γ *is reduced divisible (resp., divisible and* $\lambda = 1$*), and* $\lambda \prod_i c_i$ *is a* Γ*-factorization.*

Then $\lambda \prod_i c_i$ *is a* Γ-fact(H)*-refinement of* $\lambda \prod_i a_i$*.*

Proof. Aside from the inevitable notation issues, this is simple. Rewriting the notation, we have factorizations $\lambda \prod_i a_i$, $a_i = \prod_j b_{i,j}$, $b_{i,j} = \prod_k c_{i,j,k}$, $\lambda \prod_{i,j} b_{i,j}$, and $\lambda \prod_{i,j,k} c_{i,j,k}$. We need to show that $\lambda \prod_{i,j,k} c_{i,j,k}$ is a Γ-fact(H)-refinement of $\prod_i a_i$, which means we need to show that each $a_i = \prod_{j,k} c_{i,j,k}$ is a $(\Gamma \cup \text{trfact}(H))$-factorization. We will go through each condition in turn.

(1) Assume that Γ is refinable, each $a_i = \prod_j b_{i,j}$ and $b_{i,j} = \prod_k c_{i,j,k}$ are $(\Gamma \cup \text{trfact}(H))$-factorizations, and $\lambda \prod_i a_i$ is a Γ-factorization. Let a_i be any Γ-factor in this last Γ-factorization. If $a_i = \prod_j b_{i,j}$ is a Γ-factorization, then by refinability we see that $a_i = \prod_{j,k} c_{i,j,k}$ is a Γ-factorization. On the other hand, if $a_i = \prod_j b_{i,j}$ is a reduced trivial factorization, then $a_i = \prod_{j,k} c_{i,j,k}$ is simply the $(\Gamma \cup \text{trfact}(H))$-factorization $a_i = b_{i,1} = \prod_k c_{i,1,k}$.

(2) If Γ is reduced divisible (resp., divisible and $\lambda = 1$) and $\lambda \prod_{i,j,k} c_{i,j,k}$ is a Γ-factorization, then $a_i = \prod_{j,k} c_{i,j,k}$ is a Γ-factorization.

$\square$

Theorem 2.1.6 tells us that if Γ is refinable or reduced divisible, then any Γ-sequence can be shortened to a Γ-sequence of length at most 2 with the same initial and terminal Γ-factorizations. We add the additional observation that the $\lambda = 1$ case in the theorem tells us that if Γ is divisible, then any Γ-sequence can be shortened to a Γ-sequence of length at most 3 with the same initial and terminal Γ-factorizations. One can adjust the leading units in Example 2.1.5 to show that the bound of 3 cannot be improved.

2.2 ψ-factorization

Perhaps the first question that comes up when thinking of ψ-factorization is what properties characterize the factorization systems of the form Γ_ψ. The following theorem provides the answer.

Theorem 2.2.1. *Let Γ be a factorization system on a monoid H. The following are equivalent.*

(1) *There is a relation ψ on H with $\Gamma = \Gamma_\psi$.*

(2) *The system Γ is (weakly) divisible and (weakly) pseudo-transitive.*

If Γ is reduced normal, then the relation ψ in (1) is unique in the sense that if $\Gamma = \Gamma_\psi = \Gamma_{\psi'}$, then ψ and ψ' have the same intersection with $(H^\times \times H_0^\#) \cup (H^\# \times H^\#)$.

For this reason, we call a (weakly) divisible and (weakly) pseudo-transitive factorization system *relational*.

Proof. (1) $\Rightarrow$ (2): Clear. (2) $\Rightarrow$ (1): Assume Γ is weakly divisible and weakly pseudo-transitive. Let ψ be the set of $(x, y) \in H \times H$ such that there is a Γ-factorization

$\lambda a_1 \cdots a_n$ with $x = \lambda$ and $y = a_i$ for some i or with $x = a_i$ and $y = a_j$ for some $i < j$. Observe that $\Gamma \subseteq \Gamma_\psi$. Now take any ψ-factorization $\lambda a_1 \cdots a_n$. For each $i = 1, \ldots, n$, by weak divisibility and the fact that $\lambda \psi a_i$, the factorization λa_i is a Γ-factorization. So let us assume $n \geq 2$. Note that $\lambda a_1 \cdots a_{n-2} a_{n-1}$ and $\lambda a_1 \cdots a_{n-2} a_n$ are ψ-factorizations, so they are Γ-factorizations by induction. By weak divisibility and the fact that $a_{n-1} \psi a_n$, there is a $\mu \in H^\times$ such that $\mu a_{n-1} a_n$ is a Γ-factorization. Thus $\lambda a_1 \cdots a_n$ is a Γ-factorization by weak pseudo-transitivity.

The uniqueness statement is easily seen from the definitions. $\square$

Let Γ be a factorization system on a monoid H. We will denote the relational closure of Γ by Γ_{rel}. More explicitly, we have $\Gamma_{rel} = \Gamma_\psi$, where ψ is a relation on H defined as in the proof of "(3) $\Rightarrow$ (1)" above.

Of course, reduced normal and unital implies normal, but in general the converse is false. However, it is true for ψ-factorization. We record this fact and some other obvious observations about the (reduced) normal and unital properties in the following theorem.

Theorem 2.2.2.

(1) *The following are equivalent.*

  (a) *The system ψ is unital.*

  (b) *For every $\lambda, \lambda' \in H^\times$ and $a \in H_0^\#$, we have $\lambda \psi a \Leftrightarrow \lambda' \psi a$.*

(2) *The factorization system ψ is reduced normal if and only if $\{1\} \times H_0^\# \subseteq \psi$.*

(3) *The following are equivalent.*

(a) *The system ψ is normal.*

(b) *The system ψ is unital and reduced normal.*

(c) $H^\times \times H_0^\# \subseteq \psi$.

As we will soon see, many of the basic ψ-factorization properties can be related to fairly simple properties of the relation ψ. When we get to τ-factorization, we will see that the relationship is much nicer still. One argument that applies to several different properties is worth abstracting as a lemma.

Lemma 2.2.3. *Let H be a monoid, ρ be a reflexive relation on $H^\#$, and ψ be a relation on H. The following are equivalent.*

(1) *The factorization system ψ is ρ-preserving.*

(2) *Whenever $\lambda \in H^\times$, $a, b, a' \in H^\#$, and $a'\rho a$:*

(a) $\lambda\psi a \Rightarrow \lambda\psi a'$,

(b) $\lambda\psi a$, $\lambda\psi b$, *and* $a\psi b \Rightarrow a'\psi b$, *and*

(c) $\lambda\psi a$, $\lambda\psi b$, *and* $b\psi a \Rightarrow b\psi a'$.

(3) *Whenever $\lambda \in H^\times$, $a, b, a', b' \in H^\#$, $a'\rho a$, and $b'\rho b$:*

(a) $\lambda\psi a \Rightarrow \lambda\psi a'$, *and*

(b) $\lambda\psi a$, $\lambda\psi b$, *and* $a\psi b \Rightarrow a'\psi b'$.

Proof. (1) $\Rightarrow$ (3): Assume that the factorization system ψ is ρ-preserving and that $\lambda \in H^\times$, $a, b, a', b' \in H^\#$, $a'\rho a$, and $b'\rho b$. If $\lambda\psi a$, then λa is a ψ-factorization, so $\lambda a'$ is a ψ-factorization by the ρ-preserving property, and hence $\lambda\psi a'$. If $\lambda\psi a$, $\lambda\psi b$, and $a\psi b$, then λab is a ψ-factorization, so by the ρ-preserving property we see that

$\lambda a'b'$ is a ψ-factorization, and hence $a'\psi b'$. (3) $\Rightarrow$ (2): Follows from the fact that ρ is reflexive. (2) $\Rightarrow$ (1): Assume (2). Let $\lambda a_1 \cdots a_n$ be any ψ-factorization and assume that some $b\rho a_i$. Then $\lambda\psi a_i$, so $\lambda\psi b$. Also, for each $j < i$ (resp., $j > i$) we have $\lambda\psi a_j$ and $a_j\psi a_i$ (resp., $a_i\psi a_j$), so $a_j\psi b$ (resp., $b\psi a_j$). Therefore $\lambda a_1 \cdots a_{i-1} b a_{i+1} \cdots a_n$ is a ψ-factorization. □

Theorem 2.2.4. *Let H be a monoid and ψ be a relation on H.*

(1) *The following are equivalent.*

  (a) *The system ψ is associate-preserving (resp., strong associate-preserving, divisive).*

  (b) *For every $\lambda \in H^\times$ and $a_1, a_2, a_1', a_2' \in H^\#$ with each $a_i' \sim a_i$ (resp., $a_i' \approx a_i$, $a_i' \mid a_i$):*

    (1) *$\lambda\psi a_1 \Rightarrow \lambda\psi a_1'$, and*

    (2) *$\lambda\psi a_1$, $\lambda\psi a_2$, and $a_1\psi a_2 \Rightarrow a_1'\psi a_2'$.*

(2) *The following are equivalent.*

  (a) *The system ψ is combinable.*

  (b) *Whenever $\lambda \in H^\times$, $a, b, c \in H^\#$, $\lambda\psi a, \lambda\psi b, \lambda\psi c, a\psi b, a\psi c$, and $b\psi c$, we have $\lambda\psi bc$, $\lambda\psi ab$, $a\psi bc$, and $ab\psi c$.*

(3) *The following are equivalent.*

  (a) *The system ψ is (weakly) multiplicative.*

  (b) *For every $\lambda \in H^\times$ and $a, b, c \in H^\#$ with $\lambda\psi a$, $\lambda\psi b$, and $\lambda\psi c$:*

    (1) *$a\psi b$ and $a\psi c \Rightarrow \lambda\psi bc$ and $a\psi bc$, and*

(2) *$a\psi c$ and $b\psi c \Rightarrow \lambda\psi ab$ and $ab\psi c$.*

(4) *The system ψ satisfies:*

(a) *the symmetric property $\Leftrightarrow$ it is a symmetric relation on $H^{\#}$,*

(b) *the transitive property $\Leftrightarrow$ $a \in H^*$, $b, c \in H^{\#}$, $a\psi b$, and $b\psi c$ implies $a\psi c$, and*

(c) *the reflexive property $\Leftrightarrow$ it is reduced normal and is a reflexive relation on $H^{\#}$.*

Proof. Part (4) is clear, while (1) follows from Lemma 2.2.3.

(2) (a) $\Rightarrow$ (b): Assume ψ is combinable. Take any $a, b, c \in H^{\#}$ and $\lambda \in H^{\times}$ with $\lambda\psi a, \lambda\psi b, \lambda\psi c, a\psi b, a\psi c$, and $b\psi c$. Then λabc is a ψ-factorization, so by the combinable property $\lambda a(bc)$ and $\lambda(ab)c$ are ψ-factorizations. Therefore $\lambda\psi ab$, $\lambda\psi bc$, $a\psi bc$, and $ab\psi c$. (b) $\Rightarrow$ (a): Assume (b), let $\lambda a_1 \cdots a_n$ be a ψ-factorization, and assume $1 \leq i \leq n-1$. For each $j < i$ (resp., $j > i+1$) we have $\lambda\psi a_j$, $\lambda\psi a_i$, $\lambda\psi a_{i+1}$, $a_j\psi a_i$ (resp., $a_i\psi a_j$), $a_j\psi a_{i+1}$ (resp., $a_{i+1}\psi a_j$), and $a_i\psi a_{i+1}$. So for $j < i$ (resp., $j > i+1$) we have $\lambda\psi a_i a_{i+1}$ and $a_j\psi a_i a_{i+1}$ (resp., $a_i a_{i+1}\psi a_j$). Therefore $\lambda a_1 \cdots a_{i-1}(a_i a_{i+1})a_{i+2} \cdots a_n$ is a ψ-factorization.

(3) (a) $\Rightarrow$ (b): Assume ψ is weakly multiplicative, $\lambda \in H^{\times}$, $a, b, c \in H^{\#}$, and $\lambda\psi a$, $\lambda\psi b$, and $\lambda\psi c$. If $a\psi b$ and $a\psi c$, then λab and λac are ψ-factorizations, so by the weakly multiplicative property $\lambda a(bc)$ is a ψ-factorization, and hence $\lambda\psi bc$ and $a\psi bc$. Similarly, if $a\psi c$ and $b\psi c$, then $\lambda\psi ab$ and $ab\psi c$. (b) $\Rightarrow$ (a): Assume (b). Let $\lambda a_1 \cdots a_n$ and $\lambda b_1 \cdots b_n$ be ψ-factorizations with $a_i = b_i$ for

each i except possibly $i = j$. Then for each $i < j$ (resp., $i > j$) we have $\lambda \psi a_i$, $\lambda \psi a_j$, $\lambda \psi b_j$, $a_i \psi a_j$ (resp., $a_j \psi a_i$), and $a_i \psi b_j$ (resp., $b_j \psi a_i$), so $\lambda \psi a_j b_j$ and $a_i \psi a_j b_j$ (resp., $a_j b_j \psi a_i$). Therefore $\lambda a_1 \cdots a_{i-1}(a_j b_j)a_{i+1} \cdots a_n$ is a ψ-factorization. So we have shown that ψ is weakly multiplicative, which is equivalent to it being multiplicative by Lemma 2.1.2 part (2).

□

We can observe that comparing parts (2) and (3) of the above theorem immediately shows that (weakly) multiplicative implies combinable in the case of ψ-factorization. This is basically the course that the authors of [6] took when proving that multiplicative implies combinable for τ-factorization, but we already have the more general fact that a multiplicative and divisible factorization system is combinable from Theorem 2.1.3 part (6).

For general factorization systems, the properties of refinable and divisive have little to do with each other. It is an interesting fact that the two properties become intricately related when we specialize to ψ-factorization.

Theorem 2.2.5. *Let ψ be a relation on a monoid H. Then the factorization system ψ is refinable if and only if it is ψ_r-divisive.*

Proof. ($\Rightarrow$): Theorem 2.1.3 part (8). ($\Leftarrow$): Assume ψ is ψ_r-divisive. Let $\lambda a_1 \cdots a_n$ be any ψ-factorization and $a_i = b_1 \cdots b_m$ be a reduced ψ-factorization of some a_i. Then $b_i \psi b_j$ for $i < j$. By the fact that ψ is ψ_r-divisive, each $\lambda a_1 \cdots a_{i-1} b_j a_{i+1} \cdots a_m$ is a ψ-factorization, so $\lambda \psi b_j$, $a_k \psi b_j$ for $k \leq i-1$, and $b_j \psi a_k$ for $k \geq i+1$. Putting everything together, we see that $\lambda a_1 \cdots a_{i-1} b_1 \cdots b_m a_{i+1} \cdots a_n$ is a ψ-factorization. $\square$

2.3 τ-factorization

With our previous work on ψ-factorization, most of the work on τ-factorizations is easy. We again start with a characterization of which factorization systems fall under the category of τ-factorization.

Theorem 2.3.1. *The following are equivalent for a factorization system Γ on a monoid H.*

(1) *There is a unique relation τ on $H^{\#}$ with $\Gamma = \Gamma_\tau$.*

(2) *The system Γ is normal and there is a relation ψ on H with $\Gamma = \Gamma_\psi$.*

(3) *The system Γ is weakly divisible, weakly pseudo-transitive, and normal.*

(4) *The system Γ is unital, (reduced) divisible, pseudo-transitive, and normal.*

(5) *The system Γ is normal and relational.*

Proof. (1) $\Rightarrow$ (2): Clear. (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4): Theorem 2.2.1. Recall from Theorem 2.1.3 that a normal pseudo-transitive factorization system is automatically unital, and note that the divisible and reduced divisible properties are equivalent in the presence of the unital property. (4) $\Rightarrow$ (1): By Theorem 2.2.1 there is a relation ψ on H with $\Gamma = \Gamma_\psi$. Because ψ is normal we have $\lambda \psi a$ for each $\lambda \in H^\times$ and $a \in H_0^{\#}$. Therefore $\Gamma_\psi = \Gamma_\tau$, where $\tau = \psi \cap (H^{\#} \times H^{\#})$. Theorem 2.2.1 also gives us uniqueness. (3) $\Leftrightarrow$ (5): This is just the definition of "relational". $\square$

It is worthwhile to go through a few theorems again for the τ-factorization case. Some of the messiness is lost because we do not need to worry so much about the leading units. Many of the formulations improve further in an obvious way if we further assume symmetry, which is usually the case in practice.

Theorem 2.3.2 ([21, Lemma 4.3])**.** *Let H be a monoid and τ be a relation on $H^{\#}$. The following are equivalent.*

(1) *The equality $\tau = \tau_H$ holds.*

(2) *The equality $\Gamma_\tau = \mathrm{fact}(H)$ holds.*

(3) *The system τ is reflexive and divisive.*

Theorem 2.3.3. *Let H be a monoid and τ be a relation on $H^{\#}$.*

(1) *The system Γ_τ is symmetric (resp., transitive, reflexive) if and only if τ is a symmetric (resp., transitive, reflexive) relation.*

(2) *Let ρ be a reflexive relation on $H^{\#}$. The following are equivalent.*

  (a) *The factorization system τ is ρ-preserving.*

  (b) *$a'\rho a$, $b'\rho b$, and $a\tau b \Rightarrow a'\tau b'$.*

(3) *The factorization system τ is associate-preserving (resp., strong associate-preserving, divisive) if and only if for every $a_1, a_2, a_1', a_2' \in H^{\#}$ with each $a_i' \sim a_i$ (resp., $a_i' \approx a_i$, $a_i' \mid a_i$), we have $a_1\tau a_2 \Rightarrow a_1'\tau a_2'$.*

(4) *The system τ is combinable if and only if $a\tau b$, $a\tau c$, and $b\tau c$ implies $a\tau bc$ and $ab\tau c$.*

(5) *The following are equivalent.*

  (a) *The system τ is (weakly) multiplicative.*

  (b) (1) *$a\tau b$ and $a\tau c \Rightarrow a\tau bc$, and*

(2) *$a\tau c$ and $b\tau c \Rightarrow ab\tau c$.*

Proof. Part (1) is clear, part (2) follows from Lemma 2.2.3, and parts (3)-(5) are immediate from Theorem 2.2.4. □

We remark that the above equivalent characterizations of associate-preserving, divisive, and multiplicative are the original definitions for the τ-factorization case given in [6]. This shows that our Γ-factorization versions of these definitions are appropriate generalizations.

2.4 Factorization and Closures, I

In this section we will begin to study how certain properties are changed when we replace a factorization system with one of its closures. We will revisit this topic multiple times during the remainder of the thesis, studying how newly introduced concepts are affected by taking closures. These sections will generalize many results given in [21] and [7] for reduced τ-factorization. Sometimes the easiest path to prove some general result is to prove it for reduced Γ-factorization and then apply results about closures to get the general Γ-factorization version after adding suitable hypotheses like unital and so on. This approach is also arguably more general than simply proving theorems for general factorization systems with the necessary extra hypotheses.

Theorem 2.4.1. *Let P and Q be factorization system properties that have closures. For a factorization system Γ, let Γ_P (resp., Γ_Q) denote the P closure (resp., Q closure) of Γ. The following are equivalent.*

(1) *The containment* $(\Gamma_P)_Q \supseteq (\Gamma_Q)_P$ *holds for every factorization system* Γ.

(2) *The property* P *is always preserved by the* Q *closure.*

Proof. (1) $\Rightarrow$ (2): If (1) holds and Γ is any factorization system with property P, then $(\Gamma_Q)_P \supseteq \Gamma_Q = (\Gamma_P)_Q \supseteq (\Gamma_Q)_P$, so $\Gamma_Q = (\Gamma_Q)_P$ possesses property P. (2) $\Rightarrow$ (1): If (2) holds and Γ is any factorization system, then $(\Gamma_P)_Q = ((\Gamma_P)_Q)_P \supseteq (\Gamma_Q)_P$. $\square$

Corollary 2.4.2. *In the notation of Theorem 2.4.1, the following are equivalent.*

(1) *The equality* $(\Gamma_P)_Q = (\Gamma_Q)_P$ $(= \Gamma_{P,Q})$ *holds for every factorization system* Γ.

(2) *The property* P *is always preserved by the* Q *closure, and vice versa.*

If the above conditions hold, we say that the P and Q closures *commute.* The next theorem gives several of the most important cases of commuting closures.

Theorem 2.4.3. *Let* H *be a monoid and* ρ *be a reflexive relation on* $H^\#$.

(1) *Every closure preserves the (reduced) normal property.*

(2) *The unital,* ρ*-preserving, symmetric, divisible, and normal closures all commute with each other.*

(3) *The reduced normal closure commutes with the* ρ*-preserving, symmetric, (reduced) divisible, relational, and normal closures.*

(4) *The relational closure commutes with the unital, symmetric, divisible, and (reduced) normal closures. The relational closure preserves the* ρ*-preserving and combinable properties.*

(5) *The combinable closure commutes with the unital, strong associate-preserving, and (reduced) normal closures. The combinable closure preserves the symmetric*

and divisible properties, and, if $a\rho b$ and $c\rho d$ implies $ac\rho bd$, then the ρ-preserving closure preserves the combinable property. In particular, the (strong) associate-preserving and divisive closures preserve the combinable property.

Proof. We prove (4) and the last half of (5). Part (1) is obvious, and the verification of the rest is left to the reader. It can be done by a routine calculation from the definitions and some of the previously mentioned forms of the closures, using whichever characterization of commuting closures in Corollary 2.4.2 is most convenient.

(4) The fact that the relational and divisible closures commute is clear. If $\Gamma = \Gamma_\psi$ for some relation ψ on H, then $\Gamma_u = \Gamma_{\psi_u}$, $\Gamma_s = \Gamma_{\psi_s}$, $\Gamma_{rnl} = \Gamma_{\psi_{rnl}}$, and $\Gamma_{nl} = \Gamma_{\psi_{nl}}$, where

$$\psi_u = \psi \cup \{(\lambda, a) \mid \lambda \in H^\times, a \in H_0^\#, (H^\times \times \{a\}) \cap \psi \neq \emptyset\},$$

$$\psi_s = \psi \cup \{(b, a) \mid (a, b) \in \psi\},$$

$$\psi_{rnl} = \psi \cup (\{1\} \times H_0^\#),$$

and

$$\psi_{nl} = \psi \cup (H^\times \times H_0^\#).$$

Using the previously given form of the relational closure, one can use Theorem 2.2.2 (resp., Theorem 2.2.4 part (4a), Lemma 2.2.3, Theorem 2.2.4 part (2)) to show that the unital (resp., symmetric, ρ-preserving, combinable) property is preserved by the relational closure. The fact that the (reduced) normal property is preserved by the relational closure is a special case of part (1).

(5) Assume $a\rho b$ and $c\rho d$ implies $ac\rho bd$. Let Γ be a combinable factorization system on H. To show that Γ_ρ is combinable, it will suffice to show that $\Gamma_{\rho,k}$ is combinable for each $k \geq 1$. So assume $k \geq 1$ and $\lambda a_1 \cdots a_n$ is a $\Gamma_{\rho,k}$-factorization. Then there is a $\Gamma_{\rho,k-1}$-factorization $\lambda b_1 \cdots b_n$ with each $a_i \rho b_i$. For any $i \in \{1, \ldots, n-1\}$, induction shows that $\lambda b_1 \cdots b_{i-1}(b_i b_{i+1}) b_{i+2} \cdots b_n$ is a $\Gamma_{\rho,k-1}$-factorization, and $a_i a_{i+1} \rho b_i b_{i+1}$ by hypothesis, and hence $\lambda a_1 \cdots a_{i-1}(a_i a_{i+1}) a_{i+2} \cdots a_n$ is a $\Gamma_{\rho,k}$-factorization, as desired.

$\square$

We call the relation ψ_u (resp., ψ_s, ψ_c, ψ_{rnl}, ψ_{nl}) constructed in the proof to part (4) above the *unital* (resp., *symmetric, combinable, reduced normal, normal*) *closure* of the relation ψ, because it is the smallest relation bounded below by ψ whose factorization system has that property. Here there is no ambiguity caused by our convention of using the relation as a shorthand for the factorization system associated with it, since the above proof shows that this factorization system is the appropriate closure of Γ_ψ. Similar constructions can be made with other closures of relations, but with these closures we must refrain from the aforementioned convention. For instance, in Example 2.4.4 below we will see a case where $(\Gamma_\psi)_{ap} \neq \Gamma_{\psi_{ap}}$. Generalizing these comments, if P is one of the properties with a closure, then $\Gamma_{\psi_P} = (\Gamma_\psi)_{rel,P} \supseteq (\Gamma_\psi)_P$. The containment can be strict, but we have equality if the P closure preserves the relational property. These same comments translate directly to (reduced) τ-factorization, since any closure preserves the (reduced) normal property.

We give examples showing that none of the cases where we stated that one closure preserves a certain property can be improved to a statement about commuting closures.

Example 2.4.4. The (strong) associate-preserving closure need not preserve the relational property. Let H be any cancellative monoid satisfying $\{1\} \subsetneq H^\times \subsetneq H^*$. Pick $a \in H^\#$ and $1 \neq \mu \in H^\times$, and let τ be the symmetric relation on $H^\#$ determined by $a\tau\mu a$. Then $\Gamma_\tau = \{a(\mu a)\}_{u,nl,s}$, and $(\Gamma_\tau)_{ap} = \{a^2\}_{u,ap,nl,s}$, which is not pseudo-transitive (hence not relational) since it contains a^2 but not a^3.

Example 2.4.5. The divisive closure need not preserve the relational property. Let H be any monoid with a non-idempotent nonzero nonunit a. Let τ be the symmetric relation on $H^\#$ determined by $a\tau a^2$. Then the divisive closure of Γ_τ is not pseudo-transitive (hence not relational), since it contains a^2 but not a^3.

Example 2.4.6. The combinable closure need not preserve the relational property. Let H be any cancellative monoid that is not a groupoid, and pick $a \in H^\#$. Let τ be the symmetric relation on $H^\#$ determined by $a\tau a^2$, $a\tau a^3$, and $a^2\tau a^3$. Then $\Gamma_\tau = \{a(a^2), a(a^3), (a^2)(a^3), a(a^2)(a^3)\}_{u,nl,s}$ and $(\Gamma_\tau)_c = \Gamma_\tau \cup \{(a^3)^2, a(a^5)\}_{u,nl,s}$. The latter factorization system is not pseudo-transitive (hence not relational) since it contains $(a^3)^2$ but not $(a^3)^3$.

Example 2.4.7. The symmetric closure need not preserve the combinable property. Let τ be the relation on $\mathbb{Z}^\#$ given by $2\tau 3$, $2\tau 4$, $3\tau 4$, $6\tau 4$, and $2\tau 12$. Then $\Gamma_\tau = \{2\cdot 3, 2\cdot 4, 3\cdot 4, 2\cdot 3\cdot 4, 6\cdot 4, 2\cdot 12\}_{u,nl}$, and we can note that τ is combinable. However, the factorization $2\cdot 4\cdot 3$ is a τ_s-factorization but $8\cdot 3$ is not, so τ_s is not combinable.

Example 2.4.8. The divisible closure need not preserve the combinable property. Obtain a factorization system Γ on $\mathbb{Z}$ by taking the combinable closure of $\{2 \cdot 3 \cdot 4 \cdot 5\}$. Then the divisible closure of Γ is not combinable, since it contains $2 \cdot 4 \cdot 5$ but not $8 \cdot 5$. We note that it would be impossible to construct such an example if we started with a symmetric factorization system. Since the divisible and truncatable closures coincide for symmetric factorization systems, this observation follows from the fact that the combinable and truncatable closures commute. We leave the simple proof to the reader.

Example 2.4.9. The combinable closure need not preserve the associate-preserving property. In order to create this example, we first need to find an example of a ring with elements a and b so that ab has an associate c that cannot be written in the form $c = a'b'$, where $a' \sim a$ and $b' \sim b$. Once we achieve this, the rest is fairly simple. Let D be an integral domain and $R = D[X, Y, Z]/I$, where $I = (X^2 - X^2YZ)$. For $f \in D[X, Y, Z]$, let $\bar{f} = f + I$. It is readily seen that $\bar{f} \sim \bar{g} \Leftrightarrow (f, X^2 - X^2YZ) = (g, X^2 - X^2YZ)$. Because $\bar{X}^2 = \overline{X^2Y}\bar{Z}$, we have $\bar{X}^2 \sim \overline{X^2Y}$. However, we claim that $\overline{X^2Y}$ cannot be written in the form $\overline{X^2Y} = \bar{f}\bar{g}$ with $\bar{f} \sim \bar{g} \sim \bar{X}$. Suppose to the contrary that it can be written so. Then $(f, X^2 - X^2YZ) = (g, X^2 - X^2YZ) = (X, X^2 - X^2YZ) = (X)$, so we may write $f = Xf_0$ and $g = Xg_0$, where $(f_0, X - XYZ) = (g_0, X - XYZ) = D[X, Y, Z]$. Now, since $(X^2 - X^2YZ) \mid (X^2Y - fg)$, we can cancel the X^2 to obtain $(1 - YZ) \mid (Y - f_0g_0)$, say $f_0g_0 = Y + (1 - YZ)h$. Thus $(Y + (1 - YZ)h, X(1 - YZ)) = (f_0g_0, X - XYZ) = D[X, Y, Z]$, so there are $p, q \in D[X, Y, Z]$ with $(Y + (1 - YZ)h)p + X(1 - YZ)q = 1$. Evaluating at

$X = 0$, we obtain $(Y + (1 - YZ)h(0, Y, Z))p(0, Y, Z) = 1$. If $h(0, Y, Z) = 0$, then $Y = Y + (1 - YZ)h(0, Y, Z)$ is a unit, a contradiction. Therefore $h(0, Y, Z) \neq 0$, so $Y + (1 - YZ)h(0, Y, Z)$ is a unit whose degree in Z is at least 1, achieving the desired contradiction. Now, let τ be the associate-preserving symmetric relation on $R^{\#}$ given by $F\tau G \Leftrightarrow F \sim G \sim \bar{X}$. Then $(\Gamma_\tau)_c$ is the union of tfact(R) with the set of factorizations $\lambda F_1 \cdots F_n$, where each F_i is a product of associates of $\bar{X}$. (Interestingly, this is one case where we do have $(\Gamma_\tau)_c = \Gamma_{\tau_c}$, where the combinable closure τ_c of the relation τ is given by $F\tau_c G \Leftrightarrow F$ and G are products of associates of $\bar{X}$.) Thus $\overline{X^2}\bar{X}$ is a τ_c-factorization and $\overline{X^2} \sim \overline{X^2 Y}$, but by our earlier claim we see that $\overline{X^2 Y}\bar{X}$ is not a τ_c-factorization.

Example 2.4.10. The combinable closure need not preserve the divisive property. Let R be a ring and $D = R[X^2, X^3]$. Let τ be the divisive symmetric relation on $D^{\#}$ determined by $X^2 \tau X^2$ and $X^3 \tau X^3$. Then the combinable closure of Γ_τ is not divisive since it contains the factorization $(X^2)(X^6)$ but not $(X^2)(X^3)$.

In order to get an example like Example 2.4.10 where we started with a divisive and divisible factorization system, we needed to start with a monoid whose factorizations were not too well-behaved. More precisely, in monoids with the property that $a \mid bc$ implies $a = b'c'$ for some $b' \mid b$ and $c' \mid c$, the combinable closure preserves the property of being divisive and divisible. Integral domains with this property are called *pre-Schreier domains*, and *Schreier domains* are integrally closed pre-Schreier domains. The (pre-)Schreier domains have become an important part of standard factorization theory. In general, UFD $\Rightarrow$ GCD domain $\Rightarrow$ Schreier $\Rightarrow$ pre-Schreier, but none of the implications reverse. One possible starting point for further reading is [11]. We analogously extend the definition of pre-Schreier to monoids. We will return to this topic in Chapter 5.

Theorem 2.4.11. *In a pre-Schreier monoid, the combinable closure preserves the property of being divisive and divisible.*

Proof. Let Γ be any divisive and divisible factorization system on a pre-Schreier monoid H. By Theorem 2.4.3 part (5) it will suffice to show that Γ_c is divisive. Let $\lambda a_1 \cdots a_n$ be any Γ-factorization, and let $\lambda b_1 \cdots b_m$ be ay partition of it, where each b_i is of the form $b_i = a_{k_i} a_{k_i+1} \cdots a_{k_{i+1}-1}$. Let c be any nonzero nonunit divisor of some b_i. We can write $c = a'_{k_i} \cdots a'_{k_{i+1}-1}$, where each $a'_j \mid a_j$, and we can arrange for the units in the product to be 1. Using the divisive and divisible properties, we see that $\lambda a_1 \cdots a_{k_i-1} a'_{k_i} \cdots a'_{k_{i+1}-1} a_{k_{i+1}} \cdots a_n$ is a Γ-factorization (omitting any 1's from the product), and thus $\lambda b_1 \cdots b_{i-1} c b_{i+1} \cdots c_n$ is a Γ_c-factorization, as desired. $\square$

CHAPTER 3
COMPLETENESS

The notion of an atomic factorization plays a central role in standard factorization theory, and in abstract factorization its analogue is just as important. A nonzero nonunit a of a cancellative monoid is called *irreducible* or an *atom* if it has no nontrivial factorizations, or, equivalently, if $a = bc$ implies that a is a (strong) associate of b or c, or, equivalently, if (a) is maximal among the proper principal ideals. In possibly non-cancellative monoids (more specifically, in non-présimplifiable monoids), those four statements are no longer equivalent in general, giving rise to four distinct notions of "atomicity", and when one generalizes further to abstract factorization the situation gets more intricate still. The paper [9] gives a survey of these topics in the ring context, and, as usual, much of the basic theory translates to the more general monoid setup with identical proofs. In a cancellative monoid, an *atomic factorization* is a factorization whose factors are atoms, or, equivalently, a factorization with no proper refinements, and a cancellative monoid is called *atomic* if every (nonzero) nonunit has an atomic factorization. (As expected, in a possibly non-cancellative setup the situation gets more complicated, and again [9] is our reference for this.) We will study abstract factorization generalizations of atomic factorization only in the simplified cancellative monoid case. The thesis [20] has carried out the generalization to the non-cancellative case in the context of τ-factorization in commutative rings, and we refer the interested reader there for an idea of how the topics we will discuss could be extended further to (not necessarily cancellative) monoids.

The two different ways we stated the definition of an atomic factorization lead to different approaches to generalizing atomic factorization: we could study Γ-factorizations whose factors are Γ-atoms (in our cancellative monoid setup, a Γ-*atom* or Γ-*irreducible* is a nonunit with no nontrivial Γ-factorizations), or we could study Γ-factorizations that have no proper Γ-refinements. A factorization of the former type is called a Γ-*atomic factorization*, while one of the latter type is called a Γ-*complete factorization*. Clearly, a Γ-atomic factorization is Γ-complete, and we show in the next chapter that the converse is true for Γ refinable, unital, and associate-preserving. This chapter will be concerned with Γ-complete factorization, while the next one will discuss Γ-atomic factorization. For the reader interested in generalizing by removing the cancellative assumption, we again mention [20] and generalizing the work here in analogy with how the author generalized τ-factorization. We make a couple remarks about this process of generalization. The great majority of the theory of Γ-complete factorization translates to non-cancellative monoids with no change. There are three things to watch out for worth mentioning: (1) sometimes a formerly optional "nonzero" qualifier in a definition or result becomes non-optional, (2) when generalizing a theorem or definition that references the associate relation one needs to take care in choosing which "associate" relation to replace it with, and (3) the proof of the implication "(1) $\Rightarrow$ (2)" in Theorem 3.3.3 only applies to présimplifiable monoids. On the other hand, working without a cancellative (or at least présimplifiable) assumption tremendously increases the complexity of the theory of Γ-atomic factorization. For those familiar with the varying levels of "irreducibility" in non-cancellative monoids, our work in the next chapter carries over very well to the theory concerning the "very strongly irreducible" notion (with similar comments to the above about translating), but that is just a fraction of the total theory.

The definition of τ-complete factorization was already present in the thesis [14] that introduced τ-factorization. The author proved sufficient conditions for the τ-complete and τ-atomic factorizations to coincide, but focused primarily on τ-atomic factorization. The subsequent literature has extended the study of the complete concepts a little, but has largely ignored them in comparison to the atomic concepts. For example, the thesis [21] that introduced Γ-factorization does not define "Γ-complete factorization". This chapter will be devoted to rectifying this omission. In fact, this thesis will make an argument for the systematic study of the complete concepts prior to an in-depth look at the atomic concepts, since in the following chapter we will see that many of the theorems about the complete concepts are generalizations of and easier to prove than previous ones for their atomic counterparts.

The first section of this chapter will be devoted to definitions and a few other preliminaries. We will find it of some interest to further abstract the concept of a Γ-complete factorization as follows: if Γ_1 and Γ_2 are factorization systems on a cancellative monoid H, then a Γ_1-Γ_2-*complete factorization* is a Γ_2-factorization with no proper Γ_1-refinements, and H is Γ_1-Γ_2-*complete* if every Γ_2-expressible (nonzero) nonunit has a Γ_1-Γ_2-complete factorization. We will go on to define various levels of "completeness", which will be categorized in the second section.

Several different kinds of atomic monoids have been defined and studied in depth, particularly in the case of integral domains. Our general reference for standard factorization theory in integral domains (and cancellative monoids by analogy) is [3]. The domains that this paper studied were generalized with τ-factorization in [6], and there the implications between them and their relationship to the standard factorization concepts was almost fully worked out. The one remaining significant piece was completed in [18], where it was shown (in our terminology) that an atomic domain need not be τ-complete even for τ both multiplicative and divisive. In the third section we will generalize these different kinds of atomic domains to different kinds of Γ_1-Γ_2-complete cancellative monoids, categorize them, and give examples showing that there are no further implications. The same process will be carried out for their stronger counterparts, the Γ_1-Γ_2-*completable* monoids, which are cancellative monoids in which every Γ_2-factorization can be sequentially Γ_1-Γ_2-refined into a Γ_1-Γ_2-complete factorization. We will see there that these monoids have quite a lot of nice properties, and in the next chapter we will see that most of the nontrivial theorems about the atomic concepts are actually special cases of the analogous results about the complete concepts.

In the final section we will begin a study of how taking various closures of a factorization system affect certain properties, with an emphasis on those related to completeness. In particular, this approach gives us a way to translate between results about Γ-factorization and reduced Γ-factorization, generalizing and extending previous theorems about reduced τ-factorization found in [21].

3.1 Basic Definitions

Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H. In analogy with the τ-factorization definition in [6], we define a Γ_1-Γ_2-*complete factorization* to be a Γ_2-factorization with no proper Γ_1-Γ_2-refinements. We say H is Γ_1-Γ_2-*complete* if every Γ_2-expressible (nonzero) nonunit has a Γ_1-Γ_2-complete factorization. Other plausible definitions are: (1) every (nonzero) nonunit is either a Γ_1-*atom* (that is, it has no nontrivial Γ_1-factorizations) or has a Γ_1-Γ_2-complete factorization, or (2) every (nonzero) nonunit has a Γ_1-Γ_2-complete factorization. All three definitions are equivalent if Γ_2 is reduced normal, but simple examples show that they differ in general. We go with our choice because it seems to lead to the cleanest theory, as we will begin to appreciate after a few theorems.

Let Γ_1, Γ_2, and Γ_3 be factorization systems on a cancellative monoid H, and let $\equiv$ be an equivalence relation on $H_0^{\#}$. We say two factorizations $\lambda a_1 \cdots a_m$ and $\mu b_1 \cdots b_n$ are $\equiv$-*equivalent* if $m = n$ and each $a_i \equiv b_i$ after a suitable reordering. Note that the notion of $\equiv$-equivalence forms an equivalence relation on $\mathrm{fact}(H)$. We say H is:

(1) Γ_1-Γ_2-Γ_3-*completable* if every Γ_2-factorization can be sequentially Γ_3-Γ_2-refined into a Γ_1-Γ_2-complete factorization,

(2) *strongly* Γ_1-Γ_2-Γ_3-*completable* if every Γ_2-factorization can be Γ_3-Γ_2-refined into a Γ_1-Γ_2-complete factorization,

(3) *(strongly)* Γ_1-Γ_2-*completable* if it is (strongly) Γ_1-Γ_2-Γ_1-completable,

(4) *weakly* Γ_1-Γ_2-*completable* if it is (strongly) Γ_1-Γ_2-$\mathrm{fact}(H)$-completable,

(5) a Γ_1-Γ_2-*complete BFM* if it is Γ_1-Γ_2-complete and there is an upper bound on the lengths of the Γ_1-Γ_2-complete factorizations of any given (nonzero) nonunit,

(6) a Γ_1-Γ_2-*complete* $FFM_{\equiv}$ if it is Γ_1-Γ_2-complete and each (nonzero) nonunit has only finitely many Γ_1-Γ_2-complete factorizations up to $\equiv$-equivalence,

(7) a Γ_1-Γ_2-*complete HFM* if it is Γ_1-Γ_2-complete and the Γ_1-Γ_2-complete factorizations of a given (nonzero) nonunit all have the same length,

(8) a Γ_1-Γ_2-*complete* $UFM_{\equiv}$ if it is Γ_1-Γ_2-complete and the Γ_1-Γ_2-complete factorizations of a given (nonzero) nonunit are unique up to $\equiv$-equivalence, and

(9) a Γ_1-Γ_2-*completable BFM* (resp., $FFM_{\equiv}$, *HFM*, $UFM_{\equiv}$) if it is Γ_1-Γ_2-completable and a Γ_1-Γ_2-complete BFM (resp., $\mathrm{FFM}_{\equiv}$, HFM, $\mathrm{UFM}_{\equiv}$).

If $\Gamma_1 = \Gamma_2 = \Gamma$, we replace the "$\Gamma_1$-$\Gamma_2$" with "$\Gamma$" in the above phrases, and if additionally $\Gamma_1 = \Gamma_2 = \mathrm{fact}(H)$ (the standard factorization case), we drop the "Γ_1-Γ_2 complete(able)". (In the next chapter we will see that the properties of Γ_1-Γ_2-complete and strongly Γ_1-Γ_2-Γ_2-completable are equivalent for Γ_2 refinable, unital, associate-preserving, and divisible.) If $\equiv = \sim$, we drop the "$\equiv$". We note that a Γ_1-Γ_2-complete(able) BFM (resp., HFM) is actually a special case of a Γ_1-Γ_2-complete(able) $\mathrm{FFM}_{\equiv}$ (resp., $\mathrm{UFM}_{\equiv}$), namely a Γ_1-Γ_2-complete(able) FFM_{ψ_H} (resp., UFM_{ψ_H}). We remark that one can define strongly or weakly Γ_1-Γ_2-completable versions of the BFM, $\mathrm{FFM}_{\equiv}$, HFM, and $\mathrm{UFM}_{\equiv}$ notions, but the former versions do not share some of the nice properties of the ones defined above (such as Theorems 3.3.1 and 3.3.3), and we will see that the latter are equivalent to their corresponding Γ_1-Γ_2-completable notions. We will later see that Γ-completable and weakly Γ-completable are equivalent if Γ is reduced divisible. We will also see in Theorem 3.2.1 below that in the case with Γ_2 reduced divisible we might as well assume that $\Gamma_1 \subseteq \Gamma_2$ in the above.

Although it is possible to develop a great deal of the theory in the full generality in which we have given the definitions, we will concentrate mostly on the case $\equiv = \sim$ for the sake of simplicity. Sometimes the same proofs work for any choice of $\equiv$, sometimes they require $\equiv$ to be bounded above or below by $\sim$, sometimes they require $\equiv$ to be partition-preserving (essentially, to be a "congruence relation" on $H_0^{\#}$), and sometimes they require precisely that $\equiv = \sim$. All in all, stating each theorem in the strongest possible form becomes quite cumbersome, but a very careful examination of the proofs given here should make it evident how certain results can be abstracted.

The acronyms "BFM", "FFM", "HFM", and "UFM" stand for "bounded factorization monoid", "finite factorization monoid", "half factorial monoid", and "unique factorization monoid", respectively. Naturally, if we want to specify that the monoid in question is actually a ring or domain, we change the "monoid" in the names to "ring" or "domain" and the "M" in the acronyms to "R" or "D" as appropriate. We will have similar conventions with all later definitions that have the word "monoid" in them.

To illustrate some of these abstract notions, we revisit some of our motivating factorization system examples.

Example 3.1.1.

(1) An excellent survey of these concepts in the case $\Gamma_1 = \Gamma_2 = \mathrm{fact}(H)$ and $\equiv = \sim$ is given in [3]. The paper [2] drops the requirement $\equiv = \sim$.

(2) A τ_d-complete domain is called a *comaximal factorization domain* (*CFD*) and a τ_d-complete UFD is called a *unique comaximal factorization domain* (*UCFD*). For example, every Noetherian domain (or more generally, any domain where each ideal has only finitely many minimal primes) is a CFD [19, Lemma 1.1], every UFD is a UCFD [19, Corollary 1.8], and of course every quasilocal domain is trivially a UCFD. The paper [19] gives a very pleasing ideal-theoretic characterization of the UCFD's as the CFD's in which every 2-generated invertible ideal is principal.

(3) Complete comaximal factorizations of ideals are always unique (up to order) when they exist [19, Theorem 5.1], so the monoid of ideals of a ring being τ_d-complete is the same as it being a τ_d-complete UFM. A sufficient condition for this to happen is for the ring in question to be Noetherian, or more generally for every ideal to have only finitely many minimal primes [19, Theorem 5.4]. This fact allows us to show that such a ring is a τ_d-complete FFR (thus improving on [19, Lemma 1.1]), since the comaximal factorizations of a nonunit are (up to associates) in a natural one-to-one correspondence with the comaximal factorizations of its principal ideal into principal ideals.

(4) Recall the construction of the $\tau_\sqcup$ factorization system: X is a set, $\mathcal{P}(X)$ is a monoid under $\cup$, and $Y \tau_\sqcup Z \Leftrightarrow Y \cap \mathbb{Z} = \emptyset$. The $\tau_\sqcup$-complete factorizations are precisely the disjoint unions of singleton subsets of X, so $\mathcal{P}(X)$ is $\tau_\sqcup$-complete if and only if it is a $\tau_\sqcup$-complete UFD if and only if X is finite. (In this example we are breaking with the convention of insisting that our monoid must

be cancellative, but as mentioned in the introduction, it really makes little difference.)

(5) The domain $\mathbb{Z}$ is a $\tau_{(n)}$-completable FFD for each $n \geq 0$. However, it is only a $\tau_{(n)}$-complete UFD in the cases $n = 0$ and $n = 1$. (See [6], [14], or [16].)

(6) Let Γ be the factorization system on $\mathbb{Z}$ consisting of factorizations with at most two even factors. The Γ-complete factorizations are (up to associates and order) precisely those factorizations of the forms $p_1 \cdots p_{n+1}$, $2p_1 \cdots p_n$, or $2^k 2^m p_1 \cdots p_n$, where $n \geq 0$, $k, m \geq 1$, and $p_1, \ldots, p_{n+1}$ are odd primes. It follows that $\mathbb{Z}$ is a Γ-complete HFD, and it is a Γ-complete FFD since every nonunit has only finitely many factorizations (see 3.3.1 below). However, it is not a Γ-complete UFD since $2 \cdot 8 = 4 \cdot 4$ are Γ-complete factorizations.

3.2 Levels of "Completeness"

Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H. Directly from the definitions we easily obtain the implications strongly Γ_1-Γ_2-completable $\Rightarrow$ Γ_1-Γ_2-completable $\Rightarrow$ weakly Γ_1-Γ_2-completable $\Rightarrow$ Γ_1-Γ_2-complete.

The special case of (reduced) τ-factorization is of considerable interest. The following theorem accomplishes a study of this case in greater generality, abstracting to the property of reduced divisibility.

Theorem 3.2.1. *Let Γ_1, Γ_2, and Γ_3 be factorization systems on a cancellative monoid H. Assume Γ_2 is reduced divisible.*

(1) *Every Γ_1-Γ_2-refinement is a $(\Gamma_1 \cap \Gamma_2)$-Γ_2-refinement.*

(2) *The* Γ_1-Γ_2*-complete factorizations and the* $(\Gamma_1 \cap \Gamma_2)$-Γ_2*-complete factorizations coincide.*

(3) *The monoid* H *is* Γ_1-Γ_2*-complete if and only if it is* $(\Gamma_1 \cap \Gamma_2)$-Γ_2*-complete.*

(4) *The following properties of* H *are equivalent:*

   (a) *(strongly)* Γ_1-Γ_2-Γ_3*-completable,*

   (b) *(strongly)* $(\Gamma_1 \cap \Gamma_2)$-Γ_2-Γ_3*-completable,*

   (c) *(strongly)* Γ_1-Γ_2-$(\Gamma_2 \cap \Gamma_3)$*-completable, and*

   (d) *(strongly)* $(\Gamma_1 \cap \Gamma_2)$-Γ_2-$(\Gamma_2 \cap \Gamma_3)$*-completable.*

(5) *The monoid* H *is a* Γ_1-Γ_2*-complete(able) BFM (resp.,* $FFM_{\equiv}$*, HFM,* $UFM_{\equiv}$*) if and only if it is a* $(\Gamma_1 \cap \Gamma_2)$-Γ_2*-complete(able) BFM (resp.,* $FFM_{\equiv}$*, HFM,* $UFM_{\equiv}$*).*

(6) *If* $\Gamma_2 \subseteq \Gamma_3$*, then the following properties of* H *are equivalent:*

   (a) *strongly* Γ_1-Γ_2-Γ_3*-completable,*

   (b) Γ_1-Γ_2-Γ_3*-completable, and*

   (c) *weakly* Γ_1-Γ_2*-completable.*

Proof. Part (1) is a simple consequence of reduced divisibility, and each of parts (2)-(5) follows nearly immediately from some combination of the parts preceding it. The implication $(a) \Rightarrow (b) \Rightarrow (c)$ in (6) is clear, and $(c) \Rightarrow (a)$ follows from part (4) by the fact that $\Gamma_2 \cap \Gamma_3 = \Gamma_2 = \Gamma_2 \cap \text{fact}(H)$ when $\Gamma_2 \subseteq \Gamma_3$. $\square$

Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H. As noted earlier, we have the implications strongly Γ_1-Γ_2-completable $\Rightarrow$ Γ_1-Γ_2-completable $\Rightarrow$

weakly Γ_1-Γ_2-completable $\Rightarrow$ Γ_1-Γ_2-complete. In the simplified case where $\Gamma_1 = \Gamma_2 = \Gamma$ is reduced divisible, this chain of implications collapses to simply Γ-completable $\Rightarrow$ Γ-complete. The rest of the section will be occupied giving examples showing that none of the implications reverse, even with several additional simplifying assumptions in place.

Example 3.2.2. An example of a τ-complete UFD that is not τ-completable, with τ an associate-preserving symmetric relation on the nonzero nonunits. Let R be an integral domain and $D = R[\{X^r, Y^r \mid r \in \mathbb{Q}^+\}]$. Let τ be the associate-preserving symmetric relation on $D^\#$ determined by $X^r \tau Y^r$ and $(XY)^r \tau (XY)^s$ for $r, s \in \mathbb{Q}^+$. Note that τ is associate-preserving, and the only τ-reducible elements are those of the form $\lambda(XY)^r$, which have unique τ-complete factorizations $\lambda(X^r)(Y^r)$ up to associates and order. Therefore H is a τ-complete UFD. However, the τ-factorization $(XY)^2$ cannot be refined into a τ-complete factorization.

Example 3.2.3. An example of a weakly Γ-completable domain that is not Γ-completable, where Γ is a symmetric, unital, and associate-preserving factorization system. Let R be an integral domain and $D = R[\{X^r \mid r \in \mathbb{Q}^+\}]$. Obtain Γ by taking the associate-preserving and unital closure of the set of factorizations of the forms $(X^{1/(2^{n+1}\cdot 3)})^6$, $(X^{1/(2^n \cdot 3)})^{2^n \cdot 3}$, $(X^{1/2^{n+1}})^2$, and $(X^{1/2^{m_1}}) \cdots (X^{1/2^{m_k}})$, where $n \geq 0$ and $m_1, \ldots, m_k \geq 1$ satisfy $\frac{1}{2^{m_1}} + \cdots + \frac{1}{2^{m_k}} = 1$. Observe that Γ is symmetric and the only Γ-reducible elements are those of the form $\lambda X^{1/2^n}$, where $\lambda \in D^\times$ and $n \geq 0$. Therefore, for $n \geq 0$ and $\lambda \in D^\times$, the Γ-factorizations $\lambda X^{1/2^n} = \lambda(X^{1/(2^n \cdot 3)})^6$ and $X = (X^{1/(2^n \cdot 3)})^{2^n \cdot 3}$ are Γ-complete. Also, for $\lambda \in D^\times$ and $n \geq 0$, the Γ-factorization

$\lambda(X^{1/2^{n+1}})^2$ has a refinement $\lambda(X^{1/(2^{n+1}\cdot 3)})^3(X^{1/(2^{n+1}\cdot 3)})^3 = (X^{1/(2^{n+1}\cdot 3)})^6$, which is a Γ-complete factorization. Finally, for $\lambda \in D^\times$, $n \geq 0$, and $m_1, \ldots, m_k \geq 1$ with $\frac{1}{2^{m_1}} + \cdots + \frac{1}{2^{m_k}} = 1$, the Γ-factorization $\lambda(X^{1/2^{m_1}}) \cdots (X^{1/2^{m_k}})$ can be refined to the Γ-complete factorization $\lambda(X^{1/(2^m\cdot 3)})^{2^{m-m_1}\cdot 3} \cdots (X^{1/(2^m\cdot 3)})^{2^{m-m_k}\cdot 3} = \lambda(X^{1/(2^m\cdot 3)})^{2^m\cdot 3}$, where $m = \max(m_1, \ldots, m_k)$. This suffices to show that H is weakly Γ-completable.

We now show that D is not Γ-completable. Pick $k \geq 2$ and $m_1, \ldots, m_k \geq 1$ with $\frac{1}{2^{m_1}} + \cdots + \frac{1}{2^{m_k}} = 1$ and $m_1 < m_k$. For each i, the fact that $\frac{1}{2^{m_i}} < 1$ implies that the only Γ-factorizations of $X^{1/2^{m_i}}$ are (up to associates and order) $(X^{1/2^{m_i+1}\cdot 3})^6$ and $(X^{1/2^{m_i+1}})^2$. Because $m_1 + 1 < m_k + 1$, we are unable to achieve a Γ-refinement by replacing each $X^{1/2^{m_i}}$ with $(X^{1/(2^{m_i+1}\cdot 3)})^6$. So the only proper Γ-refinement (up to associates and order) of the Γ-factorization $(X^{1/2^{m_1}}) \cdots (X^{1/2^{m_k}})$ is $(X^{1/2^{m_1+1}})^2 \cdots (X^{1/2^{m_k+1}})^2$, which is not Γ-complete, and we note that $2k \geq 4 > 2$, $\frac{1}{2^{m_1+1}} + \frac{1}{2^{m_1+1}} + \frac{1}{2^{m_2+1}} + \frac{1}{2^{m_2+1}} + \cdots + \frac{1}{2^{m_k+1}} + \frac{1}{2^{m_k+1}} = 1$, and $m_1 + 1 < m_k + 1$. The preceding argument shows that $(X^{1/2})(X^{1/4})(X^{1/4})$ is a Γ-factorization that cannot be sequentially Γ-refined to a Γ-complete factorization.

A Γ-completable domain that is not strongly Γ-completable is exhibited in Example 2.1.5.

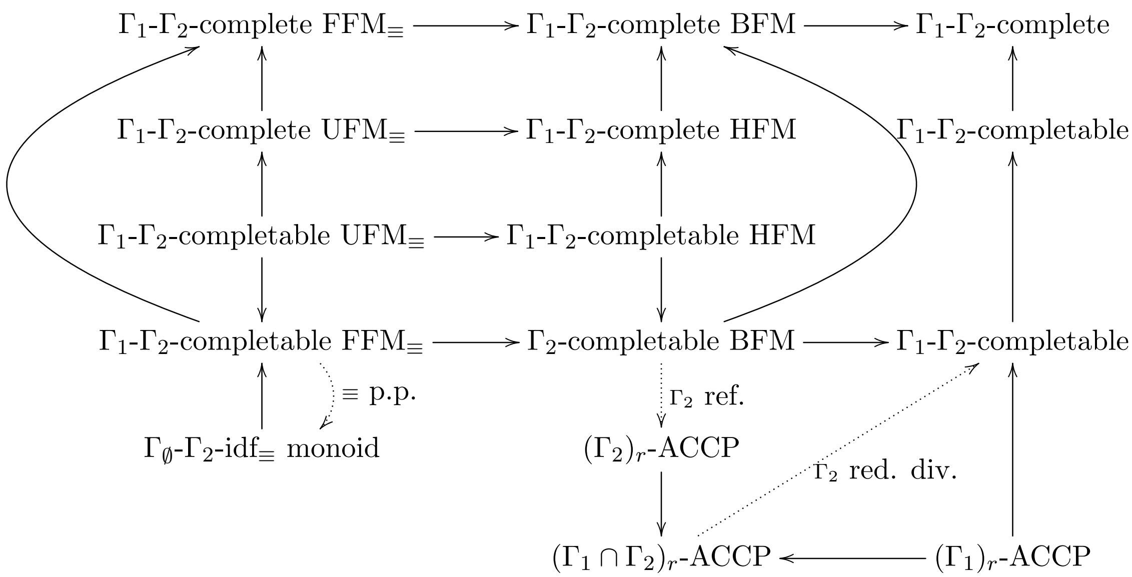
3.3 Classifying the "Complete" Cancellative Monoids

The purpose of this section is to classify all of the various kinds of Γ_1-Γ_2-complete cancellative monoids that we have defined, completely determining the hierarchy between them. The results of this effort are shown in the diagram of implications given in Figure 3.1. A dotted line indicates that the implication holds if some additional assumption is made, and that assumption is listed alongside the line. The "p.p." stands for the "partition-preserving" property we will define below. For now, it will suffice to mention that $\sim$ is partition-preserving.

There are two more terms appearing in Figure 3.1 that need to be defined. Let H be a cancellative monoid, Γ and Γ' be factorization systems on H, and $\equiv$ be an equivalence relation on $H_0^{\#}$. For $X \subseteq H^{\#}$, we call H an *X-Γ-df$_{\equiv}$ monoid* if each (nonzero) nonunit has only finitely many Γ-divisors in X up to $\equiv$-equivalence. The "df" is an acronym for "divisor finite". We define the following additional abbreviations. A *Γ'-Γ-idf$_{\equiv}$ monoid* is an atom(Γ')-Γ-df$_{\equiv}$-monoid (where the "i" stands for "irreducible" and atom(Γ') denotes the set of Γ'-atoms), a *Γ-idf$_{\equiv}$ monoid* is a Γ-Γ-idf$_{\equiv}$ monoid, an *idf$_{\equiv}$ monoid* is a fact(H)-idf$_{\equiv}$ monoid, and an *X-df$_{\equiv}$ monoid* is an X-fact(H)-df$_{\equiv}$ monoid. If $\equiv = \sim$, then we drop the "$\equiv$" from the above names.

Let H be a cancellative monoid, Γ be a factorization system on H, and ρ be a relation on H. We say H satisfies the *Γ-ascending chain condition up to ρ* (*Γ-ACC$_\rho$*) if whenever $\{a_n\}_{n=1}^{\infty}$ is a sequence of (nonzero) nonunits with each $a_{n+1} \mid_{\Gamma} a_n$, then there is an $N \geq 1$ with $a_{k+1} \rho a_k$ for $k \geq N$. Again we drop the "Γ" in the case $\Gamma = \text{fact}(H)$. The *Γ-ascending chain condition on principal ideals* (*Γ-ACCP*) is the Γ-ACC$_\sim$, and we will be focusing on this case. We note that the ascending chain conditions arising from each of the three kinds of "associate" relations are important in a non-cancellative setup.

Figure 3.1: Classifying Γ_1-Γ_2-complete Cancellative Monoids



Most of the implications in Figure 3.1 are obvious from the definitions. The rest will follow from theorems that we give below. The reader may have noticed that the "Γ_2-completable BFM" entry is missing the "Γ_1" and wonder if a typo was made. Interestingly, it turns out that that property is completely independent of the choice of Γ_1. On a similar note, in the usual case where $\equiv$ is partition-preserving, the "Γ_1" is also redundant in "Γ_1-Γ_2-completable FFM$_\equiv$". Figure 3.2 gives a simplified τ-factorization version of Figure 3.1. (More generally, this same simplification could be made for any reduced divisible factorization system.)

Figure 3.2: Classifying τ-complete Cancellative Monoids

$$
\begin{tikzcd}
\tau\text{-complete FFM}_{\equiv} \arrow[r] & \tau\text{-complete BFM} \arrow[r] & \tau\text{-complete} \\
\tau\text{-complete UFM}_{\equiv} \arrow[r] \arrow[u] & \tau\text{-complete HFM} \arrow[u] & \\
\tau\text{-completable UFM}_{\equiv} \arrow[r] \arrow[u] \arrow[d] & \tau\text{-completable HFM} \arrow[u] \arrow[d] & \\
\tau\text{-completable FFM}_{\equiv} \arrow[r] \arrow[uuu, bend left=60] \arrow[d, dotted, bend left, "\equiv \text{ p.p.}"] & \tau\text{-completable BFM} \arrow[r] \arrow[uuu, bend right=60] \arrow[d, dotted, "\tau \text{ ref.}"] & \tau\text{-completable} \arrow[uuu] \\
\tau_{\emptyset}\text{-}\tau\text{-idf}_{\equiv} \text{ monoid} \arrow[u] & \tau_r\text{-ACCP} \arrow[ur] &
\end{tikzcd}
$$

Specializing still further to the case of standard factorization in domains, we obtain Figure 3.3. This special case was originally given in [3], and the authors gave examples showing that no nontrivial implications could be added, which goes a long way towards showing that no nontrivial implications can be added to Figures 3.1 and 3.3. During the remainder of this section we will see examples that finish the proof of the fact that none of the figures can be improved.

Figure 3.3: Classifying Atomic Domains

$$
\begin{tikzcd}
 & \text{UFD} \arrow[r] \arrow[d] & \text{HFD} \arrow[d] & & \\
\text{atomic idf domain} \arrow[r, leftrightarrow] & \text{FFD} \arrow[r] & \text{BFD} \arrow[r] & \text{ACCP} \arrow[r] & \text{atomic}
\end{tikzcd}
$$

Our first step to better understand the various "completable" cancellative monoids is to give alternative characterizations of the "BFM" case. Strictly speaking, it is marginally more efficient to prove the "FFM" version of this theorem first and derive the "BFM" version as an immediate corollary, but the latter version is so much simpler that we find it more illuminating to prove it directly.

Theorem 3.3.1. *Let* Γ_1 *and* Γ_2 *be factorization systems on a cancellative monoid* H*. The following properties of* H *are equivalent:*

(1) $\Gamma_\emptyset$-Γ_2*-complete BFM, i.e., every (nonzero) nonunit has an upper bound on the lengths of its* Γ_2*-factorizations;*

(2) Γ_1-Γ_2*-completable BFM;*

(3) *weakly* Γ_1-Γ_2*-completable BFM, i.e., a* Γ_1-Γ_2*-complete BFM that is weakly* Γ_1-Γ_2*-complete; and*

(4) Γ_2*-completable BFM.*

Proof. It will suffice to show the equivalence of $(1) - (3)$. $(1) \Rightarrow (2)$: The only nontrivial part is showing that (1) implies that H is Γ_1-Γ_2-completable. For this, we note that if there is a Γ_2-factorization such that any Γ_1-Γ_2-sequence starting at that Γ_2-factorization has no Γ_1-Γ_2-complete elements, then one can construct an arbitrarily long Γ_1-Γ_2-sequence starting at that Γ_2-factorization such that the lengths of the Γ_2-factorizations in the Γ_1-Γ_2-sequence are strictly increasing. (2) $\Rightarrow$ (3): Clear. (3) $\Rightarrow$ (1): If H is weakly Γ_1-Γ_2-completable, then every Γ_2-factorization can be refined into a Γ_1-Γ_2-complete factorization, which is necessarily at least as long. $\square$

We wish to prove a similar theorem to the above for the "FFM" case, but in order to do that, we will have to make some sort of assumption about the equivalence relation. The additional property that we are looking for is the "partition-preserving" property of the next lemma.

Lemma 3.3.2. *Let H be a monoid and $\equiv$ be an equivalence relation on $H_0^{\#}$. The following are equivalent.*

1. *If two factorizations are $\equiv$-equivalent, then any partition of the first is $\equiv$-equivalent to the corresponding partition of a reordering of the second.*
2. *If two factorizations are $\equiv$-equivalent, then any partition of the first is $\equiv$-equivalent to some partition of a reordering of the second.*
3. *If $a, b, c, d \in H^{\#}$, $a \equiv c$, and $b \equiv d$, then $ab \equiv cd$.*
4. *If $a_1, \ldots, a_n, b_1, \ldots, b_n \in H^{\#}$ and each $a_i \equiv b_i$, then $a_1 \cdots a_n \equiv b_1 \cdots b_n$.*
5. *The relation $\equiv$ can be extended to a congruence relation on H.*

Proof. (2) $\Rightarrow$ (3): Assume (2) and take any $a, b, c, d \in H^{\#}$ with $a \equiv c$ and $b \equiv d$. Then ab and cd are $\equiv$-equivalent factorizations, so some partition of cd must be $\equiv$-equivalent to the partition (ab) of cd. This forces $ab \equiv cd$. (3) $\Rightarrow$ (4): Follows by an easy induction. (4) $\Rightarrow$ (1) $\Rightarrow$ (2): Clear. (3) $\Leftrightarrow$ (5): The implication (5) $\Rightarrow$ (3) is obvious, while, if (3) holds, then it is routine to check that $\equiv \cup (H^\times \times H^\times)$ is a congruence relation on H. $\square$

We call an equivalence relation satisfying the equivalent conditions of Lemma 3.3.2 *partition-preserving*. In particular, any congruence relation is partition-preserving.

Theorem 3.3.3. *Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H, and let $\equiv$ be an equivalence relation on H. Consider the following statements.*

1. *The monoid H is a $\Gamma_\emptyset$-Γ_2-idf$_\equiv$ monoid, i.e., each (nonzero) nonunit has only finitely many Γ_2-divisors up to $\equiv$-equivalence.*
2. *The monoid H is a $\Gamma_\emptyset$-Γ_2-complete FFM$_\equiv$, i.e., each (nonzero) nonunit has only finitely many Γ_2-factorizations up to $\equiv$-equivalence.*
3. *The monoid H is a Γ_1-Γ_2-completable FFM$_\equiv$.*
4. *The monoid H is a weakly Γ_1-Γ_2-completable FFM$_\equiv$, i.e., a Γ_1-Γ_2-complete FFM$_\equiv$ that is weakly Γ_1-Γ_2-completable.*
5. *The monoid H is a Γ_2-completable FFM$_\equiv$.*

Then $(1), (5) \Leftarrow (2) \Rightarrow (3) \Leftrightarrow (4)$. If $\equiv \leq \sim$, then $(1) \Leftrightarrow (2)$. If $\equiv$ is partition-preserving, then $(2) - (5)$ are equivalent. Thus all five statements are equivalent if $\equiv \leq \sim$ and $\equiv$ is partition-preserving.

Proof. For the first statement, it will suffice to prove (1) $\Leftarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4). (2) $\Rightarrow$ (1): By contradiction. Suppose that H is a $\Gamma_\emptyset$-Γ_2-complete FFM$_\equiv$ and there is a nonzero nonunit a with an infinite sequence $a_1, a_2, \ldots$ of non-$\equiv$-related Γ_2-divisors. For each i pick some Γ_2-factorization F_i of a containing a_i. By assumption, there are only finitely many Γ_2-factorizations of a up to $\equiv$-equivalence, so there must be some Γ_2-factorization that is $\equiv$-equivalent to infinitely many F_i's. This factorization must contain factors $\equiv$-equivalent to infinitely many a_i's, and these factors are necessarily distinct, a contradiction. (2) $\Rightarrow$ (3): The only nontrivial part is proving that H is Γ_1-Γ_2-completable if it is a $\Gamma_\emptyset$-Γ_2-complete FFM$_\equiv$. But in this case each nonzero nonunit has an upper bound on the lengths of its Γ_2-factorizations, so Theorem 3.3.1 gives us our desired conclusion. (3) $\Rightarrow$ (4): Clear. (4) $\Rightarrow$ (3): Theorem 3.3.1.

Now assume $\equiv \leq \sim$. (1) $\Rightarrow$ (2): We adapt the proof of [3, Theorem 5.1]. Assume that H is a $\Gamma_\emptyset$-Γ_2-idf$_\equiv$ monoid and take any Γ_2-expressible nonzero nonunit a. Let $a_1, \ldots, a_m$ be representatives of the finitely many $\equiv$-equivalence classes of the Γ_2-divisors of a. Then each Γ_2-factorization of a is $\equiv$-equivalent (hence $\sim$-equivalent) to a factorization of the form $\lambda a_1^{n_1} \cdots a_m^{n_m}$, where $\lambda \in H^\times$ and each $n_i \geq 0$, and $\lambda a_1^{n_1} \cdots a_m^{n_m} \sim a$ by the fact that $\sim$ is a congruence relation. Suppose that some n_i, say n_1, is not bounded. Then we have factorizations $\lambda_k a_1^{n_{1,k}} \cdots a_m^{n_{m,k}}$ $\equiv$-equivalent to Γ_2-factorizations of a such that $n_{1,1} < n_{1,2} < n_{1,3} < \cdots$. Suppose that $\{n_{i,k}\}_{k=1}^\infty$ is bounded for each $i \geq 2$. Then there are only finitely many ways to choose $(n_{2,k}, n_{3,k}, \ldots, n_{m,k})$, so there must be some $k < j$ with $(n_{2,k}, \ldots, n_{m,k}) = (n_{2,j}, \ldots, n_{m,j})$. We have $a_1^{n_{1,k}} \cdots a_m^{n_{m,k}} \sim a \sim a_1^{n_{1,j}} \cdots a_m^{n_{m,j}}$, and canceling yields

$a_1^{n_{1,j}-n_{1,k}} \sim 1$ and $n_{1,j} - n_{1,k} > 0$, a contradiction. Therefore after a suitable re-indexing we have $\{n_{2,k}\}_{k=1}^{\infty}$ unbounded. We can adjust by taking subsequences so that $n_{1,1} < n_{1,2} < n_{1,3} < \cdots$ and $n_{2,1} < n_{2,2} < n_{2,3} < \cdots$. Recursively carrying out this process allows us to adjust things so that $n_{i,1} < n_{i,2} < \cdots$ for each i. But then $a_1^{n_{1,1}} \cdots a_1^{n_{m,1}} \sim a_1^{n_{1,2}} \cdots a_m^{n_{m,2}}$, and canceling yields $a_1^{n_{1,2}-n_{1,1}} \cdots a_m^{n_{m,2}-n_{m,1}} \sim 1$ with each $n_{i,2} - n_{i,1} > 0$, a contradiction. Therefore there is a bound on each n_i, so a has only finitely many Γ_2-factorizations up to $\equiv$-equivalence.

Now assume that $\equiv$ is partition-preserving. It will suffice to prove the equivalence of $(2) - (4)$. $(4) \Rightarrow (2)$: Assume that H is a weakly Γ_1-Γ_2-completable $\mathrm{FFM}_{\equiv}$. Let a be any Γ_2-expressible nonzero nonunit and let $F_1, \ldots, F_n$ denote representatives of the finitely many $\equiv$-equivalence classes of the Γ_1-Γ_2-complete factorizations of a. Because H is weakly Γ_1-Γ_2-completable, every Γ_2-factorization of a is a partition of some Γ_1-Γ_2-complete factorization of a, and hence $\equiv$-equivalent to some partition of some F_i by the fact that $\equiv$ is partition-preserving. Therefore a has only finitely many Γ_2-factorizations up to $\equiv$-equivalence. $\square$

Neither of the assumptions about $\equiv$ can be dropped from Theorem 3.3.3. One cannot drop the "$\equiv \leq \sim$" requirement, since using $\Gamma = \mathrm{fact}(H)$ and $\equiv = \psi_H$ yields the false result that every monoid is a BFM. The following example shows that the partition-preserving requirement cannot be dropped.

Example 3.3.4. An example of a $\mathrm{UFD}_{\equiv}$ D that is not a $\tau_{\emptyset}$-τ_D-$\mathrm{idf}_{\equiv}$ domain. Let K be any algebraically closed field, let $D = K[x]$, and let $\equiv$ be the equivalence relation on D given by $f \equiv g \Leftrightarrow f = g$ or $\deg f = \deg g \in \{0, 1\}$. (Note that $\equiv$ is not partition-preserving since for $\alpha \in K^* \setminus \{1\}$ we have $x \equiv x$, $x \equiv \alpha x$, and $x^2 \not\equiv \alpha x^2$.) By the fact that K is algebraically closed, each nonzero nonunit in $K[x]$ splits into linear factors, and this is certainly its unique atomic factorization up to $\equiv$-equivalence, since any two same-length factorizations with all linear factors are $\equiv$-equivalent. Therefore D is a UFD$_\equiv$. However, the nonzero nonunit x^3 has an infinite family $\{\alpha x^2 \mid \alpha \in K\}$ of non-$\equiv$-equivalent divisors, so D is not a $\tau_\emptyset$-τ_H-idf$_\equiv$ domain.

Theorem 3.3.3 shows that the notions of a Γ-completable FFM$_\equiv$ and a $\Gamma_\emptyset$-Γ-idf$_\equiv$ are equivalent if $\equiv$ is partition-preserving and $\equiv \leq \sim$. We get a somewhat weaker variant with no assumptions on $\equiv$.

Theorem 3.3.5. *Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H. If H is a Γ_1-Γ_2-completable FFM$_\equiv$, then it is a Γ_1-Γ_2-idf$_\equiv$ monoid.*

Proof. By contradiction. Suppose that H is a Γ_1-Γ_2-completable FFM$_\equiv$ and there is an $a \in H^\#$ with an infinite sequence $a_1, a_2, \ldots$ of non-$\equiv$-related Γ_1-irreducible Γ_2-divisors. Each a_i appears in some Γ_2-factorization of a, which we may sequentially Γ_1-Γ_2-refine to obtain a Γ_1-Γ_2-complete factorization F_i, which necessarily has a_i as a Γ_2-factor by the fact that a_i is Γ_1-irreducible. Because there are only finitely many Γ_1-Γ_2-complete factorizations of a up to $\equiv$-equivalence, there must be some Γ_2-factorization that is $\equiv$-equivalent to infinitely many F_i's. This factorization must contain factors $\equiv$-equivalent to infinitely many a_i's, and these factors are necessarily distinct, a contradiction. $\square$

We have stated Theorem 3.3.3 in its most general version in order to give some idea of how the properties of $\sim$ come into play, but from now on we will be dealing only with the following simplified case.

Corollary 3.3.6. *Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H. The following properties of H are equivalent:*

(1) *$\Gamma_\emptyset$-Γ_2-idf monoid,*

(2) *$\Gamma_\emptyset$-Γ_2-complete FFM,*

(3) *Γ_1-Γ_2-completable FFM,*

(4) *weakly Γ_1-Γ_2-completable FFM, and*

(5) *Γ_2-completable FFM.*

At this point it is useful to bring in the ascending chain conditions. The paper [6] showed that a domain is τ-completable if it satisfies τ-ACCP. The following is a generalization.

Theorem 3.3.7. *Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H. Then H is Γ_1-Γ_2-completable if either of the following hold.*

(1) *The monoid H satisfies $(\Gamma_1)_r$-ACCP.*

(2) *The system Γ_2 is reduced divisible and H satisfies $(\Gamma_1 \cap \Gamma_2)_r$-ACCP.*

Proof. By contradiction. Suppose that (1) (resp., (2)) holds and that there is a Γ_2-factorization $\lambda a_1 \cdots a_n$ that cannot be sequentially Γ_1-Γ_2-refined into a Γ_1-Γ_2-complete factorization. Then there is an infinite sequence of Γ_2-factorizations starting at $\lambda a_1 \cdots a_n$ with each a proper Γ_1-Γ_2-refinement of the last. Then for some a_i there must be an infinite sequence of reduced Γ_1-factorizations starting at a_i with each a proper Γ_1-refinement of the last (resp., and by the reduced divisibility of Γ_2 each is in fact a proper $(\Gamma_1 \cap \Gamma_2)$-refinement of the last). Let $b_1 = a_i$ and carry out the following recursive construction. Given $b_1, \ldots, b_k$ with each b_{j+1} a proper $(\Gamma_1)_r$-divisor (resp., $(\Gamma_1 \cap \Gamma_2)_r$-divisor) of b_j and an infinite sequence of reduced Γ_1-factorizations (resp., $(\Gamma_1 \cap \Gamma_2)$-factorizations) starting at b_k with each a proper Γ_1-refinement (resp., $(\Gamma_1 \cap \Gamma_2)$-refinement) of the last, we have a nontrivial reduced Γ_1-factorization (resp., $(\Gamma_1 \cap \Gamma_2)$-factorization) $b_k = c_1 \cdots c_l b_{k+1} c_{l+1} \cdots c_m$ where b_{k+1} has in turn an infinite sequence of reduced Γ_1-factorizations (resp., $(\Gamma_1 \cap \Gamma_2)$-factorizations) starting at it with each a proper Γ_1-refinement (resp., $(\Gamma_1 \cap \Gamma_2)$-refinement) of the last. We now have an infinite sequence $\{b_k\}_{k=1}^{\infty}$ in $H^{\#}$ with each b_{k+1} a proper $(\Gamma_1)_r$-divisor (resp., $(\Gamma_1 \cap \Gamma_2)_r$-divisor), a contradiction. $\square$

The following example shows that the hypothesis that Γ_2 is reduced divisible in (2) above cannot be dropped.

Example 3.3.8. An example of a domain satisfying Γ-ACCP but not being τ-Γ-complete, with τ and Γ having some nice properties. Let R be an integral domain and $D = R[Y, \{X^r \mid r \in \mathbb{Q}^+\}]$. Define a relation τ on $D^{\#}$ by $\lambda X^r \tau \mu X^s$ for $r, s \in \mathbb{Q}^+$ and note that τ is symmetric, multiplicative, divisive, and transitive. Obtain a factorization system Γ on D by taking the symmetric, unital, associate-preserving, and normal closure of the set of factorizations of the form $(X^{r_1}) \cdots (X^{r_k})Y$, where $k \geq 0$ and $r_1, \ldots, r_k \in \mathbb{Q}^+$. We may observe that Γ is refinable and divisive. Also, any element appearing in a nontrivial Γ-factorization is a Γ-atom, so D satisfies Γ-ACCP. Finally, the nonzero nonunit XY has no τ-Γ-complete factorization.

The paper [6] proved that a τ-completable BFD satisfies τ-ACCP for τ divisive. The following is a generalization.

Theorem 3.3.9. *Let Γ be a refinable factorization system on a cancellative monoid H. If H is a Γ_r-completable BFM, then it satisfies Γ_r-ACCP.*

Proof. Assume H is a Γ_r-completable BFM and let $\{a_n\}_{n=1}^{\infty}$ be any sequence in $H^{\#}$ with each $a_{n+1} \mid_{\Gamma_r} a_n$. If there are infinitely many values of n with a_{n+1} a proper Γ_r-divisor of a_n, then by refinability we can use these reduced Γ-factorizations to obtain arbitrarily long reduced Γ-factorizations of a_1, a contradiction to Theorem 3.3.1. $\square$

The next example shows that the refinability assumption cannot be dropped.

Example 3.3.10. An example of a τ-completable UFD that does not satisfy τ-ACCP, where τ is a symmetric and associate-preserving relation on the nonzero nonunits. Let R be an integral domain and $D = R[\{X^r \mid r \in \mathbb{Q}^+\}]$. Let τ be the associate-preserving and symmetric relation on $D^{\#}$ determined by $X^{3/2^{n+2}} \tau X^{1/2^{n+2}}$ for $n \geq 0$. We observe that the only non-trivial τ-factorizations are (up to associates and order) those of the form $\lambda(X^{3/2^{n+2}})(X^{1/2^{n+2}})$, which are necessarily τ-complete since there are no τ-factorizations of length greater than 2. It follows that H is a τ-completable UFD. However, we have an infinite sequence $\{X^{1/2^{2n}}\}_{n=0}^{\infty}$ in $D^{\#}$ where each $X^{1/2^{2(n+1)}} = X^{1/2^{2n+2}}$ is a proper τ-divisor of $X^{1/2^{2n}} = (X^{3/2^{n+2}})(X^{1/2^{2n+2}})$, so D does not satisfy τ-ACCP.

We end this section with a look at several theorems where a certain factorization property is automatically inherited if we have the corresponding one with some different factorization systems.

The following is a generalization of [21, Theorem 4.11].

Theorem 3.3.11. *Let H be a cancellative monoid, $\Gamma \subseteq \Gamma'$ be factorization systems on H, and $\rho \geq \rho'$ be relations on H. If H satisfies Γ'-$ACC_{\rho'}$, then it satisfies Γ-ACC_{ρ}.*

Proof. Assume H satisfies Γ'-$\mathrm{ACC}_{\rho'}$. If $\{x_n\}_{n=1}^{\infty}$ is a sequence with each $x_{n+1} \mid_{\Gamma} x_n$, then each $x_{n+1} \mid_{\Gamma'} x_n$, so there is an $N \geq 1$ with $x_{k+1}\rho' x_k$ (hence $x_{k+1}\rho x_k$) for $k \geq N$. $\square$

Theorem 3.3.12. *Let $\Gamma_1 \subseteq \Gamma_1'$ and $\Gamma_3 \supseteq \Gamma_3'$ be factorization systems on a cancellative monoid H. If H is Γ_1'-Γ_2-complete, then it is Γ_1-Γ_2-complete. If H is (strongly) Γ_1'-Γ_2-Γ_3'-completable, then it is (strongly) Γ_1-Γ_2-Γ_3-completable.*

Proof. Any Γ_1'-Γ_2-complete factorization is Γ_1-Γ_2-complete, and any Γ_3'-Γ_2-refinement is a Γ_3-Γ_2-refinement. $\square$

Theorem 3.3.13. *Let $\Gamma \subseteq \Gamma'$ be factorization systems on a cancellative monoid H. If H is a Γ'-completable BFM (resp., FFM), then it is a Γ-completable BFM (resp., FFM). In particular, a BFM (resp., FFM) is a Γ-completable BFM (resp., FFM).*

Proof. Follows from Theorem 3.3.1 (resp., Corollary 3.3.6). $\square$

Theorem 3.3.14. *Let $\Gamma_1 \supseteq \Gamma_1'$ and Γ_2 be factorization systems on a cancellative monoid H. If H is a Γ_1'-Γ_2-completable HFM (resp., UFM), then it is a Γ_1-Γ_2-completable HFM (resp., UFM).*

Proof. The only nontrivial part is showing that a Γ_1'-Γ_2-completable HFM (resp., UFM) is Γ_1-Γ_2-completable, but this is taken care of by Theorem 3.3.1. $\square$

There is more to be said about unique factorization inheritance, but at this point it would involve too great of an excursion from the main themes of this chapter. We leave this discussion for a later chapter after we have introduced Γ-primes and other useful tools.

We end this section with diagrams of implications summarizing the most important inheritance properties that follow from the results of this section.

Figure 3.4: Inheritance of "Completable" Properties with $\Gamma_2 \subseteq \Gamma_2'$

$$
\begin{tikzcd}
\Gamma_1'\text{-}\Gamma_2'\text{-completable UFM} \arrow[d] & \Gamma_1\text{-}\Gamma_2\text{-completable UFM} \arrow[d] \\
\Gamma_2'\text{-completable FFM} \arrow[r] \arrow[d] & \Gamma_2\text{-completable FFM} \arrow[d] \\
\Gamma_2'\text{-completable BFM} \arrow[r] \arrow[d, dotted, "\Gamma_2' \text{ ref.}"] \arrow[dd, bend right] & \Gamma_2\text{-completable BFM} \arrow[d, dotted, "\Gamma_2 \text{ ref.}"'] \arrow[dd, bend left] \\
(\Gamma_2')_r\text{-ACCP} \arrow[r] \arrow[d, dotted, "\Gamma_2' \text{ red. div.}"] & (\Gamma_2)_r\text{-ACCP} \arrow[d, dotted, "\Gamma_2 \text{ red. div.}"'] \\
\Gamma_1'\text{-}\Gamma_2'\text{-completable} & \Gamma_1\text{-}\Gamma_2\text{-completable}
\end{tikzcd}
$$

Figure 3.5: Inheritance of "Completable" Properties with $\Gamma \subseteq \Gamma'$

$$\begin{tikzcd}
\Gamma'\text{-completable UFM} \arrow[d] & \Gamma\text{-completable UFM} \arrow[d] \\
\Gamma'\text{-completable FFM} \arrow[r] \arrow[d] & \Gamma\text{-completable FFM} \arrow[d] \\
\Gamma'\text{-completable BFM} \arrow[r] \arrow[d, dotted, "r' \text{ ref.}"] \arrow[dd, bend right=60] & \Gamma\text{-completable BFM} \arrow[d, dotted, "r \text{ ref.}"'] \arrow[dd, bend left=60] \\
(\Gamma')_r\text{-ACCP} \arrow[r] \arrow[d] & \Gamma_r\text{-ACCP} \arrow[d] \\
\Gamma'\text{-completable} & \Gamma\text{-completable}
\end{tikzcd}$$

3.4 Factorization and Closures, II

In this section we will continue our study of the various closures of factorization systems, primarily focusing on how the complete concepts are affected. In the next chapter there will be an analogous section focusing on the atomic concepts.

Theorem 3.4.1. *Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H, and let Γ be the factorization system consisting of the Γ_1-Γ_2-complete factorizations.*

(1) *If Γ_2 is unital, then so is Γ.*

(2) *If $(\Gamma_1)_r$ and Γ_2 are associate-preserving, then so is Γ.*

Proof.

(1) Assume that Γ_2 is unital. Let $\lambda a_1 \cdots a_n$ be any Γ_1-Γ_2-complete factorization, and take any $\mu \in H^\times$. The factorization $\mu a_1 \cdots a_n$ is a Γ_2-factorization by the

unital property, and we must show that it has no proper Γ_1-Γ_2-refinements. But this is evident, since the unital property gives an obvious length-preserving one-to-one correspondence between the Γ_1-Γ_2-refinements of the aforementioned factorizations.

(2) Assume that $(\Gamma_1)_r$ and Γ_2 are associate-preserving. Take any Γ_1-Γ_2-complete factorization $\lambda a_1 \cdots a_n$ and $\mu_1, \ldots, \mu_n \in H^\times$. Then $\lambda(\mu_1 a_1) \cdots (\mu_n a_n)$ is a Γ_2-factorization by the associate-preserving property of Γ_2. We must show that it is Γ_1-Γ_2-complete, so let $\mu_1 a_1 = b_{1,1} \cdots b_{1,m_1}$, $\ldots$, $\mu_n a_n = b_{n,1} \cdots b_{n,m_n}$ be any $((\Gamma_1)_r \cup \operatorname{trfact}(H))$-factorizations with $\lambda b_{1,1} \cdots b_{n,m_n}$ a Γ_2-factorization. By the fact that $(\Gamma_1)_r$ is associate-preserving, each $a_i = (\mu_i^{-1} b_{i,1}) b_{i,2} \cdots b_{i,m_i}$ is a $((\Gamma_1)_r \cup \operatorname{trfact}(H))$-factorization, and $\lambda(\mu_1^{-1} b_{1,1}) b_{1,2} \cdots b_{1,m_1} \cdots (\mu_n^{-1} b_{n,1}) b_{n,2} \cdots b_{n,m_n}$ is a Γ_1-Γ_2-refinement of $\lambda a_1 \cdots a_n$ by the fact that Γ_2 is associate-preserving. By Γ_1-Γ_2-completeness, each $m_i = 1$, as desired.

□

Theorem 3.4.2. *Let Γ be a unital or strong associate-preserving factorization system on a monoid H.*

(1) *For every $\Gamma_{u,sap}$-factorization, there is a Γ-factorization of the same element of the same length whose factors are strong associates of the corresponding factors in the original factorization.*

(2) *The Γ-expressible and $\Gamma_{u,sap}$-expressible elements coincide.*

(3) *Every $\Gamma_{u,sap}$-divisor of a nonunit is a strong associate of a Γ-divisor of that element.*

Proof. It will suffice to prove (1), since (2)-(3) will then become obvious. Let $a = \lambda a_1 \cdots a_n$ be a $\Gamma_{u,sap}$-factorization. Then there are $\nu, \mu_1, \ldots, \mu_n \in H^\times$ with $\nu(\mu_1 a_1) \cdots (\mu_n a_n)$ a Γ-factorization. If Γ is unital, then $a = (\lambda \mu_1^{-1} \cdots \mu_n^{-1})(\mu_1 a_1) \cdots (\mu_n a_n)$ is a Γ-factorization, while if Γ is strong associate-preserving, then $a = \nu(\nu^{-1}\lambda a_1)a_2 \cdots a_n$ is a Γ-factorization.

□

Corollary 3.4.3. *Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H, with Γ_1 associate-preserving and Γ_2 unital or associate-preserving. Then every Γ_1-$(\Gamma_2)_{u,ap}$-complete factorization is $\sim$-equivalent to a Γ_1-Γ_2-complete factorization of the same element.*

Proof. Theorem 3.4.2 part (1) and Theorem 3.4.1. □

Lemma 3.4.4. *Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H.*

(1) *If H is Γ_1-Γ_2-complete and every $(\Gamma_2)_u$-expressible nonzero nonunit is Γ_2-expressible, then H is Γ_1-$(\Gamma_2)_u$-complete.*

(2) *If H is a Γ_1-$(\Gamma_2)_u$-completable BFM (resp., FFM, HFM, UFM), then it is a Γ_1-Γ_2-completable BFM (resp., FFM, HFM, UFM).*

Proof.

(1) Follows from the observation that a Γ_1-Γ_2-complete factorization is Γ_1-$(\Gamma_2)_u$-complete.

(2) Assume H is a Γ_1-$(\Gamma_2)_u$-completable BFM (resp., FFM, HFM, UFM). By Theorem 3.3.13, we know that H is Γ_1-Γ_2-completable, and all that remains is to show

that the other half of the Γ_1-Γ_2-completable BFM (resp., FFM, HFM, UFM) definition is satisfied. For this it suffices to note that every Γ_1-Γ_2-complete factorization is Γ_1-$(\Gamma_2)_u$-complete.

$\square$

The implications in (1) and (2) above do not reverse.

Example 3.4.5. An example of a τ-complete UFD that is not τ_r-complete, with τ a symmetric relation on the nonzero nonunits. Let $D = \mathbb{C}[\{X^r \mid r \in \mathbb{Q}^+\}]$ and τ be the symmetric relation on $D^\#$ determined by $X^r \tau i X^r$. Then the τ-reducible nonzero nonunits are precisely those of the form λX^r, which have unique (up to order) τ-complete factorizations $(-i\lambda)(X^{r/2})(iX^{r/2})$. However, the nonzero nonunit X^r has no τ_r-complete factorization.

Example 3.4.6. An example of a τ_r-completable UFD that is not a τ-complete BFD. Let $D = \mathbb{R}[\{X^r \mid r \in \mathbb{Q}^+\}]$ and τ be the symmetric relation on $D^\#$ given by $2X^r \tau 2X^r$. Note that D is τ_r-completable since every nontrivial reduced τ-factorization is already τ-complete. Moreover, the τ_r-reducible nonzero nonunits are precisely those of the form $2^n X^r$, which have unique τ_r-complete factorizations $(2X^{r/n})^n$, so D is a τ_r-completable UFD. However, since X has arbitrarily long τ-complete factorizations, namely those of the form $X = 2^{-n}(2X^{r/n})^n$, the domain D is not a τ-complete BFD.

Theorem 3.4.7. *Let H be a cancellative monoid, Γ and Γ' be factorization systems on H, and $X \subseteq H^\#$. Assume Γ is unital or associate-preserving and Γ' is associate-preserving.*

(1) *The monoid H satisfies Γ-ACCP if and only if it satisfies $\Gamma_{u,ap}$-ACCP.*

(2) *The monoid H is a Γ-completable BFM (resp., FFM) if and only if it is a $\Gamma_{u,ap}$-completable BFM (resp., FFM).*

(3) *If H is a Γ'-Γ-completable HFM (resp., UFM), then it is a Γ'-$\Gamma_{u,ap}$-completable HFM (resp., UFM). In particular, a Γ-completable HFM (resp., UFM) is a $\Gamma_{u,ap}$-completable HFM (resp., UFM).*

(4) *The monoid H is an X-Γ-df monoid if and only if is an X-$\Gamma_{u,ap}$-df monoid.*

(5) *If H is Γ'-$\Gamma_{u,ap}$-complete, then it is Γ'-Γ-complete.*

Proof.

(1) ($\Rightarrow$): Assume H satisfies Γ-ACCP and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonzero nonunits with each $a_{n+1} \mid_{\Gamma_{u,ap}} a_n$. Let $b_1 = a_1$ and carry out the following recursive construction. Given $b_1, \ldots, b_k$ with each $b_i \sim a_i$ and each $b_{i+1} \mid_\Gamma b_i$, note that $a_{k+1} \mid_{\Gamma_{u,ap}} a_k$ and hence $a_{k+1} \mid_{\Gamma_{u,ap}} b_k$ by the unital property, and Theorem 3.4.2 part (3) implies that there is a $b_{k+1} \sim a_{k+1}$ with $b_{k+1} \mid_\Gamma b_k$. So we have constructed a sequence $\{b_n\}_{n=1}^{\infty}$ with each $b_{n+1} \mid_\Gamma b_n$ and $b_n \sim a_n$. By the Γ-ACCP property, there is an N with $b_{k+1} \sim b_k$ (and hence $a_{k+1} \sim a_k$) for $k \geq N$. ($\Leftarrow$): Theorem 3.3.11.

(2) By Corollary 3.3.6, the "FFM" case is just the special case $X = H^{\#}$ of part (4) below. The "BFM" case follows from Theorem 3.4.2 part (1) and Theorem 3.3.1.

(3) The second statement will follow from combining the first with Theorem 3.3.14.

Now assume that H is a Γ'-Γ-completable HFM (resp., UFM). By part (2) and Theorem 3.3.1, we know H is Γ'-$\Gamma_{u,ap}$-completable, so all that remains is to show the Γ'-$\Gamma_{u,ap}$-complete factorizations of a given nonzero nonunit are all of the same length (resp., are all $\sim$-equivalent). For this it will suffice to show that every Γ'-$\Gamma_{u,ap}$-complete factorization is $\sim$-equivalent to a Γ'-Γ-complete factorization of the same nonzero nonunit, which is accomplished in Corollary 3.4.3.

(4) ($\Rightarrow$): Theorem 3.4.2 part (3). ($\Leftarrow$): Clear.

(5) Assume H is Γ'-$\Gamma_{u,ap}$-complete and take any Γ-expressible $a \in H^{\#}$. Then a is certainly $\Gamma_{u,ap}$-expressible, and thus has a Γ'-$\Gamma_{u,ap}$-complete factorization $a = \lambda a_1 \cdots a_n$. By Theorem 3.4.2 part (1), there is a Γ-factorization $a = \lambda' a_1' \cdots a_n'$ with each $a_i' \sim a_i$, and this is a Γ'-Γ-complete factorization by Theorem 3.4.1.

□

The following examples show that the implications in (3) and (5) do not reverse.

Example 3.4.8. An example of a τ_{ap}-completable UFD that is not a τ_{ap}-τ-complete HFD (hence not a τ-complete HFD), with τ a symmetric relation on the nonzero nonunits with $(\Gamma_\tau)_{ap} = \Gamma_{\tau_{ap}}$. Let τ be the symmetric relation on $\mathbb{Z}^{\#}$ given by $a\tau b \Leftrightarrow ab > 0$. Note that $(\Gamma_\tau)_{ap} = \Gamma_{\tau_{ap}} = \text{fact}(\mathbb{Z})$, and $\mathbb{Z}$ is certainly a UFD. But $\mathbb{Z}$ is not a $\tau_{\mathbb{Z}}$-τ-complete HFD, since $(-4)(-2) = 2^3$ are two different length $\tau_{\mathbb{Z}}$-τ-complete factorizations of the same element.

Example 3.4.9. An example of a domain that is both a τ-complete UFD and a τ_{ap}-τ-complete UFD, but not τ-τ_{ap}-complete (hence not τ_{ap}-complete), where τ is a symmetric relation on the nonzero nonunits with $(\Gamma_\tau)_{ap} = \Gamma_{\tau_{ap}}$. Let $D = \mathbb{C}[\{X^r \mid r \in \mathbb{Q}^+\}]$ and τ be the symmetric relation on $D^\#$ determined by $\pm X^r \tau X^r$, $\pm X^r \tau i X^r$, and $-iX^r \tau iX^r$, where $r \in \mathbb{Q}^+$. Observe that $(\Gamma_\tau)_{ap} = \Gamma_{\tau_{ap}} = \operatorname{tfact}(D) \cup \{\lambda(\mu_1 X^r) \cdots (\mu_n X^r) \mid \lambda, \mu_1, \ldots, \mu_n \in \mathbb{C}^\times\}$. The nontrivial τ-factorizations are (up to order) the factorizations of the forms $\lambda(X^r)^n$, $\lambda(X^r)^n(-X^r)$, $\lambda(X^r)^n(iX^r)$, $\lambda(X^r)^n(-X^r)(iX^r)$, $\lambda(-X^r)(iX^r)$, and $\lambda(iX^r)(-iX^r)$. The last is τ_{ap}-τ-complete, while the rest have proper τ-refinements $\lambda(X^{r/2})^{2n}$, $\lambda(X^{r/2})^{2n+1}(-X^{r/2})$, $\lambda(X^{r/2})^{2n+1}(iX^{r/2})$, $\lambda(X^{r/2})^{2n+1}(-X^{r/2})(X^{r/2})(iX^{r/2})$, and $\lambda(X^{r/2})(-X^{r/2})(X^{r/2})(iX^{r/2})$, respectively. The τ-reducible nonzero nonunits are those of the form λX^r, which have unique (up to order) τ-complete factorizations $\lambda(iX^{r/2})(-iX^{r/2})$, which are in fact τ_{ap}-τ-complete. Therefore D is both a τ-complete UFD and a τ_{ap}-τ-complete UFD. However, we can use the reduced τ-factorizations $X^r = (X^{r/2})^2$, $-X^r = (X^{r/2})(-X^{r/2})$, $iX^r = (X^{r/2})(iX^{r/2})$, and $-iX^r = (-X^{r/2})(iX^{r/2})$ to obtain a proper τ-τ_{ap}-refinement of any nontrivial τ_{ap}-factorization, so D is not τ-τ_{ap}-complete.

Some of these results can be improved if we strengthen “unital or associate-preserving” to “associate-preserving”.

Theorem 3.4.10. *Let Γ_1 and Γ_2 be associate-preserving factorization systems on a cancellative monoid H.*

(1) *The monoid H is Γ_1-Γ_2-complete if and only if it is Γ_1-$(\Gamma_2)_u$-complete.*

(2) *The monoid H is a Γ_1-Γ_2-complete(able) BFM (resp., FFM, HFM, UFM) if and only if it is a Γ_1-$(\Gamma_2)_u$-complete(able) BFM (resp., FFM, HFM, UFM).*

Proof.

(1) ($\Rightarrow$): Theorem 3.4.2 part (2) and Lemma 3.4.4 part (1). ($\Leftarrow$): Theorem 3.4.7 part (5).

(2) The "$\Leftarrow$" direction for the "completable" case is Lemma 3.4.4 part (2), and the "$\Rightarrow$" direction for that case is covered in Theorem 3.4.7. So we move on to the "complete" case. ($\Rightarrow$): Assume H is a Γ_1-Γ_2-complete BFM (resp., FFM, HFM, UFM). By part (1), we know H is Γ_1-$(\Gamma_2)_u$-complete, and it suffices to prove the other half of the Γ_1-$(\Gamma_2)_u$-complete BFM (resp., FFM, HFM, UFM) definition is satisfied, and for this it suffices to show that every Γ_1-$(\Gamma_2)_u$-complete factorization is $\sim$-equivalent to a Γ_1-Γ_2-complete factorization of the same element. For this we cite Corollary 3.4.3. ($\Leftarrow$): Analogously to the "$\Rightarrow$" direction, it suffices to note that every Γ_1-Γ_2-complete factorization is Γ_1-$(\Gamma_2)_u$-complete.

□

While we can obtain several results about reduced Γ-factorization as a special case of the very general setup of Theorem 3.4.2, there are some that require something more specific about reduced Γ-factorization, namely the leading unit being constant. Let H be a monoid, Γ be a factorization system on H, and $\nu \in H^\times$. We use $\Gamma_{r,\nu}$ to denote the factorization system consisting of Γ-factorizations with ν as the leading unit. For lack of a better term, we call a $\mathrm{fact}(H)_{r,\nu}$-factorization a *ν-reduced* *factorization*. Of course, the 1-reduced Γ-factorizations are simply the reduced Γ-factorizations. The following observations are noteworthy, as they allow us to derive immediate ν-reduced Γ-factorization corollaries from many general results: (i) if ρ is a relation on $H^{\#}$ and Γ is ρ-preserving, then so is $\Gamma_{r,\nu}$, and (ii) if Γ is unital, then $(\Gamma_{r,\nu})_u = \Gamma = \Gamma_u$ and $(\Gamma_{r,\nu})_{u,sap} = \Gamma_{sap} = \Gamma_{u,sap}$.

Lemma 3.4.11. *Let Γ_1, Γ_2, and Γ_3 be factorization systems on a cancellative monoid H with Γ_2 unital, and let $\nu \in H^{\times}$.*

(1) *If H is Γ_1-$(\Gamma_2)_{r,\nu}$-complete, then it is Γ_1-Γ_2-complete.*

(2) *The monoid H is (strongly) Γ_1-Γ_2-Γ_3-completable if and only if it is (strongly) Γ_1-$(\Gamma_2)_{r,\nu}$-Γ_3-completable.*

(3) *If H is a Γ_1-Γ_2-completable BFM (resp., FFM, HFM, UFM), then it is a Γ_1-$(\Gamma_2)_{r,\nu}$-completable BFM (resp., FFM, HFM, UFM).*

Proof.

(1) Lemma 3.4.4 part (1).

(2) ($\Rightarrow$): Clear. ($\Leftarrow$): We will prove the "non-strongly" case. The "strongly" case is similar. Assume H is Γ_1-$(\Gamma_2)_{r,\nu}$-Γ_3-completable. Let $\lambda a_1 \cdots a_n$ be any Γ_2-factorization. We can sequentially Γ_3-$(\Gamma_2)_{r,\nu}$-refine the $(\Gamma_2)_{r,\nu}$-factorization $\nu a_1 \cdots a_n$ to a Γ_1-$(\Gamma_2)_{r,\nu}$-complete (hence Γ_1-Γ_2-complete) factorization $\nu b_1 \cdots b_m$. By the fact that Γ_2 is unital, this means we can sequentially Γ_3-Γ_2-refine $\lambda a_1 \cdots a_n$ to $\lambda b_1 \cdots b_m$, which is Γ_1-Γ_2-complete by Theorem 3.4.1.

(3) Follows from part (2) and the observation that every Γ_1-$(\Gamma_2)_{r,\nu}$-complete fac-

torization is Γ_1-Γ_2-complete.

$\square$

Example 3.4.6 shows that the converse to (3) is false, while the following example shows that the converse to (1) is false. Also, these examples show that there is no hope of making a version of (3) with the "completable" monoids replaced with the weaker "complete" ones.

Example 3.4.12. An example of a τ-completable UFD that is not τ_r-complete, with τ a symmetric relation on the nonzero nonunits. Let D and τ be as in Example 3.4.8 above. Then D is a τ-completable UFD, but it is not τ_r-complete since iX has no τ_r-complete factorization.

Theorem 3.4.13. *Let H be a cancellative monoid, Γ_1 and Γ_2 be factorization systems on H, and $\nu \in H^{\times}$. Assume Γ_1 is associate-preserving and Γ_2 is unital and associate-preserving.*

(1) *The monoid H is Γ_1-Γ_2-complete if and only if it is Γ_1-$(\Gamma_2)_{r,\nu}$-complete.*

(2) *The monoid H is a Γ_1-Γ_2-complete(able) BFM (resp., FFM, HFM, UFM) if and only if it is a Γ_1-$(\Gamma_2)_{r,\nu}$-complete(able) BFM (resp., FFM, HFM, UFM). In particular, the monoid H is a Γ_2-completable BFM (resp., FFM, HFM, UFM) if and only if it is a $(\Gamma_2)_{r,\nu}$-completable BFM (resp., FFM, HFM, UFM).*

Proof.

(1) ($\Rightarrow$): Use Theorem 3.4.1 to construct Γ_1-$(\Gamma_2)_{r,\nu}$-complete factorizations of elements out of their Γ_1-Γ_2-complete factorizations. ($\Leftarrow$): Lemma 3.4.11.

(2) In view of part (1), it will suffice to show that every Γ_1-Γ_2-complete factorization is $\sim$-equivalent to a Γ_1-$(\Gamma_2)_{r,\nu}$-complete factorization of the same element, and this follows from Corollary 3.4.3.

$\square$

Let H be a cancellative monoid and τ be an associate-preserving relation on $H^\#$. Applying the results of this section to τ_r and $(\tau_r)_u = \tau$, or to τ and τ_r, we can obtain several results about reduced τ-factorization. This sort of comment will apply to many future theorems as well (and obviously its applications extend well beyond reduced τ-factorization). This generalizes work done in [21] on the relationship between τ-factorization and reduced τ-factorization, and also makes more explicit the general convention of [6] of working with reduced τ-factorization whenever τ is associate-preserving. For convenience, we now restate several of the above results in a (reduced) τ-factorization context.

Corollary 3.4.14. *Let H be a cancellative monoid and τ be a relation on $H^\#$.*

(1) *If H satisfies τ-ACCP, then it satisfies τ_r-ACCP. If τ is associate-preserving, then the converse is true.*

(2) *If H is τ_r-complete, then it is τ-complete. If τ is associate-preserving, then the converse is true.*

(3) *The monoid H is τ-completable if and only if it is τ_r-completable.*

(4) *If H is a τ-completable BFM (resp., FFM, HFM, UFM), then it is a τ_r-completable BFM (resp., FFM, HFM, UFM). If τ is associate-preserving, then*

the converse is true.

Another important closure to consider is the combinable closure.

Theorem 3.4.15. *Let Γ be a factorization system on a monoid H.*

(1) *A nonunit has a Γ-factorization of length at least n if and only if it has a Γ_c-factorization of length at least n.*

(2) *The Γ-expressible elements of $H_0^\#$ coincide with the Γ_c-expressible elements.*

(3) $(\Gamma_c)_u = (\Gamma_u)_c$.

(4) $((\Gamma_c)_{sap})_u = (\Gamma_{u,sap})_c$.

(5) *If Γ is associate-preserving (resp., symmetric, reflexive, transitive, divisive, (reduced) normal, (reduced) divisible, (reduced) divisible, relational), then so is Γ_c.*

Proof. Parts (1)-(3) are clear, and (4) is a straightforward calculation. We will do the relational case of (5) and leave the rest to the reader. In this case, we may write $\Gamma = \Gamma_\psi$ for some relation ψ on H by Theorem 2.2.1. Define $\lambda\psi_c a$ for $\lambda \in H^\times$ and $a \in H_0^\#$ if and only if a has a factorization $a = a_1 \cdots a_n$, where $\lambda\psi a_i$ for each i and $a_i\psi a_j$ for each $i < j$. Define $a\psi_c b$ for $a, b \in H^\#$ if and only if a and b have factorizations $a = a_1 \cdots a_m$ and $b = a_{m+1} \cdots a_n$, where $a_i\psi a_j$ for $i < j$. It is easily verified that $\Gamma_c = \Gamma_{\psi_c}$. □

Theorem 3.4.16. *Let H be a cancellative monoid and Γ be a factorization system on H.*

(1) *The monoid H is a Γ-completable BFM if and only if it is a Γ_c-completable BFM.*

(2) *If $\equiv$ is partition-preserving, then H is a Γ-completable FFM$_\equiv$ if and only if it is a Γ_c-completable FFM$_\equiv$.*

Proof.

(1) Theorem 3.4.15 part (1) and Theorem 3.3.1.

(2) Assume $\equiv$ is partition-preserving. ($\Rightarrow$): Assume H is a Γ-completable FFM$_\equiv$. Every Γ_c-factorization of an nonzero nonunit is a partition of a Γ-factorization of that element. As in the proof of Theorem 3.3.3, this implies that each nonzero nonunit has only finitely many Γ_c-factorizations up to $\equiv$-equivalence. We apply Theorem 3.3.3 to conclude that H is a Γ_c-completable FFM$_\equiv$. ($\Leftarrow$): Theorem 3.3.13.

□

Theorem 3.4.17. *Let Γ be a reduced divisible factorization system on a monoid H.*

(1) *Every* fact(H)-Γ_c*-refinement of a factorization may be Γ-refined to a Γ-refinement of the original factorization.*

(2) *The Γ-complete factorizations coincide with the $(\Gamma)_c$-complete factorizations.*

Proof.

(1) Let $\lambda a_1 \cdots a_n$ be any factorization and $\lambda b_{1,1} \cdots b_{n,m_n}$ be any fact(H)-Γ_c-refinement of it, where each $a_i = b_{i,1} \cdots b_{i,m_i}$ is a reduced Γ_c-factorization. We may write each $b_{i,j}$ as a reduced factorization $b_{i,j} = c_{i,j,1} \cdots c_{i,j,k_{i,j}}$, where

$\lambda c_{1,1,1} \cdots c_{n,m_n,k_{n,m_n}}$ is a Γ-factorization. By the reduced divisibility of Γ, each $a_i = c_{i,1,1} \cdots c_{i,m_i,k_{i,m_i}}$ is a Γ-factorization and each $b_{i,j} = c_{i,j,1} \cdots c_{i,j,k_{i,j}}$ is a Γ-factorization. Therefore the Γ-factorization $\lambda c_{1,1,1} \cdots c_{n,m_n,k_{n,m_n}}$ is a Γ-refinement of both $\lambda b_{1,1} \cdots b_{n,m_n}$ and $\lambda a_1 \cdots a_n$.

(2) Any Γ-complete factorization is Γ_c-complete by part (2). It follows from the reduced divisible property that any (Γ_c)-complete factorization must be Γ-complete.

□

We will finish this section with some results about normal closures.

Theorem 3.4.18. *Let H be a cancellative monoid, and let $\Gamma, \Gamma', \Gamma_i$, and Γ_i' be factorization systems on H for $i = 1, 2, 3$. Assume $\Gamma \subseteq \Gamma' \subseteq \Gamma \cup \mathrm{tfact}(H)$, and $\Gamma_i \subseteq \Gamma_i' \subseteq \Gamma_i \cup \mathrm{tfact}(H)$ for each i.*

(1) *The Γ-$\mathrm{fact}(H)$-refinements of a factorization coincide with its Γ'-$\mathrm{fact}(H)$-refinements.*

(2) *A Γ_1-Γ_2-complete factorization is Γ_1'-Γ_2'-complete.*

(3) *A factorization of a non-Γ-expressible nonzero nonunit has no proper $\mathrm{fact}(H)$-Γ'-refinements.*

(4) *If H is Γ_1-Γ_2-complete, then it is Γ_1'-Γ_2'-complete.*

(5) *If H is (strongly) Γ_1'-Γ_2'-Γ_3'-completable, then it is (strongly) Γ_1-Γ_2-Γ_3-completable.*

(6) *The monoid H is a Γ-completable BFM (resp., FFM) if and only if it is a Γ'-completable BFM (resp., FFM).*

(7) *If H satisfies Γ'-ACCP, then it satisfies Γ-ACCP. If Γ is unital or associate-preserving, then the converse is true.*

Proof.

(1) This follows from the fact that $\Gamma_r \cup \operatorname{trfact}(H) = (\Gamma')_r \cup \operatorname{trfact}(H)$.

(2) A Γ_1-Γ_2-complete factorization is Γ_1-Γ_2'-complete because every nontrivial Γ_2'-factorization is a Γ_2-factorization, and hence Γ_1'-Γ_2'-complete by part (2).

(3) If $a \in H^\#$ has a factorization that can be properly refined to a Γ'-factorization, then that Γ'-factorization is nontrivial and hence a Γ-factorization.

(4) Assume H is Γ_1-Γ_2-complete and let $a \in H^\#$ be any Γ_2'-expressible nonzero nonunit. If a is not Γ_2-expressible, then any Γ_2'-factorization of a is Γ_1'-Γ_2'-complete by part (4). On the other hand, if a is Γ_2-expressible, then it has a Γ_1-Γ_2-complete factorization, which is Γ_1'-Γ_2-complete by part (2), which is in fact Γ_1'-Γ_2'-complete since all of the nontrivial Γ_2'-factorizations are Γ_2-factorizations.

(5) We prove the "non-strongly" case; the "strongly" case is similar. Assume H is Γ_1'-Γ_2'-Γ_3'-completable. Then any Γ_2-factorization can be sequentially Γ_3'-Γ_2'-refined to a Γ_1'-Γ_2'-complete factorization. But any sequential Γ_3'-Γ_2'-refinement of a Γ_2-factorization is in fact a sequential Γ_3-Γ_2-refinement, so any Γ_2-factorization can be sequentially Γ_3-Γ_2-refined to a Γ_1-Γ_2-complete factorization.

(6) Follows from Theorem 3.3.1 (resp., 3.3.3) since each nonzero nonunit has at most one Γ'-factorization that is not a Γ-factorization.

(7) ($\Rightarrow$): Theorem 3.3.11. ($\Leftarrow$): Assume that Γ is unital or associate-preserving and that H satisfies Γ-ACCP. We proceed by contradiction. Suppose that there is a sequence $\{a_n\}_{n=1}^{\infty}$ in $H^{\#}$ with each $a_{n+1} \mid_{\Gamma'} a_n$ and $a_{k+1} \not\sim a_k$ for infinitely many k's. By the associate-preserving property of $(\Gamma')_{ap}$, we can take subsequences and start our notation over again with each a_{n+1} a proper $(\Gamma')_{ap}$-divisor of a_n, hence a proper $\Gamma_{u,ap}$-divisor of a_n. But this contradicts Theorem 3.4.7's assertion that H satisfies $\Gamma_{u,ap}$-ACCP.

$\square$

Theorem 3.4.19. *Let Γ, Γ', and Γ_3 be factorization systems on a cancellative monoid H. Assume Γ is unital and associate-preserving, and that $\Gamma \subseteq \Gamma' \subseteq \Gamma \cup \operatorname{tfact}(H)$.*

(1) *The Γ-complete and Γ'-complete factorizations of Γ-expressible elements of $H^{\#}$ coincide.*

(2) *The monoid H is Γ-complete if and only if it is Γ'-complete.*

(3) *If $\Gamma_3 \supseteq \Gamma$, then H is Γ-Γ-Γ_3-completable if and only if it is Γ'-Γ'-Γ_3'-completable.*

(4) *The monoid H is a Γ-complete(able) BFM (resp., HFM) if and only if it is a Γ'-complete(able) BFM (resp., HFM).*

(5) *If H is a Γ'-complete(able) FFM$_{\equiv}$ (resp., UFM$_{\equiv}$), then it is a Γ-complete(able) FFM$_{\equiv}$ (resp., UFM$_{\equiv}$). If $\sim\leq\equiv$, then the converse is true.*

Proof.

(1) It will suffice to show that any trivial Γ'-complete factorization λa of a Γ-expressible nonzero nonunit is Γ-complete. Since Γ is unital and associate-

preserving, we know a has a reduced Γ-factorization, which we can use to get a Γ-refinement of λa. This refinement cannot be proper, so λa is in fact a Γ-complete factorization.

(2) Follows from part (1).

(3) Assume $\Gamma_3 \supseteq \Gamma$. ($\Rightarrow$): Assume H is Γ-Γ-Γ_3-completable. Any Γ-factorization can be sequentially Γ_3-Γ-refined, hence sequentially Γ'_3-Γ'-refined, into a Γ-complete factorization, which is Γ'-complete by Theorem 3.4.18 part (3). Any Γ'-factorization that is not a Γ-factorization is a trivial factorization λa. If λa is not Γ-expressible, then this factorization is Γ'-complete by part (1), so let us assume λa is Γ-expressible, and hence a has a reduced Γ-factorization $a = b_1 \cdots b_n$ by the fact that Γ is associate-preserving. So $a = \lambda b_1 \cdots b_n$ is a Γ-refinement of the factorization λa, and this Γ-refinement may in turn be sequentially Γ_3-Γ-refined into a Γ-complete factorization, which is Γ'-complete by Theorem 3.4.18 part (3). ($\Leftarrow$): Theorem 3.4.18 part (6).

(4) Follows from parts (1)-(3).

(5) ($\Rightarrow$): Follows from parts (1)-(3). ($\Leftarrow$): If $\sim \leq \equiv$, then each nonzero nonunit that is Γ'-expressible but not Γ-expressible has exactly one Γ'-factorization up to $\equiv$-equivalence, and the remainder of the proof is accomplished by parts (1)-(3).

$\square$

CHAPTER 4
ATOMICITY

In this chapter, we will consider the other natural generalization of atomic factorization, namely Γ-atomic factorization. Recall that, in a cancellative monoid, an *atom* or *irreducible* is a nonunit with no nontrivial factorizations, or, equivalently, a nonunit whose non-unit divisors are all (strongly) associate to it. The equivalence of these statements also holds for a présimplifiable monoid, but in general monoids they are not necessarily equivalent, giving rise to three different kinds of "irreducibles": the *very strong irreducibles*, *strong irreducibles*, and *irreducibles*, in order from strongest to weakest. So in general monoids we would need three separate analogous notions of "Γ-irreducible", and we will simplify matters by restricting our consideration to cancellative monoids. In a cancellative monoid, a Γ-*atom* or Γ-*irreducible* is a nonunit with no nontrivial Γ-factorizations. (Since the monoid is cancellative, this is equivalent to a nonunit whose Γ-divisors are all (strongly) associate to it.) Again returning to past remarks, the thesis [20] carries out the intricate work of generalizing τ-atomic factorization from domains to commutative rings with zero divisors, and we refer the reader there for an idea of how the atomic topics discussed here could be analogously generalized to non-cancellative monoids. The work contained in this thesis for the most part generalizes to the "very strongly irreducible" portion of the most general theory with trivial changes, since the definition we have given for a Γ-atom in a cancellative monoid is the definition of a *very strong* Γ-*atom* in a general monoid.

The first four sections of this chapter will be analogous to those of the previous chapter, focusing on the atomic concepts analogous to those complete ones studied in the corresponding sections. Attention will be given to the relationship between the atomic and complete concepts, demonstrating all of the nontrivial implications between them and showing that no others exist. One can generalize the Γ-atomic factorizations to the Γ_1-Γ_2-*atomic factorizations*, which are Γ_2-factorizations whose factors are Γ_1-atoms, and one can further generalize these to X-Γ-*factorizations*, which are Γ-factorizations whose factors are taken from some set X of distinguished elements. We will see in the first section that X is the set of Γ'-atoms for some factorization system Γ' if and only if X contains all of the irreducible elements of H. It is easy to see that, if every nonzero nonunit has an X-Γ-factorization and X is *associate-preserving* (i.e., if $a \sim a'$, then $a \in X \Leftrightarrow a' \in X$), then these equivalent conditions hold, so there is little lost in restricting ourselves to the study of Γ_1-Γ_2-atomic factorization.

In the fifth section we see all of our work on the atomic and complete concepts come together when we study how refinability (along with some other mild hypotheses) makes the complete and atomic concepts equivalent. We will show how most of the known theorems for the atomic concepts are special cases of the complete ones. At the end we show that many of the Γ-atomic factorization properties can be characterized by chain conditions if Γ has some particularly nice properties.

One thing that we have left out of this chapter and the last is an in-depth study of unique factorization, opting to leave this for the final chapter when it can be studied in conjunction with generalized primes.

4.1 Preliminaries

Let Γ be a factorization system on a cancellative monoid H. A nonunit is Γ-*irreducible* or a Γ-*atom* if it has no nontrivial Γ-factorizations. We denote the set of Γ-atoms by $\operatorname{atom}(\Gamma)$. The $\operatorname{fact}(H)$-atoms are called simply *atoms* or *irreducibles*, and we abbreviate $\operatorname{atom}(\operatorname{fact}(H)) = \operatorname{atom}(H)$. The following proposition makes explicit some fairly obvious facts that we will be using.

Proposition 4.1.1. *Let Γ be a factorization system on a cancellative monoid H.*

(1) *A nonunit is Γ-irreducible if and only if all its Γ-divisors are associate to it.*

(2) *If Γ is unital or associate-preserving, then a nonunit is Γ-irreducible if and only if it is an associate of a Γ-atom.*

(3) *If Γ is combinable, then a nonunit is Γ-irreducible if and only if it has no Γ-factorization of length 2.*

(4) *If $\Gamma \subseteq \Gamma'$, then $\operatorname{atom}(\Gamma') \subseteq \operatorname{atom}(\Gamma)$.*

(5) *If $\{\Gamma_\alpha\}_{\alpha \in J}$ is any family of factorization systems, then $\operatorname{atom}(\bigcup_\alpha \Gamma_\alpha) = \bigcap_\alpha \operatorname{atom}(\Gamma_\alpha)$ and $\operatorname{atom}(\bigcap_\alpha \Gamma_\alpha) \supseteq \bigcup_\alpha \operatorname{atom}(\Gamma_\alpha)$.*

Proof. Parts (1), (3), and (4) are clear, and part (5) is [7, Theorem 4.6(13),(14)]. For part (2), assume Γ is unital (resp., associate-preserving), and let a be any Γ-atom and let $\mu \in H^\times$. If $\mu a = \lambda a_1 \cdots a_n$ is a nontrivial Γ-factorization, then so is $a = (\mu^{-1}\lambda) a_1 \cdots a_n$ (resp., $a = \lambda(\mu^{-1} a_1) a_2 \cdots a_n$), a contradiction. Therefore μa is a Γ-atom. $\square$

In standard factorization, the most important case of writing an element as a product of some sort of distinguished elements is undoubtedly factorization into atoms. Similarly, our main example in this chapter will be Γ-factorization into Γ-atoms. However, we will find that we can accomplish most of our goals in much greater generality by replacing the set of Γ-atoms with an arbitrary distinguished set of nonunits.

We will find it useful to define several properties that such a set of distinguished elements can have. Let H be a monoid and ρ be a relation on $H_0^\#$. We call $X \subseteq H_0^\#$ *ρ-preserving* if $a \rho b \in X \Rightarrow a \in X$. Note that the ρ-preserving property is preserved by arbitrary intersections. A set of nonunits is called *associate-preserving* (resp., *strong associate-preserving, divisive*) if it is $\sim$-preserving (resp., $\approx$-preserving, $|$-preserving).

Let Γ be a factorization system on a cancellative monoid H. By Proposition 4.1.1 part (2), the set $\text{atom}(\Gamma)$ is associate-preserving if Γ is unital or associate-preserving. A natural thing to ask is what sets of nonunits correspond to the Γ-atoms of some factorization system Γ. For this question and many others, the following construction will prove extremely useful. Given $X \subseteq H_0^\#$ and a factorization system Γ, we define a new factorization system $\Gamma(X)$ obtained by removing the nontrivial factorizations of elements of X from Γ.

Theorem 4.1.2. *Let H be a cancellative monoid, $X \subseteq H_0^\#$, and Γ be a factorization system on H. Then $\Gamma(X)$ is the unique largest factorization system contained in Γ with $X \subseteq \text{atom}(\Gamma(X))$. Also, $\text{atom}(\Gamma(X)) \subseteq X \cup \text{atom}(\Gamma)$, so $X = \text{atom}(\Gamma(X))$ if $\text{atom}(\Gamma) \subseteq X$. If Γ is symmetric (resp., combinable, normal, reduced normal), then so is $\Gamma(X)$. If X is associate-preserving, then $\Gamma(X)$ is unital (resp., (weakly)* *associate-preserving) if* Γ *is.*

Proof. The fact that $\Gamma(X)$ is the unique largest factorization system contained in Γ with $X \subseteq \text{atom}(\Gamma(X))$ is clear. If a Γ-reducible element is not in X, then its nontrivial Γ-factorizations are $\Gamma(X)$-factorizations. Hence $\text{atom}(\Gamma(X)) \subseteq X \cup \text{atom}(\Gamma)$. Because no trivial factorizations were removed from Γ in constructing $\Gamma(X)$, the system $\Gamma(X)$ is (reduced) normal if Γ is. The permutations (resp., partitions) of a factorization are still factorizations of the same element, and it follows that $\Gamma(X)$ is symmetric (resp., combinable) if Γ is. If X is associate-preserving, then in constructing $\Gamma(X)$ from Γ we must remove all the factorizations that are equal up to associates with any factorization that we remove, and the last claim follows. $\square$

Let H be a cancellative monoid and $X \subseteq H_0^{\#}$. Theorem 4.1.2 and Proposition 4.1.1 part (4) provide a nice answer to the above question: we have $X = \text{atom}(\Gamma)$ for some factorization system Γ if and only if X contains all of the atoms. More specifically, if Γ' is a factorization system on H, then $X = \text{atom}(\Gamma)$ for some factorization system $\Gamma \subseteq \Gamma'$ if and only if $\text{atom}(\Gamma') \subseteq X$. In this case, the unique largest such Γ is $\Gamma'(X)$.

Let Γ be a factorization system on a cancellative monoid H, and let $X \subseteq H_0^{\#}$. A Γ-factorization into elements of X is called an X-Γ-*factorization*. We say H is X-Γ-*atomic* if every Γ-expressible nonzero nonunit has an X-Γ-factorization. Other plausible definitions are: (1) every nonzero nonunit not in X has an X-Γ-factorization, or (2) every nonzero nonunit has an X-Γ-factorization. All three definitions are equivalent if Γ is reduced normal, but simple examples show that they differ in general.

We go with our choice because it seems to lead to the cleanest factorization theory, as we will begin to appreciate after a few theorems. The definition and most of the general theory of the X-Γ-factorizations could be adapted to non-cancellative monoids with essentially trivial changes (similar comments to adapting the theory of the complete concepts apply), but we will work in a cancellative context since the main example we have in mind is when X is the set of "irreducibles" for some factorization system.

Theorem 4.1.3. *Let H be a cancellative monoid, Γ be a factorization system on H, and $X \subseteq H_0^{\#}$. Assume H is X-Γ-atomic and that X contains all the non-Γ-expressible nonunits. Then* $\operatorname{atom}(\Gamma) \subseteq H^{\times}X$. *So, if X is associate-preserving, then* $\operatorname{atom}(\Gamma) \subseteq X$ *and* $X = \operatorname{atom}(\Gamma(X))$.

Proof. Every Γ-expressible Γ-atom must have a trivial X-Γ-factorization, and thus must be an associate of an element of X. The last statement follows by Theorem 4.1.2. □

Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H. A Γ_1-Γ_2-*atomic factorization* is an $\operatorname{atom}(\Gamma_1)$-$\Gamma_2$-factorization, and we say that H is Γ_1-Γ_2-*atomic* if it is $\operatorname{atom}(\Gamma_1)$-$\Gamma_2$-atomic. Theorem 4.1.3 above shows that there is little lost in restricting ourselves to the study of Γ_1-Γ_2-atomic factorizations instead of the slightly more general X-Γ-factorizations of the previous paragraph, so we will make this simplification from now on.

With these definitions in mind, we are ready to introduce some more advanced concepts. Let Γ_1, Γ_2, and Γ_3 be factorization systems on a cancellative monoid H. We call H:

(1) Γ_1-Γ_2-Γ_3-*atomicable* if every Γ_2-factorization can be sequentially Γ_3-Γ_2-refined into a Γ_1-Γ_2-atomic factorization,

(2) *strongly* Γ_1-Γ_2-Γ_3-*atomicable* if every Γ_2-factorization can be Γ_3-Γ_2-refined into a Γ_1-Γ_2-atomic factorization,

(3) *(strongly)* Γ_1-Γ_2-*atomicable* if it is (strongly) Γ_1-Γ_2-Γ_1-atomicable,

(4) *weakly* Γ_1-Γ_2-*atomicable* if it is (strongly) Γ_1-Γ_2-fact(H)-atomicable,

(5) a Γ_1-Γ_2-*BFM* if it is Γ_1-Γ_2-atomic and each (nonzero) nonunit has a finite upper bound on the lengths of its Γ_1-Γ_2-atomic factorizations,

(6) a Γ_1-Γ_2-*FFM*$_\equiv$ if it is Γ_1-Γ_2-atomic and each (nonzero) nonunit has only finitely many Γ_1-Γ_2-atomic factorizations up to $\equiv$-equivalence,

(7) a Γ_1-Γ_2-*HFM* if it is Γ_1-Γ_2-atomic and any two Γ_1-Γ_2-atomic factorizations of a given (nonzero) nonunit have the same length,

(8) a Γ_1-Γ_2-*UFM*$_\equiv$ if it is Γ_1-Γ_2-atomic and the Γ_1-Γ_2-atomic factorizations of a given (nonzero) nonunit are unique up to $\equiv$-equivalence, and

(9) a Γ_1-Γ_2-*atomicable BFM* (resp., *FFM*$_\equiv$, *HFM*, *UFM*$_\equiv$) if it is Γ_1-Γ_2-atomicable and a Γ_1-Γ_2-BFM (resp., -FFM$_\equiv$, -HFM, -UFM$_\equiv$).

When $\equiv$ is the relation $\sim$, we drop the "$\equiv$" from the above names. If $\Gamma_1 = \Gamma_2 = \Gamma$, then we replace the "$\Gamma_1$-$\Gamma_2$" in the above names with simply "Γ", and if additionally $\Gamma = \text{fact}(H)$, then we drop the "Γ_1-Γ_2" entirely. As will be more formally demonstrated later, the properties of atomicable and atomic are equivalent, so we would never write "atomicable UFM" but simply "UFM", and so on. One potential area of confusion is what "$\emptyset$-Γ" means in the above, as we get radically different meanings depending on whether we interpret the "$\emptyset$" as the empty set of nonunits or as the empty factorization system. Of course, all of the above definitions are utterly trivial with the former interpretation of "$\emptyset$-Γ", so it can always be reasonably assumed that the latter is meant. However, to achieve perfect clarity, we would suggest writing the second interpretation in one of the following equivalent ways: "$H_0^\#$-Γ", "$\psi_\emptyset$-Γ", "$\tau_\emptyset$-Γ", or "$\Gamma_\emptyset$-Γ".

As it turns out, under some mild hypotheses we only need to study Γ_1-Γ_2-atomic factorizations for $\Gamma_1 \subseteq \Gamma_2$ for most purposes. We will take the point of view that the situations in which we cannot reduce our considerations to the case $\Gamma_1 \subseteq \Gamma_2$ are not of sufficient interest to inconvenience ourselves writing slightly stronger versions of theorems to include them. For example, sometimes a theorem may be proved for Γ_1-Γ_2-atomic factorizations where $\Gamma_1 \supseteq \Gamma_2$, but we will just state it for the case $\Gamma_1 = \Gamma_2 = \Gamma$.

To illustrate some of these abstract notions, we revisit some of our motivating factorization system examples.

Example 4.1.4.

(1) We will see later in this chapter that, when the factorization systems in question are refinable, unital, and associate-preserving, then the "complete" and "atomic" concepts coincide. (Or "complete" and "very strongly irreducible"

when the monoid in question is not présimplifiable, but we will not be getting too deeply into this.) For example, this is the case for the fact(H), τ_d, $\tau_{\sqcup}$, $\tau_{[\,]}$, and $\tau_{(2)}$ factorization systems.

(2) A (very strong) fact(H)-atom is simply a (very strongly) irreducible element.

(3) The τ_d-atoms are the *pseudo-irreducibles* defined by [19]. It turns out that pseudo-irreducibles and "very strongly pseudo-irreducibles" are the same thing. (This terminology applies to either elements or ideals of a ring.)

(4) Recall the construction of the $\tau_{\sqcup}$ factorization system: X is a set, $\mathcal{P}(X)$ is a monoid under $\cup$, and $Y \tau_{\sqcup} Z \Leftrightarrow Y \cap \mathbb{Z} = \emptyset$. The (very strong) $\tau_{\sqcup}$-atoms are the singleton subsets of X.

(5) The thesis [16] has shown that the values of $n \geq 0$ for which $\mathbb{Z}$ is $\tau_{(n)}$-atomic are precisely $0 \leq n \leq 6$, $n = 8$, and $n = 10$. (The original claim of the thesis that $\mathbb{Z}$ is $\tau_{(12)}$-atomic is false, since the element $2^4 \cdot 3^3$ has no $\tau_{(12)}$-atomic factorization.) Observe that the same statement with "$\tau_{(n)}$-atomic" replaced with "a $\tau_{(n)}$-FFD" must also be true. However, the domain $\mathbb{Z}$ is only a $\tau_{(n)}$-UFD in the cases $n = 0$ and $n = 1$. (See [6], [14], or [16].)

(6) Let Γ be the factorization system on $\mathbb{Z}$ consisting of factorizations with at most two even factors. The Γ-atoms are simply the atoms, so the Γ-atomic factorizations are (up to associates and order) precisely those factorizations of the forms $p_1 \cdots p_{n+1}$, $2p_1 \cdots p_n$, or $2 \cdot 2p_1 \cdots p_n$, where $n \geq 0$, $k, m \geq 1$, and $p_1, \ldots, p_{n+1}$ are odd primes. So $\mathbb{Z}$ is not Γ-atomic, since any nonzero multiple of 8 has no Γ-atomic factorization.

Example 4.1.5. Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H. In analogy with [21], we can define a Γ_1-Γ_2-*bi-atomic factorization* to be an $(\mathrm{atom}(\Gamma_1)\cap\mathrm{atom}(\Gamma_2))$-$\Gamma_2$-factorization, and H to be Γ_1-Γ_2-*bi-atomic* if it is $(\mathrm{atom}(\Gamma_1)\cap \mathrm{atom}(\Gamma_2))$-$\Gamma_2$-atomic. We can similarly define "bi-atomic" versions of the rest of the concepts we have defined above. Now we consider the case where each Γ_i is unital or associate-preserving (as would be the case in a τ-factorization setup, for instance). Then Proposition 4.1.1 tells us that $\mathrm{atom}(\Gamma_1) \cap \mathrm{atom}(\Gamma_2)$ is associate-preserving, and it follows from Theorem 4.1.3 that if H is Γ_1-Γ_2-bi-atomic, then $\mathrm{atom}(\Gamma_2) = \mathrm{atom}(\Gamma_1) \cap \mathrm{atom}(\Gamma_2)$. Thus we have the following result that unfortunately brings a rather quick resolution to the study of Γ_1-Γ_2-bi-atomic factorization: H is Γ_1-Γ_2-bi-atomic if and only if it is Γ_2-atomic and $\mathrm{atom}(\Gamma_2) \subseteq \mathrm{atom}(\Gamma_1)$, in which case the Γ_2-atomic factorizations and the Γ_1-Γ_2-bi-atomic factorizations coincide.

4.2 Levels of "Atomicity"

As an easy consequence of the definitions along with the observation that a $(\Gamma_1)_r$-Γ_2-atomic factorization (hence a Γ_1-Γ_2-atomic factorization) is necessarily Γ_1-Γ_2-complete, we obtain the implications illustrated in Figure 4.1.

Figure 4.1: Levels of Atomicity and Completeness in Cancellative Monoids

$$\begin{tikzcd}
\text{strongly } \Gamma_1\text{-}\Gamma_2\text{-atomicable} \arrow[r] \arrow[d] & \text{strongly } \Gamma_1\text{-}\Gamma_2\text{-completable} \arrow[d] \\
\Gamma_1\text{-}\Gamma_2\text{-atomicable} \arrow[r] \arrow[d] & \Gamma_1\text{-}\Gamma_2\text{-completable} \arrow[d] \\
\text{weakly } \Gamma_1\text{-}\Gamma_2\text{-atomicable} \arrow[r] \arrow[d] & \text{weakly } \Gamma_1\text{-}\Gamma_2\text{-completable} \arrow[d] \\
\Gamma_1\text{-}\Gamma_2\text{-atomic} \arrow[r] & \Gamma_1\text{-}\Gamma_2\text{-complete}
\end{tikzcd}$$

If $\Gamma_1 = \Gamma_2 = \Gamma$ is reduced divisible, then Figure 4.1 simplifies to Figure 4.2, which was already demonstrated for the special case of τ-factorization in [14].

Figure 4.2: Levels with Γ Reduced Divisible

$$\begin{tikzcd}
\Gamma\text{-atomicable} \arrow[r] \arrow[d] & \Gamma\text{-completable} \arrow[d] \\
\Gamma\text{-atomic} \arrow[r] & \Gamma\text{-complete}
\end{tikzcd}$$

We already know from Theorem 3.2.1 that the completeness half reduces as shown. Proceeding analogously with the atomic concepts yields the following theorem, giving us the reduction in the atomic half.

Theorem 4.2.1. *Let* Γ_1, Γ_2, *and* Γ_3 *be factorization systems on a cancellative monoid*

H be a cancellative monoid, with Γ_2 reduced divisible.

(1) *H is (strongly) Γ_1-Γ_2-Γ_3-atomicable if and only if it is (strongly) Γ_1-Γ_2-($\Gamma_2 \cap \Gamma_3$)-atomicable.*

(2) *If $\Gamma_2 \subseteq \Gamma_3$, then the following properties of H are equivalent:*

   (a) *strongly Γ_1-Γ_2-Γ_3-atomicable,*

   (b) *Γ_1-Γ_2-Γ_3-atomicable, and*

   (c) *weakly Γ_1-Γ_2-atomicable.*

We end this section by giving examples showing that no nontrivial implications can be added to Figures 4.1 or 4.2.

Example 4.2.2. An example of a τ-completable UFD that is not τ-atomic, where τ is an associate-preserving and symmetric relation on the nonzero nonunits. Let τ be the associate-preserving and symmetric relation on $\mathbb{Z}^{\#}$ determined by $2\tau 6$ and $2\tau 3$. Then the only τ-reducible elements of $\mathbb{Z}$ are ± 12 and ± 6, which have unique (up to order and associates) τ-complete factorizations $\pm 12 = \pm 2 \cdot 6$ and $\pm 6 = \pm 2 \cdot 3$. Therefore $\mathbb{Z}$ is a τ-complete UFD. In fact, every nontrivial τ-factorization is τ-complete, so $\mathbb{Z}$ is a τ-completable UFD. However, the nonzero nonunit 12 has no τ-atomic factorization.

The remainder of the necessary examples have in fact already been given, but as examples of other things. We collect a list of them here and leave it up to the reader to reread the old examples and verify that each does indeed serve this new additional purpose.

(1) Example 2.1.5 exhibits a Γ-atomicable domain that is not strongly Γ-completable.

(2) Example 3.2.2 exhibits a τ-UFD that is not τ-completable, with τ an associate-preserving symmetric relation on the nonzero nonunits.

(3) Example 3.2.3 exhibits a weakly Γ-atomicable but not Γ-completable domain, with Γ a symmetric and associate-preserving factorization system.

4.3 Classifying the "Atomic" Monoids

Simply following the definitions leads us directly to the implications depicted in Figure 4.3.

Figure 4.3: Classifying the "Atomic" and "Complete" Cancellative Monoids

$$
\begin{tikzcd}
\Gamma_1\text{-}\Gamma_2\text{-UFM}_{\equiv} \arrow[r] \arrow[d] & \Gamma_1\text{-}\Gamma_2\text{-HFM} \arrow[d] & \\
\Gamma_1\text{-}\Gamma_2\text{-FFM}_{\equiv} \arrow[r] & \Gamma_1\text{-}\Gamma_2\text{-BFM} \arrow[r] & \Gamma_1\text{-}\Gamma_2\text{-atomic} \arrow[dd] \\
\Gamma_1\text{-}\Gamma_2\text{-complete UFM}_{\equiv} \arrow[r] \arrow[d] & \Gamma_1\text{-}\Gamma_2\text{-complete HFM} \arrow[d] & \\
\Gamma_1\text{-}\Gamma_2\text{-complete FFM}_{\equiv} \arrow[r] & \Gamma_1\text{-}\Gamma_2\text{-complete BFM} \arrow[r] & \Gamma_1\text{-}\Gamma_2\text{-complete}
\end{tikzcd}
$$

We now show that no nontrivial implications can be added to Figure 4.3, even in the simplified case of τ-atomic/complete factorization. Most of the work has already been done, either in previous examples in this thesis, or in [3] when it was shown that Figure 3.3 cannot be improved. The remainder is taken care of by the following example.

Example 4.3.1. An example of a τ-UFD that is not a τ-complete BFD, with τ an associate-preserving symmetric relation on the nonzero nonunits. Let R be an integral domain and $D = R[\{X^r, Y^r \mid n \in \mathbb{Q}^+\}]$. Let τ be the associate-preserving and symmetric relation on $D^\#$ determined by $X^r \tau Y^s$, $X^r \tau X^r Y^s$, $Y^s \tau X^r Y^s$, and $X^r Y^s \tau X^r Y^s$ for $r, s \in \mathbb{Q}^+$. The only τ-reducible elements are those of the form $\lambda X^r Y^s$, which have unique (up to order and associates) τ-atomic factorizations $\lambda(X^r)(Y^s)$ and arbitrarily long τ-complete factorizations of the form $\lambda(X^{r/(n+1)})(Y^{s/(n+1)})(X^{r/(n+1)}Y^{s/(n+1)})^n$. Therefore D is a τ-UFD but not a τ-complete BFD.

We now move on to studying the different kinds of "atomicable" monoids. Our first step is to show that the "atomicable" monoids are stronger versions of their "completable" counterparts.

Theorem 4.3.2. *Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H, and let $\equiv$ be an equivalence relation on $H_0^\#$. Then H is a Γ_1-Γ_2-atomicable BFM (resp., FFM$_\equiv$, HFM$_\equiv$, UFM$_\equiv$) if and only if it is Γ_1-Γ_2-atomicable and a Γ_1-Γ_2-complete(able) BFM (resp., FFM$_\equiv$, HFM$_\equiv$, UFM$_\equiv$).*

Proof. Follows immediately after observing that, if H is Γ_1-Γ_2-atomicable, then the Γ_1-Γ_2-atomic and Γ_1-Γ_2-complete factorizations coincide. □

The relationship between the "FFM" and "idf monoid" concepts is interesting. In [3, Theorem 5.1] it is shown that an FFD is equivalent to an atomic idf domain. Later, the thesis [14] noted this is a special case of how the notions of a τ-atomic

τ-idf domain and a τ-FFD coincide for τ divisive. In the refinability section we will come to understand that the following theorem is a generalization of this last fact.

Theorem 4.3.3. *Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H, and let $\equiv$ be an equivalence relation on $H_0^{\#}$. If H is a Γ_1-Γ_2-atomicable FFM$_{\equiv}$, then it is a Γ_1-Γ_2-atomicable Γ_1-Γ_2-idf$_{\equiv}$ monoid. If $\equiv\leq\sim$, then the converse is true.*

Proof. ($\Rightarrow$): Theorems 4.3.2 and 3.3.5. ($\Leftarrow$): Adapt the proof of "(1) $\Rightarrow$ (2)" in Theorem 3.3.3. $\square$

The thesis [21] has done extensive work on τ_1-τ_2-atomic factorizations, and it notes that a τ_1-τ_2-atomic τ_1-τ_2-idf domain is a τ_1-τ_2-FFD. A slight modification of the proof of "(1) $\Rightarrow$ (2)" in Theorem 3.3.3 yields the generalization that a Γ_1-Γ_2-atomic Γ_1-Γ_2-idf$_{\equiv}$ monoid is a Γ_1-Γ_2-FFM$_{\equiv}$. Theorem 4.3.3 shows the partial converse that a weakly Γ_1-Γ_2-atomicable FFM$_{\equiv}$ is a Γ_1-Γ_2-idf monoid. However, the following example spoils our hopes of a full converse.

Example 4.3.4. An example of a τ-UFD that is not a fact(H)-τ-idf domain (hence not a τ-idf domain), with τ an associate-preserving symmetric relation on the nonzero nonunits. We modify Example 4.1(a) in [3]. Let $D = \mathbb{R} + X\mathbb{C}[X]$ and τ be the symmetric and associate-preserving relation on $D^{\#}$ determined by $(r+i)X\tau\frac{X^4}{r+i}$, $\frac{X}{r+i}\tau X^3$, and $X^2\tau X^3$ for $r \in \mathbb{R}$. The only τ-reducible elements are those of the forms $\lambda X^5 = \lambda(X^2)(X^3)$ and $\frac{\lambda}{r+i}X^4 = \lambda(\frac{1}{r+i}X)(X^3)$, which also happen to be their unique τ-atomic factorizations up to associates and order. Therefore D is a τ-UFD. However, it is not a fact(H)-τ-idf domain because the set $\{(r+i)X \mid r \in \mathbb{R}\}$ forms an infinite family of non-associate irreducible τ-divisors of x^5.

In [21], the author stated that the following are equivalent: (1) The domain D is a τ_1-τ_2-FFD, (2) Every element of $D^\#$ has only finitely many τ_2-factorizations into τ_1-atoms, and (3) D is a τ_1-τ_2-atomic τ_1-τ_2-idf domain. We have seen that (3) $\Rightarrow$ (1) $\Rightarrow$ (2) holds, but (1) $\Rightarrow$ (3) is false by the above example. An integral domain that is not a field and has no atoms (see [13] or [5] for examples) vacuously has each element having only finitely many factorizations into atoms but is not atomic, so (2) $\Rightarrow$ (1) is false even for standard factorization. The equivalence (1) $\Leftrightarrow$ (3) is true if one assumes that τ_2 is associate-preserving and refinable, as we will see in the refinability section.

One difference between the Γ-atomicable and -completable concepts comes up in their relationship to the Γ-ACCP property. Example 3.3.10 shows that a τ-completable UFD need not satisfy τ-ACCP. However, we have the following.

Theorem 4.3.5. *Let Γ be a reduced divisible factorization system on a cancellative monoid H. If H is a Γ-atomicable HFM, then it satisfies Γ-ACCP.*

Proof. Assume H is a Γ-atomicable HFM, and let $\{a_n\}_{n=1}^{\infty}$ be any sequence in $H^\#$ with each $a_{n+1} \mid_\Gamma a_n$. For any $n \geq 1$ with a_{n+1} a proper Γ-divisor of a_n, we can Γ-refine a nontrivial Γ-factorization of a_n in which a_{n+1} appears to obtain a Γ-atomic factorization of a_n that is strictly larger than the common length of the Γ-atomic factorizations of a_{n+1}. This observation implies that we can have at most $m - 1$ instances where a_{n+1} properly divides a_n, where m is the common length of the Γ-atomic factorizations of a_1. $\square$

A similar proof to that of the above theorem shows that it is true with "Γ" replaced by "Γ_r" and "ACCP" replaced by "ACC_ρ" for any reflexive relation ρ on H. Also, adjusting the proof *mutatis mutandis* gives the following generalization. Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H. If H is strongly Γ_1-Γ_2-Γ_2-atomicable and a Γ_1-Γ_2-HFM, then it satisfies Γ_2-ACCP.

The following example shows that there is no improvement on the above theorem even for the simplified case of τ-factorization.

Example 4.3.6. An example of a τ-atomicable FFD that does not satisfy τ-ACCP, with τ an associate-preserving symmetric relation on the nonzero nonunits. Let R be an integral domain and $D = R[\{X^r \mid r \in \mathbb{Q}^+\}]$. Let τ be the associate-preserving symmetric relation on $D^\#$ determined by $X^{3/2^{n+2}} \tau X^{1/2^{n+2}}$, $X^{3/2^{n+2}} \tau X^{5/2^{n+6}}$, $X^{3/2^{n+2}} \tau X^{11/2^{n+6}}$, and $X^{5/2^{n+6}} \tau X^{11/2^{n+6}}$ for $n \geq 0$. It follows from Theorem 3.3.11 and Example 3.3.10 that D does not satisfy τ-ACCP. The only nontrivial τ-factorizations (up to associates and order) are those of the forms $\lambda X^{1/2^n} = \lambda(X^{3/2^{n+2}})(X^{1/2^{n+2}})$, $\lambda X^{1/2^n} = \lambda(X^{3/2^{n+2}})(X^{5/2^{n+6}})(X^{11/2^{n+6}})$, $\lambda X^{53/2^{n+6}} = \lambda(X^{3/2^{n+2}})(X^{5/2^{n+6}})$, $\lambda X^{59/2^{n+4}} = \lambda(X^{3/2^{n+2}})(X^{11/2^{n+6}})$, and $\lambda X^{1/2^{n+2}} = (X^{5/2^{n+6}})(X^{11/2^{n+6}})$ for $\lambda \in H^\times$ and $n \geq 0$. By inspection, we can see that every nonzero nonunit has a τ-atomic factorization. The only nontrivial non-τ-atomic factorizations are those of the form $\lambda(X^{3/2^{n+2}})(X^{1/2^{n+2}})$ for $\lambda \in D^\times$ and $n \geq 0$, which can be τ-refined into the τ-atomic factorizations $\lambda(X^{3/2^{n+2}})(X^{5/2^{n+6}})(X^{11/2^{n+6}})$. Also, each nonzero nonunit has a unique τ-atomic factorization (up to associates and order), except for those of the form $\lambda X^{1/2^n}$ for $n \geq 2$, which have exactly two τ-atomic factorizations (up to associates and order), namely $\lambda X^{1/2^n} = (X^{5/2^{n+4}})(X^{11/2^{n+4}})$ and $\lambda X^{1/2^n} = \lambda(X^{3/2^{n+2}})(X^{5/2^{n+6}})(X^{11/2^{n+6}})$. From the above observations it follows that D is a τ-atomicable FFD.

Some of the work in length functions for standard factorization translates over to abstract factorization. Let Γ be a factorization system on a cancellative monoid H. We define a length function $L_\Gamma : H \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$ as follows. For $a \in H^\times$, we set $L_\Gamma(a) = 0$. For a non-Γ-expressible nonunit a, we set $L_\Gamma(a) = 1$. Otherwise $L_\Gamma(a)$ is the supremum of the lengths of the Γ-factorizations of a. We abbreviate $L = L_{\text{fact}(H)}$. Note that $L_\Gamma(a) = 1$ if and only if a is a Γ-atom. Talking about length functions for non-cancellative monoids is a more complicated matter, since in a non-présimplifiable monoid one is confronted with the question of what to do with the "redundant" factorizations of the form $\lambda a_1 \cdots a_n b_1 \cdots b_m = \lambda a_1 \cdots a_n$, and it would be a good idea to define an alternate length function that only considers "irredundant" factorizations. We refer the interested reader to [1] for more information.

The following is an abstract factorization version of a fact noted in [3].

Theorem 4.3.7. *Let Γ be a factorization system on a cancellative monoid H. Assume H is Γ-atomicable. The following are equivalent.*

(1) *The monoid H is a Γ-HFM.*

(2) *There is a function $l : H_0^\# \to \mathbb{Z}^+$ such that $l(a) = 1$ if (and only if) a is a Γ-atom, and $l(\lambda a_1 \cdots a_n) = l(a_1) + \cdots + l(a_n)$ whenever $\lambda a_1 \cdots a_n$ is a Γ-factorization.*

(3) *For each Γ-factorization $\lambda a_1 \cdots a_n$, we have $L_\Gamma(\lambda a_1 \cdots a_n) = L_\Gamma(a_1) + \cdots +$*

$L_\Gamma(a_n)$.

If atom(Γ) *is associate-preserving and* Γ *is combinable and reduced divisible, then we can add:*

(4) There is a function $l : H_0^\# \to \mathbb{Z}^+$ *such that* $l(a) = 1$ *if (and only if)* a *is a* Γ*-atom, and* $l(\lambda ab) = l(a) + l(b)$ *whenever* λab *is a* Γ*-factorization.*

(5) We have $L_\Gamma(\lambda ab) = L_\Gamma(a) + L_\Gamma(b)$ *whenever* λab *is a* Γ*-factorization.*

When the equivalent conditions hold, we may take $l = L_\Gamma$.

Proof. (3) ⇒ (2): Use $l = L_\Gamma$. The inclusion $L_\Gamma(H_0^\#) \subseteq \mathbb{Z}^+$ follows from (3) and the fact that H is Γ-atomic. (2) ⇒ (1): If the formally weaker version of (2) holds and $\lambda a_1 \cdots a_m = \mu b_1 \cdots b_n$ are two Γ-atomic factorizations of the same nonzero nonunit, then $m = l(a_1) + \cdots + l(a_m) = l(\lambda a_1 \cdots a_m) = l(\mu b_1 \cdots b_n) = l(b_1) + \cdots + l(b_n) = n$, showing H to be a Γ-HFM. (1) ⇒ (3): Assume H is a Γ-HFM. Take any Γ-factorization $a = \lambda a_1 \cdots a_n$. By the fact that H is Γ-atomicable, we may Γ-refine this to a Γ-atomic factorization of the form $\lambda b_{1,1} \cdots b_{n,n_m}$, where each $a_i = b_{i,1} \cdots b_{i,n_i}$ is a $(\Gamma \cup \mathrm{trfact}(H))$-atomic factorization. Thus $L_\Gamma(a) = n_1 + \cdots + n_m = L_\Gamma(a_1) + \cdots + L_\Gamma(a_n)$. (2) ⇒ (4) and (3) ⇒ (5): Clear.

Now assume that atom(Γ) is associate-preserving and Γ is combinable and reduced divisible. (4) ⇒ (2): Assume (4) and let $\lambda a_1 \cdots a_n$ be any Γ-atomic factorization. We have $l(\lambda a_1) = 1$ by the fact that atom(Γ) is associate-preserving, so let us assume $n \geq 2$. Then $\lambda a_1(a_2 \cdots a_n)$ and $a_2 \cdots a_n$ are Γ-factorizations by the combinable and reduced divisible properties, so $l(\lambda a_1 \cdots a_n) = l(\lambda a_1) + l(a_2 \cdots a_n) = 1 + (n-1) = n$ by induction. (5) ⇒ (3): Similar to (4) ⇒ (2). □

The following examples show that none of the three extra hypotheses necessary for the addition of (4) and (5) can be dropped.

Example 4.3.8. An example of a ψ-atomicable FFD that satisfies (5) above but is not a ψ-HFD, with ψ a combinable and refinable symmetric relation on the domain. Let ψ be the symmetric relation on $\mathbb{Z}$ determined by $(-1)\psi(\pm 2)$, $(-1)\psi(-4)$, and $2\psi(-2)$. Then $\Gamma_\psi = \{(-1)(\pm 2), (-1)(-4), (-1)(\pm 2)(\mp 2)\}$, and it is easily verified that ψ is combinable and refinable, and that every ψ-factorization is ψ-atomic, so $\mathbb{Z}$ is trivially ψ-atomicable. Also, the elements ± 2 have unique ψ-atomic factorizations $\pm 2 = (-1)(\mp 2)$, while 4, the only other ψ-expressible element, has exactly two ψ-atomic factorizations up to order and associates, namely $4 = (-1)(-4)$ and $4 = (-1)(2)(-2)$. Thus $\mathbb{Z}$ is a ψ-atomicable FFD but not a ψ-HFD. Checking that $L_\psi((-1)(\pm 2)(\mp 2)) = 2 = 1+1 = L_\psi(\pm 2)+L_\psi(\mp 2)$ shows that (5) above is satisfied. Note that this example was made possible by atom(ψ) failing to be associate-preserving, with -4 being a ψ-atom but its associate 4 being ψ-reducible.

Example 4.3.9. An example of a τ-FFD that satisfies (5) above but is not a τ-HFD, with τ a divisive symmetric relation on the nonzero nonunits. (We will later see that the corresponding atomic, atomicable, complete, and completable concepts are equivalent for refinable and associate-preserving factorization systems, so for τ divisive a τ-FFD is the same thing as a τ-atomicable FFD.) Let R be an FFD, $D = R[X^3, X^4]$, and τ be the symmetric relation on $\mathbb{Z}^\#$ given by $\lambda X^3 \tau \mu X^3$ and $\lambda X^4 \tau \mu X^4$ for $\lambda, \mu \in R^\times$. Note that τ is divisive, and that (5) above is satisfied by L_τ. Now, the domain $R[X]$ is an FFD by [3, Proposition 5.3]. Using the equivalence in the parenthetical remark at the start of the example, one can easily use the work in the previous chapter to show that D is a τ-FFD. It is also routine to verify that (5) holds. However, the τ-atomic factorizations $(X^3)^4 = (X^4)^3$ show that D is not a τ-HFD. Note that this example was made possible by τ not being combinable.

Example 4.3.10. An example of a Γ-FFD that satisfies (5) above but is not a Γ-HFD, with Γ a refinable, unital, associate-preserving, combinable, normal, and symmetric factorization system. Let $\Gamma = \text{fact}(\mathbb{Z}) \setminus \{\lambda ab \mid a, b \in \text{atom}(\mathbb{Z})\}$, and note that Γ has the stated properties. By the equivalence mentioned at the beginning of the previous example, we immediately see that $\mathbb{Z}$ is a Γ-FFD. However, the Γ-atomic factorizations $(2^2) \cdot 2 = 2^3$ show that $\mathbb{Z}$ is not a Γ-HFD. Note that this example was made possible by Γ failing to be reduced divisible.

We could similarly prove an analogous version of Theorem 4.3.7 for reduced Γ-factorization, and in that theorem one could drop the assumption that $\text{atom}(\Gamma_r)$ is associate-preserving. However, Example 4.3.8 shows that the corresponding assumption cannot be dropped for (-1)-reduced Γ-factorization.

It is useful to review some special cases of the hypothesis that H is Γ-atomicable in Theorem 4.3.7. If Γ is reduced divisible, then a weakly Γ-atomicable monoid is Γ-atomicable by Theorem 3.2.1. We will later see in Theorem 4.5.5 that if Γ is refinable and associate-preserving, then a Γ-complete monoid is Γ-atomicable. It is not hard to see that (2) above implies that the Γ-factorizations of any given nonunit have an upper bound on their lengths, and hence that H is a weakly Γ-completable BFM by Theorem 3.3.1. With these observations, we can get another version of the theorem by replacing the assumption that H is Γ-atomicable with the assumption that Γ is refinable and associate-preserving.

Let Γ be a factorization system on a cancellative monoid H. A question one might ask is what sort of subset of $\mathbb{Z}^+ \cup \{\infty\}$ the set $L_\Gamma(H_0^\#)$ is. For standard factorization, we have $L(H_0^\#) = \mathbb{Z}^+$ if and only if H is a BFM. This result obviously does not carry over to general factorization systems (for instance, any quasi-local domain is a UCFD with $L_{\tau_d}(H_0^\#) = \{1\}$), but it is interesting to see how close we can get with certain hypotheses on Γ and H. Here are a few simple observations.

Theorem 4.3.11. *Let Γ be a factorization system on a cancellative monoid H.*

(1) *If Γ is unital or associate-preserving, then $L_\Gamma = L_{\Gamma_{u,ap}}$.*

(2) *The monoid H is a Γ-completable BFM if and only if $L_\Gamma(H_0^\#) \subseteq \mathbb{Z}^+$.*

(3) *If Γ is (reduced) divisible and H is a Γ-atomicable HFM, then $L_\Gamma(H^\#)$ is either $\mathbb{Z}^+$ or an interval $\{1, \ldots, N\}$.*

(4) *If Γ_r is divisible, refinable, and combinable, then $L_{\Gamma_r}(H^\#)$ is either $\{\infty\}$, $\mathbb{Z}^+$, $\mathbb{Z}^+ \cup \{\infty\}$, an interval $\{1, \ldots, N\}$, or the union of such an interval with $\{\infty\}$.*

Proof.

(1) Theorem 3.4.2 part (1).

(2) Theorem 3.3.1.

(3) Assume Γ is (reduced) divisible and H is a Γ-atomicable HFM. Then H is a Γ-completable BFM, so by part (1) we have $L_\Gamma(H^\#) \subseteq \mathbb{Z}^+$ and it will suffice to show that for every $n \in L_\Gamma(H^\#) \setminus \{1\}$ we have $n - 1 \in L_\Gamma(H^\#)$. Recall from

the proof of the previous theorem that for a Γ-expressible $a \in H^{\#}$, the value of $L_\Gamma(a)$ is the common length of a's Γ-atomic factorizations. Therefore for any $n \in L_\Gamma(H^{\#})\setminus\{1\}$ there is a Γ-atomic factorization $\lambda a_1 \cdots a_n$ of length n, and by (reduced) divisibility $\lambda a_1 \cdots a_{n-1}$ (resp., $a_1 \cdots a_{n-1}$) is a Γ-atomic factorization, so $n-1 \in L_\Gamma(H^{\#})$.

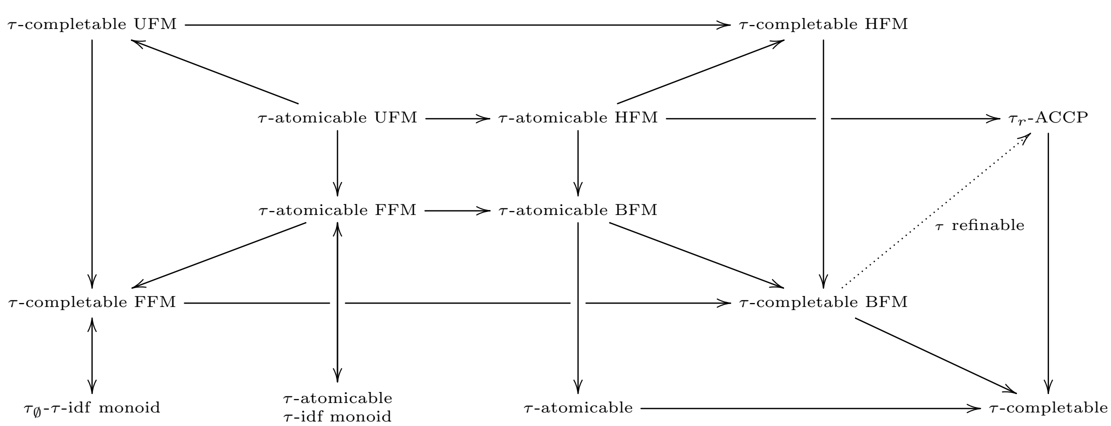
(4) Assume Γ_r is divisible, refinable, and combinable. It will suffice to show that for every $n \in L_{\Gamma_r}(H^{\#}) \setminus \{1,\infty\}$ we have $n-1 \in L_{\Gamma_r}(H^{\#})$. Pick such an n and let $a \in H^{\#}$ be an element with $L_{\Gamma_r}(a) = n$, say $a = a_1 \cdots a_n$ is a reduced Γ-factorization of a of maximum length. By divisibility, the factorization $a_1 \cdots a_{n-1}$ is a reduced Γ-factorization, so $L_{\Gamma_r}(a_1 \cdots a_{n-1}) \geq n-1$. If $b_1 \cdots b_m$ is a reduced Γ-factorization of $a_1 \cdots a_{n-1}$, then by combinability and refinability $(a_1 \cdots a_{n-1})a_n = (b_1 \cdots b_m)a_n = b_1 \cdots b_m a_n$ are reduced Γ-factorizations of a, so $m \leq n-1$. Hence $L_{\Gamma_r}(a_1 \cdots a_{n-1}) = n-1$.

□

The following is one of the more important special cases of (1) above. We have $\Gamma_{u,ap} = (\Gamma_r)_{u,ap}$. So if Γ_r is associate-preserving, then $L_{\Gamma_{u,ap}} = L_{(\Gamma_r)_{u,ap}} = L_{\Gamma_r} \leq L_\Gamma \leq L_{\Gamma_{u,ap}}$ and hence $L_{\Gamma_r} = L_\Gamma = L_{\Gamma_{u,ap}}$. By a similar sort of argument, we arrive at the following generalization. If for $i = 1, 2$ we have Γ_i unital or associate-preserving, $\Gamma_i \subseteq \Gamma'_i \subseteq (\Gamma_i)_{u,ap}$, and $(\Gamma_1)_{u,ap} = (\Gamma_2)_{u,ap}$, then $L_{\Gamma'_1} = L_{\Gamma'_2}$. As a consequence of all this, if we additionally assume that Γ_r is associate-preserving in (4), then we can replace "$L_{\Gamma_r}(H^{\#})$" in the conclusion with "$L_{\Gamma'}(H^{\#})$", where Γ' is any factorization system with $\Gamma_r \subseteq \Gamma' \subseteq \Gamma_{u,ap}$.

We have the following diagram of implications summing up our work on classifying the Γ_1-Γ_2-atomicable and -completable monoids in the special case of τ-factorization. Replacing "τ" with "Γ_1-Γ_2" as appropriate generalizes the diagram to the case where Γ_2 is reduced divisible. We invite the reader to use the results of this section to construct a diagram for the most general case with no assumptions on Γ_2. It is not a hard task, but the diagram itself is slightly too large to display here.

Figure 4.4: Classifying τ-atomicable and τ-completable Monoids



Examples have been given to show that we did not omit any nontrivial implications.

The following theorems list some simple results about inheritance of the "atomic(able)" property. They follow straight from the definitions, and we have in fact already implicitly used them a couple times.

Theorem 4.3.12. *Let $\Gamma_1 \subseteq \Gamma_1'$ and $\Gamma_2 \supseteq \Gamma_2'$ be factorization systems on a cancellative monoid H. Assume that every Γ_2-expressible nonzero nonunit is Γ_2'-expressible. If H is Γ_1'-Γ_2'-atomic, then it is Γ_1-Γ_2-atomic.*

Theorem 4.3.13. *Let $\Gamma_1 \subseteq \Gamma_1'$, Γ_2, and $\Gamma_3 \supseteq \Gamma_3'$ be factorization systems on a cancellative monoid H. If H is (strongly) Γ_1'-Γ_2-Γ_3'-atomicable, then it is (strongly) Γ_1-Γ_2-Γ_3-atomicable.*

4.4 Factorization and Closures, III

In this section we will give some conditions under which the Γ_1-Γ_2-atomic factorization concepts will coincide with the ones where we replace Γ_1 or Γ_2 or both with various closures. This will be highly convenient when we study the effects of refinability in the next section, because the easiest path is to prove the results for reduced factorization systems and then apply the results of this section to get the Γ-factorization versions of the theorems after adding suitable hypotheses like unital and so on.

The following simple but fundamental result follows immediately from Theorem 3.4.2 part (1).

Theorem 4.4.1. *Let Γ be a unital or associate-preserving factorization system on a cancellative monoid H. Then* $\operatorname{atom}(\Gamma) = \operatorname{atom}(\Gamma_{u,ap})$.

Let Γ_1, Γ_2, Γ_1', and Γ_2' be factorization systems on a cancellative monoid H. If every Γ_2'-expressible element is Γ_2-expressible, then Γ_1-Γ_2-atomic implies Γ_1'-Γ_2'-atomic for any $\Gamma_1' \subseteq \Gamma_1$ and $\Gamma_2' \supseteq \Gamma_2$. If not every Γ_2'-expressible element is Γ_2-expressible, the above conclusion does not hold in general. However, the desired result does work for unital and associate-preserving closures, as we will see in the next theorem.

Theorem 4.4.2. *Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H. In the following statements, we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3).*

(1) *H is Γ_1-Γ_2-atomic.*

(2) *H is Γ_1-$(\Gamma_2)_u$-atomic.*

(3) *H is Γ_1-$(\Gamma_2)_{u,ap}$-atomic.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2): Assume H is Γ_1-Γ_2-atomic and let $a \in H^{\#}$ be any $(\Gamma_2)_u$-expressible element. There is a $\lambda \in H^{\times}$ with λa Γ_2-expressible, so there is a Γ_1-Γ_2-atomic factorization $\lambda a = \mu a_1 \cdots a_n$, and $a = (\lambda^{-1}\mu)a_1 \cdots a_n$ is a Γ_1-$(\Gamma_2)_u$-atomic factorization.

(2) $\Rightarrow$ (3): Theorems 4.3.12 and 3.4.2. $\square$

Theorem 4.4.3. *Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H, each of which is unital or associate-preserving. Let $\equiv$ be an equivalence relation on $H_0^{\#}$.*

(1) *For every Γ_1-$(\Gamma_2)_{u,ap}$-atomic factorization, there is a Γ_1-Γ_2-atomic factorization of the same element of the same length whose factors are associates of the corresponding factors in the original factorization.*

(2) *The monoid H is Γ_1-Γ_2-atomic if and only if it is Γ_1-$(\Gamma_2)_{u,ap}$-atomic.*

(3) *The monoid H is a Γ_1-Γ_2-BFM (resp., -HFM) if and only if it is a Γ_1-$(\Gamma_2)_{u,ap}$-BFM (resp., -HFM).*

(4) *If H is a Γ_1-$(\Gamma_2)_{u,ap}$-FFM$_{\equiv}$ (resp., -UFM$_{\equiv}$), then it is a Γ_1-Γ_2-FFM$_{\equiv}$ (resp., -UFM$_{\equiv}$). If $\sim\leq\equiv$, then the converse is true.*

Proof.

(1) Follows from the proof of Lemma 3.4.2 part (1) and the fact that an associate of a Γ_1-atom is Γ_1-irreducible.

(2) ($\Rightarrow$): Lemma 4.4.2. ($\Leftarrow$): Follows from part (1).

(3) Parts (1) and (2).

(4) By part (2), we only need to consider the uniqueness aspects. ($\Rightarrow$): Clear. ($\Leftarrow$): If $\sim\leq\equiv$, then by part (1) the existence of a set of m non-$\equiv$-equivalent Γ_1-$(\Gamma_2)_{u,ap}$-atomic factorizations of some $a \in H^{\#}$ implies the existence of a set of m non-$\equiv$-equivalent Γ_1-Γ_2-atomic factorizations of a.

$\square$

Lemma 4.4.4. *Let Γ_1, Γ_2, and Γ_3 be factorization systems on a cancellative monoid H, with Γ_2 unital. For $\nu \in H^{\times}$, the monoid H is (strongly) Γ_1-$(\Gamma_2)_{r,\nu}$-Γ_3-atomicable if and only if it is (strongly) Γ_1-Γ_2-Γ_3-atomicable.*

Proof. Similar to the proof of Lemma 3.4.11 part (2). $\square$

Lemma 4.4.5. *Let Γ_1, Γ_2, and Γ_3 be factorization systems on a cancellative monoid H, with $(\Gamma_3)_r$ associate-preserving and Γ_1 unital or associate-preserving. If H is (strongly) Γ_1-Γ_2-Γ_3-atomicable, then it is (strongly) Γ_1-$(\Gamma_2)_{u,ap}$-Γ_3-atomicable.*

Proof. Assume H is Γ_1-Γ_2-Γ_3-atomicable, and let $\lambda a_1 \cdots a_n$ be any $(\Gamma_2)_{u,ap}$-factorization. Then there are $\mu_1, \ldots, \mu_n \in H^{\times}$ with $\nu(\mu_1 a_1) \cdots (\mu_n a_n)$ a Γ_2-factorization, so it may be sequentially Γ_3-Γ_2-refined to a Γ_1-Γ_2-atomic factorization. If none of these refinements are proper, then $\nu(\mu_1 a_1)\cdots(\mu_n a_n)$ is a Γ_1-Γ_2-atomic factorization, and hence $\lambda a_1 \cdots a_n$ is a Γ_1-$(\Gamma_2)_{u,ap}$-atomic factorization by the fact that atom(Γ_1) is associate-preserving. So we may assume that the sequence of Γ_3-Γ_2-refinements consists of some positive number N of proper Γ_3-Γ_2-refinements. Let us say the first of these is $\nu c_{1,1} \cdots c_{n,m_n}$, where each $\mu_i a_i = c_{i,1} \cdots c_{i,m_i}$ is a reduced Γ_3-factorization. By the fact that $(\Gamma_3)_r$ is associate-preserving, for each i we have $a_i = (\mu_i^{-1} c_{i,1}) c_{i,2} \cdots c_{i,m_i}$ a reduced Γ_3-factorization. We can use these to get a Γ_3-$(\Gamma_2)_{u,ap}$-refinement of $\lambda a_1 \cdots a_n$ whose factors are each associates of the corresponding ones in the Γ_3-Γ_2-refinement $\nu c_{1,1} \cdots c_{n,m_n}$ of $\nu(\mu_1 a_1)\cdots(\mu_n a_n)$. If $N = 1$, then we have Γ_3-$(\Gamma_2)_{u,ap}$-refined $\lambda a_1 \cdots a_n$ into a Γ_1-$(\Gamma_2)_{u,ap}$-atomic factorization by the fact that atom(Γ_1) is associate-preserving. If $N > 1$, then by induction we can sequentially Γ_3-$(\Gamma_2)_{u,ap}$-refine this Γ_3-$(\Gamma_2)_{u,ap}$-refinement into a Γ_1-$(\Gamma_2)_{u,ap}$-atomic factorization. It follows that H is Γ_1-$(\Gamma_2)_{u,ap}$-Γ_3-atomicable. If H is strongly Γ_1-Γ_2-Γ_3-atomicable, then the above argument shows that it is strongly Γ_1-$(\Gamma_2)_{u,ap}$-Γ_3-atomicable, since we may choose our sequence so that $N = 1$ in the above. □

We finish this section with some results about the combinable and normal closures.

Theorem 4.4.6. *Let* Γ, Γ_1, *and* Γ_2 *be factorization systems on a cancellative monoid* H.

(1) atom(Γ) = atom(Γ_c).

(2) *If* H *is* Γ_1-Γ_2-*atomic, then it is* Γ_1-$(\Gamma_2)_c$-*atomic.*

(3) *If* Γ *is reduced divisible, then the* Γ*-atomic factorizations coincide with the* Γ_c*-atomic factorizations.*

Proof. Parts (1) and (2) are clear, and hence every Γ-atomic factorization is a Γ_c-atomic factorization. Now assume Γ is reduced divisible. Then every Γ_c-factorization has a $\mathrm{fact}(H)$-Γ-refinement, and hence a Γ-refinement by reduced divisibility. Since a Γ_c-atomic factorization has no nontrivial Γ-refinements by part (1), every Γ_c-atomic factorization must be Γ-atomic. $\square$

Theorem 4.4.7. *Let* Γ *and* Γ' *be factorization systems on a cancellative monoid* H*, with* $\Gamma \subseteq \Gamma' \subseteq \Gamma \cup \mathrm{tfact}(H)$*. Then* $\mathrm{atom}(\Gamma) = \mathrm{atom}(\Gamma')$.

Proof. Follows after observing that Γ and Γ' have the same nontrivial factorizations. $\square$

Theorem 4.4.8. *Let* Γ *and* Γ' *be factorization systems on a cancellative monoid* H*, with* Γ *unital and associate-preserving and* $\Gamma \subseteq \Gamma' \subseteq \Gamma \cup \mathrm{tfact}(H)$*.*

(1) *The* Γ*-atomic and* Γ'*-atomic factorizations of* Γ*-expressible nonzero nonunits coincide. Every* Γ'*-factorization of a non-*Γ*-expressible nonzero nonunit is a trivial* Γ'*-atomic factorization.*

(2) *The monoid* H *is* Γ*-atomic if and only if it is* Γ'*-atomic.*

(3) *If* $\Gamma \subseteq \Gamma_3 \subseteq \mathrm{fact}(H)$*, then* H *is* Γ*-*Γ*-*Γ_3*-atomicable if and only if it is* Γ'*-*Γ'*-*Γ_3'*-atomicable.*

(4) *The monoid* H *is a* Γ*-(atomicable)BFM (resp., -(atomicable) HFM) if and only if it is a* Γ'*-(atomicable)BFM (resp., -(atomicable) HFM).*

(5) *If H is a Γ'-(atomicable)FFM$_\equiv$ (resp., -(atomicable)UFM$_\equiv$), then it is a Γ-(atomicable)FFM$_\equiv$ (resp., -(atomicable)UFM$_\equiv$). If $\sim \leq \equiv$, then the converse is true.*

Proof. Adjust the proof of Theorem 3.4.19. □

4.5 The Effect of Refinability

In [14] it was shown that the properties of τ-atomic, τ-complete, τ-completable, and τ-atomicable are equivalent for τ divisive. This section will be devoted to developing a generalization involving the Γ_1-Γ_2-complete and Γ_1-Γ_2-atomic factorizations. We will then derive generalized versions of all the main theorems about the τ-atomic monoids as corollaries of this and previous results.

The following two theorems should be compared with Theorem 3.2.1, which also gives conditions where we can lump different levels of completeness together.

Theorem 4.5.1. *Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H, with Γ_2 refinable. Assume further that every non-$(\Gamma_2)_r$-expressible nonzero nonunit is Γ_1-irreducible. The following properties of H are equivalent:*

(1) *strongly Γ_1-$(\Gamma_2)_r$-Γ_2-atomicable,*

(2) *Γ_1-$(\Gamma_2)_r$-Γ_2-atomicable,*

(3) *weakly Γ_1-$(\Gamma_2)_r$-atomicable, and*

(4) *Γ_1-$(\Gamma_2)_r$-atomic.*

Also, the monoid H is a Γ_1-$(\Gamma_2)_r$-BFM (resp., -FFM$_\equiv$, -HFM, -UFM$_\equiv$) if and only if it is a Γ_1-$(\Gamma_2)_r$-atomicable BFM (resp., -FFM$_\equiv$, -HFM, -UFM$_\equiv$). (Here $\equiv$ is an *equivalence relation on $H_0^\#$.)*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4): Clear. (4) $\Rightarrow$ (1): Assume H is Γ_1-$(\Gamma_2)_r$-atomic and let $a_1 \cdots a_n$ be any reduced Γ_2-factorization. If some $a_i = 0$, then $n = 1$ and the factorization is already Γ_1-$(\Gamma_2)_r$-atomic, so let us assume that each $a_i \in H^\#$, and hence either Γ_1-irreducible or $(\Gamma_2)_r$-expressible. Either way, each a_i has a Γ_1-$((\Gamma_2)_r \cup \operatorname{trfact}(H))$-atomic factorization, and we can use these along with the refinability of Γ_2 to Γ_2-refine $a_1 \cdots a_n$ into a Γ_1-$(\Gamma_2)_r$-atomic factorization. $\square$

In the special case $\Gamma_1 = \Gamma_2 = \Gamma$ of the above theorem, we need only assume that Γ is refinable, since every non-Γ_r-expressible nonzero nonunit is automatically a Γ_r-atom.

Theorem 4.5.2. *Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H, with Γ_2 refinable and reduced divisible. Let $\equiv$ be an equivalence relation on $H_0^\#$. The following properties of H are equivalent:*

1. *strongly Γ_1-$(\Gamma_2)_r$-Γ_2-completable,*
2. *Γ_1-$(\Gamma_2)_r$-Γ_2-completable,*
3. *weakly Γ_1-$(\Gamma_2)_r$-completable, and*
4. *Γ_1-$(\Gamma_2)_r$-complete.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4): Clear. (4) $\Rightarrow$ (1): Assume H is Γ_1-$(\Gamma_2)_r$-complete and let $a_1 \cdots a_n$ be any reduced Γ_2-factorization. By the fact that Γ_2 is reduced divisible, each a_i is Γ_r-expressible and thus has a Γ_1-$(\Gamma_2)_r$-complete factorization. We can use these and the refinability of Γ_2 to Γ_2-refine $a_1 \cdots a_n$. If this Γ_2-refinement has a proper

Γ_1-Γ_2-refinement, then by divisibility some reduced Γ_1-Γ_2-complete factorization of some a_i has a proper Γ_1-Γ_2-refinement, a contradiction. Therefore this Γ_2-refinement is Γ_1-Γ_2-complete. $\square$

Corollary 4.5.3. *Let Γ be a refinable and reduced divisible factorization system on a cancellative monoid H. Let $\equiv$ be an equivalence relation on $H_0^{\#}$. The monoid H is Γ_r-complete $\Leftrightarrow$ it is (strongly) Γ_r-completable $\Leftrightarrow$ it is weakly Γ_r-completable. Furthermore, it is a a Γ_r-complete BFM (resp., FFM$_\equiv$, HFM, UFM$_\equiv$) if and only if it is a Γ_r-completable BFM (resp., FFM$_\equiv$, HFM, UFM$_\equiv$).*

If we add the hypothesis that Γ_1 is associate-preserving, we may replace each "$(\Gamma_1)_r$" with "Γ_1" in the following theorem.

Theorem 4.5.4. *Let Γ_1 and Γ_2 be factorization systems on a cancellative monoid H, with Γ_2 Γ_1-refinable. Let $\equiv$ be an equivalence relation on $H_0^{\#}$.*

(1) *The Γ_1-Γ_2-complete factorizations and the $(\Gamma_1)_r$-Γ_2-atomic factorizations coincide.*

(2) *The monoid H is a Γ_1-Γ_2-complete(able) BFM (resp., FFM$_\equiv$, HFM, UFM$_\equiv$) if and only if it is a $(\Gamma_1)_r$-Γ_2-(atomicable)BFM (resp., -(atomicable)FFM$_\equiv$, -(atomicable)HFM, -(atomicable)UFM$_\equiv$).*

Proof.

(1) Let $\lambda a_1 \cdots a_n$ be any Γ_1-Γ_2-complete factorization. If some a_i has a nontrivial reduced Γ_1-factorization $a_i = b_1 \cdots b_m$, then $\lambda a_1 \cdots a_{i-1} b_1 \cdots b_m a_{i+1} \cdots a_n$ is a proper Γ_1-Γ_2-refinement of $\lambda a_1 \cdots a_n$ by the fact that Γ_2 is Γ_1-refinable, a

contradiction. Therefore $\lambda a_1 \cdots a_n$ is $(\Gamma_1)_r$-Γ_2-atomic.

(2) Obvious by part (1).

$\square$

Theorem 4.5.5. *Let Γ be a refinable factorization system on a cancellative monoid H.*

(1) *The following properties of H are equivalent:*

- (a) *strongly Γ_r-atomicable,*
- (b) *Γ_r-atomicable,*
- (c) *weakly Γ_r-atomicable,*
- (d) *Γ_r-atomic,*
- (e) *strongly Γ_r-completable,*
- (f) *Γ_r-completable,*
- (g) *weakly Γ_r-completable, and*
- (h) *Γ_r-complete.*

(2) *The following properties of H are equivalent:*

- (a) *Γ_r-complete BFM (resp., FFM$_\equiv$, HFM, UFM$_\equiv$),*
- (b) *Γ_r-completable BFM (resp., FFM$_\equiv$, HFM, UFM$_\equiv$),*
- (c) *Γ_r-BFM (resp., FFM$_\equiv$, HFM, UFM$_\equiv$), and*
- (d) *Γ_r-atomicable BFM (resp., FFM$_\equiv$, HFM, UFM$_\equiv$).*

Proof.

(1) We have (a) $\Leftrightarrow$ (e) and (d) $\Leftrightarrow$ (h) by Theorem 4.5.4. The implications (a) $\Rightarrow$

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) and (e) $\Rightarrow$ (f) $\Rightarrow$ (g) $\Rightarrow$ (h) are clear. Finally, we have (d) $\Rightarrow$ (a) by Theorem 4.5.1.

(2) We have (a) $\Leftrightarrow$ (c) and (b) $\Leftrightarrow$ (d) by Theorem 4.5.4, and (c) $\Leftrightarrow$ (d) by Theorem 4.5.1.

$\square$

We briefly consider the special case $\Gamma = \Gamma_\tau$ of the above theorem. If τ is divisive (or more generally refinable and associate-preserving), then all the hypotheses of the above theorem are satisfied, and since τ is unital and associate-preserving we may replace each "τ_r" with "τ" and arrive at the result of [14].

Let $\Gamma \subseteq \Gamma'$ be factorization systems on a cancellative monoid H. We have the following diagram of implications, where a single dotted line indicates that the factorization systems involved are refinable, unital, and associate-preserving, and where two parallel dotted lines indicate that we additionally have Γ divisible and divisive. We will prove that this latter implication holds in the next chapter, while the remainder of the implications are just a special case of Figure 3.4.

Figure 4.5: Inheritance of "Atomic" Properties with $\Gamma \subseteq \Gamma'$

$$\begin{tikzcd}
\Gamma'\text{-UFM} \arrow[r] \arrow[d, Rightarrow, dotted] & \Gamma'\text{-FFM} \arrow[r] \arrow[d, dotted] & \Gamma'\text{-BFM} \arrow[r, dotted] \arrow[d, dotted] \arrow[rr, bend left] & \Gamma'\text{-ACCP} \arrow[r, dotted] \arrow[d] & \Gamma'\text{-atomic} \\
\Gamma\text{-UFM} \arrow[r] & \Gamma\text{-FFM} \arrow[r] & \Gamma\text{-BFM} \arrow[r, dotted] \arrow[rr, bend right] & \Gamma\text{-ACCP} \arrow[r, dotted] & \Gamma\text{-atomic}
\end{tikzcd}$$

Examples from [14] and [18] show that no nontrivial implications can be added to the above diagram, even in the special case of τ-factorization inheritance from standard factorization in integral domains.

The situation is simplified if we are working in a reduced factorization setup, where the unital and associate-preserving assumptions of the last diagram can be dropped.

Figure 4.6: Inheritance of "Atomic" Properties with $\Gamma_r \subseteq (\Gamma')_r$

$$\begin{tikzcd}
(\Gamma')_r\text{-UFM} \arrow[r] \arrow[d, dotted] & (\Gamma')_r\text{-FFM} \arrow[r] \arrow[d, dotted] & (\Gamma')_r\text{-BFM} \arrow[r, dotted] \arrow[d, dotted] \arrow[rr, bend left] & (\Gamma')_r\text{-ACCP} \arrow[r, dotted] \arrow[d] & (\Gamma')_r\text{-atomic} \\
\Gamma_r\text{-UFM} \arrow[r] & \Gamma_r\text{-FFM} \arrow[r] & \Gamma_r\text{-BFM} \arrow[r, dotted] \arrow[rr, bend right] & \Gamma_r\text{-ACCP} \arrow[r, dotted] & \Gamma_r\text{-atomic}
\end{tikzcd}$$

The above diagram for the special case of divisive $\tau_1 \leq \tau_2$ was demonstrated in [21].

The thesis [21] gives results slightly different than ours.

(1) If Γ is unital and refinable, then a Γ-BFD satisfies Γ-ACCP.

(2) If Γ is unital and refinable, then a domain satisfying Γ-ACCP is Γ-atomic.

(3) If Γ is unital and refinable, then a BFD (resp., FFD) is a Γ-BFD (resp., FFD).

(4) If Γ is unital, associate-preserving, refinable, and divisive, then a UFD is a Γ-UFD.

Results (1)-(3) are the results given here with the "associate-preserving" requirement dropped, while (4) has "divisible" left out from our corresponding version. We note that in that thesis the author defines Γ-atomic to mean that every element in $H^{\#}$ has a Γ-atomic factorization. So "H is Γ-atomic (resp., a Γ-BFM, a Γ-FFM, a Γ-HFM, a Γ-UFM)" in that thesis is equivalent to "Γ is expressive and H is Γ-atomic (resp., a Γ-BFM, a Γ-FFM, a Γ-HFM, a Γ-UFM)" in our terminology. The following three examples show that the above statements are unfortunately incorrect with either definition of Γ-atomic.

Example 4.5.6. An example of a τ-UFD that does not satisfy τ-ACCP, with τ refinable. Let $D = \mathbb{Z}[\{X^r \mid r \in \mathbb{Q}^+\}]$, and let τ be the symmetric relation on $D^{\#}$ determined by $(-X^{1/2^{n+2}})\tau(-X^{3/2^{n+2}})$ and $X^{3/2^{n+3}}\tau X^{5/2^{n+3}}$ for $n \geq 0$. The only τ-reducible nonzero nonunits are those of the form $\pm X^{1/2^n}$ for $n \geq 0$, which have unique τ-atomic factorizations $\pm(X^{3/2^{n+3}})(X^{5/2^{n+3}})$. Also, the system τ is refinable because the only τ-reducible nonzero nonunits that appear in nontrivial τ-factorizations are those of the form $-X^{1/2^{n+2}}$ for $n \geq 0$, and they are τ_r-irreducible. However, we have a sequence $\{-X^{1/2^{2n}}\}_{n=0}^{\infty}$ of nonzero nonunits with each $-X^{1/2^{2(n+1)}} = -X^{1/2^{2n+2}}$ a proper τ-divisor of $-X^{1/2^{2n}} = -(-X^{1/2^{2n+2}})(-X^{3/2^{2n+2}})$, so D does not satisfy τ-ACCP.

Example 4.5.7. An example of a UFD that is not τ-atomic, with τ refinable. Let τ be the symmetric relation on $\mathbb{Z}^{\#}$ determined by $3\tau 4$ and $2\tau(-2)$. The only thing we need to do to verify that τ is refinable is note that 4 is τ_r-irreducible. Finally, the domain $\mathbb{Z}$ is a UFD but 12 has no τ-atomic factorization.

Example 4.5.8. An example of a UFD that is not a Γ-UFD, with Γ a refinable, divisive, associate-preserving, normal, symmetric, and transitive factorization system.

Let D be a domain and Γ be the normal closure of the set of factorizations in D of the form $\lambda a_1 \cdots a_n$ where $(a_1, \ldots, a_n) = D$. Note that Γ satisfies all of the above properties.

We claim that D is a Γ-UFD if and only if it is quasi-local. The fact that D is a Γ-UFD if it is quasi-local is clear. Now suppose that D is a Γ-UFD that is not quasi-local. Then there are $x, y \in D^{\#}$ with $(x, y) = D$, and we may arrange for x and y to be Γ-atoms. If $b \in H^{\#}$ is a reducible Γ-atom, say $b = cd$, then $bxy = cdxy$ are two Γ-factorizations, and using reduced Γ-atomic factorizations of c and d, we may Γ-refine the right-hand side to a Γ-atomic factorization of a longer length than the right-hand-side, a contradiction. So $\operatorname{atom}(\Gamma) = \operatorname{atom}(H)$. If $b_1 \cdots b_m = c_1 \cdots c_k$ are two atomic factorizations, then $b_1 \cdots b_m xy = c_1 \cdots c_k xy$ are two Γ-atomic factorizations, so by uniqueness of Γ-atomic factorization it follows that $m = k$ and each $b_i \sim c_i$ after a suitable reordering. Therefore H is a UFD. Since $\operatorname{atom}(H) = \operatorname{atom}(\Gamma)$, every Γ-atomic factorization is an atomic factorization, and by uniqueness of atomic factorizations it follows that x^2 has no Γ-atomic factorization, a contradiction.

So, by taking D to be any non-quasi-local UFD, we get a UFD that is not a Γ-UFD.

Refinability allows us to characterize the Γ-BFM's with length functions in the same way that was done for BFD's in [3, Theorem 2.4].

Theorem 4.5.9. *Let Γ be a refinable factorization system on a cancellative monoid*

H. The following are equivalent.

1. *The monoid H is a Γ_r-BFM.*
2. *There is a function $l : H^{\#} \to \mathbb{Z}^+$ with $l(a_1 \cdots a_n) \geq l(a_1) + \cdots + l(a_n)$ whenever $a_1 \cdots a_n$ is a reduced Γ-factorization.*
3. *$L_{\Gamma_r}(H^{\#}) \subseteq \mathbb{Z}^+$ and $L_{\Gamma_r}(a_1 \cdots a_n) \geq L_{\Gamma_r}(a_1) + \cdots + L_{\Gamma_r}(a_n)$ whenever $a_1 \cdots a_n$ is a reduced Γ-factorization.*

If Γ is combinable and reduced divisible, then we can add the following statements to the equivalence.

4. *There is a function $l : H^{\#} \to \mathbb{Z}^+$ with $l(ab) \geq l(a) + l(b)$ whenever ab is a reduced Γ-factorization.*
5. *$L_{\Gamma_r}(H^{\#}) \subseteq \mathbb{Z}^+$ and $L_{\Gamma_r}(ab) \geq L_{\Gamma_r}(a) + L_{\Gamma_r}(b)$ whenever ab is a reduced Γ-factorization.*

Proof. (3) $\Rightarrow$ (2): Clear. (2) $\Rightarrow$ (1): If $a = a_1 \cdots a_n$ is a reduced Γ-factorization, then $n \leq l(a_1) + \cdots + l(a_n) \leq l(a_1 \cdots a_n) = l(a)$, showing that H is a Γ_r-completable BFM by Theorem 3.3.1, or equivalently a Γ_r-BFM by Theorem 4.5.5. (1) $\Rightarrow$ (3): If H is a Γ_r-BFM, then $L_{\Gamma_r}(H^{\#}) \subseteq \mathbb{Z}^+$ by Theorem 4.3.11 part (2), and it follows from refinability that for any reduced Γ-factorization $a_1 \cdots a_n$ we have $L_{\Gamma_r}(a_1 \cdots a_n) \geq L_{\Gamma_r}(a_1) + \cdots + L_{\Gamma_r}(a_n)$. (3) $\Rightarrow$ (5) $\Rightarrow$ (4): Clear.

Now assume Γ is combinable and reduced divisible. (4) $\Rightarrow$ (2): Assume (4) and let $a_1 \cdots a_n$ be any reduced Γ-factorization. The case $n = 1$ is trivial, so let us assume $n \geq 2$. Then $a_1(a_2 \cdots a_n)$ and $a_2 \cdots a_n$ are reduced Γ-factorizations by combinability and reduced divisibility, so we have $l(a_1 \cdots a_n) = l(a_1(a_2 \cdots a_n)) \geq$

$l(a_1) + l(a_2 \cdots a_n) \geq l(a_1) + l(a_2) + \cdots + l(a_n)$ by induction. $\square$

Let Γ be a refinable factorization system on a cancellative monoid H. We can define a strict partial order $<_{\Gamma_r}$ on $H^\#$ by defining $a <_{\Gamma_r} b$ if and only if b is a proper Γ_r-divisor of a. We call a nonempty subset of $H^\#$ a Γ_r-*chain* if $<_{\Gamma_r}$ is a strict total order on it. The *length* of a Γ_r-chain is the number of elements in it. We say that two finite Γ_r-chains $a_1 <_{\Gamma_r} \cdots <_{\Gamma_r} a_m$ and $b_1 <_{\Gamma_r} \cdots <_{\Gamma_r} b_n$ are $\equiv$-*equivalent* if $m = n$ and each $a_i \equiv b_i$. For $a \in H^\#$ we use $C_\Gamma(a)$ to denote the set of finite Γ_r-chains with smallest element a. We call a Γ_r-chain C' an *extension* of a Γ_r-chain C if C' and C have the same minimal element and $C \subseteq C'$; it is *proper* if $C \neq C'$. A *maximal* element of $C_\Gamma(a)$ is one with no proper extensions. For each $a \in H^\#$, we may associate the $\sim$ equivalence classes of elements of $C_{\text{fact}(H)}(a)$ with finite chains of principal ideals starting at (a). Without much effort, one can translate some results given in [3] into our terminology. The monoid H is:

(1) atomic if and only if each $C_{\text{fact}(H)}(a)$ has a maximal element,

(2) a BFM if and only if each $C_{\text{fact}(H)}(a)$ has an upper bound on the lengths of its elements,

(3) an FFM if and only if each $C_{\text{fact}(H)}(a)$ has only finitely many elements up to associates, and

(4) an HFM if and only if each $C_{\text{fact}(H)}(a)$ has a maximal element and any two such maximal elements have the same length.

Under suitable hypotheses, we can characterize the corresponding Γ_r-atomic concepts in the same fashion.

Theorem 4.5.10. *Let Γ be a symmetric, refinable, combinable, and divisible factorization system on a cancellative monoid H.*

(1) *For each $a \in H^{\#}$, there is a length-preserving bijection $F(a)$ between $\Gamma_r(a) \cup \operatorname{trfact}(a)$ and $C_\Gamma(a)$, given by taking $a = a_1 \cdots a_n$ to*

$$a_1 \cdots a_n <_{\Gamma_r} a_2 \cdots a_n <_{\Gamma_r} \cdots <_{\Gamma_r} a_{n-1}a_n <_{\Gamma_r} a_n;$$

its inverse map $F(a)^{-1}$ takes $a = b_1 <_{\Gamma_r} \cdots <_{\Gamma_r} b_m$ to $a = (b_1b_2^{-1}) \cdots (b_{m-1}b_m^{-1})b_m$.

(2) *The image under $F(a)$ of a (proper) Γ_r-refinement of a $(\Gamma_r \cup \operatorname{trfact}(H))$-factorization of a is a (proper) extension of the image of the original factorization.*

(3) *The image under $F(a)^{-1}$ of a (proper) extension of a Γ_r-chain in $C_\Gamma(a)$ is a (proper) Γ_r-refinement of the image of the original Γ_r-chain.*

(4) *The map $F(a)$ gives a one-to-one correspondence between the Γ_r-$(\Gamma_r \cup \operatorname{trfact}(H))$-atomic factorizations of a and the maximal elements of $C_\Gamma(a)$.*

(5) *The following are equivalent.*

  (a) *The monoid H is Γ_r-atomic.*

  (b) *For every $a \in H^{\#}$, every element of $C_\Gamma(a)$ can be extended to a maximal element.*

  (c) *For every $a \in H^{\#}$, the set $C_\Gamma(a)$ has a maximal element.*

(6) *The monoid H is a Γ_r-BFM if and only if for each $a \in H^{\#}$ there is an upper*

bound on the lengths of the Γ_r-chains in $C_\Gamma(a)$.

(7) *If $\equiv$ is a partition-preserving equivalence relation on $H_0^{\#}$, then H is a Γ_r-FFM$_{\equiv}$ if and only if for every $a \in H^{\#}$ there are only finitely many elements of $C_\Gamma(a)$ up to $\equiv$-equivalence.*

(8) *The monoid H is a Γ_r-HFM if and only if for every $a \in H^{\#}$ the set of maximal elements of $C_\Gamma(a)$ form a non-empty set of Γ_r-chains of the same length.*

Proof.

(1) Take any $a \in H^{\#}$. If the proposed maps are well-defined, then a simple computation shows that they are indeed inverses. The fact that the map from $\Gamma_r(a) \cup \text{trfact}(a)$ is well-defined follows by the combinable and divisible properties, while the fact that the proposed inverse map is well-defined follows from the combinable, refinable, and symmetric properties. The fact that $F(a)$ is length-preserving is clear.

(2)-(3) These are straightforward consequences of the definitions.

(4) Because $(\Gamma_r \cup \text{trfact}(H))$ is Γ_r-refinable, by 4.5.4 it will suffice to show that $F(a)(A) = B$, where A is the set of $(\Gamma_r \cup \text{trfact}(H))$-factorizations of a with a proper Γ_r-refinement and B is the set of Γ_r-chains in $C_\Gamma(a)$ with a proper extension. By part (2) we have $F(a)(A) \subseteq B$, and by part (3) we have $F(a)^{-1}(B) \subseteq A$. Hence $B = F(a)(F(a)^{-1}(B)) \subseteq F(a)(A) \subseteq B$, as desired.

(5) (a) $\Rightarrow$ (b): Assume H is Γ_r-atomic. Then it is Γ_r-atomicable by Theorem 4.5.1. Take any $a \in H^{\#}$. If a is not Γ_r-expressible, then $C_\Gamma(a) = \{\{a\}\}$, so let us

assume a is Γ_r-expressible. Then a has a Γ_r-atomic factorization, so the Γ_r-chain $\{a\}$ can be extended to a maximal element of $C_\Gamma(a)$ by part (4). On the other hand, any Γ_r-chain in $C_\Gamma(a)$ with more than two elements gets mapped by $F(a)^{-1}$ to a reduced Γ-factorization of a, which can be Γ_r-refined to a Γ_r-atomic factorization, whose image under $F(a)$ is an extension of the original Γ_r-chain that is a maximal element of $C_\Gamma(a)$ by part (4). (b) $\Rightarrow$ (c): If (b) holds, then for every $a \in H^\#$, the Γ_r-chain $\{a\}$ can be extended to a maximal element of the set $C_\Gamma(a)$. (c) $\Rightarrow$ (a): Assume (c) and take any Γ_r-expressible $a \in H^\#$. If a is a Γ_r-atom, then $a = a$ must be a Γ_r-atomic factorization, so let us assume a is Γ_r-reducible. Let $a = a_1 \cdots a_n$ be the image of a maximal element of $C_\Gamma(a)$. By part (4), this is a Γ_r-($\Gamma_r \cup \mathrm{trfact}(H)$)-atomic factorization. By the fact that a is Γ_r-reducible, $n \geq 2$, so this is in fact a Γ_r-atomic factorization.

(6) By Theorems 3.3.1 and 4.5.5, the monoid H is a Γ_r-BFM if and only if each element of $H^\#$ has an upper bound on the lengths of its Γ_r-factorizations. The result is now immediate from part (1).

(7) Assume $\equiv$ is partition-preserving. Let us introduce some temporary terminology and call two factorizations $\lambda a_1 \cdots a_m$ and $\mu b_1 \cdots b_n$ *strongly $\equiv$-equivalent* if $m = n$, $\lambda \equiv \mu$, and each $a_i \equiv b_i$. It is not hard to see that we could equivalently define Γ_r-FFM with "strong $\equiv$-equivalence" in place of "$\equiv$-equivalence". ($\Rightarrow$): By contrapositive. Assume there is an $a \in H^\#$ with an infinite sequence $C_1, C_2, \ldots$ of non-$\equiv$-equivalent Γ_r-chains in $C_\Gamma(a)$. We may choose such a sequence with each C_i not the trivial Γ_r-chain $\{a\}$, and hence each $F(a)^{-1}(C_i)$ a reduced Γ-

factorization of a. If $F(a)^{-1}(C_i)$ and $F(a)^{-1}(C_j)$ are strongly $\equiv$-equivalent for some $i \neq j$, then it follows by the partition-preserving property that C_i and C_j are $\equiv$-equivalent, a contradiction. Therefore $F(a)^{-1}(C_1), F(a)^{-1}(C_2), \ldots$ is an infinite sequence of non-strongly $\equiv$-equivalent reduced Γ-factorizations of a, and thus H is not a Γ_r-completable FFM$_\equiv$ by Theorem 3.3.3, hence not a Γ_r-FFM$_\equiv$ by Theorem 4.5.5. ($\Leftarrow$): By contrapositive. Assume H is not a Γ_r-FFM$_\equiv$. By Theorems 3.3.3 and 4.5.5, there is an $a \in H^\#$ with an infinite sequence of non-$\equiv$-related Γ_r-divisors $a_1, a_2, \ldots$, and $a <_{\Gamma_r} a_1$, $a <_{\Gamma_r} a_2$, $\ldots$ forms an infinite sequence of non-$\equiv$-related elements of $C_\Gamma(a)$.

(8) Let a be any Γ_r-expressible nonzero nonunit. If a is a Γ_r-atom, then $a = a$ is a Γ_r-atomic factorization. Therefore the Γ_r-($\Gamma_r \cup \text{trfact}(H)$)-atomic factorizations and Γ_r-atomic factorizations coincide, so by parts (1) and (4) the map $F(a)$ is a length-preserving bijection between the Γ_r-atomic factorizations of a and the maximal elements of $C_\Gamma(a)$. The equivalence now follows.

$\square$

CHAPTER 5
GENERALIZED PRIMES AND PRIMALS

In this chapter we will study generalized primes and primals and their effects on unique factorization. In order to avoid spending excessive effort dealing with trivialities, all factorization systems in this chapter will be reduced normal.

5.1 Generalized Primals

Let Γ, Γ_1, Γ_2, and Γ_3 be factorization systems on a monoid H. We call a nonunit Γ_1-Γ_2-Γ_3-*superprimal* if whenever it Γ_2-divides a Γ_1-factorization $\lambda a_1 \cdots a_n$, then there are $1 \leq i_1 < \cdots < i_k \leq n$ such that it has a Γ_1-factorization $\mu a'_{i_1} \cdots a'_{i_k}$, where each $a'_{i_j} \mid_{\Gamma_3} a_{i_j}$. A Γ_1-Γ_2-Γ_3-*primal* is defined by restricting to the case $n \leq 2$ in the above definition. Observe that 0 is necessarily Γ_1-Γ_2-Γ_3-(super)primal. A nonunit is *completely* Γ_1-Γ_2-Γ_3-*(super)primal* if all of its Γ_1-divisors are Γ_1-Γ_2-Γ_3-(super)primal. A Γ-*(super)primal* is a Γ-fact(H)-fact(H)-(super)primal, a $|_\Gamma$-*(super)primal* is a Γ-Γ-Γ-(super)primal, a *half* $|_\Gamma$-*(super)primal* is a Γ-Γ-fact(H)-(super)primal, and a *(super)primal* is a fact(H)-(super)primal. We call H Γ_1-Γ_2-Γ_3-*pre-Schreier* if every (nonzero) nonunit is Γ_1-Γ_2-Γ_3-superprimal, and Γ_1-Γ_2-Γ_3-*Schreier* if it is additionally integrally closed; similar abbreviations as those used for the special kinds of Γ_1-Γ_2-Γ_3-superprimals apply. The thesis [21] uses "τ_1-τ_2-τ_3-primal" to mean what we would call "τ_1-τ_2-τ_3-superprimal"; we have made the above changes so that the special case $\tau = \tau_H$ is consistent with the classic definitions of primal and (pre-)Schreier. (More specifically, we will see shortly that superprimal is in general a stronger property than primal. Traditionally, in defining "(pre-)Schreier", one uses "primal" in place of "superprimal", but we will see that the two definitions are equivalent, since completely primal implies superprimal.) We have introduced the definitions of the "half" versions, which are motivated by the half $|_\tau$-primes studied in [18]. We will get to these and the other generalizations of prime elements later, but we start with the more general theory of the various "superprimals". The reader is referred to the classic paper [12] for some basics on Schreier domains and on primals. The usual example of a Schreier domain is a GCD domain. (See [12, Theorem 2.4].)

We start by exploring how the superprimal property is related to the classic primal and completely primal properties. Let H be a monoid. For $n \geq 1$, we call a nonunit a *n-primal* if whenever it divides a length n product $b_1 \cdots b_n$, we can write $a = a_1 \cdots a_n$, where each $a_i \mid b_i$. (Of course, any element is trivially 1-primal.) Thus a superprimal is an element that is n-primal for all n, and a primal is a 2-primal. For each $n \geq 1$ we have: completely primal $\Rightarrow$ superprimal $\Rightarrow$ $(n+1)$-primal $\Rightarrow$ n-primal. (The first implication follows from a simple induction.) We give examples to show that none of the implications can be reversed.

Example 5.1.1. An example of a superprimal that is not completely primal. Let R be any non-pre-Schreier domain (for example, any atomic domain that is not a UFD), let K be its quotient field, and let $D = R + K[X]$. In [23] it is shown that the element X of D is primal but not completely primal. We will extend this result a little by showing that X is superprimal. We make the following observation: for a monomial f and a polynomial $g \neq 0$ in D, $f \mid g$ if and only if f divides g's nonzero term of lowest degree. So to prove that X is superprimal it will suffice to consider the case where $X \mid (\alpha_1 X^{r_1}) \cdots (\alpha_n X^{r_n})$ for some $n \geq 3$, $\alpha_1, \ldots, \alpha_n \in K$, and $r_1, \ldots, r_n \geq 0$. Because X is primal, we may reduce to the case where X divides no length 2 subproduct of $(\alpha_1 X^{r_1}) \cdots (\alpha_n X^{r_n})$, which forces (without loss of generality) $r_1 = 1$ and $r_2 = \cdots = r_n = 0$. Then the fact that $X \mid (\alpha_1 X)\alpha_2\alpha_3 \cdots \alpha_n$ implies that $\alpha_1 \cdots \alpha_n \in R$, and we can write $X = ((\alpha_2 \cdots \alpha_n)^{-1} X)\alpha_2 \cdots \alpha_n$, where $(\alpha_2 \cdots \alpha_n)^{-1} X \mid \alpha_1 X$, as desired.

Example 5.1.2. An example of a domain that has, for each $n \geq 1$, an n-primal that is not $(n+1)$-primal. Let $K \subsetneq L$ be any fields so that for each $n \geq 1$ there is a $c \in L$ with $c^n \notin K$, and let $D = K + XL[X]$. (For example, we could take $K = \mathbb{R}$ and $L = \mathbb{C}$, or K to be any field and form L by adjoining a transcendental element.) Consider the element $X^n \in D$. In [10, Example 3.4] it is shown that X^2 is primal but not completely primal. We will extend this result by showing that X^n is n-primal but not $(n+1)$-primal.

As in the previous example, a monomial f divides a polynomial $g \neq 0$ if and only if f divides g's nonzero term of lowest degree. So, to show that X^n is n-primal it will suffice to consider the case where $X^n \mid (\alpha_1 X^{r_1}) \cdots (\alpha_n X^{r_n})$ for some $\alpha_1, \ldots, \alpha_n \in L$ and $r_1, \ldots, r_n$ with $0 \leq r_n \leq \cdots \leq r_1 \leq n$. If $r_1 + \cdots + r_n = n$, then $r_1 \geq 1$, $\alpha_1 \cdots \alpha_n \in K$, and $X^n = ((\alpha_2 \cdots \alpha_n)^{-1} X^{r_1})(\alpha_2 X^{r_2})(\alpha_3 X^{r_3}) \cdots (\alpha_n X^{r_n})$, where $(\alpha_2 \cdots \alpha_n)^{-1} X^{r_1} \mid \alpha_1 X^{r_1}$. So let us assume $r_1 + \cdots + r_n \geq n+1$. Then $r_1 \geq 2$, and there are $i_2, \ldots, i_n$ with each $i_k \in \{0, \ldots, r_k\}$ such that $(r_1 - 1) + i_2 + \cdots + i_n = n$. For each $k = 2, \ldots, n$, let $\beta_k = 1$ if $i_k = 0$ and $\beta_k = \alpha_k$ otherwise. Then $X^n = ((\beta_2 \cdots \beta_n)^{-1} X^{r_1 - 1})(\beta_2 X^{i_2}) \cdots (\beta_n X^{i_n})$, where $(\beta_2 \cdots \beta_n)^{-1} X^{r_1 - 1}$ is an element of D dividing $\alpha_1 X^{r_1}$, and for $k \geq 2$ we have $\beta_k X^{i_k}$ an element of D dividing $\alpha_k X^{r_k}$, as desired.

Now suppose that X^n is $(n+1)$-primal. Find $c \in L$ with $c^n \notin K$. Then the fact that $X^n \mid (cX)^{(n+1)}$ implies that $X^n = (a_1 X) \cdots (a_n X)(a_1 \cdots a_n)^{-1}$ for some $a_1, \ldots, a_n \in L$ with $a_1 \cdots a_n \in K$ and each $\frac{c}{a_i} \in K$. Therefore $c^n = (\frac{c}{a_1}) \cdots (\frac{c}{a_n})(a_1 \cdots a_n) \in K$, a contradiction.

In this chapter we will mostly restrict our study to the various "Γ-(super)primals", since the primary thing we are interested in is unique Γ-atomic or Γ-complete factorization, but many of the results we will prove do generalize. It is intuitively clear that there are countless technicalities that we avoid by dealing only with reduced normal factorization systems in this chapter, since certain proofs will depend on being able to assert that $a = a$ is a Γ-factorization, or that $a_{i_j} \mid_\Gamma a_{i_j}$.

We begin our study with some basic properties. We will focus on the more important "superprimal" case, but we note that a great deal of the proofs apply to the generalized primals *mutandis mutatis.*

Theorem 5.1.3. *Let* Γ *be a unital, divisible, and divisive factorization system on a monoid* H*. Then a nonunit is* Γ*-superprimal (resp.,* $|_\Gamma$*-superprimal, half* $|_\Gamma$*-superprimal) if and only if whenever it divides (resp.,* Γ*-divides,* Γ*-divides) a* Γ*-factorization* $\lambda a_1 \cdots a_n$*, then there are* $1 \leq i_1 < \cdots < i_k \leq n$ *such that it has a factorization* $\mu a'_{i_1} \cdots a'_{i_k}$*, where each* a'_{i_j} *divides (resp.,* Γ*-divides, divides)* a_{i_j}*.*

Proof. This is a simple consequence of the definitions. $\square$

Theorem 5.1.4. *Let Γ be a unital and strong associate-preserving factorization system on a monoid H.*

(1) *The Γ-superprimals (resp., half $|_\Gamma$-superprimals) and Γ_r-superprimals (resp., half $|_{\Gamma_r}$-superprimals) coincide.*

(2) *A $|_{\Gamma_r}$-superprimal is $|_\Gamma$-superprimal.*

Proof.

(1) First let a be any Γ-superprimal (resp., half $|_\Gamma$-superprimal), and let $a_1 \cdots a_n$ be any reduced Γ-factorization that it divides (resp., Γ_r-divides). Then there is a Γ-factorization $a = \lambda a'_{i_1} \cdots a'_{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq n$ and each a'_{i_j} divides a_{i_j}. Since Γ is unital and strong associate-preserving, the factorization $a = (\lambda a'_{i_1})a'_{i_2} \cdots a'_{i_k}$ is a reduced Γ-factorization, and of course $\lambda a'_{i_1} \mid a_{i_1}$.

Now let a be any Γ_r-superprimal (resp., half $|_{\Gamma_r}$-superprimal) and let $\lambda a_1 \cdots a_n$ be any Γ factorization that it divides (resp., Γ-divides). Since Γ is unital and strong associate-preserving, we know that $(\lambda a_1)a_2 \cdots a_n$ is a reduced Γ-factorization, and we know that a divides (resp., Γ-divides) it. If $a \approx a_1 \cdots a_n$, then by the unital property there is a Γ-factorization $a = \mu a_1 \cdots a_n$, so let us assume that $a \not\approx a_1 \cdots a_n$. Then a divides (resp., Γ_r-divides) the aforementioned reduced Γ-factorization, and the rest is simple.

(2) Proceed as in the second paragraph of the proof of part (1), making the necessary small adjustments.

□

We note that the $|_{\Gamma_r}$-superprimal property is hardly ever satisfied. For example, if H is a monoid, $1 \neq \lambda \in H^\times$, and $a, b \in H^\#$ are relatively prime, then a is not $|_{\mathrm{rfact}(H)}$-superprimal. This can be seen by considering the reduced factorization $(\lambda a)(\lambda^{-1} b)$.

Theorem 5.1.5. *Let Γ be a factorization system on a monoid H.*

(1) *If Γ is unital or strong associate-preserving, then a strong associate of a Γ-superprimal is Γ-superprimal.*

(2) *If Γ is unital and strong associate-preserving, then a strong associate of a (half) $|_\Gamma$-superprimal is (half) $|_\Gamma$-superprimal.*

Proof.

(1) Assume that Γ is unital or strong associate-preserving. Let a be Γ-superprimal and take any $\nu \in H^\times$ and any Γ-factorization $\lambda a_1 \cdots a_n$ that νa divides. Then $a \mid \lambda a_1 \cdots a_n$, so there is a Γ-factorization $a = \mu a'_{i_1} \cdots a'_{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq n$ and each $a'_{i_j} \mid a_{i_j}$. If Γ is unital, then $\nu a = (\nu\mu) a'_{i_1} \cdots a'_{i_k}$ is a Γ-factorization, while, if Γ is strong associate-preserving, then $\nu a = \mu(\nu a'_{i_1}) a'_{i_2} \cdots a'_{i_k}$ is a Γ-factorization and $\nu a'_{i_1} \mid a_{i_1}$. Either way, we have shown that νa is Γ-superprimal.

(2) Assume that Γ is unital and strong associate-preserving. Let a be (half) $|_\Gamma$-superprimal and take any $\nu \in H^\times$ and any Γ-factorization $\lambda a_1 \cdots a_n$ that νa Γ-divides. Because Γ is strong associate-preserving, we know that a Γ-divides $(\nu^{-1}\lambda) a_1 \cdots a_n$, which is a Γ-factorization by the unital property. Since a is (half) $|_\Gamma$-superprimal, there are $1 \leq i_1 < \cdots < i_k \leq n$ and a Γ-factorization

$a = \mu a'_{i_1} \cdots a'_{i_k}$ with each a'_{i_j} Γ-dividing (dividing) a_{i_j}. Again using the unital property, we obtain a Γ-factorization $\nu a = (\nu\mu)a'_{i_1} \cdots a'_{i_k}$, as desired.

□

It is immediate from the definitions that, if $\Gamma_2 \subseteq \Gamma'_2$ and $\Gamma_3 \supseteq \Gamma'_3$, then a Γ_1-Γ'_2-Γ'_3-(super)primal is Γ_1-Γ_2-Γ_3-(super)primal. If Γ_1 is divisible and divisive, then it is not hard to use a generalization of Theorem 5.1.3 part (1) to show that, if $\Gamma_1 \subseteq \Gamma'_1$, then a Γ'_1-Γ'_2-Γ'_3-(super)primal is Γ_1-Γ_2-Γ_3-(super)primal. The analogous result for a τ-factorization setup was stated in [21], but with the divisive requirement dropped. Unfortunately, the following example shows that this is false.

Example 5.1.6. An example of a completely primal element that is not τ-primal, where τ is symmetric, refinable, and associate-preserving. Let R be an integral domain and $D = R[X, Y]$. Let τ be the symmetric and associate-preserving relation on $D^\#$ determined by $X^2 \tau Y^2$. Note that τ is refinable but not divisive. The element XY is completely primal by [12, Lemma 2.5] since it is a product of prime (hence completely primal) elements, but it is not τ-primal, since it divides the τ-factorization $(X^2)(Y^2)$, but has no τ-factorization of the required form.

In [21, Lemma 2.8(1)] it is shown that a τ-superprimal is $|_\tau$-superprimal for τ symmetric, divisive, and multiplicative. The following is a generalization.

Theorem 5.1.7. *Let Γ be a symmetric, unital, divisive, and combinable factorization system on a cancellative monoid H. Then the $|_\Gamma$-superprimals and half $|_\Gamma$-superprimals coincide.*

Proof. It is clear that every $|_\Gamma$-superprimal is half $|_\Gamma$-superprimal. Now let a be any nonzero nonunit half $|_\Gamma$-superprimal and let $\lambda a_1 \cdots a_n$ be any Γ-factorization that it Γ-divides, say $\lambda a_1 \cdots a_n = \mu ab$ are Γ-factorizations. (The fact that Γ is symmetric and combinable ensures that we can write the right-hand side in the given form.) Since a is half $|_\Gamma$-superprimal, there is a Γ-factorization $a = \nu a'_{i_1} \cdots a'_{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq n$ and each $a'_{i_j} \mid a_{i_j}$, say $a_{i_j} = a'_{i_j} c_{i_j}$. Since Γ is unital and strong associate-preserving, we may arrange for each invertible c_{i_j} to be 1. Canceling yields $c_{i_1} \cdots c_{i_k} \mid b$, so by divisiveness each $\mu a'_{i_j} c_{i_j}$ is a Γ-factorization (ignoring any factors of 1), so that each $a_{i_j} = a'_{i_j} c_{i_j}$ is a Γ-factorization (ignoring any factors of 1) by the unital property, as desired. $\square$

Thus we have $|_\Gamma$-superprimal $\Rightarrow$ half $|_\Gamma$-superprimal $\Leftarrow$ Γ-superprimal, and the second implication reverses if Γ is symmetric, unital, divisive, and combinable. The same implications hold with "pre-Schreier" in place of "superprimal".

5.2 Generalized Primes

Let Γ_1, Γ_2, and Γ_3 be factorization systems on a monoid H. In analogy with [6] and [21], we call a nonunit Γ_1-Γ_2-Γ_3-*prime* if whenever it Γ_2-divides a Γ_1-factorization, then it Γ_3-divides some factor. A Γ-*prime* is a Γ-fact(H)-fact(H)-prime, a $|_\Gamma$-*prime* is a Γ-Γ-Γ-prime, and a *half* $|_\Gamma$-*prime* is a Γ-Γ-fact(H)-prime. A *prime* is a fact(H)-prime; Theorem 5.2.4 part (1) shows that this is equivalent to the usual definition of a prime element.

The following characterization of the Γ_1-Γ_2-Γ_3-primals in cancellative monoids shows us how they generalize the Γ_1-Γ_2-Γ_3-primes.

Theorem 5.2.1. *Let Γ_1, Γ_2, and Γ_3 be factorization systems on a cancellative monoid H. A Γ_1-Γ_2-Γ_3-prime is a Γ_1-irreducible Γ_1-Γ_2-Γ_3-superprimal. If Γ_2 is unital and associate-preserving, then the converse is true.*

Proof. ($\Rightarrow$): Let p be a Γ_1-Γ_2-Γ_3-prime. If $p = \lambda a_1 \cdots a_n$ is a Γ_1-factorization, then p divides some a_i, and the présimplifiable property forces $n = 1$. Therefore p is Γ_1-irreducible. Now let $\lambda a_1 \cdots a_n$ be any Γ_1-factorization that p Γ_2-divides. Then $p = p$ is a Γ_1-factorization and p Γ_3-divides some a_i, showing that p is Γ_1-Γ_2-Γ_3-primal. ($\Leftarrow$): Let p be a Γ_1-irreducible Γ_1-Γ_2-Γ_3-primal, and let $\lambda a_1 \cdots a_n$ be any Γ_1-factorization that it Γ_2-divides. Then there is a Γ_1-factorization $p = \mu a'_{i_1} \cdots a'_{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq n$ and each $a'_{i_j} \mid_{\Gamma_2} a_{i_j}$. Since p is Γ_1-irreducible, we have $p = \mu a'_{i_1}$, which Γ_2-divides a_{i_1} by the fact that Γ_2 is unital and associate-preserving. □

Corollary 5.2.2. *Let Γ be a factorization system on a cancellative monoid H. The Γ-primes are precisely the Γ-irreducible Γ-superprimals, and, if Γ is unital and associate-preserving, then the (half) $|_\Gamma$-primes are precisely the Γ-irreducible (half) $|_\Gamma$-superprimals.*

Corollary 5.2.3. *Let Γ be a symmetric, unital, divisive, and combinable factorization system on a cancellative monoid H. Then the $|_\Gamma$-primes and half $|_\Gamma$-primes coincide.*

Proof. Theorem 5.1.7 and Corollary 5.2.2. □

So $|_\Gamma$-primal $\Rightarrow$ half $|_\Gamma$-primal $\Leftarrow$ Γ-primal, and the first implication reverses for Γ symmetric, unital, divisive, and combinable.

Analogously to the previous section, we will mostly content ourselves with studying the various kinds of "Γ-primes". We continue giving some basic properties.

Theorem 5.2.4. *Let Γ be a factorization system on a monoid H.*

(1) *If Γ is divisible, unital, strong associate-preserving, and combinable, then a nonunit is Γ-prime (resp., $|_\Gamma$-prime) if and only if whenever it divides (resp., Γ-divides) a length 2 reduced Γ-factorization, then it divides (resp., Γ-divides) some factor.*

(2) *A strong associate of a Γ-prime is Γ-prime.*

(3) *If Γ is unital and strong associate-preserving, then a strong associate of a (half) $|_\Gamma$-prime is (half) $|_\Gamma$-prime.*

Proof.

(1) ($\Rightarrow$): Clear. ($\Leftarrow$): Assume that Γ is divisible, unital, strong associate-preserving, and combinable. Let a be any nonunit with the stated property, and let $a = \lambda a_1 \cdots a_n$ be any Γ-factorization that it divides (resp., Γ-divides). If $n = 1$, then $a = \lambda a_1$, which divides (resp., Γ-divides) a_1 by the fact that $\mathrm{fact}(H)$ (resp., Γ) is normal. So let us assume $n \geq 2$. By the unital and strong associate-preserving properties, the factorization $a = a_1(\lambda a_2 \cdots a_n)$ is a length 2 reduced Γ-factorization, so a divides (resp., Γ-divides) a_1 or $\lambda a_2 \cdots a_n$. In the former case we are done, while in the latter case the proof is finished by induction since $\lambda a_2 \cdots a_n$ is a Γ-factorization by divisibility.

(2) Let a be a Γ-prime, $\nu \in H^\times$, and $\lambda a_1 \cdots a_n$ be any Γ-factorization that νa

divides. Then $a \mid \lambda a_1 \cdots a_n$, so a divides some a_i, and hence $\mu a \mid a_i$.

(3) Adjust the proof of part (2), using the unital and strong associate-preserving properties where necessary.

$\square$

5.3 Generalized Primes and Unique Factorization

Let Γ be a factorization system on a cancellative monoid H. The purpose of this section is to thoroughly investigate the implications between the following statements.

(1) The monoid H is a Γ-UFM.

(2) Every (nonzero) nonunit has a Γ-factorization into $|_\Gamma$-primes.

(3) The monoid H is Γ-atomic and every Γ-atom is $|_\Gamma$-prime.

(2′) Every (nonzero) nonunit has a Γ-factorization into Γ-primes.

(3′) The monoid H is Γ-atomic and every Γ-atom is Γ-prime.

(2″) Every (nonzero) nonunit has a Γ-factorization into half $|_\Gamma$-primes.

(3″) The monoid H is Γ-atomic and every τ-atom is half $|_\Gamma$-prime.

The equivalence (2′) $\Leftrightarrow$ (3′) follows quickly from the fact that an associate of a Γ-prime is Γ-prime, and, if Γ is unital and associate-preserving, then (2) $\Leftrightarrow$ (3) and (2″) $\Leftrightarrow$ (3″) follow similarly. Corollary 5.4.5 will give the perhaps unexpected result that, if $\Gamma = \Gamma_\tau$ is divisive, then "Γ-factorization" can equivalently be replaced with "factorization" in (2′).

The paper [6] was the first to bring up this type of question, when it proved that $(2') \Leftrightarrow (3') \Rightarrow (1) \Leftrightarrow (2) \Leftrightarrow (3)$ for the special case where $\Gamma = \Gamma_\tau$ is divisive (examining the proofs allows us to weaken "divisive" to "refinable and associate-preserving"), and the authors posed the question of whether all five statements are in fact equivalent in this case. The τ-factorization versions of $(2'')$ and $(3'')$ were added to the list in [18, Section 4], where the work of [6] was extended to show that $(1) - (3)$, $(2'')$, and $(3'')$ are equivalent in the refinable and associate-preserving case, and that the other properties are not equivalent, even for τ both multiplicative and divisive. We will extend these results to a Γ-factorization framework, as well as investigate the implications between the statements with various different assumptions on the factorization systems.

We will give examples to show that our results about statements $(1) - (3)$, $(2')$, and $(3')$ are the best possible under these possible hypotheses on τ: no hypotheses, τ associate-preserving, τ divisive, τ both multiplicative and divisive, and $\tau = \tau_d$. In particular, we answer the above question from [6] in the negative. However, the half $|_\tau$-primes are still somewhat mysterious and not well understood when τ is not associate-preserving.

Let Γ be a factorization system on a monoid H. It is obvious from the definitions that any two Γ-factorizations of the same element into half $|_\Gamma$-primes are *homomorphic* (i.e., each factor in one divides some factor in the other). If H is cancellative, then homomorphic factorizations are $\sim$-equivalent (see, for example, the proof of [18, Theorem 4.1]), and we have established the theorem below.

Theorem 5.3.1. *Let Γ be a factorization system on a cancellative monoid H. Any two Γ-factorizations of the same element into half $|_\Gamma$-primes are equal up to order and associates.*

We can prove stronger results for $|_\Gamma$-primes if Γ is reasonably well-behaved, with the following series of results extending and generalizing [21, Theorem 2.4].

Theorem 5.3.2. *Let Γ be a unital and divisible factorization system on a cancellative monoid H. Let $\lambda a_1 \cdots a_m = \mu b_1 \cdots b_n$ be Γ-atomic factorizations with at most one a_i not $|_\Gamma$-prime. Then $m = n$ and each $a_i \sim b_i$ after a suitable reordering.*

Proof. If $m = 1$, then by the fact that a_1 is a Γ-atom we have $n = 1$ and $a_1 \sim b_1$. So let us assume that $m \geq 2$. After a suitable reordering, a_m is $|_\Gamma$-prime and $a_m |_\Gamma b_n$, so $a_m \sim b_n$. Canceling, we obtain Γ-atomic factorizations $\lambda' a_1 \cdots a_{m-1} = \mu' b_1 \cdots b_{n-1}$, and by induction $m - 1 = n - 1$ (hence $m = n$) and $a_i \sim b_i$ for $i = 1, \ldots, m - 1$ after a suitable reordering. $\square$

Adding a divisive requirement makes the situation even nicer.

Theorem 5.3.3. *Let Γ be a unital, divisive, and divisible factorization system on a cancellative monoid H. Let $x_1 \cdots x_n = \mu p_1 \cdots p_m$ be Γ-factorizations with possibly some of the x_i's units, and the p_i's Γ-atoms, all but at most one of which is half $|_\Gamma$-prime. Then there is a partition of $\{1, \ldots, m\}$ into (possibly empty) disjoint sets $B_1, \ldots, B_n$ such that each $\prod_{i \in B_j} p_i \sim x_j$.*

Proof. If $m = 1$, then by the fact that p_1 is a Γ-atom we have some $x_i \sim p_1$ and the other x_j's are units, so let us assume $m \geq 2$. Without loss of generality, assume p_1

is half $|_\Gamma$-prime and $p_1 \mid x_1$, say $x_1 = p_1 a$. Canceling yields $ax_2 \cdots x_n = \mu p_2 \cdots p_m$, which are Γ-factorizations by divisibility and divisiveness (with a and some of the x_i's possibly units). By induction, there is a partition of $\{2, \ldots, m\}$ into disjoint sets $B, B_2, \ldots, B_n$ such that $\prod_{i \in B} p_i \sim a$ and each $\prod_{i \in B_j} p_i \sim x_j$ for $2 \leq j \leq n$. Let $B_1 = B \cup \{1\}$. Then $B_1, \ldots, B_n$ form a partition of $\{1, \ldots, m\}$ and $\prod_{i \in B_1} p_i = p_1 \prod_{i \in B} p_i \sim p_1 a = x_1$, as desired. □

Corollary 5.3.4. *Let Γ be a unital, divisive, and divisible factorization system on a cancellative monoid H. Let $\lambda x_1 \cdots x_n = \mu p_1 \cdots p_m$ be Γ-factorizations, where the p_i's are Γ-atoms, all but at most one of which is half $|_\Gamma$-prime. Then:*

(1) *$m \geq n$.*

(2) *If each x_i is Γ-irreducible, then $n = m$ and each $x_i \sim p_i$ after a suitable reordering.*

(3) *If $n = m$, then each $x_i \sim p_i$ after a suitable reordering.*

Proof. Partition $\{1, \ldots, m\}$ as in Theorem 5.3.3. Because each x_j is not a unit, each $|B_j| \geq 1$, so $m = |B_1| + \cdots + |B_n| \geq n$. If each x_i is Γ-irreducible, then each $|B_i| = 1$ so $m = n$. If $n = m$, this forces each $|B_i| = 1$. □

We are now ready for our Γ-factorization generalization of [6, Theorem 2.7].

Theorem 5.3.5. *Let Γ be a factorization system on a cancellative monoid H. In the statements at the beginning of the section, $(2) \Leftarrow (3) \Rightarrow (3'') \Rightarrow (1)$ and $(2') \Leftrightarrow (3') \Rightarrow (3'') \Rightarrow (2'') \Leftarrow (2)$. If Γ is unital and associate-preserving, then $(2'') \Leftrightarrow (3'')$ and $(2) \Leftrightarrow (3)$. If Γ is unital and divisible, then $(2) \Rightarrow (1)$. If Γ is refinable, unital,* *and associate-preserving, then* (1) − (3)*,* (2″)*, and* (3″) *are equivalent.*

Proof. The non-obvious implications in the first four sentences are (2) ⇒ (1) and (3″) ⇒ (1), which follow from Theorems 5.3.2 and 5.3.1, respectively. Now assume that Γ is refinable, unital, and associate-preserving, and that H is a Γ-UFM. Let a be any Γ-atom and $\lambda a_1 \cdots a_n$ be any Γ-factorization that it Γ-divides, say $\mu a x_1 \cdots x_k = \lambda a_1 \cdots a_n$ are Γ-factorizations. By Theorem 4.5.5 and Lemma 4.4.4, the monoid H is a Γ-atomicable UFM, so we can Γ-refine both Γ-factorizations into Γ-atomic factorizations. By uniqueness, the element a is an associate of some Γ-factor of the Γ-refined right-hand side, which in turn is a Γ-divisor of some a_i. Because Γ is unital and associate-preserving, we have $a \mid_\Gamma a_i$, as desired. □

The following diagram of implications summarizes Theorem 5.3.5.

Figure 5.1: Statements about Generalized Primes and Unique Factorization

$$\begin{tikzcd}
3 \arrow[rr] \arrow[dd] & & 2 \arrow[dd] \arrow[dl, dotted, "\text{u., divisible}"] \arrow[ll, bend right, dotted, "\text{u., a.p.}"'] \\
 & 1 \arrow[ul, dotted, "\text{ref., u., a.p.}"] & \\
3'' \arrow[rr] \arrow[ur] & & 2'' \arrow[ll, bend left, dotted, "\text{u., a.p.}"] \\
3' \arrow[u] \arrow[rr, leftrightarrow] & & 2' \arrow[u]
\end{tikzcd}$$

We have the following slightly simplified diagram that applies to τ-factorization (or more generally, to a unital and divisible factorization system).

Figure 5.2: Statements in a τ-factorization Context

$$\begin{tikzcd}
3 \arrow[rr] \arrow[dd] & & 2 \arrow[dl] \arrow[dd] \arrow[ll, dotted, bend right, "\text{a.p.}"'] \\
& 1 \arrow[ul, dotted, "\text{ref., a.p.}"] & \\
3'' \arrow[ur] \arrow[rr] & & 2'' \arrow[ll, dotted, bend left, "\text{a.p.}"] \\
3' \arrow[u] \arrow[rr, leftrightarrow] & & 2' \arrow[u]
\end{tikzcd}$$

We will now give several examples to show that Figure 5.2 is at least close to the best possible.

Example 5.3.6 ([18, Example 4.3])**.** In a τ-UFD, an atom (hence a τ-atom) need not be τ-prime, even if τ is both multiplicative and divisive. Let R be an integral domain and $D = R[X^2, Y^2, XY]$. Define τ to be the symmetric and associate-preserving relation on $D^\#$ determined by $(X^{2m})\tau(Y^{2n})$ for $m, n \geq 1$. Note that τ is divisive and multiplicative. The only nonzero nonunits that are not τ-atoms are those of the form $\lambda(X^{2m})(Y^{2n})$, which also happens to be their unique τ-atomic factorization (up to associates and order). So D is a τ-UFD. Now, the atom XY divides the τ-factorization $(X^2)(Y^2)$, but it does not divide X^2 or Y^2, so XY is not τ-prime.

Example 5.3.7. In a τ-UFD, an atom need not be an associate of a half $|_\tau$-prime, even if τ is associate-preserving. Let R be an integral domain, $D = R[X, Y^2, XY]$, and τ be the symmetric and associate-preserving relation on $D^\#$ determined by $X^2 \tau Y^2$, $XY \tau XY$, and $X \tau X$. The only τ-reducible elements are those of the forms $\lambda(XY)^n$ and λX^n ($n \geq 2$), which also happens to be their unique τ-atomic factorizations. (Note that the τ-factorization $(X^2)(Y^2)$ is not τ-atomic.) Because $(XY)^2 = (X^2)(Y^2)$ are τ-factorizations and XY does not divide X^2 or Y^2, the atom XY is not half $|_\tau$-prime. Because τ is associate-preserving, we conclude that XY is not an associate of a half $|_\tau$-prime.

Example 5.3.8. An example of a domain where every nonzero nonunit is a τ-product of prime $|_\tau$-primes (hence τ-primes), but not every prime (hence not every τ-atom) is $|_\tau$-prime. Let $D = \mathbb{Z}$ and define $a \tau b \Leftrightarrow a, b > 0$ or $ab = 12$. Observe that every positive prime is $|_\tau$-prime, so every nonzero nonunit is a τ-product of prime $|_\tau$-primes. Now, the element -2 is prime and $3 \cdot 4 = (-2)(-6)$ are τ-factorizations, but -2 does not τ-divide either 3 or 4, so -2 is not $|_\tau$-prime.

Example 5.3.9. An example where τ is associate-preserving, D is a τ-UFD, and every τ-atom is τ-prime, but not every prime (hence not every τ-atom) is $|_\tau$-prime. Let $D = \mathbb{Z}$ and let τ be the symmetric relation on $D^\#$ determined by $(\pm 2)\tau(\pm 24)$, $(\pm 2)\tau(\pm 3)$, $(\pm 2)\tau(\pm 4)$, $(\pm 3)\tau(\pm 4)$, and $(\pm 4)\tau(\pm 4)$. Note that τ is associate-preserving. We may observe that the only τ-reducible nonzero nonunits are those elements of the form $\pm 2^{2n}$, $\pm 2^{2n+1}$, $\pm 3 \cdot 2^{2n}$, and $\pm 3 \cdot 2^{2n+1}$ for some $n \in \mathbb{Z}^{\geq 0}$, which have unique τ-atomic factorizations $\pm 4^n$, $\pm 2 \cdot 4^n$, $\pm 3 \cdot 4^n$, and $\pm 2 \cdot 3 \cdot 4^n$, respectively, so D is a

τ-UFD. (Note that $2 \cdot 24$ is not a τ-atomic factorization since $24 = 2 \cdot 3 \cdot 4$ is a nontrivial τ-factorization of 24. So $48 = 3 \cdot 4^2$ is indeed the unique τ-atomic factorization of 48.) To show that every τ-atom is τ-prime, it will suffice to show that every τ-atom $x \in \mathbb{Z}^{\geq 2}$ dividing something of the form $2 \cdot 3 \cdot 4^n$ is τ-prime. Such an x must be either 2, 3, or 4. 2 and 3 are prime (hence τ-prime), and any τ-factorization that 4 divides must be of the form $\pm 4^n$, $\pm 2 \cdot 4^n$, $\pm 3 \cdot 4^n$, or $\pm 2 \cdot 3 \cdot 4^n$ for some $n \in \mathbb{Z}^+$, and in any case 4 divides one of the terms of that τ-factorization. So every τ-atom is τ-prime. However, 2 is prime and 2 τ-divides $48 = 3 \cdot 4^2 = 2 \cdot 24$, but 2 does not τ-divide 3 or 4, so 2 is not $|_\tau$-prime.

For $\tau = \tau_d$, all seven of the above statements are equivalent, but we have seen that most of the implications between them do not hold for a general symmetric relation τ. In the following tables, we will attempt to gain some insight into what is happening by showing the truth or falsity of each implication as we impose progressively stronger conditions on τ: τ any symmetric relation, τ an associate-preserving relation, τ divisive, τ both divisive and multiplicative, $\tau = \tau_\star$, and $\tau = \tau_d$. The entries will indicate whether the row implies the column. "T" indicates that the implication is true by Theorem 5.3.5, a reference to a theorem indicates that it is true by the theorem referenced, a reference to one of the above examples indicates that it is false and that example provides a counterexample, and a question mark indicates that the truth of the implication is unknown to us. The question marks in the $\tau = \tau_\star$ chart were conjectured to be true in [6].

Table 5.1: Implications with τ any relation

	(1)	(2)	(2′)	(3)	(3′)	(2″)	(3″)
(1)	T	Ex 5.3.7	Ex 5.3.6	Ex 5.3.7	Ex 5.3.6	Ex 5.3.7	Ex 5.3.7
(2)	T	T	Ex 5.3.6	Ex 5.3.8	Ex 5.3.6	T	?
(2′)	T	Ex 5.3.9	T	Ex 5.3.9	T	T	T
(3)	T	T	Ex 5.3.6	T	Ex 5.3.6	T	T
(3′)	T	Ex 5.3.9	T	Ex 5.3.9	T	T	T
(2″)	?	Ex 5.3.9	Ex 5.3.6	Ex 5.3.8	Ex 5.3.6	T	?
(3″)	T	Ex 5.3.9	Ex 5.3.6	Ex 5.3.9	?	T	T

Table 5.2: Implications with τ associate-preserving

	(1)	(2)	(2′)	(3)	(3′)	(2″)	(3″)
(1)	T	Ex 5.3.7	Ex 5.3.6	Ex 5.3.7	Ex 5.3.6	Ex 5.3.7	Ex 5.3.7
(2)	T	T	Ex 5.3.6	T	Ex 5.3.6	T	T
(2′)	T	Ex 5.3.9	T	Ex 5.3.9	T	T	T
(3)	T	T	Ex 5.3.6	T	Ex 5.3.6	T	T
(3′)	T	Ex 5.3.9	T	Ex 5.3.9	T	T	T
(2″)	T	Ex 5.3.9	Ex 5.3.6	Ex 5.3.9	Ex 5.3.6	T	T
(3″)	T	Ex 5.3.9	Ex 5.3.6	Ex 5.3.9	Ex 5.3.6	T	T

Table 5.3: Implications with τ divisive

	(1)	(2)	(2′)	(3)	(3′)	(2″)	(3″)
(1)	T	T	Ex 5.3.6	T	Ex 5.3.6	T	T
(2)	T	T	Ex 5.3.6	T	Ex 5.3.6	T	T
(2′)	T	T	T	T	T	T	T
(3)	T	T	Ex 5.3.6	T	Ex 5.3.6	T	T
(3′)	T	T	T	T	T	T	T
(2″)	T	T	Ex 5.3.6	T	Ex 5.3.6	T	T
(3″)	T	T	Ex 5.3.6	T	Ex 5.3.6	T	T

The above diagram remains unchanged if we strengthen "divisive" to "both multiplicative and divisive", or if we weaken it to "refinable and associate-preserving".

Table 5.4: Implications for $\tau = \tau_\star$

	(1)	(2)	(2′)	(3)	(3′)	(2″)	(3″)
(1)	T	T	?	T	?	T	T
(2)	T	T	?	T	?	T	T
(2′)	T	T	T	T	T	T	T
(3)	T	T	?	T	?	T	T
(3′)	T	T	T	T	T	T	T
(2″)	T	T	?	T	?	T	T
(3″)	T	T	?	T	?	T	T

Table 5.5: Implications for $\tau = \tau_d$

	(1)	(2)	(2′)	(3)	(3′)	(2″)	(3″)
(1)	T	T	$[6, 4.1]$	T	$[6, 4.1]$	T	T
(2)	T	T	$[6, 4.1]$	T	$[6, 4.1]$	T	T
(2′)	T	T	T	T	T	T	T
(3)	T	T	$[6, 4.1]$	T	$[6, 4.1]$	T	T
(3′)	T	T	T	T	T	T	T
(2″)	T	T	$[6, 4.1]$	T	$[6, 4.1]$	T	T
(3″)	T	T	$[6, 4.1]$	T	$[6, 4.1]$	T	T

5.4 Properties of Products of Elements

The theory of abstract factorization has several theorems of the form "if Γ has property P_1, a is a nonunit with properties P_2, and b is a nonunit with property P_3, then either ab is a Γ-factorization or ab has property P_4". In this section, we will derive several results of this type and investigate some of their applications.

Let D be an integral domain and τ be a relation on $D^{\#}$. In [6], the first theorem of the above form was proved: if τ is symmetric and divisive, a is τ-prime, and b is a τ-atom, then either $a\tau b$ or ab is a τ-atom. The thesis [21] expanded on this work with some more theorems of this type. We intend to review some of these theorems, generalize, and make additions. However, we would first like to correct two mistakes. In [21, Lemma 2.18(2),(4)], it is stated that the following for τ symmetric and divisive and $a, b \in D^{\#}$ with $a \not\tau b$:

(1) If a is a τ-atom and b is τ-superprimal, then ab is a τ-atom.

(2) If a is τ-prime and b is a τ-superprimal relatively prime to a, then ab is τ-prime.

Unfortunately, however, both statements are false, as the following counterexample shows.

Example 5.4.1. An example where τ is a multiplicative and divisive relation on the nonzero nonunits of a domain D, $a \in D^\#$ is prime, $b \in D^\#$ is a primal relatively prime to a, $a \nmid b$, and ab is not a τ-atom. Let R be an integral domain, let $D = R[X, Y, Z]$, and let τ be the symmetric and associate-preserving relation on $D^\#$ determined by $X^k \tau Y^m Z^n$ for $k \geq 1$, and $m, n \geq 0$ with $m+n \geq 1$. Note that τ is both multiplicative and divisive, Y is prime, the element XZ is primal and relatively prime to Y, $Y \nmid XZ$, and $Y(XZ) = X(YZ)$ is not a τ-atom. We can additionally arrange for D to have nice properties like being a UFD and so on by choosing R appropriately.

With that counterexample out of the way, we are ready to give our first abstraction of [6, Theorem 4.12].

Theorem 5.4.2. *Let Γ be a symmetric, unital, divisive, and divisible factorization system on a cancellative monoid H. Let a be a Γ-atom and b be Γ-prime. Then ab is a Γ-factorization or a Γ-atom.*

Proof. Assume ab has a nontrivial Γ-factorization $ab = c_1 \cdots c_n$. Because b is Γ-prime, it divides some c_i, say $c_1 = br$. Canceling yields $a = rc_2 \cdots c_n$, where the right-hand side is a Γ-factorization (with r possibly a unit) by divisiveness or by divisibility and the unital property. Because a is Γ-irreducible, we have $r \in H^\times$, $n = 2$, and $a \sim c_2$. Canceling now gives $b \sim c_1$, so $c_1c_2 \sim ba = ab$ are Γ-factorizations by the associate-preserving and symmetric properties. $\square$

Theorem 5.4.3. *Let Γ be a unital, divisive, and divisible factorization system on a monoid H.*

(1) *Any product of (completely) Γ-superprimal elements is (completely) Γ-superprimal.*

(2) *If Γ is additionally symmetric, refinable, and combinable, then any Γ-product of (completely) $|_\Gamma$-superprimal (resp., half $|_\Gamma$-superprimal) elements is (completely) $|_\Gamma$-superprimal (resp., half $|_\Gamma$-superprimal).*

Proof.

(1) Let a and b be Γ-superprimal, and let $c_1 \cdots c_n$ be any reduced Γ-factorization that ab divides. Since $b \mid c_1 \cdots c_n$, we have a reduced Γ-factorization $b = d_1 \cdots d_n$ (with some of the d_i's possibly 1) where each $d_i \mid c_i$, say $c_i = d_i x_i$. Canceling gives $a \mid x_1 \cdots x_n$, the latter being a Γ-factorization (with the units collected together) by divisiveness and the unital property. So we have a Γ-factorization $a = y_1 \cdots y_n$ (with some of the y_i's possibly 1), where each $y_i \mid x_i$, say $x_i = y_i z_i$. So $ab = (d_1 y_1) \cdots (d_n y_n)$ and each $d_i y_i \mid (d_i y_i) z_i = d_i x_i = c_i$. By Theorem 5.1.3, the proof of the "non-completely" case is complete.

Now assume that a and b are completely Γ-superprimal, and let $ab = f_1 \cdots f_k$ be any reduced Γ-factorization. We need to show that each f_i is Γ-superprimal. We have a reduced Γ-factorization $a = f'_1 \cdots f'_k$ (with some of the factors possibly 1) with each $f'_i \mid f_i$, say $f_i = f'_i g_i$. Canceling yields $b = g_1 \cdots g_k$, which is a Γ-factorization (collecting all the units together) by the divisive, divisible, and unital properties. Since a and b are completely Γ-superprimal, each nonunit f'_i and g_i are Γ-superprimal, so the product of all those elements is Γ-

superprimal by the previous paragraph, and ab is Γ-superprimal since associates of Γ-superprimals are Γ-superprimal.

(2) Assume Γ is symmetric, refinable, and combinable. In view of Theorem 5.1.7, combinability, and induction, we only need to show that any Γ-product ab of two (completely) $|_\Gamma$-superprimal elements is (completely) $|_\Gamma$-superprimal. Let $c_1 \cdots c_n$ be any reduced Γ-factorization that ab Γ-divides. Since Γ is refinable, unital, and associate-preserving, we have $b \mid_\Gamma c_1 \cdots c_n$, so there is a reduced Γ-factorization $b = d_1 \cdots d_n$ (with some of the d_i's possibly 1) where each nonunit $d_i \mid_\Gamma c_i$, say $c_i = d_i x_i$. (Set $x_i = c_i$ when $d_i = 1$.) Use these Γ-factorizations to Γ-refine $c_1 \cdots c_n$ and one of its Γ-factorizations with ab as a Γ-factor, and then cancel and apply the divisible and unital properties to see that $a \mid_\Gamma x_1 \cdots x_n$. So we have a Γ-factorization $a = y_1 \cdots y_n$ (with some of the y_i's possibly 1), where each nonunit $y_i \mid_\Gamma x_i$, say $x_i = y_i z_i$. (Set $z_i = x_i$ when $y_i = 1$.) So $ab = (d_1 y_1) \cdots (d_n y_n)$. By refinability and combinability, the factorizations $c_i = d_i x_i = d_i(y_i z_i) = d_i y_i z_i = (d_i y_i) z_i$ are Γ-factorizations, so each $d_i y_i \mid_\Gamma c_i$ and we have shown that ab is $|_\Gamma$-superprimal by Theorem 5.1.3.

If a and b are completely $|_\Gamma$-superprimal, then we can show that ab is completely $|_\Gamma$-superprimal by a minor modification to the second paragraph of the proof of part (1).

□

The following generalizes [21, Lemma 2.18(3)].

Corollary 5.4.4. *Let Γ be a unital, divisive, and divisible factorization system on a cancellative monoid H. Any product of two Γ-primes is Γ-prime or a Γ-factorization.*

Proof. Theorems 5.4.2, 5.4.3, and Corollary 5.2.2. $\square$

We note that the following result is one of only a handful of abstract factorization theorems that make use of the relational property; most theorems about τ-factorization can be proven just as easily for any unital, divisible, and normal factorization system.

Corollary 5.4.5. *Let H be a cancellative monoid and τ be a symmetric and divisive relation on $H^{\#}$. Then any product of τ-primes is (after a suitable reordering) a refinement of a τ-product of τ-primes.*

Proof. Let $\lambda a_1 \cdots a_n$ be any product of τ-primes that is not a τ-product. Reorder if necessary so that $a_1 \not\tau a_2$. Then $\lambda(a_1 a_2)a_2 \cdots a_n$ is a product of τ-primes by by Corollary 5.4.4, which by induction is a refinement of a τ-product of τ-primes. $\square$

We should remark that the above result only elaborates very slightly on [21, Lemma 4.9], which already showed that an element that, for τ symmetric and divisive, an element that can be written as a product of τ-primes can be written as a τ-product of τ-primes.

We end this thesis with a theorem collecting some of our main characterizations of Γ-UFM's, and a corollary showing how to apply these characterizations to obtain the results about unique factorization inheritance that we have long been promising. The following is a summary of the history of the theory of complete factorization inheritance. In [6] it was shown that a UFD is a τ-UFD for τ divisive. The proof was very involved, but a much simpler one was later discovered: the result is immediate from Corollary 5.4.5 and Theorem 5.3.5, since a prime is clearly τ-prime. Later, the thesis [21] extended unique factorization inheritance further by showing that a τ_2-UFD is a τ_1-UFD for divisive $\tau_1 \leq \tau_2$; the proof essentially adapted the one of [6], and, as expected, was even more difficult than the original. At this point, it would be natural to wonder if there could be a simplified proof analogous to the simple proof of a UFD being a τ-UFD, but Example 5.3.6 ruins all hope of such an approach working. A simplified proof of the result of [21] is given in [18, Theorem 4.4], where as an added bonus the divisive requirement on τ_2 was weakened to refinable and associate-preserving. (The divisive requirement on τ_1 cannot be similarly weakened, since the UFD $\mathbb{Z}$ is not a $\tau_{(2)}$-UFD even though $\tau_{(2)}$ is refinable and associate-preserving.) Our proof of the analogous Γ-factorization theorem will essentially be the same. Finally, as a side note, one interesting proof that a UFD is a Γ-UFD for Γ unital, associate-preserving, refinable, divisible, and divisive is given in [21, Theorem 6.16(1)]. A UFD is Schreier, and thus Γ-Schreier by Theorem 5.1.3. On the other hand, a UFD is Γ-atomic by Figure 4.6, and every Γ-atom is Γ-primal and hence Γ-prime, so the result now follows from Theorem 5.3.5.

Theorem 5.4.6. *Let Γ be a unital, associate-preserving, refinable, and divisible factorization system on a cancellative monoid H. The following are equivalent.*

(1) *The monoid H is a Γ-UFM.*

(2) *The monoid H is Γ-atomic and (half) $|_\Gamma$-pre-Schreier.*

(3) *The monoid H is Γ-atomic and every Γ-atom is (half) $|_\Gamma$-prime.*

Proof. (1) $\Leftrightarrow$ (3): Theorem 5.3.5. (2) $\Rightarrow$ (3): Corollary 5.2.2. (1) $\Rightarrow$ (2): Assume that H is a Γ-UFM, and let $\lambda a_1 \cdots a_m = \mu b_1 \cdots b_n$ be any Γ-factorizations of the same element. We need to show that each a_i has a Γ-factorization of the form $a_i = b'_{i_1} \cdots b'_{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq n$ and each $b'_{i_j} \mid_\Gamma b_{i_j}$. This is accomplished by Γ-refining the two Γ-factorizations into Γ-atomic factorizations and applying uniqueness. $\square$

Corollary 5.4.7. *Let $\Gamma \subseteq \Gamma'$ be unital, associate-preserving, refinable, and divisible factorization systems on a cancellative monoid H. Additionally assume that Γ is divisive. If H is a Γ'-UFM, then it is a Γ-UFM.*

Proof. Assume H is a Γ'-UFM. Then it is half $|_{\Gamma'}$-pre-Schreier (hence half $|_\Gamma$-pre-Schreier) by Theorem 5.4.6. From Figure 4.6, we see that H is also Γ-atomic, so it is a Γ-UFM by Theorem 5.4.6. $\square$

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