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Extension of positive definite functions

Robert Niedzialomski
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EXTENSION OF POSITIVE DEFINITE FUNCTIONS

by

Robert Niedzialomski

<u>An Abstract</u>

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

May 2013

Thesis Supervisors: Professor Palle Jorgensen
Professor Lihe Wang

ABSTRACT

Let $\Omega \subset \mathbb{R}^n$ be an open and connected subset of $\mathbb{R}^n$ and let $F\colon \Omega - \Omega \to \mathbb{C}$, where $\Omega - \Omega = \{x - y\colon x, y \in \Omega\}$, be a continuous positive definite function. We give necessary and sufficient conditions for F to have an extension to a continuous and positive definite function defined on the entire Euclidean space $\mathbb{R}^n$. The conditions are formulated in terms of strong commutativity of a system of certain unbounded selfadjoint operators defined on a Hilbert space associated to our function.

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Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the May 2013 graduation.

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ABSTRACT

Let $\Omega \subset \mathbb{R}^n$ be an open and connected subset of $\mathbb{R}^n$ and let $F\colon \Omega - \Omega \to \mathbb{C}$, where $\Omega - \Omega = \{x - y\colon x, y \in \Omega\}$, be a continuous positive definite function. We give necessary and sufficient conditions for F to have an extension to a continuous and positive definite function defined on the entire Euclidean space $\mathbb{R}^n$. The conditions are formulated in terms of strong commutativity of a system of certain unbounded selfadjoint operators defined on a Hilbert space associated to our function.

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CHAPTER 1
INTRODUCTION

Positive definite functions play an important role in many aspects of pure and applied mathematics. They are of interest in probability theory, stochastic processes, representation theory, harmonic analysis, complex analysis, approximation theory, information theory, and machine learning. In the next chapter, we will give many examples of positive definite functions and positive definite kernels to, at least partially, validate the above statement.

The approach to introduce positive definite functions, that we take here, is to look at them as a subcollection of the collection of positive definite kernels. In this way, we naturally arrive at the notion of a reproducing kernel Hilbert space. The study and the construction of the reproducing kernel Hilbert space associated to a positive definite kernel will be of great importance for us.

Let us introduce the notion of a positive definite function. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $F\colon \Omega - \Omega \to \mathbb{C}$ be a function, where $\Omega - \Omega = \{x - y\colon x, y \in \Omega\}$. We say that F is positive definite if for any $x_1, \ldots, x_m \in \Omega$ and any $c_1, \ldots, c_m \in \mathbb{C}$

$$\sum_{j,k=1}^{m} F(x_j - x_k) c_j \overline{c_k} \geq 0.$$

Remark 1.0.1. The assumption that Ω is an open set is not necessary for the definition, and the definition works for any set $\Omega \subset \mathbb{R}^n$, or even, for example, any subset of an abelian group. See the next chapter.

The this thesis we consider the extension problem for continuous positive definite function defined on an open subset of the Euclidean space $\mathbb{R}^n$. By extension we always mean an extension to a continuous and positive definite function defined on the entire space $\mathbb{R}^n$. It is worth mentioning here that, according to a theorem of Bochner, any continuous positive definite function defined on the entire Euclidean space, call it $F\colon \mathbb{R}^n \to \mathbb{C}$, is of the form

$$F(x) = \widehat{\mu}(x) = \int_{\mathbb{R}^n} e^{it\cdot x} d\mu(t),$$

where μ is a finite, positive Borel measure on $\mathbb{R}^n$ and $\widehat{\mu}$ the Fourier transform of μ, also called the characteristic function of μ. Our main theorem gives necessary and sufficient conditions for F to have an extension. The conditions are formulated in terms of strong commutativity of some certain selfadjoint operators defined on a reproducing kernel Hilbert space associated to our positive definite function. Therefore we relate two extension problems, which on the surface might seem to be unrelated. One is the problem of extending positive definite functions defined on an open subset of $\mathbb{R}^n$, and the second one is the problem of existence of strongly commuting selfadjoint extensions of n Hermitian operators that initially commute on a dense domain in a Hilbert space. In the earlier investigations, the question of existence of strongly commuting selfadjoint extensions played a role in harmonic analysis. For example, in [Fug], Fuglede showed that existence of certain strongly commuting selfadjoint partial differential operators is directly related to the following two existence questions. The first one is existence of geometric tiling by translations, and the second one is existence of an orthonormal basis for $L^2(\Omega)$ consisting of Fourier frequencies.

As we have already mentioned above, positive definite functions constitute a big class of positive definite kernels. Let us recall that a function $K\colon X\times X\to\mathbb{C}$, where X is any set, is called a positive definite kernel if for any points $x_1,\ldots,x_m\in X$ and any $c_1,\ldots,c_m\in\mathbb{C}$ we have that $\sum_{j,k=1}^m K(x_j,x_k)c_j\overline{c_k}\geq 0$. Therefore we could say that a function $F\colon\Omega-\Omega\to\mathbb{C}$, where $\Omega\subset\mathbb{R}^n$ is an open set, is positive definite if $K_F\colon\Omega\times\Omega\to\mathbb{C}$ given by $K(x,y)=F(x-y)$ is a positive definite kernel.

One main result is motivated, in part, by recent development in the study of Gaussian stochastic processes, see [AJL], [AJ]. Recall that the covariance function of a stationary stochastic process in a positive definite function (see example 2.1.15), and that every positive definite kernel defined on a set $S\times S$, where S is any set, is the covariance function of a Gaussian stochastic process indexed by S ([PS], see also [P1, P2, PV]). Therefore problem of extending positive definite functions is related to possibility of extending stochastic processes.

Let us precisely state the main theorem of this thesis. Let $\Omega\subset\mathbb{R}^n$ be an open set. For $\varphi\in C_0^\infty(\Omega)$, where $C_0^\infty(\Omega)$ denotes the space of smooth complex-valued functions with supports contained in Ω, we define a function $F_\varphi\colon\Omega\to\mathbb{C}$ by

$$F_\varphi(x)=\int_\Omega F(x-y)\varphi(y)dy,$$

and for two functions $\varphi,\psi\in C_0^\infty(\Omega)$ we put

$$\langle F_\varphi,F_\psi\rangle=\int_\Omega\int_\Omega F(y-x)\varphi(x)\overline{\psi(y)}dxdy.$$

We define

$$\mathcal{W}=\{F_\varphi\colon\varphi\in C_0^\infty(\Omega)\}$$

Then $\mathcal{W}$ is a complex vector space and $\langle\cdot,\cdot\rangle$ is a complex inner-product on $\mathcal{W}$. We complete the complex inner-product space $(\mathcal{W}, \langle\cdot,\cdot\rangle)$ to get a complex Hilbert space $\mathcal{H}_F$. For $j = 1, \dots, n$ we consider densely defined operators $-i\frac{\partial}{\partial x_j} \colon \mathcal{H}_F \to \mathcal{H}_F$, with the common dense domain $\mathcal{W}$, given by

$$\left(-i\frac{\partial}{\partial x_j}\right) F_\varphi = -iF_{\frac{\partial \varphi}{\partial x_j}},$$

where, as we mentioned above, $F_\varphi \in \mathcal{W}$. The main theorem that we prove is the following.

Main Theorem. Let $F \colon \Omega - \Omega \to \mathbb{C}$ be a continuous positive definite function, where Ω is an open and connected subset of $\mathbb{R}^n$. Then there exists an extension of F to a continuous positive definite function on the entire Euclidean space $\mathbb{R}^n$ if and only if there exist selfadjoint strongly commuting extensions of the operators $-i\frac{\partial}{\partial x_j}$, $j = 1, \dots, n$.

Let's recall that strong commutativity is understood in the following sense. If A is a densely defined selfadjoint operator on a Hilbert space H, then there is a unique spectral measure i.e. a Borel "measure" E on $\mathbb{R}$ with values in the space of orthogonal projections on H such that

$$A = \int_{\mathbb{R}} z dE(z).$$

If we have a finite collection $A_1, \dots, A_n$ of densely defined selfadjoint operators on a Hilbert space H and if $E_1, \dots, E_n$ are the associated spectral measures, then these operators strongly commute if for any Borel sets $M, N \subset \mathbb{R}$ and any indices $i, j = 1, \dots, n$ operators $E_i(M)$ and $E_j(N)$ commute i.e.

$$E_i(M)E_j(N) = E_j(N)E_i(M).$$

Let us shortly recall the main known results concerning extension of continuous positive definite functions. Before we do that let us mention that if $\Omega \subset \mathbb{R}^n$ is a symmetric ($x \in \Omega \Rightarrow -x \in \Omega\}$) and convex set, then $\Omega - \Omega = 2\Omega = \{2x \colon x \in \Omega\}$. Krein ([Kr]) studies this extension problem in the case when Ω is a bounded interval $(-r, r)$, $r > 0$. He proves that a positive definite extension always exists, but is not necessarily unique. Later Devinatz ([De]) gives sufficient conditions for existence of extensions when the domain is a rectangle $(-r_1, r_1) \times (-r_2, r_2) \subset \mathbb{R}^2$. In [R2] Rudin extends the result of M. Krein and proves the following. Every radial positive definite function defined on a ball $B(0, r) = \{x \in \mathbb{R}^n \colon |x| < r\}$ extends to a radial positive definite function defined on the entire space $\mathbb{R}^n$. Nussbaum (see [N1],[N2]) gives a characterization (similar to Bochner's theorem) of continuous positive definite functions relative to the orthogonal group obtaining the result of Rudin as a corollary. Moreover, in [R1], the author gives an example of a continuous positive definite function defined on the unit cube $I^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \colon |x_i| < 1, i = 1, \ldots, n\}$ that cannot be extended. Therefore in order to be guaranteed existence of an extension we have to impose some symmetry conditions on the function or on the domain of the function. For a nice and brief overview of positive definite functions and of the extension problem (not only in the class of continuous functions) we refer the reader to survey articles [St] and [S].

This thesis is organized as follows. In the first chapter we gather some well-known facts about positive definite kernels, positive definite functions, and reproducing kernel Hilbert spaces. We focus on examples to give the reader good feeling of how positive functions behave and to convince him/her that these functions play important role in many subjects of mathematics. The presentation is highly influenced by the references [A], [BCR], [H], [J2011], [P], and [Sz]. The second chapter is devoted to stating and proving the main theorem of the thesis. We start with an integral characterization of continuous positive definite functions defined on an open subset of the Euclidean space $\mathbb{R}^n$ (theorem 3.0.15). This fact is well-know. However, as we were not able to find a reference with a proof, we provide our own justification. Than we fix a positive definite function defined on an open subset of $\mathbb{R}^n$ and using the above characterization we associate a reproducing kernel Hilbert space to this function. We study this Hilbert space, we state necessary and sufficient conditions for existence of an extension of our positive definite function, and we follow with a proof (see [JN]). Finally, we derive the result of Krein using our theorem and we give an example showing that the assumption of connectedness in the main theorem is crucial. In the last chapter we talk about possible future research projects connected with the extension problem for positive definite functions, and we show that the Hilbert space $\mathcal{H}_F$ is isometrically isomorphic to the standard reproducing kernel Hilbert space associated to our positive function F. In the appendix we collect some facts about spectral measures; the tool that we rely heavily on when proving the main theorem of the thesis.

CHAPTER 2
POSITIVE DEFINITE KERNELS AND FUNCTIONS, REPRODUCING KERNEL HILBERT SPACES

2.1 Positive definite kernels

Let X be a set and let $K\colon X \times X \to \mathbb{C}$ be a mapping. We say that K is a positive definite kernel if for any $m \in \mathbb{N}$ and any $x_1, \ldots, x_m \in X$ the matrix $[K(x_j, x_k)]_{j,k=1}^{m}$ is positive definite i.e. for any complex numbers $c_1, \ldots, c_m \in \mathbb{C}$ we have that

$$\sum_{j,k=1}^{m} K(x_j, x_k) c_j \overline{c_k} \geq 0.$$

We recall that if $A = [a_{jk}]$ is a square matrix with entries being complex numbers which is positive definite, then it is Hermitian ($a_{jk} = \overline{a_{kj}}$) and the eigenvalues of A are real non-negative numbers.

Let $K\colon X \times X \to \mathbb{C}$ be a positive definite kernel. Then for any $x \in X$ we get that $K(x,x) \geq 0$. Moreover, by the above remark, for any $x, y \in X$ the matrix

$$\begin{pmatrix} K(x,x) & K(x,y) \\ K(y,x) & K(y,y) \end{pmatrix}$$

is Hermitian with positive determinant. Thus

$$K(x,y) = \overline{K(y,x)} \tag{2.1}$$

and

$$|K(x,y)|^2 \leq K(x,x)K(y,y) \tag{2.2}$$

for any $x, y \in X$.

The following theorem of Moore and Aronszajn holds true.

Theorem 2.1.1. *(Moore, Aronszajn) Let X be a set and let $K\colon X \times X \to \mathbb{C}$ be a mapping. Then K is a positive definite kernel if and only if there exists a complex Hilbert space H and there exists a mapping $T\colon X \to H$ such that for any $x, y \in X$*

$$K(x,y) = \langle Tx, Ty\rangle. \tag{2.3}$$

Proof. ($\Leftarrow$) Let $x_1, \ldots, x_m \in X$ and let $c_1, \ldots, c_m \in \mathbb{C}$. Then

$$\sum_{j,k=1}^{m} K(x_j, x_k) c_j \overline{c_k} = \sum_{j,k=1}^{m} \langle Tx_j, Tx_k\rangle c_j \overline{c_k} = \| \sum_{j=1}^{m} c_j T x_j \|^2 \geq 0.$$

($\Rightarrow$) Let $K\colon X \times X \to \mathbb{C}$ be a positive definite kernel. For each $x \in X$ denote by φ_x the function from X to $\mathbb{C}$ such that $\varphi_x(y) = 0$ for $y \neq x$ and $\varphi_x(x) = 1$. Let V be the complex vector space all functions that are finite combinations of functions φ_x, $x \in X$, i.e

$$V = \{\varphi = \sum_{j=1}^{m} a_j \varphi_{x_j} \colon m \in \mathbb{N}, x_1, \ldots, x_m \in X, a_1, \ldots, a_m \in \mathbb{C}\}.$$

For two elements $\varphi, \psi \in V$ we define

$$\langle \varphi, \psi\rangle = \sum_{x,y \in X} K(x,y)\varphi(x)\overline{\psi(y)}.$$

In other words, if $\varphi = \sum_{j=1}^{m} a_j \varphi_{x_j} \in V$ and $\psi = \sum_{j=1}^{m} b_j \varphi_{x_j} \in V$, with $a_1, \ldots, a_m \in \mathbb{C}$ and $x_1, \ldots, x_m \in X$, then

$$\langle \varphi, \psi\rangle = \sum_{j,k=1}^{m} a_j \overline{b_k} K(x_j, x_k).$$

Then $\langle \cdot, \cdot\rangle\colon V \times V \to \mathbb{C}$ is a mapping that is complex linear in the first slot, conjugate complex linear in the second slot, and is positive definite in the sense that $\langle \varphi, \varphi\rangle \geq 0$

for any $\varphi \in V$. Therefore, if we define U to be the subspace of V spanned by those $\varphi \in V$ such that $\langle \varphi, \varphi \rangle = 0$ and if we set $W = V/U$ to be the quotient space of V by U, then $\langle \cdot, \cdot \rangle$ "projected" to W gives W a structure of a complex pre-Hilbert space. The completion of this pre-Hilbert space W is a Hilbert space, which we denote H. The mapping $T\colon X \to H$ given by $Tx = [\varphi_x]$ is a desired mapping i.e. $K(x,y) = \langle Tx, Ty \rangle$ for any $x, y \in X$. $\square$

Remark 2.1.2. Let X be a set, let $K\colon X \times X \to \mathbb{C}$ be a positive definite kernel, let H be a complex Hilbert space, and let $T\colon X \to H$ be a mapping such that (2.3) holds. Let $\{e_s\colon s \in S\}$ be an orthonormal basis (ONB) of H. Then for any $x, y \in X$

$$K(x,y) = \sum_{s \in S} \langle Tx, e_s \rangle \overline{\langle Ty, e_s \rangle}.$$

Therefore, if we define $\varphi_s\colon X \to \mathbb{C}$ by

$$\varphi_s(x) = \langle Tx, e_s \rangle,$$

then we can write

$$K(x,y) = \sum_{s \in S} \varphi_s(x) \overline{\varphi_s(y)}$$

for any $x, y \in X$.

We provide some examples of positive definite kernels.

Example 2.1.3. Let X be a set and let $f\colon X \to \mathbb{C}$ be a function. We define $K\colon X \times X \to \mathbb{C}$ by

$$K(x,y) = f(x)\overline{f(y)} \tag{2.4}$$

Then K is a positive definite kernel. Indeed, let $x_1, \ldots, x_m \in X$ and let $c_1, \ldots, c_m \in \mathbb{C}$. We have

$$\sum_{j,k=1}^{m} K(x_j, x_k) c_j \overline{c_k} = \sum_{j,k=1}^{m} f(x_j) c_j \overline{f(x_k)} \overline{c_k} = \left| \sum_{j=1}^{m} f(x_j) c_j \right|^2 \geq 0$$

Example 2.1.4. Let X be a set, let $K \colon X \times X \to \mathbb{C}$ be a positive definite kernel, and let $f \colon X \to \mathbb{C}$ be a function. Then $L \colon X \times X \to \mathbb{C}$ given by

$$L(x, y) = f(x) K(x, y) \overline{f(y)}$$

is a positive definite kernel. Take $x_1, \ldots, x_m \in X$ and $c_1, \ldots, c_m \in \mathbb{C}$ and compute

$$\sum_{j,k=1}^{m} L(x_j, x_k) c_j \overline{c_k} = \sum_{j,k=1}^{m} K(x_j, x_k) c_j f(x_j) \overline{c_k f(x_k)} \geq 0$$

Example 2.1.5. Let X be a set, let $K, L \colon X \times X \to \mathbb{C}$ be two positive definite kernels, and let $a, b > 0$. Then the mapping

$$aK + bL \colon X \times X \to \mathbb{C}$$

is a positive definite kernel.

Example 2.1.6. Let X be a set. If $f_1, \ldots, f_n \colon X \to \mathbb{C}$ are functions, then $M \colon X \times X \to \mathbb{C}$ given by

$$M(x, y) = f_1(x) \overline{f_1(y)} + \ldots + f_n(x) \overline{f_n(y)}$$

is a positive definite kernel.

Example 2.1.7. The pointwise limit of positive definite kernels is a positive definite kernel. More precisely, if X is a set and if $K_n \colon X \times X \to \mathbb{C}$, $n \in \mathbb{N}$, is a positive definite kernel such that for any $x, y \in X$ the limit

$$K(x,y)\colon = \lim_{n\to\infty} K_n(x,y)$$

exists, then $K\colon X \times X \to \mathbb{C}$ is a positive definite kernel.

Example 2.1.8. Let $(M, \mathfrak{m}, \mu)$ be a measure space, and let $\{k_\omega\colon X \times X \to \mathbb{C}\colon \omega \in M\}$ be a collection of positive definite kernels such that $M \ni \omega \mapsto k_\omega(x,y) \in \mathbb{C}$ is a μ-integrable function; that is $k.(x,y) \in L^1(d\mu)$; for any $x, y \in X$. Then

$$k(x,y) = \int_M k_\omega(x,y) d\mu(\omega),$$

where $x, y \in X$, is a positive definite kernel.

Example 2.1.9. Let $K, L\colon X \times X \to \mathbb{C}$ be two positive definite kernels. Then the product

$$K \cdot L\colon X \times X \to \mathbb{C}$$

is a positive definite kernel. If follows from the following well-known fact, which is usually attributed to Schur. Let $A = [a_{jk}]$ and $B = [a_{jk}]$ be two positive definite matrices. We define a matrix $A \star B$ by

$$(A \star B)_{jk} = a_{jk} b_{jk}.$$

Then $A \star B$ is a positive definite matrix.

Example 2.1.10. Let X be a set and let $K\colon X \times X \to \mathbb{C}$ be a positive definite kernel and let $f(z) = \sum_{n=1}^{\infty} a_n z^n$, with $a_n \geq 0$, be an analytic function in a disk

$C(R) = \{z \in \mathbb{C} \colon |z| < R\}$, where $R \in (0, \infty]$. Suppose that $|K(x,y)| < R$ for $x, y \in X$. Then

$$f \circ K \colon X \times X \to \mathbb{C}$$

is a positive definite kernel. In particular, if $L \colon X \times X \to \mathbb{C}$ is a positive definite kernel, then so is $e^L \colon X \times X \to \mathbb{C}$.

Example 2.1.11. Let X be a set and let $K \colon X \times X \to \mathbb{R}$ be a mapping. Then K is a positive definite kernel if and only if it is symmetric; meaning that $K(x,y) = K(y,x)$ for any $x, y \in X$; and for any $x_1, \ldots, x_m \in X$ and any $c_1, \ldots, c_m \in \mathbb{R}$ the following condition holds

$$\sum_{j,k=1}^{m} K(x_j, x_k) c_j c_k \geq 0 \tag{2.5}$$

($\Rightarrow$) Clear.

($\Leftarrow$) Let $K \colon X \times X \to \mathbb{R}$ be a symmetric mapping satisfying (2.5). Let $x_1, \ldots, x_m \in X$ and let $c_1 = a_1 + ib_1, \ldots, c_m = a_m + ib_m \in \mathbb{C}$. Then

$$\begin{aligned}\sum_{j,k=1}^{m} K(x_j, x_k) c_j \overline{c_k} = &\sum_{j,k=1}^{m} K(x_j, x_k)(a_j a_k + b_j b_k) \\ &+ i \sum_{j,k=1}^{m} K(x_j, x_k)(b_j a_k - a_j b_k)\end{aligned}$$

Since the last summation vanishes (by the symmetry of K), we are done.

Example 2.1.12. Let X be set, Y a subset of X, and let $K \colon X \times X \to \mathbb{C}$ be a positive definite kernel. Then K restricted to $Y \times Y$ is a positive definite kernel.

Example 2.1.13. Let X be a set, let $(M, \mathfrak{m}, \mu)$ be a measure space, and let $k \colon X \times M \to \mathbb{C}$ be a function such that for any $x \in X$ the function $k(x, \cdot) \colon M \to \mathbb{C}$ is

μ-square integrable i.e. $k(x,\cdot) \in L^2(d\mu)$. We define a map $K\colon X \times X \to \mathbb{C}$ as follows

$$K(x,y) = \int_M k(x,s)\overline{k(y,s)}d\mu(s). \tag{2.6}$$

Then K is a positive definite kernel. Let $x_1, \dots, x_m \in X, c_1, \dots, c_m \in \mathbb{C}$. Then

$$\begin{aligned}\sum_{j,k=1}^{m} K(x_j,x_k)c_j\overline{c_k} &= \int_M \sum_{j,k=1}^{m} k(x_j,s)\overline{k(x_k,s)}c_j\overline{c_k}d\mu(s)\\ &= \int_M \left|\sum_{j=1}^{m} k(x_j,s)c_j\right|^2 d\mu(s) \geq 0.\end{aligned}$$

We note that any positive definite kernel can be written in the form (2.6). Let $L\colon X \times X \to \mathbb{C}$ be a positive definite kernel. Then there is a complex Hilbert space H and there is a mapping $T\colon X \to H$ such that $L(x,y) = \langle Tx, Ty\rangle$ for any $x, y \in X$. Therefore, if $\{e_s\colon s \in S\}$ is an ONB of H, and if we put $l(x,s) = \langle Tx, e_s\rangle$ for $x \in X$ and $s \in S$, then

$$L(x,y) = \sum_{s\in S} l(x,s)\overline{l(x,s)}$$

for any $x, y \in X$. In particular, if we define $(M, \mathfrak{m}, \mu)$ by $M = S$, $\mathfrak{m} = 2^S$ and $\mu\colon \mathfrak{m} \to [0,\infty]$ is the counting measure, then we get that

$$L(x,y) = \int_S l(x,s)\overline{l(x,s)}d\mu(s).$$

Example 2.1.14. (Brownian motion) Let $(\Omega, \mathfrak{m}, \mu)$ be a probability space and let $T \subset \mathbb{R}$. A stochastic process is a measurable function $X\colon T \times \Omega \to \mathbb{R}$. An example of a stochastic process is a Brownian motion. A Brownian motion is a stochastic process $B\colon [0,\infty) \times \Omega \to \mathbb{R}$ such that

1. $\mu\{\omega \in \Omega\colon B(0,\omega) = 0\} = 1$,

2. For ant $0 \leq s < t$ the random variable $B(t) - B(s)$ is normally distributed with mean 0 and variance $t - s$; this means that the push-forward measure $(B(t) - B(s))\sharp\mu$ of the measure μ by the map $B(t) - B(s)$ is absolutely continuous with respect to the Lebesgue measure and the Radon-Nikodym derivative $d[(B(t) - B(s))\sharp\mu]/dx$ satisfies

$$\frac{d[(B(t) - B(s))\sharp\mu]}{dx} = \frac{1}{\sqrt{2\pi(t-s)}} e^{-x^2/2(t-s)},$$

where the push-forward measure $(B(t) - B(s))\sharp\mu$ is defined as follows: for any Borel set $B \subset \mathbb{R}^n$

$$(B(t) - B(s))\sharp\mu(B) = \mu((B(t) - B(s))^{-1}(B)).$$

3. For any partition $0 \leq t_1 < \ldots < t_m$ the random variables

$$B(t_1), B(t_2) - B(t_1), \ldots, B(t_m) - B(t_{m-1})$$

are independent,

4. $\mu\{\omega \in \Omega \colon B(\cdot, \omega) \text{ is continuous}\} = 1$.

If $B \colon [0, \infty) \times \Omega \to \mathbb{R}$ is a Brownian motion, then it is well-known that

$$\int_\Omega B(t, \omega) B(s, \omega) d\mu(\omega) = \min(s, t).$$

This shows that the function

$$K(s, t) = \min(s, t),$$

where $s, t \in [0, \infty)$, is a positive definite kernel.

Example 2.1.15. (Covariance Function) Let $(\Omega, \mathfrak{m}, \mu)$ be a probability space and let $T \subset \mathbb{R}^n$. Let X be a stochastic process on T i.e. X is a measurable function $X\colon T \times \Omega \to \mathbb{R}$. Assume that for any $s \in T$ we have that $X(s) \in L^2(d\mu)$. We define the expectation $EX(s)$ of X at s to be the quantity

$$EX(s) = \int_\Omega X(s) d\mu.$$

The covariance function $C\colon T \times T \to \mathbb{R}$ is given by

$$C(s,t) = E((X(s) - EX(s))(X(t) - EX(t))).$$

Hence the covariance function is a positive definite kernel. We say that our stochastic process T is stationary if there exists a function $F\colon T - T \to \mathbb{R}$ such that for $s, t \in T$ we have

$$C(s,t) = F(s-t).$$

Then necessarily F is a positive definite function. We note that if $0 \in T$, then for any $s \in T$ we get that $s \in T - T$ and $F(s) = F(s-0) = C(s,0)$. Therefore F is uniquely determined by C. Moreover, for $s, t \in T$ and $h \in \mathbb{R}^n$ such that $s + h, t + h \in T$ we obtain

$$C(s+h, t+h) = F(s+h-t-h) = F(s-t) = C(s,t).$$

We see that extension of stationary stochastic processes is closely related to extension problem for positive definite functions.

Remark 2.1.16. (**Negative definite kernels**) There is a notion of a negative definite kernel, which is an analogue of the notion of a positive definite kernel. Let X be a set and let $N\colon X \times X \to \mathbb{C}$ be a mapping. We say that N is a negative definite kernel if $N(x,y) = \overline{N(y,x)}$ for $x, y \in X$ and if for any $x_1, \ldots, x_m \in X$ and any $c_1, \ldots, c_m \in \mathbb{C}$ with $c_1 + \ldots + c_m = 0$ the following holds

$$\sum_{j,k=1}^{m} N(x_j, x_k) c_j \overline{c_k} \leq 0.$$

An example of a negative definite kernel is the norm squared associated to an inner product. More precisely, if H is a Hilbert space, then the mapping $N\colon H \times H \to \mathbb{C}$ given by

$$N(x,y) = \|x - y\|^2$$

is a negative definite kernel. Let $x_1, \ldots, x_m \in H$ and let $c_1, \ldots, c_m \in \mathbb{C}$ with $c_1 + \ldots + c_m = 0$. Then

$$\begin{aligned}\sum_{j,k=1}^{m} N(x_j, x_k) c_j \overline{c_k} &= \sum_{j,k=1}^{m} \|x_j - x_k\|^2 c_j \overline{c_k} \\ &= (\sum_{j=1}^{m} \|x_j\|^2 c_j)(\sum_{k=1}^{m} \overline{c_k}) - 2\|\sum_{j=1}^{m} c_j x_j\|^2 + (\sum_{j=1}^{m} c_j)(\sum_{k=1}^{m} \|x_k\|^2 \overline{c_k}) \\ &= -2\|\sum_{j=1}^{m} c_j x_j\|^2 \leq 0.\end{aligned}$$

Hence the notion of a negative definite kernel is closely related to the problem of existence of isometric embeddings of metric spaces into Hilbert spaces. The following holds.

Theorem 2.1.17. *(Schoenberg) Let (X, d) be a metric space. Then there exists a Hilbert space H and an isometric embedding $I\colon X \to H$ (by isometric embedding we mean $d(x,y) = \|Ix - Iy\|$ for $x, y \in X$) if and only if the square of the distance function i.e the mapping $d^2\colon X \times X \to \mathbb{R} \subset \mathbb{C}$, is a negative definite kernel.*

Moreover there is a relation between positive definite and negative definite kernels.

Theorem 2.1.18. *A kernel N is negative definite if and only if for any $t > 0$ the kernel e^{-tN} is positive definite.*

For proofs of above theorems and for more on negative definite kernels we encourage the reader to look at [BCR], [BL], and [P].

2.2 Positive definite functions

Let $(X, +)$ be an abelian group and let $Z \subset X$ be a subset of X. Let $f \colon Z - Z \to \mathbb{C}$ be a function, where

$$Z - Z = \{x - y \in X \colon x, y \in Z\}.$$

We note that $0 \in Z - Z$ and that $Z - Z$ is a symmetric subset of X i.e. $z \in Z - Z$ implies that $-z \in Z - Z$. We say that f is a positive definite function if the mapping

$$K(x, y) = f(x - y), \quad x, y \in Z,$$

is a positive definite kernel.

Let $f \colon Z - Z \to \mathbb{C}$ be a positive definite function. Then, using the properties of positive definite kernels, we obtain that $f(0) \geq 0$ and that for any $z \in Z - Z$

$$|f(z)| \leq f(0) \quad \text{and} \quad f(-z) = \overline{f(z)}. \tag{2.7}$$

Moreover, there exists a complex Hilbert space H and a mapping $T \colon Z \to H$ such that for any $x, y \in Z$

$$f(x - y) = \langle Tx, Ty \rangle.$$

Hence $f(0) = \|Tx\|^2$ for any $x \in Z$. Also, for any $x, y, a \in Z$ we obtain

$$\begin{aligned} |f(x-a) - f(y-a)|^2 &= |\langle Tx - Ty, Ta\rangle|^2 \\ &\leq \|Ta\|^2 \|Tx - Ty\|^2 \\ &= 2f(0)(f(0) - \mathrm{Re} f(x-y)). \end{aligned}$$

We deduce the following.

Theorem 2.2.1. *Let $Z \subset \mathbb{R}^n$ be an open set and let $F\colon Z - Z \to \mathbb{C}$ be a positive definite function. If the real part of F is continuous at 0, the F is continuous everywhere.*

We give some examples of positive definite functions (compare with examples of positive definite kernels).

Example 2.2.2. The product of positive definite functions is a positive definite function.

Example 2.2.3. If f, g are two positive definite functions and $a, b \geq 0$, then $af + bg$ is a positive definite function.

Example 2.2.4. If $(f_n)_{n\in\mathbb{N}}$ is a sequence of positive definite functions, then the pointwise limit $f = \lim_{n\to\infty} f_n$, if it exists, is a positive definite function.

Example 2.2.5. If $f\colon Z - Z \to \mathbb{C}$ is a positive definite function, then so is $\overline{f}$. Indeed, let $x_1, \ldots, x_m \in Z$ and let $c_1, \ldots, c_m \in \mathbb{C}$. Then by (2.7)

$$\sum_{j,k=1}^{m} \overline{f}(x_j - x_k) c_j \overline{c_k} = \sum_{j,k=1}^{m} f(x_k - x_j) \overline{c_k} \cdot \overline{\overline{c_j}} \geq 0.$$

Since f and $\overline{f}$ are positive definite, we conclude that $|f|^2$ and Ref are positive definite functions. We just write $|f|^2 = f\overline{f}$ and Re$f = (1/2)(f + \overline{f})$.

Example 2.2.6. The function $f(x) = e^{i\langle t,x\rangle}$, where $x \in \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ denotes the standard inner-product in $\mathbb{R}^n$, is positive definite for any fixed $t \in \mathbb{R}^n$. The computation goes as follows. Let $x_1, \ldots, x_m \in \mathbb{R}^n$ and let $c_1, \ldots, c_m \in \mathbb{C}$. Then

$$\begin{aligned}\sum_{j,k=1}^{m} f(x_j - x_k)c_j\overline{c_k} &= \sum_{j,k=1}^{m} e^{i\langle x_j - x_k, t\rangle} c_j\overline{c_k} \\ &= \sum_{j,k=1}^{m} c_j e^{i\langle x_j, t\rangle}\overline{c_k e^{i\langle x_k, t\rangle}} = \left|\sum_{j=1}^{m} c_j e^{i\langle x_j, t\rangle}\right|^2 \geq 0.\end{aligned}$$

Example 2.2.7. (Theorem of Bochner) Let $M(\mathbb{R}^n)$ be the set of all finite positive Borel measures on $\mathbb{R}^n$. For a measure $\mu \in M(\mathbb{R}^n)$ we define its Fourier transform $\widehat{\mu}\colon \mathbb{R}^n \to \mathbb{C}$ by

$$\widehat{\mu}(t) = \int_{\mathbb{R}^n} e^{i\langle t,x\rangle} d\mu(x).$$

Then $\widehat{\mu}$ is a positive definite and bounded continuous function. It is clear that $\widehat{\mu}$ is bounded; for any $t \in \mathbb{R}^n$

$$|\widehat{\mu}(t)| \leq \widehat{\mu}(0) = \mu(\mathbb{R}^n);$$

and it is well-known that $\widehat{\mu}$ is continuous. The computation that $\widehat{\mu}$ is positive is similar to the computation in the previous example. A theorem of Bochner says that any continuous and bounded positive definite function defined on the Euclidean space $\mathbb{R}^n$ is the Fourier transform of some finite positive Borel measure on $\mathbb{R}^n$ (see [Boch], [BW], and [Ka]).

Example 2.2.8. The function $f(x) = \cos x$, $x \in \mathbb{R}$, is a positive definite function, since it is the Fourier transform of the measure $\mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$. The fact that f is a positive definite function follows, also, directly from the definition. For $x_1, \dots, x_m \in \mathbb{R}$ and $c_1, \dots, c_m \in \mathbb{C}$ we have

$$\begin{aligned}\sum_{j,k=1}^{m} \cos(x_j - x_k) c_j \overline{c_k} &= \sum_{j,k=1}^{m} (\cos x_j \cos x_k + \sin x_j \sin x_k) c_j \overline{c_k} \\ &= \left| \sum_{j=1}^{m} c_j \cos x_j \right|^2 + \left| \sum_{j=1}^{m} c_j \sin x_j \right|^2 \geq 0.\end{aligned}$$

Example 2.2.9. (Gaussian Distributions) The function

$$\rho(x) = e^{-|x|^2},$$

where $x \in \mathbb{R}^n$, is positive definite. This follows from the fact that ρ is the Fourier transform of the measure defined by the density function $\mathbb{R}^n \ni x \mapsto e^{-|x|^2/4} \in \mathbb{R}$ (see also a theorem of Polya below).

Example 2.2.10. The function

$$\theta(x) = \frac{\sin x}{x},$$

where $x \in \mathbb{R}$, is positive definite. Indeed, let ν be the uniformly distributed probability measure on the interval $[-1, 1]$. Then $\widehat{\nu} = \theta$. Thus θ is a positive definite function.

Example 2.2.11. Let H be a complex Hilbert space and let $(U_t)_{t \in \mathbb{R}}$ be a one-parameter group of unitary operators on H i.e. each operator U_t, $t \in \mathbb{R}$, is unitary, $U_0 = I$ and $U_s U_t = U_{s+t}$ for all $s, t \in \mathbb{R}$. Then for a fixed $x \in H$ the function $f(t) = \langle U_t x, x \rangle$, $t \in \mathbb{R}$, is positive definite. Let $t_1, \dots, t_m \in \mathbb{R}$ and let $c_1, \dots, c_m \in \mathbb{C}$.

Then we have

$$\begin{aligned}\sum_{j,k=1}^{m}\langle U_{t_j-t_k}x, x\rangle c_j\overline{c_k} &= \sum_{j,k=1}^{m}\langle U_{t_j}U_{-t_k}x, x\rangle c_j\overline{c_k} \\ &= \sum_{j,k=1}^{m}\langle U_{-t_k}x, U_{-t_j}x\rangle c_j\overline{c_k} = \|\sum_{j=1}^{m}\overline{c_j}U_{-t_j}x\|^2,\end{aligned}$$

since $U_t^\star = U_t^{-1} = U_{-t}$ for any $t \in \mathbb{R}$.

Example 2.2.12. (Theorem of Polya) Here we provide the reader with a big class of positive definite functions that was discovered by Polya. We start with a function $\phi\colon [0,\infty) \to [0,1]$ satisfying the following conditions:

1. ϕ is continuous with $\phi(0) = 1$,
2. $\lim_{x\to\infty}\phi(x) = 0$,
3. ϕ is convex.

Then the even extension of ϕ to the real line is a positive definite function. In other words, the function $f\colon \mathbb{R} \to [0,1]$ given by

$$f(x) = \phi(|x|)$$

is positive definite (see [L]).

Example 2.2.13. Using the theorem of Polya we see that the functions

$$f(x) = \max(0, 1-|x|) \quad \text{and} \quad g(x) = e^{-|x|},$$

where $x \in \mathbb{R}$, are positive definite.

2.3 Reproducing kernel Hilbert spaces

Let X be a set and let $\mathcal{F}(X)$ be the complex vector space of all $\mathbb{C}$-valued functions on X with operations given by the standard addition and multiplication by scalars. A Hilbert space H is called a reproducing kernel Hilbert space (shortly RKHS) if $H \subset \mathcal{F}(X)$ as a subspace and if for each $x \in X$ the evaluation mapping $E_x \colon H \to \mathbb{C}$ given by

$$E_x(f) = f(x)$$

is a bounded functional.

Let H be a reproducing kernel Hilbert space. Then, by the Riesz representation theorem, for each point $x \in X$ there exists a unique element $k_x \in H$ such that

$$f(x) = E_x(f) = \langle f, k_x \rangle, \quad f \in H.$$

We call k_x the reproducing kernel for the point x and we call the above identity the reproducing identity. We define $K \colon X \times X \to \mathbb{C}$ by

$$K(x, y) = \langle k_y, k_x \rangle \tag{2.8}$$

and we call K the reproducing kernel for H. Then $K(x, y) = \overline{K(y, x)}$ and $K(x, x) = \|k_x\|^2 \geq 0$ for $x, y \in X$. We, also, get for $x, y \in X$

$$k_x(y) = \langle k_x, k_y \rangle = K(y, x),$$

and, again by the Riesz representation theorem,

$$\|E_x\|^2 = \|k_x\|^2 = K(x, x) = k_x(x).$$

We note that if we denote the map $X \ni x \mapsto k_x \in H$ by T, then

$$K(x, y) = \langle Ty, Tx \rangle.$$

Thus K is a positive definite kernel.

Lemma 2.3.1. *Let H be a reproducing kernel Hilbert space with kernel K. Then*

$$W = \operatorname{span}\{k_x \colon x \in X\}$$

is a dense subset of H.

Proof. Let $f \in H$. Suppose that $\langle f, k_x \rangle = 0$ for any $x \in X$. Then $f = 0$, since for any $x \in X$ we have that $f(x) = \langle f, k_x \rangle = 0$. This finishes the proof. □

We will prove that convergence in a reproducing kernel Hilbert space implies pointwise convergence.

Lemma 2.3.2. *Let H be a reproducing kernel Hilbert space and let f and f_n, $n \in \mathbb{N}$, be elements of H. Then the following holds:*

$$f_n \to f \text{ in } H \quad \Rightarrow \quad f_n(x) \to f(x) \text{ for every } x \in X.$$

Proof. We have for any $x \in X$ with the help of the Cauchy-Schwarz inequality

$$|f_n(x) - f(x)| = |\langle f_n - f, k_x \rangle| \leq \|f - f_n\| \|k_x\| \to 0.$$

□

Corollary 2.3.3. *Let H_1 and H_2 be two reproducing kernel Hilbert spaces build over a set X. Let K_1 and K_2 be the associated kernels. If $K_1 = K_2$, then $H_1 = H_2$ as Hilbert spaces.*

Proof. Let $k_x^1(y) = K_1(y,x)$ and $k_x^2(y) = K_2(y,x)$ for $x, y \in X$ and let $W_1 = \operatorname{span}\{k_x^1 \colon x \in X\}$ and $W_2 = \operatorname{span}\{k_x^2 \colon x \in X\}$. Then by the assumption $k_x^1 = k_x^2 (=: k_x)$ for any $x \in X$. Therefore $W_1 = W_2 (=: W)$. Moreover for $f = \sum_{j=1}^m a_j k_{x_j}$, where $a_j \in \mathbb{C}$ and $x_j \in X$ for $j = 1, \dots, m$, we have

$$\begin{aligned} \|f\|_1^2 &= \sum_{j,l=1}^m a_j \overline{a_l} \langle k_{x_j}, k_{x_l} \rangle_1 = \sum_{j,l=1}^m a_j \overline{a_l} K_1(x_l, x_j) \\ &= \sum_{j,l=1}^m a_j \overline{a_l} K_2(x_l, x_j) = \sum_{j,l=1}^m a_j \overline{a_l} \langle k_{x_j}, k_{x_l} \rangle_2 = \|f\|_2^2. \end{aligned}$$

Now let $f \in H_1$ and let $f_n \in W$, $n \in \mathbb{N}$, be such that $f_n \to f$ in H_1. Then f_n is a Cauchy sequence in H_1, and, since $\|f_n - f_m\|_1 = \|f_n - f_m\|_2$, it is a Cauchy sequence in H_2. Therefore it converges in H_2 to some element $g \in H_2$. For any $x \in X$ we have

$$f(x) = \lim_{n \to \infty} f_n(x) = g(x).$$

Hence $f = g \in H_2$. This shows that $H_1 \subset H_2$. A similar argument gives $H_2 \subset H_1$, thus $H_1 = H_2$ as sets. Let $f \in H_1 = H_2$. We have shown that there exist elements $f_n \in W$, $n \in \mathbb{N}$, such that $f_n \to f$ in H_1 and in H_2. Therefore

$$\|f\|_1 = \lim_{n \to \infty} \|f_n\|_1 = \lim_{n \to \infty} \|f_n\|_2 = \|f\|_2.$$

This finishes the proof. $\square$

We will show that any positive definite kernel K determines a unique reproducing kernel Hilbert space H_K such that its kernel is exactly K.

Let X be a set and let $K \colon X \times X \to \mathbb{C}$ be a positive definite kernel. For $x \in X$ let $k_x \colon X \to \mathbb{C}$ be given by $k_x(y) = K(y,x)$. Let W be the complex vector space of finite combinations of the functions k_x, $x \in X$ i.e.

$$W = \{\varphi = \sum_{j=1}^{m} a_j k_{x_j} \colon m \in \mathbb{N}, x_1, \ldots, x_m \in X, a_1, \ldots, a_m \in \mathbb{C}\}.$$

To be precise, the elements of the space W are functions and NOT formal finite combinations of elements k_x, where $x \in X$. For two functions $\varphi = \sum_{j=1}^{m} a_j k_{x_j}$ and $\psi = \sum_{j=1}^{m} b_j k_{x_j}$ we define

$$\langle \varphi, \psi \rangle = \sum_{j,k=1}^{m} a_j \overline{b_k} K(x_k, x_j).$$

We need two show that $\langle \cdot, \cdot \rangle$ is well-defined; which means that it does not depend on the way we write φ and ψ as linear combinations of functions k_x, $x \in X$. Let

$$\varphi = \sum_{j=1}^{m} a_j k_{x_j} = \sum_{j=1}^{m} \alpha_j k_{x_j} \quad \text{and} \quad \psi = \sum_{j=1}^{m} b_j k_{x_j} = \sum_{j=1}^{m} \beta_j k_{x_j}.$$

Then

$$\begin{aligned}
\varphi(x_k) &= \sum_{j=1}^{m} a_j K(x_k, x_j) = \sum_{j=1}^{m} \alpha_j K(x_k, x_j), \\
\overline{\psi(x_j)} &= \sum_{k=1}^{m} \overline{b_k} K(x_k, x_j) = \sum_{k=1}^{m} \overline{\beta_j} K(x_k, x_j).
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{j,k=1}^{m} a_j \overline{b_k} K(x_k, x_j) &= \sum_{k=1}^{m} \overline{b_k} \left(\sum_{j=1}^{m} a_j K(x_k, x_j) \right) \\
&= \sum_{k=1}^{m} \overline{b_k} \left(\sum_{j=1}^{m} \alpha_j K(x_k, x_j) \right) \\
&= \sum_{j=1}^{m} \alpha_j \left(\sum_{k=1}^{m} \overline{b_k} K(x_k, x_j) \right) \\
&= \sum_{j=1}^{m} \alpha_j \left(\sum_{k=1}^{m} \overline{\beta_k} K(x_k, x_j) \right) \\
&= \sum_{j,k=1}^{m} \alpha_j \overline{\beta_k} K(x_k, x_j).
\end{aligned}$$

Hence $\langle\cdot,\cdot\rangle$ is well-defined. It is complex linear in the first slot and conjugate complex linear in the second slot. Moreover, it is a positive definite form, for if $\varphi = \sum_{j=1}^m a_j k_{x_j}$, then

$$\langle\varphi,\varphi\rangle = \sum_{j,k=1}^m K(x_k,x_j)a_j\overline{a_k} \geq 0.$$

We can state the following reproducing property. For any $\varphi \in W$ and any $z \in X$ we have that

$$\varphi(z) = \langle\varphi, k_z\rangle.$$

Indeed, if $\varphi = \sum_{j=1}^m a_j k_{x_j}$, then

$$\varphi(z) = \sum_{j=1}^m a_j k_{x_j}(z) = \sum_{j=1}^m a_j K(z,x_j) = \langle\varphi,k_z\rangle.$$

We will show that if $\varphi = \sum_{j=1}^m a_j k_{x_j} \in W$ satisfies $\langle\varphi,\varphi\rangle = 0$, then $\varphi = 0$. Let $\varphi = \sum_{j=1}^m a_j k_{x_j} \in W$ be such that $\langle\varphi,\varphi\rangle = 0$. Fix $w \in W$ and let $\lambda \in \mathbb{C}$. Since K is a positive definite kernel, we have

$$\begin{aligned}0 &\leq \langle\varphi+\lambda w, \varphi+\lambda w\rangle \\ &= \langle\varphi,\varphi\rangle + \overline{\lambda}\langle\varphi,w\rangle + \lambda\langle w,\varphi\rangle + |\lambda|^2\langle w,w\rangle \\ &= 2\mathrm{Re}(\overline{\lambda}\langle\varphi,w\rangle) + |\lambda|^2\langle w,w\rangle.\end{aligned}$$

Taking $\Theta \in [0,2\pi)$ such that $e^{-i\Theta}\langle\varphi,w\rangle \geq 0$ and putting $\lambda = te^{i\Theta}$ for $t \in \mathbb{R}$, we get that

$$0 \leq 2t|\langle\varphi,w\rangle| + t^2\langle w,w\rangle.$$

Therefore, since $t \in \mathbb{R}$ was arbitrary, we conclude that $\langle\varphi,w\rangle = 0$. In particular, for any $z \in X$ we have that $0 = \langle\varphi,k_z\rangle = \varphi(z)$. Therefore $\varphi = 0$. Thus $\langle\cdot,\cdot\rangle$ is a complex inner-product on W. Now we complete this complex pre-Hilbert space to obtain a Hilbert space, which we denote by $\mathcal{H}_K$. We note that the map $T\colon X \to \mathcal{H}_K$ given by $Tx = k_x$ satisfies

$$K(y,x) = \langle Tx, Ty\rangle$$

for any $x, y \in X$.

The next step is to show that the space $\mathcal{H}_K$, which is the completion of the pre-Hilbert space W of functions, consists of functions. We first recall the construction of the completion of a metric space.

Let W be a pre-Hilbert space. We consider the set $CS(W)$ of all Cauchy sequences in W and we introduce the following relation on the set $CS(W)$:

$$(x_n) \sim (y_n) \Leftrightarrow \|x_n - y_n\| \to 0.$$

Then $\sim$ is an equivalence relation. We denote by H the set of all equivalence classes. We introduce an inner-product on H by

$$\langle [(x_n)]_\sim, [(y_n)]_\sim \rangle = \lim_{n\to\infty} \langle x_n, y_n\rangle.$$

Then H is a Hilbert space and the mapping $W \ni x \mapsto [(x)]_\sim \in H$ is an isometric embedding of W into H with W being dense in H.

Let (φ_n) be a Cauchy sequence in $\mathcal{H}_K$ consisting of elements of the dense subspace W. Then, using the Cauchy-Schwarz inequality, for any $x \in X$ and for any $m, n \in \mathbb{N}$ we have

$$|\varphi_n(x) - \varphi_m(x)| = |\langle \varphi_n - \varphi_m, k_x\rangle| \le \|\varphi_n - \varphi_m\| \|k_x\|.$$

Therefore for any $x \in X$ the sequence $\varphi_n(x)$ is Cauchy, thus it converges to some complex number $\varphi(x)$. We get a function $\varphi\colon X \to \mathbb{C}$. Thus we have assigned a function $\varphi\colon X \to \mathbb{C}$ to any Cauchy sequence (φ_n) of elements of W. Suppose we have two Cauchy sequences (φ_n) and (ψ_n) with corresponding functions φ and ψ and suppose that $(\varphi_n) \sim (\psi_n)$. We want to show that $\varphi = \psi$. Let $x \in X$. Since

$$\varphi_n(x) = \langle \varphi_n, k_x \rangle \quad \text{and} \quad \psi_n(x) = \langle \psi_n, k_x \rangle,$$

we get that

$$|\varphi_n(x) - \psi_n(x)| = |\langle \varphi_n - \psi_n, k_x \rangle| \leq \|\varphi_n - \psi_n\| \|k_x\| \to 0.$$

Therefore $\varphi(x) = \psi(x)$. We have proven that we have a well-defined assignment of a function φ to an equivalence class represented by a Cauchy sequence (φ_n). We see that to a constant sequence (φ), with $\varphi \in W$, is assigned the same function φ. It is not difficult to show that this assignment is linear. We will show that it is injective. Suppose that we have two equivalence classes $(\varphi_n)_\sim$ and $(\psi_n)_\sim$ such that $\varphi = \psi$. Since (φ_n) is a Cauchy sequence in W, hence a Cauchy sequence in $\mathcal{H}_K$ (after identification of W with a subspace of $\mathcal{H}_K$), it converges to some element $\Phi \in \mathcal{H}_K$. Similarly, (ψ_n) converges to some element $\Psi \in \mathcal{H}_K$. For any $x \in X$ we have

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} \langle \varphi, k_x \rangle,$$

$$\psi(x) = \lim_{n\to\infty} \psi_n(x) = \lim_{n\to\infty} \langle \psi, k_x \rangle.$$

Since $\varphi(x) = \psi(x)$ we deduce that for any $x \in X$

$$\langle \varphi_n - \psi_n, k_x \rangle \to 0.$$

By continuity of the inner product for any $x \in X$

$$\langle \Phi - \Psi, k_x \rangle = 0.$$

Therefore $\Phi = \Psi$. Finally

$$\|\varphi_n - \psi_n\| \leq \|\varphi_n - \Phi\| + \|\Phi - \Psi\| + \|\Psi - \psi_n\| \to 0.$$

Thus the assignment $(\varphi_n)_\sim \mapsto \varphi$ is injective. On the complex vector space of functions φ, denote if by H_K, we define an inner-product by

$$\langle \varphi, \psi \rangle_K = \langle (\varphi_n)_\sim, (\psi_n)_\sim \rangle = \lim_{n\to\infty} \langle \varphi_n, \psi_n \rangle.$$

Then this space becomes a Hilbert space isometrically isomorphic with the Hilbert space $\mathcal{H}_K$. Moreover, for any $x \in X$ and any φ

$$\varphi(x) = \lim_{n\to\infty} \langle \varphi_n, k_x \rangle = \langle \varphi, k_x \rangle_K.$$

Thus the evaluation map $\varphi \mapsto \varphi(x)$ is a bounded functional and so it is a reproducing kernel Hilbert space. Its kernel is exactly the map $K \colon X \times X \to \mathbb{C}$ we started with.

We will now express the kernel of a separable reproducing kernel Hilbert space H in terms of an orthonormal basis of H.

Let H be a reproducing kernel Hilbert space with kernel K and let $\{\varphi_n \colon n \in \mathbb{N}\}$ be an orthonormal basis of H. We recall that for any $\varphi \in H$ we have

$$\varphi = \sum_{n\in\mathbb{N}} \langle \varphi, \varphi_n \rangle \varphi_n \quad \text{and} \quad \|\varphi\|^2 = \sum_{n\in\mathbb{N}} |\langle \varphi, \varphi_n \rangle|^2.$$

Lemma 2.3.4. *The following holds:*

$$K(x, y) = \sum_{n\in\mathbb{N}} \varphi_n(x)\overline{\varphi_n(y)}, \tag{2.9}$$

where the convergence is pointwise.

Proof. For fixed $y \in X$ and $n \in \mathbb{N}$ we obtain

$$\langle k_y, \varphi_n \rangle = \overline{\langle \varphi_n, k_y \rangle} = \overline{\varphi_n(y)}.$$

Therefore

$$k_y = \sum_{n \in \mathbb{N}} \overline{\varphi_n(y)} \varphi_n \quad \text{(convergence in } H\text{)}.$$

Since convergence in H implies pointwise convergence, we get

$$K(x,y) = k_y(x) = \sum_{n \in \mathbb{N}} \varphi_n(x) \overline{\varphi_n(y)}.$$

This finishes the proof. $\square$

We are ready to give some examples of reproducing kernel Hilbert spaces.

Example 2.3.5. (The Bergman space) Let $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ and let $L^2(\mathbb{D}, \frac{1}{\pi} dA)$ be the space of square-integrable complex valued functions with respect to the normalized Lebesgue probability measure on $\mathbb{D}$. We denote this measure by $\frac{1}{\pi} dA$. Then the subspace B^2 of square-integrable complex valued functions with respect to this measure, which are analytic functions is a closed subspace of $L^2(\mathbb{D}, dA)$ (see the inequality below). Therefore B^2 is a complex Hilbert space of functions on $\mathbb{D}$. To show that it is a reproducing kernel Hilbert space we need to proof that for each $z \in \mathbb{D}$ the evaluation functional $E_z(f) = f(z)$ is bounded. This follows from the following inequality. For $z \in \mathbb{D}$ and $f \in B^2$ we have

$$|E_z(f)| = |f(z)| \leq \frac{1}{1 - |z|} \|f\|_{B^2}.$$

Thus B^2 as a reproducing kernel Hilbert space has its kernel. The goal is to find out what the kernel looks like. Since the functions $\mathbb{D} \ni z \mapsto \sqrt{n+1} z^n \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$, form an orthonormal basis of B^2, by the above lemma, we get that the kernel $B = B(z,w)$ of B^2 has the form

$$B(z,w) = \sum_{n=0}^{\infty}(n+1)z^n\overline{w}^n.$$

After a little bit of work we can rewrite the above as follows

$$B(z,w) = \frac{1}{(1-z\overline{w})^2}.$$

This kernel is called the Bergman kernel. For details see for example [McC].

Example 2.3.6. (The Hardy space) We denote by H^2 the complex vector space of analytic functions $f = \sum_{n=0}^{\infty} a_n z^n$ in the unit disc $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ satisfying the condition that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. We call this space the Hardy space. It is a Hilbert space with the inner-product given by

$$\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}.$$

We note that the Hardy space is isometrically isomorphic to the "little l2" space $l_2(\mathbb{N} \cup \{0\})$. It is a reproducing kernel Hilbert space, since for any $w \in \mathbb{D}$ and any $f = \sum_{n=0}^{\infty} a_n z^n \in H^2$

$$\begin{aligned} |f(w)| &\leq \sum_{n=0}^{\infty} |a_n||w|^n \\ &\leq \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{1/2} \left(\sum_{n=0}^{\infty} |w|^{2n}\right)^{1/2} = \frac{1}{\sqrt{1-|w|^2}}\|f\|_{H^2}. \end{aligned}$$

Since the functions $\mathbb{D} \ni z \mapsto z^n \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$, form an orthonormal basis of H^2 we get that the kernel $S = S(z,w)$ associated to the reproducing kernel Hilbert space

H^2 has the form

$$S(z,w) = \sum_{n=0}^{\infty} z^n \overline{w}^n,$$

which simplifies to the following

$$S(z,w) = \frac{1}{1 - z\overline{w}}.$$

This kernel is called the Szego kernel. Again, for details see for example [McC].

Example 2.3.7. (Band-limited L^2 functions - Shannon Sampling, see [J2011]) For a function $u \in L^1(\mathbb{R})$ define $\widehat{u}\colon \mathbb{R} \to \mathbb{C}$ by

$$\widehat{u}(\xi) = \int_{\mathbb{R}} e^{-i\,x\xi} u(x) dx.$$

Then the mapping $u \mapsto \widehat{u}$ can be "extended" to the action on square integrable functions (see [Ka] or [Rud]). We define

$$L^2_B = \{u \in L^2(\mathbb{R})\colon \operatorname{supp} \widehat{u} \subset [0,1]\}.$$

The inverse Fourier formula says that the elements of the space L^2_B are continuous functions. This follows from the fact that a square-integrable function with bounded support is integrable. The space L^2_B is a Hilbert space. The elements of L^2_B are called band-limited L^2 functions. We will show that L^2_B is a reproducing kernel Hilbert space. We define $K\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$K(x,y) = \frac{\sin(x-y)}{\pi(x-y)}.$$

Then K is a positive definite kernel and for any $x \in \mathbb{R}$ and $u \in L^2_B$ we have with

$k_x = K(\cdot, x)$

$$\begin{aligned}
\langle u, k_x \rangle &= \frac{1}{\pi} \int_{\mathbb{R}} u(y) \frac{\sin(y-x)}{y-x} dy \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} u(y) \int_{-1}^{1} e^{-it(y-x)} dt dy \\
&= \frac{1}{2\pi} \int_{-1}^{1} \int_{\mathbb{R}} e^{-ity} u(y) dy e^{itx} dt \\
&= \frac{1}{2\pi} \int_{-1}^{1} e^{itx} \widehat{u}(t) dt \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{itx} \widehat{u}(t) dt \\
&= u(x),
\end{aligned}$$

where the last equality follows by the Fourier inversion formula. Therefore L^2_B is the reproducing kernel Hilbert space associated to the kernel $K(x, y) = \sin(x-y)/(\pi(x-y))$, $x, y \in \mathbb{R}$. The Shannon sampling theorem says that for any $u \in L^2_B$

$$u(x) = \sum_{k=-\infty}^{\infty} u(k) \frac{\sin(x-k)}{\pi(x-k)}, \quad x \in \mathbb{R},$$

with the series converging absolutely and locally uniformly.

Example 2.3.8. (see [J2011]) Let H be the space of real-valued absolutely continuous functions $u\colon [0, A] \to \mathbb{R}$ defined on an interval $[0, A]$, where $A > 0$, such that $u' \in L^2(0, A)$ and $u(0) = 0$. Then H is a reproducing kernel Hilbert space with the inner-product given by

$$\langle u, v \rangle = \int_0^A u'(t) v'(t) dt.$$

To see that let $x \in [0, A]$ and let $u \in H$. Since

$$u(x) = \int_0^x u(t) dt,$$

we use the Cauchy-Schwarz inequality to obtain

$$|u(x)| \leq \int_0^x |u(t)|dt \leq \|u\|\sqrt{x}.$$

It is not so difficult to see what the kernel associated to this reproducing kernel Hilbert space is. For $x \in [0, A]$ define $k_x \colon [0, A] \to \mathbb{R}$ by

$$k_x(y) = \min(x, y) = \begin{cases} y & y \leq x \\ x & y > x. \end{cases}$$

Then $k'_x = \chi_{[0,x]}$ almost everywhere; where $\chi_{[0,x]}$ is the characteristic function of the interval $[0, x]$. Therefore for $x \in [0, A]$ and $u \in H$

$$u(x) = \int_0^x u'(t)dt = \int_0^A u'(t)k'_x(t)dt = \langle u, k_x \rangle.$$

Therefore the reproducing kernel of H takes the form

$$K(x, y) = \min(x, y).$$

for $x, y \in [0, 1]$. Compare with the example about Brownian motion.

Example 2.3.9. (The Energy Hilbert space, see [JP],[J2010],[J2011]) We will associate a reproducing kernel Hilbert space to a resistance network (a simple connected weighted graph). Recall that a graph G is a pair $G = (V, E)$, where V is a set called the vertex set and $E \subset 2^V$ is a set of 2-element subsets of V, which we call the set of edges. We call a set $\{x, y\} \in E$ the edge between x and y. A path joining two vertices $x, y \in V$ is a finite sequence $x_1 = x, \ldots, x_m = y$ of vertices such that for each $i = 1, \ldots, m-1$ we have that $\{x_i, x_{i+1}\} \in E$. We write $x \sim y$ if $\{x, y\} \in E$. We require our graphs to be connected i.e. any two points can be connected by a path. We don't assume that graphs are finite or that they are locally finite. A conductance function on a graph $G = (V, E)$ is a function $c\colon V \times V \to [0, \infty)$ such that

1. $c_{xy} > 0$ if and only if $x \sim y$. In particular $c_{xx} = 0$ for any $x \in V$.

2. $c_{xy} = c_{yx}$.

3. for any $x \in V$ the quantity $c(x) = \sum_{y \sim x} c_{xy}$ is finite

A pair (G, c), where G is a graph and c is a conductance function, is called a resistance network.

Let (G, c) be a resistance networks. The Laplacian Δ is an operator acting on functions $v\colon V \to \mathbb{R}$ as follows

$$(\Delta v)(x) = \sum_{y \sim x} c_{xy}(v(y) - v(x)) = -c(x)v(x) + \sum_{y \sim x} c_{xy} v(y),$$

provided that for any $x \in V$ the quantity $\sum_{y \sim x} c_{xy} v(y)$ is finite. We call a function $v\colon V \to \mathbb{R}$ harmonic if $\Delta v = 0$. We define the energy of a pair of functions $u, v\colon V \to \mathbb{R}$, provided it is finite, by

$$\mathcal{E}(u, v) = \frac{1}{2} \sum_{x,y \in V} c_{xy}(u(x) - u(y))(v(x) - v(y)).$$

We set $\mathrm{Dom}(\mathcal{E}) = \{u\colon V \to \mathbb{R}\colon \mathcal{E}(u, u) < \infty\}$. We see that a function $u\colon V \to \mathbb{R}$ is in the kernel of $\mathcal{E}$, which means that $\mathcal{E}(u, u) = 0$, if and only if u is a constant function. We fix a vertex $a \in V$ an we define

$$\mathcal{H}_{\mathcal{E}} = \{u \in \mathrm{Dom}\mathcal{E}\colon u(a) = 0\}.$$

Then $\mathcal{H}_{\mathcal{E}}$ with the inner-product

$$\langle u, v\rangle_{\mathcal{E}} = \mathcal{E}(u, v),$$

where $u, v \in \mathcal{H}_{\mathcal{E}}$, becomes a Hilbert space. We will show that $\mathcal{H}_{\mathcal{E}}$ is a reproducing kernel Hilbert space. For $x \in V$ we define $E_x\colon \mathcal{H}_{\mathcal{E}} \to \mathbb{R}$ by $E_x(u) = u(x)$. Note that $E_a = 0$. We need to show that E_x is a bounded functional. Let $x = x_0, \ldots, a = x_m$ be a path connecting x with a and put

$$M = \left(\sum_{i=1}^{m} \frac{1}{c_{x_{i-1}x_i}}\right)^{\frac{1}{2}}.$$

Then, by the Cauchy-Schwarz inequality,

$$\begin{aligned} u(x) &= u(x) - u(a) \\ &= \sum_{i=1}^{m} u(x_{i-1}) - u(x_i) \\ &= \sum_{i=1}^{m} \frac{1}{\sqrt{c_{x_{i-1}x_i}}} \cdot \sqrt{c_{x_{i-1}x_i}}(u(x_{i-1}) - u(x_i)) \\ &\le M\|u\|_{\mathcal{E}}. \end{aligned}$$

Thus $\mathcal{H}_{\mathcal{E}}$ is a reproducing kernel Hilbert space.

Remark 2.3.10. (Relative reproducing kernel Hilbert spaces, see [J2010]) In the last two examples in order to obtain RKHS we needed to add an assumption that the functions we considered took value 0 at some fixed point. These examples fit better into the category of relative reproducing kernel Hilbert spaces. The motivation for introducing this notion was the fact that in many cases we are interested in the difference of function values $f(x) - f(y)$, and not in the values $f(x)$ of a function as indicated in [J2010] by Palle Jorgensen. Let H be a Hilbert space of complex-valued functions defined on a set X. We say that H is a relative reproducing kernel Hilbert space (RRKHS) if there is a mapping $k\colon X \times X \to H$ such that for any $x, y \in X$ and any $u \in H$

$$u(x) - u(y) = \langle u, k(x, y)\rangle.$$

We see that the mapping k is uniquely determined and is called the relative reproducing kernel of H. We note that for fixed $x, y \in X$ the element $k(x, y)$ is a complex-valued function on X. Now we see that the Hilbert spaces

$$H = \{u \in AC[0, A] \colon u' \in L^2(0, A)\} \quad \text{and} \quad \mathrm{Dom}(\mathcal{E})$$

are relative reproducing kernel Hilbert spaces.

Example 2.3.11. (RKHS associated to a kernel $K(x, y) = f(x)\overline{f(y)}$) Let X be a set and let $f\colon X \to \mathbb{C}$ be a non-zero function. Define $K(x, y) = f(x)\overline{f(y)}$, for $x, y \in X$. Then K is a positive definite kernel. Moreover, for any $x, y \in X$ we see that $k_x(y) = K(y, x) = \overline{f(x)}f(y)$. Therefore $k_x = \overline{f(x)}f$, and thus the reproducing kernel Hilbert space H_K is one-dimensional with f being the generator. Moreover, the inner-product satisfies $\|f\| = 1$.

Remark 2.3.12. A reproducing kernel Hilbert space defined on a set X determines a distance function on X as follows. Let H be a reproducing kernel Hilbert space over a set X with kernel $K(x, y) = \langle k_y, k_x\rangle$. Suppose that H separates points; that is for any $x, y \in X$ there exists a function $f \in H$ such that $f(x) \neq f(y)$. We put for

$x, y \in X$

$$d(x,y) = \inf\{|f(x) - f(y)| \colon \|f\| \leq 1\}$$
$$= \inf\{\langle f, k_x - k_y\rangle \colon \|f\| \leq 1\}.$$

Then $d\colon X \times X \to [0, \infty)$ is a distance function on X. Moreover $d(x,y) \leq \|k_x - k_y\|$ for any $x, y \in X$.

In particular any positive definite kernel on a set X gives rise to a distance function on X by the procedure:

Positive definite kernel $K \to$ RKHS $H_K \to$ Distance function

Remark 2.3.13. Let $H \subset \mathcal{F}(X)$ be a reproducing kernel Hilbert space built over a set X and let $K\colon X \times X \to \mathbb{C}$ be the associated kernel. We recall that for any $x, y \in X$

$$|K(x,y)| \leq K(x,x)K(y,y).$$

We let $K_D\colon X \to \mathbb{C}$ be the restriction of the kernel K to the diagonal $D = \{(x,x) \in X \times X \colon x \in X\}$; that is

$$K_D(x) = K(x,x).$$

Therefore K is bounded if and only if K_D is bounded. Moreover, for any $x, y \in X$ and any $f \in H$ we have

$$|f(x) - f(y)| = |\langle f, k_x - k_y\rangle| \leq \|f\|\|k_x - k_y\|$$
$$\leq \|f\|\sqrt{K(x,x) + K(y,y) - K(x,y) - K(y,x)}$$

and

$$|f(x)| = |\langle f, k_x\rangle| \leq \|f\|\sqrt{K(x,x)}.$$

In particular

$$\|f\|_\infty \leq \sup_{x\in X} \sqrt{K(x,x)}\|f\|.$$

We also have the following facts.

1. Let (X,d) be a metric space, let $K\colon X \times X \to \mathbb{C}$ be a continuous positive definite kernel, and let H_K be the reproducing kernel Hilbert space associated to K. Then the elements of H_K are continuous functions on X.

2. Let X and K be as above, and let, in addition, K be bounded. Then convergence in H_K implies uniform convergence.

3. Let $\Omega \subset \mathbb{C}^n$ be an open set and let $K\colon \Omega \times \Omega \to \mathbb{C}$ be a holomorphic function and a positive definite kernel. Then elements of the associated reproducing kernel Hilbert space H_K are holomorphic functions.

CHAPTER 3
EXTENSION OF POSITIVE DEFINITE FUNCTIONS

Let $\Omega \subset \mathbb{R}^n$ be an open set. We put $\Omega - \Omega = \{x - y \colon x, y \in \Omega\}$. Then $\Omega - \Omega$ is an open, symmetric set containing 0.

Remark 3.0.14. We note that if Ω is a symmetric ($x \in \Omega \Rightarrow -x \in \Omega$) and convex ($x, y \in \Omega \Rightarrow (1/2)x + (1/2)y \in \Omega$) set, then $\Omega - \Omega = 2\Omega = \{2x \colon x \in \Omega\}$.

Let $F \colon \Omega - \Omega \to \mathbb{C}$ be a positive complex-valued function. We recall that F is positive definite if for any $x_1, \ldots, x_m \in \Omega$ and any $c_1, \ldots, c_m \in \mathbb{C}$

$$\sum_{j,k=1}^{m} F(x_j - x_k) c_j \overline{c_k} \geq 0.$$

As before, we see that if $F \colon \Omega - \Omega \to \mathbb{C}$ is a positive definite function, then for any $z \in \Omega - \Omega$

$$|F(z)| \leq F(0) \quad \text{and} \quad \overline{F(z)} = F(-z).$$

We start with the following well-known characterization of continuous positive definite functions defined on an open subset of $\mathbb{R}^n$. We provide a proof for completeness and because of the fact that we couldn't find a reference, which contains a proof. For an open set $\Omega \subset \mathbb{R}^n$ we denote by $C_0(\Omega)$ the space of all complex-valued continuous functions $\varphi \colon \Omega \to \mathbb{C}$ with compact support, and by $C_0^\infty(\Omega)$ the space of all complex-valued smooth (C^∞) functions $\varphi \colon \Omega \to \mathbb{C}$ with compact support.

Theorem 3.0.15. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $F \colon \Omega - \Omega \to \mathbb{C}$ be a continuous function. Then F is positive definite if and only if for any $\varphi \in C_0(\Omega)$ the following* *holds*

$$\mathcal{L}_F(\varphi) = \int_\Omega \int_\Omega F(y-x)\varphi(x)\overline{\varphi(y)}dxdy \geq 0. \tag{3.1}$$

Remark 3.0.16. The theorem holds true if we replace the class $C_0(\Omega)$ by the class $C_0^\infty(\Omega)$.

Proof. ($\Leftarrow$) Let $F\colon \Omega - \Omega \to \mathbb{C}$, where $\Omega \subset \mathbb{R}^n$ is an open set, be a continuous function and let (3.1) hold for any $\varphi \in C_0(\Omega)$. We will first show that (3.1) holds true for any $\varphi \in L^1(\Omega)$ with compact support (we write $\varphi \in L^1_0(\Omega)$). We say that $\varphi \in L^1(\Omega)$ has compact support if there exists a compact set $D \subset \Omega$ such that the set of points outside of D where φ is not zero has zero Lebesgue measure. Let $\varphi \in L^1_0(\Omega)$, let D be as above and let $\varphi_m \in C_0(\Omega)$, $m \in \mathbb{N}$, be such that $\varphi_n \to \varphi$ in L^1 with $\text{supp}\varphi_m \subset D$. We want to show that

$$\mathcal{L}_F(\varphi_m) \to \mathcal{L}_F(\varphi).$$

We denote $M = \sup_{x,y\in D} |F(x-y)|$. We note that $D - D$ is a compact subset of $\Omega - \Omega$, hence M is finite, due to the fact that F is a continuous function. We compute

$$\begin{aligned}
&|\mathcal{L}_F(\varphi_m) - \mathcal{L}_F(\varphi)| \\
&= \left| \int_\Omega \int_\Omega F(y-x)[\varphi(x)(\overline{\varphi(y)} - \overline{\varphi_m(y)}) + \overline{\varphi_m(y)}(\varphi(x) - \varphi_m(x))]dxdy \right| \\
&\leq M(\|\varphi\|_{L^1}\|\varphi - \varphi_m\|_{L^1} + \|\varphi_m\|_{L^1}\|\varphi - \varphi_m\|_{L^1}).
\end{aligned}$$

Thus $\mathcal{L}_F(\varphi_m) \to \mathcal{L}_F(\varphi)$. Finally $\mathcal{L}(\varphi_m) \geq 0$, $m \in \mathbb{N}$, implies that $\mathcal{L}_F(\varphi) \geq 0$.

We are ready to show that F is a positive definite function. Let $x_1, \ldots, x_m \in$

$\mathbb{R}^n$ and let $c_1, \ldots, c_m \in \mathbb{C}$. We need to show that

$$\sum_{j,k=1}^{m} F(x_j - x_k) c_j \overline{c_k} \geq 0.$$

We may assume that $\sum_{j=1}^{m} |c_j|^2 > 0$. For $r > 0$ sufficiently small we put

$$\varphi_r = \sum_{j=1}^{m} \frac{c_j}{|B(x_j, r)|} \chi_{B(x_j,r)},$$

where $|A|$ denotes the Lebesque measure of a Lebesgue measurable set $A \subset \mathbb{R}^n$, χ_A is the characteristic function of the set A i.e. $\chi_A(y) = 1$ for $y \in A$ and $\chi_A(y) = 0$ for $y \notin A$, and $B(x, r) \subset \mathbb{R}^n$ is the open ball centered at x with radius r, $x \in \mathbb{R}^n$, $r > 0$. Then $\mathcal{L}_F(\varphi_r) \geq 0$ for any $r > 0$ and

$$\mathcal{L}_F(\varphi_r) \to \sum_{j,k=1}^{m} F(x_j - x_k) c_j \overline{c_k} \quad \text{as } r \to 0. \tag{3.2}$$

To see (3.2) let $\epsilon > 0$. Choose $\delta > 0$ small enough such that for any $j, k = 1, \ldots, m$ and any $x \in B(x_j, \delta)$, $y \in B(x_k, \delta)$ we have that

$$|F(x - y) - F(x_j - x_k)| < \frac{\epsilon}{(\sum_{j=1}^{m} |c_j|)^2}.$$

Then for any $r < \delta$ we obtain

$$\begin{aligned}
&\left| \sum_{j,k=1}^{n} c_j \overline{c_k} F(x_j - x_k) - \mathcal{L}_F(\varphi_r) \right| \leq \\
&\sum_{j,k=1}^{m} \frac{|c_j||c_k|}{|B(x_j, r)||B(x_k, r)|} \int_{B(x_j,r)} \int_{B(x_k,r)} |F(x - y) - F(x_j - x_k)| dx dy \\
&\leq \epsilon.
\end{aligned}$$

This proves (3.2). Since $\mathcal{L}(\varphi_r) \geq 0$ for any $r > 0$ we conclude that

$$\sum_{j,k=1}^{m} F(x_j - x_k) c_j \overline{c_k} \geq 0,$$

thus F is a positive definite function.

($\Rightarrow$) Let $\varphi \in C_0(\mathbb{R}^n)$ with $\varphi \neq 0$ and let $F\colon \Omega - \Omega \to \mathbb{C}$, where $\Omega \subset \mathbb{R}^n$ is an open set, be a continuous positive definite function. We denote

$$M = |\operatorname{supp}\varphi| \quad \text{and} \quad K = \sup\{|F(x-y)|\colon x, y \in \operatorname{supp}\varphi\}.$$

We need to show that (3.1) holds. Suppose, by contradiction, that $\mathcal{L}_F(\varphi) < 0$. Choose $\epsilon > 0$ such that $\mathcal{L}_F(\varphi) + \epsilon < 0$. The function $(x, y) \mapsto F(x-y)$ is uniformly continuous on the set $\operatorname{supp}\varphi \times \operatorname{supp}\varphi$. Therefore there exists $\delta > 0$ such that for any $(x, y), (a, b) \in \operatorname{supp}\varphi \times \operatorname{supp}\varphi$ with $|x-a| < \delta$ and $|y-b| < \delta$ we have that

$$|F(x,y) - F(a,b)| < \frac{\epsilon}{2M^2\|\varphi\|_\infty^2}.$$

We divide the set $\operatorname{supp}\varphi$, if necessary, into disjoint subsets A_j, $j = 1, \ldots, m$, so that $\operatorname{diam} A_j \leq \delta$. We choose points $x_1 \in A_1, \ldots, x_m \in A_m$ and consider the following function

$$\psi = \sum_{j=1}^{m} \varphi(x_j)\chi_{A_j}.$$

Then we get

$$\begin{aligned}
&\Big|\mathcal{L}_F(\psi) - \sum_{j,k=1}^{m} F(x_j - x_k)(\varphi(x_j)|A_j|)(\overline{\varphi(x_k)}|A_k|)\Big| \\
&\qquad \leq \|\varphi\|_\infty^2 \sum_{j,k=1}^{m} \int_{A_j}\int_{A_k} |F(x-y) - F(x_j - x_k)|dxdy \\
&\qquad \leq \|\varphi\|_\infty^2 \int_{\operatorname{supp}\varphi}\int_{\operatorname{supp}\varphi} \frac{\epsilon}{2} dxdy < \frac{\epsilon}{2}.
\end{aligned}$$

Moreover, making $\delta > 0$ small enough so that for any $j, k = 1, \ldots, m$ and any $x \in A_j$ and $y \in A_k$

$$|\varphi(x_j)\overline{\varphi(x_k)} - \varphi(x)\overline{\varphi(y)}| < \frac{\epsilon}{2KM^2},$$

we obtain

$$\begin{aligned}
&|\mathcal{L}_F(\psi) - \mathcal{L}_F(\varphi)| \\
&\leq \sum_{j,k=1}^{m} \int_{A_j} \int_{A_k} |F(x-y)||\varphi(x_j)\overline{\varphi(x_k)} - \varphi(x)\overline{\varphi(y)}|dxdy \\
&< \frac{\epsilon}{2}.
\end{aligned}$$

Therefore, if we put $c_j = \varphi(x_j)|A_j|$ for $j = 1, \dots, m$, then

$$\begin{aligned}
&\sum_{j,k=1}^{m} F(x_j - x_k)c_j\overline{c_k} \\
&= \sum_{j,k=1}^{m} F(x_j - x_k)c_j\overline{c_k} - \mathcal{L}_F(\psi) + \mathcal{L}_F(\psi) - \mathcal{L}_F(\varphi) + \mathcal{L}_F(\varphi) \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} + \mathcal{L}_F(\varphi) < 0.
\end{aligned}$$

Contradiction with the fact that F is a positive definite function. Therefore the assumption that $\mathcal{L}_F(\varphi) < 0$ is false. This proves the theorem. $\square$

Remark 3.0.17. Let $F\colon \mathbb{R}^n \to \mathbb{C}$ be a continuous positive definite function and let $\mu \in M(\mathbb{R}^n)$ be a finite positive Borel measure such that $\widehat{\mu} = f$. Then for any $\varphi \in C_0(\mathbb{R}^n)$

$$\begin{aligned}
\mathcal{L}_F(\varphi) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widehat{\mu}(y-x)\varphi(x)\overline{\varphi(y)}dxdy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle y-x,z\rangle} d\mu(z) \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-i\langle x,z\rangle}\varphi(x)dx \right) \left(\overline{\int_{\mathbb{R}^n} e^{-i\langle y,z\rangle}\varphi(y)dy} \right) d\mu(z) \\
&= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{-i\langle x,z\rangle}\varphi(x)dx \right|^2 d\mu(z) \geq 0.
\end{aligned}$$

We denote the function

$$\mathbb{R}^n \ni z \to \int_{\mathbb{R}^n} e^{-i\langle x,z\rangle}\varphi(x)dx$$

by $\widehat{\varphi}$. Therefore we can write

$$\mathcal{L}_F(\varphi) = \|\widehat{\varphi}\|^2_{L^2(d\mu)}. \tag{3.3}$$

The above suggests that we could associate to a continuous positive definite function defined on the entire space $\mathbb{R}^n$ a complex inner-product on the space $C_0(\mathbb{R}^n)$. Let $F\colon \mathbb{R}^n \to \mathbb{C}$ be a continuous positive definite function. We define $\langle\cdot,\cdot\rangle_F\colon C_0(\mathbb{R}^n)\times C_0(\mathbb{R}^n) \to \mathbb{C}$ by

$$\langle\varphi,\psi\rangle_F = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(y-x)\varphi(x)\overline{\psi(y)}dxdy \tag{3.4}$$

Then

1. $\langle\cdot,\cdot\rangle_F$ is complex linear in the first slot and conjugate complex linear in the second slot i.e. for any $\varphi,\psi \in C_0(\mathbb{R}^n)$ and any $a\in\mathbb{C}$

$$\langle a\varphi,\psi\rangle_F = a\langle\varphi,\psi\rangle_F \quad \text{and} \quad \langle\varphi,a\psi\rangle_F = \overline{a}\langle\varphi,\psi\rangle_F.$$

2. $\langle\cdot,\cdot\rangle_F$ is conjugate symmetric i.e. for any $\varphi,\psi\in C_0(\mathbb{R}^n)$

$$\langle\varphi,\psi\rangle_F = \overline{\langle\psi,\varphi\rangle_F}.$$

3. $\langle\cdot,\cdot\rangle_F$ is positive definite in the sense that for any $\varphi\in C_0(\mathbb{R}^n)$

$$\langle\varphi,\varphi\rangle_F = \mathcal{L}_F(\varphi) \geq 0.$$

We would like the map $\langle\cdot,\cdot\rangle_F\colon C_0(\mathbb{R}^n)\times C_0(\mathbb{R}^n)\to\mathbb{C}$ to be a complex inner-product. We need the following

4. if $\varphi \in C_0(\mathbb{R}^n)$ is such that $\langle \varphi, \varphi \rangle_F = 0$, then $\varphi = 0$.

This, however, follows immediately from (3.3). We can complete this complex inner-product space $(C_0(\mathbb{R}^n), \langle \cdot, \cdot \rangle_F)$ to obtain a complex Hilbert space H_F associated to our positive definite function F. Since the map

$$C_0(\mathbb{R}^n) \ni \varphi \mapsto \widehat{\varphi} \in L^2(d\mu)$$

is an isometry between inner-product spaces $(C_0(\mathbb{R}^n), \langle \cdot, \cdot \rangle_F)$ and $L^2(d\mu)$ i.e. it is complex linear and satisfies

$$\langle \varphi, \psi \rangle_F = \langle \widehat{\varphi}, \widehat{\psi} \rangle_{L^2(d\mu)},$$

we can think of H_F as a closed subspace of the Hilbert space $L^2(d\mu)$.

□

3.1 The Hilbert space $\mathcal{H}_F$

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $F \colon \Omega - \Omega \to \mathbb{C}$ be a continuous positive definite function. For $\varphi \in C_0^\infty(\Omega)$ we define a function $F_\varphi \colon \Omega \to \mathbb{C}$ by

$$F_\varphi(x) = \int_\Omega F(x-y)\varphi(y)dy,$$

and for $\varphi, \psi \in C_0^\infty(\Omega)$ we put

$$\langle F_\varphi, F_\psi \rangle = \int_\Omega \int_\Omega F(y-x)\varphi(x)\overline{\psi(y)}dxdy.$$

We denote by F^e the extension of F to $\mathbb{R}^n$ by putting $F^e = 0$ in the complement of $\Omega - \Omega$. Then F_φ is the convolution of F^e with φ i.e. for $x \in \Omega$

$$F_\varphi(x) = F^e \star \varphi(x) = \int_{\mathbb{R}^n} F^e(x-y)\varphi(y)dy$$

and for $\varphi, \psi \in C_0^\infty(\Omega)$

$$\langle F_\varphi, F_\psi \rangle = \int_{\mathbb{R}^n} F^e \star \varphi(y) \overline{\psi(y)} dx = \int_{\mathbb{R}^n} F_\varphi(x) \overline{\psi(x)} dx. \tag{3.5}$$

Moreover, using the fact that $F(x-y) = \overline{F(y-x)}$ for any $x, y \in \Omega$, we obtain for $\varphi, \psi \in C_0^\infty(\Omega)$

$$\begin{aligned}
\langle F_\varphi, F_\psi \rangle &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F^e(x-y)\varphi(y)\overline{\psi(x)} dx dy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \overline{F^e(y-x)}\varphi(y)\overline{\psi(x)} dx dy \\
&= \int_{\mathbb{R}^n} \overline{F^e \star \psi(y)} \varphi(y) dy \\
&= \int_{\mathbb{R}^n} \varphi(y) \overline{F_\psi(y)} dy.
\end{aligned}$$

Therefore for any $\varphi, \psi \in C_0^\infty(\Omega)$

$$\langle F_\varphi, F_\psi \rangle = \int_{\mathbb{R}^n} F_\varphi(x) \overline{\psi(x)} dx = \int_{\mathbb{R}^n} \varphi(y) \overline{F_\psi(y)} dy. \tag{3.6}$$

We are ready to show that the form $\langle \cdot, \cdot \rangle$ is well defined. Take four functions $\alpha, \beta, \varphi, \psi \in C_0^\infty(\Omega)$ and let $F_\alpha = F_\varphi$ and $F_\beta = F_\psi$. We want to show that

$$\langle F_\varphi, F_\psi \rangle = \langle F_\alpha, F_\beta \rangle. \tag{3.7}$$

This will follow by (3.6). Firstly, we have

$$\begin{aligned}
\langle F_\varphi, F_\psi \rangle &= \int_{\mathbb{R}^n} F_\varphi(x) \overline{\psi(x)} dx \\
&= \int_{\mathbb{R}^n} F_\alpha(x) \overline{\psi(x)} dx = \langle F_\alpha, F_\psi \rangle.
\end{aligned}$$

Secondly,

$$\begin{aligned}
\langle F_\alpha, F_\psi \rangle &= \int_{\mathbb{R}^n} \alpha(y) \overline{F_\psi(y)} dy \\
&= \int_{\mathbb{R}^n} \alpha(y) \overline{F_\beta(y)} dy = \langle F_\alpha, F_\beta \rangle.
\end{aligned}$$

We note that for any $\varphi \in C_0^\infty(\Omega)$ the function $F^e \star \varphi$, and hence the function F_φ, is smooth and it has compact support when Ω is a bounded set.

We denote by $\mathcal{W}$ the complex vector space of all functions F_φ with $\varphi \in C_0^\infty(\Omega)$, i.e.

$$\mathcal{W} = \{F_\varphi \colon \varphi \in C_0^\infty(\Omega)\}.$$

Then the pair $(\mathcal{W}, \langle \cdot, \cdot \rangle)$ is a complex inner-product space, which we prove in the lemma below.

Let (ϕ_ϵ) be a smoothing kernel meaning that we take a smooth non-negative function $\phi \colon \mathbb{R}^n \to [0, \infty)$ with support contained in the unit ball $B(0,1) = \{x \in \mathbb{R}^n \colon |x| < 1\}$ satisfying $\int \phi = 1$, and we define $\phi_\epsilon(x) = 1/\epsilon^n \phi(x/\epsilon)$ for $\epsilon > 0$. Then for any $\epsilon > 0$

$$\operatorname{supp} \phi_\epsilon \subset B(0, \epsilon) \quad \text{and} \quad \int_{\mathbb{R}^n} \phi_\epsilon(x) dx = 1.$$

In particular, ϕ_ϵ converges to the delta Dirac measure supported at 0 when ϵ goes to 0, where the convergence is understood in the sense of distributions.

Moreover, for any $a \in \mathbb{R}^n$ we define translations of ϕ_ϵ by a, that is we put $\phi_{a,\epsilon}(x) = \phi_\epsilon(x - a)$ for $x \in \mathbb{R}^n$ and $\epsilon > 0$.

Lemma 3.1.1. *The form $\langle \cdot, \cdot \rangle$ is a complex inner-product on the space $\mathcal{W}$.*

Proof. We just need to show that for $F_\varphi \in \mathcal{W}$ the following holds

$$\langle F_\varphi, F_\varphi \rangle = 0 \Rightarrow F_\varphi = 0.$$

Let $F_\varphi \in \mathcal{W}$ be such that $\langle F_\varphi, F_\varphi \rangle = 0$. We take $\lambda \in \mathbb{C}$, $F_\psi \in \mathcal{W}$, and we consider the quantity

$$\langle F_\varphi + \lambda F_\psi, F_\varphi + \lambda F_\psi \rangle,$$

which we know is non-negative. Now we proceed as in the classical proof of the Cauchy-Schwarz inequality. We will arrive at the following fact:

$$\langle F_\varphi, F_\psi \rangle = 0. \tag{3.8}$$

Indeed, we compute

$$\begin{aligned} 0 &\leq \langle F_\varphi + \lambda F_\psi, \varphi + \lambda F_\psi \rangle \\ &= \langle F_\varphi, F_\varphi \rangle + \overline{\lambda} \langle F_\varphi, F_\psi \rangle + \lambda \langle F_\psi, F_\varphi \rangle + |\lambda|^2 \langle F_\psi, F_\psi \rangle \\ &= 2\mathrm{Re}(\overline{\lambda} \langle F_\varphi, F_\psi \rangle) + |\lambda|^2 \langle F_\psi, F_\psi \rangle. \end{aligned}$$

Taking $\Theta \in [0, 2\pi)$ such that $e^{-i\Theta} \langle F_\varphi, F_\psi \rangle \geq 0$ and putting $\lambda = te^{i\Theta}$, where $t \in \mathbb{R}$, we get that

$$0 \leq 2t|\langle F_\varphi, F_\psi \rangle| + t^2 \langle F_\psi, F_\psi \rangle.$$

Therefore, since $t \in \mathbb{R}$ is arbitrary, we conclude that $\langle F_\varphi, F_\psi \rangle = 0$. It remains to show that if (3.8) holds for any $F_\psi \in \mathcal{W}$, then $F_\varphi = 0$. This follows from the lemma below. $\square$

Lemma 3.1.2. *For $a \in \Omega$ and $\epsilon > 0$ small enough put $F_{a,\epsilon} = F_{\phi_{a,\epsilon}}$. Then for any $\varphi \in C_0^\infty(\Omega)$*

$$\langle F_\varphi, F_{a,\epsilon} \rangle \to F_\varphi(a) \quad \textit{as } \epsilon \to 0. \tag{3.9}$$

Proof. Let $\varphi \in C_0^\infty(\Omega)$. We may assume that $\varphi \neq 0$. Then

$$\begin{aligned}
&|\langle F_\varphi, F_{a,\epsilon}\rangle - F_\varphi(a)| \\
&= \left| \int_\Omega \int_\Omega F(y-x)\varphi(x)\phi_{a,\epsilon}(y)dxdy - \int_\Omega F(a-x)\varphi(x)dx \right| \\
&= \left| \int_\Omega \int_\Omega (F(y-x) - F(a-x))\varphi(x)\phi_{a,\epsilon}(y)dxdy \right|
\end{aligned}$$

Now we use continuity of the function F. Let $\eta > 0$. Since the function

$$\Omega \times \Omega \ni (x,y) \mapsto F(y-x) - F(a-x)$$

is continuous, it is uniformly continuous on the set $\operatorname{supp}\varphi \times B(0,\epsilon)$. Therefore there exists $\delta > 0$ such that for any $(x,y),(x,a) \in \operatorname{supp}\varphi \times B(0,\epsilon)$ with $|y-a| < \delta$ we have

$$|F(y-x) - F(a-x)| \leq \frac{\eta}{\|\varphi\|_\infty \cdot |\operatorname{supp}\varphi|}.$$

Hence for $\epsilon < \delta$ we obtain

$$|\langle F_\varphi, F_{a,\epsilon}\rangle - F_\varphi(a)| \leq \eta.$$

This finishes the proof. $\square$

We complete the space $(\mathcal{W}, \langle\cdot,\cdot\rangle)$ and we obtain a Hilbert space, which we denote by $\mathcal{H}_F$. This Hilbert space, as the completion of the space $(\mathcal{W}, \langle\cdot,\cdot\rangle)$, consists of equivalence classes of Cauchy sequences of $\mathcal{W}$, where the equivalence relation is given by

$$(F_{\varphi_n}) \sim (F_{\psi_n}) \Leftrightarrow \|F_{\varphi_n} - F_{\psi_n}\| \to 0.$$

for two Cauchy sequences (F_{φ_n}) and (F_{ψ_n}). The inner-product in $\mathcal{H}_F$ is given by

$$\langle [(F_{\varphi_n})]_\sim, [(F_{\psi_n})]_\sim \rangle = \lim_{n\to\infty} \langle F_{\varphi_n}, F_{\psi_n}\rangle.$$

Moreover, the space $\mathcal{W}$ is a dense subset of $\mathcal{H}_F$ under the identification of an element $F_\varphi \in \mathcal{W}$ with the constant Cauchy sequence (F_φ). The following holds.

Lemma 3.1.3. *For any $a \in \Omega$ the sequence $(F_{a,1/k})_{k\in\mathbb{N}}$ is Cauchy in $\mathcal{W}$. We denote $\gamma_a = [(F_{a,1/k})]_\sim \in \mathcal{H}_F$.*

Proof. The proof is similar to the proof of the lemma 3.1.2. Let $\eta > 0$. For $k, m \in \mathbb{N}$ we have

$$\begin{aligned}\langle F_{a,1/k}, F_{a,1/m}\rangle &= \int_\Omega \int_\Omega F(y-x)\phi_{a,1/k}(x)\phi_{a,1/m}(y)dxdy \\ &= \int_\Omega \int_\Omega (F(y-x) - F(0))\phi_{a,1/k}(x)\phi_{a,1/m}(y)dxdy \\ &+ F(0).\end{aligned}$$

Therefore for $k, m > K$

$$\begin{aligned}&\|F_{a,1/k} - F_{a,1/m}\|^2 \\ &= \|F_{a,1/k}\|^2 + \|F_{a,1/m}\|^2 - \langle F_{a,1/k}, F_{a,1/m}\rangle - \langle F_{a,1/m}, F_{a,1/k}\rangle \\ &= \int_\Omega \int_\Omega (F(y-x) - F(0)) \cdot [\phi_{a,1/k}(x)\phi_{a,1/k}(y) + \phi_{a,1/m}(x)\phi_{a,1/m}(y) \\ &- \phi_{a,1/k}(x)\phi_{a,1/m}(y) - \phi_{a,1/m}(x)\phi_{a,1/k}(y)]dxdy.\end{aligned}$$

Now we use continuity of F at 0. We note that $B(a, 1/k) + B(a, 1/m) \subset B(0, 1/k + 1/m)$. Choose $\delta > 0$ such that $B(0, \delta) \subset \Omega - \Omega$ and such that for $z \in B(0, \delta)$ we have that $|F(z) - F(0)| \leq \eta/4$. Let $K \in \mathbb{N}$ be such that $2/K \leq \delta$. Then for $k, m > K$ we have that $\min\{2/k, 2/m, 1/k + 1/m\} < 2/K \leq \delta$ and

$$\|F_{a,1/k} - F_{a,1/m}\|^2 \leq \eta.$$

This finishes the proof. $\square$

Combining lemma 3.1.2 and lemma 3.1.3 we get that for any $F_\varphi \in \mathcal{W}$ and any $a \in \Omega$

$$F_\varphi(a) = \langle F_\varphi, \gamma_a \rangle. \tag{3.10}$$

We call (3.10) the reproducing identity and we call the collection $\{\gamma_a \colon a \in \Omega\}$ the reproducing kernel for $\mathcal{H}_F$. We deduce that the set span $\{\gamma_a \colon a \in \Omega\}$ is a dense subset of $\mathcal{H}_F$. We also get that

$$\langle \gamma_b, \gamma_a \rangle = F(a-b) \tag{3.11}$$

for any $a, b \in \Omega$. Indeed, by (3.10), for $a, b \in \Omega$ and $k \in \mathbb{N}$ we have

$$F_{b,1/k}(a) = \langle F_{b,1/k}, \gamma_a \rangle.$$

The right had side of the above equality converges to $\langle \gamma_b, \gamma_a \rangle$ when $k \to \infty$. We just need to show that $F_{a,1/k}(b)$ converges to $F(a-b)$. We compute

$$\begin{aligned} F_{b,1/k}(a) &= \int_\Omega F(a-y)\phi_{b,1/k}(y)dy \\ &= \int_\Omega (F(a-y) - F(a-b))\phi_{b,1/k}(y)dy + F(a-b). \end{aligned}$$

The desired result follows by continuity of F at $a-b$.

3.2 Main Theorem

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $F \colon \Omega - \Omega \to \mathbb{C}$ be a continuous positive definite function. For $j = 1, \ldots, n$ we consider densely defined operators $-i\frac{\partial}{\partial x_j} \colon \mathcal{H}_F \to \mathcal{H}_F$, with common domain $\mathcal{W}$, given by

$$\left(-i\frac{\partial}{\partial x_j}\right) F_\varphi = -iF_{\frac{\partial \varphi}{\partial x_j}},$$

where, as we already mentioned, $F_\varphi \in \mathcal{W}$.

The main theorem that we prove relates the problem of existence of an extension of our positive definite function F to the problem of existence of strongly commuting selfadjoint extensions of our operators $-i\frac{\partial}{\partial x_j}$, $j = 1, \dots, n$. However, before we state and prove the main theorem, we need to study operators $-i\frac{\partial}{\partial x_j}$, $j = 1, \dots, n$, more closely. We start with the following fact.

Lemma 3.2.1. *The operators $-i\frac{\partial}{\partial x_j} : \mathcal{W} \to \mathcal{W}$, $j = 1, \dots, n$, are Hermitian; that is for any $F_\varphi, F_\psi \in \mathcal{W}$*

$$\left\langle \left(-i\frac{\partial}{\partial x_j}\right) F_\varphi, F_\psi \right\rangle = \left\langle F_\varphi, \left(-i\frac{\partial}{\partial x_j}\right) F_\psi \right\rangle. \tag{3.12}$$

Proof. We compute using the integration by parts formula

$$\begin{aligned}
\left\langle \left(-i\frac{\partial}{\partial x_j}\right) F_\varphi, F_\psi \right\rangle &= -i \int_{\mathbb{R}^n} \left(F^e \star \frac{\partial \varphi}{\partial x_j}(x) \right) \overline{\psi(x)} dx \\
&= -i \int_{\mathbb{R}^n} \frac{\partial \varphi}{\partial x_j} (F^e \star \varphi(x)) \overline{\psi(x)} dx \\
&= i \int_{\mathbb{R}^n} (F^e \star \varphi(x)) \overline{\frac{\partial \psi}{\partial x_j}(x)} dx \\
&= \left\langle F_\varphi, \left(-i\frac{\partial}{\partial x_j}\right) F_\psi \right\rangle.
\end{aligned}$$

Above we used (3.5) and the fact that for any $\varphi \in C_0^\infty(\Omega)$ the function $F^e \star \varphi$ is smooth ($F^e \star \varphi \in C^\infty(\mathbb{R}^n)$) and

$$\frac{\partial}{\partial x_j}(F^e \star \varphi) = F^e \star \frac{\partial \varphi}{\partial x_j}$$

for any $j = 1, \dots, n$. $\square$

We fix $j = 1, \dots, n$ and we suppose that $A_j \colon D_j \subset \mathcal{H}_F \to \mathcal{H}_F$, with $\mathcal{W} \subset D_j$, is a selfadjoint extension of the operator

$$-i\frac{\partial}{\partial x_j} \colon \mathcal{W} \subset \mathcal{H}_{\mathcal{F}} \to \mathcal{W} \subset \mathcal{H}_{\mathcal{F}}.$$

Let E_j be the spectral measure associated to A_j and let $\{U_j(t) \colon t \in \mathbb{R}\}$ be the strongly continuous one-parameter unitary group defined by A_j i.e.

$$U_j(t) = e^{itA_j} = \int_{\mathbb{R}} e^{itx} dE_j(x)$$

for any $t \in \mathbb{R}$. We recall that A_j is uniquely determined by U_j and that for $f \in D_j$ and $t \in \mathbb{R}$ we have $U_j(t)f \in D_j$. Moreover,

$$D_j = \{f \in \mathcal{H}_F \colon \frac{d}{dt}|_{t=0} U(t)f = \lim_{t \to 0} \frac{U(t)f - f}{t} \text{ exists}\},$$

and for $f \in D_j$ and $s \in \mathbb{R}$

$$A_j f = -i\frac{d}{dt}|_{t=0} U_j(t)f \quad \text{and} \quad \frac{d}{dt}|_{t=s} U_j(t)f = iU_j(s)A_j f = iA_j U_j(s)f$$

(see appendix for more details). We denote by $\{e_1, \dots, e_n\}$ the standard basis of $\mathbb{R}^n$. The lemma below shows that the strongly continuous one-parameter unitary group of operators $\{U_j(t) \colon t \in \mathbb{R}\}$ acts by translations. More precisely, the following holds.

Lemma 3.2.2. *Let $a, b \in \mathbb{R}$ with $a < b$ be such that $[ae_j, be_j] = \{a(1-t)e_j + bte_j \colon 0 \leq t \leq 1\} \subset \Omega$. Then*

$$U_j(b-a)\gamma_{ae_j} = \gamma_{be_j}. \tag{3.13}$$

Proof. Let $a, b \in \mathbb{R}$ be such that $[ae_j, be_j] \subset \Omega$, let $F_\varphi \in \mathcal{W}$, and let (ϕ_ϵ) be as before.

Assume that $a < b$. For each $\epsilon > 0$ we consider a function $g_\epsilon : [0, b-a] \to \mathbb{C}$ given by

$$
\begin{aligned}
g_\epsilon(t) &= \langle F_\varphi, U_j(b-a-t)F_{(a+t)e_j,\epsilon}\rangle \\
&= \langle U_j(t-b+a)F_\varphi, F_{(a+t)e_j,\epsilon}\rangle.
\end{aligned}
$$

We will show that g is a differentiable function with $g'_\epsilon(t) = 0$ for any $t \in (0, b-a)$. We compute

$$
\begin{aligned}
g'_\epsilon(t) &= \langle \frac{d}{dt}U_j(t-b+a)F_\varphi, F_{(a+t)e_j,\epsilon}\rangle + \langle U_j(t-b+a)F_\varphi, \frac{d}{dt}F_{(a+t)e_j,\epsilon}\rangle \\
&= \langle iA_jU_j(t-b+a)F_\varphi, F_{(a+t)e_j,\epsilon}\rangle + \langle U_j(t-b+a)F_\varphi, \frac{\partial}{\partial x_j}F_{(a+t)e_j,\epsilon}\rangle \\
&= -\langle U_j(t-b+a)F_\varphi, \frac{\partial}{\partial x_j}F_{(a+t)e_j,\epsilon}\rangle + \langle U_j(t-b+a)F_\varphi, \frac{\partial}{\partial x_j}F_{(a+t)e_j,\epsilon}\rangle \\
&= 0.
\end{aligned}
$$

Above we used the fact that

$$
\frac{d}{dt}F_{(a+t)e_j,\epsilon} = \frac{\partial}{\partial x_j}F_{(a+t)e_j,\epsilon}. \tag{3.14}
$$

To see this we need to show that

$$
\frac{1}{h}\left(F_{(a+t+h)e_j,\epsilon} - F_{(a+t)e_j,\epsilon}\right) \to \frac{\partial}{\partial x_j}F_{(a+t)e_j,\epsilon} \quad \text{when } h \to 0,
$$

where the convergence is in $\mathcal{H}_F$. This follows, by the dominated convergence theorem, from the fact that for any x

$$
\frac{\phi_{(a+t+h)e_j,\epsilon}(x) - \phi_{(a+t)e_j,\epsilon}(x)}{h} \to \frac{\partial}{\partial x_j}\phi_{(a+t)e_j,\epsilon}(x) \quad \text{when } h \to 0.
$$

Therefore our function g_ϵ is constant. In particular, $g(b-a) = g(0)$. This means that

$$
\langle F_\varphi, F_{be_j,\epsilon} - U_j(b-a)F_{ae_j,\epsilon}\rangle = 0. \tag{3.15}
$$

Since (3.15) holds for any $F_\varphi \in \mathcal{W}$ we deduce that

$$F_{be_j,\epsilon} = U_j(b-a)F_{ae_j,\epsilon}. \tag{3.16}$$

We take the limit in (3.16) when $\epsilon \to 0$ to obtain (3.13). $\square$

Let, for each $j = 1, \dots, n$, an operator $A_j \colon D_j \subset \mathcal{H}_F \to \mathcal{H}_F$, with $\mathcal{W} \subset D_j$, be a selfadjoint extension of the operator

$$-i\frac{\partial}{\partial x_j} \colon \mathcal{W} \subset \mathcal{H}_\mathcal{F} \to \mathcal{W} \subset \mathcal{H}_\mathcal{F},$$

let E_j be the spectral measure associated to A_j, and let $\{U_j(t)\colon t \in \mathbb{R}\}$ be the strongly continuous one-parameter unitary group of operators defined by A_j. Suppose that operators $A_1, \dots, A_n$ strongly commute. This implies that for any $j, k = 1, \dots, n$ and any $s, t \in \mathbb{R}$ the unitary operators $U_j(s)$ and $U_k(t)$ commute in the sense that

$$U_j(s)U_k(t) = U_k(t)U_j(s).$$

For each $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ we define a unitary operator $U(t)$ by

$$U(t) = U(t_1) \cdot \ldots \cdot U(t_n).$$

Then for any $c_1, c_2 \in \mathbb{R}^n$ we have that

$$U(c_1 + c_2) = U(c_1)U(c_2) = U(c_2)U(c_1).$$

The following fact holds true.

Corollary 3.2.3. *Let $a, b \in \Omega$ with Ω connected. Then*

$$U(a-b)\gamma_b = \gamma_a. \tag{3.17}$$

Proof. Let $c_1, \dots, c_m \in \Omega$ be such that the vectors $a - c_1, c_2 - c_1, \dots, c_m - c_{m-1}, b - c_m \in \mathbb{R}^n$ are parallel to the coordinate axis and such that the intervals

$$[a, c_1], [c_1, c_2], \dots, [c_{m-1}, c_m], [c_m, b]$$

are contained in Ω. Then, by the previous lemma,

$$U(a-b)\gamma_b = U(a-c_1)U(c_1-c_2)\dots U(c_m-b)\gamma_b = \gamma_a.$$

□

We are ready to state and prove our main theorem. We denote by $\mathcal{B}_n$ the σ-algebra of Borel subsets of $\mathbb{R}^n$.

Main Theorem. Let $F\colon \Omega - \Omega \to \mathbb{C}$ be a continuous positive definite function, where Ω is an open and connected subset of $\mathbb{R}^n$. Then there exists an extension of F to a continuous positive definite function defined on the entire Euclidean space $\mathbb{R}^n$ if and only if there exist selfadjoint strongly commuting extensions of the operators $-i\frac{\partial}{\partial x_j}$, $j = 1, \dots, n$.

Proof of the Main Theorem. ($\Leftarrow$) Suppose that there exist strongly commuting selfadjoint operators $A_j\colon D_j \subset \mathcal{H}_F \to \mathcal{H}_F$ that extend operators $-i\frac{\partial}{\partial x_j}\colon \mathcal{W} \subset \mathcal{H}_{\mathcal{F}} \to \mathcal{W} \subset \mathcal{H}_{\mathcal{F}}$, $j = 1, \dots, n$. Let E_j be the spectral measure associated to A_j. We define, as above, the following strongly continuous n-parameter unitary group of transformations of $\mathcal{H}_F$. For $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ put

$$U(t) = \prod_{j=1}^{n} U_j(t_j) = \prod_{j=1}^{n} \int_{\mathbb{R}} e^{it_j x_j} dE_j(x_j) = \int_{\mathbb{R}^n} e^{i\langle t, x\rangle} dE(x),$$

where $E = E_1 \times \ldots \times E_n$ is the product spectral measure. We fix $x_0 \in \Omega$ and we define a function $\widetilde{F}\colon \mathbb{R}^n \to \mathbb{C}$ by

$$\widetilde{F}(t) = \langle \gamma_{x_0}, U(t)\gamma_{x_0} \rangle.$$

Hence, if we denote by μ the finite positive Borel measure

$$\mathcal{B}_n \ni \Delta \to \langle \gamma_{x_0}, E(\Delta)\gamma_{x_0} \rangle,$$

then

$$\widetilde{F}(t) = \int_{\mathbb{R}^n} e^{i\langle t,x \rangle} d\mu(x) = \hat{\mu}(t).$$

Therefore $\widetilde{F}$ is a continuous positive definite function. We will show that $\widetilde{F}$ is an extension of our function F. For $a - b \in \Omega - \Omega$ we obtain

$$\begin{aligned}
\widetilde{F}(a-b) &= \langle \gamma_{x_0}, U(a-b)\gamma_{x_0} \rangle \\
&= \langle U(b)\gamma_{x_0}, U(a)\gamma_{x_0} \rangle \\
&= \langle U(x_0)U(b-x_0)\gamma_{x_0}, U(x_0)U(x_0-a)\gamma_{x_0} \rangle \\
&= \langle U(x_0)\gamma_b, U(x_0)\gamma_a \rangle \\
&= \langle \gamma_b, \gamma_a \rangle \\
&= F(a-b).
\end{aligned}$$

This proves that $\widetilde{F}\colon \mathbb{R}^n \to \mathbb{C}$ is a continuous positive definite extension of our function F.

($\Rightarrow$) Let $\widetilde{F}\colon \mathbb{R}^n \to \mathbb{C}$ be a continuous positive definite function that extends our function F. Then, by a theorem of Bochner, there exists a finite positive Borel measure μ such that $F = \hat{\mu}$. For any $F_\varphi, F_\psi \in \mathcal{W}$ we have

$$\begin{aligned}\langle F_\varphi, F_\psi \rangle &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(y-x)\varphi(x)\overline{\psi(y)}dxdy \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-i\langle t,x \rangle} \varphi(x)dx \right) \left(\overline{\int_{\mathbb{R}^n} e^{-i\langle t,y \rangle} \psi(y)dy} \right) d\mu(t).\end{aligned}$$

We denote for $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\widehat{\varphi}(t) = \int_{\mathbb{R}^n} e^{-i\langle t,x \rangle} \varphi(x)dx,$$

where $t \in \mathbb{R}^n$. Then

$$\langle F_\varphi, F_\psi \rangle = \int_{\mathbb{R}^n} \widehat{\varphi}(t)\overline{\widehat{\psi}(t)}d\mu(t). \tag{3.18}$$

Therefore the map

$$\mathcal{W} \ni F_\varphi \mapsto \widehat{\varphi} \in L^2(d\mu)$$

is a linear isometry, which can be extended to a linear isometry $W\colon \mathcal{H}_F \to L^2(d\mu)$. We denote by $W^*\colon L^2(d\mu) \to \mathcal{H}_F$ the adjoint operator to W. Notice that the range $R(W)$ of W is a closed subspace of $L^2(d\mu)$. We denote $W^{-1}\colon R(W) \to \mathcal{H}_F$ the inverse of W. Since $N(W^*) = R(W)^\perp$ we deduce the following. Let $y = u + v \in R(W) \otimes R(W)^\perp = L^2(d\mu)$. Then $W^*y = W^{-1}u$.

For $t \in \mathbb{R}^n$ we consider the multiplication operator on $L^2(d\mu)$ by the bounded function $x \mapsto e^{i\langle t,x \rangle}$, which we denote by $V(t)$ i.e.

$$V(t) = e^{i\langle t,\cdot \rangle}\colon L^2(d\mu) \to L^2(d\mu). \tag{3.19}$$

We obtain a strongly continuous n-parameter unitary group $\{V(t)\colon t \in \mathbb{R}^n\}$ of transformations on $L^2(d\mu)$. Therefore $\{W(t) = W^\star V(t) W\colon t \in \mathbb{R}^n\}$ is a strongly continuous n-parameter unitary group of transformations on $\mathcal{H}_F$, since for any $t \in \mathbb{R}^n$ and any $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$V(t)\widehat{\varphi} = \widehat{\varphi(\cdot - t)}.$$

In particular, $V(t)$ preserves the range of W i.e. $V(t)R(W) \subset R(W)$.

For a fixed $j = 1, \ldots, n$ we put

$$D_j = \{f \in \mathcal{H}_F \colon \frac{\partial}{\partial t_j}|_{t=0} W(t)f = \lim_{h \to 0} \frac{W(he_j)f - f}{h} \text{ exists}\},$$

and we define an operator $A_j \colon D_j \to \mathcal{H}_F$ by

$$A_j f = -i\frac{\partial}{\partial t_j}|_{t=0} W(t)f.$$

Then, it is well-known, that the operators $A_1, \ldots, A_n$ are selfadjoint and that they strongly commute (see appendix). We will show that operators $A_1, \ldots, A_n$ extend differential operators $-i\frac{\partial}{\partial x_j} \colon \mathcal{W} \to \mathcal{W}$, $j = 1, \ldots, n$. We divide the proof into three steps. Fix $j = 1, \ldots, n$.

Step 1. We will prove that $\mathcal{W} \subset D_j$. We need to show that for any $F_\varphi \in \mathcal{W}$ the limit

$$\lim_{h \to 0} \frac{W(he_j)F_\varphi - F_\varphi}{h} \tag{3.20}$$

exists. We compute

$$\begin{aligned} \frac{W(he_j)F_\varphi - F_\varphi}{h} &= \frac{W^\star V(t)\widehat{\varphi} - W^\star \widehat{\varphi}}{h} \\ &= W^\star \left(\frac{V(he_j) - 1}{h} \widehat{\varphi} \right) \end{aligned}$$

We just need to show that

$$\frac{V(he_j) - 1}{h} \widehat{\varphi} \tag{3.21}$$

converges in $L^2(d\mu)$ when $h \to 0$. For any $x \in \mathbb{R}^n$

$$\frac{V(he_j) - 1}{h}\widehat{\varphi}(x) = \frac{e^{ihx_j} - 1}{h}\widehat{\varphi}(x)$$

converges to $ix_j\widehat{\varphi}(x)$ when $h \to 0$. Note that the function $\mathbb{R}^n \ni x \mapsto ix_j\widehat{\varphi}(x)$ is bounded. By the dominated convergence theorem we deduce that the limit (3.21) exists in $L^2(d\mu)$ and that it equals $\mathbb{R}^n \ni x \mapsto ix_j\widehat{\varphi}(x)$.

Step 2. Let $F_\varphi \in \mathcal{W}$. We will show that the function $\mathbb{R}^n \ni x \mapsto ix_j\widehat{\varphi}(x)$ is in the range $R(W)$ of W. We have for $x \in \mathbb{R}^n$

$$\begin{aligned} ix_j\widehat{\varphi}(x) &= \widehat{\frac{\partial\varphi}{\partial x_j}}(x) \\ &= (WF_{\frac{\partial\varphi}{\partial x_j}})(x). \end{aligned}$$

Step 3. We will show that the operator A_j extends operator $-i\frac{\partial}{\partial x_j}$. We compute for $F_\varphi \in \mathcal{W}$

$$\begin{aligned} A_jF_\varphi &= -i\frac{\partial}{\partial t_j}|_{t=0}W(t)F_\varphi \\ &= -iW^\star WF_{\frac{\partial\varphi}{\partial x_j}} \\ &= -iW^{-1}WF_{\frac{\partial\varphi}{\partial x_j}} \\ &= -i\frac{\partial}{\partial x_j}F_\varphi. \end{aligned}$$

This finishes the proof. □

Corollary 3.2.4. *A continuous positive definite function $F\colon \Omega - \Omega \to \mathbb{C}$, with $\Omega = (p, r) \subset \mathbb{R}$ being an open interval, has always an extension to a continuous positive definite function defined on the entire real line $\mathbb{R}$.*

Proof. We need to show that the operator

$$-i\frac{d}{dx}\colon \mathcal{W} \subset \mathcal{H}_F \to \mathcal{W} \subset \mathcal{H}_F$$

has a selfadjoint extension. We define $J\colon (p,r) \to (p,r)$ by $J(t) = -t + p + r$. For $\varphi \in C_0^\infty(\Omega)$ put $\varphi_J\colon \Omega \to \mathbb{C}$ by $\varphi_J(t) = \overline{\varphi(J(t))}$ and then set $JF_\varphi = F_{\varphi_J}$. Then $J\colon \mathcal{W} \to \mathcal{W}$ is a conjugation i.e. J is conjugate-linear $J(iF_\varphi) = -iJF_\varphi$, $J^2 = \mathrm{Id}$, and $\|JF_\varphi\| = \|F_\varphi\|$ for any $F_\varphi \in \mathcal{W}$. Moreover $J \circ (-i\frac{d}{dx}) = (-i\frac{d}{dx}) \circ J$. This follows from the fact that J is conjugate-linear and that

$$\frac{d\varphi_J}{dx} = -\left(\frac{d\varphi}{dx}\right)_J.$$

Thus $-i\frac{d}{dx}$, as a Hermitian operator that commutes with a conjugation, has an extension to a selfadjoint operator, by a theorem of von Neumann (see appendix). $\square$

Example 3.2.5. Here we give an example of a continuous function positive definite function defined on an open, but not connected subset of $\mathbb{R}$ that cannot be extended to a continuous positive definite function defined on the entire $\mathbb{R}$. This shows that we cannot drop the assumption of connectedness of a set Ω in the statement of the main theorem.

We put $\Omega = (-\frac{1}{4}, \frac{1}{4}) \cup (\frac{3}{4}, 1)$. We also denote $\Omega_A = (-\frac{1}{4}, \frac{1}{4})$ and $\Omega_B = (\frac{3}{4}, 1)$. Since Ω_A is a symmetric and convex set, we deduce that $\Omega_A - \Omega_A = 2\Omega_A = (-\frac{1}{2}, \frac{1}{2})$. Moreover,

$$\Omega_A - \Omega_B = \left(-1, -\frac{3}{4}\right) + \left(-\frac{1}{4}, \frac{1}{4}\right) = \left(-\frac{5}{4}, -\frac{1}{2}\right),$$

$\Omega_A - \Omega_B = (\frac{1}{2}, \frac{5}{4})$, and

$$\Omega_B - \Omega_B = \left(-1, -\frac{3}{4}\right) + \left(\frac{3}{4}, 1\right) = \left(-\frac{1}{4}, \frac{1}{4}\right).$$

Therefore $\Omega - \Omega = (-\frac{5}{4}, -\frac{1}{2}) \cup (-\frac{1}{2}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{5}{4})$. We define a function $F\colon \Omega - \Omega \to [0,1]$ as follows.

$$F(x) = \begin{cases} 1 - |x| & x \in (-\frac{1}{2}, \frac{1}{2}) \\ \\ 0 & x \in (-\frac{5}{4}, -\frac{1}{2}) \cup (\frac{1}{2}, \frac{5}{4}). \end{cases}$$

We will show that F is a positive definite function. Let $x_1, \ldots, x_m \in \Omega$ and let $c_1, \ldots, c_m \in \mathbb{C}$. Then

$$\sum_{j,k=1}^{m} F(x_j - x_k) c_j \overline{c_k} = \sum_{x_j, x_k \in \Omega_A} F(x_j - x_k) c_j \overline{c_k} + \sum_{x_j, x_k \in \Omega_B} F(x_j - x_k) c_j \overline{c_k}$$

Both summations above are nonnegative, since the function $\mathbb{R} \ni x \to \max(0, 1 - |x|)$ is positive definite (see example 1.31). Clearly F has no extension to a continuous function defined on the entire real line.

CHAPTER 4
REMARKS

We start this section with two open problems. Then, we follow with a remark where we show that the reproducing kernel Hilbert space associated to a continuous positive definite function F defined on an open subset of $\mathbb{R}^n$ is unitary isomorphic to the Hilbert space $\mathcal{H}_F$.

Question 1. In this thesis we study extension problem for continuous positive definite functions from a connected open set to the entire Euclidean space $\mathbb{R}^n$. A closely related problem is the following. Under what conditions on the set or the function do we have uniqueness of an extension of a positive definite function. Also, in the case of non-uniqueness of extension, can we give a (geometric) description of the set of all possible extensions?

We should mention that we already know, that in general, there is no uniqueness. This follows from Polya's criterion (see example 1.30). A brief account of some results in this direction can be found in [S].

Question 2. Using the integral characterization of continuous positive definite functions (theorem 3.1) one can define the notion of a positive definite distribution (generalized function) (see [St]). Therefore the extension problem can be studies in the setting of distributions. In [G] the author proves that an extension always exists for positive definite distributions defined on a subset of the real line. This generalizes the result of Krein [Kr]. However, the extension problem for positive definite distributions in higher dimensions is untouched, and it would be interesting to see what happens there.

Remark 4.0.6. In the first chapter we gave a construction that associate a reproducing kernel Hilbert space to a positive definite kernel. Therefore we can associate a reproducing kernel Hilbert space to a positive definite function. Let $F\colon \Omega - \Omega \to \mathbb{C}$ be a continuous positive definite function, where Ω is an open subset of $\mathbb{R}^n$. We briefly recall the construction of the reproducing kernel Hilbert space corresponding to F. For each $a \in \Omega$ we define a function $k_a\colon \Omega \to \mathbb{R}$ by

$$k_a(x) = F(x-a).$$

Then the space $W_F = \operatorname{span}\{k_a\colon a \in \Omega\}$ together with the inner-product

$$\langle \sum_{j=1}^{m} c_j k_{a_j}, \sum_{j=1}^{m} d_j k_{a_j} \rangle_F = \sum_{j,k=1}^{m} F(a_k - a_j) c_j \overline{d_k}$$

is a complex pre-Hilbert space. Its completion H_F is called the reproducing kernel Hilbert space associated to F. We note that

$$\langle k_a, k_b \rangle_F = F(b-a).$$

We also recall that we associated a Hilbert space $\mathcal{H}_F$ to our function F and we had elements $\gamma_a \in \mathcal{H}_F$, where $a \in \Omega$ such that $\langle \gamma_a, \gamma_b \rangle = F(b-a)$. Therefore we conclude the following. Since

$$\sum_{j=1}^{m} c_j \gamma_{a_j} = 0 \Leftrightarrow \sum_{j,k=1}^{m} F(a_k - a_j) c_j \overline{c_k} = 0 \Leftrightarrow \sum_{j=1}^{m} c_j k_{a_j} = 0,$$

the assignment

$$\gamma_a \mapsto k_a$$

gives rise to a well-defined map between Hilbert spaces $\mathcal{H}_F$ and H_F and this map is a unitary isomorphism between Hilbert spaces $\mathcal{H}_F$ and H_F.

APPENDIX A
SPECTRAL MEASURES

In this section we gather some facts about spectral theory of unbounded operators defined on a complex Hilbert space. We follow closely the book ([Rud]) of Rudin and the recent book ([Sch]) of Schmudgen.

A.1 The adjoint operator

Let H be a complex Hilbert space. An operator on H is a linear map

$$T\colon D(T) \subset H \to H,$$

where $D(T)$ is a subspace of H. We call $D(T)$ the domain of T. We say that an operator T is densely defined if $D(T)$ is a dense subspace of H, and we say that T is closed if the graph $G(T) = \{(x, Tx) \in H \times H\colon x \in D(T)\}$ is a closed subspace of $H \times H$. We view $H \times H$ as a Hilbert space with inner-product given by $\langle (a,b),(x,y)\rangle = \langle a,x\rangle + \langle b,y\rangle$ for $a,b,x,y \in H$. For two operators S and T on H we write $S \subset T$ if $G(S) \subset G(T)$. We denote the set of all operators on H be $Op(H)$. We note that for two operators $S,T \in Op(H)$ we have operators $S+T$ and ST, where the domain of $S+T$ is $D(S+T) = D(S) \cap D(T)$ and the domain of ST is $D(ST) = \{x \in D(T)\colon Tx \in D(S)\}$. Then for three operators R,S and T we have

$$(R+S)+T = R+(S+T) \quad \text{and} \quad (RS)T = R(ST)$$

and

$$(R+S)T = RT + ST \quad \text{and} \quad R(T+S) \supset RT + ST.$$

An operator T on H is called bounded if $D(T) = H$ and if there exists a constant $C > 0$ such that for any $x \in H$ we have that $\|Tx\| \leq C\|x\|$. We denote the set of all bounded operator on H by $B(H)$. It is a Banach space with the norm given by $\|T\| = \sup_{\|x\|=1} \|Tx\|$.

For an operator $T \in Op(H)$ we put

$$R(T) = \{Tx \in H \colon x \in D(T)\} \quad \text{and} \quad N(T) = \{x \in D(T) \colon Tx = 0\}.$$

We call $R(T)$ the range of T and $N(T)$ the kernel of T.

The adjoint operator to a bounded operator $T \in B(H)$ is the unique operator $T^\star \in B(H)$ satisfying

$$\langle Tx, y \rangle = \langle x, T^\star y \rangle$$

for $x, y \in H$. Then

$$\|T\| = \|T^\star\| \quad \text{and} \quad \|TT^\star\| = \|T\|^2$$

We say that $T \in B(H)$ is normal if $TT^\star = T^\star T$; selfadjoint if $T^\star = T$; positive definite if $\langle Tx, x \rangle \geq 0$ for all $x \in H$; unitary if $TT^\star = I = T^\star T$; a projection if $T^2 = T$.

Theorem A.0.7. *Let $T \in B(H)$ be a projection. Then the following statements are equivalent:*

1. *T is normal.*

2. *T is selfadjoint.*

3. *$R(T) = N(T)^\perp$.*

4. For any $x \in H$ we have that $\langle Tx, x\rangle = \|Tx\|^2$.

Moreover if $P, Q \in B(H)$ are selfadjoint projections, then the following holds:

$$PQ = 0 \Leftrightarrow R(P) \perp R(Q).$$

The definition of the adjoint operator associated to any arbitrary operator is a little bit more involved. Let H be a complex Hilbert space. Let T be a densely defined operator on H. We define the set $D(T^\star)$ as follows. We say that $y \in D(T^\star)$ if the functional

$$D(T) \ni x \mapsto \langle Tx, y\rangle$$

is bounded. Then, it has a unique extension to a bounded functional S_y defined on the entire space H. By the Riesz representation theorem we get a unique element $T^\star y \in H$ such that

$$S_y(x) = \langle x, T^* y\rangle \quad x \in H.$$

In particular, for any $x \in D(T)$

$$\langle Tx, y\rangle = \langle x, T^\star y\rangle.$$

The set $D(T^\star)$ is a subspace of H and the map T^* defined by the assignment

$$D(T^\star) \ni y \mapsto T^\star y \in H$$

is a linear operator, called the adjoint operator to T.

Let T be an operator on H. We say that T is selfadjoint if T is densely defined and if $T^\star = T$; symmetric if for any $x, y \in D(T)$

$$\langle Tx, y\rangle = \langle x, Ty\rangle.$$

Therefore, if T is a densely defined operator, then T is symmetric if and only if $T \subset T^{\star}$. A densely defined closed operator T is called normal if $T^{\star}T = TT^{\star}$.

A.1 Von Neumann Extension Theorem

Let H be a complex Hilbert space. We are interested in the question when a symmetric operator can be extended to a selfadjoint operator. We start with the following.

Theorem A.1.1. *Let T be a symmetric operator on H. Then*

1. *For any $x \in D(T)$*

$$\|Tx + ix\|^2 = \|x\|^2 + \|Tx\|^2 = \|Tx - ix\|^2.$$

2. *T is closed if and only if $R(T + iI)$ is a closed set and, similarly, T is closed if and only if $R(T - iI)$ is a closed set.*

3. *Both operators $T + iI$ and $T - iI$ are one-to-one.*

Hence if T is a symmetric operator, then we can define the following isometry $U\colon R(T + iI) \to R(T - iI)$, called the Cayley transform,

$$U(Tx + ix) = Tx - ix$$

for $x \in D(T)$. By "isometry" we mean that $\|U(z)\| = \|z\|$ for any $z \in D(U) = R(T + iI)$. Then we have.

Theorem A.1.2. *Let T be a symmetric operator with the corresponding Cayley transform U. Then*

1. *U is closed if and only if T is closed.*

2. *U is unitary if and only if T is unitary.*

3. *$D(T) = R(I - U)$, $I - U$ is one-to-one and*

$$T = i(I + U)(I - U)^{-1}.$$

Let T be a symmetric operator on H. The dimensions of the spaces

$$R(T + iI)^{\perp} \quad \text{and} \quad R(T - iI)^{\perp}$$

are called the deficiency indices of T, and we denote them by d_+ and d_- respectively.

Theorem A.1.3. *(von Neumann) Let T be a symmetric operator. Then*

1. *T is selfadjoint if and only if the deficiency indices of T are zero.*

2. *T has a selfadjoint extension if and only if the deficiency indices of T are equal i.e. $d_+ = d_-$.*

We finish this section with a useful criterion for existence of selfadjoint extensions. A conjugation on H is a conjugate linear map $J\colon H \to H$, that is

$$J(ax + by) = \overline{a}Jx + \overline{b}Jy$$

for $a, b \in \mathbb{C}$, $x, y \in H$, such that $\langle Jx, Jy\rangle = \langle y, x\rangle$ for any $x, y \in H$ and such that $J^2 = I$.

Theorem A.1.4. *Let J be a conjugation and let T be a symmetric operator such that $JD(T) \subset D(T)$ and such that $JTx = TJx$ for $x \in D(T)$. Then T has an extension to a selfadjoint operator.*

A.2 Spectral measures

Let H be a complex Hilbert space and let $(X, \mathfrak{m})$ be a measure space i.e. X is a set and $\mathfrak{m}$ is a σ-algebra of subset of X. A map $E\colon \mathfrak{m} \to B(H)$ is called a spectral measure if

1. $E(\emptyset) = 0$, $E(X) = I$.

2. $E(A)$ is a selfadjoint projection for any $A \in \mathfrak{m}$.

3. $E(A \cap B) = E(A)E(B)$ for any $A, B \in \mathfrak{m}$.

4. For any disjoint sets $A_n \in \mathfrak{m}$, $n \in \mathbb{N}$, and any $x \in H$

$$\sum_{n=1}^{\infty} E(A_n)x = E(\cup_{n=1}^{\infty} A_n)x.$$

Let $E\colon \mathfrak{m} \to B(H)$ be a spectral measure. For any $x, y \in H$ we define $E_{x,y}\colon \mathfrak{m} \to \mathbb{C}$ by

$$E_{x,y}(A) = \langle E(A)x, y\rangle$$

Then $E_{x,y}$ is a complex measure with total variation bounded by $\|x\|\|y\|$. Moreover $E_{x,x}$ is a finite positive measure with $E_{x,x}(X) = \|x\|^2$. If the measure space $(X, \mathfrak{m})$ is a Borel measure space i.e. X is a topological space and $\mathfrak{m}$ is the σ-algebra of Borel subsets of X, then as part of the defition of a spectral measure we require that the measures $E_{x,y}$ are Borel regular. We have the following.

Theorem A.2.1. *Let $X_1, \ldots, X_n$ be locally compact Hausdorff spaces, which are second countable, and let E_j be a Borel spectral measure on X_j, $j = 1, \ldots, n$. Suppose* *that the spectral measures* $E_1, \ldots, E_n$ *commute; that is for any* $i, j = 1, \ldots, n$ *and any Borel sets* $A \subset X_i$ *and* $B \subset X_j$

$$E_i(A)E_j(B) = E_j(B)E_i(A). \tag{A.1}$$

Then there exists a unique Borel spectral measure E *on* $X_1 \times \ldots \times X_n$ *such that*

$$E(A_1 \times \ldots \times A_n) = E(A_1) \ldots E(A_n)$$

for any Borel sets $A_1 \subset X_1$, $A_n \subset X_n$. *This unique spectral measure* E *is called the product measure and is denoted by* $E_1 \times \ldots \times E_n$.

We define the integral with respect to a spectral measure. Let E be a spectral measure and let $f\colon X \to \mathbb{C}$ be a bounded measurable function. We define a form $B\colon H \times H \to \mathbb{C}$ by

$$B(x, y) = \int_X f(s) dE_{x,y}(s).$$

Then

$$|B(x, y)| \leq \|f\|_\infty \|x\| \|y\|.$$

Therefore there is a unique operator $\pi(f) \in B(H)$ such that

$$B(x, y) = \langle \pi(f)x, y \rangle$$

for any $x, y \in H$. We call $\pi(f)$ the integral of f with respect to the spectral measure E. We will often write

$$\pi(f) = \int_X f(s) dE(s).$$

Then the integral is linear; that is

$$\pi(f + g) = \pi(f) + \pi(g) \quad \text{and} \quad \pi(af) = a\pi(f),$$

where $f, g\colon X \to \mathbb{C}$ are bounded measurable functions and $a \in \mathbb{C}$. Moreover for $A \in \mathfrak{m}$

$$\pi(\chi_A) = E(A),$$

$\pi(f)$ is a normal operator, and if f is real valued, the $\pi(f)$ is a selfadjoint operator. Furthermore, for any $x, y \in H$ and any bounded functions $f, g\colon X \to \mathbb{C}$

$$\langle \pi(f)x, \pi(g)y \rangle = \int_X f(s)\overline{g}(s)dE_{x,y}(s). \tag{A.2}$$

In particular,

$$\|\pi(f)x\|^2 = \langle \pi(f)x, x \rangle = \int_X |f|^2(s)dE_{x,x}(s). \tag{A.3}$$

We now define the integral with respect to a spectral measure of any (not necessary bounded) measurable function. Let $(X, \mathfrak{m})$ be a measure space and let $E\colon \mathfrak{m} \to B(H)$ be a spectral measure. Let $f\colon X \to \mathbb{C} \cup \{\infty\}$ be a measurable function with $E\{s \in X\colon f(s) = \infty\} = 0$. For any $n \in \mathbb{N}$ we put

$$M_n = \{s \in X\colon |f(s)| \leq n\}.$$

We define a subspace $D(\pi(f))$ of H as follows. We say that $x \in D(\pi(f))$ if the sequence $(\pi(\chi_{M_n} f)x)_{n\in\mathbb{N}}$ converges in H. We set for $x \in D(\pi(f))$

$$\pi(f)x = \lim_{n\to\infty} \pi(\chi_{M_n} f)x$$

and we call $\pi(f)\colon D(\pi(f)) \subset H \to H$ the integral of f with respect to the spectral measure E. Again, we often write

$$\pi(f) = \int_X f(s)dE(s).$$

Then, as before, this integral is linear, $\pi(f)$ is a normal operator (in particular it is a closed operator), if f is real valued, then $\pi(f)$ is a selfadjoint operator, and identities (A.2), and (A.3) hold true. Moreover,

$$D(\pi(f)) = \{x \in H \colon \int_X |f(s)|^2 dE_{x,x}(s) < \infty\}.$$

We are ready to state the spectral theorem for selfadjoint operators. Let H be a complex space and let T be a selfadjoint operator on H. Then there exists a unique spectral measure E defined on the σ-algebra of Borel subsets of the real line such that

$$T = \int_{\mathbb{R}} z dE(z).$$

In particular,

$$D(T) = \{x \in H \colon \int_{\mathbb{R}} |z|^2 E_{x,x}(z) < \infty\}.$$

We say that selfadjoint operators $T_1, \ldots, T_n$ strongly commute if the corresponding spectral measures commute. Therefore, if selfadjoint operators $T_1, \ldots, T_n$ strongly commute and if $E_1, \ldots, E_n$ are corresponding spectral measures, then with respect to the product measure $E = E_1 \times \ldots \times E_n$ we have that

$$T_j = \int_{\mathbb{R}^n} z_j dE(z_1, \ldots, z_n),$$

where $j = 1, \ldots, n$.

A.3 Strongly continuous unitary groups of operators

Let H be a complex Hilbert space. A collection $\{U(t) \colon t \in \mathbb{R}^n\}$ is called a strongly continuous n-parameter unitary group (of transformations) on H if for any $t \in \mathbb{R}$ the operator $U(t)$ is unitary and if

1. For any $t, s \in \mathbb{R}^n$

$$U(t)U(s) = U(t+s).$$

2. For any $x \in H$ and $t \in \mathbb{R}^n$

$$\lim_{h \to 0} U(t+h)x = U(t)x.$$

We often say "a unitary representation of $\mathbb{R}^n$ on H" instead of saying "a strongly continuous n-parameter unitary group on H". From the above definition we deduce that $U(0) = I$ and that $U(-t) = U(t)^{-1} = U(t)^\star$ for any $t \in \mathbb{R}^n$.

Let A be a selfadjoint operator on H and let E_A be the spectral measure associated to A. For each $t \in \mathbb{R}$ we define a bounded operator e^{itA} by

$$e^{itA} = \int_{\mathbb{R}^n} e^{it\xi} dE_A(\xi).$$

Theorem A.3.1. *The collection of operators $\{U(t) = e^{itA} \colon t \in \mathbb{R}\}$ is a strongly continuous one-parameter unitary group on H. Moreover,*

$$D(A) = \{x \in H \colon \frac{d}{dt}|_{t=0} U(t)x = \lim_{t \to 0} \frac{U(t) - I}{t} \text{ exists}\} \tag{A.4}$$

$$= \{x \in H \colon \frac{d^+}{dt}|_{t=0} U(t)x = \lim_{t \to 0^+} \frac{U(t) - I}{t} \text{ exists}\} \tag{A.5}$$

and we can recover A from the collection $U(t)$ by the formula

$$iAx = \frac{d}{dt} U(t)x = \frac{d^+}{dt} U(t)x, \tag{A.6}$$

where $x \in D(A)$. What is more, for any $x \in D(A)$ and any $t \in \mathbb{R}$ we have that $U(t)x \in D(A)$ and

$$\frac{d}{dt} U(t)x = iAU(t)x = iU(t)Ax.$$

A theorem of Stone says that any strongly continuous one-parameter unitary group on H is of the form $\{e^{itA} \colon t \in \mathbb{R}\}$ for some uniquely determined selfadjoint operator A. More precisely, A is defined by (A.4) and (A.6).

Moreover, the following holds true.

Theorem A.3.2. *Let S and T be two selfadjoint operators on H. Then they strongly commute if and only if for any $s, t \in \mathbb{R}$*

$$e^{isS}e^{itT} = e^{itT}e^{isS}.$$

Hence, if $A_1, \ldots, A_n$ are strongly commuting selfadjoint operators then

$$\{e^{it_1A_1} \ldots e^{it_nA_n} \colon t = (t_1, \ldots, t_n) \in \mathbb{R}^n\}$$

is a strongly continuous n-parameter unitary group and for each $j = 1, \ldots, n$

$$D(A_j) = \{x \in H \colon \frac{\partial}{\partial t_j}|_{t=0}U(t)x = \lim_{h \to 0} \frac{U(he_j) - I}{h} \text{ exists}\} \tag{A.7}$$

$$= \{x \in H \colon \frac{\partial^+}{\partial t}|_{t=0}U(t)x = \lim_{h \to 0^+} \frac{U(he_j) - I}{h} \text{ exists}\} \tag{A.8}$$

and we can recover A_j from the collection $U(t)$ by the formula

$$iA_jx = \frac{\partial}{\partial t_j}U(t)x = \frac{\partial^+}{\partial t_j}U(t)x, \tag{A.9}$$

where $x \in D(A_j)$. Conversely, if $\{U(t) \colon t \in \mathbb{R}^n\}$ is a strongly continuous n-parameter unitary group, then the operators $A_1, \ldots, A_n$ defined by (A.9) and (A.7) are selfadjoint, strongly commute and $U(t) = e^{it_1A_1} \ldots e^{it_nA_n}$ for any $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$.

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