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Operations on Infinite x Infinite Matrices and Their Use in Dynamics and Spectral Theory

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OPERATIONS ON INFINITE $\times$ INFINITE MATRICES, AND THEIR USE IN DYNAMICS AND SPECTRAL THEORY

by

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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To
Mom and Dad

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ABSTRACT

By first looking at the orthonormal basis:

$$\Gamma = \{\sum_i 4^i b_i : b_i \in \{0, 1\}, \text{ finite sums } \}$$

and the related orthonormal basis:

$$5\Gamma = \{5 \sum_i 4^i b_i : b_i \in \{0, 1\}, \text{ finite sums } \}$$

we find several interesting relationship with the unitary matrix $U_{\alpha,\beta}$ arising from the operator $U : \Gamma \to 5\Gamma$. Further, we investigate the relationships between U and the operators $S_0 : \Gamma \to 4\Gamma$ defined by $S_0 e_\gamma = e_{4\gamma}$ where $e_\gamma = e^{2\pi i \gamma x}$ and $S_1 : \Gamma \to 4\Gamma + 1$ defined by $S_1 e_\gamma = e_{4\gamma+1}$.

Most intriguing, we found that when taking powers of the aforementioned $U_{\alpha,\beta}$ matrix that although there are infinitely many 1's occurring in the entries of $U_{\alpha,\beta}$ only one such 1 occurs in the subsequent higher powers $U^k_{\alpha,\beta}$. This means that there are infinitely many $\gamma \in \Gamma \cap 5\Gamma$, but only one such γ in the intersection of Γ and $5^k\Gamma$ for $k \geq 2$.

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CHAPTER 1
INTRODUCTION

1.1 Motivation

The general theme of this thesis is the scaling in a Fourier duality of certain infinite Bernoulli convolutions. We will pay special attention to the scaling by $\frac{1}{4}$. The Bernoulli measure $\mu_{1/4}$ (from now on referred to as just μ) is supported on the Cantor set obtained by dividing the line segment $[0,1]$ into 4 equal intervals and retaining only the first and third intervals. The process is repeated infinitely many times. By [21], we know that $L^2(\mu)$ has an orthogonal Fourier basis, $\{e_\lambda : \lambda \in \Gamma\}$ where $\Gamma = \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0,1\}, \text{finite sums}\}$. The pair (μ, Γ) is called a spectral pair. Its rigid structure will be studied.

In earlier papers we learned that scaling this spectral pair by 5 opens up interesting spectral theoretic problems for the initial Fourier duality problem. It is surprising that if we scale the set Γ by 5, turning it into 5Γ that it is once again an orthogonal Fourier basis in $L^2(\mu)$. Therefore we can introduce a unitary operator U in $L^2(\mu)$, such that $U : \Gamma \to 5\Gamma$. The aim of this thesis is to study this operator's spectral properties as they relate to ergodic theory of the initial spectral pair. Specifically, if $\Gamma = \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0,1\}, \text{finite sums}\} = \{0, 1, 4, 5, \cdots\}$ (see [21]) is the natural Fourier basis in $L^2(\mu)$, then the orthonormal property is preserved under scaling by 5; meaning $5\Gamma = \{0, 5, 20, 25, \cdots\}$ is also a Fourier basis (see [11]). This is somewhat surprising since 5Γ seems "smaller" or more "thin" than Γ.

Although U is a unitary operator, it cannot be induced by a measure-preserving transformation in the measure space (X, μ). In fact, as it turns out, the spectral representation, and the spectral resolution, for U is surprisingly subtle. There is a large literature on spectral theory for affine dynamical systems. For example, we point to related papers, by Jorgensen with co-authors, S. Pedersen, D. Dutkay, K.A. Kornelson; J.-L. Li and others; see the reference list.

1.2 Overview

In Chapter 2, we will review the definitions and basic operations of infinite matrices. Although similar to the finite dimensional case, we must pay special attention to whether such operations are well-defined. Chapter 3 will expound upon the orthonormal basis $\Gamma = \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0, 1\}, \text{finite sums}\} = \{0, 1, 4, 5, \cdots\}$, focusing on the paper, [21], by P. Jorgensen and S. Pedersen. In Chapter 4, we will introduce four operators $S_0 : \Gamma \to 4\Gamma$, $S_1 : \Gamma \to 4\Gamma + 1$, $M_1 : \Gamma \to \Gamma + 1$ and $U : \Gamma \to 5\Gamma$ and discover relationships between their infinite matrix representations. Finally, in Chapter 5, we will focus solely on properties of U. The big theorem we will set out to prove is discovering where entries of 1 occur in the matrix $U_{\alpha,\beta}$ and powers of $U_{\alpha,\beta}$, $U^k_{\alpha,\beta}$.

CHAPTER 2
INFINITE MATRICES

In Chapters 4 and 5, we will be looking at $U_{\alpha,\beta}$, the infinite matrix representation of the operator $U : \Gamma \to 5\Gamma$ with respect to the orthonormal basis, $\Gamma = \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0,1\}, \text{finite sums}\}$. This chapter presents the definitions and lemmas needed to understand $U_{\alpha,\beta}$.

Henri Poincaré is given the credit of originating the theory of infinite matrices in 1884. The study was furthered by Helge von Koch (1893) and David Hilbert (1906). Back in the nineteenth century, mathematicians thought of infinite matrices in terms of determinants. Today, however, we think of these in terms of subspaces and linear operators. [For more information on the history of infinite matrices see [30] and [4]].

2.1 Inner Product

Before we start performing operations with infinite matrices, we need to establish some basic definitions and lemmas that we will be using throughout the following chapters. Let's first have a reminder of some basic properties of the inner product. The following definitions and propositions can be found in [24].

Definition 2.1. Let X be a vector space over $\mathbb{C}$. An *inner product* is a map $< \cdot, \cdot >$: $X \times X \to \mathbb{C}$ satisfying, for x, y and z in X and scalars $c \in \mathbb{C}$.

1. $< x, y >= \overline{< y, x >}$

2. $< x, x > \geq 0$ with $< x, x >= 0$ if and only if $x = 0$

3. $< x + y, z > = < x, z > + < y, z >$

4. $< cx, y > = c < x, y >$

The next Lemma is the Cauchy-Schwarz Inequality (see [7]), which we will use in the proof of the major theorem of this thesis, Theorem 5.12.

Lemma 2.2. *(Cauchy-Schwarz Inequality) If $< \cdot, \cdot >$ is an inner product on a vector space X, then for all x and y in X we have*

$$| < x, y > |^2 \leq < x, x > < y, y >$$

The norm of an element $x \in X$ can be defined in the following way:

Lemma 2.3. *(norm) If $< \cdot, \cdot >$ is an inner product on a vector space X, then*

$$||x|| := < x, x >^{1/2}$$

is a norm on X.

Proof. Proposition 1.15 in [24] □

The following theorem (especially parts (2), (3), and (5)) will also be used in the proof of theorems in Chapter 5.

Theorem 2.4. *If $\{e_n\}$ is an orthonormal sequence in a Hilbert space $\mathcal{H}$, then the following conditions are equivalent*

1. *$\{e_n\}$ is an orthonormal basis*

2. *If $h \in \mathcal{H}$ and $h \perp e_n$ for all n, then $h = 0$*

3. *For every* $h \in \mathcal{H}$, $h = \sum < e_n, h > e_n$*: equality here means the convergence in the norm of* $\mathcal{H}$ *of the partial sums to* h

4. *For every* $h \in \mathcal{H}$, *there exists complex numbers* a_n *so that* $h = \sum a_n e_n$

5. *For every* $h \in \mathcal{H}$, $\sum | < h, e_n > |^2 = ||h||^2$

6. *For all* h *and* g *in* $\mathcal{H}$, $\sum < h, e_n >< e_n, g >=< h, g >$

Proof. Theorem 1.33 in [24] □

With these properties in mind, we can now move on to defining infinite matrices.

2.2 From Bounded Linear Operators to Infinite Matrices

This section covers the basic definition of an infinite matrix and serves as the basis for all computation used in Chapters 4 and 5. To begin, in order to create a 'nice' infinite matrix (meaning one that is well-defined for such operations such as multiplication), we first need a (bounded) linear operator and an orthonormal basis. The following definitions from [27] and [30] describe a bounded linear operator and the basic definition of an infinite matrix:

Definition 2.5. A linear operator $A : X \to Y$ with X and Y normed spaces with norm

$$||A|| = \sup\{||Ax|| : x \in X, ||x|| \leq 1\}$$

is bounded if $||A|| < \infty$

Definition 2.6. Given a bounded linear operator A on a Hilbert space $\mathcal{H}$ and $\{e_n\}_{n\in\mathcal{I}}$ an orthonormal basis for $\mathcal{H}$, then the matrix that arises from A and the orthonormal basis is denoted $A_{m,n} = (a_{mn})$ where $a_{mn} =< e_m, Ae_n >$.

$$A_{m,n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where the inner product on $L^2(X, \mu)$ for a positive measure space (X, μ) is

$$< f, g >= \int_X \bar{f} g d\mu$$

Remark 2.7. In this paper, we will denote operators with capital letters, such as A, U or S_0. When referring to the infinite matrix that arises from these operators with respect to an orthonormal basis, we will use capital letters along subscripts, such as $A_{i,j}$, $U_{\alpha,\beta}$, or $S_{0\alpha,\beta}$. This not only distinguishes the operator from the infinite matrix, it has the added benefit of reiterating the importance of the orthonormal basis (which i, j or α, β come from). Finally, if we discuss a particular element of the matrix, we will use lower case letters with subscripts, such as a_{ij}, $u_{\alpha\beta}$ or $s_{0\alpha\beta}$. Again, i, j or α, β come from the orthonormal basis and refer to the (row, column) entry of the matrix.

2.2.1 Matrices and Unbounded Operators

Notice that in the above definition (Definition 2.6), we are defining how a matrix arises from a bounded operator. Yet, it is possible to have an infinite $\times$ infinite matrix that does not arise from a bounded operator (see [16],[1]).

The following is an example of such a case (see [1]):

Example 2.1. Consider the matrix where each $(i,j)th$ entry is $A_{i,j} = 1$

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & \cdots \\ 1 & 1 & 1 & \cdots & 1 & \cdots \\ 1 & 1 & 1 & \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2.1}$$

Notice that if we take $x = \{1, 0, 0, \cdots\}$ we have that $||x|| \leq 1$ but

$$||Ax|| = \sqrt{1^2 + 1^2 + \cdots + 1^2 + \cdots} \to \infty$$

so the operator A is unbounded. ⊞

Matrices arising from unbounded operators will cause problems in the area of multiplication, as we will see later.

2.3 Boundedness of a Matrix

How can we tell if a matrix arises from a bounded operator? The answer is not succinct. For instance, it is *necessary* for each row and each column of the matrix to be square summable (in other words all rows and columns must be in ℓ^2). This is because if we let $e_i = \{0, 0, \cdots, 1, 0 \cdots 0\}$ be a vector with 1 in the i-th position, then if A is a bounded operator, by Definition 2.5 $||Ae_i|| < \infty$ and Ae_i refers to the i-th column. So each column must be in ℓ^2. Similarly, if A is bounded then A^* is bounded, so each row is also in ℓ^2.

Although it is a necessary condition, it is not *sufficient* as we can see in the following example from [16].

Example 2.2. Consider the matrix

$$A_{i,j} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 2 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 3 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where, for each row n, the only non-zero entry is n on the diagonal. Note that each row and each column is square summable. If we let the vector $f = \left(0, \frac{1}{2}, 0, \frac{1}{4}, 0, 0, 0, \frac{1}{8}, 0 \cdots\right)$, where for $m = 1, 2, 3, \cdots$ each 2^m term is $\frac{1}{2^m}$, we have that $||f|| \leq 1$ but $||A_{i,j}f|| = \infty$.

⊞

Now that we have a necessary condition for a matrix to arise from a bounded operator, what is a sufficient condition? In [16], a sufficient condition is stated that for a matrix (a_{ij}) to arise from a bounded operator A we must have $\sum_i \sum_j |a_{ij}|^2 < \infty$. To see this, consider a vector f with $||f|| \leq 1$. We have that

$$|\sum_j a_{ij} < f, e_j > |^2 \leq \sum_j |a_{ij}|^2 ||f||^2$$

for each i and f. Then, by taking the sum over all the i's,

$$||\sum_i \sum_j a_{ij} < f, e_j > e_i||^2 \leq \sum_i \sum_j |a_{ij}|^2 \cdot ||f||^2.$$

Since we have that $\sum_i \sum_j |a_{ij}|^2 < \infty$ and $||f|| \leq 1$, we get that $||A|| \leq \infty$. However, this is not necessary as is seen in the identity matrix.

2.3.1 Hilbert Matrix

As we have seen, when given an infinite matrix, it is difficult to tell whether it arises from a bounded operator or not. The Hilbert matrix

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & 0 & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & 0 & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & 0 & \cdots \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is probably the most famous example of a matrix that arises from a bounded operator. In fact, it arises from an operator A with $||A|| \leq \pi$ (the details can be seen in [16] and a different proof can be seen in [5]). Also of interest, is the matrix referred to as the exponential Hilbert matrix.

Example 2.3. The exponential Hilbert matrix (as it is called in [16]) is an example of a matrix with a bounded operator:

$$a_{ij} = 2^{-(i+j+1)}, \text{ where } i, j = 0, 1, 2, \cdots.$$

It is also a Hankel matrix of the form $\dfrac{1}{x+1}$ (a Hankel matrix is a matrix $A_{i,j} = (a_{ij})$ where $a_{ij} = i + j$). To find the norm of this matrix, consider that the rows, r_i, of the matrix are all multiples of $r_0 = (2^{-(0+j+1)})_j = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\right)$. Then $Ar = 2 < r, r_0 > r_0$ and $||A|| = 2||r_0||^2 = 2\sum \frac{1}{4^n} = \frac{2}{3}$ (see [16] for more information). $\boxplus$

With these basic definitions in hand, we are now ready to start making computations with these infinite matrices.

2.4 Basic Operations on Infinite Matrices

The basic operations of infinite matrices that we will be using in the last chapter are multiplication, inverse, and the conjugate transpose. Although we would like infinite matrices to work exactly like finite matrices, there are some complications to consider such as existence.

2.4.1 Conjugate and Transpose

Remark 2.8. We will be using the notation $\overline{A}_{i,j} = (\overline{a}_{ij})$ to represent the conjugate of the matrix $A_{i,j}$, $A'_{i,j} = (a_{j,i}) = A_{j,i}$ to represent the transpose of the matrix $A_{i,j}$ and $A^*_{i,j}$ to represent the conjugate transpose of $A_{i,j}$, where $A^*_{i,j} = (\overline{a}_{ji})$.

Two other definitions that come up later on in this thesis are Hermitian and symmetric matrices (both definitions are from [7]):

Definition 2.9. An infinite matrix $A_{i,j} = (a_{ij})$ is said to be symmetric if $a_{ij} = a_{ji}$

Example 2.4. An example of a symmetric matrix in infinite dimensions is the Hilbert matrix. The infinite Hilbert Matrix is made up of reciprocals of natural numbers,

$$a_{i,j} = \frac{1}{i+j+1} \tag{2.2}$$

with $i, j = 0, 1, 2, \cdots$:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & 0 & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & 0 & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & 0 & \cdots \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

⊞

An extension of symmetric matrices into $\mathbb{C}$-space are Hermitian matrices.

Definition 2.10. $A_{i,j}$ is said to be Hermitian if $A_{i,j}^* = A_{i,j}$

Example 2.5. An example of a Hermitian matrix in finite-dimensional space is

$$\begin{pmatrix} 1 & 1+i & 2+i \\ 1-i & 2 & 3+i \\ 2-i & 3-i & 3 \end{pmatrix}$$

Notice that in the case of Hermitian, the diagonal entries must be real since they must equal their own conjugate ($a_{ii} = \overline{a}_{ii}$). ⊞

Example 2.6. The (infinite and finite) identity matrix is an example of both a symmetric and a Hermitian matrix. ⊞

2.4.2 Addition

The addition of infinite matrices is exactly like the arithmetic of finite matrices. Let $A_{i,j} = (a_{ij})$ where a_{ij} represents the $(i,j)th$ entry of the infinite matrix $A_{i,j}$ and $B_{i,j} = (b_{ij})$ where b_{ij} represents the $(i,j)th$ entry of the infinite matrix $B_{i,j}$, then we can add component wise with matrix $(A+B)_{i,j} = (c_{ij})$ can be determined in the $(i,j)th$ entry as

$$c_{ij} = a_{ij} + b_{ij} \tag{2.3}$$

We do not have to worry whether or not $A_{i,j}$ and $B_{i,j}$ arise from unbounded or bounded operators since if $A_{i,j}$ and $B_{i,j}$ exist then $(A+B)_{i,j}$ exists. Here, *exist* means that each entry is finite.

2.4.3 Multiplication

Multiplication will be the operation we will use the most in discovering relationships with the matrix $U_{\alpha,\beta}$. Multiplying infinite matrices is done in a similar fashion to multiplying finite matrices. Given the same matrices $A_{i,j}$ and $B_{i,j}$ as above, multiplication can be defined component wise like multiplication of finite matrices with the $(i,j)th$ entry of matrix $(AB)_{i,j}$ as

$$\sum_k a_{i,k} b_{k,j} \tag{2.4}$$

Although this would seem to be just an extension of the finite multiplication operation, for infinite matrices we need to be concerned with multiplication being well defined. It is possible that the matrices $A_{i,j}$ and $B_{i,j}$ exist, but $(AB)_{i,j}$ does not exist (in other words $\sum_k a_{ik} b_{kj}$ diverges). The following example is an example of such a case:

Example 2.7. Consider

$$A_{i,j} = B_{i,j} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & \cdots \\ 1 & 1 & 1 & \cdots & 1 & \cdots \\ 1 & 1 & 1 & \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Then $(AB)_{ij}$ is not defined since the (i,j) entry $(1,1)$ is

$$\sum_k a_{ik} b_{kj} = \sum_k 1 \to \infty$$

⊞

If both $A_{i,j}$ and $B_{i,j}$ arise from bounded operators, then $(AB)_{i,j}$ exists. We need to check for absolute convergence of the matrix product because when we have convergent sums for each entry, then the matrix is well defined.The following lemma from [20] details this added requirement:

Lemma 2.11. *Let A and B be linear operators densely defined on $\ell^2(\mathbb{N}_0)$ and A^* represent the conjugate transpose of A, such that Ae_j, A^*e_j, and Be_j are defined and in $\ell^2(\mathbb{N}_0)$ for every element of the standard orthonormal basis $\{e_j\}_{j\in\mathbb{N}_0}$. Then $A_{i,j}$ and $B_{i,j}$, the infinite matrix representation of A and B respectively, are defined and the matrix product $(AB)_{i,j}$ is well defined.*

Proof. [20] □

2.4.4 Inverse

Multiplication of infinite matrices leads to the question of how to tell whether the matrix $A_{i,j}$ has an inverse. In finite dimensions, a square matrix A has an inverse if $\det A \neq 0$. In fact, determinants play a large roll in calculating the matrix A^{-1} in finite dimensions. Perhaps it is not surprising that back in the nineteenth century, mathematicians thought of infinite matrices in terms of determinants. As time went on, the study of infinite matrices became less about determinants and more about subspaces and linear operators (see [4]).

In the case of infinite matrices, we can define (formally) one-to-one and onto as stated in [5].

Definition 2.12. An infinite matrix $A_{i,j} = (a_{ij})$ is one-to-one if the trivial sequence is the only sequence (x_k) such that $\sum_j a_{ij}x_j = 0$

Definition 2.13. An infinite matrix $A_{i,j} = (a_{ij})$ is onto if for each (y_k) there exists a (x_k) such that $\sum_j a_{ij}x_j = y_j$

Not all books agree on the definition of inverse for infinite matrices (see [20] vs [5] vs [30]). We will be using the following definition from [5] which is very similar to the finite case.

Definition 2.14. An infinite matrix A_{ij} has an inverse if there exists $B_{i,j}$ such that $(AB)_{ij} = (BA)_{ij} = I_{ij}$, where I_{ij} is the infinite identity matrix,

$$I_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

There is no actual correlation between one-to-one, onto, and inverse (see [5]) as one can see in the following example of the infinite Hilbert Matrix from [5].

Example 2.8. The infinite Hilbert Matrix is made up of reciprocals of natural numbers,

$$A_{i,j} = \frac{1}{i+j+1} \tag{2.5}$$

with $i, j = 0, 1, 2, \cdots$. So, the matrix looks like

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & 0 & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & 0 & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & 0 & \cdots \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

In [5] we discover that this matrix is formally one-to-one, meaning $\sum_j a_{ij}\alpha_j = 0$ only for the trivial sequence (α_j) and it is not onto since if A maps an infinite sequence $(b_1, b_2, b_3, \cdots)$ to $(1, 0, 0, \cdots)$, then A maps $(0, b_1, b_2, \cdots)$ to $(0, 0, 0, \cdots)$, therefore $(1, 0, 0, \cdots)$ is not in the range of A. Which also means that A does not have an inverse. $\boxplus$

2.5 Unitary Operators and Matrices

In this section we will discuss the unitary matrices which we will use in discussing the matrix $U_{\alpha,\beta}$.

Definition 2.15. An operator A is unitary if $A^*A = I = AA^*$ where A^* is the conjugate transpose of A

Another way to state this definition (as is seen in [30]) is that an operator $A : H \to K$ is unitary if

$$< Ax, Ay >=< x, y > \text{ for all } x, y \in H.$$

From this definition we can see that A is unitary as an operator if and only if its infinite matrix representation $A_{i,j}$ is unitary (see [30] and [20]

Since $U : \Gamma \to 5\Gamma$, both orthonormal basis, we have that U is unitary which means that $U_{\alpha,\beta}$ is a unitary infinite matrix.

2.6 Hadamard Product

We will be looking at the results of the Hadamard product in relation to the matrix $U_{\alpha,\beta}$ in Chapter 5.

The Hadamard product, or Schur product as it is sometimes called, is a different way for multiplying matrices that is by term. It is based on the fact discovered by Schur in 1911 that if (a_{ij}) and (b_{ij}) are bounded matrix operators on ℓ_2 then $||(a_{ij}b_{ij})|| \leq ||(a_{ij})|| \cdot ||(b_{ij})||$ (see [29] for more details). We define the Hadamard product as follows:

Definition 2.16. Given matrices $A_{i,j} = (a_{ij})$ and $B_{i,j} = (b_{ij})$ the Hadamard product of the two is the termwise multiplication:

$$(A * B)_{i,j} = (a_{ij}b_{ij})$$

To see the Hadamard product in action, consider a finite dimension example first.

Example 2.9. Given the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \tag{2.6}$$

The Hadamard product $A * B = \begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}$ ⊞

The Hadamard product is commutative, associative, and distributive. Commutative since $(A * B)_{i,j} = (a_{ij}b_{ij}) = (b_{ij}a_{ij}) = (B * A)_{i,j}$. Associative since $((A * B) * C)_{i,j} = ((a_{ij}b_{ij})c_{ij}) = (a_{ij}(b_{ij}c_{ij})) = (A * (B * C))_{i,j}$. Distributive since $(A * (B + C))_{i,j} = (a_{ij}) * (b_{ij} + c_{ij}) = (a_{ij}(b_{ij} + c_{ij})) = (a_{ij}b_{ij} + a_{ij}c_{ij}) = (A * B)_{i,j} + (A * C)_{i,j}$.

The identity under the Hadamard product is different from the normal identity. If we define the identity as $E_{i,j} = (e_{ij})$ such that for any matrix $A_{i,j} = (a_{ij})$ we have that $(A * E)_{i,j} = (a_{ij}e_{ij}) = (a_{ij})$ and $(E * A)_{i,j} = (e_{ij}a_{ij}) = (a_{ij})$, then E must be the matrix with 1 in all the entries. This means that the the matrix $A_{i,j}$ has an inverse iff it contains nonzero entries.

Example 2.10. The following is an example in the finite dimensional case. The inverse of $\begin{pmatrix} 1 & 4 \\ \\ \frac{2}{3} & 7 \end{pmatrix}$ is $\begin{pmatrix} 1 & \frac{1}{4} \\ \\ \frac{3}{2} & \frac{1}{7} \end{pmatrix}$ $\boxplus$

CHAPTER 3
INTRODUCTION TO ORTHONORMAL BASIS Γ

In this chapter, we will look at $\Gamma = \{\sum_i 4^i b_i : b_i \in \{0,1\},\ \text{finite sums}\ \}$. Summarizing parts of the papers [11] and [21], we will show that Γ is an orthonormal basis for $L^2(\mu)$ as described below. The goal of this chapter is to give all the background necessary to understand

$$U : \Gamma \to 5\Gamma$$

where $5\Gamma = \{0, 5, 20, 25, \cdots\}$ is an orthonormal basis for $L^2(\mu)$.

3.1 Measure

If μ is a Lebesgue measure on $I = [0,1]$, then

$$\{e_n\} = \{e^{2\pi i n x} : n \in \mathbb{N}_0\}$$

spans the Hardy space, H_2, of analytic functions on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\{e_n\}$ is an orthonormal basis for $L_2(I)$ with normalized Lebesgue measure.

It is known (see [18]) that there is a special probability measure μ such that

$$\int f d\mu = \frac{1}{2}\left(\int f\left(\frac{x}{4}\right) d\mu(x) + \int f\left(\frac{x}{4} + \frac{1}{2}\right) d\mu(x)\right)$$

for all continuous f on $\mathbb{R}$ with compact support on the Cantor set obtained by dividing the line segment $[0,1]$ into 4 equal intervals and retaining only the first and third intervals. This chapter will focus on showing that

$$\Gamma = \{0, 1, 4, 5, \cdots\} = \{\sum 4^i b_i : b_i \in \{0,1\},\ \text{finite sums}\ \}$$

is an orthonormal basis for $L^2(\mu)$ (see [21]).

First, we will look into a more general case in ν dimensions. In [21] a system (R, B, L) is considered, where R is a real $\nu \times \nu$ matrix with integer entries and B and L are subspaces of $\mathbb{R}^\nu$. The subset B is required to satisfy an open set condition, namely that for $x \in \mathbb{R}^\nu$,

$$\sigma_b(x) = R^{-1}x + b. \tag{3.1}$$

B is a subset of $\mathbb{R}^\nu$ which is finite and the number of elements in B is N. Also, $RB \subset \mathbb{Z}^\nu$ and $0 \in B$. Further, the difference between any two elements, b_1 and b_2, in B is not in $\mathbb{Z}^\nu$ when $b_1 \neq b_2$. L is another subset of $\mathbb{R}^\nu$ where $L \subset \mathbb{Z}^\nu$ and $0 \in L$. The number of elements in L is the same as the number of elements in B. Finally, the Hadamard matrix $H_{BL} = N^{-1/2}\left(e^{2i\pi b\cdot l}\right)$ where $b \in B$ and $l \in L$ (with N being the number of elements in B the same as the number of elements in L) is a unitary complex matrix.

It is known, by [17], that there is a unique probability measure μ on $\mathbb{R}^\nu$ of compact support such that

$$\int f d\mu = \frac{1}{N}\sum_{b\in B}\int f\left(\sigma_b(x)\right) d\mu(x) \tag{3.2}$$

where $\sigma_b(x) = R^{-1}x + b$, $x \in \mathbb{R}^\nu$ and N and B as defined above (see [21] for more information). Further calculation leads to the following equality (see [21]):

$$\begin{aligned}
\int f d\mu &= \frac{1}{N} \sum_{b \in B} \int f\left(\sigma_b(x)\right) d\mu(x) \\
&= \frac{1}{N} \sum_{b \in B} \int e^{i2\pi t \cdot (\sigma_b(x))} d\mu(x) \\
&= \frac{1}{N} \sum_{b \in B} \int e^{i2\pi t \cdot \left(R^{-1}x + b\right)} d\mu(x) \\
&= \frac{1}{N} \sum_{b \in B} \int e^{i2\pi {R^*}^{-1} t \cdot x} e^{2\pi i t \cdot b} d\mu(x) \\
&= \frac{1}{N} \sum_{b \in B} e^{2\pi i t \cdot b} \int e^{i2\pi {R^*}^{-1} t \cdot x} d\mu(x)
\end{aligned}$$

If we set

$$e_t(x) := e^{2\pi i t \cdot x},\ (t, x \in \mathbb{R}^\nu)$$

we get the following definition that defines what we mean by Γ being orthogonal in $L^2(\mu)$ (from [11]):

Definition 3.1. Let $\Gamma \in \mathbb{R}^\nu$ be some discrete subset and let

$$E(\Gamma) := \{e_\gamma : \gamma \in \Gamma\}.$$

We say that Γ is orthogonal in $L^2(\mu)$ iff the functions in $E(\Gamma)$ are orthogonal. That is,

$$< e_{\gamma_1}, e_{\gamma_2} >= \begin{cases} 0 \text{ for all } \gamma_1 \neq \gamma_2 \in \Gamma \\ \\ 1 \text{ if } \gamma_1 = \gamma_2 \in \Gamma \end{cases}$$

where $\hat{\mu}$ is defined as the Fourier transform

$$\hat{\mu}(t) = \int e^{i2\pi t \cdot x} d\mu(x) \tag{3.3}$$

where $t \cdot x = \sum_{i=1}^{n} t_i x_i$.

From [21], we have the following lemma introduces a set P such that

$$\{e_\lambda : \lambda \in P\}$$

are mutually orthogonal in $L^2(\mu)$. This is the general case in ν dimensions described above. We will use it as a basis to introduce the orthonormal basis Γ that we desire.

Lemma 3.2. *With the assumptions introduced above for the (R, B, L) system, set*

$$P := \{l_0 + R^* l_1 + \cdots : l_i \in L, \text{ finite sums}\}$$

Then the functions $\{e_\lambda : \lambda \in P\}$ are mutually orthogonal in $L^2(\mu)$ where $e_\lambda := e^{i2\pi\lambda \cdot x}$

Proof. Lemma 3.1 in [21] $\square$

In particular, we are interested in the case where $N = 2$, and $(R, B, L) = \left(4, \left\{0, \frac{1}{2}\right\}, \{0, 1\}\right)$ in $\mathbb{R}$. Directly from the previous lemma, we get the following corollary in which we introduce the orthonormal basis Γ:

Corollary 3.3. *Let μ be the measure on the line $\mathbb{R}$ given by*

$$\int f d\mu = \frac{1}{2}\left(\int f\left(\frac{x}{4}\right) d\mu(x) + \int f\left(\frac{x}{4} + \frac{1}{2}\right) d\mu(x)\right) \text{ for all continuous } f$$

with Hausdorff dimension $d_H = \frac{1}{2}$. (We have $R = 4$, $B = \{0, \frac{1}{2}\}$ and $L = \{0, 1\}$)

Then

$$\Gamma := \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0,1\}, \textit{finite sums}\}$$

and $\{e_\lambda : \lambda \in \Gamma\}$ *is an orthonormal subset of* $L^2(\mu)$*.*

Proof. Corollary 3.2 in [21] □

Of course, this only shows that it is an orthonormal subset of $L^2(\mu)$, but it is proven in [21], that Γ is indeed an orthonormal basis for $L^2(\mu)$.

3.2 5Γ

Throughout the rest of the paper we will working with the orthonormal basis found in [21]. From this point forward it will be referred to as Γ:

$$\begin{aligned}
\Gamma :&= \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0,1\}, \text{finite sums}\} \\
&= \{\sum_i 4^i b_i : b_i \in \{0,1\}, \text{ finite sums } \} \\
&= \{0, 1, 4, 5, 16, 17, 20, \cdots\}
\end{aligned}$$

This thesis will look into what happens if we look at 5Γ

$$\begin{aligned}
5\Gamma :&= \{5(l_0 + 4l_1 + 4^2 l_2 + \cdots) :: l_i \in \{0,1\}, \text{finite sums}\} \\
&= \{0, 5, 20, 25, \cdots\}
\end{aligned}$$

From the paper by D.E. Dutkay and P.E.T. Jorgensen, (see [11]) we know that $5^k\Gamma$ is an orthonormal basis for for $L^2(\mu)$ for $k = 1, 2, 3\cdots$. With this, we can create a unitary operator

$$U : \Gamma \to 5\Gamma.$$

This operator is of interest as it illustrates what happens when mapping between two different orthonormal basis (one is not a subset of the other).

CHAPTER 4
MORE INFINITE MATRICES

This chapter is an extension of the orthonormal basis and system (R, B, L) from [21] described in the last chapter: $N = 2$, and $(R, B, L) = \left(4, \left\{0, \frac{1}{2}\right\}, \{0, 1\}\right)$ in $\mathbb{R}$ and $\Gamma = \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0, 1\}, \text{finite sums}\}$. Much of the work in this chapter will be with the Fourier transform $\hat{\mu}$ from the last chapter (described in [21]).

In the proofs throughout this chapter we will using the fact that

$$\hat{\mu}(\text{odd number}) = 0$$

and

$$\text{for } \gamma \in \Gamma,\ \hat{\mu}(\gamma) = 0 \text{ unless } \gamma = 0.$$

To see this, it is easier to use the definition for $\hat{\mu}$ as follows:

$$\hat{\mu}(t) = \prod_{n=0}^{\infty} \frac{1}{2} \left(1 + e^{i \frac{i\pi}{4^n}}\right) = e^{i\pi \frac{2t}{3}} \prod_{n=0}^{\infty} \cos\left(\frac{\pi t}{2 \cdot 4^n}\right) \tag{4.1}$$

The following lemma gives a short proof of why this is true.

Lemma 4.1. *With the above, (R, B, L) and Γ, $\hat{\mu}(t) = 0$ if $t \in \Gamma - \{0\}$ or $t \in \{4^k \times n | n \in 2\mathbb{Z} + 1, k = \{0, 1, 2, \cdots\}\}$*

Proof. We have $R = 4$, $N = 2$, and $B = \left\{0, \frac{1}{2}\right\}$, using the definitions in chapter 3 we get that:

$$\begin{aligned}\hat{\mu}(t) &= \left(\frac{1}{N}\sum_{b\in B} e^{2i\pi b\cdot t}\right)\hat{\mu}\left(R^{*-1}t\right)\\ &= \frac{1}{2}\left(1+e^{i\pi t}\right)\hat{\mu}\left(\frac{t}{4}\right)\end{aligned}$$

By using the same process again, but this time for $\hat{\mu}(\frac{t}{4})$ we get:

$$= \frac{1}{2}\left(1+e^{i\pi t}\right)\frac{1}{2}\left(1+e^{i\pi t/4}\right)\hat{\mu}\left(\frac{t}{4^2}\right)$$

Continuing this process infinitely many times, we end up with

$$\begin{aligned}&= \prod_{n=0}^{\infty}\frac{1}{2}\left(1+e^{i\pi t/4^n}\right)\\ &= \prod e^{i\pi t/(2\cdot 4^n)}\left(\frac{e^{-i\pi t/(2\cdot 4^n)}+e^{i\pi t/(2\cdot 4^n)}}{2}\right)\\ &= e^{1/2 i\pi t\sum\frac{1}{4^n}}\prod\cos\left(\frac{\pi t}{2\cdot 4^n}\right)\\ &= e^{i\pi t 2/3}\prod\cos\left(\frac{\pi t}{2\cdot 4^n}\right)\end{aligned}$$

So, $\hat{\mu}(t) = 0$ if $t \in \Gamma - \{0\}$ or $t \in \{4^k \times n | n \in 2\mathbb{Z}+1, k = \{0,1,2,\cdots\}\}$. $\square$

Remark 4.2. Notice that by the definition of $\hat{\mu}(t)$ (see Equation 4.1) we can both take out powers of 4 and multiply by powers of 4 without changing $\hat{\mu}(\cdot)$. In other words $\hat{\mu}(4^k t) = \hat{\mu}(t)$.

Remark 4.3. We will be looking at three major operators:

$$S_0 : \Gamma \to 4\Gamma,\ S_1 : \Gamma \to 4\Gamma + 1$$

and $U : \Gamma \to 5\Gamma$. Before we start delving into the operators, it will be useful to make note of the fact that since 4Γ (and $4\Gamma+1$) are subsets of Γ, an orthonormal basis, for $\alpha, \beta \in \Gamma$ we have that

$$< e_\alpha, e_{4\beta} >= \begin{cases} 1 & \text{if } 4\beta = \alpha \\ \\ 0 & \text{else} \end{cases}$$

(follows similarly for $4\beta+1$). It is also know, by [11], that 5Γ is an orthonormal basis. In fact $5^k\Gamma$ is an orthonormal basis for $k = 0, 1, 2, \cdots$ (see [11]).

4.1 Operators S_0, S_1 and U

Now, we will formally introduce our three major operators. Let (μ, Γ) be a spectral pair.

$$\hat{\mu}(t) = e^{i\pi t 2/3} \prod \cos\left(\frac{\pi t}{2 \cdot 4^n}\right)$$

and

$$\Gamma = \left\{ \sum_0^{\text{finite}} b_i 4^i | b_i \in \{0, 1\} \right\}$$

We are going to be looking at three major operators. Again, we will refer to the operators by capital letters (such as U) and to the infinite matrices by capital letters and subscripts (such as $U_{\alpha,\beta}$), and to the entries of the infinite matrices with lower case letters (such as $u_{\alpha\beta}$). First, consider

$$S_0 : L^2(\mu) \to L^2(\mu)$$

be determined by

$$S_0 e_\lambda = e_{4\lambda}$$

Notice that $S_0 : \Gamma \to 4\Gamma$ and that $4\Gamma \subset \Gamma$. We can consider the infinite matrix of S_0 together with the orthonormal basis Γ to be $S_{0\alpha,\beta} = (s_{0\alpha\beta})$ where

$$s_{0\alpha\beta} =< e_\alpha, S_0 e_\beta >= \hat{\mu}(4\beta - \alpha) \tag{4.2}$$

The matrix representation is:

$$S_{0\alpha,\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Similarly, define

$$S_1 : L^2(\mu) \to L^2(\mu)$$

to be determined by

$$S_1 e_\lambda = e_{4\lambda+1}$$

Notice that $S_1 : \Gamma \to 4\Gamma + 1$ and that $4\Gamma + 1 \subset \Gamma$. If we combine this operator with the orthonormal basis Γ, then we can create the infinite matrix $S_{1\alpha,\beta} = (s_{1\alpha\beta})$ where

$$s_{1\alpha\beta} =< e_\alpha, S_1 e_\beta >= \hat{\mu}(4\beta + 1 - \alpha) \qquad (4.3)$$

The matrix itself looks like:

$$S_{1\alpha,\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Finally, we will introduce our last major operator, U. Let

$$U : L^2(\mu) \to L^2(\mu)$$

be determined by

$$Ue_\lambda = e_{5\lambda}$$

and $U_{\alpha,\beta} = (u_{\alpha\beta})$ where

$$u_{\alpha\beta} =< e_\alpha, Ue_\beta >= \hat{\mu}(5\beta - \alpha) \tag{4.4}$$

Notice that $U : \Gamma \to 5\Gamma$. Unlike 4Γ and $4\Gamma + 1$, 5Γ is not a subset of Γ. For example $Ue_5 = e_{25}$ but $25 = 1 + 2 \cdot 4 + 4^2$ and so is not in Γ. Therefore, the matrix of $U_{\alpha,\beta}$ is a bit more complicated than that of $S_{0\alpha,\beta}$ or $S_{1\alpha,\beta}$. As we can see from the following matrix representation, there are entries of the matrix that are neither 0 nor 1.

$$U_{\alpha,\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \hat{\mu}(6) & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \hat{\mu}(2) & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

4.2 Commuting between $S_{0\alpha,\beta}$, $S_{1\alpha,\beta}$ and $U_{\alpha,\beta}$

Looking at the three infinite matrices $S_{0\alpha,\beta}$, $S_{1\alpha,\beta}$ and $U_{\alpha,\beta}$, one of the first questions that arises is whether these matrices commute. As the following lemmas show, only one of the pairings actually commutes.

Lemma 4.4. *The infinite matrices $S_{0\alpha,\beta}$ and $S_{1\alpha,\beta}$ as described in the above section, do not commute.*

Proof. First, consider $(S_0S_1)_{\alpha,\beta}$. For $\gamma \in \Gamma$, each entry in the infinite matrix can be represented by

$$(s_0s_1)_{\alpha\beta} = \sum_{\gamma} \hat{\mu}(4\gamma - \alpha)\hat{\mu}(4\beta + 1 - \gamma)$$

From Remark 4.3 we have that $\hat{\mu}(4\beta+1-\gamma) = 0$ unless $4\beta+1 = \gamma$. This means that

$$\begin{aligned}(s_0s_1)_{\alpha\beta} &= \hat{\mu}(16\beta + 4 - \alpha)\\ &= \begin{cases} 1 & \text{if } (\alpha,\beta) = (16\beta+4,\beta) \\ 0 & \text{else} \end{cases}\end{aligned}$$

On the other hand, for $(S_1S_0)_{\alpha,\beta}$ each entry can be represented by

$$(s_1s_0)_{\alpha\beta} = \sum_{\gamma} \hat{\mu}(4\gamma + 1 - \alpha)\hat{\mu}(4\beta - \gamma)$$

From Remark 4.3 we have that $\hat{\mu}(4\beta - \gamma) = 0$ unless $4\beta = \gamma$. This means that

$$\begin{aligned}(s_0s_1)_{\alpha\beta} &= \hat{\mu}(16\beta + 1 - \alpha)\\ &= \begin{cases} 1 & \text{if } (\alpha,\beta) = (16\beta+1,\beta) \\ 0 & \text{else} \end{cases}\end{aligned}$$

In particular, $(s_0s_1)_{4,0} = 1$ while $(s_1s_0)_{4,0} = 0$. Therefore, they do not commute. $\square$

Another way to look at the previous lemma is to look at the operators S_0 and S_1 and how they together act on an element in the orthonormal basis $\{e_\gamma\}$: $S_0S_1e_\gamma \to S_0e_{4\gamma+1} \to e_{16\gamma+4}$ while $S_1S_0e_\gamma \to S_1e_{4\gamma} \to e_{16\gamma+1}$.

When dealing with the operator U, we run into difficulties since $U : \Gamma \to 5\Gamma$ and 5Γ is not a subset of Γ.

Lemma 4.5. *The infinite matrices $U_{\alpha,\beta}$ and $S_{0\alpha,\beta}$ as described above with the orthonormal basis $\Gamma = \{l_0 + 4l_1 + 4^2l_2 + \cdots : l_i \in \{0,1\}, \text{finite sums}\}$ commute*

Proof. Let's first look at $(US_0)_{\alpha,\beta}$ where each entry can be written as

$$(us_0)_{\alpha,\beta} = \sum_\gamma \hat{\mu}(5\gamma - \alpha)\hat{\mu}(4\beta - \gamma)$$

Since 4β and γ are in Γ by Remark 4.3 we have that

$$\hat{\mu}(4\beta - \gamma) = \begin{cases} 1 \text{ if } 4\beta = \gamma \\ \\ 0 \text{ else} \end{cases}$$

which means that

$$\begin{aligned}(us_0)_{\alpha,\beta} &= \hat{\mu}(5(4\beta) - \alpha) \\ &= \hat{\mu}(20\beta - \alpha)\end{aligned}$$

On the other hand, we have

$$(s_0u)_{\alpha,\beta} = \sum_\gamma \hat{\mu}(4\gamma - \alpha)\hat{\mu}(5\beta - \gamma)$$

We can use Lemma 4.1 and its following remark to get

$$= \sum_\gamma \hat{\mu}(4\gamma - \alpha)\hat{\mu}(20\beta - 4\gamma)$$

Again, by Remark 4.3 we have that $\hat{\mu}(4\gamma - \alpha) = 0$ unless $4\gamma = \alpha$. So,

$$(s_0 u)_{\alpha,\beta} = \hat{\mu}(20\beta - \alpha)$$

$$= (u s_0)_{\alpha,\beta}$$

Therefore, $U_{\alpha,\beta}$ and $S_{0\alpha,\beta}$ commute. □

Since $U_{\alpha,\beta}$ and $S_{0\alpha,\beta}$ commute, one might assume that $U_{\alpha,\beta}$ and $S_{1\alpha,\beta}$ also commute; however as the following lemma shows, this is not the case:

Lemma 4.6. *The infinite matrices $S_{1\alpha,\beta}$ and $U_{\alpha,\beta}$ do not commute*

Proof. Let's first look at the infinite matrix $(US_1)_{\alpha,\beta}$ where each entry is:

$$(us_1)_{\alpha\beta} = \sum_{\gamma} \hat{\mu}(5\gamma - \alpha)\hat{\mu}(4\beta + 1 - \alpha)$$

By Remark 4.3 we have that:

$$= \hat{\mu}(5(4\beta + 1) - \alpha)$$

$$= \hat{\mu}(20\beta + 5 - \alpha)$$

At this point, we can tell that nonzero terms of (us_1) occur when $\alpha \in 4\Gamma + 1$.

Now, let's look at $(S_1 U)_{\alpha,\beta}$ with entries:

$$(s_1 u)_{\alpha\beta} = \sum_{\gamma} \hat{\mu}(4\gamma + 1 - \alpha)\hat{\mu}(5\gamma - \beta)$$

By Remark 4.3 we have that nonzero terms only occur when $4\gamma + 1 = \alpha$. We can rewrite $(5\gamma - \beta)$ as $20\gamma - 4\beta$ by Remark 4.2. Also,

$$20\gamma - 4\beta = 5(4\gamma) - 4\beta$$

$$= 5(4\gamma + 1) - 5 - 4\beta$$

With this notation, we can conclude that

$$(s_1u)_{\alpha\beta} = \hat{\mu}(5(4\gamma+1) - 5 - 4\beta)$$
$$= \hat{\mu}(5\alpha - 5 - 4\beta)$$

Since $(us_1)_{1,0} = \hat{\mu}(5-1) = 0$ and $(s_1u)_{1,0} = \hat{\mu}(5-5-0) = 1$ then $(US_1)_{\alpha,\beta} \neq (S_1U)_{\alpha,\beta}$ □

4.3 Operator M_k

In this section we introduce a new operator

$$M_k : \Gamma \to \Gamma + k$$

for $\gamma \in \Gamma = \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0,1\}, \text{finite sums}\}$ such that

$$M_k e_\gamma = e_{\gamma+k}$$

With this operator we can find even more relations between U, S_0 and S_1. First we will look at $M_1 : \Gamma \to \Gamma + 1$ where $M_1 e_\gamma = e_{\gamma+1}$ with respect to the orthonormal basis Γ. We define the matrix $M_{1\alpha,\beta} = (m_{1\alpha\beta})$ of the operator M_1 with respect to Γ to consist of entries

$$m_{1\alpha\beta} =< e_\alpha, M_1 e_\beta >= \hat{\mu}(\beta + 1 - \alpha)$$

which has entries that are neither 0 nor 1 (unlike $S_{0\alpha,\beta}$ and $S_{1\alpha,\beta}$).

$$M_{1\alpha,\beta} = \begin{pmatrix} 0 & \hat{\mu}(2) & 0 & \hat{\mu}(6) & 0 & \hat{\mu}(18) & 0 & \hat{\mu}(22) & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \hat{\mu}(2) & 0 & \hat{\mu}(2) & 0 & \hat{\mu}(14) & 0 & \hat{\mu}(18) & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \hat{\mu}(14) & 0 & \hat{\mu}(10) & 0 & \hat{\mu}(2) & 0 & \hat{\mu}(6) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The first thing we will consider is what happens when we take powers of the matrix $M_{1\alpha,\beta}$. In the best of worlds, the powers of $M_{1\alpha,\beta}$, $M^k_{1\alpha,\beta}$ will be the equivalent to $M_{k\alpha,\beta}$ where each entry of the matrix is $m_{k\alpha\beta} = \hat{\mu}(\beta + k - \alpha)$. At first it looks promising, as we look at the case when $k = 2$.

Lemma 4.7. *Let $M_k : \Gamma \to \Gamma + k$ where $M_k e_\gamma = e_{\gamma+k}$ with the orthonormal basis $\Gamma = \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0, 1\}, \text{finite sums}\}$ represent the matrix $M_{k\alpha,\beta} = (m_{k\alpha\beta})$ where $m_{k\alpha\beta} =< e_\alpha, M_k e_\beta >= \hat{\mu}(\beta + k - \alpha)$. Then we have that $M^2_{1\alpha,\beta} = M_{2\alpha,\beta}$*

Proof. Each (α, β) element in the matrix $(M_1^2)_{\alpha\beta}$ is

$$= \textstyle\sum_\gamma \hat{\mu}(\gamma + 1 - \alpha)\hat{\mu}(\beta + 1 - \gamma)$$

Notice that $\hat{\mu}(\gamma + 1 - \alpha) = 0$ unless $\gamma \in 4\Gamma + 1$. Let $\gamma = 4\xi + 1$ where $\xi \in \Gamma$. Then

$$= \sum_\xi \hat{\mu}(4\xi + 2 - \alpha)\hat{\mu}(\beta - 4\xi)$$

By Remark 4.3, $\hat{\mu}(\beta - 4\xi) = 0$ unless $\beta = 4\xi$ where we have $\hat{\mu}(\beta - 4\xi) = \hat{\mu}(0) = 1$

Therefore, each (α, β) entry of the matrix

$$= (\hat{\mu}(\beta + 2 - \alpha))$$

$$= (m_{2\alpha\beta})$$

Therefore $(M_1^2)_{\alpha,\beta} = M_{2\alpha\beta}$. $\square$

Unfortunately, although it works out in the case of $k = 2$, in general, $(M_1^k)_{\alpha,\beta} \neq M_{k\alpha,\beta}$ as is seen in the following lemma:

Lemma 4.8. *With $M_{k\alpha,\beta}$ as described in Lemma 4.7, $(M_1^3)_{\alpha,\beta} \neq M_{3\alpha,\beta}$*

Proof. By Lemma 4.7 we know that $(M_1^2)_{\alpha,\beta} = M_{2\alpha,\beta}$, so we have that that each (α, β) entry in the matrix $(M_1^3)_{\alpha,\beta}$ is

$$= \sum_{\gamma} m_{2\alpha,\gamma} m_{1\gamma,\beta}$$

$$= \sum_{\gamma} \hat{\mu}(\gamma + 2 - \alpha)\hat{\mu}(\beta + 1 - \gamma)$$

Case 1: If $\beta \in 4\Gamma$ we have that by Lemma 4.1 $\hat{\mu}(\beta + 1 - \gamma) = \begin{cases} 1 \text{ if } \gamma = \beta + 1 \\ 0 \text{ else} \end{cases}$ So, we have that for $\beta \in 4\Gamma$, $m_{1\alpha\beta}^3 = \hat{\mu}(\beta + 3 - \alpha) = m_{3\alpha,\beta}$.

Case 2: But, if $\beta \in 4\Gamma + 1$ we have that by Lemma 4.1, $\gamma \in 4\Gamma$ and $(M_1^3)_{\alpha,\beta} = \sum_{\gamma} \hat{\mu}(4\gamma + 2 - \alpha)\hat{\mu}(\beta + 1 - 4\gamma)$ and, therefore, for $\alpha \in 4\Gamma$ we have nonzero terms.

In particular, ${(m_1)^3}_{0,1} = \sum \gamma\hat{\mu}(\gamma + 2)\hat{\mu}(2 - \gamma)$, while $m_{3(0,1)} = \hat{\mu}(1 + 3 - 0) = 0$. $\square$

Now that we have established what $M_{1\alpha,\beta}^k$ looks like, we will now see how it relates to $S_{0\alpha,\beta}$, $S_{1\alpha,\beta}$ and $U_{\alpha,\beta}$.

4.4 Relationships between $S_{0\alpha,\beta}$, $S_{1\alpha,\beta}$ and $U_{\alpha,\beta}$ and $M_{1\alpha,\beta}$

First, we will look at which of the matrices $S_{0\alpha,\beta}$, $S_{1\alpha,\beta}$ and $U_{\alpha,\beta}$ commute with $M_{1\alpha,\beta}$. This will give us insight into other ways that the first three operators relate. Since $M_1 : \Gamma \to \Gamma + 1$ and $\Gamma + 1$, much like 5Γ, is not a subset of Γ we anticipate that not all of the matrices will commute with $M_{1\alpha,\beta}$. However, it may be surprising that none of them actually commute.

Lemma 4.9. *The infinite matrices with respect to the orthonormal basis*

$$\Gamma = \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0,1\}, \textit{finite sums}\},$$

$S_{1\alpha,\beta}, S_{0\alpha,\beta}$, and $U_{\alpha,\beta}$, do not commute with $M_{1\alpha,\beta}$

Proof. For $(S_1 M_1)_{\alpha,\beta}$, the entries of the matrix will be:

$$\sum_{\gamma} s_{1\alpha\gamma} m_{1\gamma\beta} = \sum_{\gamma} \hat{\mu}(4\gamma + 1 - \alpha)\hat{\mu}(\beta + 1 - \gamma)$$

since $\hat{\mu}(4\gamma + 1 - \alpha) = 0$ for all but $\alpha = 4\gamma + 1$, we get

$$= \hat{\mu}(4\beta + 4 - 4\gamma)$$

$$= \hat{\mu}(4\beta + 4 - (\alpha - 1))$$

$$= \hat{\mu}(4\beta + 3 - \alpha)$$

Now let's consider $(M_1 S_1)_{\alpha,\beta}$ where each (α, β) entry in the matrix is:

$$\sum_{\gamma} m_{1\alpha\gamma} s_{1\gamma\beta} = \sum_{\gamma} \hat{\mu}(\gamma + 1 - \alpha)\hat{\mu}(4\beta + 1 - \gamma)$$

Since $\hat{\mu}(4\beta + 1 - \gamma) = 0$ except when $\gamma = 4\beta + 1$

$$= \hat{\mu}(4\beta + 2 - \alpha)$$

In particular, we have that $(s_1 m_1)_{0,5} = 0$, while $(m_1 s_1)_{0,5} = \hat{\mu}(22) \neq 0$

Next, let's look at $(S_0 M_1)_{\alpha,\beta}$ where each (α, β) entry is:

$$\sum_{\gamma} \hat{\mu}(4\gamma - \alpha)\hat{\mu}(\beta + 1 - \gamma) = \hat{\mu}(4\beta + 4 - \alpha)$$

since $\hat{\mu}(4\gamma - \alpha) = 0$ unless $4\gamma = \alpha$.

On the other hand, each (α, β) entry of $(M_1 S_0)_{\alpha,\beta}$ is:

$$\sum_{\gamma} \hat{\mu}(\gamma + 1 - \alpha)\hat{\mu}(4\beta - \gamma) = \hat{\mu}(4\beta + 1 - \alpha)$$

since $\hat{\mu}(4\beta - \gamma) = 0$ unless $4\beta = \gamma$. Notice that this shows that $(M_1 S_0)_{\alpha,\beta} = S_{1\alpha,\beta}$. So, $S_{0\alpha,\beta}$ and $M_{1\alpha,\beta}$ do not commute since, in particular, we have that $(s_0 m_1)_{0,1} = \hat{\mu}(2)$, while $(m_1 s_0)_{0,1} = 0$

Finally, let's look at $(M_1 U)_{\alpha,\beta}$ where each (α, β) entry is:

$$\sum_{\gamma} \hat{\mu}(\gamma + 1 - \alpha)\hat{\mu}(5\beta - \gamma)$$

On the other hand, each entry of $(U M_1)_{\alpha,\beta}$ is

$$\sum_{\gamma} \hat{\mu}(5\gamma - \alpha)\hat{\mu}(\beta + 1 - \gamma)$$

In particular, $(m_1 u)_{1,0} = \sum_{\gamma} \hat{\mu}(\gamma + 1 - 1)\hat{\mu}(0 - \gamma) = 1$ since Γ is an orthonormal basis and $(m_1 u)_{1,0} = \sum_{\gamma} \hat{\mu}(5\gamma - 1)\hat{\mu}(1 - \gamma) = \hat{\mu}(5 - 1) = 0$ $\square$

Although none of the matrices commute, as we have seen in the proof of Lemma 4.9, there is a relationship between $S_{0\alpha,\beta}$, $S_{1\alpha,\beta}$ and $M_{1\alpha,\beta}$:

Lemma 4.10. *Given S_1, S_0 and M_1 with respect to the orthonormal basis*

$$\Gamma = \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0, 1\}, \textit{finite sums}\},$$

The matrix $S_{1\alpha,\beta} = (M_1 S_0)_{\alpha,\beta}$. *In other words, for each* $\alpha, \beta \in \Gamma$, *we get* $s_{1\alpha,\beta} = \sum_\gamma m_{1\alpha,\gamma} s_{0\gamma,\beta}$

Proof. Shown in the proof of Lemma 4.9 □

Remark 4.11. Another way of looking at the above Lemma is to consider that for any $\gamma \in \Gamma$, $M_1 S_0 e_\gamma = M_1 e_{4\gamma}$. Since $4\gamma \in \Gamma$, we get that $M_1 e_{4\gamma} = e_{4\gamma+1} = S_1 e_\gamma$.

4.4.1 Block Matrix

Another interesting way to relate $U_{\alpha,\beta}$, $S_{0\alpha,\beta}$ and $S_{1\alpha,\beta}$ is where the zero and nonzero entries occur in the block matrix:

$$\begin{pmatrix} S_0^* M_1 S_0 & S_0^* M_1 S_1 \\ S_1^* M_1 S_0 & S_1^* M_1 S_1 \end{pmatrix} \tag{4.5}$$

Before we start calculating the matrix, we will need the following lemma:

Lemma 4.12. *Given* S_0 *and* S_1 *with orthonormal basis*

$$\Gamma = \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0, 1\}, \textit{finite sums}\},$$

for the corresponding infinite matrices $S^*_{0\alpha,\beta} = \left(s^*_{0\alpha\beta}\right)$ *and* $S_{1\alpha,\beta} = (s_{1\alpha\beta})$ *we have that* $(S_0^* S_1)_{\alpha,\beta} = (0)_{\alpha,\beta}$ *the zero matrix.*

Proof. For the infinite matrix $(S_0^* S_1)_{\alpha,\beta}$ each entry is

$$\begin{aligned} \sum_\gamma \sum_\gamma s^*_{0\alpha\gamma} s_{1\gamma\beta} &= \sum_\gamma \hat{\mu}(4\alpha - \gamma)\hat{\mu}(4\beta + 1 - \gamma) \\ &= \hat{\mu}(4\alpha - (4\beta + 1)) \\ &= 0 \end{aligned}$$

since both 4α and $4\beta + 1$ are in Γ □

With the previous lemma in place, we can now discover where the zero terms of the complicated block matrix, Matrix 4.5.

Theorem 4.13. *The matrix*

$$\begin{pmatrix} (S_0^* M_1 S_0)_{\alpha,\beta} & (S_0^* M_1 S_1)_{\alpha,\beta} \\ \\ (S_1^* M_1 S_0)_{\alpha,\beta} & (S_1^* M_1 S_1)_{\alpha,\beta} \end{pmatrix} = \begin{pmatrix} 0 & v \\ \\ w & 0 \end{pmatrix}$$

where v and w are nonzero.

Proof. First, consider $(S_0^* M_1 S_0)_{\alpha,\beta}$. By Lemma 4.10 and Lemma 4.12 we have that $(S_0^* M_1 S_0)_{\alpha,\beta} = (S_0^* S_1)_{\alpha,\beta} = (0)_{\alpha,\beta}$.

Next, we have that $(S_1^* M_1 S_1)_{\alpha,\beta} = (0)_{\alpha,\beta}$ because each entry of $(S_1^* M_1 S_1)_{\alpha,\beta}$ is:

$$\sum_{\gamma} \sum_{\xi} \hat{\mu}(4\alpha + 1 - \gamma)\hat{\mu}(\xi + 1 - \gamma)\hat{\mu}(4\beta + 1 - \xi) = \sum_{\gamma} \hat{\mu}(4\alpha + 1 - \gamma)\hat{\mu}(4\beta + 2 - \gamma)$$

since $4\hat{\mu}(4\beta + 1 - \xi) = 0$ unless $4\beta + 1 = \xi$. So each (α, β) entry becomes

$$= \sum_{\gamma} \hat{\mu}(4\alpha + 1 - 4\gamma)\hat{\mu}(4\beta + 2 - 4\gamma)$$

$$= \sum_{\gamma} \hat{\mu}(4(\alpha - \gamma) + 1)\hat{\mu}(4\beta + 2 - 4\gamma) = 0$$

Also, $(S_0^* M_1 S_1)_{\alpha,\beta}$ has nonzero entries since

$$\sum_{\gamma} \sum_{\xi} \hat{\mu}(4\alpha - \gamma)\hat{\mu}(\xi + 1 - \gamma)\hat{\mu}(4\beta + 1 - \xi) = \sum_{\gamma} \hat{\mu}(4\alpha - \gamma)\hat{\mu}(4\beta + 2 - \gamma)$$

$$= \hat{\mu}(4\beta + 2 - 4\alpha)$$

which is nonzero since $(S_0^* M_1 S_1)_{1,1} = \hat{\mu}(2)$.

Finally, $S_1^* M_1 S_0$ is nonzero because $(S_1^* M_1 S_0)_{\alpha,\beta} = (S_1^* S_1)_{\alpha,\beta} = I_{\alpha,\beta}$ □

When considering powers of the above matrix, we will have

$$\begin{pmatrix} 0 & S_0^* M_1 S_1 \\ S_1^* M_1 S_0 & 0 \end{pmatrix}^2 = \begin{pmatrix} S_0^* M_1 S_1 S_1^* M_1 S_0 & 0 \\ 0 & S_1^* M_1 S_0 S_0^* M_1 S_1 \end{pmatrix}$$

$$= \begin{pmatrix} S_0^* (M_1)^2 S_0 & 0 \\ 0 & S_1^* (M_1)^2 S_1 \end{pmatrix}$$

From the above lemma 4.10 we have that $M_1 S_0 = S_1$ so that $S_0^* M_1^{k+1} S_0 = S_0^* M_1^k S_1$

Lemma 4.14. *For $k \in Z$, the infinite matrix $\left(S_0{}^* M_1^k S_1\right)_{\alpha,\beta}$ with respect to the orthonormal basis $\Gamma = \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0,1\}, \text{finite sums}\}$ is equivalent to the zero matrix,$(0)_{\alpha,\beta}$ if k is even and$\left(S_0{}^* M_1^k S_1\right)_{\alpha,\beta} \neq (0)_{\alpha,\beta}$ if k is odd.*

Proof. For the case where k is even, consider first the case when $k = 2$ $M_1^2 = M_2$ we have

$$(S_0^* M_1^2 S_1)_{\alpha,\beta} = (S_0^* M_2 S_1)_{\alpha,\beta}$$

Each entry (α, β) will be

$$= \sum_{\gamma_1} \sum_{\gamma_2} \hat{\mu}(4\alpha - \gamma_1) \hat{\mu}(\gamma_2 + 2 - \gamma_1) \hat{\mu}(4\beta + 1 - \gamma_2)$$

So we have that $4\beta + 1 = \gamma_2$ and $4\alpha = \gamma_1$ so each entry in $(S_0^* M_1^2 S_1)_{\alpha,\beta}$ is $= \hat{\mu}(4\beta + 3 - 4\alpha) = 0$ since $4\beta + 3 - 4\alpha$ is odd.

In general, $(S_0^* M_1^{2k} S_1)_{\alpha,\beta} = (S_0^* M_2^k S_1)_{\alpha,\beta}$, so each entry (α, β) will be

$$= \sum_{\gamma_1, \gamma_2, \cdots, \gamma_{k+1}} \hat{\mu}(4\alpha - \gamma_1) \hat{\mu}(\gamma_2 + 2 - \gamma_1) \hat{\mu}(\gamma_3 + 2 - \gamma_2) \cdots \hat{\mu}(\gamma_{k+1} + 2 - \gamma_k) \hat{\mu}(4\beta + 1 - \gamma_{k+1})$$

. Therefore, any nonzero terms will occur when $4\alpha = \gamma_1$, so we get

$$= \sum_{\gamma_2,\gamma_3,\cdots\gamma_{k+1}} \hat{\mu}(\gamma_2 + 2 - 4\alpha)\hat{\mu}(\gamma_3 + 2 - \gamma_2)\cdots\hat{\mu}(\gamma_{k+1} + 2 - \gamma_k)\hat{\mu}(4\beta + 1 - \gamma_{k+1})$$

This means that $\gamma_2 \in 4\Gamma$ and $\gamma_3, \gamma_4 \cdots, \gamma_{k+1} \in 4\Gamma$. But, then since $\gamma_{k+1} \in 4\Gamma$ we get $\hat{\mu}(4\beta + 1 - \gamma_{k+1}) = 0$ by Lemma 4.1.

We have already shown in Theorem 4.13 that $(S_0^* M_1 S_1)_{\alpha,\beta}$ has nonzero entries. For the general case, consider $(S_0^* M_1^{2k+1} S_1)_{\alpha,\beta}$. The entries (α, β) will be:

$$= \sum_{\gamma_1,\gamma_2,\cdots,\gamma_{2k+2}} \hat{\mu}(4\alpha - \gamma_1)\hat{\mu}(\gamma_2 + 1 - \gamma_1)\cdots\hat{\mu}(\gamma_{2k+2} + 1 - \gamma_{2k+1})\hat{\mu}(4\beta + 1 - \gamma_{2k+2})$$

since this is nonzero only when $4\alpha = \gamma_1$, we get:

$$= \sum_{\gamma_2,\cdots,\gamma_{2k+2}} \hat{\mu}(\gamma_2 + 1 - 4\alpha)\hat{\mu}(\gamma_3 + 1 - \gamma_2)\cdots\hat{\mu}(\gamma_{2k+2} + 1 - \gamma_{2k+1})\hat{\mu}(4\beta + 1 - \gamma_{2k+2})$$

This is nonzero only when $\gamma_2 \in 4\Gamma + 1$, so we get that

$$= \sum_{\gamma_2,\cdots,\gamma_{2k+2}} \hat{\mu}(4\gamma_2 + 2 - 4\alpha)\hat{\mu}(\gamma_3 - 4\gamma_2)\cdots\hat{\mu}(\gamma_{2k+2} + 1 - \gamma_{2k+1})\hat{\mu}(4\beta + 1 - \gamma_{2k+2})$$

This means that $\gamma_3 = 4\gamma_2$ in order for nonzero terms. Continuing on in this manner, we get:

$$= \sum_{\gamma_2,\gamma_4,\cdots,\gamma_{2k+2}} \hat{\mu}(4\gamma_2 + 2 - 4\alpha)\hat{\mu}(4\gamma_4 + 2 - 4\gamma_2)\cdots\hat{\mu}(4\beta + 2 - 4\gamma_{2k+2})$$

which has nonzero terms. $\square$

4.4.2 Hadamard Product

Previously, we have been viewing the results normal matrix multiplication. In this section, we will look at what happens when we use the Hadamard product to multiply the matrices. Recall that the Hadamard product, unlike regular matrix multiplication, multiplies just the entries together. In this case, when a 1 occurs in the Hadamard product, that means that since $|\hat{\mu}(\cdot)| \leq 1$, that both matrices have a 1 in the exact same spot. It is an interesting way to examine where the related nonzero terms of the matrices are located.

Lemma 4.15. *Given the matrices $U_{\alpha,\beta}$ and $S_{0\alpha,\beta}$ the Hadamard product of the two is*

$$(U * S_0)_{\alpha,\beta} = (u_{\alpha\beta}s_{0\alpha\beta}) \text{ where } u_{\alpha\beta}s_{0\alpha\beta} = \begin{cases} 1 & \text{at } (0,0) \\ 0 & \text{else} \end{cases}$$

Proof. $u_{\alpha\beta}s_{0\alpha,\beta} = \hat{\mu}(5\beta - \alpha)\hat{\mu}(4\beta - \alpha)$. Since $s_{0\alpha\beta} = 0$ for all (α, β) except when $\alpha = 4\beta$, then

$$\begin{aligned} (u * s_0)_{\alpha,\beta} &= \hat{\mu}(5\beta - \alpha)\hat{\mu}(4\beta - \alpha) \\ &= \hat{\mu}(5\beta - 4\beta) \\ &= \hat{\mu}(\beta) \\ &= \begin{cases} 1 & \text{when } \beta = 0 \\ 0 & \text{else} \end{cases}. \end{aligned}$$

So, the only nonzero entry is $(\alpha, \beta) = (0, 0)$ □

Remark 4.16. The conjugate transpose entries of $S_{0\alpha,\beta}$ are $s^*_{0\alpha,\beta} = \hat{\mu}(4\alpha - \beta)$ and the conjugate transpose entries of $S_{1\alpha,\beta}$ are $s^*_{1\alpha,\beta} = \hat{\mu}(4\alpha + 1 - \beta)$

Lemma 4.17. *Given the matrices $U_{\alpha,\beta}$ and $S^*_{0\alpha,\beta}$ the Hadamard product of the two is $(U * S^*_0)_{\alpha,\beta} = (u_{\alpha\beta}s^*_{\alpha\beta})$ where*

$$u_{\alpha\beta}s^*_{0\alpha\beta} = \begin{cases} 1 & at\ (0,0) \\ \\ 0 & else \end{cases}$$

Proof.

$$u_{\alpha\beta}s^*_{0\alpha\beta} = \hat{\mu}(5\beta - \alpha)\hat{\mu}(4\alpha - \beta)$$

Notice that $\hat{\mu}(4\alpha - \beta) = 0$ unless $4\alpha = \beta$ in which case $\hat{\mu}(4\alpha - \beta) = 1$. So, we have $\beta = 4\alpha$ which implies that $u_{\alpha\beta}s^*_{0\alpha\beta} = \hat{\mu}(19\alpha)$. If α is odd, then $\hat{\mu}(19\alpha) = 0$. But, if $\alpha \neq 0$ is even, we get that $\alpha = \sum_i 4^i a_i$ where $a_0 = 0$. Considering $\hat{\mu}(19\sum_i 4^i a_i)$, by Remark 4.2 we can take out the smallest power of 4, leaving us with $\hat{\mu}(19\sum_i 4^i a_i)$ where $a_0 = 1$, which means $\hat{\mu}(19\alpha) = 0$. So, we have that

$$\hat{\mu}(19\alpha) = \begin{cases} 1 & \text{if } \alpha = 0 \\ \\ 0 & \text{else} \end{cases}.$$

So, when $(\alpha, \beta) = (0, 0)$, we get $1 \cdot 1 = 1$, otherwise we get 0. ☐

Lemma 4.18. *Given the matrices $U_{\alpha,\beta}$ and $S_{1\alpha,\beta}$ the Hadamard product of the two is $(U * S_1)_{\alpha,\beta} = (u_{\alpha\beta}s_{1\alpha\beta})$ where*

$$u_{\alpha\beta}s_{1\alpha\beta} = \begin{cases} 1 & at\ (5,1) \\ \\ 0 & else \end{cases}$$

Proof. Each entry $u_{\alpha\beta}s_{1\alpha\beta} = \hat{\mu}(5\beta - \alpha)\hat{\mu}(4\beta + 1 - \alpha)$. The only time $s_{1\alpha\beta}$ is nonzero is when $\alpha = 4\beta + 1$, which means $u_{\alpha\beta}s_{1\alpha\beta} = \hat{\mu}(5\beta - (4\beta + 1)) = \hat{\mu}(\beta - 1)$. Since $1 \in \Gamma$, $\hat{\mu}(\beta - 1) = 1$ for $\beta = 1$ and is zero everywhere else. Since $\alpha = 4\beta + 1 = 5$, the only nonzero entry is at $(\alpha, \beta) = (5, 1)$. ☐

So far, these results have not been too exciting. The last Hadamard product we will take is $U_{\alpha\beta}$ and $S^*_{1\alpha,\beta}$. Surprisingly, unlike the previous results, the Hadamard product of these two has infinitely many nonzero terms:

Lemma 4.19. *Given the matrices $U_{\alpha,\beta}$ and $S^*_{1\alpha,\beta}$ the Hadamard product of the two is $(U * S^*_1)_{\alpha,\beta} = (u_{\alpha\beta}s^*_{1\alpha\beta})$. $(U * S^*_1)_{\alpha,\beta}$ has infinitely many nonzero entries.*

Remark 4.20. For the following proof we will be making use of the fact that since $\Gamma = \{\sum_i 4^i b_i : b_i = \{0,1\},\ \text{finite sums}\ \}$ every $\gamma \in \Gamma$ can be written in base 4 notation with only zeros and ones occurring. For example, $20 \in \Gamma$ and $20 = 4^2 + 4 = (110)_4$. If a number, x has a 2 or 3 occurring in its base 4 representation, then $x \notin \Gamma$.

Proof. The entries in $(U * S^*_1)_{\alpha,\beta}$ are

$$u_{\alpha\beta}s^*_{1\alpha\beta} = \hat{\mu}(5\beta - \alpha)\hat{\mu}(4\alpha + 1 - \beta)\ (S^*_1)_{\alpha,\beta} = 1$$

if $\beta = 4\alpha + 1$ and zero otherwise. So, $(U * S^*_1)_{\alpha,\beta} = \hat{\mu}(5(4\alpha + 1) - \alpha) = \hat{\mu}(19\alpha + 5)$, which is zero is α is even. However, if α is odd, $19 = (103)_4$ multiply by a number that ends in $(01)_4$ the last two numbers are $(03)_4$ If we add $5 = (11)_4$ we end up with a number ending in $(20)_4$ which is not in Γ. Since there are infinitely many $\gamma \in \Gamma$ that end in $(01)_4$ there are infinitely many nonzero entries in $(U * S^*_1)_{\alpha,\beta}$. ∎

Now that we have clarified the relationships between $S_{0\alpha,\beta}$, $S_{1\alpha,\beta}$, $M_{1\alpha,\beta}$ and $U_{\alpha,\beta}$ we will turn our attention in the next chapter solely to $U_{\alpha,\beta}$.

CHAPTER 5
THE $U_{\alpha,\beta}$ MATRIX

In this chapter, we will look in depth at the matrix $U_{\alpha,\beta}$ and the subsequent powers, $U^k_{\alpha,\beta}$. In doing so, we will also be discovering what $\gamma \in \Gamma \cap 5^k\Gamma$ for $k = 1, 2, 3 \cdots$.

5.1 The 1's of $U_{\alpha,\beta}$

An interesting question is where do 1's occur in the entries of the infinite matrix $U_{\alpha,\beta}$; in detail, given $U_{\alpha,\beta} = (u_{\alpha\beta})$, for what (row , column) = (α, β) does $u_{\alpha\beta} = 1$. Since $u_{\alpha\beta} = \hat{\mu}(5\beta - \alpha)$ if $u_{\alpha\beta} = 1$, then $5\beta = \alpha$. In other words, we are finding $\gamma \in \Gamma$ such that $\gamma \in 5\Gamma \cap \Gamma$. The answer to this question is complicated by the fact that $U : \Gamma \to 5\Gamma$ and $5\Gamma \not\subset \Gamma$ (as is seen by $Ue_5 = e_{25}$ and $25 \notin \Gamma$).

To get a sense of how this question can be answered, let's first look at the entries of the infinite matrices $S_{0\alpha,\beta}$ and $S_{1\alpha,\beta}$. These should be easier to find since, for example, $s_{0\alpha\beta} = \hat{\mu}(4\beta - \alpha)$. Therefore, if we find where $s_{0\alpha\beta} = 1$ we are finding $\gamma \in \Gamma$ such that $\gamma \in 4\Gamma \cap \Gamma$. Since $4\Gamma \subset \Gamma$, we can already know the answer to this question.

Lemma 5.1. *For $\alpha, \beta \in \Gamma$, $S_{0\alpha,\beta} = (\hat{\mu}(4\beta + 1 - \alpha))$ has infinitely many entries of 1. These 1's of the matrix $S_{0\alpha,\beta}$ occur at entries (row, column)= $(4\beta, \beta)$.*

Proof. This can be easily seen as each entry, $s_{0\alpha\beta}$, of the infinite matrix $S_{0\alpha,\beta}$ is

$$s_{0\alpha\beta} = \hat{\mu}(4\beta - \alpha)$$

Since $4\beta - \alpha \in \Gamma$, $\hat{\mu}(4\beta - \alpha) = 0$ unless $4\beta = \alpha$, in which case $\hat{\mu}(4\beta - \alpha) = 1$. Since $4\beta \in \Gamma$ for all β there are infinitely such (α, β). □

Remark 5.2. The 1's of the matrix $S_{0\alpha,\beta}$ occur at $(0, 0)$, $(4, 1)$, $(16, 4)$, etc. A 1 occurs once in every even row. All other entries are 0. Again, another way of looking at these points (α, β) is that we found all values of α such that $\alpha \in \Gamma \cap 4\Gamma$.

Lemma 5.3. *For $\alpha, \beta \in \Gamma$, $S_{1\alpha,\beta} = (\hat{\mu}(4\beta + 1 - \alpha))$ has infinitely many entries of 1. The 1's of the matrix $S_{1\alpha,\beta}$ occur at entries $(4\beta + 1, \beta)$.*

Proof. Very similar to the previous lemma, we have that each entry of $S_{1\alpha,\beta}$ is

$$s_{1\alpha\beta} = \hat{\mu}(4\beta + 1 - \alpha)$$

which is zero unless $4\beta + 1 = \alpha$, in which case $\hat{\mu}(4\beta + 1 - \alpha) = 1$. Since $4\Gamma + 1 \subset \Gamma$ there are infinitely such entries (α, β). □

Remark 5.4. The 1's of the matrix $S_{1\alpha,\beta}$ occur at $(1, 0)$, $(5, 1)$, $(17, 4)$, etc. In other words, a 1 occurs once in every odd row. Again, by looking at all these points (α, β), we have found all values α such that $\alpha \in \Gamma \cap 4\Gamma + 1$.

Finding the entries of 1 in these infinite matrices was made easier by the fact that these operators map Γ back into Γ. Now, let's take a look at what happens with the infinite matrix $U_{\alpha,\beta}$.

Theorem 5.5. *Let (μ, Γ) be a spectral pair.*

$$\hat{\mu}(t) = e^{i\pi t 2/3} \prod \cos\left(\frac{\pi t}{2 \cdot 4^n}\right)$$

and

$$\Gamma = \left\{ \sum_0^{finite} b_i 4^i | b_i \in \{0,1\} \right\}$$

Let $U : L^2(\mu) \to L^2(\mu)$ *be determined by*

$$Ue_\lambda = e_{5\lambda}$$

For $\alpha, \beta \in \Gamma$, $U_{\alpha,\beta} = (\hat{\mu}(5\beta - \alpha))$ *has infinitely many entries of 1's and they occur at* (α, β) *where* $\alpha = 5\beta$ *when for* $c_i = 0$ *or* 1, $\beta = \sum_i c_i 4^i$ *such that if* $c_i = 1$ *then* $c_{i+1} = 0$.

Proof. If $\hat{\mu}(5\beta - \alpha) = 1$ then $5\beta = \alpha$. It might be more helpful if we look at it more generally as: $5\beta \in \Gamma$

Then we have that since $\beta \in \Gamma$,

$$5\beta = 5\sum_i c_i 4^i = (1+4)\sum_i c_i 4^i = \sum_i c_i 4^i + \sum_i c_i 4^{i+1}$$

Since we want $5\beta \in \Gamma$, we need $\sum_i c_i 4^i + \sum_i c_i 4^{i+1} = \sum_j a_j 4^j$ where $a_j = 0$ or 1. Notice that $\sum_i c_i 4^i + \sum_i c_i 4^{i+1} = c_0 + 4(c_0 + c_1) + 4^2(c_1 + c_2) + \cdots + 4^k(c_{k-1} + c_k)$. This is in Γ if $c_i = 1$ then $c_{i+1} = 0$. Now we will look at $\alpha = 5\beta$. As stated above

$$\alpha = \sum_j a_j 4^j = c_0 + 4(c_0 + c_1) + 4^2(c_1 + c_2) + \cdots + 4^k(c_{k-1} + c_k)$$

where if $c_i = 1$ then $c_{i+1} = 0$ for $i = 0, 1, 2, 3, \cdots, n$ and $j = 0, 1, 2, 3, \cdots, m$ such that if $c_k = 1$, then $a_k = 1$ and $a_{k+1} = 1$, otherwise $a_i = 0$. (One way to see this is to consider that if $1 = \beta$, then $c_0 = 1$. We would then have $a_0 = 1$ and $a_1 = 1$, so that $\alpha = 1 + 4 \cdot 1 = 5$ and so we would have$\hat{\mu}(5\beta - \alpha) = 1$). By construction, there are infinitely many such entries (α, β). $\square$

Remark 5.6. Some of the (α, β) such that $u_{\alpha,\beta} = 1$ are $(0,0)$, $(5,1)$, $(20,4)$, $(80,16)$, etc. We asked earlier what $\gamma \in \Gamma$ are also in 5Γ. The answer is the α described in the proof of Theorem 5.5: $\alpha \in \Gamma \cap 5\Gamma$ (notice that $0, 5, 20, \cdots \in 5\Gamma \cap \Gamma$).

5.2 Powers of $U_{\alpha,\beta}$

The next question to ask is where the 1's occur in the powers of the $U_{\alpha,\beta}$ matrix, $U^k_{\alpha,\beta}$. This question is more complicated than it may first appear.

5.2.1 Hadamard Product

It is not as easy as it would be looking at, say, the Hadamard product. By construction of the Hadamard product, $(U * U)_{\alpha,\beta} = (u_{\alpha\beta}u_{\alpha\beta}) = 1$ when $u_{\alpha\beta} = 1$. Consider the case of $(U * U^*)_{\alpha,\beta}$.

Lemma 5.7. *The Hadamard product of $U_{\alpha,\beta}$ and $U^*_{\alpha,\beta}$, $(U * U^*)_{\alpha,\beta} = 1$ only at $(0,0)$*

Proof. $u_{\alpha\beta} * u^*_{\alpha\beta} = \hat{\mu}(5\beta - \alpha)\hat{\mu}(5\alpha - \beta) = 1$ iff both $\hat{\mu}(5\beta - \alpha)$ and $\hat{\mu}(5\alpha - \beta)$ are 1 (since $\hat{\mu}(\gamma) \leq 1$ by definition). This happens only if $5\alpha = \beta$ and $5\beta = \alpha$. So, when $25\beta = \beta$ which happens only at $\beta = 0$. □

Remark 5.8. It is not nonzero for all other entries. For example for all entries $(\alpha, 5\alpha)$ we get that the entries of $U * U^*$ are

$$u_{\alpha\beta}u^*_{\alpha\beta} = \hat{\mu}(24\alpha) = \hat{\mu}(6\alpha)$$

which is nonzero for all α

5.2.2 Squaring $S_{0\alpha,\beta}$ and $U_{\alpha,\beta}$

Now, let's go back to regular matrix multiplication and look at $S^2_{0\alpha,\beta}$.

Lemma 5.9. *$S^2_{0\alpha,\beta}$ has infinitely many entries of 1, occurring at $(16\beta, \beta)$.*

Proof. Each entry of $S^2_{0\alpha,\beta}$ can be represented by $s_{0\alpha\beta} = \sum_\gamma \hat{\mu}(4\gamma - \alpha)\hat{\mu}(4\beta - \gamma)$. Since $4\gamma, 4\beta, \alpha, \beta \in \Gamma$, the only nonzero terms are when $4\gamma = \alpha$ and $4\beta = \gamma$. In other words, when $4^2\beta = \alpha$. When this happens $\hat{\mu}(4\gamma - \alpha) = \hat{\mu}(4\beta - \gamma) = 1$. Since $16\beta \in \Gamma$ for all β, there are infinitely such entries. $\square$

What worked out nicely in this case is that 4β and 4γ were elements of an orthonormal basis Γ (see remark 4.3). We cannot use the same quick solution in the case of $U^2_{\alpha,\beta} = (u_{2\alpha\beta})$ since

$$u_{2\alpha,\beta} = \sum_\gamma \hat{\mu}(5\gamma - \alpha)\hat{\mu}(5\beta - \gamma)$$

and 5γ and 5β may or may not be in Γ. Before we can use a similar method to discover where the 1's of the matrix $U^k_{\alpha,\beta} = (u_{k\alpha\beta})$ we will need a way to check when

$$\sum_{\gamma_1,\gamma_2,\cdots\gamma_{k-1}} \hat{\mu}(5\gamma_1 - \alpha)\hat{\mu}(5\gamma_2 - \gamma_1)\cdots\hat{\mu}(5\beta - \gamma_{k-1}) = 1$$

First, though, we will need the following lemma:

Lemma 5.10. *Given $\Gamma = \{\sum_i 4^i b_i : b_i \in \{0,1\},$ finite sums $\}$ and the set $25\Gamma = \{25\sum_i 4^i b_i : b_i \in \{0,1\},$ finite sums $\} = \{0, 25, 100, 125, \cdots\}$, $\beta \in \Gamma \cap 25\Gamma$ iff $\beta = 0$.*

Remark 5.11. From [11] we know that $5^k\Gamma$ is an orthonormal basis of $L^2(\mu)$ for all $k = 0, 1, 2, 3, \cdots$.

Proof. Note that $0 \in \Gamma$ and since $25 \cdot 0 = 0$, $0 \in 25\Gamma$. If we break down 5^2 into $\sum_i 4^i c_i$ form we get:

$$5^2 = 4^2 + 4 \cdot 2 + 1$$

First, since $\beta \in \Gamma$ then $\beta = \sum_i 4^i c_i$ where $c_i = 0$ or 1. Therefore, we have that:

$$\begin{aligned} 5^2\beta &= \left(4^2 + 4 \cdot 2 + 1\right)\beta \\ &= \left(4^2 + 4 \cdot 2 + 1\right)\sum c_i 4^i \\ &= \sum 4^{i+2}c_i + 2\sum c_i 4^{i+1} + \sum 4^i c_i \end{aligned}$$

where each of the $c_i = 0$ or 1. We want this to be in Γ so we want

$$\sum 4^{i+2}c_i + 2\sum c_i 4^{i+1} + \sum 4^i c_i = \sum 4^j a_j \text{ for } a_j = 0 \text{ or } 1. \tag{5.1}$$

If we rewrite the left hand side of the equation by expanding the sums, we get

$$\begin{aligned} &= 4^2c_0 + 2 \cdot 4c_0 + c_0 + 4^3c_1 + 2 \cdot 4^2c_1 + 4c_1 + 4^4c_2 + 2 \cdot 4^3c_2 + 4^2c_2 \cdots \\ &= c_0 + 4(2x_0 + c_1) + 4^2(c_0 + 2c_1 + c_2) + 4^3(c_1 + 2c_2 + c_3) + \cdots \\ &= c_0 + 4(2c_0 + c_1) + \sum_{i=2} 4^i(c_{i-2} + 2c_{i-1} + c_i) \end{aligned}$$

Let's first consider the case where $c_0 = 1$. This means that $2c_0 + c_1 = 2 + c_1$. Since $c_1 = 0$ or 1, $2 + c_1 = 2$ or 3. Conflicts with equation 5.1. So, $c_0 \neq 1$.

Next, consider the case where $c_0 = 0$. Then, $2c_0 + c_1 = c_1$, so $c_1 = 0$ or 1. If $c_1 = 1$, then as in the first case we get that $c_0 + 2c_1 + c_2 = 2 + c_2 = 2$ or 3. So $c_1 \neq 1$. If $c_1 = 0$, then $c_0 + 2c_1 + c_2 = c_2 = 0$ or 1.

Suppose that c_k is the first $c_i \neq 0$. As we have shown, $k \geq 2$. So we have

$$4^k(c_k) + 4^{k+1}(2c_k + c_{k+1}) + 4^{k+2}(c_k + 2c_{k+1} + c_{k+2}) + \cdots$$

But, this means that $2c_k + c_{k+1} = 2 + c_{k+1} = 2$ or 3, which cannot happen. Therefore, the only time $5^2\beta \in \Gamma$ is when $\beta = \sum_i 4^i c_i = 0$. $\square$

Theorem 5.12. *Given the operator U along with the orthonormal basis*

$$\Gamma = \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0,1\}, \textit{finite sums}\},$$

the (α, β) entry of the infinite matrix $U^k_{\alpha,\beta}$ represented by

$$\sum_{\gamma_1,\gamma_2,\cdots\gamma_{k-1}} \hat{\mu}(5\gamma_1 - \alpha)\hat{\mu}(5\gamma_2 - \gamma_1)\cdots\hat{\mu}(5\beta - \gamma_{k-1}) = 1 \tag{5.2}$$

iff each $\hat{\mu}(\cdot) = 1$.

Proof. One way is trivial since if each $\hat{\mu}(5\lambda_i - \xi_i) = 1$ then $5\lambda_i = \xi_i$ for each i and we get that $1 \cdot 1 \cdots 1 = 1$.

For the other direction, assume $\sum_{\gamma_1,\gamma_2,\cdots\gamma_{k-1}} \hat{\mu}(5\gamma_1 - \alpha)\hat{\mu}(5\gamma_2 - \gamma_1)\cdots\hat{\mu}(5\beta - \gamma_{k-1}) = 1$ Since Γ is an orthonormal basis we have that $\sum_{\xi\in\Gamma} |\hat{\mu}(t-\xi)|^2 = 1$.

Let's first look at the case when $k = 2$.

By Cauchy-Schwarz inequality (Lemma 2.2) we know that since we have an orthonormal basis,

$$|u_{2\alpha\beta}|^2 \leq \sum_{\gamma} \left(|\hat{\mu}(5\beta - \gamma)|^2\right)^{\frac{1}{2}} \left(|\hat{\mu}(5\gamma - \alpha)|^2\right)^{\frac{1}{2}} \tag{5.3}$$

We want to find when an entry in $|u_{2\alpha\beta}|^2$ is 1, so we want the above inequality to be equality. In order for equality to occur in Cauchy-Schwarz the vectors need to be aligned. This means that

$$\hat{\mu}(5\gamma - \alpha) = c_{\alpha\beta}\hat{\mu}(5\beta - \gamma) \tag{5.4}$$

for all $\gamma \in \Gamma$ where α, β are fixed and $c_{\alpha\beta}$ depends on $\alpha, \beta \in \Gamma$ and $|c_{\alpha\beta}| = 1$.

Substitute back into Equation 5.11 to get

$$|u_{2\alpha\beta}|^2 = \sum_\gamma c_{\alpha\beta} \left(\hat{\mu}(5\beta - \gamma)\right)^2$$

Since c does not depend on γ we can take it out of the sum:

$$|u_{2\alpha\beta}|^2 = c_{\alpha\beta} \sum_\gamma \left(\hat{\mu}(5\beta - \gamma)\right)^2 \tag{5.5}$$

Notice that we no longer have the right hand side depending on α so

$$c_{\alpha\beta} = c_\beta.$$

We can rewrite Equation 5.12 as

$$< e_\alpha, e_{5\gamma} > = c_\beta < e_\gamma, e_{5\beta} > \tag{5.6}$$

where $|c_\beta| = 1$.

The question is: Is there more than one term that is nonzero? If there is not, then the theorem is true.

Let's look at the case when $\gamma_0 = 0$ in Equation 5.6:

$$< e_\alpha, e_0 > = c_\beta < e_0, e_{5\beta} >$$

Since $< e_\alpha, e_0 >= 0$ unless $\alpha = 0$, and $< e_0, e_{5\beta} > = 0$ unless $\beta = 0$, then $\alpha = \beta = 0$ is the only nonzero term.

Now, let's look at the case of γ_1. Then $< e_\alpha, e_5 > = 0$ unless $\alpha = 5$ and $< e_{5\beta}, e_1 > < 1$ so there are no nonzero terms.

Let's consider the general case of $\gamma \in \Gamma$. Let γ_1 be the first $\gamma \in \Gamma$ such that both

$$|\hat{\mu}(5\gamma - \alpha)| < 1 \text{ and } |\hat{\mu}(5\beta - \gamma)| < 1. \tag{5.7}$$

where the equation

$$\hat{\mu}(5\gamma - \alpha) = c_{\alpha\beta}\hat{\mu}(5\beta - \gamma) \tag{5.8}$$

holds. So, we have multiple $\gamma_i : \gamma_1 < \gamma_2 < \gamma_3 < \cdots$ such that

$$e_{5\beta} = Ae_{\gamma_1} + Be_{\gamma_2} + \cdots$$

$$e_{\alpha} = c_{\beta}\left(Ae_{5\gamma_1} + Be_{5\gamma_2} + \cdots\right)$$

Therefore, we have the equation

$$Ue_{5\beta} = c_{\beta}e_{\alpha} \tag{5.9}$$

which means that

$$\beta \to 5\beta \to \alpha \to 5\alpha \tag{5.10}$$

which implies that

$$25\beta = \alpha$$

But since $\alpha \in \Gamma$, by Lemma 5.10 we know that this only happens when $\beta = 0 = \alpha$.

Now let's return to:

$$\begin{aligned} \hat{\mu}(5\gamma - \alpha) &= c_{\beta}\hat{\mu}(5\beta - \gamma) \\ < e_{\alpha}, e_{5\gamma} > \; &= c_{\beta} < e_{\gamma}, e_{5\beta} > \\ c_{\beta}^{*} < e_{\alpha}, e_{5\gamma} > \; &= < e_{\gamma}, e_{5\beta} > \\ e_{5\beta} &= c_{\beta}^{*}\sum_{\gamma} < e_{\alpha}, e_{5\gamma} > e_{\gamma} \\ c_{\beta}e_{5\beta} &= \sum_{\gamma} < e_{\alpha}, e_{5\gamma} > e_{\gamma} \end{aligned}$$

Notice that the left side is only dependent on β and the right side is only dependent on α and since there is only one nonzero term and $\hat{\mu}(5\gamma-\alpha)\hat{\mu}(5\beta-\gamma) = 1$ only when $\hat{\mu}(5\gamma - \alpha) = 1$ and $\hat{\mu}(5\beta - \gamma) = 1$

Now, suppose that for k, $U^k_{\alpha,\beta}$ has an entry of one iff each $\hat{\mu}(\cdot) = 1$ of

$$u_{k\alpha\beta} = \sum_{\gamma_1,\gamma_2,\ldots\gamma_k} \hat{\mu}(5\beta - \gamma_k)\hat{\mu}(5\gamma_k - 5\gamma_{k-1})\ldots\hat{\mu}(5\gamma_1 - \alpha).$$

Looking at $k+1$:

For $U^{k+1}_{\alpha,\beta} = U^k_{\alpha,\gamma_k} U_{\gamma_k,\beta}$,

By Cauchy-Schwarz inequality (Lemma 2.2) we know that since we have an orthonormal basis,

$$|u_{(k+1)\alpha\beta}|^2 \leq \sum_{\gamma} \left(|u_{k\alpha\gamma_k}|^2\right)^{\frac{1}{2}} \left(|\hat{\mu}(5\gamma_k - \alpha)|^2\right)^{\frac{1}{2}} \tag{5.11}$$

We want to find when an entry in $|u_{(k+1)\alpha\beta}|^2$ is 1, so we want the inequality to be equality. In order for equality to occur in Cauchy-Schwarz the vectors need to be aligned. This means that

$$u_{k\alpha\gamma_k} = c_{\alpha\beta}\hat{\mu}(5\beta - \gamma) \tag{5.12}$$

for all $\gamma \in \Gamma$ where α, β are fixed and $c_{\alpha\beta}$ depends on $\alpha, \beta \in \Gamma$ and $|c_{\alpha\beta}| = 1$.

Substitute back to get

$$|u_{(k+1)\alpha\beta}|^2 = \sum_{\gamma} c_{\alpha\beta} \left(\hat{\mu}(5\beta - \gamma)\right)^2$$

And so, we end up with the same scenario we had for $U^2_{\alpha,\beta}$. So in order for $|u_{(k+1)\alpha\beta}| = 1$, we must have that $\hat{\mu}(5\beta - \gamma_k) = \hat{\mu}(5\gamma_k - \gamma_{k-1}) = \cdots = \hat{\mu}(5\gamma_1 - \alpha) = 1$. $\square$

Now, finally, let's find the 1's of the matrix $U^k_{\alpha,\beta}$ for $k = 2, 3, 4, \cdots$.

Theorem 5.13. *Consider the infinite matrix $U^k_{\alpha,\beta}$ with respect to the orthonormal basis*

$$\Gamma = \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0,1\}, \textit{finite sums}\}.$$

Each (α,β) entry of $U^k_{\alpha,\beta} = (u_{k\alpha\beta})$ will be represented by $u_{k\alpha\beta}$. For $k \geq 2$, $u_{k\alpha,\beta} = 1$ if and only if $(\alpha,\beta) = (0,0)$

Proof. To begin, let's look at the case where $k = 2$: Let $\alpha, \beta \in \Gamma$. Since

$$u_{\alpha,\beta} =< e_\alpha, Ue_\beta >= \sum \hat{\mu}(5\beta - \alpha) \text{ then each entry of } U^2_{\alpha,\beta}$$

will be

$$u_{2\alpha,\beta} = \textstyle\sum_\gamma \hat{\mu}(5\gamma - \alpha)\hat{\mu}(5\beta - \gamma)$$

If $(\alpha,\beta) \neq (0,0)$ then $5^2\beta \in \Gamma$ since in order for $U^2_{\alpha,\beta} = \sum \hat{\mu}(5\gamma - \alpha)\hat{\mu}(5\beta - \gamma)$ by Theorem 5.12 we need $5\gamma = \alpha$ and $5\beta = \gamma$. This means that $5(5\beta) = \alpha$. By Lemma 5.10 we know this happens only when $\beta = 0 = \alpha$.

Now, let's look at the general k:

If $(\alpha,\beta) = (0,0)$ then the entry of $U^k_{\alpha,\beta}$ is:

$$\begin{aligned}
&= \sum_{\gamma_1,\gamma_2,\cdots,\gamma_{k-1}} \hat{\mu}(5\gamma_1 - \alpha)\hat{\mu}(5\gamma_2 - \gamma_1)\cdots\hat{\mu}(5\beta - \gamma_{k-1}) \\
&= \sum_{\gamma_1,\gamma_2,\cdots,\gamma_{k-1}} \hat{\mu}(5\gamma_1)\hat{\mu}(5\gamma_2 - \gamma_1)\cdots\hat{\mu}(5\gamma_{k-1} - \gamma_{k-2})\hat{\mu}(\gamma_{k-1})
\end{aligned}$$

Since $\gamma_{k-1} \in \Gamma$, in order for $\hat{\mu}(\gamma_{k-1}) \neq 0$, then $\gamma_{k-1} = 0$. In a similar fashion, $\gamma_{k-1} = 0 \Rightarrow 0 = \gamma_{k-2} = \gamma_{k-3} = \gamma_{k-4} = \cdots = \gamma_1$. So that the $(\alpha,\beta) = (0,0)$ entry of $U^k_{\alpha,\beta}$ is 1.

Suppose there is a k such that an element of the matrix $U^k_{\alpha,\beta}$, $u_{k\alpha\beta} = 1$, then

$$u_{k\alpha\beta} = \sum_{\gamma_1,\gamma_2,\cdots,\gamma_{k-1}} \hat{\mu}(5\gamma_1 - \alpha)\hat{\mu}(5\gamma_2 - \gamma_1)\cdots\hat{\mu}(5\beta - \gamma_{k-1})$$

Using the results from Theorem 5.12 we have that all $\hat{\mu}(\cdot) = 1$, which means that $\hat{\mu}(5\beta - \gamma_{k-1}) = 1$ so that $5\beta = \gamma_{k-1}$ and $\hat{\mu}(5\gamma_{k-1} - \gamma_{k-2}) = 1$ which means that $5\gamma_1 = \gamma_{k-2}$ then we have that $25\beta \in \Gamma$. Based on , Lemma 5.10, $\beta = 0$. If $\beta = 0$, then $0 = 5\beta = \gamma_{k-1}$ and since $\gamma_{k-1} = 0$, then $\gamma_{k-2} = 0$ and so on until $\gamma_1 = 0 \Rightarrow \alpha = 0$. $\square$

This somewhat surprising result. It means that there are infinitely many γ such that $\gamma \in \Gamma \cap 5\Gamma$, but only $0 \in \Gamma \cap 25\Gamma$. In fact, only $0 \in \Gamma \cap 5^k\Gamma$ for all $k \geq 2$.

5.3 Conclusion

By first looking at the orthonormal basis found in [21]:

$$\Gamma = \{\sum_i 4^i b_i : b_i \in \{0,1\}, \text{finite sums}\}$$

and the related orthonormal basis found in [11]

$$5\Gamma = \{5\sum_i 4^i b_i : b_i \in \{0,1\},\ \text{finite sums }\}$$

we found several interesting relationship with the unitary matrix $U_{\alpha,\beta}$ arising from the operator $U : \Gamma \to 5\Gamma$. Investigating the relationships between $S_0 : \Gamma \to 4\Gamma$, $S_1 : \Gamma \to 4\Gamma + 1$, and $M_1 : \Gamma \to \Gamma + 1$ we discovered that $U_{\alpha,\beta}$ commutes with $S_{0\alpha,\beta}$ although it does not commute with $S_{1\alpha,\beta}$ nor $M_{1\alpha,\beta}$.

Most intriguing, when we searched for 1's in the infinite matrix $U^k_{\alpha,\beta}$ we have found that given

$$u_{k\alpha\beta} = \sum_{\gamma_1,\gamma_2,\cdots,\gamma_{k-1}} \hat{\mu}(5\gamma_1 - \alpha)\hat{\mu}(5\gamma_2 - \gamma_1)\cdots\hat{\mu}(5\beta - \gamma_{k-1})$$

in order for $u_k = 1$, we must have each $\hat{\mu}(\cdot) = 1$. Although there are infinitely many 1's occurring in the entries of $U_{\alpha,\beta}$, only one such 1 occurs in the higher powers of U^k. This means that there are infinitely many $\gamma \in \Gamma \cap 5\Gamma$, but $\gamma \in \Gamma \cap 5^k\Gamma = \{0\}$ for $k \geq 2$.

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