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On Radio Labeling of Diameter N-2 and Caterpillar Graphs

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ON RADIO LABELING OF DIAMETER $N-2$ AND CATERPILLAR GRAPHS

by

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CERTIFICATE OF APPROVAL

PH.D. THESIS

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ABSTRACT

Radio labeling of graphs is a specific type of graph labeling. The basic type of graph labeling is vertex coloring; this is where the vertices of a graph G are assigned different colors so that adjacent vertices are not given the same color. A k-coloring of a graph G is a coloring that uses k colors. The chromatic number of a graph G is the minimum value for k such that a k-coloring exists for G [2].

Radio labeling is a type of graph labeling that evolved as a way to use graph theory to try to solve the channel assignment problem: how to assign radio channels to radio transmitters so that two transmitters that are relatively close to one another do not have frequencies that cause interference between them. This problem of channel assignment was first put into a graph theoretic context by Hale [6]. In terms of graph theory, the vertices of a graph represent the locations of the radio transmitters, or radio stations, with the labels of the vertices corresponding to channels or frequencies assigned to the stations.

Different restrictions on labelings of graphs have been studied to address the channel assignment problem. Radio labeling of a simple connected graph G is a labeling $f : V(G) \to \mathbb{Z}^+$ such that for every pair of distinct vertices u and v of G, $\text{distance}(u, v) + |f(u) - f(v)| \geq \text{diameter}(G) + 1$. The radio number of G is the smallest number m such that there exists a radio labeling f with $f(v) \leq m$ for all v in $V(G)$. The radio numbers of certain families of graphs have already been found. Bounds and radio numbers of some tree graphs have been determined. Daphne Der-

Fen Liu and Xuding Zhu determined the radio number of paths [9], Daphne Der-Fen Liu found a general lower bound for the radio number of trees [8], and Xiangwen Li, Vicky Mak, Sanming Zhou determined the radio number of complete m-ary trees [7]. Ruxandra Marinescu-Ghemeci found the radio number for some thorn graphs, one of which is a particular type of caterpillar graph [10].

This thesis builds off of work done on paths and trees in general to determine an improved lower bound or the actual radio number of certain types of caterpillar graphs. This thesis includes joint work with Matthew Porter and Maggy Tomova on determining the radio numbers of graphs with n vertices and diameter $n-2$, a subcase of which is a particular caterpillar. This thesis also establishes the radio number of some specific caterpillar graphs as well as an improved lower bound for the radio number of more general caterpillar graphs.

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INTRODUCTION AND BACKGROUND

1.1 Motivation

Radio labeling of simple connected graphs is a specific type of graph labeling. The basic type of graph labeling is vertex coloring; this is where the vertices of a graph G are assigned different colors so that adjacent vertices are not given the same color. A k-coloring of a graph G is a coloring that uses k colors. The chromatic number of a graph G is the minimum value for k such that a k-coloring exists for G [2].

A famous graph coloring problem is the Four Color Theorem. The idea of this problem starts with having a map that is divided into countries, or sections of some kind. Colors are assigned to each country so that countries which share a border are given a different color. The goal is to color the map with the least number of colors possible. This problem is translated to graph theory by letting each country be represented by a vertex in a graph with two vertices adjacent in the graph if their corresponding countries share a border. Then the goal of determining the fewest number of colors needed to color the countries of a map is the same as finding the chromatic number of the graph corresponding to the map. The statement that any planar graph can be colored with four or fewer colors is what is known as the Four Color Theorem [2].

Radio labeling is another type of graph labeling that evolved as a way to use graph theory to try to solve a problem. Radio labeling addresses the channel assignment problem: how to assign radio channels or frequencies to different radio transmitters in an optimal way. This means we want to assign radio channels so that two radio transmitters that are geographically close to one another do not have channels with frequencies that interfere with one another. This problem of channel assignment was first put into a graph theoretic context by Hale [6]. In terms of graph theory, the vertices of a simple connected graph represent the locations of the radio transmitters, or radio stations, with the labels of the vertices corresponding to channels or frequencies assigned to the stations.

1.2 Graph Theory Definitions and Notation

There are some basic graph theoretic definitions and notation that will be used throughout this thesis. The graphs considered in this thesis are *simple connected* graphs. This means there are no loops (edges from one vertex back to itself), no multiple edges between two vertices, and between every pair of distinct vertices, there exists a path. Throughout this thesis, let G be a simple connected graph with n vertices. Let $V(G)$ denote the vertex set of G and $E(G)$ denote the edge set of G. We say an edge e is *incident* to a vertex v if one of the endpoints of e is v. The number of edges incident to a vertex v is called the *degree* of v. For a given set S, let the *order* of S, denoted $|S|$, be the number of elements in S. Similarly, for a component C of G, let $|V(C)|$ denote the order of the vertex set of C and $|E(C)|$ denote the order of the edge set of C. For two distinct vertices u and v of G, let

$d(u, v)$ denote the *distance* between u and v, which is the length of the shortest path between u and v. If $d(u, v) = 1$, we say u and v are *adjacent*. The *diameter* of G, denoted by D, is the maximum distance between two vertices in G. A *labeling*, or coloring, f of G is a function from the vertex set of G to the positive integers.

1.3 k-radio Labeling

There have been various restrictions used on labeling, or coloring, graphs in an effort to model the channel assignment problem. Chartrand and Zhang discussed the use of k-radio coloring of graphs and distance 2 labeling [3]. The k-radio coloring condition of graphs is when, given a graph G with diameter D and $1 \leq k \leq D$ with $f : V(G) \rightarrow \mathbb{Z}^+$ a coloring, the inequality

$$d(u, v) + |f(u) - f(v)| \geq 1 + k$$

is satisfied for all vertices u, v in G. The largest number used as a label under the labeling f is called the span of f. When k-radio labeling a graph, one tries to minimize the span of that particular graph.

When $k = 1$, k-radio coloring can be used to determine the chromatic number of a graph G. If f is a 1-radio labeling for G, then for adjacent vertices x and y, the condition that needs to be satisfied becomes $1 + |f(x) - f(y)| \geq 1 + 1$ which implies that $|f(x) - f(y)| \geq 1$. This means that adjacent vertices cannot have the same label. Also, for any two vertices of G that have distance two or greater, the 1-radio coloring condition is satisfied even if those vertices have the same label. Thus, minimizing the largest label given to a vertex of G such that the labeling satisfies the inequality

$d(u,v) + |f(u) - f(v)| \geq 2$ for all $u, v \in V(G)$ gives the chromatic number of G.

Roberts suggested a variation of this type of labeling by determining the labeling based on when transmitters are considered to be close or very close to one another [5]. The labeling that resulted from that distinction of closeness of stations is distance-2 labeling. This labeling, denoted $L(2,1)$ is a labeling f of a graph G such that

$$|f(u) - f(v)| \geq \begin{cases} 2 \text{ if } d(u,v) = 1 \\ \\ 1 \text{ if } d(u,v) = 2 \end{cases} \quad \text{for } u, v \in V(G).$$

It can be seen that $L(2,1)$ labeling is k-radio coloring with $k = 2$. Minimizing the span of a distance-2 labeling has been studied quite thoroughly as a way to address the channel assignment problem. A variation on this type of labeling is a $L(j,k)$-labeling, which is mentioned in a discussion on radio labeling of m-ary trees by Li, Mak, and Zhou [7]. This is a labeling where adjacent vertices have labels that have absolute difference at least j and vertices distance 2 apart have labels with absolute difference at least k.

Using different k values can help when considering how the distance between two stations could affect how close their corresponding frequencies could be. This is what led to radio labeling, which is the specific k-radio coloring when k is the diameter of a graph G. Studying this type of k-radio coloring has been helpful in trying to solve the channel assignment problem. As Liu and Zhu mention, in practical applications, interference between channels may occur between stations greater than distance two apart[9]. This leads to the definition of a radio labeling, or multilevel distance labeling, of a graph G.

1.4 Radio Labeling

A radio labeling is a labeling $f : V(G) \rightarrow \mathbb{Z}^+$[1] such that the *radio condition* is satisfied: for all pairs of vertices $u, v \in V(G)$,

$$d(u,v) + |f(u) - f(v)| \geq D + 1.$$

The largest value given in a labeling is called the *span* of that labeling. The *radio number* of a graph G, denoted $rn(G)$, is the smallest possible span of a radio labeling of G. Equivalently, the radio number of G is the smallest integer m such that there exists a radio labeling f of G with $f(v) \leq m$ for all $v \in V(G)$ [9].

Work has been done to determine the radio number of various families of graphs. Some of the graphs whose radio numbers have been determined are paths, k-partite graphs, cycles, n-cubes, certain types of trees, certain types of spider graphs, m-ary trees, some thorn graphs, complete graphs, stars, wheels, gear graphs, and Cartesian products of complete graphs [3, 4, 6, 7, 8, 9, 10, 11].

1.5 Previous Results for Tree Graphs

A particular type of simple connected graph is a tree graph. This is a graph with no cycles. Work has been done to determine the radio numbers of various different types of tree graphs, including [7, 8, 9, 10].

In this thesis, we mostly look at particular types of tree graphs whose radio

[1] Some authors allow 0 as a label. In this thesis, we do not allow 0 to be a label and have adjusted all the formulas of cited results accordingly.

numbers are not yet known. In particular, we look at improving the lower bounds of the radio number for these tree graphs that was established in [8]. As we discuss in Chapter 3, in work with Maggy Tomova and Matthew Porter, we not only improve the lower bound, but find the radio number for all simple connected graphs with n vertices and diameter $n-2$. We establish improved bounds for the radio number of some more general trees in Chapter 4. In that chapter, we also determine the radio number for an edge-balanced caterpillar that satisfies specific conditions.

CHAPTER 2
TECHNIQUES FOR BOUNDS OF RADIO NUMBERS

To determine the radio number of a given graph G, we must find a labeling that produces a relatively small span for the graph. We also must prove that it is not possible to have a smaller span for a radio labeling of that particular graph. In essence, this means we must prove an upper bound and a lower bound for the radio number are equal. Finding this upper bound usually involves establishing an algorithm to determine the order the vertices of G should be labeled to produce the smallest possible span in a radio labeling of G. In this chapter we establish some techniques to help in finding and proving a lower bound of the radio number that equals an established upper bound.

2.1 Lower Bound Techniques

In this section we develop some general techniques for determining a good lower bound for the radio number of a graph. First we establish some terminology and notation we will use throughout this thesis to help when relating the order vertices are labeled and a particular labeling function of a given graph G. Some of the results in this section are from joint work with Maggy Tomova and Matthew Porter that can be found in [1].

Definition. An *ordering* of the vertices of a graph G with n vertices is a bijection of the vertices of G to the set $\{x_1, \ldots, x_n\}$ where the subscript denotes the order the vertices are labeled.

Definition. Given an ordering $x_1, \ldots, x_n$ of the n vertices of a simple connected graph G let the *associated* radio labeling be a function f with $f(x_1) = 1$ and defined inductively so that $f(x_i)$ is the smallest integer so that the radio condition is satisfied for all pairs x_i and x_j with $j < i$.

For the rest of this thesis, unless otherwise indicated, for a graph G with n vertices, we refer to $x_1, \ldots, x_n$ as the ordering of the vertices of G and call the associated radio labeling f.

Now consider the process in labeling vertices of a graph G so that the radio condition is satisfied. Since a radio labeling f is a function from the vertices of G to the positive integers, we let $f(x_1) = 1$. As we label the rest of the vertices, at each step, we choose $f(x_i)$ to be the smallest integer that satisfies the radio condition with all vertices $x_1, x_2, \ldots, x_{i-1}$. When labeling x_i, a reasonable first consideration for $f(x_i)$ is the positive integer z such that

$$z = D + 1 + f(x_{i-1}) - d(x_{i-1}, x_i).$$

Notice that if $f(x_i) = z$, then the radio condition between the successively labeled vertices x_{i-1} and x_i is an equality. However, this value might not satisfy the radio condition with x_j for some $1 \leq j \leq i-2$. If this is the case, we having the following:

$$z < D + 1 + f(x_j) - d(x_j, x_i)$$

$$\Rightarrow z + J_f(x_{i-1}, x_i) = D + 1 + f(x_j) - d(x_j, x_i)$$

for some $J_f(x_{i-1}, x_i) \in \mathbb{Z}^+$. Thus, for the radio condition to be satisfied for all pairs of vertices, we need to increase the value of $f(x_i)$ so that $f(x_i) = z + J_f(x_{i-1}, x_i)$. Then when considering $f(x_i)$ in terms of the successively labeled vertices x_{i-1} and x_i, we have the following:

$$f(x_i) = D + 1 + f(x_{i-1}) - d(x_{i-1}, x_i) + J_f(x_{i-1}, x_i).$$

In this case, the radio condition is satisfied with a strict inequality for the pair of vertices x_{i-1} and x_i. This need to have a strict inequality for the radio condition between successively labeled vertices is what we will refer to as needing *jumps*. This is because we need to make an increase, or jump, in the value of $f(x_i)$ beyond what is required when just considering the radio condition between the successively labeled vertices x_{i-1} and x_i. More formally, we have the following:

Definition. As in [7], let $J_f(x_i, x_{i+1})$ be a non-negative integer such that

$$d(x_i, x_{i+1}) + f(x_{i+1}) - f(x_i) = D + 1 + J_f(x_i, x_{i+1}).$$

We call $J_f(x_i, x_{i+1})$ the *jump of f from x_i to x_{i+1}*.

Definition. Given an ordering $x_1, \ldots, x_n$ of the vertices of a graph G and the associated radio labeling f, we say that f *requires jumps* if $\sum_{i=1}^{n-1} J_f(x_i, x_{i+1}) \geq 1$.

Proposition 1. *Let G be a simple connected graph with n vertices and let $x_1, \ldots, x_n$ be any ordering of the vertices of G with f the associated radio labeling. Then,*

$$f(x_n) = (n-1)(D+1) + f(x_1) - \sum_{i=1}^{n-1} d(x_i, x_{i+1}) + \sum_{i=1}^{n-1} J_f(x_i, x_{i+1}).$$

Proof. The result is obtained by adding up the equations

$$d(x_1, x_2) + f(x_2) - f(x_1) = D + 1 + J_f(x_1, x_2),$$

$$d(x_2, x_3) + f(x_3) - f(x_2) = D + 1 + J_f(x_2, x_3),$$

$$\vdots$$

$$d(x_{n-1}, x_n) + f(x_n) - f(x_{n-1}) = D + 1 + J_f(x_{n-1}, x_n).$$

□

Proposition 2. *Let G be a simple connected graph with n vertices. Then*

$$rn(G) \geq (n-1)(D+1) + f(x_1) - \max_p \sum_{i=1}^{n-1} d(x_i, x_{i+1})$$

where the maximum is taken over all possible bijections p from $V(G)$ to $\{x_1, \ldots, x_n\}$.

Proof. This result follows directly from minimizing the right side of the equation in Proposition 1. □

From Proposition 2 we see that finding $\max_p \sum_{i=1}^{n-1} d(x_i, x_{i+1})$ for a graph G will give a lower bound for the radio number of G. As we will refer to this occurrence of maximizing $\sum_{i=1}^{n-1} d(x_i, x_{i+1})$, we have the following definition:

Definition. We call any ordering $x_1, \ldots, x_n$ of the vertices of a graph G for which $\max_p \sum_{i=1}^{n-1} d(x_i, x_{i+1})$ is achieved a *distance maximizing ordering.* If $(\max_p \sum_{i=1}^{n-1} d(x_i, x_{i+1})) - 1$ is achieved, we will call the ordering an *almost distance maximizing ordering.*

Notice that when G is a tree, Proposition 2 gives a preliminary lower bound for the radio number of a tree. This is the same lower bound as given by Liu in Theorem 3 of [8] but with different notation. In Liu's proof, she shows that $\sum_{i=0}^{m-2} d(u_{i+1}, u_i) \leq 2\omega(T) - 1$ where $\omega(T) = \min\{\sum_{u \in V(G)} d(w, u) : w \in V(G)\}$ is the weight of the tree T. The sum $\sum_{i=0}^{m-2} d(u_{i+1}, u_i)$ in [8] is equivalent to $\sum_{i=1}^{n-1} d(x_i, x_{i+1})$ in this thesis. Therefore, according to Liu's proof, $\max_p \sum_{i=1}^{n-1} d(x_i, x_{i+1}) = 2\omega(G) - 1$ where the maximum is taken over all possible bijections p from $V(G)$ to $\{x_1, \ldots, x_n\}$. Making this substitution, exchanging variables to match this thesis' notation, and adjusting for that fact that Liu uses 0 as the first label in her labelings shows that the bound given in Theorem 3 of [8] is the same as the bound given in Proposition 2. In this thesis, we improve this bound for some particular types of tree graphs; in Section 3.2 we improve this bound for spire graphs and in Chapter 4, we improve this bound for some other caterpillar graphs.

The following lemma will be useful in techniques we develop to determine the value of $\sum_{i=1}^{n-1} d(x_i, x_{i+1})$ for particular graphs.

Lemma 1. *Let G be a graph with vertices $v_1, ..., v_n$ and edges $e_1, ..., e_m$. Let p be a bijection from the vertices of G to the set $\{x_1, ..., x_n\}$. Let P_j be a fixed shortest path from x_j to x_{j+1}. Let $n(e_i)$ be the number of paths P_j that contain the edge e_i. Then the following hold:*

1. *Each edge can appear in any path P_j at most once.*

2. *Let $\{e^k_{i_1}, ..., e^k_{i_r}\}$ be the set of all the edges incident to x_k. Then $n(e^k_{i_1}) + ... + n(e^k_{i_r})$*

is even unless $k = 1$ *or* $k = n$ *in which case the sum is odd.*

3. *Suppose* e_i *is an edge so that removing it from the graph gives a disconnected two component graph where the two components are denoted* A *and* B*. Furthermore assume that if* x_j *and* x_{j+1} *are both contained in the same component, then so is* P_j*. Then* $n(e_i) \leq 2min\{|V(A)|, |V(B)|\}$*.*

4. *Let* $\{e_{i_1}, ..., e_{i_r}\}$ *be a set of edges so that no two of them are ever contained in the same* P_j*. Then* $n(e_{i_1}) + ... + n(e_{i_r}) \leq n - 1$*.*

Proof. The first conclusion follows from the fact that P_j is a shortest path so it cannot contain any cycles.

The second conclusion follows from the fact that if x_k is not the endpoint of a path P_j but the vertex is included in this path, two of its incident edges belong to the path. If x_k is the endpoint of a path, then exactly one of its incident edges is part of the path. For $1 < k < n$, x_k is the endpoint of exactly two paths while each of x_1 and x_n is an endpoint of exactly one of the paths.

A path P_j contains the edge e_i if and only if its endpoints are in different components of the graph obtained by deleting e_i. This observation verifies the third conclusion.

The final conclusion follows from the fact that there are $n - 1$ paths and any edge can appear in a path at most once. □

Sometimes we will need a generalization of the third condition of Lemma 1, i.e., we will need to simultaneously remove multiple edges to disconnect a graph.

The following lemma describes the corresponding result in this case. In this thesis, this generalization will only be needed when we consider graphs with n vertices and diameter $n-2$ that are not tree graphs in section 3.3.

Lemma 2. *Let G be a graph with vertices $v_1, ..., v_n$ and edges $e_1, ..., e_m$. Let p be a bijection from the vertices of G to the set $\{x_1, ..., x_n\}$. Let P_j be a fixed shortest path from x_j to x_{j+1}. Let $n(e_i)$ be the number of paths P_j that contain the edge e_i. Let $\{e_{i_1}, ..., e_{i_r}\}$ be a set of edges so that removing all of them from the graph gives a disconnected two component graph, with the components denoted A and B. Furthermore assume that*

- *If x_j and x_{j+1} are both contained in the same component, then so is P_j, and*
- *Each path P_j contains at most one of the edges $\{e_{i_1}, ..., e_{i_r}\}$.*

Then $n(e_{i_1}) + ... + n(e_{i_r}) \leq 2min\{|V(A)|, |V(B)|\}$.

Proof. By the first condition a path P_j can contain one of the edges $\{e_{i_1}, ..., e_{i_r}\}$ only if its endpoints are in different components of the disconnected graph. Thus there are at most $2\min\{|V(A)|, |V(B)|\}$ paths that contain one of these edges. By the second condition each path can contain at most one of the edges so $n(e_{i_1}) + ... + n(e_{i_r}) \leq 2\ min\{|V(A)|, |V(B)|\}$. ∎

Remark 1. *Let G be a graph with vertices $v_1, ..., v_n$ and edges $e_1, ..., e_m$. Let $N(e_i)$ be the maximal value of $n(e_i)$ allowable under the conditions of Lemmas 1 and 2. Then $\max_p \sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \sum_{j=1}^{m} N(e_j)$ where the maximum is taken over all bijections p from the vertices of G to $\{x_1, \ldots, x_n\}$.*

Now we introduce notation similar to that of Lemmas 1 and 2 for a particular ordering of the vertices of a graph G.

Notation. Let G be a graph with an ordering $x_1, \ldots, x_n$ of its vertices. As in Lemma 1, let P_j be a fixed shortest path from x_j to x_{j+1}. For an edge $e \in G$, let $n_x(e)$ denote the number of paths P_j that contain the edge e under the ordering $x_1, \ldots, x_n$.

2.1.1 Techniques for Trees

In this thesis, the majority of the graphs considered are particular types of tree graphs. Note that this means for a tree G with n vertices, there are $n-1$ edges. Unless otherwise indicated, in this thesis, we will denote the edges of a tree G as $e_1, \ldots, e_{n-1}$.

We make the following observations that result from Lemma 1 when G is a tree graph.

Remark 2. *Let G be a tree:*

1. *Removing one edge will result in a disconnected graph of two components and removing more than one edge will result in a disconnected graph with three or more components. Thus, in a tree, removing just one edge, e_i will result in two disjoint components, A_i and B_i. Then for a given edge e_i of G, (3) of Lemma 1 gives that $n(e_i) \leq 2\min\{|V(A_i)|, |V(B_i)|\}$. Also, (4) of Lemma 1 shows that $N(e_i) \leq n-1$ for all edges e_i. It follows that the maximum possible value for*

$n_x(e_i)$ for all possible orderings $x_1, \ldots, x_n$ and edges e_i is

$$N(e_i) = \begin{cases} n-1 & \textit{if } \min\{|V(A_i)|, |V(B_i)|\} = \frac{n}{2} \\ 2\min\{|V(A_i)|, |V(B_i)|\} & \textit{else} \end{cases}.$$

2. *Note that from (2) of Lemma 1, for a specific ordering $x_1, \ldots, x_n$ of the vertices of G, there needs to be at least one edge e_i in G such that $n_x(e_i)$ is odd.*

3. *Let $x_1, \ldots, x_n$ be an ordering of the vertices of G. Suppose x_1 and x_n are not adjacent. Let $\{e^1_{i_1}, \ldots, e^1_{i_r}\}$ be the set of edges incident to x_1 and $\{e^n_{i_1}, \ldots, e^n_{i_s}\}$ be the set of edges incident to x_n. By (2) of Lemma 1, $\sum_{j=1}^{r} n_x(e^1_{i_j})$ and $\sum_{j=1}^{s} n_x(e^n_{i_j})$ must both be odd. Also, since x_1 and x_n are not adjacent, the sets $\{e^1_{i_1}, \ldots e^1_{i_r}\}$ and $\{e^n_{i_1}, \ldots e^n_{i_s}\}$ do not have any common members. Thus, there must be at least two edges e_j such that $n_x(e_j)$ is odd when x_1 and x_n are not adjacent. Note, this also means that when there is only one $n_x(e_i)$ value that is odd, then x_1 and x_n are adjacent under the ordering $x_1, \ldots, x_n$ and both are incident to the edge e_k such that $n_x(e_k)$ is odd.*

The following proposition determines a way to describe $\sum_{i=1}^{n-1} d(x_i, x_{i+1})$ for a tree graph G and ordering $x_1, \ldots, x_n$ in terms of the $n_x(e_i)$ values for the edges e_i of G.

Proposition 3. *Let G be a tree with ordering $x_1, x_2, \ldots, x_n$ of the vertices of G. Let $e_1, e_2, \ldots, e_{n-1}$ be the edges of G. Then $\sum_{i=1}^{n-1} d(x_i, x_{i+1}) = \sum_{i=1}^{n-1} n_x(e_i)$.*

Proof. Consider a fixed shortest path P_j between x_j and x_{j+1}. Suppose this path is of length k. Since the length of this path is the shortest length of a path between x_j and x_{j+1}, it follows that $d(x_j, x_{j+1}) = k$. Thus, $d(x_j, x_{j+1})$ contributes k to the total $\sum_{i=1}^{n-1} d(x_i, x_{i+1})$.

Also, since there are k edges in P_j, this path contributes 1 to the $n_x(e_i)$ value for each of the k edges e_i in the path. Therefore, P_j contributes k to the total sum $\sum_{i=1}^{n-1} n_x(e_i)$.

Since the above arguments are true for each j, $1 \leq j \leq n-1$, it follows that $\sum_{i=1}^{n-1} d(x_i, x_{i+1}) = \sum_{i=1}^{n-1} n_x(e_i)$. □

Note that for a tree graph G, Proposition 3 implies that for $x_1, \ldots, x_n$ to be a distance maximizing ordering, $\sum_{e \in E(G)} n_x(e)$ is maximized.

The following definition for trees in general will help us divide caterpillar graphs, a particular type of tree graph, into different cases to consider in Chapter 4 when we work to improve the lower bound of their radio numbers.

Definition. Let G be a simple connected tree on n vertices with edges $e_1, \ldots, e_{n-1}$. Let $N(e_i)$ denote the maximum $n(e_i)$ value for edge e_i allowable under the conditions of Lemma 1. A *center edge*, e_c, is an edge with largest $N(e_i)$ value for the graph G. The removal of a center edge results in a disconnected graph with two components, A and B.

CHAPTER 3
GRAPHS WITH N VERTICES AND DIAMETER $N-2$

The radio number of paths, trees with n vertices and diameter $n-1$, has been determined by Liu and Zhu in [9]. In this chapter, we determine the radio number of all graphs with n vertices and diameter $n-2$. The results of this chapter are from joint work with Maggy Tomova and Matthew Porter that can be found in [1].

Much of this chapter will be devoted to studying a family of graphs which we call spire graphs, which are paths with an extra leg vertex. More formally, we have the following:

Definition. Let $n, s \in \mathbb{Z}$ where $n \geq 4$ and $2 \leq s \leq n-2$. The spire graph $S_{n,s}$ is the graph with vertices $v_1, ..., v_n$ and edges $\{(v_i, v_{i+1}) | i = 1, 2, ..., n-2\}$ together with the edge (v_s, v_n). The vertex v_n is called *the spire*. Without loss of generality we will always assume that $s \leq \lfloor \frac{n}{2} \rfloor$. See Figure 3.1.

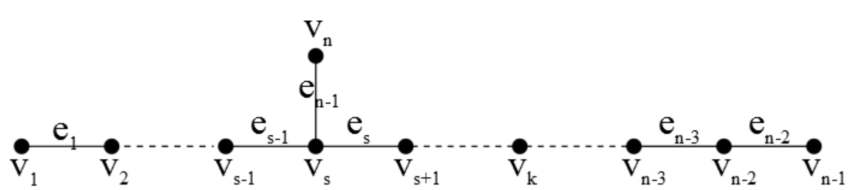

Figure 3.1: $S_{n,s}$.

We will show that:

Theorem (Radio Number of $S_{n,s}$) *Let $S_{n,s}$ be a spire graph, where* $2 \leq s \leq$

$\lfloor \frac{n}{2} \rfloor$. *Then,*

$$rn(S_{n,s}) = \begin{cases} 2k^2 - 4k + 2s + 3 & \text{if } n = 2k \text{ and } 2 \leq s \leq k-2, \\ 2k^2 - 2k & \text{if } n = 2k \text{ and } s = k-1, \\ 2k^2 - 2k + 1 & \text{if } n = 2k \text{ and } s = k, \\ 2k^2 - 2k + 2s & \text{if } n = 2k+1. \end{cases}$$

Based on this result, in Section 3.3 we will also determine the radio numbers of all other graphs with n vertices and diameter $n-2$.

As mentioned in Section 2.1, Liu establishes bounds for the radio numbers of trees in [8]. In particular she determines the exact radio numbers of spire graphs with an odd number of vertices and of spire graphs when the spire is very close to the middle of the path. Although our techniques easily cover these cases as well, in the interest of brevity we will quote Liu's results whenever feasible.

3.1 Radio Number of Spire Graphs–Upper Bound

In this section, we present algorithms for finding specific orderings of the vertices of spire graphs. The associated radio labeling of these orderings gives an upper bound for the radio number of these graphs. In Section 3.2, we find a lower bound for the radio number of graphs which matches the upper bound found in this section to establish the radio number of spire graphs.

Theorem 3 (Upper bound for $S_{n,s}$). *Let $S_{n,s}$ be a spire graph, where $2 \leq s \leq \lfloor \frac{n}{2} \rfloor$. Then,*

$$rn(S_{n,s}) \leq \begin{cases} 2k^2 - 4k + 2s + 3 & \textit{if } n = 2k \textit{ and } 2 \leq s \leq k-2, \\ 2k^2 - 2k & \textit{if } n = 2k \textit{ and } s = k-1, \\ 2k^2 - 2k + 1 & \textit{if } n = 2k \textit{ and } s = k, \\ 2k^2 - 2k + 2s & \textit{if } n = 2k+1. \end{cases}$$

Proof. To establish this bound we define a labeling with the appropriate span. The cases for n even and n odd are discussed separately.

Case I: First consider the case when $n = 2k$ for some $k \in \mathbb{Z}$. The upper bounds for cases when $k < 7$ that are not included in this proof are shown explicitly in Appendix A.

Subcase A: $2 \leq s \leq k-2$ and $k \geq 7$. Order the vertices of $S_{n,s}$ into three groups as follows:

Group I: $v_k, v_{2k}, v_{k+4}, v_5, v_{k+3}, v_3, v_{k+2}, v_4,$

Group II: $v_{k+5}, v_6, v_{k+6}, v_7, \ldots, v_{k+m}, v_{m+1}, \ldots, v_{k+(k-3)}, v_{k-2},$

Group III: $v_{2k-2}, v_2, v_{k+1}, v_1, v_{2k-1}, v_{k-1}.$

In this ordering Group I always contains the same 8 vertices and Group III always contains the same 6 vertices. Group II follows the indicated pattern and contains $n - 14$ vertices.

Now, rename the vertices of $S_{n,s}$ in the above ordering by $x_1, x_2, \ldots, x_n$ where $x_1 = v_k$, $x_2 = v_{2k}$, etc. In Table 3.1 we define a labeling f of $S_{n,s}$. We will let $f(x_1) = 1$. The first column in the table gives the order in which the vertices are labeled, i.e., the inequality $f(x_i) > f(x_{i-1})$ always holds. The second column reminds the reader which vertex we are labeling. In the third column we have computed the distance between x_i and x_{i+1}. Finally in the last column we give the difference between the labels $f(x_i)$ and $f(x_{i+1})$. Given that $f(x_1) = 1$, one can use the last column to compute $f(x_i)$ by summing the first $i-1$ entries of the column and then adding one to this sum.

Claim: The function f defined in Table 3.1 is a radio labeling on $S_{n,s}$.

x_i	Vertex Names	$d(x_i, x_{i+1})$	$f(x_{i+1}) - f(x_i)$
x_1	v_k	$k-s+1$	$k+s-2$
x_2	v_{2k}	$k-s+5$	$k+s-6$
x_3	v_{k+4}	$k-1$	k
x_4	v_5	$k-2$	$k+1$
x_5	v_{k+3}	k	$k-1$
x_6	v_3	$k-1$	k
x_7	v_{k+2}	$k-2$	$k+1$
x_8	v_4	$k+1$	$k-2$
x_9	v_{k+5}	$k-1$	k
x_{10}	v_6	k	$k-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{2m-1}	v_{k+m}	$k-1$	k
x_{2m}	v_{m+1}	k	$k-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{n-7}	$v_{k+(k-3)}$	$k-1$	k
x_{n-6}	v_{k-2}	k	$k-1$
x_{n-5}	v_{2k-2}	$2k-4$	4
x_{n-4}	v_2	$k-1$	k
x_{n-3}	v_{k+1}	k	$k-1$
x_{n-2}	v_1	$2k-2$	2
x_{n-1}	v_{2k-1}	k	$k-1$
x_n	v_{k-1}	n/a	n/a

Table 3.1: Radio Labeling f on $S_{n,s}$ where $n = 2k, 2 \leq s \leq k-2, k \geq 7$.

Proof of claim: To prove that f is a radio labeling, we need to verify that the radio condition holds for all vertices $x_i, x_j \in V(S_{n,s})$. In this case, the diameter of $S_{n,s}$ is $2k-2$ so we must show that for every i, j with $j > i$, $d(x_i, x_j)+f(x_j)-f(x_i) \geq 2k-1$.

Case 1: $j = i+1$. To verify the radio condition it suffices to add the entries in the 3^{rd} and 4^{th} columns of the i^{th} row of Table 3.1 and check that this sum is always at least $2k-1$.

Case 2: $j = i+2$. Note that $f(x_j) - f(x_i)$ is equal to the sum of the entries in the last column of rows i and $i+1$ of Table 3.1. One can quickly check that in most cases $f(x_j) - f(x_i) \geq 2k-2$ and therefore $d(x_i, x_j) + f(x_j) - f(x_i) \geq 1 + 2k - 2$. It is less clear that the inequality $d(x_i, x_j) + f(x_j) - f(x_i) \geq 2k-1$ holds for the following six pairs of vertices: $\{x_3, x_1\}, \{x_4, x_2\}, \{x_{n-4}, x_{n-6}\}, \{x_{n-3}, x_{n-5}\}, \{x_{n-1}, x_{n-3}\}$, and $\{x_n, x_{n-2}\}$. In Table 3.2 we compute the distance between vertices and the difference between their labels for five of those vertex pairs. The reader can easily verify that these pairs satisfy the radio condition.

Vertex pair	$d(x_i, x_{i+2})$	$f(x_{i+2}) - f(x_i)$
$\{x_3, x_1\}$	4	$2k+2s-8$
$\{x_{n-4}, x_{n-6}\}$	$k-4$	$k+3$
$\{x_{n-3}, x_{n-5}\}$	$k-3$	$k+4$
$\{x_{n-1}, x_{n-3}\}$	$k-2$	$k+1$
$\{x_n, x_{n-2}\}$	$k-2$	$k+1$

Table 3.2: Radio labeling f found in Table 3.1: Verifying radio condition for $\{x_i, x_j\}$ with $j = i+2$.

For the pair $\{x_4, x_2\}$, note that the vertex incident to the spire is v_s. We consider two cases:

(1) If $s < 5$, then $d(x_2, x_4) + c(x_4) - c(x_2) = d(v_n, v_5) + 2k + s - 6 = 5 - s + 1 + 2k + s - 6 = 2k$.

(2) If $s \geq 5$ then $c(x_4) - c(x_2) = 2k + s - 6 \geq 2k + 5 - 6 = 2k - 1$.

In both cases the radio condition is satisfied.

Case 3: $j \geq i + 3$. Note that $f(x_j) - f(x_i)$ is at least equal to the sum of the entries in the last column of rows i, $i+1$ and $i+2$ in Table 3.1. As the sum of any three consecutive entries in the column is at least $2k - 2$, in this case the radio condition is always satisfied.

And thus the claim has been proven.

Letting $f(x_1) = 1$, the largest number in the range of the radio labeling f is $f(x_n)$ and is therefore equal to the sum of the entries in the last column of Table 3.1 plus one. Since the sums of Group I, Group II, and Group III are $8k + 2s - 9$, $(k-7)(2k-1)$, and $3k+4$, respectively, we conclude that $\mathrm{rn}(S_{n,s}) \leq 2k^2 - 4k + 2s + 3$ as desired.

Subcase B: $s = k - 1$ and $k \geq 3$. As this algorithm is similar to the previous one but simpler, we summarize the algorithm directly in Table 3.3.

By adding the third and fourth entries in each row of Table 3.3, we can verify that $d(x_i, x_{i+1}) + f(x_{i+1}) - f(x_i) \geq 2k - 1$ for all i. In this case it is also easy to check that $f(x_{i+j}) - f(x_i)$ is at least $2k - 2$ for all i and all $j \geq 2$ so the radio condition is always satisfied. Adding one to the sum of the values in the last column of Table 3.3

x_i	Vertex Names	$d(x_i, x_{i+1})$	$f(x_{i+1}) - f(x_i)$
x_1	v_{k-1}	k	$k-1$
x_2	v_{2k-1}	$k+1$	$k-1$
x_3	v_{2k}	k	$k-1$
x_4	v_{2k-2}	k	$k-1$
x_5	v_{k-2}	$k-1$	k
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{2m}	v_{2k-m}	k	$k-1$
x_{2m+1}	v_{k-m}	$k-1$	k
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{n-2}	$v_{2k-(k-1)}$	k	$k-1$
x_{n-1}	$v_{k-(k-1)}$	$k-1$	k
x_n	v_k	n/a	n/a

Table 3.3: Radio Labeling f on $S_{n,s}$ where $n = 2k$, $s = k-1$ and $k \geq 3$.

gives the desired upper bound for the radio number in this case.

Subcase C: $s = k$ and $k \geq 2$.

Table 3.4 corresponds to the labeling algorithm. As in Subcase B, checking that f is a radio labeling is trivial. Again the sum of the values in the last column plus one gives the desired upper bound for the radio number.

Case II: Now suppose that $n = 2k+1$ for some $k \in \mathbb{Z}$. Order the vertices of $S_{n,s}$ as follows:

$$\text{Group I: } v_{k-1}, v_{2k-1}, v_{k-2}, v_{2k-2}, v_{k-3}, v_{2k-3}, \ldots, v_{k+3}, v_2, v_{k+2},$$

$$\text{Group II: } v_{2k+1}, v_{k+1}, v_1, v_{2k}, v_k.$$

In this ordering Group I always contains $n-5$ vertices and Group II always contains the same 5 vertices. Now, rename the vertices of $S_{n,s}$ in the above ordering

x_i	Vertex Names	$d(x_i, x_{i+1})$	$f(x_{i+1}) - f(x_i)$
x_1	v_k	$k-1$	k
x_2	v_1	k	$k-1$
x_3	v_{k+1}	$k-1$	k
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{2m}	v_m	k	$k-1$
x_{2m+1}	v_{k+m}	$k-1$	k
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{n-2}	v_{k-1}	k	$k-1$
x_{n-1}	v_{2k-1}	k	$k-1$
x_n	v_{2k}	n/a	n/a

Table 3.4: Radio Labeling f on $S_{n,s}$ where $n = 2k, s = k, k \geq 2$.

by $x_1, x_2, \ldots, x_n$. This is the label order of the vertices of $S_{n,s}$.

Claim: The function f defined in Table 3.5 is a radio labeling on $S_{n,s}$.

Proof of claim: To prove that f is a radio labeling, we need to verify that the radio condition holds for all vertices $x_i, x_j \in S_{n,s}$, i.e., we must show that for every i, j with $j > i$, $d(x_i, x_j) + f(x_j) - f(x_i) \geq 2k$.

Case 1: $j = i + 1$. To verify the radio condition it suffices to add the entries in the 3^{rd} and 4^{th} column of the i^{th} row of Table 3.5 and check that this sum is always at least $2k$.

Case 2: $j = i + 2$. Note that $f(x_j) - f(x_i)$ is equal to the sum of the entries in the last column of rows i and $i + 1$ in Table 3.5. One can quickly check that in most cases $f(x_j) - f(x_i) \geq 2k - 1$ and therefore $d(x_i, x_j) + f(x_j) - f(x_i) \geq 1 + 2k - 1 = 2k$. It is less clear that $d(x_i, x_j) + f(x_j) - f(x_i) \geq 2k$ holds for the following five pairs of vertices $\{u, v\}$: $\{x_{n-4}, x_{n-6}\}, \{x_{n-3}, x_{n-5}\}, \{x_{n-2}, x_{n-4}\}, \{x_{n-1}, x_{n-3}\}$, and $\{x_n, x_{n-2}\}$.

x_i	Vertex Names	$d(x_i, x_{i+1})$	$f(x_{i+1}) - f(x_i)$
x_1	v_{k-1}	k	k
x_2	v_{2k-1}	$k+1$	$k-1$
x_3	v_{k-2}	k	k
x_4	v_{2k-2}	$k+1$	$k-1$
x_5	v_{k-3}	k	k
x_6	v_{2k-3}	$k+1$	$k-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{n-8}	v_3	k	k
x_{n-7}	v_{k+3}	$k+1$	$k-1$
x_{n-6}	v_2	k	k
x_{n-5}	v_{k+2}	$k+3-s$	$k-3+s$
x_{n-4}	v_{2k+1}	$k+2-s$	$k-2+s$
x_{n-3}	v_{k+1}	k	k
x_{n-2}	v_1	$2k-1$	1
x_{n-1}	v_{2k}	k	k
x_n	v_k	n/a	n/a

Table 3.5: Radio Labeling f on $S_{n,s}$ where $n = 2k+1$.

In Table 3.6 we compute the distance between vertices and difference between their labels for these vertex pairs. The reader can verify that these pairs of vertices satisfy the radio condition keeping in mind that $s \geq 2$.

Vertex pair	$d(x_i, x_{i+2})$	$f(x_{i+2}) - f(x_i)$
$\{x_{n-4}, x_{n-6}\}$	$s-1$	$2k-3+s$
$\{x_{n-3}, x_{n-5}\}$	1	$2k-5+2s$
$\{x_{n-2}, x_{n-4}\}$	s	$2k-2+s$
$\{x_{n-1}, x_{n-3}\}$	$k-1$	$k+1$
$\{x_n, x_{n-2}\}$	$k-1$	$k+1$

Table 3.6: Radio Labeling f on $S_{n,s}$ where $n = 2k+1$: Verifying radio condition for $\{x_i, x_j\}$ with $j = i+2$.

Case 3: $j \geq i+3$. Note that $f(x_j) - f(x_i)$ is at least equal to the sum of the entries in the last column of rows i, $i+1$ and $i+2$ in Table 3.5. As the sum of any three consecutive entries in the column is at least $2k$, in this case the radio condition is always satisfied.

And thus the claim has been proven.

The largest number in the range of the radio labeling c is then $f(x_n)$ and is therefore equal to the sum of the entries in the last column of Table 3.5 plus one. Since the sums of Group I and Group II are $(k-3)(2k-1)+2k-3+s$ and $3k-1+s$, respectively, we conclude that $rn(G) \leq 2k^2 - 2k + 2s$ as desired. □

3.2 Radio Number of Spire Graphs–Lower Bound

We can now prove that the upper bound for $rn(S_{n,s})$ found in Section 3.1 is also a lower bound. The result for odd values of n follows from [8]. The proof for even values of n is done in two steps. First we will compute a lower bound using Proposition 2 by determining $\max_p \sum_{i=1}^{n-1} d(x_i, x_{i+1})$ where p is a bijection from $V(S_{n,s})$ to the set $\{x_1, ..., x_n\}$. However this bound is not sharp so the second part of the proof shows how to improve the bound so it reaches the upper bound we established in Section 3.1.

Theorem 4 (Lower bound for $S_{n,s}$). *Let $S_{n,s}$ be a spire graph, where $2 \leq s \leq \lfloor \frac{n}{2} \rfloor$. Then,*

$$rn(S_{n,s}) \geq \begin{cases} 2k^2 - 4k + 2s + 3 & \textit{if } n = 2k \textit{ and } 2 \leq s \leq k-2, \\ 2k^2 - 2k & \textit{if } n = 2k \textit{ and } s = k-1, \\ 2k^2 - 2k + 1 & \textit{if } n = 2k \textit{ and } s = k, \\ 2k^2 - 2k + 2s & \textit{if } n = 2k+1. \end{cases}$$

Proof. If $n = 2k + 1$ the desired lower bound follows directly from Corollary 5 of [8]: we observe that $S_{n,s}$ is a spider (a tree with at most one vertex of degree more than two) so

$$rn(S_{n,s}) \geq 2k^2 - 2k + 2s.$$

Similarly if $n = 2k$, and $s = k - 1$ or $s = k$, the desired bound follows from Theorem 12 of [8].

Assume then that $n = 2k$, and $2 \leq s \leq k - 2$. First we determine $\max_p \sum_{i=1}^{n-1} d(x_i, x_{i+1})$ where p is a bijection from $V(S_{n,s})$ to the set $\{x_1, ..., x_n\}$.

Name the edges of $S_{2k,s}$ so that for $1 \leq i \leq n-2$, e_i is the edge between v_i and v_{i+1} and let e_{n-1} be the edge between v_s and v_n. The distance between x_j and x_{j+1} is the number of edges in the shortest path P_j between these two vertices in the graph. Note that removing any edge e_i from $S_{2k,s}$ results in a disconnected graph of two components. By the third and fourth conclusions of Lemma 1, (see also Figure 3.1), it follows that:

$$N(e_i) = \begin{cases} 2i & \text{if } i \leq s-1, \\ 2i+2 & \text{if } s \leq i \leq k-2, \\ 2k-1 & \text{if } i = k-1, \\ 2(2k-1-i) & \text{if } k \leq i \leq 2k-2, \\ 2 & \text{if } i = 2k-1. \end{cases}$$

So $\max_p \sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \sum_{i=1}^{n-1} N(e_i) = 2k^2 - 2s + 1$. Thus we substitute this sum into the maximum distance lower bound to find that

$$rn(S_{2k,s}) \geq 2k^2 - 4k + 2s + 1.$$

We now argue that this lower bound for $rn(S_{2k,s})$ can be increased by 2. Recall that if $\tilde{x}_1, \ldots, \tilde{x}_n$ is an ordering of the vertices of $S_{2k,s}$ with $\tilde{f}$ the associated radio labeling, then for each $i \in \{1, ..., n-1\}$ there is a non-negative integer $J_{\tilde{f}}(\tilde{x}_i, \tilde{x}_{i+1})$ such that $d(\tilde{x}_i, \tilde{x}_{i+1}) + \tilde{f}(\tilde{x}_{i+1}) - \tilde{f}(\tilde{x}_i) = n - 1 + J_{\tilde{f}}(\tilde{x}_i, \tilde{x}_{i+1})$. We will show that if $\tilde{x}_1, \ldots, \tilde{x}_n$ is a distance maximizing ordering, then $\sum_{i=1}^{n-1} J_{\tilde{f}}(\tilde{x}_i, \tilde{x}_{i+1}) \geq 2$ and if $\tilde{x}_1, \ldots, \tilde{x}_n$ is an almost distance maximizing ordering, then $\sum_{i=1}^{n-1} J_{\tilde{f}}(\tilde{x}_i, \tilde{x}_{i+1}) \geq 1$. In either case we conclude that

$$rn(S_{2k,s}) \geq (2k^2 - 4k + 2s + 1) + 2.$$

<u>Claim:</u> Let $x_1, \ldots, x_n$ be an ordering of the vertices of G with f the associated radio labeling and let $\{x_{i-1}, x_i, x_{i+1}\}$ be three consecutively labeled vertices such that $f(x_{i-1}) < f(x_i) < f(x_{i+1})$. Assume that $x_{i-1}, x_{i+1} \in \{v_1, v_2, ..., v_s, ...v_{k-1}, v_n\}$ and

$x_i \in \{v_k, v_{k+1}, ..., v_{2k-1}\}$. Let α denote x_{i-1} or x_{i+1}, whichever has smaller distance to x_i, and let β denote the one with the larger distance to x_i (the only case in which the two distances are equal is when $x_{i-1} = v_n$ and $x_{i+1} = v_{s-1}$ (or vice versa); in this case let α be v_n). Let $J_f(x_i, \alpha)$ and $J_f(x_i, \beta)$ be non-negative integers such that

$$d(x_i, \alpha) + |f(x_i) - f(\alpha)| = n - 1 + J_f(x_i, \alpha) \text{ and}$$

$$d(x_i, \beta) + |f(x_i) - f(\beta)| = n - 1 + J_f(x_i, \beta).$$

Then

$$J_f(x_i, \alpha) + J_f(x_i, \beta) \geq \begin{cases} 2(d(x_i, \alpha)) - n + 1 & \alpha \neq v_n, \\ 2(d(x_i, \alpha)) - n - 1 & \alpha = v_n, \end{cases}$$

Proof of Claim:

Let $\{x_{i-1}, x_i, x_{i+1}\}$ be a triple of vertices satisfying the hypotheses of the claim. We observe that

$$d(\alpha, \beta) = \begin{cases} d(x_i, \beta) - d(x_i, \alpha) & \alpha \neq v_n, \\ d(x_i, \beta) - d(x_i, \alpha) + 2 & \alpha = v_n. \end{cases}$$

We will prove the claim in detail in the case when $f(\alpha) < f(x_i) < f(\beta)$ and $\alpha \neq v_n$. For the other cases we only present the final result and let the interested reader verify the details of the computations.

The radio condition applied to the pair of vertices α and β gives

$$n - 1 \leq d(\alpha, \beta) + f(\beta) - f(\alpha).$$

We substitute $d(\alpha, \beta) = d(x_i, \beta) - d(x_i, \alpha)$ in the above equation and add and subtract $f(x_i)$ to obtain

$$n - 1 \leq d(x_i, \beta) - d(x_i, \alpha) + f(\beta) - f(\alpha) + f(x_i) - f(x_i).$$

Recall that

$$d(x_i, \alpha) + f(x_i) - f(\alpha) = n - 1 + J_f(x_i, \alpha) \text{ and}$$

$$d(x_i, \beta) + f(\beta) - f(x_i) = n - 1 + J_f(x_i, \beta)$$

where $J_f(x_i, \alpha)$ and $J_f(x_i, \beta)$ are non-negative integers. We now make a series of substitutions to obtain a lower bound for $J_f(x_i, \alpha) + J_f(x_i, \beta)$. First, we substitute $d(x_i, \beta) + f(\beta) - f(x_i) = n - 1 + J_f(x_i, \beta)$ and add and subtract $J_f(x_i, \alpha)$ to obtain

$$n - 1 \leq n - 1 + J_f(x_i, \beta) - d(x_i, \alpha) - f(\alpha) + f(x_i) + J_f(x_i, \alpha) - J_f(x_i, \alpha).$$

Now, we substitute $n - 1 + J_f(x_i, \alpha) = d(x_i, \alpha) + f(x_i) - f(\alpha)$, which yields, after canceling $d(x_i, \alpha)$,

$$n - 1 \leq 2(f(x_i) - f(\alpha)) + J_f(x_i, \beta) - J_f(x_i, \alpha).$$

Solving for $f(x_i) - f(\alpha)$ and multiplying through by (-1) shows that

$$f(\alpha) - f(x_i) \leq \tfrac{1}{2}(-n + 1 + J_f(x_i, \beta) - J_f(x_i, \alpha)).$$

Then

$$\begin{aligned} d(x_i, \alpha) + f(x_i) - f(\alpha) &= n - 1 + J_f(x_i, \alpha) \\ \implies d(x_i, \alpha) &= n - 1 + J_f(x_i, \alpha) + f(\alpha) - f(x_i) \\ \implies d(x_i, \alpha) &\leq n - 1 + J_f(x_i, \alpha) + \tfrac{1}{2}(-n + 1 + J_f(x_i, \beta) - J_f(x_i, \alpha)) \\ &= \tfrac{1}{2}(n - 1 + J_f(x_i, \alpha) + J_f(x_i, \beta)) \\ \implies J_f(x_i, \alpha) + J_f(x_i, \beta) &\geq 2(d(x_i, \alpha)) - n + 1, \end{aligned}$$

and we have obtained the desired lower bound for $J_f(x_i, \alpha) + J_f(x_i, \beta)$. Making similar series of substitutions in the other three cases depending on the label order of α, x_i and β and on whether or not $\alpha = v_n$ shows that

$$J_f(x_i,\alpha)+J_f(x_i,\beta)\geq \begin{cases} 2(d(x_i,\alpha))-n+1 & \alpha\neq v_n, \\ 2(d(x_i,\alpha))-n-1 & \alpha=v_n. \end{cases}$$

And this completes the proof of the claim.

From these two inequalities, we construct Table 3.7, in which each entry gives the lower bound for the $J_f(x_i,\alpha)+J_f(x_i,\beta)$ associated to the corresponding $x_i \in \{v_k, ..., v_{2k-1}\}$ and $\alpha \in \{v_1, ..., v_{k-1}, v_n\}$ based on the equation above.

	v_k	v_{k+1}	v_{k+2}	...	v_{2k-3}	v_{2k-2}	v_{2k-1}
v_1	0	1	3	...	$2k-7$	$2k-5$	$2k-3$
v_2	0	0	1	...	$2k-9$	$2k-7$	$2k-5$
v_3	0	0	0	...	$2k-11$	$2k-9$	$2k-7$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$
v_{k-3}	0	0	0	...	1	3	5
v_{k-2}	0	0	0	...	0	1	3
v_{k-1}	0	0	0	...	0	0	1
v_n	≥ 0	≥ 0	≥ 0	...	≥ 0	≥ 1	≥ 3

Table 3.7: Lower bound for $J_f(x_i,\alpha)+J_f(x_i,\beta)$ associated to corresponding $x_i \in \{v_k,\ldots,v_{2k-1}\}$ and $\alpha \in \{v_1,\ldots,v_{k-1},v_n\}$.

Suppose $x_1,\ldots,x_n$ is any distance maximizing ordering of the vertices of $S_{2k,s}$ with associated radio labeling f. Note that in this case $n_x(e_{k-1}) = 2k-1$ so by conclusions 3 and 4 of Lemma 1 if x_i is in the set $\{v_k, ..., v_{2k-1}\}$, then x_{i-1} and x_{i+1} are in the set $\{v_1, ..., v_{k-1}, v_n\}$ so the hypotheses of the claim are satisfied for the triple $\{x_{i-1}, x_i, x_{i+1}\}$. By the claim a lower bound for $J_f(x_i,\alpha)+J_f(x_i,\beta)$ is given by Table 3.7. Let m be such that $x_m = v_{2k-1}$. In any distance maximizing ordering,

$n_x(e_{2k-2}) = 2$. By conclusion 2 of Lemma 1, as $n_x(e_{2k-2})$ is even, v_{2k-1} is not the first or last labeled vertex. Therefore $1 < m < n$ and we can use Table 3.7 to compute a lower bound of 1 for $J_f(x_m, \alpha) + J_f(x_m, \beta)$.

If $J_f(x_m, \alpha)+J_f(x_m, \beta) > 1$ then $\sum_{i=1}^{n-1} J_f(x_i, x_{i+1}) \geq 2$ as desired. If $J_f(x_m, \alpha)+J_f(x_m, \beta) = 1$ then either x_{m-1} or x_{m+1}, whichever is closest to v_{2k-1}, is v_{k-1}, as this is the only row with an entry less than 2 in the last column of Table 3.7. In any distance maximizing ordering, v_{k-1} must be the first or last vertex labeled because $n_x(e_{k-2}) + n_x(e_{k-1})$ is odd. Without loss of generality assume that v_{k-1} is the first labeled vertex and so $m = 2$. Since $x_1, \ldots, x_n$ is a distance maximizing ordering, it follows that $x_3 \in \{v_1, \ldots, v_{k-1}, v_n\}$. Now consider the vertex v_{2k-2} which corresponds to some x_r with $r \geq 4$. Therefore $r - 1 \geq 3$ so in particular $x_{r-1}, x_{r+1} \neq v_{k-1}$. Thus $J_f(x_{r-1}, x_r) + J_f(x_r, x_{r+1}) \geq 1$ and so $\sum_{i=1}^{n-1} J_f(x_i, x_{i+1}) \geq 2$ as desired.

Now we consider when $x_1, \ldots, x_n$ is an almost distance maximizing ordering of the vertices of $S_{2k,s}$. As the ordering is almost distance maximizing exactly one of the $n_x(e_i)$ values considered above is exactly one less. If this value is $n_x(e_{k-1})$, then all values for $n_x(e_i)$ would be even, contradicting conclusion 2 of Lemma 1. Thus $n_x(e_{k-1}) = 2k - 1$ in this case too, so by conclusion 2 of Lemma 1 if x_i is in the set $\{v_k, ..., v_{2k-1}\}$, then the hypotheses of the claim are satisfied for the triple $\{x_{i-1}, x_i, x_{i+1}\}$. Therefore the above argument when $x_m = v_{2k-1}$ still holds and so $\sum_{i=1}^{n-1} J_f(x_i, x_{i+1}) \geq 1$.

In conclusion, we have shown that if an ordering $x_1, \ldots, x_n$ of vertices is distance maximizing then $\sum_{i=1}^{n-1} J_f(x_i, x_{i+1}) \geq 2$ and if the ordering is almost distance maximizing then $\sum_{i=1}^{n-1} J_f(x_i, x_{i+1}) \geq 1$. In either case by Proposition 1 we conclude that $rn(S_{2k,s}) \geq 2k^2 - 4k + 2s + 3$. If $x_1, \ldots, x_n$ is neither distance maximizing, nor almost distance maximizing then by Proposition 1 it follows that $rn(S_{2k,s}) \geq 2k^2 - 4k + 2s + 3$ as $\sum_{i=1}^{n-1} J_f(x_i, x_{i+1})$ is always non-negative. □

3.3 Radio number of all other diameter $n-2$ graphs

In this section we will determine the radio number of all other diameter $n-2$ graphs. We start with some definitions.

Definition. Let $n, s \in \mathbb{Z}$ where $n \geq 4$ and $2 \leq s \leq n$. We define the graph $S^1_{n,s}$ with vertices $v_1, ..., v_n$ and edges $\{(v_i, v_{i+1}) | i = 1, 2, ..., n-2\}$ together with the edges (v_s, v_n) and (v_{s-1}, v_n). Without loss of generality we will always assume that $s \leq \lfloor \frac{n+1}{2} \rfloor$. See Figure 3.2.

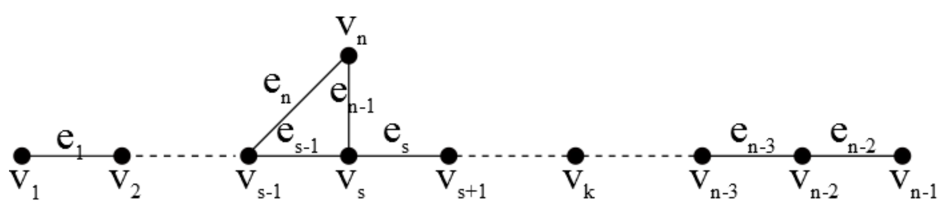

Figure 3.2: $S^1_{n,s}$.

Definition. Let $n, s \in \mathbb{Z}$ where $n \geq 4$ and $3 \leq s \leq n$. We define the graph $S^2_{n,s}$ with vertices $v_1, ..., v_n$ and edges $\{(v_i, v_{i+1}) | i = 1, 2, ..., n-2\}$ together with the edges (v_s, v_n) and (v_{s-2}, v_n). Without loss of generality we will always assume that

$s \leq \lfloor \frac{n+2}{2} \rfloor$. See Figure 3.3.

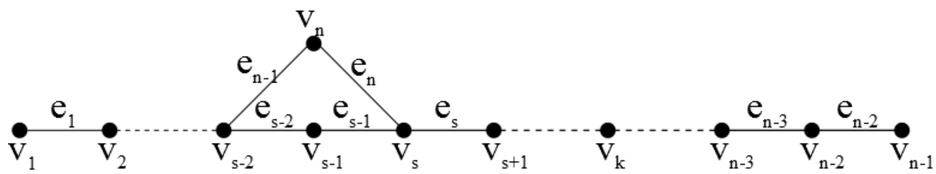

Figure 3.3: $S^2_{n,s}$.

Definition. Let $n, s \in \mathbb{Z}$ where $n \geq 4$ and $3 \leq s \leq n$. We define the graph $S^{1,2}_{n,s}$ with vertices $v_1, ..., v_n$ and edges $\{(v_i, v_{i+1}) | i = 1, 2, ..., n-2\}$ together with the edges (v_s, v_n), (v_{s-1}, v_n), and (v_{s-2}, v_n). Without loss of generality we will always assume that $s \leq \lfloor \frac{n+2}{2} \rfloor$. See Figure 3.4.

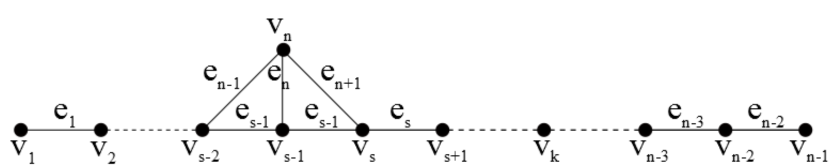

Figure 3.4: $S^{1,2}_{n,s}$.

Note that other than the complete graph K_3, these and spire graphs are all possible n-vertex graphs with diameter $n-2$. Such a graph must contain a path of diameter $n-2$ leaving one available vertex that is necessarily not part of the path. If this vertex is adjacent to two vertices on the path, these two vertices must be a distance of at most 2 from each other along the path as otherwise the diameter of the graph will be less than $n-2$. The complete graph K_3 also has diameter $n-2$, but since the radio number of K_3 is known and a proof of it is found in [4], we do not discuss it here.

To determine the radio numbers of these graphs, we begin with the following remark:

Remark 5. *Suppose a connected graph G' results from removing one or more edges from a connected graph G where D' is the diameter of G' and D is the diameter of G. If $D' = D$, then $rn(G') \leq rn(G)$.*

Theorem 6. *For $2 \leq s \leq \lfloor \frac{n}{2} \rfloor$, $rn(S^*_{n,s}) = rn(S_{n,s})$ where $rn(S^*_{n,s})$ is any one of $rn(S^1_{n,s})$, $rn(S^2_{n,s})$ or $rn(S^{1,2}_{n,s})$.*

Proof. For $2 \leq s \leq \lfloor \frac{n}{2} \rfloor$ the graph $S_{n,s}$ results from removing an edge from either $S^1_{n,s}$ or $S^2_{n,s}$, both of which result from removing an edge from $S^{1,2}_{n,s}$. Since all the graphs have diameter $n-2$, by Remark 5

$$rn(S_{n,s}) \leq rn(S^1_{n,s}) \leq rn(S^{1,2}_{n,s}), \text{ and}$$
$$rn(S_{n,s}) \leq rn(S^2_{n,s}) \leq rn(S^{1,2}_{n,s}).$$

By the above discussion, we only need to show that $rn(S_{n,s}) \geq rn(S^*_{n,s})$. We will do that by demonstrating that the radio labeling for $S_{n,s}$ given in Theorem 3 induces a radio labeling for $S^*_{n,s}$ with the same span. Let $v_1, ..., v_n$ be the vertices of $S_{n,s}$ and let $v^*_1, ..., v^*_n$ be the vertices of $S^*_{n,s}$. Let $f^* : V(S^*_{n,s}) \to \mathbb{Z}^+$ be given by $f^*(v^*_i) = f(v_i)$ where f is the function in Theorem 3 (for the corresponding case).

Notice that $d(v_i^*, v_j^*) = d(v_i, v_j)$ for all $j > i$ except possibly when $j = n$ and $i \leq s - 1$. Thus to verify that f^* is a radio labeling, we only need to verify the radio condition for the pairs $\{v_i^*, v_n^*\}$, where $i \leq s - 1$.

Case I: $n = 2k$ and $s \leq k - 2$.

By Theorem 3 we have that $f^*(v_n^*) = f(x_2)$ so we verify the radio condition for all pairs $\{x_i, x_2\}$. Recall that we are assuming that $s \geq 2$ and so $k \geq 4$. By adding the entries in the 2^{nd}, 3^{rd}, and 4^{th} rows of the last column of Table 3.1, we calculate that for all $i \geq 5$, $f^*(x_i) - f^*(x_2) \geq 3k + s - 5 \geq 2k - 1$.

Thus regardless of the value of s, the radio condition is satisfied for all $i \geq 5$. Note that x_1 corresponds to v_k^*, and x_3 corresponds to v_{k+4}^*. As $s \leq k - 2$, $d(v_i, v_n) = d(v_i^*, v_n^*)$ for $i = k, k+4$ so the radio condition is satisfied for these pairs. Finally we consider the pair $\{x_4, x_2\}$. Noting that x_4 corresponds to v_5^*, we have that $d(v_5, v_n) = d(v_5^*, v_n^*)$ if $s \leq 5$ and the radio condition is satisfied. If $s \geq 6$, then by adding the entries in the 2^{nd} and 3^{rd} rows of the last column of Table 3.1, we calculate that $f^*(x_4) - f^*(x_2) = 2k + s - 6 \geq 2k + 6 - 6 = 2k$, and the radio condition is satisfied.

Case II: $n = 2k$, and $s = k - 1$ or $s = k$.

As these cases are straightforward, we leave it to the reader to check them using Tables 3.3 and 3.4.

Case III: $n = 2k + 1$ and $2 \leq s \leq k$.

The reader can check these using Tables 3.5 and 3.6. □

Notice that the reasoning in Case I of the above proof applies to the graphs whose upper bounds are shown in Appendix A. This shows that the labelings given for $S_{n,s}$ in that section are also radio labelings for $rn(S^*_{n,s})$ when such a graph exists.

Theorem 6 leaves out only a few graphs with diameter $n-2$. The following theorem establishes the radio number in those cases:

Theorem 7. $rn(S^1_{2k+1,k+1}) = 2k^2+1$.

$rn(S^{1,2}_{2k+1,k+1}) = 2k^2+1$.

$rn(S^2_{2k+1,k+1}) = 2k^2$.

$rn(S^{1,2}_{2k,k+1}) = 2k^2-2k+2$.

$rn(S^2_{2k,k+1}) = 2k^2-2k+1$.

Proof. **Case I:** $S^1_{2k+1,k+1}$.

We first prove that $2k^2+1$ is an upper bound for $rn(S^1_{2k+1,k+1})$. Order the vertices of $S^1_{2k+1,k+1}$ into three groups as follows:

Group I: v_k, v_{2k+1},

Group II: $v_{2k}, v_{k-1}, v_{2k-1}, v_{k-2}, ..., v_{k+2}, v_1$,

Group III: v_{k+1}.

Now, rename the vertices of $S^1_{2k+1,k+1}$ in the above ordering by $x_1, x_2, \ldots, x_n$. This is the label order of the vertices of $S^1_{2k+1,k+1}$.

Claim: The function f defined in Table 3.8 is a radio labeling on $S^1_{2k+1,k+1}$.

Proof of Claim: We let the reader verify that the radio condition holds for all vertices $x_i, x_j \in V(S^1_{2k+1,k+1})$. In this case, the diameter of $S^1_{2k+1,k+1}$ is $2k-1$ so for every i, j with $j > i$, $d(x_i, x_j) + f(x_j) - f(x_i) \geq 2k$ must hold.

x_i	Vertex Names	$d(x_i, x_{i+1})$	$f(x_{i+1}) - f(x_i)$
x_1	v_k	1	$2k-1$
x_2	v_{2k+1}	k	k
x_3	v_{2k}	$k+1$	$k-1$
x_4	v_{k-1}	k	k
x_5	v_{2k-1}	$k+1$	$k-1$
x_6	v_{k-2}	k	k
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{n-4}	v_{k+3}	$k+1$	$k-1$
x_{n-3}	v_2	k	k
x_{n-2}	v_{k+2}	$k+1$	$k-1$
x_{n-1}	v_1	k	k
x_n	v_{k+1}	n/a	n/a

Table 3.8: Radio Labeling f on $S^1_{2k+1,k+1}$.

Letting $f(x_1) = 1$, the largest number in the range of the radio labeling f is then $f(x_n)$ and is therefore equal to the sum of the entries in the last column of Table 3.8 plus one. We let the reader verify that $rn(S^1_{2k+1,k+1}) \leq 2k^2 + 1$ as desired. And thus the claim has been proven.

Claim: $rn(S^1_{2k+1,k+1}) \geq 2k^2 + 1$.

Proof of Claim: We find a lower bound for $rn(S^1_{2k+1,k+1})$ by using Proposition 2 and determining $\max_p \sum_{i=1}^{n-1} d(x_i, x_{i+1})$. For $1 \leq i \leq 2k-1$ let e_i be the edge between v_i and v_{i+1}. Let e_{2k} and e_{2k+1} be the two edges incident to v_{2k+1} (see Figure 3.2). We will use the terminology established in Lemma 1. Using the third conclusion of that lemma, it follows that

$$N(e_i) \leq \begin{cases} 2i & \text{if } i \leq k-1, \\ 2(2k-i) & \text{if } k+1 \leq i \leq 2k-1. \end{cases}$$

Furthermore note that any path P_j contains at most one of e_k, e_{2k} and e_{2k+1}.

As there are a total of $2k$ paths P_j, it follows that $N(e_k)+N(e_{2k})+N(e_{2k+1})\leq 2k$. Therefore $\max_p\sum_{i=1}^{n-1} d(x_i,x_{i+1}) \leq \sum_{i=1}^{2k+1} N(e_i) \leq 2k^2$, and Lemma 2 shows that $rn(S^1_{2k+1,k+1})\geq 4k^2+1-2k^2=2k^2+1$ as desired. Thus, the claim has been proven.

Case II: $S^{1,2}_{2k+1,k+1}$.

Note that $S^1_{2k+1,k+1}$ results from removing an edge from $S^{1,2}_{2k+1,k+1}$ (and the graphs have the same diameter), so by Remark 5 and Case 1, $rn(S^1_{2k+1,k+1})=2k^2+1\leq rn(S^{1,2}_{2k+1,k+1})$. We leave it to the reader to verify that the same labeling in Table 3.8 is valid.

Case III: $S^2_{2k+1,k+1}$.

Notice that $S_{2k+1,k+1}=S_{2k+1,k}$ by symmetry. Then since $S_{2k+1,k+1}$ results from removing an edge from $S^2_{2k+1,k+1}$ (and the graphs have the same diameter), we have by Remark 5, Theorem 3, and Theorem 4 that $rn(S_{2k+1,k+1})=rn(S_{2k+1,k})=2k^2\leq rn(S^2_{2k+1,k+1})$. We use the labeling of Table 3.8 making the change that $f(x_2)-f(x_1)=2k-2$ since now $d(x_1,x_2)=2$ to conclude that $rn(S^2_{2k+1,k+1})\leq 2k^2$.

Case IV: $S^{1,2}_{2k,k+1}$.

We first prove that $2k^2-2k+2$ is an upper bound for $rn(S^{1,2}_{2k,k+1})$. Order the vertices of $S^{1,2}_{2k,k+1}$ into three groups as follows:

Group I: v_k, v_{2k}, v_{2k-1},

Group II: $v_1, v_{k+1}, v_2, v_{k+2}, ..., v_{k-2}, v_{2k-2}$,

Group III: v_{k-1}.

Now, rename the vertices of $S^{1,2}_{2k,k+1}$ in the above ordering by $x_1, x_2, \ldots, x_n$. This is the label order of the vertices of $S^{1,2}_{2k,k+1}$.

Claim: The function f defined in Table 3.9 is a radio labeling on $S^{1,2}_{2k,k+1}$.

x_i	Vertex Names	$d(x_i, x_{i+1})$	$f(x_{i+1}) - f(x_i)$
x_1	v_k	1	$2k-2$
x_2	v_{2k}	$k-1$	k
x_3	v_{2k-1}	$2k-2$	1
x_4	v_1	k	$k-1$
x_5	v_{k+1}	$k-1$	k
x_6	v_2	k	$k-1$
x_7	v_{k+2}	$k-1$	k
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{n-4}	v_{k-3}	k	$k-1$
x_{n-3}	v_{2k-3}	$k-1$	k
x_{n-2}	v_{k-2}	k	$k-1$
x_{n-1}	v_{2k-2}	$k-1$	k
x_n	v_{k-1}	n/a	n/a

Table 3.9: Radio Labeling f on $S^{1,2}_{2k,k+1}$.

Proof of Claim: We let the reader verify that the radio condition holds for all vertices $x_i, x_j \in V(S^{1,2}_{2k,k+1})$. In this case, the diameter of $S^{1,2}_{2k,k+1}$ is $2k-2$ so for every i, j with $j > i$, $d(x_i, x_j) + f(x_j) - f(x_i) \geq 2k-1$ must hold.

Letting $f(x_1) = 1$, the largest number in the range of the radio labeling f is then $f(x_n)$ and is therefore equal to the sum of the entries in the last column of Table 3.9 plus one. We let the reader verify that $rn(S^{1,2}_{2k,k+1}) \leq 2k^2 - 2k + 2$ as desired. Thus the claim has been proven.

Claim: $rn(S^{1,2}_{2k,k+1}) \geq 2k^2 - 2k + 2$.

Proof of Claim: We find a lower bound for $rn(S^{1,2}_{2k,k+1})$ by using Proposition 2 and determining $\max_p \sum_{i=1}^{n-1} d(x_i, x_{i+1})$. For $1 \leq i \leq 2k-2$ let e_i be the edge between v_i

and v_{i+1}. Let e_{2k-1}, e_{2k} and e_{2k+1} be the three edges incident to v_{2k} where e_{2k-1} is incident to v_{k-1}, e_{2k} is incident to v_k, and e_{2k+1} is incident to v_{k+1} (see Figure 3.4). By the third conclusion of Lemma 1 it follows that

$$N(e_i) \leq \begin{cases} 2i & \text{if } 1 \leq i \leq k-2, \\ 2(2k-1-i) & \text{if } k+1 \leq i \leq 2k-2. \end{cases}$$

Furthermore by Lemma 2 it follows that $N(e_{k-1}) + N(e_{2k-1}) \leq 2(k-1)$ and $N(e_k) + N(e_{2k+1}) \leq 2(k-1)$. Finally, for any ordering $x_1, \ldots, x_n$ of the vertices, $n_x(e_{2k}) \leq 1$ as it is only contained in a path with endpoints v_k and v_{2k}. Note that if all three of these inequalities are equalities, then v_k and v_{2k} correspond to x_1 and x_{2k} by the first conclusion of Lemma 1 as these are the only vertices for which the sum of the $n_x(e_i)$ for the incident edges may be odd. At the same time v_k and v_{2k} must correspond to x_i and x_{i+1} for some i as $n_x(e_{2k}) = 1$. This is a contradiction. Therefore $n_x(e_{k-1})+n_x(e_k)+n_x(e_{2k-1})+n_x(e_{2k})+n_x(e_{2k+1}) \leq 4(k-1)$. Thus $\max_p \sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq 2k^2 - 2k$, and Lemma 2 shows that $rn(S^{1,2}_{2k,k+1}) \geq 4k^2 - 4k + 2 - 2k^2 + 2k = 2k^2 - 2k + 2$.

Case V: $S^2_{2k,k+1}$.

We use the labeling of Table 3.9 making the change that $f(x_2) - f(x_1) = 2k-3$ since now $d(x_1, x_2) = 2$ to conclude that $rn(S^2_{2k,k+1}) \leq 2k^2 - 2k + 1$. For $1 \leq i \leq 2k-2$ let e_i be the edge between v_i and v_{i+1}. Let e_{2k-1} and e_{2k} be the edges incident to v_{2k} where e_{2k-1} is incident to v_{k-1}, and e_{2k} is incident to v_{k+1}. As in the previous case it follows that

$$N(e_i) \leq \begin{cases} 2i & \text{if } i \leq k-2, \\ 2(2k-1-i) & \text{if } k+1 \leq i \leq 2k-2. \end{cases}$$

Unlike in the previous case, here exactly one path may contain e_{k-1} and e_{2k-1} or it may contain e_k and e_{2k}. This would be the path (if such a path exists) with endpoints v_k and v_{2k}. Without loss of generality we can assume that this path contains e_{k-1} and e_{2k-1}. Therefore in this case $n_x(e_{k-1}) + n_x(e_{2k-1}) \leq 2(k-1)+1$ and $n_x(e_k) + n_x(e_{2k}) \leq 2(k-1)$. Thus $\max_p \sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq 2k^2 - 2k + 1$, and Lemma 2 shows that $rn(S^{1,2}_{2k,k+1}) \geq 4k^2 - 4k + 2 - 2k^2 + 2k - 1 = 2k^2 - 2k + 1$. □

CHAPTER 4
CATERPILLAR GRAPHS

In this chapter, we use techniques from Chapter 2 to improve the lower bound of the radio number of certain tree graphs as well as determine the radio number of some specific tree graphs. To begin, we define the general type of graph this chapter will address.

Definition. Let $n, s, l \in \mathbb{Z}^+$ with $n = s + l$. A caterpillar graph G is a tree graph with n vertices, $v_1, v_2, \ldots, v_{s+l}$. The *spine* of G consists of vertices $v_1, v_2, \ldots, v_s$ along with edges (v_i, v_{i+1}) for $i = 1, \ldots, s-1$. A *leg vertex* is a degree one vertex adjacent to v_i for some i, $2 \leq i \leq s-1$. See an example in Figure 4.1.

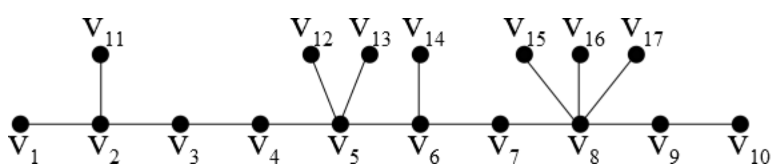

Figure 4.1: A Caterpillar with $s = 10$ vertices on the spine and $l = 7$ leg vertices.

4.1 Caterpillar Preliminaries

In this section, we establish notation for specific types of caterpillar graphs as well as determine properties of those graphs that build off of techniques for trees in general that were discussed in Chapter 2.

There are four main categories of caterpillar graphs in terms of the center edge definition from Section 2.1.1. There could be one center edge e_c where $N(e_c)$ is odd, there could be one center edge e_c where $N(e_c)$ is even, there could be two center edges, or there could be multiple center edges. Notice that the only way for a caterpillar graph to have multiple center edges is if the caterpillar is a star graph. Then every edge e is such that $N(e) = 2$. Since the radio number of star graphs has been determined in [4], we will not consider this last case in this thesis.

Since are considering caterpillar graphs with one or two center edges, we use the following notation for the rest of the thesis:

Notation. If G is a caterpillar with one center edge, e_c, let v_{c_a} and v_{c_b} be the vertices incident to e_c with v_{c_a} in A and v_{c_b} in B. Let e_{c_a} be the edge on the spine in A that is incident to v_{c_a}. Similarly, let e_{c_b} be the edge on the spine in B that is incident to v_{c_b}. See Figure 4.2.

If G is a caterpillar and there are two center edges, call the center edges e_{c_a} and e_{c_b}. Let v_c be the vertex incident to both e_{c_a} and e_{c_b}. Let v_{c_a} and v_{c_b} be the vertices on the spine of G adjacent to v_c such that v_{c_a} is incident to e_{c_a} and v_{c_b} is incident to e_{c_b}. Let A be the component v_{c_a} is in when e_{c_a} is removed from G and B be the component v_{c_b} is in when e_{c_b} is removed from G. See Figure 4.3.

Now we use ideas from Section 2.1 along with the structure of caterpillars in regard to their center edge(s) to determine when $\sum_{i=1}^{n-1} n_x(e_i)$ is maximized for a specific ordering $x_1, \ldots, x_n$ of the vertices of a caterpillar graph.

Proposition 4. *Let G be a caterpillar with n vertices. Let $x_1, \ldots, x_n$ be an ordering*

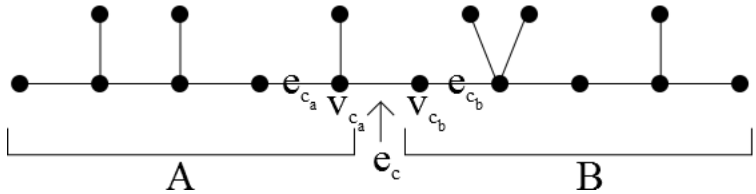

Figure 4.2: A caterpillar G with one center edge e_c.

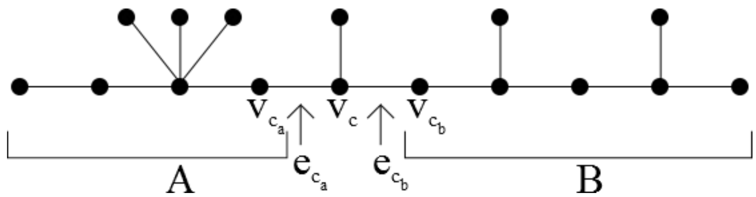

Figure 4.3: A caterpillar G with two center edges e_{c_a} and e_{c_b}.

of the vertices of G*. Let* $e_1, e_2, \ldots, e_{n-1}$ *be the edges of* G*. Then we have the following:*

- *If there is one center edge and* $N(e_c)$ *is odd, the sum* $\sum_{i=1}^{n-1} n_x(e_i)$ *is maximized when* $n_x(e_i) = N(e_i)$ *for all* $e_i \in E(G)$*.*

- *If there is either one center edge with* $N(e_c)$ *even or there are two center edges, the sum* $\sum_{i=1}^{n-1} n_x(e_i)$ *is maximized when*

$$\begin{cases} n_x(e_k) = N(e_k) - 1 & \textit{for some edge } e_k \in E(G) \\ n_x(e_i) = N(e_i) & \textit{for all edges } e_i \neq e_k \in E(G). \end{cases}$$

When this maximized sum occurs, there is only one edge in $E(G)$ *such that* $n_x(e)$ *is odd.*

Proof. First, we consider when G has one center edge and $N(e_c)$ is odd.

By (1) of Remark 2, the only time $N(e)$ is odd for some edge e in a tree is when $N(e) = n - 1$. Thus, in this case, $N(e_c) = n - 1$. Note that $N(e)$ for all other edges of G is even. Remark 2 (2) indicates that for the ordering $x_1, \ldots, x_n$, there must be at least one edge e with $n_x(e)$ odd. Since $n_x(e_c)$ is already odd, $\sum_{i=1}^{n-1} n_x(e_i)$ is maximized when $n_x(e_i) = N(e_i)$ for all $e_i \in E(G)$ for the ordering $x_1, \ldots, x_n$.

Next, we consider when G has one center edge with $N(e_c)$ even or when G has two center edges. In each of these cases, all of the $N(e_i)$ values are even. Thus, by Remark 2 (2), there has to be at least one edge e_k such that $n_x(e_k) \neq N(e_k)$ because $n_x(e_k)$ must be odd. To maximize $\sum_{i=1}^{n-1} n_x(e_i)$, it follows that there is exactly one edge e_k such that $n_x(e_k) \neq N(e_k)$. Specifically, $n_x(e_k) = N(e_k) - 1$. Therefore, $\sum_{i=1}^{n-1} n_x(e_i)$ is maximized when $\begin{cases} n_x(e_k) = N(e_k) - 1 & \text{for some edge } e_k \in E(G) \\ n_x(e_i) = N(e_i) & \text{for all edges } e_i \neq e_k \in E(G) \end{cases}$ for the ordering $x_1, \ldots, x_n$.

In both of these cases, when $\sum_{i=1}^{n-1} n_x(e_i)$ was maximized, there was only one edge with an odd $n_x(e)$ value. □

Remark 8. *Notice that Propositions 3 and 4 show that in a distance maximizing ordering $x_1, \ldots, x_n$ of the vertices of a caterpillar G, there is only one edge e such that $n_x(e)$ is odd. Then, from (3) of Remark 2, it follows that in a distance maximizing ordering of the vertices of G, the vertices x_1 and x_n are adjacent.*

Notation. For a distance maximizing ordering $x_1, \ldots, x_n$ of the vertices of a caterpillar G, let e_* denote the edge that is incident to both x_1 and x_n.

In Section 4.2, we consider a specific type of caterpillar. In order to define this particular caterpillar, we need the following proposition.

Proposition 5. *Let G be a tree with n vertices and one center edge e_c. The value of $N(e_c)$ is odd if and only if n is even and $|V(A)| = |V(B)| = \frac{n}{2}$.*

Proof. ($\Leftarrow$) Suppose e_c is the only center edge and $|V(A)| = \frac{n}{2} = |V(B)|$. Note that since $n = 2(\frac{n}{2})$ is the total number of vertices in the graph, n is even.

Since $|V(A)| = \frac{n}{2} = |V(B)|$, it follows that $2\min\{|V(A)|, |V(B)|\} = 2(\frac{n}{2}) = n$. By Lemma 1 (4), $N(e_c) \neq n = 2\min\{|V(A)|, |V(B)|\}$. Therefore, $N(e_c) = n - 1$ which is odd.

($\Rightarrow$) First note that there is no tree with n odd where the removal of an edge will result in two disconnected components A and B such that $|V(A)| = \frac{n}{2} = |V(B)|$. Thus, n is even.

Let A and B be the components of G after the removal of e_c. Suppose by way of contradiction that $|V(A)| \neq |V(B)|$. Without loss of generality, suppose $|V(A)| > |V(B)|$. Notice that $|V(B)| < \frac{n}{2}$. Since

$$N(e_i) = \begin{cases} n - 1 & \text{if } \min\{|V(A_i)|, |V(B_i)|\} = \frac{n}{2} \\ 2\min\{|V(A_i)|, |V(B_i)|\} & \text{else,} \end{cases}$$

$N(e_c) = 2|V(B)|$ which is even, contradicting the assumption that $N(e_c)$ is odd. Therefore, $|V(A)| = \frac{n}{2} = |V(B)|$. □

Applying results from Proposition 5 to caterpillars, we have the following definition:

Definition. A caterpillar is *edge-balanced* if there is an edge so that removing this edge results in exactly two components with an equal number of vertices. By Proposition 5, this is a caterpillar with one center edge where $N(e_c)$ is odd. Let G be an edge-balanced caterpillar with n vertices (note that by Proposition 5, n is necessarily even). Name the vertices of G as follows: The vertices of the spine will be denoted $u_1, ..., u_s$ (note that $D = s - 1$). If there are t leg vertices adjacent to u_r, we will denote them $l_{r-1}^1, ..., l_{r-1}^t$ if they are to the left of the center edge and $l_{r+1}^1, ..., l_{r+1}^t$ if they are to the right. See Figure 4.4 for an example.

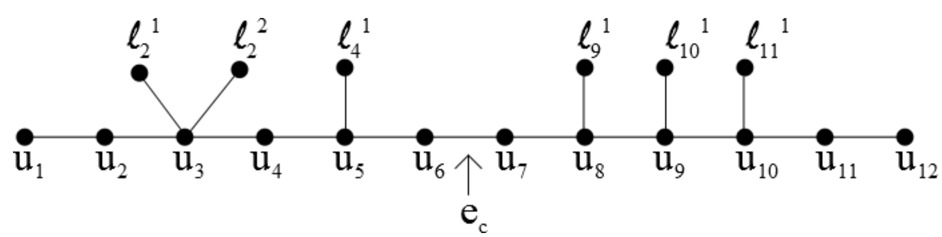

Figure 4.4: An edge-balanced caterpillar with nine vertices on each side of the center edge e_c.

Note that the distance between any two vertices on opposite sides of the center edge is given by the absolute difference of their subscripts.

To stay consistent with earlier notation, for an edge-balanced caterpillar G, let u_{c_a} and u_{c_b} be the vertices on the spine of G incident to e_c. This means $1 \leq c_a < c_b \leq s$ with $c_a + 1 = c_b$. Notice that this means we refer to A as the component to the left of the center edge and B as the component to the right of the center edge.

4.2 Algorithm for Edge-Balanced Caterpillars

In this section, we determine an algorithm for an ordering of the vertices of an edge-balanced caterpillar to provide an optimal radio labeling of that caterpillar.

Consider Table 4.1. We will construct this type of table to help us determine an ordering for a radio labeling of an edge-balanced caterpillar.

<table>
<tr><th colspan="2">Group 1</th><th colspan="2">Group 2</th></tr>
<tr><th>Column 1</th><th>Column 2</th><th>Column 3</th><th>Column 4</th></tr>
<tr><td>2</td><td>1</td><td>$n-1$</td><td>n</td></tr>
<tr><td>6</td><td rowspan="2">7</td><td>3</td><td rowspan="2">4</td></tr>
<tr><td>8</td><td>5</td></tr>
<tr><td>12</td><td rowspan="2">13</td><td>9</td><td rowspan="2">10</td></tr>
<tr><td>14</td><td>11</td></tr>
<tr><td>18</td><td rowspan="2">19</td><td>15</td><td rowspan="2">16</td></tr>
<tr><td>20</td><td>17</td></tr>
<tr><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td></tr>
<tr><td>$j+3$</td><td rowspan="2">$j+4$</td><td>j</td><td rowspan="2">$j+1$</td></tr>
<tr><td>$j+5$</td><td>$j+2$</td></tr>
<tr><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td></tr>
</table>

Table 4.1: Grid for Edge-Balanced Caterpillars.

For a particular edge-balanced caterpillar G, a table can be constructed in the same manner as Table 4.1. The last number placed in the table is $n-2$. Notice that $n-2$ will be the first column or the fourth column. We will use two copies of Table 4.1 to determine two orderings of a given edge-balanced caterpillar G using the algorithm below.

Algorithm 1. Consider an edge-balanced caterpillar G with n vertices. Construct two tables like Table 4.1 with n numbered cells. Call these tables Table A and Table B.

Place the names of the vertices of G in Table A as follows: Vertices that are to the left of the center edge are consecutively inserted into Column 1 starting with u_1 with non-decreasing subscripts where leg vertices are inserted after spine vertices with the same subscript. Similarly vertices to the right of the center edge are consecutively inserted in Column 3 starting with u_s with non-increasing subscripts where leg vertices are inserted after spine vertices with the same subscript. In Column 2, consecutively insert vertices from the right side of the center edge starting with u_{c_b} keeping the subscripts in non-decreasing order and inserting leg vertices before spine vertices with the same subscript. Finally, in Column 4, insert vertices to the left of the center edge, starting with u_{c_a} with subscripts in non-increasing order and inserting leg vertices before spine vertices with the same subscript.

Next, place the names of the vertices of G into Table B as follows: Vertices to the right of the center edge are consecutively inserted into Column 1 starting with u_s with non-increasing subscripts where leg vertices are inserted after spine vertices with the same subscript. Similarly vertices to the left of the center edge are consecutively inserted into Column 3 starting with u_1 with non-decreasing subscripts where leg vertices are inserted after spine vertices with the same subscript. In Column 2, consecutively insert vertices from the left of the center edge starting with u_{c_a}, keeping the subscripts in non-increasing order and inserting leg vertices before spine vertices with the same subscript. Finally, in Column 4, insert vertices to the right of the center edge, starting with u_{c_b} with subscripts in non-decreasing order and inserting leg vertices before spine vertices with the same subscript.

Algorithm 1 provides two tables corresponding to a given edge-balanced caterpillar. For each table, when the table has been completely filled in, each vertex of G is contained in exactly one numbered cell of the table.

For Table A, all the vertices to the right of the center edge are in Columns 2 and 3 while vertices to the left of the center edge are in Columns 1 and 4. For Table B, all vertices to the left of the center edge are in Columns 2 and 3 while vertices to the right of the center edge are in Columns 1 and 4. Since the center edge divides G into two components with $\frac{n}{2}$ vertices each, this means that the total number of vertices in the middle two columns is $\frac{n}{2}$ and the total number of vertices in the outside columns is $\frac{n}{2}$.

The numbers in the cells with the names of the vertices are the subscripts i for the orderings of the vertices given by Algorithm 1.

Applying the process of Algorithm 1 to the caterpillar in Figure 4.4 gives Tables 4.2 and 4.3.

We introduce the following definitions and notation to help us determine when Algorithm 1 gives an ordering which corresponds to a radio labeling of G that gives the radio number of G.

Notation. Let $x_1, \ldots, x_n$ be an ordering of the vertices of G. For a fixed i let $\alpha_{x_i}, \beta_{x_i}$ be the vertices x_{i-1} and x_{i+1} with the names chosen so that $d(x_i, \alpha_{x_i}) \leq d(x_i, \beta_{x_i})$. Note: for $i = 1$, consider x_2 as α_{x_i} and for $i = n$, consider x_{n-1} as α_{x_i}.

Group 1		Group 2	
Column 1	Column 2	Column 3	Column 4
u_1 2	u_7 1	u_{12} 17	u_6 18
u_2 6	u_8 7	u_{11} 3	u_5 4
l_2^1 8		l_{11}^1 5	
l_2^2 12	l_9^1 13	u_{10} 9	l_4^1 10
u_3 14		l_{10}^1 11	
		u_9 15	u_4 16

Table 4.2: Table A for Edge-Balanced Caterpillar of Figure 4.4 given by Algorithm 1.

Group 1		Group 2	
Column 1	Column 2	Column 3	Column 4
u_{12} 2	u_6 1	u_1 17	u_7 18
u_{11} 6	u_5 7	u_2 3	u_8 4
l_{11}^1 8		l_2^1 5	
u_{10} 12	l_4^1 13	l_2^2 9	l_9^1 10
l_{10}^1 14		u_3 11	
		u_4 15	u_9 16

Table 4.3: Table B for Edge-Balanced Caterpillar of Figure 4.4 given by Algorithm 1.

Definition. Let G be a caterpillar with an ordering $x_1, \ldots, x_n$ of its vertices. For a given i, let $t_{\alpha_{x_i}} = \begin{cases} 1 \text{ if } \alpha_{x_i} \text{ is a leg} \\ 0 \text{ otherwise.} \end{cases}$.

Definition. Let G be an edge-balanced caterpillar. Let $y_1, \ldots, y_n$ be an ordering of the vertices of G given by Algorithm 1. If the following conditions hold for at least one of the orderings given by Algorithm 1, then G is called a *jumpless caterpillar.*

1. Suppose the distance between any pair of vertices that are in horizontally ad-

jacent cells in Group 1 (respectively Group 2) is at most $\frac{D+1}{2}+t$ where t is 1 if the vertex in Column 2 (respectively Column 4) is a leg vertex and 0 otherwise.

2. Suppose $d(y_{n-2}, y_{n-3}) \leq \frac{D+1}{2} + t_{\alpha_{y_{n-2}}}$.

Remark 9. *If an edge-balanced caterpillar G is a jumpless caterpillar, we will represent G so that the ordering given by Table A in Algorithm 1 satisfies the conditions in the definition of a jumpless caterpillar. Note that a redrawing of G may be needed for this to be the case.*

For the rest of this thesis, we let $y_1, y_2, \ldots, y_n$ represent the vertices of G in the order they are labeled under Table A of Algorithm 1 and refer to this as the ordering given by Algorithm 1. This means the vertex in the cell of Table A with the number 1 in it is the vertex that is labeled first, or thought of as y_1 under the ordering given by this algorithm.

Notice that the orderings given in Tables 4.2 and 4.3 satisfy the conditions of a jumpless caterpillar. Thus, the graph G of Figure 4.4 is a jumpless caterpillar.

Proposition 6. *Let G be an edge-balanced caterpillar with $y_1, \ldots, y_n$ the ordering of vertices given by Algorithm 1. Then this ordering is a distance maximizing ordering.*

Proof. First note that the structure of an edge-balanced caterpillar G means that e_c divides G into two components, each with $\frac{n}{2}$ vertices. Thus, $N(e_c) = n - 1$.

Under Algorithm 1, y_1 and y_n are adjacent and both are incident to e_c. It can be checked that the pattern of Algorithm 1, which alternates labeling a vertex in A and then a vertex in B, causes $n_y(e) = N(e)$ for all edges in G.

Thus, by Proposition 4, $\sum_{i=1}^{n-1} n_y(e_i)$ is maximized and therefore, by Proposition 3, $\sum_{i=1}^{n-1} d(y_i, y_{i+1})$ is maximized. Thus, $y_1, \ldots, y_n$ is a distance maximizing ordering of G. □

Lemma 3. *Let y_{i-1}, y_i, y_{i+1} be a triple of vertices under the order given by Algorithm 1, with $\{y_{i-1}, y_{i+1}\} = \{\alpha_{y_i}, \beta_{y_i}\}$ such that $d(y_i, \alpha_{y_i}) \leq d(y_i, \beta_{y_i})$. When $y_i \notin \{y_1, y_{n-2}, y_n\}$, the following statements are true:*

- *If y_i is entered in Column 1 of Table 4.1, then α_{y_i} is entered in Column 2 of Table 4.1.*
- *If y_i is entered in Column 2 of Table 4.1, then α_{y_i} is entered in Column 1 of Table 4.1. In particular, $\alpha_{y_i} = y_{i+1}$.*
- *If y_i is entered in Column 3 of Table 4.1, then α_{y_i} is entered in Column 4 of Table 4.1.*
- *If y_i is entered in Column 4 of Table 4.1, then α_{y_i} is entered in Column 3 of Table 4.1. In particular, $\alpha_{y_i} = y_{i+1}$.*

When $y_i = y_1$, it follows that $\alpha_{y_1} = y_2$. When $y_i = y_{n-2}$, it follows that $\alpha_{y_{n-2}} = y_{n-3}$. When $y_i = y_n$, it follows that $\alpha_{y_n} = y_{n-1}$.

In particular, α_{y_i} is always in a cell that is horizontally adjacent to the cell for y_i where both α_{y_i} and y_i are in Group 1 or both are in Group 2 of Table 4.1.

Proof. First, we consider the case when $y_i \notin \{y_1, y_{n-2}, y_n\}$.

Case I: Suppose y_i is in Column 1 of Table 4.1.

Then, by the structure of the table, y_{i-1}, y_{i+1} are in Columns 2 and 3 of Table 4.1. Let $\{u, v\} = \{\alpha_{y_i}, \beta_{y_i}\}$ with u in Column 2 and v in Column 3. Under the process of Algorithm 1 $d(y_i, u) \leq d(y_i, v)$. When the inequality is strict, α_{y_i} is the vertex in Column 2.

If $d(y_i, u) = d(y_i, v)$, either both u and v are leg vertices or u is a leg vertex and v is on the spine of G. Note that either way, a leg vertex is in Column 2. By convention, let α_{y_i} be the leg vertex in Column 2.

Case II: Suppose y_i is in Column 2 of Table 4.1.

Then, by the structure of the table, both y_{i-1} and y_{i+1} are in Column 1. Therefore, α_{y_i} is in Column 1.

In particular, by Algorithm 1, $d(y_{i-1}, y_i) \geq d(y_i, y_{i+1})$ when y_{i-1}, y_{i+1} are in Column 1 and y_i is in Column 2 of Table 4.1. The distances are equal when both y_{i-1} and y_{i+1} are leg vertices or y_{i-1} is on the spine of G and y_{i+1} is a leg vertex. Thus, by convention, when the distances are equal, let α_{y_i} be the leg vertex entered into the $i+1$ cell of Table 4.1.

Case III: Suppose y_i is in Column 3 of Table 4.1.

The proof is analogous to the proof of Case I.

Case IV: Suppose y_i is in Column 4 of Table 4.1.

The proof is analogous to the proof of Case II.

Now we consider the case when y_i is in $\{y_1, y_{n-2}, y_n\}$.

When $y_i = y_1$, then it is not part of a triple of vertices y_{i-1}, y_i, y_{i+1}. In this case, as before, consider y_2 as α_{y_1}. Note that α_{y_1} is in Column 1 of Table 4.1.

When $y_i = y_{n-2}$, y_i is in Column 1 or Column 4 of Table 4.1. If y_{n-2} is in Column 4, then both y_{n-3} and y_{n-1} are in Column 3. If y_{n-2} is in Column 1, then either both y_{n-3} and y_{n-1} are in Column 3 or y_{n-3} is in Column 2 and y_{n-1} is in Column 3. In each case, by the process of Algorithm 1, $d(u_{c_b}, y_{n-3}) \leq d(u_{c_b}, y_{n-1})$. In the case where the distances are equal, y_{n-3} is a leg vertex and y_{n-1} is on the spine. In that case, we choose $\alpha_{y_{n-2}} = y_{n-3}$, the leg vertex. Therefore, $\alpha_{y_{n-2}}$ is y_{n-3} in all cases.

When $y_i = y_n$ then it is not part of a triple of vertices y_{i-1}, y_i, y_{i+1}. In this case, consider y_{n-1} as α_{y_n}. Note that α_{y_n} is in Column 3 of Table 4.1.

In all of the above cases, it can be checked that the cells of Table 4.1 that y_i and α_{y_i} are in are horizontally adjacent cells in Group 1 or in horizontally adjacent cells in Group 2. □

Definition. Let G be a caterpillar. Let $m_i := d(x_i, \alpha_{x_i}) - (\frac{D+1}{2} + t_{\alpha_{x_i}})$, if the quantity is positive and zero otherwise.

Theorem 10. *Let G be an edge-balanced caterpillar with ordering $y_1, y_2, \ldots, y_n$ of vertices as given by Algorithm 1. Define a labeling g such that $g(y_1) = 1$ and $g(y_{i+1}) = D + 1 - d(y_i, y_{i+1}) + g(y_i)$ for all i, $1 \leq i \leq n-1$. If G is a jumpless caterpillar, then g is a radio labeling of G and is therefore the associated radio labeling to the ordering given by Algorithm 1.*

Proof. We begin by showing that $m_i = 0$ for all i.

We start by considering when $y_i \neq y_{n-2}$. Then we have the following cases:

Case I: Consider a vertex in Column 1 or Column 3 of Table 4.1 as y_i in a triple of vertices y_{i-1}, y_i, y_{i+1}. By Lemma 3, α_{y_i} and y_i are in horizontally adjacent cells in Group 1 or in Group 2 of Table 4.1. Since G is a jumpless caterpillar, this means $d(y_i, \alpha_{y_i}) \leq \frac{D+1}{2} + t_{\alpha_{y_i}}$. Thus, $m_i = 0$ for all y_i when y_i is in Columns 1 or 3 of Table 4.1.

Case II: Consider a vertex in Column 2 or Column 4 of Table 4.1 as y_i in a triple of vertices y_{i-1}, y_i, y_{i+1}. Then we consider the following two cases:

Subcase A: Suppose y_i is on the spine of G. Then, since y_i is in Column 2 or Column 4 and G is a jumpless caterpillar, $d(y_i, y_{i-1}) \leq \frac{D+1}{2} + 0$ and $d(y_i, y_{i+1}) \leq \frac{D+1}{2} + 0$ because both y_{i-1} and y_{i+1} are in cells that are horizontally adjacent to the cell for y_i such that all three vertices are in Group 1 or all three are in Group 2 of Table 4.1. Note that this means $d(y_i, \alpha_{y_i}) \leq \frac{D+1}{2}$.

1. Suppose α_i is on the spine of G. Then $m_i = d(y_i, \alpha_{y_i}) - (\frac{D+1}{2} + 0) \leq \frac{D+1}{2} - (\frac{D+1}{2} + 0) = 0$ so by definition of m_i, it follows that $m_i = 0$.

2. Suppose α_i is a leg vertex. Then $m_i = d(y_i, \alpha_{y_i}) - (\frac{D+1}{2} + 1) \leq \frac{D+1}{2} - (\frac{D+1}{2} + 1) = -1$ so by definition of m_i, $m_i = 0$.

Therefore, when y_i is on the spine of G, it follows that $m_i = 0$.

Subcase B: Suppose y_i is a leg vertex of G. Then, since y_i is in Column 2 or Column 4, by definition of G being a jumpless caterpillar, $d(y_i, y_{i-1}) \leq \frac{D+1}{2} + 1$ and $d(y_i, y_{i+1}) \leq \frac{D+1}{2} + 1$ because both y_{i-1} and y_{i+1} are in cells that are horizontally adjacent to the cell for y_i such that all three vertices are in Group 1 or all three vertices are in Group 2 of Table 4.1. This means $d(y_i, \alpha_{y_i}) \leq \frac{D+1}{2} + 1$.

1. Suppose both y_{i-1} and y_{i+1} are on the spine of G. Then, $t_{\alpha_{y_i}} = 0$. Thus, $m_i = d(y_i, \alpha_{y_i}) - \frac{D+1}{2}$. The only time when this is not zero is if $d(y_i, \alpha_{y_i}) = \frac{D+1}{2} + 1$. If this were the case, notice that both $d(y_i, y_{i-1}) = \frac{D+1}{2} + 1$ and $d(y_i, y_{i+1}) = \frac{D+1}{2} + 1$ because if one were smaller, than $d(y_i, \alpha_{y_i})$ would be smaller. However, this cannot happen because, by Algorithm 1, y_{i-1} and y_{i+1} are in the same component of G. In a caterpillar, there is a unique vertex on the spine in component A (or component B) that is distance $\frac{D+1}{2} + 1$ from y_i. Therefore, if both y_{i-1} and y_{i+1} are on the spine, $m_i = 0$.

2. Suppose at least one of y_{i-1} or y_{i+1} is a leg. Then $m_i = d(y_i, \alpha_{y_i}) - (\frac{D+1}{2} + t_{\alpha_{y_i}})$. The only time this is not necessarily 0 is if $d(y_i, \alpha_{y_i}) = \frac{D+1}{2} + 1$ and $t_{\alpha_{y_i}} = 0$. This would mean that $d(y_{i-1}, y_i) = \frac{D+1}{2} + 1 = d(y_i, y_{i+1})$ (because otherwise $d(y_i, \alpha_{y_i}) < \frac{D+1}{2} + 1$) and that α_{y_i} is on the spine of G. However, since $d(y_{i-1}, y_i) = d(y_i, y_{i+1})$ and one of y_{i-1} or y_{i+1} is a leg vertex, as in Lemma 3, let the one that is a leg be α_{y_i}. Then, $m_i = 0$.

Now, we consider y_{n-2} in the triple of vertices $y_{n-3}, y_{n-2}, y_{n-1}$.

From Lemma 3, we know that $\alpha_{y_{n-2}} = y_{n-3}$. Thus, from condition (2) of the definition of G being a jumpless caterpillar, $m_{n-2} = d(y_{n-2}, \alpha_{y_{n-2}}) - (\frac{D+1}{2} + t_{\alpha_{y_{n-2}}}) \leq \frac{D+1}{2} + t_{\alpha_{y_{n-2}}} - (\frac{D+1}{2} + t_{\alpha_{y_{n-2}}}) = 0$.

Therefore, when G is a jumpless caterpillar, $m_i = 0$ for all i. Notice that this means $d(y_i, \alpha_{y_i}) \leq \frac{D+1}{2} + t_{\alpha_{y_i}}$ for $1 \leq i \leq n$.

Now, consider the labeling g such that $g(y_1) = 1$ and $g(y_{i+1}) = D + 1 - d(y_i, y_{i+1}) + g(y_i)$ for $1 \leq i \leq n-1$. We claim g is a radio labeling of G.

By the definition of g, the radio condition is satisfied for any pair of vertices y_i, y_{i+1}.

We will next verify the radio condition for pairs of vertices y_{i-1}, y_{i+1}. Notice that

$$d(\alpha_{y_i}, \beta_{y_i}) = d(y_i, \beta_{y_i}) - d(y_i, \alpha_{y_i}) + s_{\alpha_{y_i}}$$

where $s_{\alpha_{y_i}} = 0$ if α_{y_i} is on the spine of G and $s_{\alpha_{y_i}} = 2$ if α_{y_i} is a leg vertex.

From the definition of g it follows that,

$$d(y_i, \alpha_{y_i}) + |g(y_i) - g(\alpha_{y_i})| = D + 1 \text{ and}$$

$$d(y_i, \beta_{y_i}) + |g(y_i) - g(\beta_{y_i})| = D + 1.$$

Consider the case when $g(\alpha_{y_i}) < g(y_i) < g(\beta_{y_i})$. (The other case is proven similarly.) We start with the radio condition for the vertices α_{y_i} and β_{y_i} and make a series of substitutions as follows:

$$d(\alpha_{y_i}, \beta_{y_i}) + g(\beta_{y_i}) - g(\alpha_{y_i}) = d(y_i, \beta_{y_i}) - d(y_i, \alpha_{y_i}) + s_{\alpha_{y_i}} + g(\beta_{y_i}) - g(y_i) + g(y_i) - g(\alpha_{y_i})$$

$$= d(y_i, \beta_{y_i}) - d(y_i, \alpha_{y_i}) + s_{\alpha_{y_i}} + D + 1 - d(\beta_{y_i}, y_i) + D + 1 - d(\alpha_{y_i}, y_i)$$

$$= 2D + 2 - 2d(y_i, \alpha_{y_i}) + s_{\alpha_{y_i}}$$

$$\geq 2D + 2 - 2\left(\frac{D+1}{2} + t_{\alpha_{y_i}}\right) + s_{\alpha_{y_i}}$$

$$= 2D + 2 - D - 1 - 2t_{\alpha_{y_i}} + s_{\alpha_{y_i}}$$

$$= D + 1.$$

Therefore the radio condition is satisfied for vertices y_{i-1} and y_{i+1}.

By the definition of g, $g(y_{i+1}) - g(y_i) = D + 1 - d(y_i, y_{i+1})$. Also, from the definition of G being a jumpless caterpillar, for all pairs of vertices y_i and y_{i+1} that are in horizontally adjacent cells with both vertices in Group 1 or both vertices in Group 2 of Table 4.1, $d(y_i, y_{i+1}) \leq \frac{D+1}{2} + 1$. Thus, for y_i and y_{i+1} in horizontally adjacent cells both in Group 1 or both in Group 2 of Table 4.1, we have that

$$g(y_{i+1}) - g(y_i) = D + 1 - d(y_i, y_{i+1})$$

$$\geq D + 1 - \left(\frac{D+1}{2} + 1\right)$$

$$= D - \frac{D+1}{2}$$

$$= \frac{D-1}{2} \tag{4.1}$$

Now consider the pair of vertices y_i and y_j where $j = i + k$ for some positive integer $k \geq 3$. Then

$$g(y_j) - g(y_i) \geq g(y_{i+3}) - g(y_i)$$

$$= g(y_{i+3}) - g(y_{i+2}) + g(y_{i+2}) - g(y_{i+1}) + g(y_{i+1}) - g(y_i). \tag{4.2}$$

From Algorithm 1, two of the label differences for a pair of successively labeled vertices in (4.2) correspond to vertices that are in horizontally adjacent cells of Table 4.1 in Group 1 or horizontally adjacent cells in Group 2. For those two pairs, we get a bound from (4.1). The other label difference is at least 1 because all labels are unique. Thus, (4.1) and (4.2) give

$$g(y_j) - g(y_i) \geq \tfrac{D-1}{2} + \tfrac{D-1}{2} + 1 = D.$$

Also, since $d(y_j, y_i) \geq 1$, it follows that $g(y_j) - g(y_i) + d(y_j, y_i) \geq D + 1$. Therefore, the radio condition is satisfied for y_i and y_j whenever $|i - j| \geq 3$. Thus, g is a radio labeling of G. □

Corollary 1. *Let G be an edge-balanced caterpillar. If G is a jumpless caterpillar, then $rn(G) = g(y_n)$.*

Proof. From Proposition 6, the ordering $y_1, \ldots, y_n$ given by Algorithm 1 is a distance maximizing ordering of G. From Theorem 10, we know that when G is a jumpless caterpillar, g is a radio labeling. By how the labeling g in Theorem 10 was defined, $g(y_{i+1}) - g(y_i) = D + 1 - d(y_i, y_{i+1})$ for $1 \leq i \leq n - 1$. Summing these $n - 1$ equations and solving for $g(y_n)$ gives $g(y_n) = (n - 1)(D + 1) + 1 - \max \sum_{i=1}^{n-1} d(y_i, y_{i+1})$. From Proposition 2, it follows that $rn(G) = g(y_n)$. □

A technique used in the proof of Theorem 10 is useful when considering characteristics of a distance maximizing ordering of an edge-balanced caterpillar that does not require jumps. We include this in the next proposition.

Proposition 7. *Let G be an edge-balanced caterpillar. Let $x_1, \ldots, x_n$ be a distance maximizing ordering of the vertices of G such that the associated radio labeling f does not require jumps. Then for every triple of vertices x_{i-1}, x_i, x_{i+1}, $d(x_i, \alpha_{y_i}) \leq \frac{D+1}{2} + t_{\alpha_{y_i}}$.*

Proof. Since f does not require jumps, $\sum_{i=1}^{n-1} J_f(x_i, x_{i+1}) = 0$ which means that $J_f(x_i, x_{i+1}) = 0$ for $1 \leq i \leq n-1$. From this we have,

$$|f(x_i) - f(\alpha_{x_i})| = D + 1 - d(x_i, \alpha_{x_i}) \text{ and}$$

$$|f(x_i) - f(\beta_{x_i})| = D + 1 - d(x_i, \beta_{x_i}).$$

Notice that $d(\alpha_{x_i}, \beta_{x_i}) = d(x_i, \beta_{x_i}) - d(x_i, \alpha_{x_i}) + s_{\alpha_{x_i}}$ where $s_{\alpha_{x_i}} = 0$ if α_{x_i} is on the spine of G and $s_{\alpha_{x_i}} = 2$ if α_{x_i} is a leg vertex.

Consider the radio condition for x_{i-1} and x_{i+1}:

$$\begin{aligned}
f(x_{i+1}) - f(x_i) + f(x_i) - f(x_{i-1}) &\geq D + 1 - d(x_{i-1}, x_{i+1}) \\
\Rightarrow 2D + 2 - d(\alpha_{x_i}, x_i) - d(x_i, \beta_{x_i}) &\geq D + 1 - d(\alpha_{x_i}, \beta_{x_i}) \\
\Rightarrow D + 1 &\geq d(x_i, \alpha_{x_i}) + d(x_i, \beta_{x_i}) - d(\alpha_{x_i}, \beta_{x_i}) \\
\Rightarrow D + 1 &\geq d(x_i, \alpha_{x_i}) + d(x_i, \beta_{x_i}) \\
&\quad -[d(x_i, \beta_{x_i}) - d(x_i, \alpha_{x_i}) + s_{\alpha_{x_i}}] \\
\Rightarrow D + 1 &\geq 2d(x_i, \alpha_{x_i}) - s_{\alpha_{x_i}} \\
\Rightarrow \frac{D + 1 + s_{\alpha_{x_i}}}{2} &\geq d(x_i, \alpha_{x_i}) \\
\Rightarrow \frac{D+1}{2} + t_{\alpha_{x_i}} &\geq d(x_i, \alpha_{x_i}).
\end{aligned}$$

□

The occurrence of a vertex x_i being considered as α_{x_j} in relation to the vertex x_j is important in the arguments of the next theorem. This leads to the following definition:

Definition. Let G be an edge-balanced caterpillar. Let $x_1, \ldots, x_n$ be an ordering of the vertices of G with associated radio labeling f. For all $1 < i < n$, x_i is labeled after x_{i-1} and before x_{i+1}. Then x_i is in two triples of successively labeled vertices such that x_i is not the middle vertex of the triple, namely, the triples $\{x_{i-2}, x_{i-1}, x_i\}$ and $\{x_i, x_{i+1}, x_{i+2}\}$. Therefore, it is possible that x_i is $\alpha_{x_{i-1}}$ and/or $\alpha_{x_{i+1}}$. When x_i is considered $\alpha_{x_{i-1}}$ or $\alpha_{x_{i+1}}$, we refer to x_i as an *alpha vertex*. Notice that x_i could be considered an alpha vertex zero, one, or two times under the ordering $x_1, \ldots, x_n$ of the vertices of G.

When $i = 1$ or $i = n$, x_i is only part of one triple of successively labeled vertices. Thus, in those cases, x_i can be considered an alpha vertex either zero or one time under the ordering $x_1, \ldots, x_n$ of the vertices of G.

We will use the above definition to make arguments based on how many times certain vertices are considered to be alpha vertices under a given ordering of vertices of a caterpillar G in the proof of the following theorem.

Theorem 11. *Let G be an edge-balanced caterpillar with n vertices. If G is not a jumpless caterpillar, then $rn(G) \geq (n-1)(D+1) + 1 - \max_p(\sum_{i=1}^{n-1} d(x_i, x_{i+1})) + 1$ where the maximum is taken over all possible bijections p from $V(G)$ to $\{x_1, \ldots, x_n\}$.*

Proof. First we consider an ordering $x_1, \ldots, x_n$ of the vertices of G that is not a distance maximizing ordering. It follows that

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \max_p \left(\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \right) - 1$$

where the maximum is taken over all bijections p from the vertices of G to the set

$\{x_1, \ldots, x_n\}$. Then from Proposition 1,

$$f(x_n) \geq (n-1)(D+1) + f(x_1) - \left(\max_p \left(\sum_{i=1}^{n-1} d(x_i, x_{i+1})\right) - 1\right) + \sum_{i=1}^{n-1} J_f(x_i, x_{i+1})$$

$$= (n-1)(D+1) + f(x_1) - \max_p \left(\sum_{i=1}^{n-1} d(x_i, x_{i+1})\right) + 1 + \sum_{i=1}^{n-1} J_f(x_i, x_{i+1})$$

$$\geq (n-1)(D+1) + f(x_1) - \max_p \left(\sum_{i=1}^{n-1} d(x_i, x_{i+1})\right) + 1.$$

Next, we consider when the vertices of G have a distance maximizing ordering.

By the hypothesis, G is not a jumpless caterpillar. Then for the ordering $y_1, \ldots, y_n$ of the vertices of G given by Algorithm 1, either

(i) there exists a pair of vertices in horizontally adjacent cells of Group 1 (or Group 2) of the table given by Algorithm 1 such that their distance is greater than $\frac{D+1}{2} + t$ where t is 1 if the vertex in Column 2 (or Column 4) is a leg vertex and 0 otherwise, or

(ii) $d(y_{n-2}, y_{n-3}) > \frac{D+1}{2} + t_{\alpha_{y_{n-2}}}$.

Let h be the associated radio labeling to the ordering $y_1, \ldots, y_n$.

Case I: Suppose condition (i) is satisfied.

Consider the vertex of this pair that is in Column 1 (or Column 3) as y_i for some $i \neq n-2$. By Lemma 3, α_{y_i} is in Column 2 (or Column 4) and thus it follows that $d(y_i, \alpha_{y_i}) > \frac{D+1}{2} + t_{\alpha_{y_i}}$. By Proposition 6, $y_1, \ldots, y_n$ is a distance maximizing ordering of the vertices of G and thus by the contrapositive of Proposition 7, the associated radio labeling requires jumps. Thus, $h(y_n) \geq (n-1)(D+1) + h(y_1) - (\sum_{i=1}^{n-1}(y_i, y_{i+1})) + 1$.

Suppose by contradiction that there exists another distance maximizing ordering $x_1, \ldots, x_n$ of the vertices of G with associated radio labeling f such that f does not require jumps. From Proposition 7, this means that for all j, $d(x_j, \alpha_{x_j}) \leq \frac{D+1}{2} + t_{\alpha_{x_j}}$.

Suppose x_j is the same vertex as y_i. From the above assumptions, $d(y_i, \alpha_{y_i}) > \frac{D+1}{2} + t_{\alpha_{y_i}}$ and $d(x_j, \alpha_{x_j}) \leq \frac{D+1}{2} + t_{\alpha_{x_j}}$ where $\alpha_{x_j} \neq \alpha_{y_i}$. This means that $d(x_j, \alpha_{x_j}) \leq d(y_i, \alpha_{y_i})$.

Claim: If the pair of vertices is $\{y_1, y_2\}$ or $\{y_{n-1}, y_n\}$, by the structure of an edge-balanced caterpillar, no such α_{x_j} exists.

Proof of Claim: For the pair $\{y_1, y_2\}$, $y_2 = u_1$ is in Column 1 and $y_1 = u_{c_b}$ is in Column 2 so $y_1 = \alpha_{y_2}$ and $d(y_2, \alpha_{y_2}) > \frac{D+1}{2}$. By the structure of an edge-balanced caterpillar, every vertex w in component B is such that $d(u_1, w) > d(u_1, u_{c_b})$. Therefore, it is not possible to have $d(x_j, \alpha_{x_j}) \leq d(y_2, \alpha_{y_2})$. A similar argument shows that no such α_{x_j} exists when $i = n-1$ and thus the claim has been proven.

By the above claim, if the pair of vertices satisfying condition (i) is $\{y_1, y_2\}$ or $\{y_{n-1}, y_n\}$, we have already reached a contradiction to the assumption that f does not require jumps.

Now we consider when $i \neq 2, n-1$ and look at the following cases to reach a contradiction to the assumption that f does not require jumps.

Subcase A: $d(x_j, \alpha_{x_j}) < d(y_i, \alpha_{y_i})$.

Since the arguments for y_i in Column 1 or y_i in Column 3 of Table 4.1 are analogous, we give the argument only once. We suppose y_i is in Column 1 for this proof.

Let $\mathscr{A}$ be the set of all vertices that are entered into cells above the cell for α_{y_i} in Column 2 of Table 4.1. Let $\mathscr{B}$ be the set of all vertices that are entered into cells above the cell for y_i in Column 1 of Table 4.1.

Claim: α_{x_j} is in $\mathscr{A}$.

Proof of Claim: Under Algorithm 1, vertices are entered into Column 2 of Table 4.1 so that the subscripts of the vertices are in non-decreasing order and leg vertices are entered before spine vertices with the same subscript. A vertex v is entered in the table above α_{y_i} means $d(u_{c_b}, v) \leq d(u_{c_b}, \alpha_{y_i})$. Since $d(x_j, \alpha_{x_j}) < d(y_i, \alpha_{y_i})$, it follows that $d(\alpha_{x_j}, u_{c_b}) < d(\alpha_{y_i}, u_{c_b})$. Thus, $\alpha_{x_j} \in \mathscr{A}$ and we have proven the claim.

We consider two possible situations depending on where y_i is located in Table 4.1. Consider arbitrary entries into Columns 1 and 2 of Table 4.1: cells $m, m+1, m+2$ where m and $m+2$ denote cells in Column 1 whose entries have their associated alpha vertex in the $m+1$ cell of Column 2.

1. y_i is in the m entry of Table 4.1 (meaning that $m = i$ in this case).

By the structure of Table 4.1, we see that $|\mathscr{B}| = 2|\mathscr{A}| - 1$.

Now consider the elements in $\mathscr{A}$. In a distance maximizing ordering of the vertices of G, every element in $\mathscr{A}$ except for u_{c_b} could be an alpha vertex for two vertices in component A. The vertex u_{c_b} can be an alpha vertex for only one vertex in component A. Thus, in general, the possible number of uses of vertices in $\mathscr{A}$ as alpha vertices under a distance maximizing ordering is $2|\mathscr{A}| - 1$.

For the distance maximizing ordering $x_1, \ldots, x_n$, vertex α_{x_j} has already been used as an alpha vertex for one vertex of component A. Therefore, there are $2|\mathscr{A}|-2$ remaining possible number of uses of vertices in $\mathscr{A}$ as alpha vertices under the ordering $x_1, \ldots, x_n$. Since $|\mathscr{B}| = 2|\mathscr{A}| - 1 > 2|\mathscr{A}| - 2$, we conclude that there exists at least one vertex x_k in $\mathscr{B}$ such that α_{x_k} is not in $\mathscr{A}$ but is in component B.

By nature of how the sets $\mathscr{A}$ and $\mathscr{B}$ were formed,

$$d(u_{c_b}, \alpha_{y_i}) \leq d(u_{c_b}, \alpha_{x_k}) \text{ and}$$

$$d(y_i, u_{c_a}) \leq d(x_k, u_{c_a}). \tag{$\star$}$$

Since $d(x_k, \alpha_{x_k}) = d(x_k, u_{c_a}) + d(u_{c_a}, u_{c_b}) + d(u_{c_b}, \alpha_{x_k})$ and $d(y_i, \alpha_{y_i}) = d(y_i, u_{c_a}) + d(u_{c_a}, u_{c_b}) + d(u_{c_b}, \alpha_{y_i})$, whenever at least one of the inequalities of ($\star$) is strict, $d(x_k, \alpha_{x_k}) > d(y_i, \alpha_{y_i})$. By hypothesis, it follows that $d(x_k, \alpha_{x_k}) > d(y_i, \alpha_{y_i}) \geq \frac{D+1}{2} + 1$ which implies that $d(x_k, \alpha_{x_k}) > \frac{D+1}{2} + 1$. By contrapositive of Proposition 7, this means the associated radio labeling f for the ordering $x_1, \ldots, x_n$ requires jumps, contradicting the assumption.

To consider when the inequalities of ($\star$) are both equalities, we notice the following:

- $\alpha_{x_k} \notin \mathscr{A}$ means that α_{x_k} is entered in Column 2 below α_{y_i}, is α_{y_i}, or is entered into Column 3 of Table 4.1.

- Leg vertices are entered into Column 2 before spine vertices with the same subscript.

Note that if $\alpha_{x_k} = \alpha_{y_i}$, then since $d(x_k, \alpha_{x_k}) = d(y_i, \alpha_{y_i}) > \frac{D+1}{2} + t_{\alpha_{y_i}} = \frac{D+1}{2} + t_{\alpha_{x_k}}$, by the contrapositive of Proposition 7, f requires jumps, which is a contradiction to the assumption.

Now suppose $\alpha_{x_k} \neq \alpha_{y_i}$. Since $d(u_{c_b}, \alpha_{y_i}) = d(u_{c_b}, \alpha_{x_k})$, α_{y_i} and α_{x_k} have the same subscript in the original edge-balanced caterpillar notation. Therefore, either both α_{x_k} and α_{y_i} are leg vertices or α_{x_k} is a vertex on the spine of G while α_{y_i} is a leg vertex.

Since α_{y_i} is a leg vertex, $t_{\alpha_{y_i}} = 1$ and thus $d(y_i, \alpha_{y_i}) > \frac{D+1}{2} + 1$. Since $d(y_i, \alpha_{y_i}) = d(x_k, \alpha_{x_k})$, it follows that $d(x_k, \alpha_{x_k}) > \frac{D+1}{2} + 1 \geq \frac{D+1}{2} + t_{\alpha_{x_k}}$. Therefore, by the contrapositive of Proposition 7, the associated radio labeling f for the ordering $x_1, \ldots, x_n$ requires jumps, which is a contradiction to the assumption.

2. y_i is in the $m+2$ entry of the table (meaning $i = m+2$ in this case).

By the structure of Table 4.1, we see that $|\mathscr{B}| = 2|\mathscr{A}|$. Notice that $\mathscr{B}$ has the vertex entered in cell m which is why the set $\mathscr{B}$ in this case has one more element than the set $\mathscr{B}$ of the previous case.

By the same arguments as in the previous case, $\alpha_{x_j} \in \mathscr{A}$. Since the set $\mathscr{A}$ is the same as in the previous case, we use the same argument to see that the number of possible uses of vertices in $\mathscr{A}$ as alpha vertices that have not been used yet under the ordering $x_1, \ldots, x_n$ is $2|\mathscr{A}| - 2$. Since $|\mathscr{B}| = 2|\mathscr{A}| > 2|\mathscr{A}| - 2$, we conclude that there exists at least one vertex x_k in $\mathscr{B}$ such that α_{x_k} is not in $\mathscr{A}$ but is in component B.

The same arguments as in Case I: Subcase A:1 show that f requires a jump which is a contradiction to the assumption.

Subcase B: $d(x_j, \alpha_{x_j}) = d(y_i, \alpha_{y_i})$.

Note that the only way this can happen is if α_{y_i} is a vertex on the spine of G, α_{x_j} is a leg vertex, and $d(x_j, \alpha_{x_j}) = \frac{D+1}{2} + 1 = d(y_i, \alpha_{y_i})$. Also, this means that α_{y_i} and α_{x_j} have the same subscript in the original edge-balanced caterpillar notation.

As before, since the arguments for y_i in Column 1 or y_i in Column 3 of Table 4.1 are analogous, we give the argument only once. We suppose y_i is in Column 1 for this proof.

From Lemma 3, we know α_{y_i} is entered into Column 2 of Table 4.1. Let $\mathscr{A}$ be the set of all vertices that are entered into cells above the cell for α_{y_i} in Column 2 of Table 4.1. Let $\mathscr{B}$ be the set of all vertices that are entered into cells above the cell for y_i in Column 1 of Table 4.1 by Algorithm 1.

Algorithm 1 inserts leg vertices into Column 2 before spine vertices with the same subscript in the edge-balanced caterpillar notation. Thus, since α_{x_j} is a leg vertex and α_{y_i} is on the spine of G, it follows that $\alpha_{x_j} \in \mathscr{A}$. The proof now follows the proof of Case I: Subcase A.

In all of the above cases, we have shown that when G is not a jumpless caterpillar such that condition (i) above is satisfied, the labeling associated with an arbitrary distance maximizing ordering requires jumps. Therefore, from Propositions 1 and 2 and the definition of a labeling requiring jumps, we have that

$$rn(G) \geq (n-1)(D+1) + f(x_1) - \max_p \left(\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \right) + 1.$$

where the maximum is taken over all bijections p from the vertices of G to the set $\{x_1, \ldots, x_n\}$.

Case II: Suppose condition (ii) is satisfied.

By Lemma 3 $\alpha_{y_{n-2}} = y_{n-3}$. Condition (ii) shows that $d(y_{n-2}, \alpha_{y_{n-2}}) > \frac{D+1}{2} + t_{\alpha_{y_{n-2}}}$. By Proposition 6, $y_1, \ldots, y_n$ is a distance maximizing ordering of the vertices of G and thus by the contrapositive of Proposition 7, the associated radio labeling requires jumps. Thus, $h(y_n) \geq (n-1)(D+1) + h(y_n) - (\sum_{i=1}^{n-1} d(y_i, y_{i+1})) + 1$.

Let $x_1, \ldots, x_n$ be an arbitrary distance maximizing ordering of the vertices of G. Suppose by contradiction that the associated radio labeling f does not require jumps. From Proposition 7, this means that for all j, $d(x_j, \alpha_{x_j}) \leq \frac{D+1}{2} + t_{\alpha_{x_j}}$.

Suppose x_j is the same vertex as y_{n-2}. From the above assumptions,

$d(y_{n-2}, \alpha_{y_{n-2}}) > \frac{D+1}{2} + t_{\alpha_{y_{n-2}}}$ and $d(x_j, \alpha_{x_j}) \leq \frac{D+1}{2} + t_{\alpha_{x_j}}$. This means $d(x_j, \alpha_{x_j}) \leq d(y_{n-2}, \alpha_{y_{n-2}})$.

Now we consider the following cases to find a contradiction to the assumption that f does not require jumps.

Subcase A: $d(x_j, \alpha_{x_j}) < d(y_{n-2}, \alpha_{y_{n-2}})$.

1. y_{n-2} is in Column 1 of Table 4.1.

Notice that $y_{n-3} = \alpha_{y_{n-2}}$ could be in Column 2 or Column 3 of Table 4.1.

a) y_{n-3} is in Column 2 of Table 4.1.

This means that for cells $m, m+1, m+2$ where m and $m+2$ are in Column 1 and $m+1$ is in Column 2 of Table 4.1, y_{n-2} is in the $m+2$ entry. Therefore, the proof of the case is the same argument as Case I: Subcase A:2 with y_{n-2} as y_i.

b) y_{n-3} is in Column 3 of Table 4.1.

Let $\mathscr{A}$ be the set of all vertices entered into cells in Column 2 of Table 4.1. Let $\mathscr{B}$ be the set of all vertices entered into cells above the cell for y_{n-2} in Column 1 of Table 4.1. Note that $|\mathscr{B}| = 2|\mathscr{A}| - 1$.

Claim: α_{x_j} is in $\mathscr{A}$.

Proof of Claim: Since $d(x_j, \alpha_{x_j}) < d(y_{n-2}, \alpha_{y_{n-2}})$, it follows that $d(u_{c_b}, \alpha_{x_j}) < d(u_{c_b}, \alpha_{y_{n-2}})$. In Algorithm 1, vertices are entered into Column 3 in non-increasing order. Since $\alpha_{y_{n-2}}$ is the last vertex entered into Column 3 and $d(u_{c_b}, \alpha_{x_j}) < d(u_{c_b}, \alpha_{y_{n-2}})$, it follows that α_{x_j} is in Column 2 of Table 4.1. Therefore, α_{x_j} is in $\mathscr{A}$ and the claim has been proven.

In a distance maximizing ordering of G, every element in $\mathscr{A}$ except for u_{c_b} could be an alpha vertex for two vertices in component A. The vertex u_{c_b} can be an alpha vertex for only one vertex in component A. Thus, in general, the possible number of uses of vertices in $\mathscr{A}$ as alpha vertices under a distance maximizing ordering is $2|\mathscr{A}| - 1$.

In the distance maximizing ordering $x_1, \ldots, x_n$, the vertex α_{x_j} has already been used as an alpha vertex for one vertex in component A. Therefore, the remaining possible number of uses of vertices in $\mathscr{A}$ as alpha vertices under the ordering $x_1, \ldots, x_n$

is $2|\mathscr{A}|-2$. Since $|\mathscr{B}|=2|\mathscr{A}|-1>2|\mathscr{A}|-2$, we conclude that there exists at least one vertex x_k in $\mathscr{B}$ such that α_{x_k} is not in $\mathscr{A}$ but is in component B. By nature of how the sets $\mathscr{A}$ and $\mathscr{B}$ were formed,

$$d(u_{c_b},\alpha_{y_{n-2}}) \leq d(u_{c_b},\alpha_{x_k}) \text{ and}$$

$$d(y_{n-2},u_{c_a}) \leq d(x_k,u_{c_a}). \tag{†}$$

Since $d(y_{n-2},\alpha_{y_{n-2}}) = d(y_{n-2},u_{c_a})+d(u_{c_a},u_{c_b})+d(u_{c_b},\alpha_{y_{n-2}})$ and $d(x_k,\alpha_{x_k}) = d(x_k,u_{c_a})+d(u_{c_a},u_{c_b})+d(u_{c_b},\alpha_{x_k})$, whenever one of the above inequalities is strict, $d(y_{n-2},\alpha_{y_{n-2}}) < d(x_k,\alpha_{x_k})$. Thus, we have that $d(x_k,\alpha_{x_k}) > d(y_{n-2},\alpha_{y_{n-2}}) \geq \frac{D+1}{2}+1$ which implies that $d(x_k,\alpha_{x_k}) > \frac{D+1}{2}+t_{\alpha_{x_k}}$. Therefore, by the contrapositive of Proposition 7, the associated radio labeling f requires jumps which is a contradiction to our assumption.

If both of the inequalities of (†) are equalities, $d(x_k,\alpha_{x_k}) = d(y_{n-2},\alpha_{y_{n-2}})$. Since $\alpha_{x_k} \notin \mathscr{A}$, α_{x_k} is either the same vertex as $\alpha_{y_{n-2}}$ or is entered in Column 3 of Table 4.1 and in a cell above the cell for $y_{n-3}=\alpha_{y_{n-2}}$.

Note that if $\alpha_{x_k}=\alpha_{y_{n-2}}$, then since $d(x_k,\alpha_{x_k}) = d(y_{n-2},\alpha_{y_{n-2}}) > \frac{D+1}{2}+t_{\alpha_{y_{n-2}}} = \frac{D+1}{2}+t_{\alpha_{x_k}}$, by the contrapositive of Proposition 7, f requires jumps, which is a contradiction to the assumption.

Now, suppose $\alpha_{x_k} \neq \alpha_{y_{n-2}}$. Since $d(\alpha_{y_{n-2}},u_{c_b}) = d(\alpha_{x_k},u_{c_b})$, $\alpha_{y_{n-2}}$ and α_{x_k} have the same subscript in the original edge-balanced caterpillar notation. Also, since α_{x_k} is in Column 3 of Table 4.1 in a cell above the cell for $\alpha_{y_{n-2}}$, this means that either both α_{x_k} and $\alpha_{y_{n-2}}$ are leg vertices or $\alpha_{y_{n-2}}$ is a leg vertex and α_{x_k} is on the spine of G. Now use the same argument as in the proof of Case I: Subcase A:1 with y_{n-2} instead of y_i to contradict the assumption that f does not require jumps.

2. y_{n-2} is in Column 4 of Table 4.1.

From Lemma 3, it follows that $\alpha_{y_{n-2}}$ is in Column 3. Consider the triple of vertices $y_{n-4}, y_{n-3}, y_{n-2}$. From Lemma 3, $y_{n-2} = \alpha_{y_{n-3}}$.

We now consider the following two cases.

a) Suppose $\alpha_{y_{n-2}}$ is on the spine of G, y_{n-2} a leg vertex, and $d(y_{n-2}, \alpha_{y_{n-2}}) = \frac{D+1}{2} + 1$.

Since $\alpha_{y_{n-2}}$ is on the spine of G, $t_{\alpha_{y_{n-2}}} = 0$ so by Proposition 7, the radio labeling associated with the ordering $y_1, \ldots, y_n$ requires a jump. We would like to use the pair of vertices $y_{n-3}, \alpha_{y_{n-3}}$ to make a similar argument to that of Case I: Subcase A:1 to reach a contradiction to the assumption that f does not require jumps. However, when considering y_{n-3} and $\alpha_{y_{n-3}}$, since $\alpha_{y_{n-3}}$ is a leg vertex, $t_{\alpha_{y_{n-3}}} = 1$ so $d(y_{n-3}, \alpha_{y_{n-3}})$ does not cause the associated radio labeling to have jumps.

We now argue why the radio labeling associated with the ordering $y_1, \ldots, y_n$ given by Algorithm 1 still requires jumps in this case and then set up analogous arguments to those found in Case I to reach a contradiction to the assumption that f requires jumps.

Consider the vertices y_{n-7}, the vertex in the cell directly above the cell for y_{n-3} in Table 4.1, and y_{n-8}, the vertex in the cell directly above the cell for y_{n-2} in Table 4.1. By Lemma 3, $y_{n-8} = \alpha_{y_{n-7}}$.

Recall that Algorithm 1 enters vertices into Column 3 of Table 4.1 so that the subscripts of the vertices are non-increasing and leg vertices are inserted after spine vertices with the same subscript. Since y_{n-3} is on the spine of G and the last entry in Column 3, it follows that the subscript for y_{n-7} is exactly one more than the subscript of y_{n-3}. This means $d(u_{c_b}, y_{n-7}) = d(u_{c_b}, y_{n-3}) + 1$.

Also, Algorithm 1 enters vertices into Column 4 of Table 4.1 so that the subscripts of the vertices are non-increasing and leg vertices are inserted before spine vertices with the same subscript. Since y_{n-2} is a leg vertex and is the last entry in Column 4, either

- y_{n-8} is a leg vertex with the same subscript as y_{n-2} which means that $d(y_{n-8}, u_{c_a}) = d(y_{n-2}, u_{c_a})$, or

- y_{n-8} is on the spine of G where its subscript is exactly one more than the subscript of y_{n-2} which means that $d(y_{n-8}, u_{c_a}) = d(y_{n-2}, u_{c_a}) - 1$.

Then we get the following bounds for $d(y_{n-8}, y_{n-7})$.

If y_{n-8} is a leg vertex, $t_{\alpha_{y_{n-7}}} = 1$ and

$$\begin{aligned} d(y_{n-8}, y_{n-7}) &= d(y_{n-8}, u_{c_a}) + d(u_{c_a}, u_{c_b}) + d(u_{c_b}, y_{n-7}) \\ &= d(y_{n-2}, u_{c_a}) + d(u_{c_a}, u_{c_b}) + d(u_{c_b}, y_{n-3}) + 1 \\ &= d(y_{n-2}, y_{n-3}) + 1 \\ &= \tfrac{D+1}{2} + 2 \\ &> \tfrac{D+1}{2} + t_{\alpha_{y_{n-7}}}. \end{aligned}$$

If y_{n-8} is on the spine of G, $t_{\alpha_{y_{n-7}}} = 0$ and

$$
\begin{aligned}
d(y_{n-8}, y_{n-7}) &= d(y_{n-8}, u_{c_a}) + d(u_{c_a}, u_{c_b}) + d(u_{c_b}, y_{n-7}) \\
&= d(y_{n-2}, u_{c_a}) - 1 + d(u_{c_a}, u_{c_b}) + d(u_{c_b}, y_{n-3}) + 1 \\
&= d(y_{n-2}, y_{n-3}) \\
&= \tfrac{D+1}{2} + 1 \\
&> \tfrac{D+1}{2} + t_{\alpha_{y_{n-7}}}.
\end{aligned}
$$

This shows that the radio labeling associated with $y_1, \ldots, y_n$ requires a jump when labeling the triple of vertices $y_{n-8}, y_{n-7}, y_{n-6}$. Let x_m be the same vertex as y_{n-7}. By the assumption that f does not require jumps and Proposition 7, $d(x_m, \alpha_{x_m}) \leq \frac{D+1}{2} + t_{\alpha_{x_m}}$.

If y_{n-8} is a leg vertex, $d(x_m, \alpha_{x_m}) \leq \frac{D+1}{2} + 1 < \frac{D+1}{2} + 2 = d(y_{n-7}, y_{\alpha_{n-7}})$. The proof now follows the proof of Case I: Subcase A:2.

If y_{n-8} is on the spine of G, $d(x_m, \alpha_{x_m}) \leq \frac{D+1}{2} + 1 = d(y_{n-7}, y_{\alpha_{n-7}})$. When this is a strict inequality, the proof now follows the proof of Case I: Subcase A:2. If this is an equality, the proof now follows the proof of Case I: Subcase B.

b) Suppose it is not the case that $\alpha_{y_{n-2}}$ is on the spine of G, y_{n-2} is a leg vertex, and $d(y_{n-2}, \alpha_{y_{n-2}}) = \frac{D+1}{2} + 1$.

In this case, $d(y_{n-3}, \alpha_{y_{n-3}}) > \frac{D+1}{2} + t_{\alpha_{y_{n-3}}}$ and thus, by the contrapositive of Proposition 7, the radio labeling associated with the ordering $y_1, \ldots, y_n$ requires jumps when labeling the triple of vertices $y_{n-4}, y_{n-3}, y_{n-2}$. Let x_m be the same vertex as y_{n-3}. By assumption, f does not require jumps and therefore $d(x_m, \alpha_{x_m}) \leq \frac{D+1}{2} + t_{\alpha_{x_m}}$. We can now use an analogous argument to that of Case I: Subcase A:1 if $d(x_m, \alpha_{x_m}) < d(y_{n-3}, \alpha_{y_{n-3}})$ by considering the original y_{n-3} as y_i in a triple of vertices.

If $d(x_m, \alpha_{x_m}) = d(y_{n-3}, \alpha_{y_{n-3}})$, the proof is analogous to the proof of Case I: Subcase B. From these analogous arguments, we conclude that f would also require jumps, which is a contradiction.

Subcase B: $d(x_j, \alpha_{x_j}) = d(y_{n-2}, \alpha_{y_{n-2}})$.

Note that the only way this can happen is when $d(x_j, \alpha_{x_j}) = \frac{D+1}{2} + 1 = d(y_{n-2}, \alpha_{y_{n-2}})$ where $\alpha_{y_{n-2}} = y_{n-3}$ is on the spine of G and α_{x_j} is a leg vertex.

1. y_{n-2} is in Column 1 of Table 4.1.

a) y_{n-3} is in Column 2 of Table 4.1.

The proof of this case is the same as that of Case I: Subcase B with $i = n - 2$.

b) y_{n-3} is in Column 3 of Table 4.1.

Let $\mathscr{A}$ be the set of all vertices entered into cells in Column 2 of Table 4.1. Let $\mathscr{B}$ be the set of all vertices entered into cells above the cell for y_{n-2} in Column 1 of Table 4.1.

Claim: α_{x_j} is in $\mathscr{A}$.

Proof of Claim: Algorithm 1 inserts leg vertices into Column 3 after spine vertices with the same subscript. Since y_{n-3} is on the spine of G and is the last vertex entered in Column 3 of Table 4.1, it follows that α_{x_j}, a leg vertex with the same subscript as y_{n-3}, is in Column 2 of Table 4.1. Therefore, α_{x_j} is in $\mathscr{A}$ and the claim has been proven.

By the same argument as in the proof of Case II: Subcase A:1b, we conclude that there exists a vertex $x_k \in \mathscr{B}$ such that $\alpha_{x_k} \notin \mathscr{A}$. The same argument holds when at least one of the inequalities of (†) is strict.

<u>Claim:</u> In this case, it is not possible for both inequalities of (†) to be equal.

Proof of Claim: Suppose by contradiction that both inequalities of (†) are equalities. Algorithm 1 inserts leg vertices into Column 3 of Table 4.1 after spine vertices with the same subscript. The last vertex entered into Column 3 is y_{n-3} which is on the spine of G. Since y_{n-3} and α_{x_k} have the same subscript in the edge-balanced caterpillar notation, α_{x_k} is in $\mathscr{A}$ which is a contradiction and thus the claim has been proven.

2. y_{n-2} is in Column 4 of Table 4.1.

From Lemma 3, it follows that y_{n-3} is in Column 3. If y_{n-2} is a leg vertex, the proof is the same as the proof of Case II: Subcase A:2a. If y_{n-2} is on the spine of G, use the triple of vertices $y_{n-4}, y_{n-3}, y_{n-2}$ to reach a contradiction like in the proof of Case II: Subcase A: 2b.

In all of the above cases, we have shown that when G is not a jumpless caterpillar such that condition (ii) above is satisfied, the labeling associated with an arbitrary distance maximizing ordering requires jumps. Therefore, from Propositions 1 and 2 and the definition of a labeling requiring jumps, we have that

$$rn(G) \geq (n-1)(D+1) + f(x_1) - \max_p \left(\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \right) + 1.$$

where the maximum is taken over all bijections p from the vertices of G to the set $\{x_1, \ldots, x_n\}$.

□

4.3 Bounds for Radio Number of Other Caterpillars

In Section 4.2, we determined a specific labeling that gives the radio number of edge-balanced caterpillars that are jumpless caterpillars. However, not all edge-balanced caterpillars are jumpless caterpillars. In Section 4.3.1, we establish some definitions and propositions to help improve the lower bound of the radio number of some other edge-balanced caterpillars. Many of these propositions have analogous results for caterpillars with two center edges. Thus, we include results about some caterpillars with two center edges in this section as well. Then, in Section 4.3.2, we determine an improved lower bound for the radio number of the caterpillars discussed in Section 4.3.1. In Section 4.3.3, we discuss some results for caterpillars with one center edge where $N(e_c)$ is even. Finally, Section 4.3.4 gives conclusions from the results of these sections.

4.3.1 Preliminaries

To help refer to caterpillars with two center edges, we include the following definitions:

Definition. Let G be a caterpillar with two center edges. Let c be the number of leg vertices adjacent to v_c.

- If $c = 0$, we call G *vertex-balanced.*
- If $c \neq 0$, we call G *almost vertex-balanced.*

To help improve the lower bounds of some caterpillars, we have the following definition:

Definition. Let G be a caterpillar with n vertices and diameter D.

- If G has one center edge, a vertex v_* is a *problem vertex* if $v_* \in A$ and $d(v_*, v_{c_b}) \geq \frac{D+2}{2}$ (or $v_* \in B$ and $d(v_*, v_{c_a}) \geq \frac{D+2}{2}$).

- If G has more than one center edge, a vertex v_* is a *problem vertex* if $d(v_*, v_c) \geq \frac{D+2}{2}$.

As some of the results will rely on characteristics of caterpillars based on where legs are located on the caterpillar, we use the following notation for the rest of the paper.

Notation. Let G be a caterpillar. Let a be the number of legs in component A and let b be the number of legs in component B. If there are two center edges, let c be the number of leg vertices adjacent to v_c.

Remark 12. *For an edge-balanced, vertex-balanced, or almost vertex-balanced caterpillar, $|V(A)| = |V(B)|$. Without loss of generality, let $a \geq b$.*

The next results are useful in categorizing caterpillar graphs based on the location of their legs. This helps to determine which types of caterpillars have an improved lower bound due to a problem vertex.

Proposition 8. *Let G be a caterpillar such that G is edge-balanced, vertex-balanced, or almost vertex-balanced.*

(i) If D is odd and $a \geq b + 2$, then there exists at least one problem vertex.

(ii) If D is even and $a > b$, then there exists at least one problem vertex.

Proof. We prove this by considering the different cases of G being an edge-balanced caterpillar or G being a vertex-balanced or almost vertex-balanced caterpillar.

Case I: G is an edge-balanced caterpillar. Thus, G has one center edge and $N(e_c)$ is odd.

Then there are $\dfrac{N(e_c)+1}{2} =: w$ vertices in A and w vertices in B. This means there are $w-a$ vertices on the spine of G in A and $w-b$ vertices on the spine of G in B. Note that there exists a vertex $u \in B$ such that $d(v_{c_a}, u) = w-b$. Also, the number of vertices on the spine of G is $w-a+w-b$ and thus $D = 2w-a-b-1$.

(i) D is odd. By hypothesis, $a \geq b+2$.

Consider

$$\begin{aligned} \frac{D+2}{2} &= \frac{2w-a-b+1}{2} \\ &\leq \frac{2w-(b+2)-b+1}{2} \\ &= \frac{2w-2b-1}{2} \\ &= w-b-\tfrac{1}{2} \\ &< w-b \end{aligned}.$$

Therefore, $d(v_{c_a}, u) = w-b > \frac{D+2}{2}$. So, by definition, u is a problem vertex.

(ii) D is even. By hypothesis, $a > b$.

Consider

$$\begin{aligned} \frac{D+2}{2} &= \frac{2w-a-b+1}{2} \\ &< \frac{2w-b-b+1}{2} \\ &= \frac{2w-2b+1}{2} \\ &= w-b+\tfrac{1}{2} \end{aligned}.$$

Thus, $\frac{D+2}{2} < w-b+\frac{1}{2}$. Since D is even, $\frac{D+2}{2}$ is an integer so $\frac{D+2}{2} \leq w-b$. Therefore, $d(v_{c_a}, u) = w-b \geq \frac{D+2}{2}$. So, u is a problem vertex.

Case II: G is a vertex-balanced or almost vertex-balanced caterpillar. Thus, G has two center edges e_{c_a} and e_{c_b}.

Since G has two center edges, there are $\frac{N(e_{c_a})}{2} = \frac{N(e_{c_b})}{2} =: w$ vertices in A and w vertices in B. Then there are $w-a$ vertices in A on the spine of G and $w-b$ vertices in B on the spine of G. Note that this means there exists a vertex $u \in B$ such that $d(v_c, u) = w-b$. Also, since v_c is on the spine, there are $w-a+w-b+1$ vertices on the spine of G. So $D = 2w-a-b$.

(i) D is odd. By hypothesis, $a \geq b+2$

Consider

$$\begin{aligned} \frac{D+2}{2} &= \frac{2w-a-b+2}{2} \\ &\leq \frac{2w-(b+2)-b+2}{2} \\ &= \frac{2w-2b}{2} \\ &= w-b \end{aligned}.$$

Therefore, $d(v_c, u) = w-b \geq \frac{D+2}{2}$. So, u is a problem vertex.

(ii) D is even. By hypothesis, $a > b$.

Consider

$$\begin{aligned}\frac{D+2}{2} &= \frac{2w-a-b+2}{2} \\ &< \frac{2w-b-b+2}{2} \\ &= \frac{2w-2b+2}{2} \\ &= w-b+1\end{aligned}.$$

Since D is even, $\frac{D+2}{2}$ is an integer. So, since $\frac{D+2}{2} < w-b+1$, it follows that $\frac{D+2}{2} \leq w-b$. Therefore, $d(v_c, u) = w-b \geq \frac{D+2}{2}$. So, by definition, u is a problem vertex. □

The previous proposition showed what conditions are needed for D, a, and b for an edge-balanced, vertex-balanced, or almost vertex-balanced caterpillar G to have a problem vertex. The following propositions show what values of D, a, and b are not possible for edge-balanced, vertex-balanced and almost vertex-balanced caterpillars. The combined results of Proposition 8 and the following propositions help us determine which types of caterpillars have an improved radio number due to a problem vertex and which caterpillars could potentially be labeled without jumps.

Proposition 9. *Let G be an edge-balanced caterpillar with n vertices.*

(i) *If D is odd, then $a \neq b+1$.*

(ii) *If D is even, then $a \neq b$.*

Proof. Recall that G has one center edge with $N(e_c)$ odd and thus by Proposition 5, n is even.

(i) D is odd. Suppose by contradiction that $a = b+1$. Since D is the diameter, there are $D+1$ vertices on the spine of G. Also, since D is odd, this means that there are an even number of vertices on the spine of G. Let $D+1 = 2y+2$ for some $y \in \mathbb{Z}$. Then

$$\begin{aligned} n &= 2y+2+a+b \\ &= 2y+2+b+1+b \\ &= 2y+2+2b+1, \end{aligned}$$

which is odd, a contradiction to Proposition 5. Therefore, when D is odd, $a \neq b+1$.

(ii) D is even. Suppose by contradiction that $a = b$. By Proposition 5, $|V(A)| = |V(B)|$. Let $w := \frac{N(e_c)+1}{2} = |V(A)| = |V(B)|$. There are $w-a$ vertices on the spine in A and $w-b$ vertices on the spine in B. So, there are $w-a+w-b$ vertices on the spine of G. Thus, $D = w-a+w-b-1 = 2w-2a-1$ which is odd, a contradiction to the assumption. Therefore, when D is even, $a \neq b$. □

Proposition 10. *Let G be a vertex-balanced or almost vertex-balanced caterpillar with n vertices. If D is odd, then $a \neq b$.*

Proof. Suppose by contradiction that D is odd and $a = b$. Let $w := \frac{N(e_{c_a})}{2} = \frac{N(e_{c_b})}{2}$. Then there are $w-a$ vertices on the spine that are in A and $w-b$ vertices on the spine in B. Since v_c is also on the spine, there are $w-a+w-b+1 = 2w-2a+1$ vertices on the spine of G. Thus, $D = 2w-2a$ which is even, contradicting the fact that D is odd. Therefore, $a \neq b$. □

4.3.2 Improved Bounds

We now use results from Section 4.3.1 to improve the lower bound for the radio number of certain edge-balanced, vertex-balanced, and almost vertex-balanced caterpillars.

Proposition 11. *Let G be an edge-balanced caterpillar with a problem vertex v_*. Then G is not a jumpless caterpillar.*

Proof. Without loss of generality, assume $v_* \in B$. Note that u_{c_b} cannot be a problem vertex so $u_{c_b} \neq v_*$.

Use Algorithm 1 to place the vertices of G into Table 4.1. From Proposition 6, the corresponding ordering $y_1, \ldots, y_n$ is distance maximizing. Since this is a distance maximizing ordering and $N(e_c)$ is odd, by Propositions 3 and 4, e_c is the only edge with $n_y(e)$ odd. By Remark 8, y_1 and y_n are both incident to e_c. Thus $\{u_{c_a}, u_{c_b}\} = \{y_1, y_n\}$. Since u_{c_b} is not a problem vertex, v_* is not the first or last labeled vertex. Thus, $v_* = y_i$ for some triple of vertices y_{i-1}, y_i, y_{i+1}.

By definition of being a problem vertex, $d(v_*, u_{c_a}) \geq \frac{D+2}{2}$. Also, by the structure of an edge-balanced caterpillar, $d(u_{c_a}, v_*) \leq d(u_{c_a}, u_s)$. Therefore,

$$d(u_{c_a}, u_s) \geq d(u_{c_a}, v_*) \geq \frac{D+2}{2} > \frac{D+1}{2}$$

$$\Rightarrow d(u_{c_a}, u_s) > \frac{D+1}{2}. \tag{4.3}$$

The ordering of vertices of G given by Algorithm 1 has $y_{n-1} = u_s$ and $y_n = u_{c_a}$. Thus, by Lemma **??**, $u_{c_a} = \alpha_{y_n}$. So, α_{y_n} is a vertex on the spine of G and by (4.3), $d(y_n, \alpha_{y_n}) > \frac{D+1}{2} = \frac{D+1}{2} + t_{\alpha_{y_n}}$. Since y_n and $y_{n-1} = \alpha_{y_n}$ are in horizontally adjacent cells in Group 2 of Table 4.1, this contradicts condition (1) of the definition of a jumpless caterpillar. Therefore, G is not a jumpless caterpillar. ∎

Corollary 2. *Let G be an edge-balanced caterpillar with n vertices. Suppose G is such that either*

(i) *D is odd and $a \geq b+2$ or*

(ii) *D is even and $a > b$.*

Then

$$rn(G) \geq (n-1)(D+1) + f(x_1) - \max_p(\sum_{i=1}^{n-1} d(x_i, x_{i+1})) + 1$$

where the maximum is taken over all possible bijections p from the vertices of G to the set $\{x_1, \ldots, x_n\}$.

Proof. From Proposition 8, G has a problem vertex. So, by Proposition 11, G is not a jumpless caterpillar. Therefore, the bound follows from Theorem 11. □

We also determine an improved lower bound for some vertex-balanced and almost vertex-balanced caterpillars. The following lemmas are used to find this improved lower bound in Theorem 13.

Lemma 4. *Let G be a vertex-balanced or almost vertex-balanced caterpillar. Let $x_1, \ldots, x_n$ be a distance maximizing ordering of the vertices of G with associated radio labeling f. Then v_c has to be x_1 or x_n.*

Proof. Let $\max_{e \in E(G)} N(e) = M$. By definition of center edges, $N(e_{c_a}) = M = N(e_{c_b})$. Note that since $x_1, \ldots, x_n$ is a distance maximizing ordering, by Propositions 3 and 4 $n_x(e_*) = N(e_*) - 1$.

If $e_* \in \{e_{c_a}, e_{c_b}\}$, then we are done.

If $e_* \notin \{e_{c_a}, e_{c_b}\}$, then we consider the following cases:

Case I: G is vertex-balanced. So G has no legs incident to v_c.

Since $e_* \notin \{e_{c_a}, e_{c_b}\}$ and there are no legs incident to v_c, e_* is not incident to v_c. Thus, $n_x(e_{c_a}) = M = n_x(e_{c_b})$. This means that all vertices in A are endpoints of two paths P_j from x_j to x_{j+1} and all vertices in B are endpoints of two paths P_j. Thus none of the vertices in A or B can be the first or last labeled, giving a contradiction.

Case II: G is almost vertex-balanced. So G has legs incident to v_c.

Since $e_* \notin \{e_{c_a}, e_{c_b}\}$, $n_x(e_{c_a}) = M = n_x(e_{c_b})$. This means that all vertices in A are endpoints of two paths P_j from x_j to x_{j+1} and all vertices in B are endpoints of two paths P_j. Therefore, e_* is not in A or in B. Thus, e_* is one of the edges incident to both v_c and a leg vertex. Therefore, v_c is incident to e_*. □

Lemma 5. *Let G be a vertex-balanced or almost vertex-balanced caterpillar. Let $x_1, \ldots, x_n$ be a distance maximizing ordering of the vertices of G with associated radio labeling f. If there is a problem vertex v_*, then v_* is not x_1 or x_n.*

Proof. By Lemma 4, $v_c \in \{x_1, x_n\}$. Since $D > 0$, $d(v_c, v_*) > 1$. Since $x_1, \ldots, x_n$ is a distance maximizing ordering, by Propositions 3 and 4 there is only one edge which has an odd $n_x(e)$ value. This is the edge e_*. Since e_* is incident to v_c, and v_c and v_* are not adjacent, it follows that $v_* \notin \{x_1, x_n\}$. □

Theorem 13. *Let G be a vertex-balanced or almost vertex-balanced caterpillar with n vertices. Let $x_1, \ldots, x_n$ be an arbitrary ordering of the vertices of G with associated radio labeling f. If G has a problem vertex v_*, then*

$$rn(G) \geq (n-1)(D+1) + f(x_1) - \max_p(\sum_{i=1}^{n-1} d(x_i, x_{i+1})) + 1$$

where the maximum is taken over all possible bijections p from the vertices of G to the set $\{x_1, \ldots, x_n\}$.

Proof. First, we consider when $x_1, \ldots, x_n$ is a distance maximizing ordering. For this proof, assume $v_* \in B$ (the proof of if $v_* \in A$ is analogous). By Lemma 4, v_c is incident to e_*. Also, by Lemma 5, v_* is not the first or last labeled vertex. Thus, $v_* = x_i$ for some triple of vertices x_{i-1}, x_i, x_{i+1}. Since $x_1, \ldots, x_n$ is distance maximizing and $v_* \in B$, x_{i-1} and x_{i+1} associated with $v_* = x_i$ are not in B. Let $\{\alpha, \beta\}$ be $\{x_{i-1}, x_{i+1}\}$ with $v_* = x_i$ where $d(v_*, \alpha) \leq d(v_*, \beta)$. By the structure of G and the definition of v_* being a problem vertex,

$$d(v_*, \alpha) \geq d(v_*, v_c) \geq \frac{D+2}{2}, \tag{4.4}$$

where the first inequality is an equality only when $v_c = \alpha$. Notice that

$$d(\alpha, \beta) = \begin{cases} d(v_*, \beta) - d(v_*, \alpha) & \text{if } \alpha \text{ is on the spine of } G \\ d(v_*, \beta) - d(v_*, \alpha) + 2 & \text{if } \alpha \text{ is a leg vertex.} \end{cases}$$

Let $J_f(v_*, \alpha)$ and $J_f(v_*, \beta)$ be non-negative integers such that

$$d(v_*, \alpha) + |f(v_*) - f(\alpha)| = D + 1 + J_f(v_*, \alpha) \text{ and}$$

$$d(v_*, \beta) + |f(v_*) - f(\beta)| = D + 1 + J_f(v_*, \beta).$$

Consider the case when $f(\alpha) < f(v_*) < f(\beta)$. (The other case is proven similarly.) The radio condition applied to vertices α and β gives the following:

$$d(\alpha,\beta)+f(\beta)-f(\alpha)\geq D+1$$

$$\Rightarrow d(\alpha,\beta)+f(\beta)-f(v_*)+f(v_*)-f(\alpha)\geq D+1$$

$$\Rightarrow d(\alpha,\beta)+D+1+J_f(v_*,\beta)-d(v_*,\beta)+f(v_*)-f(\alpha)\geq D+1$$

$$\Rightarrow d(\alpha,\beta)+D+1+J_f(v_*,\beta)-d(v_*,\beta)+D+1+J_f(v_*,\alpha)-d(v_*,\alpha)+f(\alpha)-f(\alpha)\geq D+1$$

$$\Rightarrow D+1+J_f(v_*,\beta)+J_f(v_*,\alpha)\geq d(v_*,\beta)+d(v_*,\alpha)-d(\alpha,\beta) \tag{4.5}$$

Case I: Suppose α is on the spine of G.

Then $d(v_*,\beta)+d(v_*,\alpha)-d(\alpha,\beta)=2d(v_*,\alpha)$.

Then, (4.5) becomes

$$D+1+J_f(v_*,\beta)+J_f(v_*,\alpha)\geq 2d(v_*,\alpha)$$

$$\Rightarrow \frac{D+1+J_f(v_*,\beta)+J_f(v_*,\alpha)}{2}\geq d(v_*,\alpha) \tag{4.6}$$

By (4.4), $d(v_*,\alpha)\geq \frac{D+2}{2}$. This and (4.6) give the following:

$$\frac{D+1+J_f(v_*,\beta)+J_f(v_*,\alpha)}{2}\geq d(v_*,\alpha)\geq \frac{D+2}{2}$$
$$\Rightarrow \frac{D+1+J_f(v_*,\beta)+J_f(v_*,\alpha)}{2}\geq \frac{D+2}{2}$$
$$\Rightarrow J_f(v_*,\beta)+J_f(v_*,\alpha_j)\geq 1.$$

Case II: Suppose α is not on the spine of G.

Then $d(v_*, \beta) + d(v_*, \alpha) - d(\alpha, \beta) = 2d(v_*, \alpha) - 2$.

Then, (4.5) becomes

$$D + 1 + J_f(v_*, \beta) + J_f(v_*, \alpha) \geq 2d(v_*, \alpha) - 2$$

$$\Rightarrow \frac{D + 3 + J_f(v_*, \beta) + J_f(v_*, \alpha)}{2} \geq d(v_*, \alpha) \quad (4.7)$$

By (4.4) and since $d(v_c, \alpha) \geq 1$, $d(v_*, \alpha) \geq \frac{D+2}{2} + 1$. This and (4.7) give the following:

$$\frac{D + 3 + J_f(v_*, \beta) + J_f(v_*, \alpha)}{2} \geq d(v_*, \alpha) \geq \frac{D + 2}{2} + 1$$
$$\Rightarrow \frac{D + 3 + J_f(v_*, \beta) + J_f(v_*, \alpha)}{2} \geq \frac{D + 2}{2} + 1$$
$$\Rightarrow J_f(v_*, \beta) + J_f(v_*, \alpha) \geq 1.$$

Therefore, in both cases, $J_f(v_*, \beta) + J_f(v_*, \alpha) \geq 1$. Using this result along with Propositions 1 and 2, we get

$$rn(G) \geq (n - 1)(D + 1) + f(x_1) - \max_p(\textstyle\sum_{i=1}^{n-1} d(x_i, x_{i+1})) + 1.$$

Now consider when $x_1, \ldots, x_n$ is not distance maximizing. The proof is now the same as the proof of Theorem 11 when considering an ordering that is not distance maximizing. □

Corollary 3. *Let G be a vertex-balanced or almost vertex-balanced caterpillar. If G is such that either* *(i)* D *is odd and* $a \geq b + 2$ *or*

(i) D *is even and* $a > b$,

then

$$rn(G) \geq (n-1)(D+1) + f(v_1) - \max_p(\sum_{i=1}^{n-1} d(x_i, x_{i+1})) + 1$$

where the maximum is taken over all possible bijections p *from the vertices of* G *to the set* $\{x_1, \dots x_n\}$.

Proof. This follows from Proposition 8 and Theorem 13. □

4.3.3 Caterpillars with one center edge such that $N(e_c)$ is even

The remaining type of caterpillar in terms of center edges that have not yet been discussed in this thesis are caterpillars with one center edge such that $N(e_c)$ is even. Determining the radio number and bounds for the radio number of these caterpillars is slightly more complicated than the other types of caterpillars discussed previously. The following lemma helps provide a case when an improved bound for the radio number of this type of caterpillar can be found.

Lemma 6. *Let* G *be a tree with one center edge and* $N(e_c)$ *is even. Suppose component* A *has more vertices than component* B. *Let* $x_1, \dots, x_n$ *be a distance maximizing ordering of the vertices of* G *with* f *the associated radio labeling. Then* e_* *is either* e_c *or in component* A.

Proof. If $e_* = e_c$, we are done.

If $e_* \neq e_c$, then $n_x(e_c) = N(e_c)$ which is even. Since the maximum $n(e_c)$ value is achieved and B is the smaller component, all vertices in B are endpoints to two paths P_j. Thus, none of the vertices in B are x_1 or x_n. Therefore, $e_* \notin B$ and since $e_* \neq e_c$ by assumption, it follows that $e_* \in A$. □

Theorem 14. *Let G be a caterpillar with n vertices with one center edge and $N(e_c)$ is even. Let component A have more vertices than component B. Let $x_1, \ldots, x_n$ be an arbitrary ordering of the vertices of G with f the associated radio labeling. If G has a problem vertex $v_* \in B$, then*

$$rn(G) \geq (n-1)(D+1) + f(x_1) - \max_p(\textstyle\sum_{i=1}^{n-1} d(x_i, x_{i+1})) + 1$$

Proof. First, suppose $x_1, \ldots, x_n$ is a distance maximizing ordering. Since component A has more vertices than component B and $x_1, \ldots, x_n$ is a distance maximizing ordering, by Lemma 6, e_* is either e_c or in component A. This means that v_* is not x_1 or x_n. Thus, v_* is x_i in a triple of vertices x_{i-1}, x_i, x_{i+1}. Since $x_1, \ldots, x_n$ is distance maximizing,

$$n_x(e_c) = \begin{cases} N(e_c) & \text{if } e_* \neq e_c \\ N(e_c) - 1 & \text{if } e_* = e_c. \end{cases}$$

Note that in either case, v_{c_b} cannot be the vertex v_*. To ensure that this $n_x(e_c)$ value is achieved, every path with an endpoint in B must include the edge e_c. Thus, $x_{i-1}, x_{i+1} \in A$. Let $\{\alpha, \beta\}$ be $\{x_{i-1}, x_{i+1}\}$ associated with $v_* = x_i$ with α the vertex such that $d(\alpha, v_*) \leq d(\beta, v_*)$.

By the structure of G and the definition of v_* being a problem vertex,

$$d(v_*, \alpha) \geq d(v_*, v_{c_a}) \geq \frac{D+2}{2}, \tag{4.8}$$

where the first inequality is an equality only when $v_{c_a} = \alpha$.

The proof now follows the same as the proof of Theorem 13 with v_{c_a} replacing v_c when applicable. □

4.3.4 Some Conclusions about Caterpillars

Theorem 14 in Section 4.3.3 gives an improved lower bound for a particular type of caterpillar with one center edge such that $N(e_c)$ is even, but the exact radio number has not yet been determined. However, some more specific conclusions can be made about edge-balanced, vertex-balanced, and almost vertex-balanced caterpillars from Section 4.3.2

Corollaries 2 and 3 establish a way to determine when the bound for the radio number given by Proposition 2 is increased for an edge-balanced, vertex-balanced, or almost vertex-balanced caterpillar G based on the structure of G.

The results of Corollary 2 and Proposition 9 indicate that edge-balanced caterpillars with the potential to have radio labelings that require no jumps are such that D is odd and $a = b$.

When there is exactly one leg adjacent to each vertex on the spine expect for u_1 and u_s, this is a thorn graph. The radio number of this particular thorn graph has been determined in [10].

In other cases when D is odd and $a = b$, one can enter the vertices of G into Table 4.1 using Algorithm 1 to determine if G is a jumpless caterpillar. If it is, then G can be labeled without jumps and the label ordering is given by Algorithm 1.

Similarly, the results of Corollary 3 and Proposition 10 indicate that vertex-balanced and almost vertex-balanced caterpillars with the potential to have radio labelings that require no jumps are such that either D is odd and $a = b + 1$ or D is even and $a = b$.

There are a couple of these types of caterpillars whose radio number has already been found. When G has exactly one leg adjacent to each vertex on the spine (including v_c) except for v_1 and v_s, the radio number of G has been found in [10]. When G is a complete binary tree of height two, the radio number has been found in [7].

APPENDIX A
LABELINGS OF GRAPHS OF ORDER $N = 2K$ WITH $K < 7$ AND DIAMETER $N - 2$

The figures below give upper bounds for the radio number of spire graphs with $k < 7$ and $n = 2k$ since these particular cases were not covered in Theorem 3. These upper bounds match the lower bounds for these graphs found in Theorem 4 to show that these bounds are the actual radio number of the graphs.

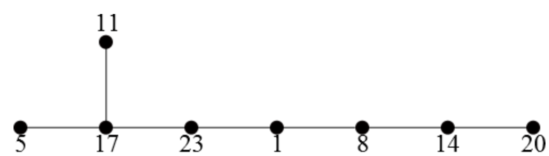

Figure A.1: $rn(S_{8,2}) \leq 23$

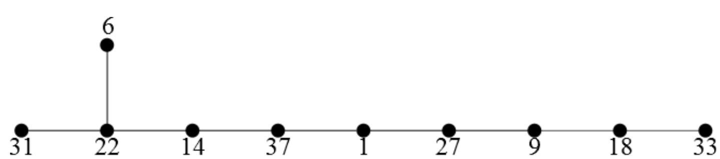

Figure A.2: $rn(S_{10,2}) \leq 37$

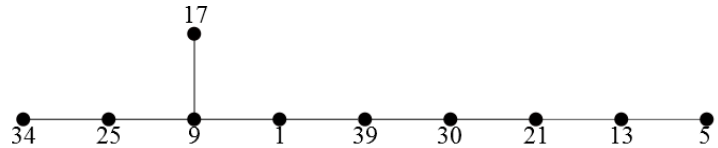

Figure A.3: $rn(S_{10,3}) \leq 39$

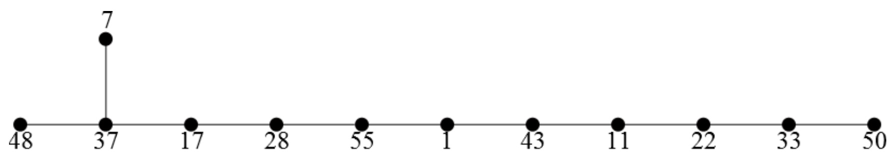

Figure A.4: $rn(S_{12,2}) \leq 55$

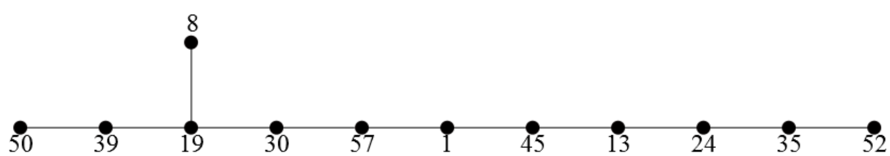

Figure A.5: $rn(S_{12,3}) \leq 57$

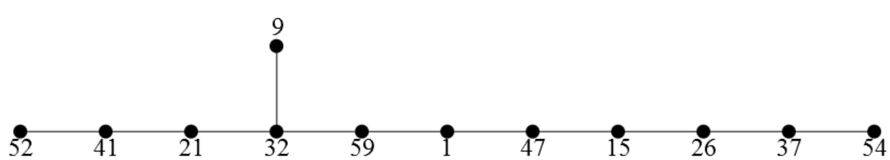

Figure A.6: $rn(S_{12,4}) \leq 59$

APPENDIX B
EDGE-BALANCED CATERPILLARS WITH $N < 8$

Theorem 11 in Chapter 4 improved the lower bound for the radio number of edge-balanced caterpillars that are not jumpless caterpillars. In some cases, the proof assumed that $n \geq 8$. Recall that for an edge-balanced caterpillar, n is even. Thus, we only need to check for edge-balanced caterpillar graphs for $n = 2, 4$, and 6. The following graphs in Figure B.1 show all the edge-balanced caterpillars such that $n < 8$. Most of these are jumpless caterpillars and thus would not be considered in Theorem 11. In all the cases shown below, whether the caterpillar is jumpless or not, the radio number of these graphs is known either from previous results or from work in this thesis.

The graph (a) in Figure B.1 is the path P_2. This is a complete graph whose radio number is known: $rn(P_2) = 2$. The graphs (b) and (c) are paths P_4 and P_6. The radio numbers for these paths were determined in [9]: $rn(P_4) = 6$ and $rn(P_6) = 14$. The graphs (d) and (e) are spire graphs, $S_{6,2}$ and $S_{6,4}$. The radio number of $S_{6,2}$ was determined in Chapter 3: $rn(S_{6,2}) = 12$. The spire $S_{6,4}$ can be redrawn as $S_{6,2}$. Thus, $rn(S_{6,4}) = 12$. Finally, it can be checked that the graph (f) of Figure B.1 is a jumpless caterpillar. Thus, using Algorithm 1 from Chapter 4, $rn(G) = 8$.

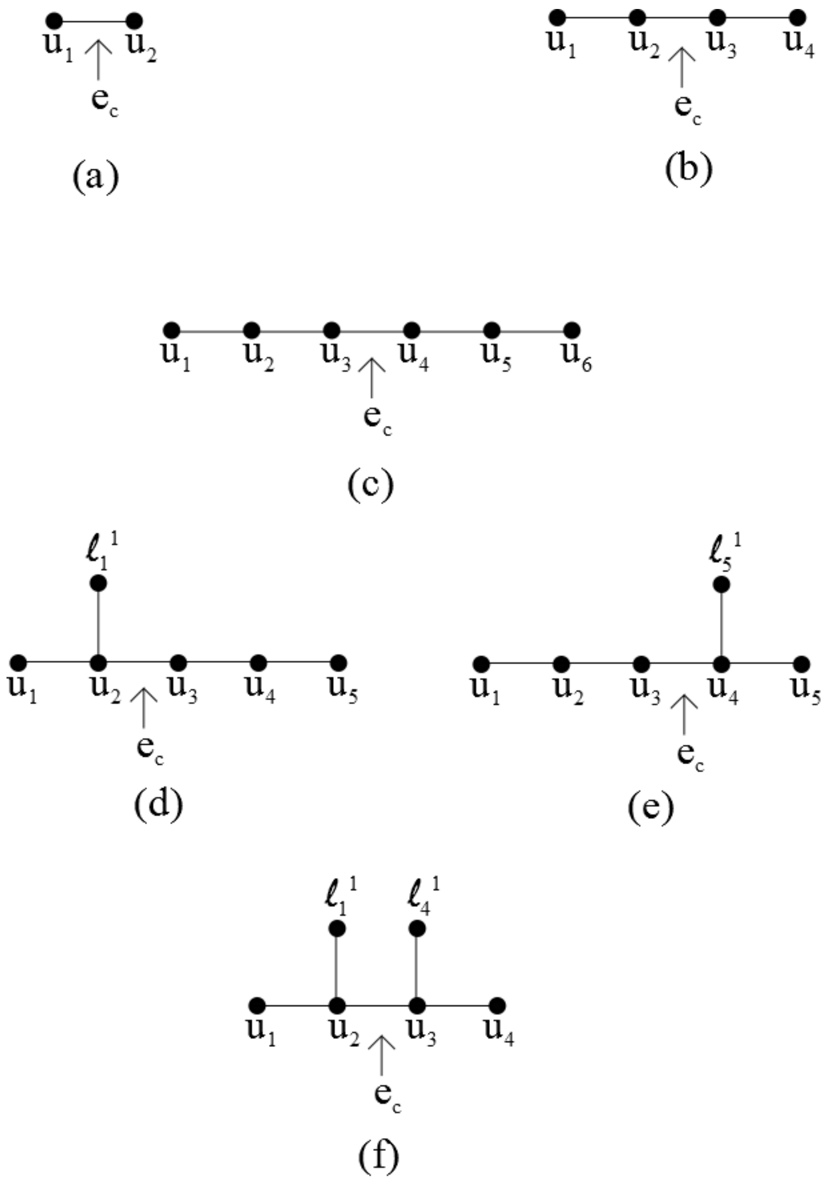

Figure B.1: Edge-Balanced Caterpillars with $n < 8$.

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