Summer 2013

Reduced [tau]_n-factorizations in Z and [tau]_n-factorizations in N

Alina Anca Florescu
University of Iowa

REDUCED τ_n-FACTORIZATIONS IN $\mathbb{Z}$ AND τ_n-FACTORIZATIONS IN $\mathbb{N}$

by

Alina Anca Florescu

A thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

August 2013

Thesis Supervisor: Professor Dan Anderson

Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Alina Anca Florescu

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the August 2013 graduation.

Thesis committee:

Dan Anderson, Thesis Supervisor

Victor Camillo

Oguz Durumeric

Surjit S. Khurana

Maggy Tomova

ACKNOWLEDGEMENTS

I owe a tremendous debt of gratitude to my advisor, Professor Dan Anderson, for his never-ending patience and wisdom. I am grateful to Suzanne Hamon for her work on τ_n-factorizations. I would like to thank my parents and John G. Altidor, my partner in life for the past eight years, for their love and support. They have been with me every step of the way despite the physical distance. I am also thankful to my friends at the University of Iowa and beyond for their encouragement.

ABSTRACT

In this dissertation we expand on the study of τ_n-factorizations or generalized integer factorizations introduced by D.D. Anderson and A. Frazier and examined by S. Hamon. Fixing a non-negative integer n, a τ_n-factorization of a nonzero nonunit integer a is a factorization of the form $a = \lambda \cdot a_1 \cdot a_2 \cdots a_t$ where $t \geq 1$, $\lambda = 1$ or -1 and the nonunit nonzero integers $a_1, a_2, \ldots, a_t$ satisfy $a_1 \equiv a_2 \equiv \ldots \equiv a_t \mod n$. The τ_n-factorizations of the form $a = a_1 \cdot a_2 \cdots a_t$ (that is, without a leading -1) are called reduced τ_n-factorizations. While similarities exist between the τ_n-factorizations and the reduced τ_n-factorizations, the study of one type of factorization does not elucidate the other. This work serves to compare the τ_n-factorizations of the integers with the reduced τ_n-factorizations in $\mathbb{Z}$ and the τ_n-factorizations in $\mathbb{N}$.

One of the main goals is to explore how the Fundamental Theorem of Arithmetic extends to these generalized factorizations. Results regarding the τ_n-factorizations in $\mathbb{Z}$ have been discussed by S. Hamon. Using different methods based on group theory we explore similar results about the reduced τ_n-factorizations in $\mathbb{Z}$ and the τ_n-factorizations in $\mathbb{N}$. In other words, we identify the few values of n for which every integer can be expressed as a product of the irreducible elements related to these factorizations and indicate when one can do so uniquely.

Using our approach the τ_n-factorizations in $\mathbb{N}$ are shown to be the easiest to describe. In $\mathbb{Z}$ the τ_n-factorizations pose less of a challenge than the reduced τ_n-factorizations.

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER 1
INTRODUCTION

The topic of τ_n-factorizations in $\mathbb{Z}$ was first introduced by D.D. Anderson and A. Frazier in [2] as a concrete example of a more general theory of factorization in integral domains. S. Hamon then provided a detailed presentation of this subject in [3]. In this dissertation we continue the study of τ_n-factorizations in order to provide a more complete picture of generalized integer factorizations. Below we relate some of the more general theory that led to this discussion.

1.1 Definitions and Examples of τ-Relations

Let D be an integral domain and $D^\#$ the set of nonzero nonunits of D. For $a, b \in D$, a *divides* b, written $a|b$, if there exists $c \in D$ satisfying $b = ac$. Elements a and b of $D^\#$ are *associates* if there exists a unit u in D so that $a = ub$. This is denoted by $a \sim b$ and is equivalent to the statement $a|b$ and $b|a$. As usual, $a \in D^\#$ is called *irreducible* or an *atom* if $a = bc$ implies that b or c is a unit. An element $a \in D^\#$ is said to be *prime* if $a|bc$ implies that $a|b$ or $a|c$.

Let τ be a relation on $D^\#$, i.e., $\tau \subseteq D^\# \times D^\#$. By convention, $a\tau b$ stands for $(a, b) \in \tau$. Throughout it is assumed that τ is symmetric, thus for $a, b \in D^\#$, $a\tau b$ implies $b\tau a$. The relation τ is called *multiplicative* if for $a, b, c \in D^\#$, $a\tau b$ and $a\tau c$ imply $a\tau bc$. If for all $a, a_1, b, b_1 \in D^\#$ with $a_1|a$ and $b_1|b$ $a\tau b$ implies $a_1\tau b_1$, then τ is said to be *divisive.* Define τ to be *associate-preserving* if for $a, b, b' \in D^\#$ with $b \sim b'$, $a\tau b$ implies $a\tau b'$. Note that a divisive relation is associate-preserving.

The following are some examples of τ-relations described in [2].

1. Let $\star$ be a star-operation on D, i.e., a closure operation on the set of fractional ideals of D satisfying $D^\star = D$ and $(aE)^\star = aE^\star$ for every nonzero a in the field of fractions of D and E a fractional ideal. Define $a\tau_\star b$ if and only if $(a,b)^\star = D$. In this case a and b are called $\star$-*comaximal*. One can check that $\tau_\star$ is multiplicative and divisive, and so associate-preserving as well.

2. Define $a\tau b$ when $\gcd(a,b) = 1$. This relation is divisive, and therefore associate-preserving. It is multiplicative if and only if the product of two primitive polynomials in $D[x]$ is primitive.

3. If J is an ideal of D, let $a\tau_J b$ whenever $a - b \in J$. For $D = \mathbb{Z}$ and $J = (n)$, this relation is denoted τ_n. Here, $a\tau_n b$ if and only if $a \equiv b \mod n$. In the case of $n = 2$, $a\tau_2 b$ if and only if the two elements have the same parity. It is easily verified that τ_2 is multiplicative and associate-preserving, although not divisive. For $n \geq 3$, τ_n is neither multiplicative, nor associate-preserving, thus also not divisive. To see why, choose $a \equiv -1 \mod n$. If τ_n were multiplicative for $n \geq 3$ then $a\tau_n a^2$, which is impossible. If τ_n were associate-preserving, then $a\tau_n(-a)$, a contradiction.

A *τ-factorization of* $a \in D^\#$ is a factorization $a = \lambda a_1 a_2 \cdots a_t$ where λ is a unit, $a_i \in D^\#$ and $a_i \tau a_j$ for $i \neq j$. The length of the factorization is t, the number of factors a_i in $D^\#$. The factor a_i is said to *τ-divide* a, denoted $a_i|_\tau a$.

A *trivial τ-factorization of* $a \in D^\#$ is a τ-factorization of length 1. Every element of $D^\#$ has a trivial τ-factorization. Whenever a admits only trivial τ-factorizations, a is called a *τ-atom* or *τ-irreducible.* The τ-factorization $a = \lambda a_1 a_2 \cdots a_t$ is said to be *τ-atomic* when every a_i is a τ-atom. Elements of $D^\#$ may have multiple τ-factorizations, but if every element has a τ-atomic factorization, the domain D is called *τ-atomic.* A *τ-unique factorization domain* (τ-UFD) is a τ-atomic domain D satisfying the condition that if $\lambda a_1 a_2 \cdots a_t = \mu b_1 b_2 \cdots b_m$ are τ-atomic factorizations, then (a) $t = m$ and (b) after reordering $a_i \sim b_i$ for each i. In [2] it is shown that if τ is divisive, a UFD is a τ-UFD.

Two notions of prime exist with τ-factorizations. First, $a \in D^\#$ is *τ-prime* if whenever $a | \mu b_1 b_2 \cdots b_m$, a τ-factorization, then $a | b_i$ for some i. Second, a is called a *τ-divide prime* (written $|_\tau$-prime) if whenever $a |_\tau \mu b_1 b_2 \cdots b_m$ then $a |_\tau b_i$ for some i. In other words, a is $|_\tau$-prime if $\mu b_1 b_2 \cdots b_m = \lambda a a_1 \cdots a_t$ implies that $b_i = \nu a c_1 \cdots c_k$ for some i, where the factorizations are τ-factorizations.

A few important observations must be made concerning τ-atoms and τ-primes. First, if $a \in D^\#$ is irreducible, respectively prime, then it is τ-irreducible, respectively τ-prime. Second, τ-primes and $|_\tau$-primes are τ-irreducible, but the converse need not hold. It is shown in [2] that when τ is both multiplicative and divisive, τ-prime implies $|_\tau$-prime. Last, associates of τ-atoms are also τ-atoms.

A *reduced τ-factorization* (denoted *${}_r\tau$-factorization*) of $a \in D^\#$ is a τ-factorization of a where $\lambda = 1$. With this definition in mind, one can extend the notions of τ-divides, τ-atom, τ-atomic factorization, τ-atomic domain, τ-UFD, τ-prime, and $|_\tau$-prime to *reduced τ-divides*, etc. Thus, if $a = a_1 a_2 \cdots a_t$ is a ${}_r\tau$-factorization, a_i is said to *reduced τ-divide* a (written *${}_r\tau$-divides*). An element a of $D^\#$ is called a *reduced*

τ*-atom* (denoted ${}_r\tau$*-atom*) if its only ${}_r\tau$-factorization is $a = a$. A *reduced* τ*-atomic factorization* (${}_r\tau$*-atomic factorization*) of $a \in D^\#$ is a ${}_r\tau$-factorization of a where each nonzero nonunit factor is a ${}_r\tau$-atom. The domain D is *reduced* τ*-atomic* (${}_r\tau$*-atomic*) if every nonzero nonunit of D has a ${}_r\tau$-atomic factorization. If, in addition, $a_1a_2\cdots a_t = b_1b_2\cdots b_m$, where the two factorizations are ${}_r\tau$-factorizations, implies that (a) $t = m$ and (b) after reordering $a_i \sim b_i$ for each i, then D is a *reduced* τ*-UFD* (written ${}_r\tau$*-UFD*). If $a|a_1a_2\cdots a_t$, where $a_1a_2\cdots a_t$ is a ${}_r\tau$-factorization, implies that $a|a_i$ for some i, a is called a *reduced* τ*-prime* (written ${}_r\tau$*-prime*). Finally, $a \in D^\#$ is a *reduced* τ*-divide prime* (denoted $|_{{}_r\tau}$*-prime*) if $a|_{{}_r\tau}b_1b_2\cdots b_m$ implies that $a|_{{}_r\tau}b_i$ for some i. This means that $a \in D^\#$ is a $|_{{}_r\tau}$-prime if $b_1b_2\cdots b_m = aa_1\cdots a_t$ implies $b_i = ac_1\cdots c_k$ for some i. Here, the three factorizations are all ${}_r\tau$-factorizations.

1.2 Overview

In Chapter 2 we outline the results related to the τ_n-factorizations in $\mathbb{Z}$ found in [3] and make a small correction to the theorem that describes when $\mathbb{Z}$ is τ_n-atomic. We show that $\mathbb{Z}$ is not τ_{12}-atomic, although it was previously thought to be. This chapter is the foundation upon which we develop the topic of reduced τ_n-factorizations in $\mathbb{Z}$ and later the τ_n-factorizations in $\mathbb{N}$ and serves as the comparison standard that we consistently refer to in Chapters 3 and 4.

Chapter 3 offers an in-depth discussion of the reduced τ_n-factorizations in $\mathbb{Z}$. The reduced τ_n-primes and reduced τ_n-divide primes classified by Hamon in [3] are reviewed. A detailed analysis of the similarities and differences between the τ_n-

factorizations and reduced τ_n-factorizations follows. Finally, using a group theoretic approach we examine when $\mathbb{Z}$ is reduced τ_n-atomic.

Using the methods from Chapter 3 similar results are produced about the τ_n-factorizations in $\mathbb{N}$ in Chapter 4. The absence of negative signs makes these factorizations less involved. The τ_n-factorizations in $\mathbb{Z}$ are more complex than the τ_n-factorization in $\mathbb{N}$ but more easily accessible than the reduced τ_n-factorizations in $\mathbb{Z}$.

CHAPTER 2
THE τ_n-FACTORIZATIONS IN $\mathbb{Z}$

In this chapter we provide an overview of the main results on τ_n-factorizations and a summary of the classification of τ_n-primes, τ_n-divide primes and τ_n-atoms. We also amend the theorem on when $\mathbb{Z}$ is τ_n-atomic and illustrate how $\mathbb{Z}$ can fail to be τ_n-atomic.

2.1 When Is $\mathbb{Z}$ τ_n-Atomic?

The first two theorems state the main results in [3].

Theorem 2.1 (Hamon [3]). *$\mathbb{Z}$ is a τ_n-UFD if and only if $n = 0, 1$.*

Theorem 2.2 (Hamon [3]). *$\mathbb{Z}$ is τ_n-atomic if and only if $n = 0, \ldots, 6$, 8, 10, 12.*

For the case $n = 12$, $\mathbb{Z}$ was erroneously thought to be τ_n-atomic. We prove this result below.

Lemma 2.3. *$\mathbb{Z}$ is not τ_{12}-atomic.*

Proof. Consider the τ_{12}-factorization $432 = 12 \cdot 36$. Both 12 and 36 are congruent to 0 mod 12, so this is indeed a τ_{12}-factorization. The second factor is not a τ_{12}-atom since $36 = 6 \cdot 6$ is a nontrivial τ_{12}-factorization of 36. Therefore $432 = 12 \cdot 36$ is not a τ_{12}-atomic factorization. Up to sign and order of factors we will show that this is the only τ_{12}-factorization of 432. (This means that $432 = (-12) \cdot (-36)$, $432 = (-1) \cdot (-12) \cdot 36$ and $432 = (-1) \cdot 12 \cdot (-36)$ are also τ_{12}-factorizations of 432.) Thus 432 has no τ_{12}-atomic factorization.

Let $432 = (\pm 1)a_1a_2\cdots a_t$ be a τ_{12}-factorization of 432. Recall that a τ_{12}-factorization of 432 requires that all factors be in the same equivalence class modulo 12. As a factor of $432 = 2^4 \cdot 3^3$, a_i is of the form $\pm 2^j 3^k$ where $0 \le j \le 4$, $0 \le k \le 3$ and $j + k \ge 1$. Therefore the equivalence class of the factors a_i can only be one of $\bar{0}$, $\bar{2}$, $\bar{3}$, $\bar{4}$, $\bar{6}$, $\bar{8}$, $\bar{9}$ or $\overline{10}$ modulo 12. Observe that $-8 \equiv 4 \mod 12$ and $-9 \equiv 3 \mod 12$. Thus if $a_i \equiv 8 \mod 12$ for all i, then $-a_i \equiv 4 \mod 12$ and if $432 = a_1a_2\cdots a_t$ is a τ_{12}-factorization, then so is $432 = (-1)^t \cdot (-a_1)(-a_2)\cdots(-a_t)$. Therefore we need only consider the equivalence classes $\bar{0}$, $\bar{2}$, $\bar{3}$, $\bar{4}$, and $\bar{6}$ modulo 12.

If each $a_i \equiv 0 \mod 12$, then every factor is a multiple of 12 and it is easy to see that up to sign and order of factors, the τ_{12}-factorization is the original one, $432 = 12 \cdot 36$.

Since 3 is a prime, it divides a_i for some i. Multiples of 3 only fall in one of $\bar{0}$, $\bar{3}$, $\bar{6}$ (or $\bar{9}$, which we no longer need to consider). Thus a_i cannot be congruent to 2 or 4 modulo 12.

Similarly, multiples of 2 do not fall in $\bar{3}$.

This leaves the τ_{12}-factorizations of 432 where $a_i \equiv 6 \mod 12$ for all i. Since there are 3 factors of 3 in the prime decomposition of $432 = 2^4 \cdot 3^3$, the length of a non-trivial τ_{12}-factorization is at most 3, i.e., $432 = a_1 \cdot a_2$ or $432 = a_1 \cdot a_2 \cdot a_3$.

There is no τ_{12}-factorization of length 2 where $a_i \equiv 6 \mod 12$. To have $a_i \equiv 6 \mod 12$ it is necessary that a_i is a multiple of 3, but not a multiple of 4 for every i (otherwise a_i is divisible by 12 and is thus $0 \mod 12$). However, since $2^4 \mid 432$ one of a_1 or a_2 has to be divisible by 4.

Finally, there cannot exist a τ_{12}-factorization of length 3 where $a_i \equiv 6 \mod 12$ since one of a_1, a_2 or a_3 must be divisible by 4.

Thus we have the following theorem.

Theorem 2.4. *$\mathbb{Z}$ is τ_n-atomic if and only if $n = 0, \ldots, 6, 8, 10$.*

2.2 Classifying the τ_n-atoms, τ_n-primes and $|_{\tau_n}$-primes

In proving that $\mathbb{Z}$ is τ_n-atomic for $n = 0-6$, Hamon [3] completely classifies the τ_n-atoms, -primes and $|_{\tau_n}$-primes for these cases. Table 2.1 summarizes these results. To distinguish the primes of $\mathbb{Z}$ from the τ_n-primes and $|_{\tau_n}$-primes, Hamon calls the primes of $\mathbb{Z}$ *standard primes* (abbreviated *std. primes*). The case $n = 1$ is omitted from the table because the τ_1-factorization is the usual factorization of $\mathbb{Z}$ with the τ_1-atoms, τ_1-primes and $|_{\tau_1}$-primes being synonymous to the standard primes. The τ_0 relation is defined by $a\tau_0 b$ if and only if $a = b$.

The study of τ_n-factorizations for the values of n in Table 2.1 uncovers some results that Hamon generalizes to any n.

Lemma 2.5 (Hamon [3]). *If a is a τ_n-atom where $a \not\equiv \pm 1 \mod n$, then $ap_1 \cdots p_t$ is also a τ_n-atom where p_i are not necessarily distinct standard primes satisfying $p_i \equiv \pm 1 \mod n$.*

Lemma 2.6 (Hamon [3]). *If p and q are standard primes such that $p \not\equiv \pm q \mod n$, then $\pm pq$ is a τ_n-atom.*

Combining the two results yields another class of τ_n-atoms.

n	τ_n-atoms	τ_n-primes	$\vert_{\tau_n}$-primes
0	$\pm p_1^{s_1} \cdots p_t^{s_t}$, p_i distinct std. primes, $s_i \geq 1$, $\gcd(s_1, \ldots, s_t) = 1$	$\pm p_1 \cdots p_t$, p_i distinct std. primes	$\pm p_1 \cdots p_t$, p_i distinct std. primes
2	$\pm p$, p std. prime	$\pm p$, p std. prime	$\pm p$, p odd std. prime
	$2m$, m odd	$\pm 2p$, p odd std. prime	
3	$\pm p$, p std. prime	$\pm p$, p std. prime	$\pm p$, $p \neq 3$ std. prime
	$3m$, $3 \nmid m$	$\pm 3p$, $p \neq 3$ std. prime	
4	$\pm p$, p std. prime	$\pm p$, p std. prime	$\pm p$, p odd std. prime
	$2m$, m odd	$\pm 2p$, p odd std. prime	
5	$\pm p$, p std. prime	$\pm p$, p std. prime	none exist
	$5m$, $\quad 5 \nmid m$		
	$\pm p_1 p_2 \cdots p_t$, $p_i \neq 5$ std. primes, $p_1 \equiv \pm 2 \mod 5$, $p_j \equiv \pm 1 \mod 5$ for all $\quad j > 1$, p_j not necessarily distinct		
6	$\pm p$, p std. prime	$\pm p$, p std. prime	$\pm p$, $p \neq 2, 3$ std. prime
	$2m$, m odd	± 2, ± 3, ± 6	
	$3m$, $3 \nmid m$, m odd	$\pm 2p$, $\pm 3p$, $\pm 6p$, $p \neq 2, 3$ std. prime	

Table 2.1: τ_n-atoms, τ_n-primes and $|_{\tau_n}$-primes for $n = 0,\ 2 - 6$

Lemma 2.7 (Hamon [3]). *Numbers of the form $ap_1 \cdots p_t q$ are τ_n-atoms where $p_i \equiv \pm 1 \mod n$, $a \not\equiv p_i \mod n$, and $\pm a p_1 \cdots p_t \not\equiv q \mod n$.*

With the following result Hamon completely classifies the τ_n-primes.

Theorem 2.8 (Hamon [3]). *A nonzero nonunit integer a is τ_n-prime if and only if it can be written as $a = \pm p_1 \cdots p_t q$ where p_i are distinct prime factors of n and q is either a unit or $q \nmid n$ is prime.*

The $|_{\tau_n}$-primes are also completely determined by Hamon. For $n \leq 6$ they are listed in Table 2.1. The remaining values of n are dealt with in the following theorem.

Theorem 2.9 (Hamon [3]). *For $n > 6$, there are no $|_{\tau_n}$-primes.*

CHAPTER 3
THE REDUCED τ_n-FACTORIZATIONS IN $\mathbb{Z}$

In this chapter we begin with a comparison and contrast between the τ_n-factorization and the reduced τ_n-factorization in $\mathbb{Z}$. We then briefly mention the reduced τ_n-primes and reduced τ_n-divide primes discussed by Hamon in [3]. Finally we conduct an in-depth study of when $\mathbb{Z}$ is reduced τ_n-atomic.

3.1 The τ_n-factorization v. the ${}_r\tau_n$-factorization in $\mathbb{Z}$

This section serves to illustrate the main differences between the τ_n-factorizations and the reduced τ_n-factorizations in $\mathbb{Z}$ and justify the need for a different approach in finding the values of n for which $\mathbb{Z}$ is reduced τ_n-atomic.

It is important to note that τ-atoms are ${}_r\tau$-atoms and that the converse does not hold. A major difference between τ-atoms and ${}_r\tau$-atoms is that while associates of τ-atoms are also τ-atoms, this property does not carry over to ${}_r\tau$-atoms. To illustrate these differences consider the τ_3-factorizations $-4 = (-1) \cdot 2 \cdot 2$ and $-4 = 2 \cdot (-2)$. The first factorization shows that -4 is not a τ_3-atom since $2 \equiv 2 \mod 3$. This factorization is a τ_3-factorization, but not a ${}_r\tau_3$-factorization since a ${}_r\tau_3$-factorization requires the leading unit to be 1. Since $-2 \not\equiv 2 \mod 3$, the second factorization is not a reduced τ_3-factorization, and thus -4 is a ${}_r\tau_3$-atom. So -4 is a ${}_r\tau_3$-atom that is not a τ_3-atom and whose associate $4 = 2 \cdot 2$ is not a ${}_r\tau_3$-atom. These statements are in fact true for $n \geq 5$.

These differences are best observed in the following discussion of why the methods used to prove that $\mathbb{Z}$ is not τ_n-atomic do not work for the reduced τ_n-factorizations. We start by showing that $\mathbb{Z}$ is not τ_7-atomic. This proof generalizes to all odd $n \geq 7$. Consider the nontrivial factorizations of 44 (up to order of factors):

$$
\begin{aligned}
44 &= 4 \cdot 11 = (-4) \cdot (-11) \\
44 &= (-1) \cdot (-4) \cdot 11 = (-1) \cdot 4 \cdot (-11) \\
44 &= 2 \cdot 2 \cdot 11 = (-2) \cdot (-2) \cdot 11 = 2 \cdot (-2) \cdot (-11) = (-1) \cdot (-2) \cdot 2 \cdot 11 \\
&= (-1) \cdot 2 \cdot 2 \cdot (-11) = (-1) \cdot (-2) \cdot (-2) \cdot (-11) \\
44 &= 2 \cdot 22 = (-2) \cdot (-22) = (-1) \cdot (-2) \cdot 22 = (-1) \cdot 2 \cdot (-22)
\end{aligned}
$$

Observe that only the factorizations in the first line are τ_7-factorizations since $4 \equiv 11 \mod 7$. The τ_7-factorization $44 = 4 \cdot 11$ is not τ_7-atomic since as a product of τ_7-atoms, $4 = 2 \cdot 2$, 4 is not a τ_7-atom. The τ_7-factorization $44 = (-4) \cdot (-11)$ is also not τ_7-atomic since -4 has the nontrivial τ_7-factorization $-4 = (-1) \cdot 2 \cdot 2$ and thus is not a τ_7-atom. Thus 44 has no τ_7-atomic factorizations.

The two τ_7-factorizations of 44 are also ${}_r\tau_7$-factorizations. While the first factorization is not ${}_r\tau_7$-atomic since $4 = 2 \cdot 2$ is not a ${}_r\tau_7$-atom, the second one is. Recall that -4 is a ${}_r\tau_n$-atom for $n = 3$ and at least 5 and that -11 is an associate of a standard prime, therefore it is τ_n-irreducible, and thus ${}_r\tau_n$-irreducible for every n. Hence, $44 = (-4) \cdot (-11)$ is a ${}_r\tau_7$-atomic factorization and this example cannot be used to prove that $\mathbb{Z}$ is not ${}_r\tau_7$-atomic.

As previously mentioned this proof generalizes to n odd and at least 7. Hamon uses Dirichlet's Theorem on Arithmetic Progression, stated below, to find a standard prime p congruent to $4 \mod n$. She then proceeds to show that the integer $4p$ has τ_n-factorizations, but none that are τ_n-atomic. This proof does not carry over to reduced τ_n-factorizations since just like in the earlier example $(-4)\cdot(-p)$ is a ${}_r\tau_n$-atomic factorization of the integer $4p$. For n even and at least 14, Hamon's proofs do not carry over for similar reasons.

Dirichlet's Theorem on Arithmetic Progression. *Given an arithmetic progression of terms $r+kn$, for $k = 1, 2, ...,$ the series contains an infinite number of primes if r and n are relatively prime and $n \geq 1$.*

3.2 Classifying the ${}_r\tau_n$-primes and $|_{r\tau_n}$-primes

Recall that if D is a domain and $a \in D^{\#}$, a is a ${}_r\tau$-prime if $a | a_1 a_2 \cdots a_t$, a ${}_r\tau$-factorization, implies $a | a_i$ for some i. From this definition it is clear that ${}_r\tau$-primes coincide with τ-primes. Due to the importance of this result we list it as a theorem.

Theorem 3.1 (Hamon [3]). *The ${}_r\tau_n$-primes are exactly the τ_n-primes.*

Hamon proves that if D is a domain with characteristic not equal to 2 and τ is reflexive, then D contains no $|_{r\tau}$-primes. Since $\mathbb{Z}$ has characteristic 0 and τ_n is reflexive, no $|_{r\tau_n}$-primes exist.

Theorem 3.2 (Hamon [3]). *There are no $|_{r\tau_n}$-primes.*

The reduced τ_n-atoms are discussed case by case in the following section.

3.3 When is $\mathbb{Z}$ ${}_r\tau_n$-atomic?

In this section we will observe that due to the associate-preserving property of the τ_1 and τ_2 relations, the reduced τ_1- and τ_2-factorizations are identical to the τ_1- and τ_2-factorizations, respectively. When $n = 0$, 3, 4 and 6 reduced τ_n-atoms exist that are not τ_n-atoms. While $\mathbb{Z}$ was τ_n-atomic for $n = 5$, 8 and 10, this no longer holds for the reduced τ_n-factorizations.

3.3.1 The Reduced τ_n-factorization for $n = 0, 1, 2$

For $a, b \in \mathbb{Z}^{\#}$, $a\tau_0 b$ is defined by $a = b$. Thus, a τ_0-factorization of a nonzero nonunit integer a is of the form $a = (\pm 1)\underbrace{b \cdots b}_{k} = (\pm 1)b^k$, where $k \geq 1$. Then a τ_0-atom is an integer of the form $(\pm 1)p_1^{k_1}p_2^{k_2}\cdots p_t^{k_t}$ with $\gcd(k_1, k_2, \ldots, k_t) = 1$, where p_i are distinct standard primes and $k_i \geq 1$. So $a = (\pm 1)b^k$, where $k \geq 1$, is a τ_0-atomic factorization of a if and only if k is maximal.

Now a ${}_r\tau_0$-factorization of a must be of the form $a = b^k$. Note that while $-4 = (-1)\cdot 2^2$ is a nontrivial τ_0-factorization, -4 is now a ${}_r\tau_0$-atom. Although previously a τ_0-UFD, $\mathbb{Z}$ is not a ${}_r\tau_0$-UFD since 16 has ${}_r\tau_0$-atomic factorizations $16 = (-4)\cdot(-4)$ and $16 = 2\cdot 2\cdot 2\cdot 2$ of different lengths.

To show that $\mathbb{Z}$ is reduced τ_0-atomic we investigate the ${}_r\tau_0$-atoms. The prime factorization of an integer $a \neq -1, 0, 1$ allows us to rewrite a as $(\pm 1)(p_1^{k_1}p_2^{k_2}\cdots p_t^{k_t})^k$ with $\gcd(k_1, k_2, \ldots, k_t) = 1$ and $k \geq 1$ (maximal). When $k = 1$, a is a τ_0-atom and thus a ${}_r\tau_0$-atom. So suppose that $k > 1$. If k is odd, then $[(\pm 1)(p_1^{k_1}p_2^{k_2}\cdots p_t^{k_t})]^k$ is a ${}_r\tau_0$-factorization of a. When a is negative, k is even and of the form 2^j for $j \geq 1$,

$a = (-1)(p_1^{k_1}p_2^{k_2}\cdots p_t^{k_t})^{2^j}$ is a ${}_r\tau_0$-atom. In the case where a is negative, k is even and of the form $2^j m$ for $j \geq 1$ and $m > 1$ odd, $a = \left[(-1)(p_1^{k_1}p_2^{k_2}\cdots p_t^{k_t})^{2^j}\right]^m$ is a nontrivial reduced τ_0-factorization of a.

Lemma 3.3. *$\mathbb{Z}$ is reduced τ_0-atomic.*

Proof. Let $a \neq -1, 0, 1$ be an integer that is not a ${}_r\tau_0$-atom. Now a can be written as $(\pm 1)(p_1^{k_1}p_2^{k_2}\cdots p_t^{k_t})^k$ with $k > 1$ maximal. If k is odd, then $[(\pm 1)(p_1^{k_1}p_2^{k_2}\cdots p_t^{k_t})]^k$ is a ${}_r\tau_0$-atomic factorization of a. This is because the maximality of k implies that $\gcd(k_1, \ldots, k_t) = 1$ and so $(\pm 1)(p_1^{k_1}p_2^{k_2}\cdots p_t^{k_t})$ is a τ_0-atom and therefore a ${}_r\tau_0$-atom. Now suppose k is even. If a is positive, then $a = (p_1^{k_1}p_2^{k_2}\cdots p_t^{k_t})^k$ is a ${}_r\tau_0$-atomic factorization of a just as before. If a is negative since a is not a ${}_r\tau_0$-atom, a is of the form $(-1)(p_1^{k_1}p_2^{k_2}\cdots p_t^{k_t})^{2^j m}$ with $j \geq 1$ and $m > 1$ odd. The factorization $a = \left[(-1)(p_1^{k_1}p_2^{k_2}\cdots p_t^{k_t})^{2^j}\right]^m$ is a nontrivial reduced τ_0-atomic factorization of a since $(-1)(p_1^{k_1}p_2^{k_2}\cdots p_t^{k_t})^{2^j}$ is a reduced τ_0-atom.

The τ_1 relation gives the usual factorization of the integers, since $a\tau_1 b$ if and only if $1|(a-b)$ for all $a, b \in \mathbb{Z}^{\#}$. The ${}_r\tau_1$-atoms and τ_1-atoms are the standard primes and their associates. Since $\mathbb{Z}$ is a UFD, it is both a τ_1-UFD and ${}_r\tau_1$-UFD.

Both the τ_1 and the τ_2 relations are associate-preserving. When τ is associate-preserving if $a = \lambda a_1 \cdots a_t$ is a τ-factorization of a nonzero nonunit element a of a domain D, then so is $a = a_1 \cdots a_{i-1}(\lambda a_i)a_{i+1} \cdots a_t$. In this case, τ-atoms and ${}_r\tau$-atoms coincide. Thus, the ${}_r\tau_2$-atoms are exactly the τ_2-atoms and $\mathbb{Z}$ is τ_2-atomic.

For $n \geq 2$, $\mathbb{Z}$ is not a ${}_r\tau_n$-UFD since for p_1, p_2 standard primes satisfying

$p_1 \equiv p_2 \equiv 1 \mod n$, $(2p_1)\cdot(2p_2)$ and $2\cdot(2p_1p_2)$ are distinct τ_n-atomic factorizations of $4p_1p_2$, and $2p_1$ and $2p_2$ are not associates of 2 or $2p_1p_2$.

In the remainder of this chapter we present a case-by-case demonstration that $\mathbb{Z}$ is not reduced τ_n-atomic for $n = 3, \ldots, 12$, 14, 15, 16, 18, 20, 24, 30, 40, 48, 60, 80, 120 and 240 and prove Theorems 3.6, 3.7, 3.8 and 3.9 to determine the following.

Theorem 3.4. *$\mathbb{Z}$ is a reduced τ_n-UFD if and only if $n = 1$.*

Theorem 3.5. *$\mathbb{Z}$ is reduced τ_n-atomic if and only if $n = 0$, 1, 2, 3, 4, 6.*

3.3.2 The Reduced τ_3-factorization

To show that $\mathbb{Z}$ is ${}_r\tau_3$-atomic we must first determine the ${}_r\tau_3$-atoms. Let a be a nonzero nonunit integer. Since a has prime factorization $a = (\pm 1)p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}$ where $t \geq 1$, $\alpha_i \geq 1$ and p_i are distinct standard primes for $1 \leq i \leq t$, to start we discuss the equivalence classes of the standard primes. Now 3 and -3 are the only standard prime and associate of a standard prime congruent to 0 mod 3. All other standard primes and their associates are congruent to $\pm 1 \mod 3$.

As an integer, a must be congruent to either 0, 1 or -1 mod 3. If $a \equiv 0 \mod 3$, then 3 divides a and we may rewrite the prime factorization of a as $a = (\pm 1)3^k p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}$ where $p_i \equiv \pm 1 \mod 3$, $k \geq 1$, $t \geq 0$ and $\alpha_i \geq 1$. Otherwise $a \equiv \pm 1 \mod 3$, 3 does not divide a and the prime factorization of a is of the form $a = (\pm 1)p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}$ with $p_i \equiv \pm 1 \mod 3$. In both cases, the p_i are (positive) standard primes.

When a is negative any reduced τ_n-factorization of a cannot contain a leading

-1, and thus one of the factors must "absorb" the negative sign. To avoid having to discuss separate cases for a positive and a negative, we adopt the following convention: let $p_i \equiv 1 \mod n$ and $q_j \equiv -1 \mod n$ be either the standard primes or the associates of standard primes in the prime factorization of a. For example, $-25 = (-5) \cdot 5$ is of the form pq where $p = -5 \equiv 1 \mod 3$ and $q = 5 \equiv -1 \mod 3$. Although some ambiguity exists (-625 is of the form $p^3q = (-5)^3 \cdot 5$ and of the form $q^3p = 5^3 \cdot (-5)$) this is not a concern for if we have found a τ_3-atomic factorization for p^3q, then we have also found one for q^3p. To reduce the number of different cases we must consider, we may replace any product of the form $q_1q_2 \cdots q_{2t}$, where q_i (not necessarily distinct) are either standard primes or associates of standard primes, by $p_1p_2 \cdots p_{2t} = (-q_1)(-q_2) \cdots (-q_{2t})$. Thus if $a \equiv \pm 1 \mod 3$, we need only consider a of the form $p_1 \cdots p_t$ and $p_1 \cdots p_t q$ where $p_i \equiv 1 \mod 3$ (not necessarily distinct), and $q \equiv -1 \mod 3$ are either standard primes or associates of standard primes.

Suppose that $a \equiv 0 \mod 3$, so a is of the form $3^k m$ where $3 \nmid m$ and $k \geq 1$. If $k = 1$, then a is a ${}_r\tau_3$-atom. If $k > 1$, then we may write a as $\underbrace{3 \cdots 3}_{k-1} \cdot (3m)$. Since $3 \equiv 3m \equiv 0 \mod 3$, in the case $k > 1$ a has a nontrivial ${}_r\tau_3$-factorization, which means that a is not a ${}_r\tau_3$-atom.

Now suppose that $a \not\equiv 0 \mod 3$. If a is of the form $p_1 \cdots p_t$ (with p_i not necessarily distinct), since $p_i \equiv 1 \mod 3$ it follows that $a \equiv 1 \mod 3$. If a is of the form $p_1 \cdots p_t q$ (again, with p_i not necessarily distinct), then $a \equiv \underbrace{1 \cdots 1}_{t}(-1) \equiv -1 \mod 3$.

If $a = p_1 \cdots p_t$ with $t > 1$, a has the nontrivial ${}_r\tau_3$-factorization $a = (p_1) \cdots (p_t)$.

Here parentheses were placed for emphasis to suggest that each p_i is a factor in the ${}_r\tau_3$-factorization of a. We will omit the parentheses if the meaning is clear from context. So $a = p$ is the only ${}_r\tau_3$-atom when $a \equiv 1 \mod 3$.

If $a = p_1 \cdots p_t q$ and $t \geq 2$, a has the nontrivial ${}_r\tau_3$-factorization $a = (-p_1)(-p_2)(qp_3 \cdots p_t)$. Given that $qp_3 \cdots p_t \equiv -1 \equiv -p_1 \equiv -p_2 \mod 3$ this is indeed a reduced τ_3-factorization. For $t = 1$ and 0, a is of the form pq and q, respectively, and is a ${}_r\tau_3$-atom.

To summarize, the ${}_r\tau_3$-atoms are the standard primes and their associates, nonzero nonunit integers of the form $3m$ where $3 \nmid m$, integers of the form p_1p_2 where p_1 and p_2 are (positive) standard primes satisfying $p_1 \equiv -p_2 \mod 3$, and integers of the form $-p_1p_2$ where $p_1 \equiv p_2 \mod 3$ (not necessarily distinct). Thus $-p^2$ is a ${}_r\tau_3$-atom where p is any standard prime other than 3. Recall that τ_n-atoms are ${}_r\tau_n$-atoms, however the converse is false. The standard primes, their associates, and integers of the form $3m$ where $3 \nmid m$ are the τ_3-atoms (see Table 2.1). The remaining ${}_r\tau_3$-atoms, integers of the form p_1p_2 where $p_1 \equiv -p_2 \mod 3$ and integers of the form $-p_1p_2$ where $p_1 \equiv p_2 \mod 3$, are not.

Lemma 3.6. *$\mathbb{Z}$ is ${}_r\tau_3$-atomic.*

Proof. Let a be a nonzero nonunit integer that is not a ${}_r\tau_3$-atom. We wish to show that a has a ${}_r\tau_3$-atomic factorization.

Suppose that $a \equiv 0 \mod 3$. Then a is of the form $3^k m$ where $k > 1$ and m is an integer that is not a multiple of 3. Then a has the ${}_r\tau_3$-atomic factorization

$\underbrace{3 \cdots 3}_{k-1} \cdot (3m)$. This is indeed a ${}_r\tau_3$-atomic factorization since 3 and $3m$ are ${}_r\tau_3$-atoms, and $3 \equiv 3m \equiv 0 \mod 3$.

Now suppose that $a \not\equiv 0 \mod 3$ and $a = p_1 \cdots p_t$ with $t > 1$, $p_i \equiv 1 \mod 3$ standard primes or associates of standard primes. Since standard primes and their associates are ${}_r\tau_3$-atoms, a has the ${}_r\tau_3$-atomic factorization $a = (p_1) \cdots (p_t)$.

Let $a = p_1 \cdots p_t q$ where $q \equiv -1 \mod 3$ and $p_i \equiv 1 \mod 3$ are standard primes or associates of standard primes. Since a is not a ${}_r\tau_3$-atom, t must be greater than 1. If t is even, then for some positive integer s, $t = 2s$ and $a = p_1 \cdots p_{2s} q$. Given that $-p_i \equiv -1 \equiv q \mod 3$ for $1 \leq i \leq 2s$, a has the ${}_r\tau_3$-atomic factorization $a = (-p_1) \cdots (-p_{2s})(q)$. If t is odd, then for some $s > 1$, $t = 2s + 1$ and $a = p_1 \cdots p_{2s+1} q$ has the ${}_r\tau_3$-factorization $a = (-p_1) \cdots (-p_{2s})(p_{2s+1} q)$. Since elements of the form pq are ${}_r\tau_3$-atoms, this is in fact a ${}_r\tau_3$-atomic factorization.

3.3.3 The Reduced τ_4-factorization

The ${}_r\tau_4$-factorization is similar to the ${}_r\tau_3$-factorization. The odd standard primes and their associates are congruent to $\pm 1 \mod 4$. We call these p and q again, with $p \equiv 1 \mod 4$ and $q \equiv -1 \mod 4$. Just as before, integers of the form p, q and pq are ${}_r\tau_4$-atoms. Integers congruent to 2 mod 4 are of the form $2m$ where m is any odd integer. They are ${}_r\tau_4$-atoms. Modulo 4, $2 \equiv -2$. Note that any nonzero nonunit a congruent to 0 mod 4 cannot be a ${}_r\tau_4$-atom. If $a \equiv 0 \mod 4$, then a is of the form $2^k m$ where $k \geq 2$ and m is any odd integer (possibly a unit). Then a has the nontrivial ${}_r\tau_4$-factorization $2 \cdot 2 \cdots 2 \cdot 2m$. (If m is negative, then $2m \equiv -2 \equiv 2$

mod 4 and a still has the same nontrivial ${}_r\tau_4$-factorization as before.)

Thus, the τ_4-atoms, $\pm p$ where p is a standard prime and integers of the form $2m$ with m odd, are ${}_r\tau_4$-atoms. The "new" ${}_r\tau_4$-atoms are integers of the form p_1p_2 where p_1 and p_2 are odd standard primes satisfying $p_1 \equiv -p_2 \mod 4$, and integers of the form $-p_1p_2$ where $p_1 \equiv p_2 \mod 4$ (not necessarily distinct). Since $-2 \equiv 2 \mod 4$, $-p^2$ is a ${}_r\tau_4$-atom only when p is an odd standard prime.

Lemma 3.7. *$\mathbb{Z}$ is ${}_r\tau_4$-atomic.*

Proof. For a a nonzero nonunit integer that is not a reduced τ_4-atom. If a is congruent to $\pm 1 \mod 4$, we refer to the proof of Lemma 3.6.

When $a \equiv 0 \mod 4$, a is of the form $2^k m$ where $k \geq 2$ and m is odd. In this case a has the reduced τ_4-atomic factorization $a = \underbrace{2 \cdots 2}_{k-1} \cdot (2m)$. Since $2m \equiv 2 \mod 4$ is a reduced τ_4-atom this is indeed a reduced τ_4-atomic factorization.

If $a \equiv 2 \mod 4$, then $2|a$, but $4 \nmid a$. Thus a is of the form $2m$ where m is any odd integer. So if $a \equiv 2 \mod 4$, then a is a ${}_r\tau_4$-atom and there is nothing to discuss.

3.3.4 The Reduced τ_5-factorization

The smallest n where the τ_n-factorization and reduced τ_n-factorization no longer agree on the atomicity of $\mathbb{Z}$ is $n = 5$.

Lemma 3.8. *$\mathbb{Z}$ is not ${}_r\tau_5$-atomic.*

Proof. Consider the nontrivial factorizations of 24 with positive factors. The ${}_r\tau_5$-

factorizations are in boldface.

$$24 = 2 \cdot 2 \cdot 2 \cdot 3$$

$$24 = 4 \cdot 2 \cdot 3 = 2 \cdot 2 \cdot 6$$

$$24 = 4 \cdot 6 = \mathbf{8} \cdot \mathbf{3} = \mathbf{2} \cdot \mathbf{12}$$

Introducing an even number of negative signs in these factorizations yields the ${}_r\tau_5$-factorizations $24 = (-8) \cdot (-3)$ and $24 = (-2) \cdot (-12)$. Since ± 8 is a cube and $12 = (-2) \cdot (-2) \cdot 3$ and $-12 = 2 \cdot 2 \cdot (-3)$ are nontrivial reduced τ_5-factorizations, none of the ${}_r\tau_5$-factorizations of 24 are atomic. Thus, although τ_5-atomic, $\mathbb{Z}$ is not ${}_r\tau_5$-atomic.

3.3.5 The Reduced τ_6-factorization

The ${}_r\tau_6$-factorization is similar to both the ${}_r\tau_3$- and ${}_r\tau_4$-factorizations. All odd standard primes other than 3 and their associates are congruent to $\pm 1 \mod 6$. As seen earlier, elements of the form p, q and pq are ${}_r\tau_6$-atoms where $p \equiv 1 \mod 6$ and $q \equiv -1 \mod 6$ are standard primes or associates. Modulo 6, $3 \equiv -3$ and every integer of the form $3m'$ with m' nonzero and not divisible by 3 is a ${}_r\tau_6$-atom. The only standard prime or associate congruent to 2 mod 6 is 2 and -2 is the only standard prime or associate congruent to 4 or -2 mod 6. Numbers of the form $2m''$ where m'' is an odd integer are ${}_r\tau_6$-atoms. Now 6 is both of the form $3m'$ and $2m''$, so 6 is a ${}_r\tau_6$-atom.

To find any remaining ${}_r\tau_6$-atoms, consider a nonzero nonunit a. If 2 and 3 do not divide a, then a is of the form $p_1 \cdots p_t$ or $p_1 \cdots p_t q$ where $p_i \equiv 1 \mod 6$ and

$q \equiv -1 \mod 6$ are standard primes or associates. As mentioned earlier, the only ${}_r\tau_6$-atoms of this form are p, q and pq.

If 3 divides a and 2 does not, then a is of the form $3^k m$ where $k \geq 1$, m is odd and not divisible by 3. Recall that if $k = 1$, then $3m$ is a ${}_r\tau_6$-atom since m is nonzero and not a multiple of 3. Now suppose $k > 1$. Since $3m \equiv 3 \equiv -3 \mod 6$, a has the non-trivial ${}_r\tau_6$-factorization $a = 3 \cdot 3 \cdots 3 \cdot 3m$.

If 2 divides a and $3 \nmid a$, then $a = 2^k m$ with $k \geq 1$, m odd and not a multiple of 3. The case $k = 1$ has already been discussed, so suppose $k \geq 2$. Since m is odd and $3 \nmid m$, $m \equiv \pm 1 \mod 6$, so m is of the form $p_1 \cdots p_t$ or $p_1 \cdots p_t q$ where $p_i \equiv 1 \mod 6$ and $q \equiv -1 \mod 6$ are standard primes or associates. When $m = p_1 \cdots p_t$, $2m \equiv 2 \mod 6$ and a has the nontrivial ${}_r\tau_6$-factorization $2 \cdot 2 \cdots 2 \cdot 2m$. Suppose $m = p_1 \cdots p_t q \equiv -1 \mod 6$. If $k = 2$, then $a = 2 \cdot 2m$ is a ${}_r\tau_6$-atom. Now let $k > 2$. Since $4m \equiv -4 \equiv 2 \mod 6$, a has the nontrivial ${}_r\tau_6$-factorization $a = \underbrace{2 \cdot 2 \cdots 2}_{k-2} \cdot 4m$.

When $6 \mid a$, a is of the form $6^k m$ where $6 \nmid m$. Let $k = 1$. Now m is either an odd multiple of 3 or just not a multiple of 3. In the latter case, m can be even or odd. If m is even and not a multiple of 3, a is a ${}_r\tau_6$-atom of the form $3m'$ with $3 \nmid m'$. When m is odd and not divisible by 3, then $m = p_1 \cdots p_t$ or $m = p_1 \cdots p_t q$ with $p_i \equiv 1 \mod 6$ and $q \equiv -1 \mod 6$ standard primes or associates. In any case, $a = 6m$ is a ${}_r\tau_6$-atom. Now suppose m is an odd multiple of 3. Then a is a ${}_r\tau_6$-atom of the form $2m''$ with m'' odd. Finally, when $k > 1$, a has the non-trivial ${}_r\tau_6$-factorization $a = \underbrace{6 \cdot 6 \cdots 6}_{k-1} \cdot 6m$.

Lemma 3.9. *$\mathbb{Z}$ is ${}_r\tau_6$-atomic.*

Proof. Let $a \neq -1, 0, 1$ be a nonzero nonunit that is not a ${}_r\tau_6$-atom. If $2, 3 \nmid a$, then a is of the form $p_1 \cdots p_t$ or $p_1 \cdots p_t q$ where $p_i \equiv 1 \mod 6$ and $q \equiv -1 \mod 6$ are standard primes or associates. As seen with the reduced τ_3- and τ_4-factorizations, a has a reduced τ_6-atomic factorization.

If 3 divides a and 2 does not, then a is of the form $3^k m$ where $k > 1$, m is odd and not divisible by 3. Then a has the reduced τ_6-atomic factorization $a = 3 \cdot 3 \cdots 3 \cdot 3m$ since $3m$ is a reduced τ_6-atom.

If 2 divides a and $3 \nmid a$, then $a = 2^k m$ with $k > 1$, m odd and not a multiple of 3. When m is of the form $p_1 \cdots p_t$, $2m \equiv 2 \mod 6$ and a has the nontrivial ${}_r\tau_6$-factorization $2 \cdot 2 \cdots 2 \cdot 2m$ since $2m$ is a reduced τ_6-atom. Otherwise, m is of the form $p_1 \cdots p_t q$. Then we must have $k > 2$. In this case a has the nontrivial ${}_r\tau_6$-atomic factorization $a = \underbrace{2 \cdot 2 \cdots 2}_{k-2} \cdot 4m$ given that $4m$ is a ${}_r\tau_6$-atom.

Finally, when $6 \mid a$, a is of the form $6^k m$ where $6 \nmid m$ and $k > 1$. Now a has the non-trivial ${}_r\tau_6$-factorization $a = \underbrace{6 \cdot 6 \cdots 6}_{k-1} \cdot 6m$. This factorization is ${}_r\tau_6$-atomic since the case $k = 1$ shows that $6m$ is a ${}_r\tau_6$-atom.

3.3.6 The Reduced τ_n-factorization for $n > 6$

Lemma 3.10. *$\mathbb{Z}$ is not ${}_r\tau_7$-atomic.*

Proof. We examine the factorizations of 162. To start, observe the nontrivial factorizations of 162 that have only positive factors, listed below in decreasing order of the length of the factorization. The ${}_r\tau_7$-factorizations are listed in boldface.

$$162 = 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3$$

$$162 = 6 \cdot 3 \cdot 3 \cdot 3 = 2 \cdot 9 \cdot 3 \cdot 3$$

$$162 = 6 \cdot 9 \cdot 3 = \mathbf{2 \cdot 9 \cdot 9} = 18 \cdot 3 \cdot 3 = 2 \cdot 3 \cdot 27$$

$$162 = 18 \cdot 9 = \mathbf{6 \cdot 27} = 54 \cdot 3 = 2 \cdot 81$$

An even number of negative signs can be inserted in these factorizations to potentially yield other ${}_r\tau_7$-factorizations. The factorizations that have factors congruent to 2 and 3 do not yield ${}_r\tau_7$-factorizations since $2 \not\equiv \pm 3 \mod 7$. Similarly, we can eliminate all factorizations except

$$162 = 2 \cdot 9 \cdot 9$$

$$162 = 18 \cdot 3 \cdot 3$$

$$162 = 6 \cdot 27$$

to get ${}_r\tau_7$-factorizations containing negative factors $162 = 18 \cdot (-3) \cdot (-3)$ and $162 = (-6) \cdot (-27)$.

Since 9 is a square and ± 27 is a cube, they are not ${}_r\tau_7$-atoms. Therefore, the ${}_r\tau_7$-factorizations $162 = 2 \cdot 9 \cdot 9$, $162 = 6 \cdot 27$, and $162 = (-6) \cdot (-27)$ are not atomic. Finally, 18 has the ${}_r\tau_7$-factorization $18 = 2 \cdot 9$, so it is not a ${}_r\tau_7$-atom. This means that the remaining ${}_r\tau_7$-factorization of 162, $162 = 18 \cdot (-3) \cdot (-3)$, is also not atomic. Thus, 162 has no ${}_r\tau_7$-atomic factorization. [Note that the τ_7-factorizations of 162 are the ${}_r\tau_7$-factorizations already found, as well as $162 = (-1) \cdot (-18) \cdot 3 \cdot 3$ and $162 = (-1) \cdot (-2) \cdot (-9) \cdot (-9)$. Also note that none are τ_7-atomic.]

The τ_n-factorization and reduced τ_n-factorization differ not only for $n = 5$, but also for $n = 8, 10$.

Lemma 3.11. *$\mathbb{Z}$ is not ${}_r\tau_8$-atomic.*

Proof. Consider the factorization $-16 = 4 \cdot (-4)$. Since $4 \equiv -4 \mod 8$, this is a ${}_r\tau_8$-factorization. Given that 4 is not a ${}_r\tau_8$-atom, it is not a ${}_r\tau_8$-atomic factorization. Up to sign and order of factors, the other factorizations of -16 are $-16 = 8 \cdot (-2)$ and $-16 = 2 \cdot 2 \cdot 2 \cdot (-2)$. Neither is a ${}_r\tau_8$-factorization. Although $\mathbb{Z}$ was τ_8-atomic, it is not ${}_r\tau_8$-atomic.

Lemma 3.12. *$\mathbb{Z}$ is not ${}_r\tau_9$-atomic.*

Proof. Consider the factorization $-81 = 9 \cdot (-9)$. Since $9 \equiv -9 \mod 9$, this is a ${}_r\tau_9$-factorization. Given that 9 is not a ${}_r\tau_9$-atom, it is not a ${}_r\tau_9$-atomic factorization. Up to sign and order of factors, the other factorizations of -81 are $-81 = 27 \cdot (-3)$ and $-81 = 3 \cdot 3 \cdot 3 \cdot (-3)$. Neither is a ${}_r\tau_9$-factorization.

We may use the factorizations of -81 to show that $\mathbb{Z}$ is also not ${}_r\tau_{10}$-atomic.

Lemma 3.13. *$\mathbb{Z}$ is not ${}_r\tau_{10}$-atomic.*

Proof. Consider the factorization $-81 = 27 \cdot (-3)$. Since $27 \equiv -3 \mod 10$, this is a ${}_r\tau_{10}$-factorization. Given that 27 is not a ${}_r\tau_{10}$-atom, it is not a ${}_r\tau_{10}$-atomic factorization. Up to sign and order of factors, the other factorizations of -81 are $-81 = 9 \cdot (-9)$ and $-81 = 3 \cdot 3 \cdot 3 \cdot (-3)$. Neither is a ${}_r\tau_{10}$-factorization.

Lemma 3.14. *$\mathbb{Z}$ is not ${}_r\tau_{11}$-atomic.*

Proof. Consider the following nontrivial factorizations of -24.

$$-24 = 2 \cdot 2 \cdot 2 \cdot (-3)$$

$$-24 = 4 \cdot 2 \cdot (-3) = 2 \cdot 2 \cdot (-6)$$

$$-24 = 4 \cdot (-6) = \mathbf{8} \cdot (-\mathbf{3}) = 2 \cdot (-12)$$

The factorization $-24 = 8 \cdot (-3)$ is the only ${}_r\tau_{11}$-factorization above. Introducing an even number of negative signs in these factorizations yields the only other ${}_r\tau_{11}$-factorization, $-24 = (-8) \cdot 3$. We have exhausted all possible nontrivial reduced τ_{11}-factorizations of -24. Since ± 8 is a cube, the two ${}_r\tau_{11}$-factorizations of -24 are not atomic.

Lemma 3.15. *$\mathbb{Z}$ is not ${}_r\tau_{12}$-atomic.*

Proof. We need only check that $432 = 12 \cdot 36$ and $432 = (-12) \cdot (-36)$ are not ${}_r\tau_{12}$-atomic factorizations of 432. For the remainder of the proof, we proceed as we did in the proof of Lemma 2.3, except we must consider factorizations where $a_i \equiv 8 \mod 12$ or $a_i \equiv 9 \mod 12$ for all i. (These are dealt with at the same time as $\bar{2}$, $\bar{3}$ and $\bar{4}$.)

From the proof of Lemma 2.3, we see that 36 has nontrivial ${}_r\tau_{12}$-factorizations $36 = 6 \cdot 6$ and $36 = (-6) \cdot (-6)$. Thus, $432 = 12 \cdot 36$ is not a ${}_r\tau_{12}$-atomic factorization of 432. Since $6 \equiv -6 \mod 12$, $-36 = 6 \cdot (-6)$ is a nontrivial ${}_r\tau_{12}$-factorization of -36. Therefore, $432 = (-12) \cdot (-36)$ is also not a ${}_r\tau_{12}$-atomic factorization.

We now proceed to show that $\mathbb{Z}$ is not reduced τ_n-atomic for $n > 12$. In the following theorem $U(n)$ denotes the group of units (also known as the multiplicative group of integers modulo n). Recall that $U(n)$ consists of the equivalence classes $\bar{x}$ modulo n where x is relatively prime to n and $1 \leq x \leq n-1$.

Theorem 3.16. *If $U(n)$ has an element of order at least 7, then $\mathbb{Z}$ is not ${}_r\tau_n$-atomic.*

Proof. Let x be an element of $U(n)$ of order at least 7. Since $x \in U(n)$, x and n are relatively prime. By Dirichlet's Theorem there exists a prime p congruent to x mod n. Thus, we may assume that x is prime. Since x has order at least 7, $x^3 \not\equiv 1 \mod n$ is a unit. Using Dirichlet's Theorem again we can find a prime y congruent to x^3 mod n. We claim that x^3y has no reduced τ_n-atomic factorization. We inspect the nontrivial factorizations of x^3y.

$$x^3y = x \cdot x \cdot x \cdot y$$

$$x^3y = x^2 \cdot x \cdot y = x \cdot x \cdot xy$$

$$x^3y = \mathbf{x^3} \cdot \mathbf{y} = x \cdot x^2y$$

Since x has order at least 7, $x \not\equiv y \equiv x^3 \mod n$, $x^2 \not\equiv y \equiv x^3 \mod n$, $x \not\equiv xy \equiv x^4 \mod n$ and $x \not\equiv x^2y \equiv x^5 \mod n$. Introducing an even number of negative signs in these factorizations yields that $x^3y = (-x^3) \cdot (-y)$ is another ${}_r\tau_n$-factorization, and possibly so is $x^3y = (-x) \cdot (-x) \cdot xy$. Since $-x^3 = (-x)(-x)(-x)$ is not a ${}_r\tau_n$-atom, $x^3y = (-x^3) \cdot (-y)$ is not a ${}_r\tau_n$-atomic factorization. If $-x \equiv xy \mod n$, then $-x \equiv x^4 \mod n$. Since x is relatively prime to n, this means that $x^3 \equiv -1 \mod n$, and thus $x^6 \equiv 1 \mod n$. Since the order of x is at least 7, $x^6 \not\equiv 1 \mod n$ and x^3y has no ${}_r\tau_n$-atomic factorizations.

The proof also works if $U(n)$ has an element of order 5. (One could modify the proof to show that $\mathbb{Z}$ is not τ_n-atomic if $U(n)$ contains an element of order 5, 7, 9 or higher, or that $\mathbb{N}$ is not τ_n-atomic if $U(n)$ contains an element of order at least 5.)

Theorem 3.17. *$\mathbb{Z}$ is not ${}_r\tau_n$-atomic if $U(n)$ contains an element of order* 5.

We wish to know when $U(n)$ has no element of order 5 or 7 or higher. Thus, we would like to know the cyclic subgroups of $U(n)$. Let C_k denote the cyclic group of order k. From number theory (see [5]) we know that $U(2) \cong C_1$, $U(4) \cong C_2$, and if p is an odd prime, $U(p^\alpha) \cong U(2p^\alpha) \cong C_{p^{\alpha-1}(p-1)}$. Also, for $\alpha \geq 3$, $U(2^\alpha) \cong C_2 \times C_{2^{\alpha-2}}$. Thus, if n is a prime larger than 7, the square of a prime larger than 3, or the cube of any odd prime, then $U(n)$ contains an element of order 5 or 7 or larger. If n is of the form 2^α and is at least 32, then $U(n)$ contains an element of order 5 or 7 or larger. Below we summarize the structure of $U(n)$ for n of the form p^α so that $U(n)$ does not contain an element of order 5, 7, or higher.

n	$U(n)$
2	C_1
4	C_2
8	$C_2 \times C_2$
16	$C_2 \times C_4$

n	$U(n)$
3	C_2
9	C_6

n	$U(n)$
5	C_4

n	$U(n)$
7	C_6

Table 3.1: Cyclic Subgroups of $U(n)$

In general, if $n = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$, then $U(n) \cong U(p_1^{\alpha_1}) \times U(p_2^{\alpha_2}) \times \ldots \times U(p_t^{\alpha_t})$.

For n the product of two or more primes, the figure above allows us to identify when $U(n)$ has no element of order 5, 7 or higher. Since $U(n)$ has an element of order 12 if it contains a subgroup isomorphic to $C_4 \times C_6$, $U(n)$ has no element of order 5, 7 or higher for $n = 6, 10, 14, 15, 21, 30, 42, 18, 63, 126, 12, 20, 28, 60, 84, 36, 252, 24, 40, 56, 120, 168, 72, 504, 48, 80, 240$. In order, all n for which $U(n)$ has no element of order 5, 7 or higher are: 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 28, 30, 36, 40, 42, 48, 56, 60, 63, 72, 80, 84, 120, 126, 168, 240, 252, and 504. Having already discussed the cases where $n \leq 12$ we introduce the following results to reveal a few of the remaining cases.

Theorem 3.18. *If $x \in U(n)$ is an element of order 6 such that $x^3 \not\equiv -1 \mod n$, then $\mathbb{Z}$ is not ${}_r\tau_n$-atomic.*

Proof. Let x be an element of $U(n)$ of order 6 such that $x^3 \not\equiv -1 \mod n$. Since $x \in U(n)$, x and n are relatively prime. By Dirichlet's Theorem there exists a prime p congruent to x mod n. Thus, we may assume that x is prime. Since the order of x is 6, $x^2 \not\equiv 1 \mod n$ is a unit. Using Dirichlet's Theorem again we can find a prime y congruent to x^2 mod n. We claim that x^4y has no reduced τ_n-atomic factorization. We inspect the nontrivial factorizations of x^4y.

$$x^4y = y \cdot x \cdot x \cdot x \cdot x$$

$$x^4y = \mathbf{y} \cdot \mathbf{x^2} \cdot \mathbf{x^2} = y \cdot x^3 \cdot x = xy \cdot x^2 \cdot x = x^2y \cdot x \cdot x$$

$$x^4y = \mathbf{x^3} \cdot \mathbf{xy} = x^2 \cdot x^2y = x \cdot x^3y$$

Since x is coprime to n, $x \equiv y \equiv x^2 \mod n$ would imply that $1 \equiv x \mod n$, however

x has order 6. Thus the first factorization above is not a reduced τ_n-factorization. Similarly, $x \not\equiv xy \equiv x^3 \mod n$, for otherwise $1 \equiv x^2 \mod n$, which is impossible. Also, $x \not\equiv x^2y \equiv x^4 \mod n$ since $1 \not\equiv x^3 \mod n$. Hence all but the first factorization in the second line are not reduced τ_n-factorizations. Finally, $x^2 \not\equiv x^2y \equiv x^4 \mod n$, and $x \not\equiv x^3y \equiv x^5 \mod n$ since x does not have order 2 or 4. The two factorizations in boldface are ${}_r\tau_n$-factorizations. Neither is ${}_r\tau_n$-atomic. Introducing an even number of negative signs we obtain the nontrivial factorizations:

$$
\begin{aligned}
x^4y &= y \cdot (-x) \cdot (-x) \cdot x \cdot x = y \cdot (-x) \cdot (-x) \cdot (-x) \cdot (-x) \\
x^4y &= y \cdot (-x^2) \cdot (-x^2) = (-y) \cdot (-x^2) \cdot x^2 \\
&= y \cdot (-x^3) \cdot (-x) = (-y) \cdot (-x^3) \cdot x = (-y) \cdot x^3 \cdot (-x) \\
&= (-xy) \cdot (-x^2) \cdot x = (-xy) \cdot x^2 \cdot (-x) = xy \cdot (-x^2) \cdot (-x) \\
&= (-x^2y) \cdot (-x) \cdot x = x^2y \cdot (-x) \cdot (-x) \\
x^4y &= (-\mathbf{x^3}) \cdot (-\mathbf{xy}) = (-x^2) \cdot (-x^2y) = (-x) \cdot (-x^3y)
\end{aligned}
$$

Now $-x \not\equiv x \mod n$, since otherwise $2x \equiv 0 \mod n$ and as a divisor of n, $x \notin U(n)$. Similarly, $-x^2 \not\equiv y \equiv x^2 \mod n$. The only interesting factorizations are $x^4y = (-x^3) \cdot (-xy)$ and $x^4y = x^2y \cdot (-x) \cdot (-x)$. The first factorization is a ${}_r\tau_n$-factorization of x^4y, but not an atomic one. The second is not a ${}_r\tau_n$-factorization since $x^2y \equiv -x \mod n$ implies $x^4 \equiv -x \mod n$, and further $x^3 \equiv -1 \mod n$ which contradicts the initial hypothesis.

Theorem 3.19. *If $U(n)$ contains a subgroup isomorphic to $C_2 \times C_6$, then $U(n)$ contains an element x of order 6 such that $x^3 \not\equiv -1 \mod n$.*

Proof. Suppose $U(n)$ contains a subgroup isomorphic to $C_2 \times C_6$. Let $< a > \cong C_2$ and $< b > \cong C_6$ where $a, b \in U(n)$ and $a \notin < b >$. Then $U(n)$ contains the three cyclic subgroups of order 6, $< b >$, $< ab >$ and $< ab^2 >$. We show that if one of the three generators cubed is congruent to $-1 \mod n$, then one of the remaining generators is the x we are looking for.

Suppose $b^3 \equiv -1 \mod n$. Then $(ab)^3 = a^3b^3 \equiv -a^3 \mod n$. Since a has order 2, this implies $(ab)^3 \equiv -a \mod n$. Also since a is of order 2, $a \not\equiv 1 \mod n$, and thus $(ab)^3 \not\equiv -1 \mod n$.

If $b^3 \not\equiv -1 \mod n$, then b is the element of $U(n)$ of order 6 that we were looking for.

For $n = 21$, 28, 36, 42, 56, 63, 72, 84, 126, 168, 252, and 504 $U(n)$ contains a subgroup isomorphic to $C_2 \times C_6$. Note that $U(63) \cong C_6 \times C_6 \cong C_2 \times C_3 \times C_6$. Thus we can state the following theorem.

Theorem 3.20. *$\mathbb{Z}$ is not ${}_r\tau_n$-atomic for $n = 21$, 28, 36, 42, 56, 63, 72, 84, 126, 168, 252, and 504.*

We must still determine whether $\mathbb{Z}$ is reduced τ_n-atomic for $n = 14$, 15, 16, 18, 20, 24, 30, 40, 48, 60, 80, 120 and 240.

Lemma 3.21. *$\mathbb{Z}$ is not ${}_r\tau_{14}$-atomic.*

Proof. For reference, we include the powers of 3 and 5, which are multiplicative units modulo 14. Here $-5 \equiv 9 \mod 14$, $-3 \equiv 11 \mod 14$, and $-1 \equiv 13 \mod 14$.

$\overline{x}$	$\overline{x^2}$	$\overline{x^3}$	$\overline{x^4}$	$\overline{x^5}$	$\overline{x^6}$
3	-5	-1	-3	5	1
5	-3	-1	-5	3	1

Consider the factorizations:

$$-1875 = 3 \cdot (-5 \cdot 5 \cdot 5 \cdot 5) \equiv 3 \cdot 5 \mod 14$$

$$-1875 = (-3 \cdot 5 \cdot 5 \cdot 5) \cdot 5 \equiv 3 \cdot 5 \mod 14$$

$$-1875 = (-3 \cdot 5 \cdot 5) \cdot (-5) \cdot (-5) \equiv (-5) \cdot (-5) \cdot (-5) \mod 14 \quad \text{not } {}_r\tau_{14}\text{-atomic}$$

$$-1875 = (3 \cdot 5 \cdot 5) \cdot (-5 \cdot 5) \equiv 5 \cdot 3 \mod 14$$

$$-1875 = -3 \cdot 5 \cdot (5 \cdot 5 \cdot 5) \equiv -3 \cdot 5 \cdot (-1) \mod 14$$

$$-1875 = (-3 \cdot 5) \cdot (5 \cdot 5 \cdot 5) \equiv (-1) \cdot (-1) \mod 14 \quad \text{not } {}_r\tau_{14}\text{-atomic}$$

$$-1875 = (3 \cdot 5) \cdot (-5 \cdot 5 \cdot 5) \equiv 1 \cdot 1 \mod 14 \quad \text{not } {}_r\tau_{14}\text{-atomic}$$

$$-1875 = (3 \cdot 5) \cdot (-5 \cdot 5) \cdot 5 \equiv 1 \cdot 3 \cdot 5 \mod 14$$

$$-1875 = (-3) \cdot (5 \cdot 5) \cdot (5 \cdot 5) \equiv (-3) \cdot (-3) \cdot (-3) \mod 14 \quad \text{not } {}_r\tau_{14}\text{-atomic}$$

$$-1875 = 3 \cdot (-5 \cdot 5) \cdot 5 \cdot 5 \equiv 3 \cdot 3 \cdot 5 \cdot 5 \mod 14$$

First observe that $-5 \cdot 5 \equiv 3 \mod 14$ is a ${}_r\tau_{14}$-atom. Since $-3 \cdot 5 \cdot 5 = 3 \cdot (-5 \cdot 5) \equiv 3 \cdot 3 \mod 14$, the ${}_r\tau_{14}$-factorization $-1875 = (-3 \cdot 5 \cdot 5) \cdot (-5) \cdot (-5)$ is not a ${}_r\tau_{14}$-atomic factorization. Given that $5 \cdot 5 \cdot 5$, $-5 \cdot 5 \cdot 5 = (-5)^3$ and $5 \cdot 5$ are not ${}_r\tau_{14}$-atoms, the remaining ${}_r\tau_{14}$-factorizations are also not ${}_r\tau_{14}$-atomic factorizations. Thus -1875 has no ${}_r\tau_{14}$-atomic factorizations.

Lemma 3.22. *$\mathbb{Z}$ is not ${}_r\tau_{15}$-atomic.*

Proof. Consider the factorization $-81 = (-3) \cdot 27$. Since $27 \equiv -3 \mod 15$, this is a ${}_r\tau_{15}$-factorization. However this factorization is not an ${}_r\tau_{15}$-factorization since 27 is a cube, and hence not an ${}_r\tau_{15}$-atom. Similarly, $-81 = 3 \cdot (-27)$ is an ${}_r\tau_{15}$-factorization, but -27 is not an ${}_r\tau_{15}$-atom. Since $9 \not\equiv -9 \mod 15$ and $3 \not\equiv \pm 9 \mod 15$, no other ${}_r\tau_{15}$-factorizations of -81 can exist.

Lemma 3.23. *$\mathbb{Z}$ is not ${}_r\tau_{16}$-atomic.*

Proof. Consider the factorization $-256 = 16 \cdot (-16)$. Since $16 \equiv 0 \equiv -16 \mod 16$, this is a ${}_r\tau_{16}$-factorization. However since $16 = 2^4$ is not an ${}_r\tau_{16}$-atom, the aforementioned factorization is not an ${}_r\tau_{16}$-factorization. To show that no other nontrivial ${}_r\tau_{16}$-atomic factorization of -256 exist, consider a general factorization $-256 = a_1 \cdots a_k$ with a_i a positive or negative multiple of 2. First suppose that for some i, $16|a_i$. Any factorization of length 3 or more cannot be a ${}_r\tau_{16}$-factorization since the remaining factors can only be ± 2, ± 4 or $\pm 8 \not\equiv 0 \equiv \pm 16 \mod 16$. Up to order, the only nontrivial ${}_r\tau_{16}$-factorization of this type is the one already mentioned. Now suppose 16 does not divide any of the factors $a_1, \ldots a_k$. Since factorizations of length 2 must contain a factor that is a multiple of 16, we need only consider factorizations of length 3 or more. If for some i, $a_i = \pm 8$, at most one other factor can be ± 8. This implies that a third factor must be ± 2 or $\pm 4 \not\equiv 8 \mod 16$. Thus no other nontrivial ${}_r\tau_{16}$-factorizations exist.

Lemma 3.24. *$\mathbb{Z}$ is not ${}_r\tau_{18}$-atomic.*

Proof. Consider the factorization $-81 = 9 \cdot (-9)$. Since $9 \equiv -9 \mod 18$, this is a ${}_r\tau_{18}$-factorization. It is not ${}_r\tau_{18}$-atomic since 9 is a square. Since the remaining factorizations of -81,

$$-81 = (-3) \cdot 3 \cdot 3 \cdot 3 = (-3) \cdot (-3) \cdot (-3) \cdot 3$$

$$-81 = (-3) \cdot 27 = 3 \cdot (-27)$$

are not ${}_r\tau_{18}$-factorizations, $\mathbb{Z}$ is not ${}_r\tau_{18}$-atomic.

Lemma 3.25. *$\mathbb{Z}$ is not ${}_r\tau_{20}$-atomic.*

Proof. Consider the ${}_r\tau_{20}$-factorizations $2000 = 20 \cdot 100$ and $2000 = (-20) \cdot (-100)$. Since $10 \equiv \pm 10 \mod 20$, ± 100 is not a ${}_r\tau_{20}$-atom and neither factorization of 2000 is ${}_r\tau_{20}$-atomic. We claim that no other ${}_r\tau_{20}$-factorizations of 2000 exist. Let $a_1 \cdots a_t$ be a ${}_r\tau_{20}$-factorization of 2000. Since $5 \,|\, 2000$, there exists an i so that $5 \,|\, a_i$. Then $a_i \equiv -5$, 0, 5, or 10 mod 20. Since $2 \,|\, 2000$, there exists some j such that $2 \,|\, a_j$. Then $a_j \equiv 0, \pm 2, \pm 4, \pm 6, \pm 8$ or 10 mod 20. Since $a_1 \cdots a_t$ is a ${}_r\tau_{20}$-factorization, $a_i \equiv a_j \mod 20$ for all $i \neq j$, and thus a_i must be congruent to either 0 or 10 mod 20 for all i. The original factorizations are the only possible ${}_r\tau_{20}$-factorizations where $a_i \equiv 0 \mod 20$. No ${}_r\tau_{20}$-factorizations where $a_i \equiv 10 \mod 20$ can exist since all a_i would have to be of the form $2 \cdot 5^k$ where $k \geq 1$. But since $2000 = 2^4 \cdot 5^3$ has more powers of 2 than of 5, such a factorization is not possible.

Lemma 3.26. *$\mathbb{Z}$ is not ${}_r\tau_{24}$-atomic.*

Proof. The factorizations $-320 = (-8) \cdot 40$ and $-320 = 8 \cdot (-40)$ are reduced τ_{24}-factorization since $-8 \equiv 40 \mod 24$. They are not reduced τ_{24}-atomic since ± 8 is a cube. We will show that no other non-trivial reduced τ_{24}-factorizations of -320 exist (up to order). Since 5 divides -320, 5 must divide one of the factors in any non-trivial reduced τ_{24}-factorization of -320. Thus one of the factors must be of the form $5 \cdot 2^k$ with $0 \leq k \leq 5$ (up to sign). Consider the congruence classes modulo 24 of these integers as well as the congruence classes mod 24 of the powers of 2.

5	2
$5 \cdot 2 = 10$	$2^2 = 4$
$5 \cdot 2^2 = 20 \equiv -4 \mod 24$	$2^3 = 8$
$5 \cdot 2^3 = 40 \equiv -8 \mod 24$	$2^4 \equiv -8$
$5 \cdot 2^4 = 80 \equiv 8 \mod 24$	$2^5 \equiv 8$
$5 \cdot 2^5 = 160 \equiv -8 \mod 24$	$2^6 \equiv -8$

Thus no reduced τ_{24}-factorization of -320 containing a factor of 5 or 10 can exist.

The factorizations $-320 = 160 \cdot (-2) \equiv (-8) \cdot (-2) \mod 24$ and $-320 = (-160) \cdot 2 \equiv 8 \cdot 2 \mod 24$ are not reduced τ_{24}-factorizations since $8 \not\equiv 2 \mod 24$.

Similarly, the factorizations

$$-320 = 80 \cdot (-4) \equiv 8 \cdot (-4) \mod 24$$

$$-320 = (-80) \cdot 4 \equiv (-8) \cdot 4 \mod 24$$

$$-320 = 80 \cdot (-2) \cdot 2 \equiv 8 \cdot (-2) \cdot 2 \mod 24$$

$$-320 = (-80) \cdot 2 \cdot 2 \equiv (-8) \cdot 2 \cdot 2 \mod 24$$

are not reduced τ_{24}-factorizations. Therefore no reduced τ_{24}-factorizations of -320 containing a factor of ± 80 exist.

Recall that $-320 = (-1) \cdot 20 \cdot (-4) \cdot (-4) \equiv (-1) \cdot (-4) \cdot (-4) \cdot (-4) \mod 24$ is not a reduced τ_{24}-factorization. Any factorization of -320 with no leading -1 and containing a factor of ± 20 must contain a factor of the form $\pm 2^j \neq -4$ with $1 \leq j \leq 4$. Thus there are no reduced τ_{24}-factorizations of -320 containing a factor of ± 20.

Finally, we have already noted that two reduced τ_{24}-factorizations of -320 containing ± 40 exist, i.e. $-320 = (-8) \cdot 40$ and $-320 = 8 \cdot (-40)$. These were not reduced τ_{24}-atomic factorizations. The other factorizations of -320 with no leading -1 and containing a factor of ± 40 also contain a factor of ± 2 or ± 4 which are not congruent to $40 \equiv -8 \mod 24$.

Lemma 3.27. *$\mathbb{Z}$ is not ${}_r\tau_{30}$-atomic.*

Proof. Consider the factorizations $-81 = (-3) \cdot 27$ and $-81 = 3 \cdot (-27)$. Since $-3 \equiv 27 \mod 30$, these are ${}_r\tau_{30}$-factorizations. Since ± 27 is a cube, it is not a ${}_r\tau_{30}$-atom. Hence the two factorizations are not ${}_r\tau_{30}$-atomic factorizations. Since $9 \not\equiv -9$

mod 30, $3 \not\equiv -3 \mod 30$ and $3 \not\equiv \pm 9 \mod 30$ the remaining factorizations of -81 are not ${}_r\tau_{30}$-factorizations.

Lemma 3.28. *$\mathbb{Z}$ is not ${}_r\tau_{40}$-atomic.*

Proof. Modulo 40, $32 \equiv -8$. Thus $(-8)\cdot 32$ and $8\cdot(-32)$ are reduced τ_{40}-factorizations of -256. These factorizations are not reduced τ_{40}-atomic since as powers of -2 and 2 the factors are not reduced τ_{40}-atoms. We will show that no other reduced τ_{40}-factorizations of $-256 = -2^8$ exist (up to order). Consider the powers of 2 mod 40:

$$\begin{array}{l} 2 \\ 2^2 = 4 \\ 2^3 = 8 \\ 2^4 = 16 \\ 2^5 \equiv -8 \end{array}$$

Any reduced τ_{40}-factorization of -256 must contain a factor of the form -2^j with $j \geq 1$. Since $2, 4 \not\equiv -2^j$ for any j, 2 and 4 cannot appear as factors in a reduced τ_{40}-factorization of -256.

Any factorization of $-256 = -2^8$ without a leading -1 can contain at most seven factors of -2. (Recall that $-256 = (-1)\underbrace{(-2)\cdots(-2)}_{8}$ is not a reduced τ_{40}-factorization because of the leading -1.) Now $-2 \equiv \pm 2^j \mod 40$ if and only if $\pm 2^j = -2$. Since every factorization of -256 without a leading -1 must contain a factor other than -2, it follows that no reduced τ_{40}-factorizations of -256 containing a factor of -2 can exist.

The same type of argument can be applied to show that -4 cannot be a factor in a reduced τ_{40}-factorization of -256. Observe that -4 is the only integer of the form $\pm 2^j$ congruent to $-4 \mod 40$ and that any reduced τ_{40}-factorization of -256 can contain at most three factors of -4. (Recall that $-256 = (-1)(-4)(-4)(-4)(-4)$ is not a reduced τ_{40}-factorization.) When a factorization of -256 (with no leading -1) contains three factors of -4, the remaining factors must be 2 and 2, or -2 and -2, or simply 4. None are congruent to $-4 \mod 40$. When a factorization of -256 (without a leading -1) contains either one or two, but not three factors of -4, the remaining factor(s) are of the form $\pm 2^j$ and not equal to -4. Thus they are not congruent to $-4 \mod 40$.

The only factorizations left to consider are

$$-256 = 8 \cdot (-32), \qquad -256 = (-8) \cdot 32, \qquad \text{and} \qquad -256 = 16 \cdot (-16)$$

The third factorization above is not a reduced τ_{40}-factorization since $16 \not\equiv -16 \mod 40$. The remaining two factorizations are the only two non-trivial reduced τ_{40}-factorizations of -256. However, they are not reduced τ_{40}-atomic factorizations as previously discussed.

Lemma 3.29. *$\mathbb{Z}$ is not ${}_r\tau_{48}$-atomic.*

Proof. Similar to the proof of Lemma 3.26 $-320 = (-8) \cdot 40$ and $-320 = 8 \cdot (-40)$ are reduced τ_{48}-factorizations, although not reduced τ_{48}-atomic factorizations. Since 5 is a standard prime, it must divide one of the factors in a non-trivial reduced τ_{48}-

factorization of -320. Thus one of the factors must be ± 5, ± 10, ± 20, ± 40, ± 80 or ± 160. Below we list the congruence classes of the positive integers just mentioned and the powers of 2 modulo 48.

5	2
$5 \cdot 2 = 10$	$2^2 = 4$
$5 \cdot 2^2 = 20$	$2^3 = 8$
$5 \cdot 2^3 = 40 \equiv -8 \mod 48$	$2^4 = 16$
$5 \cdot 2^4 = 80 \equiv -16 \mod 48$	$2^5 \equiv -16 \mod 48$
$5 \cdot 2^5 = 160 \equiv 16 \mod 48$	$2^6 \equiv 16 \mod 48$

Since no power of 2 (or associate) is congruent to 5, 10 or 20 modulo 48, the multiple of 5 that we seek in a reduced τ_{48}-factorization of -320 must be one of ± 40, ± 80 or ± 160.

The non-trivial factorizations $-320 = 160 \cdot (-2) \equiv 16 \cdot (-2) \mod 48$ and $-320 = (-160) \cdot 2 \equiv (-16) \cdot 2 \mod 48$ are not reduced τ_{48}-factorizations. Similarly, $-320 = 80 \cdot (-4) \equiv (-16) \cdot (-4) \mod 48$, $-320 = (-80) \cdot 4 \equiv 16 \cdot 4 \mod 48$, $-320 = 80 \cdot (-2) \cdot 2 \equiv 16 \cdot (-2) \cdot 2 \mod 48$ and $-320 = (-80) \cdot 2 \cdot 2 \equiv (-16) \cdot 2 \cdot 2 \mod 48$ are not reduced τ_{48}-factorizations.

Thus the only non-trivial reduced τ_{48}-factorizations of -320 contain a factor of ± 40. Since $40 \equiv -8 \mod 48$ the only reduced τ_{48}-factorizations of -320 are the ones previously mentioned. The remaining factorizations of -320 containing a factor of ± 40 have as factor(s) ± 2 or ± 4 which are not congruent to $\pm 40 \equiv \pm 8 \mod 48$.

Lemma 3.30. *$\mathbb{Z}$ is not ${}_r\tau_{60}$-atomic.*

Proof. The factorizations $544 = 68 \cdot 8$ and $-544 = (-68) \cdot (-8)$ are reduced τ_{60}-factorizations. We will show that no other non-trivial reduced τ_{60}-factorizations of $544 = 17 \cdot 2^5$ exist.

Since 17 is prime one of the factors in a non-trivial reduced τ_{60}-factorizations of 544 must be divisible by 17. Thus this factor must be one of the ones listed below on the left. On the right we list the congruence classes of the powers of 2 modulo 60.

17	2
$17 \cdot 2 = 34$	$2^2 = 4$
$17 \cdot 2^2 = 68 \equiv 8 \mod 60$	$2^3 = 8$
$17 \cdot 2^3 = 136 \equiv 16 \mod 60$	$2^4 = 16$
$17 \cdot 2^4 = 272 \equiv 32 \equiv -28 \mod 60$	$2^5 \equiv 32 \equiv -28 \mod 60$

Since 544 is the product of 17 and powers of 2 and the powers of 2 and their associates are not congruent to 17 or 34 modulo 60, any reduced τ_{60}-factorization of 544 cannot contain a factor of 17 or 34. Thus the multiple of 17 in a reduced τ_{60}-factorization of 544 must be one of ± 68, ± 136 or ± 272.

The factorizations $544 = 272 \cdot 2 \equiv 32 \cdot 2 \mod 60$ and $544 = (-272) \cdot (-2) \equiv (-32) \cdot (-2) \mod 60$ are not reduced τ_{60}-factorizations.

Similarly, any factorization containing a factor of ± 136 must also contain the factors ± 2 or ± 4 which are not congruent to $\pm 136 \equiv \pm 16 \mod 60$.

The aforementioned reduced τ_{60}-factorizations of 544 contain a factor of ± 68. The remaining factorizations of 544 containing a factor of ± 68 also contain factors of ± 2 and possibly ± 4 which are not congruent to $\pm 68 \equiv \pm 8 \mod 60$.

Lemma 3.31. *$\mathbb{Z}$ is not ${}_r\tau_{80}$-atomic.*

Proof. The reduced τ_{80}-factorizations $704 = 88 \cdot 8$ and $704 = (-88) \cdot (-8)$ are not reduced τ_{80}-atomic factorizations since ± 8 is a cube. We will show that no other reduced τ_{80}-factorizations of $704 = 11 \cdot 2^6$ exist. Since the standard prime 11 divides 704 one of the factors in a reduced τ_{80}-factorizations of 704 must be of the form $11 \cdot 2^k$ with $1 \leq k \leq 6$ (or an associate). Below we list the congruence classes of such integers and the congruence classes of the relevant powers of 2.

11	2
$11 \cdot 2 = 22$	$2^2 = 4$
$11 \cdot 2^2 = 44 \equiv -36 \mod 80$	$2^3 = 8$
$11 \cdot 2^3 = 88 \equiv 8 \mod 80$	$2^4 = 16$
$11 \cdot 2^4 = 176 \equiv 16 \mod 80$	$2^5 = 32$
$11 \cdot 2^5 = 352 \equiv 32 \mod 80$	$2^6 = 64 \equiv -16 \mod 80$

Since 11, 22 and 44 are not congruent to $\pm 2^k$ for $1 \leq k \leq 6$, they cannot appear in a reduced τ_{80}-factorization of 704.

The factorizations containing a factor of $\pm 352 \equiv \pm 32 \mod 80$ or $\pm 176 \equiv \pm 16 \mod 80$ necessarily also contain a factor of ± 2 or possibly ± 4 in the latter case. Thus these factorizations are not reduced τ_{80}-factorizations.

The two reduced τ_{80}-factorizations of 740 previously mentioned contain a factor of $88 \equiv 8 \mod 80$ or $-88 \equiv -8 \mod 80$. The remaining factorizations of 740 containing either factor must also contain ± 2 (some also contain ± 4). Therefore $704 = 88 \cdot 8$ and $704 = (-88) \cdot (-8)$ are the only reduced τ_{80}-factorizations of 704.

Lemma 3.32. *$\mathbb{Z}$ is not ${}_r\tau_{120}$-atomic.*

Proof. Consider the following factorizations of 1984:

$$1984 = 248 \cdot 8 = (-248) \cdot (-8)$$

$$1984 = 124 \cdot 4 \cdot 4$$

Since $248 \equiv 8 \mod 120$ and $124 \equiv 4 \mod 120$ the three factorizations are reduced τ_{120}-factorizations. None are reduced τ_{120}-atomic factorizations given that ± 8 is a cube and 4 is a square. We will show that no other reduced τ_{120}-factorizations of 1984 exist. Since 31 is a standard prime that divides $1984 = 31 \cdot 2^6$, it must divide one of the factors in a reduced τ_{120}-factorization of 1984. In order to explore the potential reduced τ_{120}-factorizations of 1984 it is then useful to list the positive integers of the form $2^k \cdot 31$ with $0 \leq k \leq 5$ and their congruence classes, as well as the congruence classes of the powers of 2 modulo 120.

31	2
$31 \cdot 2 = 62 \equiv -58 \mod 120$	$2^2 = 4$
$31 \cdot 2^2 = 124 \equiv 4 \mod 120$	$2^3 = 8$
$31 \cdot 2^3 = 248 \equiv 8 \mod 120$	$2^4 = 16$
$31 \cdot 2^4 = 496 \equiv 16 \mod 120$	$2^5 = 32$
$31 \cdot 2^5 = 992 \equiv 32 \mod 120$	$2^6 = 64 \equiv -56 \mod 120$

Since $\pm 2^k \not\equiv 31 \mod 120$ for $1 \leq k \leq 6$, any reduced τ_{120}-factorization of 1984 will not contain a factor of ± 31. Similarly, since $\pm 2^k \not\equiv 62 \mod 120$ for $1 \leq k \leq 5$, no factorization of 1984 containing a factor of ± 62 is a reduced τ_{120}-factorization. The factorizations of 1984 containing a factor of ± 992 are not reduced τ_{120}-factorizations since $992 = 31 \cdot 2^5 \equiv 32 \not\equiv \pm 2 \mod 120$. Also, the factorizations of 1984 containing a factor of ± 496 (where $496 = 31 \cdot 2^4$) are not reduced τ_{120}-factorizations because $496 \equiv 16 \not\equiv 2, 4 \mod 120$.

The reduced τ_{120}-factorizations containing a factor of 248 or -248 are $1984 = 248 \cdot 8$ and $1984 = (-248) \cdot (-8)$. The remaining factorizations of 1984 containing a factor of ± 248 are not reduced τ_{120}-factorizations since they must contain factors of ± 2 and possibly ± 4 which are not congruent to ± 248.

Finally, the only reduced τ_{120}-factorization of 1984 containing a factor of either 124 or -124 is $1984 = 124 \cdot 4 \cdot 4$ (up to order of factors). When we introduce two negative signs the resulting factorizations are not reduced τ_{120}-factorizations since $-124 \not\equiv 4 \mod 120$ and $-4 \not\equiv 4 \mod 120$. The factorizations of 1984 containing as factors ± 124 and also ± 2 and possibly ± 8 are not reduced τ_{120}-factorizations since $124 \not\equiv \pm 2, \pm 8 \mod 120$.

Lemma 3.33. *$\mathbb{Z}$ is not ${}_r\tau_{240}$-atomic.*

Proof. Consider the factorizations $1984 = 248 \cdot 8$ and $1984 = (-248) \cdot (-8)$. These are reduced τ_{240}-factorizations, although not reduced τ_{240}-atomic factorizations given that ± 8 is a cube. We will show that no other reduced τ_{240}-factorizations of 1984 exist. Since $1984 = 31 \cdot 2^6$ one of the factors in a reduced τ_{240}-factorization of 1984 must be divisible by 31. Below we list the congruence classes of such factors and also the congruence classes of the powers of 2.

31	2
$31 \cdot 2 = 62$	$2^2 = 4$
$31 \cdot 2^2 = 124 \equiv -116 \mod 240$	$2^3 = 8$
$31 \cdot 2^3 = 248 \equiv 8 \mod 240$	$2^4 = 16$
$31 \cdot 2^4 = 496 \equiv 16 \mod 240$	$2^5 = 32$
$31 \cdot 2^5 = 992 \equiv 32 \mod 240$	$2^6 = 64$

Given that the powers of 2 are not congruent to ± 31, ± 62 or ± 124, no reduced τ_{240}-factorization of 1984 exists containing these factors.

The factorizations of 1984 that contain a factor of $\pm 992 \equiv 32 \mod 240$ or $\pm 496 \equiv 16 \mod 240$ must also contain a factor of ± 2 or possibly ± 4. Thus these factorizations are not reduced τ_{240}-factorizations.

Finally, the two reduced τ_{240}-factorizations of 1984 contain the factors 248 and -248. The remaining factorizations of 1984 that contain either factor also contain a factor of ± 2 (some also contain ± 4). These other factorizations are also not reduced τ_{240}-factorizations.

CHAPTER 4
THE τ_n-FACTORIZATIONS IN $\mathbb{N}$

This chapter describes the τ_n-factorizations in $\mathbb{N}$. Using the methods from the previous chapter we will prove the following results.

Theorem 4.1. $\mathbb{N}$ *is a* τ_n*-UFD if and only if* $n = 0$ *or 1.*

Theorem 4.2. $\mathbb{N}$ *is* τ_n*-atomic if and only if* $n = 0$*, 1, 2, 3, 4, 6.*

Proving Theorem 4.2 is far less challenging than its analogues in the τ_n- and reduced τ_n-factorizations in $\mathbb{Z}$. However, describing the τ_n-atoms in $\mathbb{N}$ is a more elaborate process than before.

4.1 The τ_n-factorization in $\mathbb{N}$ for $n = 0, 1, 2$

Lemma 4.3. $\mathbb{N}$ *is a* τ_0*-UFD.*

Proof. Let a be a natural number greater than 1. By the Fundamental Theorem of Arithmetic a can be written as $a = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$ where $p_1, \ldots, p_t$ are distinct standard primes and $k_1, \ldots, k_t \geq 1$. If $\gcd(k_1, k_2, \ldots, k_t) = 1$ then a is a τ_0-atom. Otherwise, let $d = \gcd(k_1, k_2, \ldots, k_t) > 1$. Then $a = \left(p_1^{k_1/d} p_2^{k_2/d} \cdots p_t^{k_t/d}\right)^d$ is a τ_0-atomic factorization. Thus $\mathbb{N}$ is τ_0-atomic.

A τ_0-atomic factorization of a natural number a greater than 1 is unique by the uniqueness of prime factorizations in $\mathbb{N}$ and the uniqueness of the gcd.

The τ_1-factorization of the natural numbers is just the usual factorization. The result below follows the Fundamental Theorem of Arithmetic.

Lemma 4.4. $\mathbb{N}$ *is a* τ_1*-UFD.*

Just as with the τ_n-factorizations and reduced τ_n-factorizations of $\mathbb{Z}$, $\mathbb{N}$ is not a τ_n-UFD for $n \geq 2$. By Dirichlet's Theorem there exist $p_1, \ldots, p_t$ distinct standard primes congruent to 1 modulo n with $t > 1$. Then $2 \cdot (2p_1 p_2 \cdots p_t)$ and $(2p_1) \cdot (2p_2 \cdots p_t)$ are both τ_n-atomic factorizations of $4p_1 \cdots p_t$.

Lemma 4.5. $\mathbb{N}$ *is* τ_2*-atomic.*

Proof. The τ_2-atoms are the standard primes as well as the natural numbers of the form $2m$ with m odd. An odd natural number of the form $p_1 p_2 \cdots p_t$ with p_i not necessarily distinct odd standard primes has τ_2-atomic factorization $p_1 \cdot p_2 \cdots p_t$. A natural number of the form $2^k m$ with $k \geq 2$ and m odd has τ_2-atomic factorization $\underbrace{2 \cdots 2}_{k-1} \cdot (2m)$.

4.2 The τ_n-factorization in $\mathbb{N}$ for $n \geq 3$

For $n \geq 3$ we observe similarities between the τ_n-atoms in $\mathbb{N}$ and the reduced τ_n-atoms in $\mathbb{Z}$.

Lemma 4.6. $\mathbb{N}$ *is* τ_3*-atomic.*

Proof. The standard primes not equal to 3 are either congruent to 1 modulo 3 or 2 modulo 3. If $p_1, p_2, \ldots, p_t, q$ are standard primes with $p_i \equiv 1 \mod 3$ for $1 \leq i \leq t$ and $q \equiv 2 \mod 3$, then $qp_1 \cdots p_t$ is a τ_3-atom. The natural numbers of the form $3m$ where $3 \nmid m$ are also τ_3-atoms.

Natural numbers of the form $p_1p_2\cdots p_tq_1q_2\cdots q_s$ where $p_1,\ldots,p_t,q_1,\ldots,q_s$ are standard primes with $s>1$, $p_i\equiv 1 \mod 3$ for $1\le i\le t$ and $q_j\equiv 2 \mod 3$ for $1\le j\le s$, have τ_3-atomic factorization $(q_1p_1p_2\cdots p_t)\cdot q_2\cdots q_s$.

Positive integers of the form 3^km where $3\nmid m$ and $k>1$ have τ_3-atomic factorizations $(3m)\cdot\underbrace{3\cdots 3}_{k-1}$.

Lemma 4.7. $\mathbb{N}$ *is* τ_4*-atomic.*

Proof. Except for 2, the standard primes are in either $\bar{1}$ or $\bar{3}$ of the congruence classes modulo 4.

If $a\in\mathbb{N}$ is of the form $2m$ where m is odd, then a is in $\bar{2}$ and is a τ_4-atom. If $a=2^k\cdot m$ with m odd and $k>1$, then $a\equiv 0 \mod 4$ and a has the τ_4-atomic factorization $\underbrace{2\cdot 2\cdots 2}_{k-1}(2m)$.

If $a=qp_1\cdots p_s$ with p_i and q standard primes in $\bar{1}$ and $\bar{3}$, respectively, then a is a τ_4-atom. If $a=p_1\cdots p_sq_1\cdots q_t$ with p_i and q_j as before, then a has the τ_4-atomic factorization $q_1\cdots q_{t-1}(q_tp_1\cdots p_s)$.

The τ_4-atoms $qp_1\cdots p_s$ replace the ${}_r\tau_6$-atoms pq where p, q were standard primes or associates so that $-q\equiv p\equiv 1 \mod 6$.

Lemma 4.8. $\mathbb{N}$ *is not* τ_5*-atomic.*

Proof. Consider the factorization $116=4\cdot 29$. Since $4\equiv 29 \mod 5$, this is a τ_5-factorization. However, it is not a τ_5-atomic factorization since $4=2\cdot 2$ is not a τ_5-atom.

Lemma 4.9. $\mathbb{N}$ *is* τ_6*-atomic.*

Proof. The standard primes of $\mathbb{N}$ lie in the following congruence classes modulo 6:

- 2 is the only prime in the class $\bar{2}$ and there are no primes in $\bar{4}$
- 3 is the only prime in the class $\bar{3}$
- all other primes are in $\bar{1}$ or $\bar{5}$

The τ_6-atoms congruent to 0 modulo 6 are numbers of the form $6m$ where $6 \nmid m$. (This includes 6.) If $a \in \mathbb{N}$ is congruent to 0 modulo 6, then a is of the form $a = 6^k m$ where $6 \nmid m$ and $k \geq 1$. Thus, $\underbrace{6 \cdots 6}_{k-1}(6m)$ is a τ_6-atomic factorization of a.

Similarly, numbers of the form $3m$ where $2, 3 \nmid m$ are τ_6-atoms congruent to 3 modulo 6. If a is a natural number congruent to 3 modulo 6, $a = 3^k m$ where $2, 3 \nmid m$ and $k \geq 1$. A τ_6-atomic factorization of a is $a = \underbrace{3 \cdots 3}_{k-1}(3m)$.

The τ_6-atoms in $\bar{4}$ are numbers of the form $2p_1 \cdots p_m q_1 \cdots q_{2s+1}$.

The τ_6-atoms in $\bar{2}$ are:

- 2 and numbers of the form $2p_1 \cdots p_m$
- $2p_1 \cdots p_m q_1 \cdots q_{2s}$
- $2 \cdot (2p_1 \cdots p_m q_1 \cdots q_{2s+1}) \equiv 2 \cdot 4 = 8 \equiv 2 \mod 6$

where p_i and q_j are standard primes with $p_i \equiv 1 \mod 6$ and $q_j \equiv 5 \mod 6$.

If $a = 2^k p_1 \cdots p_m q_1 \cdots q_{2s}$ with $k \geq 2$, then $a \equiv 2 \mod 6$ and a has the τ_6-atomic factorization $a = \underbrace{2 \cdots 2}_{k-1}(2p_1 \cdots p_m q_1 \cdots q_{2s})$.

If $a = 2^k p_1 \cdots p_m q_1 \cdots q_{2s+1}$, then a is in $\bar{2}$ or $\bar{4}$. If $k \leq 2$, then a is a τ_6-atom as previously discussed. When $k > 2$, $\underbrace{2 \cdots 2}_{k-2}(2 \cdot (2p_1 \cdots p_m q_1 \cdots q_{2s+1}))$ is a τ_6-atomic factorization of a.

As in the case $n = 4$, $qp_1 \cdots p_m$ is a τ_6-atom and $a = p_1 \cdots p_m q_1 \cdots q_s$ has the τ_6-factorization

$q_1 \cdots q_{s-1}(q_s p_1 \cdots p_m)$.

The τ_6-atoms of the form $qp_1 \cdots p_m$ are both similar and different to the ${}_r\tau_6$-atoms pq where p and q were standard primes or associates of standard primes so that $-q \equiv p \equiv 1 \mod 6$. (Recall that neither of these were atoms in the τ_6-factorization in $\mathbb{Z}$). The τ_6-atoms $2 \cdot 2m$ where $m \equiv -1 \mod 6$ occur in both the τ_6-factorization in $\mathbb{N}$ and the ${}_r\tau_6$-factorization of $\mathbb{Z}$.

Lemma 4.10. *If $U(n)$ contains an element of order at least 3, then $\mathbb{N}$ is not τ_n-atomic.*

Proof. Let x be an element of $U(n)$ of order at least 3. Since $x \in U(n)$, x is relatively prime to n. By Dirichlet's Theorem there exists a prime p congruent to x mod n. So we may assume that x is prime. Since x is relatively prime to n, so is x^2. By Dirichlet's Theorem there exists a prime y congruent to x^2 mod n. Consider the factorizations:

$$\begin{aligned} yx^2 &= y \cdot x^2 \\ &= y \cdot x \cdot x \\ &= xy \cdot x \end{aligned}$$

The first factorization is a τ_n-factorization. However, since x^2 is a square, it is not a τ_n-atomic factorization. Since x has order at least 3, $x \not\equiv y \equiv x^2 \mod n$ since otherwise

$1 \equiv x \mod n$. So the second factorization is not a τ_n-factorization. Finally, the third factorization is not a τ_n-factorization because $xy \equiv x \mod n$ implies $x^3 \equiv x \mod n$ or $x^2 \equiv 1 \mod n$ which is impossible given that x has order at least 3.

According to Table 3.1 $\mathbb{N}$ is not τ_n-atomic except possibly when $n = 0-4, 6, 8, 12, 24$. Having already studied the cases $n \leq 6$, we state the result about the remaining cases below.

Lemma 4.11. *$\mathbb{N}$ is not τ_n-atomic for $n \geq 7$ except possibly for $n = 8, 12, 24$.*

Lemma 4.12. *$\mathbb{N}$ is not τ_8-atomic.*

Proof. Consider the factorizations of 48:

$$\begin{aligned}
48 &= 4 \cdot 12 \\
&= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \\
&= 4 \cdot 2 \cdot 2 \cdot 3 \\
&= 6 \cdot 2 \cdot 2 \cdot 2 \\
&= 8 \cdot 2 \cdot 3 \\
&= 12 \cdot 2 \cdot 2 \\
&= 4 \cdot 4 \cdot 3 \\
&= 4 \cdot 2 \cdot 6 \\
&= 2 \cdot 24 \\
&= 6 \cdot 8
\end{aligned}$$

Only the first factorization is a τ_8-factorization. Since 4 is a square, it is not a τ_8-atom. Thus no τ_8-atomic factorizations of 48 exist.

Lemma 4.13. $\mathbb{N}$ *is not* τ_{12}*-atomic.*

Proof. Only the following factorization of $432 = 2^4 \cdot 3^3$ is a τ_{12}-factorization:

$$432 = 12 \cdot 36$$

Given that $3^2 = 9$, $3^3 \equiv 3 \mod 12$ and $2^k \not\equiv 3, 9 \mod 12$ for $k \geq 1$, no τ_{12}-factorizations of 432 exist containing 3 or 9 as factors. Since $6 \equiv 3^k 6 \mod 12$ with $k \geq 0$ and there are more powers of 2 in 432 than of 3, there cannot be any τ_{12}-factorizations of 432 with one or more factors of 6. Finally, the only τ_{12}-factorization of 432 containing a factor of 12 is the one previously mentioned. Since 36 is a square, it is not a τ_{12}-atomic factorization.

Lemma 4.14. $\mathbb{N}$ *is not* τ_{24}*-atomic.*

Proof. Similar to the previous proof, only the following factorization of $432 = 2^4 \cdot 3^3$ is a τ_{24}-factorization:

$$432 = 12 \cdot 36$$

Given that $3^2 = 9$, $3^3 \equiv 3 \mod 12$ and $2^k \not\equiv 3, 9 \mod 24$ for $k \geq 1$, no τ_{24}-factorizations of 432 exist containing 3 or 9 as factors. Since $6 \equiv 3^{2k} 6 \mod 24$ with $k \geq 0$ and there are more powers of 2 in 432 than of 3, there cannot be any τ_{12}-factorizations of 432 with one or more factors of 6. Finally, the only τ_{12}-factorization of 432 containing a factor of 12 is the one previously mentioned. Since 36 is a square, it is not a τ_{12}-atomic factorization. Finally, $432 = 24 \cdot 18$, $432 = 24 \cdot 9 \cdot 2$, $432 = 24 \cdot 6 \cdot 3$, and $432 = 24 \cdot 2 \cdot 3 \cdot 3$ are not τ_{24}-factorizations.

CHAPTER 5
FUTURE WORK

5.1 Future Work

One of the questions on τ_n-factorizations and reduced τ_n-factorizations that has yet to be answered is whether for some n there exists an integer with a non-trivial reduced τ_n-atomic factorization but no non-trivial τ_n-atomic factorization. We know the converse to be true, for example -4 has the non-trivial τ_n-factorization $-4 = (-1) \cdot 2 \cdot 2$, but is a reduced τ_n-atom for $n = 3$ and at least 5. Since -4 is just one of the many examples of reduced τ_n-atoms that are not τ_n-atoms, it is plausible that an integer could have a reduced τ_n-atomic factorization where some factor is not a τ_n-atom and no τ_n-atomic factorization. It would also be interesting to find if such examples occur with values of n for which $\mathbb{Z}$ is both τ_n-atomic and reduced τ_n-atomic.

Other areas of interest include the study of factorizations in $\mathbb{Z}[x]$ where the coefficients must be τ_n-related.

REFERENCES

[1] D.D. Anderson, D.F. Anderson, and M. Zafrullah. Factorization in integral domains. *J. Pure Appl. Algebra*, 69:1–19, 1990.

[2] D.D. Anderson and A. Frazier. On a general theory of factorization in integral domains. *Rocky Mountain J. Math.*, 41:663–706, 2011.

[3] S.M. Hamon. *Some topics in τ-factorizations*. PhD thesis, The University of Iowa, 2007.

[4] I. Kaplansky. *Commutative Rings*. University of Chicago Press, Chicago, Ill.-London, revised edition, 1974.

[5] M.B. Nathanson. *Elementary Methods in Number Theory*. Springer, New York, 2000.