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Coisometric Extensions

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COISOMETRIC EXTENSIONS

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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ABSTRACT

There are two primary sources of motivation for the contents of this thesis. The first is an effort to generalize classical dilation theory, a brief history of which is given in Section 2.1. The second source of motivation is the study of the representation theory of tensor algebras associated to C^*-correspondences; these concepts are discussed in Sections 2.2 and 2.4. Although seemingly unrelated, there is a close connection between these two motivating theories.

The link between classical dilation theory and the representation theory of tensor algebras over C^*-correspondences was established by Muhly and Solel in their 1998 paper *Tensor Algebras over C*-Correspondences: Representations, Dilations, and C*-Envelopes* [12]. In that paper, the authors not only introduced the concept of (operator-theoretic) tensor algebras – non-selfadjoint operator algebras that generalize algebraic tensor algebras – but they also developed the representation theory of these algebras. In order to do so, they introduced and made extensive use of a generalized dilation theory for contractions on Hilbert space. In analogy with classical dilation theory, they developed notions of “isometric dilation” and “coisometric extension” for completely contractive representations of the tensor algebra. The process of forming isometric dilations proceeded smoothly, but constructing coisometric extensions proved more problematic. In contrast to the classical case, Muhly and Solel showed that there is a high degree of nonuniqueness involved when building coisometric extensions. This lack of uniqueness proved to be an impediment to developing a full generalization of the dilation and model theories of Sz.-Nagy and Foiaş.

In this thesis, we introduce a way to manage the ambiguities that arise when forming coisometric extensions. More specifically, we show that the notion of a transfer operator from classical dynamics can be adapted to this setting, and we prove that when a transfer operator is fixed in advance, every completely contractive representation of the tensor algebra admits a *unique* coisometric extension that respects the transfer operator in a fashion that we describe in Chapter 5. We also prove a commutant lifting theorem in the context of coisometric extensions.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Overview

If S is an operator[1] on a Hilbert space K and T is an operator on a subspace $H \subseteq K$, we say that S is a *dilation* of T to K and that T is the *compression* of S to H if

$$P\,S\,|_H = T, \tag{1.1}$$

where P denotes the orthogonal projection of K onto H. Additionally, if H is invariant under S (i.e., $SH \subseteq H$, so that the P in (1.1) is superfluous), we call S an *extension* of T to K, and we call T the *restriction* of S to H. An *isometric dilation* of T is a dilation S for which $S^*S = I_K$, and a *coisometric extension* of T is an extension S such that $SS^* = I_K$, where I_K denotes the identity operator on K. The difference between dilating and extending operators can be seen most clearly if the operators are represented matricially: If $S \in B(K)$ is a dilation of $T \in B(H)$, then the matrix of S with respect to the direct sum decomposition $K = H \oplus H^\perp$, where $H^\perp$ denotes the perpendicular complement of H in K, i.e., $H^\perp = K \ominus H$, can be written as

$$S = \begin{bmatrix} T & B \\ C & D \end{bmatrix},$$

[1]We adhere to the convention that "operator" means bounded linear transformation between (two possibly different) Hilbert spaces. Additionally, we denote by $B(H, K)$ the space of all operators from a Hilbert space H to a Hilbert space K, and we write $B(H)$ for $B(H, H)$.

for operators $B \in B(H^\perp, H)$, $C \in B(H, H^\perp)$, and $D \in B(H^\perp)$. Note that this formula implies that H is invariant under S if and only if $C = 0$. Thus, when S is an extension of T, we can write

$$S = \begin{bmatrix} T & B \\ 0 & D \end{bmatrix}.$$

This idea of representing dilations and extensions matricially carries over to the more general setting studied by Muhly and Solel. In fact, their proof of the existence of generalized isometric dilations [12, Theorem 3.3] (see also [13, Theorem 2.18]) was modeled on the so-called Schäffer matrix. Recall that Béla Sz.-Nagy's eponymous dilation theorem [20, Théorème I] (or Theorem 2.4 from Chapter 2) implies that every contraction T on Hilbert space has a unique *unitary (power) dilation* (i.e., a dilation S such that $S^*S = SS^* = I_K$ and such that $T^n = PS^n|_H$ for all $n \geq 1$), subject to a certain minimality condition. One of the many proofs of this historic result was provided by J.J. Schäffer [18], who presented a matricial construction of a unitary dilation for a given contraction. It turns out that restricting one's attention to the "northwest quadrant" of the Schäffer matrix for a contraction T yields an isometric dilation for T, and the "southeast quadrant" provides a coisometric extension for T. By comparison, Muhly and Solel's isometric dilation looks very similar to the northwest quadrant of the Schäffer matrix, and our construction of a coisometric extension closely resembles the southeast quadrant (see Section 5.2).

In order to describe how the southeast corner of the Schäffer matrix extends to the more general setting, we need the notion of the tensor algebra $\mathcal{T}_+(E)$ over a C^*-

correspondence E. For now, a C^*-correspondence E can be thought of as a bimodule over a C^*-algebra A that generalizes the concept of a Hilbert space. In fact, every Hilbert space is a C^*-correspondence over the C^*-algebra $A = \mathbb{C}$ of complex numbers. There are analogs of the C^*-algebra of bounded operators on a Hilbert space and its C^*-subalgebra of compact operators in this more general setting; they are denoted $\mathcal{L}(E)$ and $\mathcal{K}(E)$, respectively. The left-action of A on E is always given by a homomorphism[2] $\phi : A \to \mathcal{L}(E)$. For the reader who is familiar with the Fock space $\mathcal{F}(E)$ associated to a C^*-correspondence E ($\mathcal{F}(E)$ is itself a C^*-correspondence), the tensor algebra over E, $\mathcal{T}_+(E)$, is the norm-closed subalgebra of $\mathcal{L}(\mathcal{F}(E))$ generated by the so-called creation operators $\{T_\xi\}_{\xi \in E}$ and $\phi_\infty(A)$, where $\phi_\infty : A \to \mathcal{L}(\mathcal{F}(E))$ denotes the homomorphism that gives the left-action of A on $\mathcal{F}(E)$. For those unfamiliar with Fock space, it suffices to consider the special case where $E = \mathbb{C}^d$, $d < \infty$, in which case the tensor algebra $\mathcal{T}_+(E)$ is naturally isomorphic to a certain closure of the free algebra on d generators.

Our focus is on the theory of completely contractive representations of the tensor algebra on Hilbert space. What is meant by the adverb "completely" is explained in the next chapter; for now, what is important is that such a representation $\rho : \mathcal{T}_+(E) \to B(H)$ is a homomorphism from $\mathcal{T}_+(E)$ to $B(H)$ that does not expand the norm. Each $\rho : \mathcal{T}_+(E) \to B(H)$ determines a C^*-representation σ of A on H

[2]Whenever we speak of a "homomorphism" between C^*-algebras we always mean a $*$-homomorphism. In a similar way, when we write "representation" we always mean *non-degenerate* $*$-representation.

given by the formula

$$\sigma(a) = \rho(\phi_\infty(a)), \quad a \in A. \tag{1.2}$$

The representation σ in turn determines the so-called induced Hilbert space $E \otimes_\sigma H$, which is the Hausdorff completion of the algebraic tensor product of E and H with respect to an inner product built from σ and the inner products on E and H. The *representation of $\mathcal{L}(E)$ induced from σ*, $\sigma^E : \mathcal{L}(E) \to B(E \otimes_\sigma H)$, is defined by the formula

$$\sigma^E(X)(\xi \otimes h) = (X\xi) \otimes h, \quad X \in \mathcal{L}(E), \xi \otimes h \in E \otimes_\sigma H.$$

The terminology in this setting is due to Rieffel [16], who adopted it from group representation theory. The formula

$$\mathfrak{z}(\xi \otimes h) = \rho(T_\xi)h, \quad \xi \otimes h \in E \otimes_\sigma H, \tag{1.3}$$

defines an operator $\mathfrak{z} : E \otimes_\sigma H \to H$ that is contractive and intertwines the representations $\sigma^E \circ \phi$ and σ, that is, $||\mathfrak{z}|| \leq 1$ and $\mathfrak{z}\sigma^E(\phi(a)) = \sigma(a)\mathfrak{z}$ for all $a \in A$. We refer to the pair $(\mathfrak{z}, \sigma)$ as the *contractive intertwining pair* associated to ρ. Conversely, given a contractive intertwining pair $(\mathfrak{z}, \sigma)$ acting on a Hilbert space H, formulas (1.2) and (1.3) determine a completely contractive representation $\rho : \mathcal{T}_+(E) \to B(H)$, which we call the *integrated form* of $(\mathfrak{z}, \sigma)$ and denote by $\rho = \mathfrak{z} \times \sigma$. We say $(\mathfrak{z}, \sigma)$ is an *isometric* (respectively, *coisometric*) *intertwining pair* if $\mathfrak{z}$ is isometric (resp., coisometric). Using these definitions, we may formulate notions of isometric dilation and coisometric extension for a completely contractive representation $\rho : \mathcal{T}_+(E) \to B(H)$.

Muhly and Solel showed that in order to dilate ρ one essentially needs only to dilate the operator $\mathfrak{z}$ given by formula (1.3) to an isometric operator, along the lines of Schäffer's matrix. In the process of dilating $\mathfrak{z}$, the representation σ from formula (1.2) is extended uniquely in a natural way. On the other hand, although $\mathfrak{z}$ can be extended to a coisometry, the process of simultaneously extending σ in this setting leads to arbitrary choices, and therefore a loss of uniqueness. In the following chapters, we eliminate the arbitrariness of the choices made in extending σ by introducing to the theory the notion of a generalized transfer operator.

If $\psi : A \to B$ is a C^*-homomorphism between C^*-algebras A and B, a *generalized transfer operator* for ψ is a completely positive linear map $\tau : B \to A$ such that

$$\tau(b\psi(a)) = \tau(b)a, \quad a \in A,\, b \in B.$$

The notion of a generalized transfer operator is a natural extension of the concept of a transfer operator that arises in classical dynamics. We usually work with unital C^*-algebras, in which case a (unital) generalized transfer operator is simply a completely positive left-inverse for ϕ. Moreover, in the special case where the C^*-algebra $A = \mathbb{C}$, a unital generalized transfer operator $\tau : B \to A$ is simply a state, i.e., a positive linear functional of norm one, on the C^*-algebra B. Thus, generalized transfer operators simultaneously extend the notions of transfer operators from classical dynamics and states on C^*-algebras.

In Chapters 4 and 5, we apply the concept of a generalized transfer operator to the homomorphism ϕ that gives the left-action of A on E. Of course, there is the issue of whether a generalized transfer operator for ϕ even exists. Evidently the zero operator from $\mathcal{L}(E)$ to A satisfies all the properties required of a generalized transfer operator. Unfortunately, it need not be the case that a *nonzero* generalized transfer operator for ϕ exists. Nevertheless, there are many situations in which they do exist. In particular, nonzero generalized transfer operators always exist when $A = \mathbb{C}$. In any event, we always assume as part of our hypotheses that nonzero transfer operators exist and are *smooth* in the sense that they do not annihilate the C^*-algebra $\mathcal{K}(E)$. It is in these situations that we can state and prove our main result.

Let E be a C^*-correspondence over a unital C^*-algebra A, and let $(\mathfrak{z}, \sigma)$ be a contractive intertwining pair acting on a Hilbert space H. Our main result (Theorems 5.2 and 5.3) shows that if the left-action map $\phi : A \to \mathcal{L}(E)$ admits a smooth, unital generalized transfer operator $\tau : \mathcal{L}(E) \to A$, then there exists a coisometric extension $(\mathfrak{w}_\infty, \pi_\infty)$ of $(\mathfrak{z}, \sigma)$, acting on a Hilbert space $K \supseteq H$, that is uniquely determined by $(\mathfrak{z}, \sigma)$ and τ subject to relations that we now describe. Just as in the classical case, we form the *defect operator* $\Delta_* = (I_H - \mathfrak{z}\mathfrak{z}^*)^{1/2}$ for the operator $\mathfrak{z}^* : H \to E \otimes_\sigma H$. We compose the representation σ with the transfer operator τ to obtain a completely positive map $\sigma \circ \tau : \mathcal{L}(E) \to B(H)$. Stinespring's dilation theorem [19, Theorem 1] then applies to give a Hilbert space K_1, a representation $\rho_1 : \mathcal{L}(E) \to B(K_1)$, and an isometry $V_1 : H \to K_1$ such that

$$V_1^* \rho_1(X) V_1 = \sigma(\tau(X)), \quad X \in \mathcal{L}(E).$$

The triple (K_1, ρ_1, V_1) is unique up to unitary equivalence under a simple minimality assumption. Furthermore, since τ is a left-inverse for ϕ under the assumption that

$\tau(I_E) = 1_A$, the previous formula implies that

$$V_1^* \rho_1(\phi(a)) V_1 = \sigma(a), \quad a \in A.$$

After restricting to a subspace $\mathcal{D}_*$ of K_1, we apply Rieffel's imprimitivity theorem [16, Theorem 6.29] to produce a Hilbert space H_1, a representation $\theta_1 : A \to B(H_1)$, and a Hilbert space isomorphism of $\mathcal{D}_*$ onto $E \otimes_{\theta_1} H_1$ that implements an equivalence between ρ_1 and the induced representation $\theta_1^E : \mathcal{L}(E) \to B(E \otimes_{\theta_1} H_1)$. Moreover, by Rieffel's equivalence theorem [16, Theorem 6.23], H_1 and θ_1 are determined uniquely up to unitary equivalence. We define an operator $\mathfrak{w}_1 : (E \otimes_\sigma H) \oplus (E \otimes_{\theta_1} H_1) \to H \oplus H_1$ matricially by

$$\mathfrak{w}_1 = \begin{bmatrix} \mathfrak{z} & \Delta_* V_1^* \\ 0 & 0 \end{bmatrix},$$

and we define a representation $\pi_1 : A \to B(H \oplus H_1)$ by

$$\pi_1 = \begin{bmatrix} \sigma & 0 \\ 0 & \theta_1 \end{bmatrix}.$$

The pair $(\mathfrak{w}_1, \pi_1)$ is a *partially isometric extension* of $(\mathfrak{z}, \sigma)$ (i.e., an extension such that $\mathfrak{w}_1$ is a partial isometry) that is uniquely determined by τ up to unitary equivalence. The rest of the proof of the existence and uniqueness of $(\mathfrak{w}_\infty, \pi_\infty)$ proceeds in an inductive fashion: At each stage of the construction of $(\mathfrak{w}_\infty, \pi_\infty)$ we produce a unique partially isometric extension of the representation from the previous step. The extension depends upon the choice of transfer operator τ, but if τ is fixed at the outset, then the whole process results in a coisometric extension of $(\mathfrak{z}, \sigma)$ that is uniquely determined up to unitary equivalence.

In addition to the theorem just described, we have proved a generalized commutant lifting theorem for coisometric extensions. The proof of our version is modeled on Douglas, Muhly, and Pearcy's proof [2] of Sz.-Nagy and Foiaş's classical commutant lifting theorem. Assume as before that the left-action map $\phi : A \to \mathcal{L}(E)$ admits a smooth, unital generalized transfer operator $\tau : \mathcal{L}(E) \to A$, and assume that $(\mathfrak{z}, \sigma)$ is a contractive intertwining pair, acting on a Hilbert space H, such that

$$\mathfrak{z}\sigma^E(Y) = \sigma(\tau(Y))\mathfrak{z}, \quad Y \in \mathcal{L}(E). \tag{1.4}$$

Assume that $(\mathfrak{w}_\infty, \pi_\infty)$, acting on a Hilbert space $K \supseteq H$, is the unique coisometric extension of $(\mathfrak{z}, \sigma)$ that is built using τ. Then, given any $X \in B(H)$ that "commutes" with $(\mathfrak{z}, \sigma)$ (see Definition 4.10), there is an operator $X_\infty \in B(K)$ satisfying the following three conditions:

1. $X_\infty H \subseteq H$ and $X_\infty|_H = X$,

2. $||X_\infty|| = ||X||$, and

3. X_∞ commutes with $(\mathfrak{w}_\infty, \pi_\infty)$.

Just as in the construction of $(\mathfrak{w}_\infty, \pi_\infty)$, X_∞ is built in an inductive fashion: We first prove that an analogous result holds for $(\mathfrak{w}_1, \pi_1)$, and we then iterate to obtain X_∞. One may wonder about the added hypothesis (1.4) in our commutant lifting theorem. It turns out that this is a strictly stronger condition than $\mathfrak{z}\sigma^E(\phi(a)) = \sigma(a)\mathfrak{z}$, $a \in A$, and as we will see in Chapter 6, is a necessary assumption if we wish to lift the commutant of $(\mathfrak{z}, \sigma)$.

1.2 Outline

In Chapter 2, we review all the requisite background material. We start with a brief history of the origins of classical dilation theory, and we recall several celebrated dilation results, including those of Béla Sz.-Nagy and W.F. Stinespring. Next, we review the basics of Hilbert C^*-modules and C^*-correspondences, and we give several examples of each. This allows us to develop the notion of an induced representation and two very useful results due to Marc Rieffel, the equivalence theorem (Theorem 2.18) and the imprimitivity theorem (Theorem 2.21). We then describe the main objects of study in this thesis – representations of the tensor algebra associated to a C^*-correspondence, covariant representations of a C^*-correspondence, and intertwining pairs associated to a C^*-correspondence – and we explore relations among these three objects. Next is a brief review of directed systems and inductive limits in the contexts of Hilbert spaces, bounded linear transformations between Hilbert spaces, C^*-algebras, and C^*-representations. Finally, we define one of the key tools in our analysis, transfer operators, and we discuss several examples. The reader who is familiar with the concepts just described may wish to skip ahead to Chapter 3 after browsing Sections 2.4 and 2.6, as the definitions of intertwining pairs and generalized transfer operators are the only novel ideas introduced in Chapter 2.

In the third chapter, we merge several ideas from Chapter 2. Specifically, we observe that if Stinespring's dilation theorem is applied to a representation composed with a transfer operator, then the resulting representation, called the Stinespring representation, is an induced representation in the sense of Rieffel (see Corollary 3.6).

This follows from a more general result, namely that every representation of $\mathcal{L}(E)$ that restricts nondegenerately to $\mathcal{K}(E)$ is an induced representation.

Chapter 4 is devoted to building partially isometric extensions, first for a single operator on Hilbert space, and then for a contractive intertwining pair associated to a C^*-correspondence. In the latter case, the partially isometric extension we build is unique up to unitary equivalence if it assumed to be minimal and adapted to a transfer operator that was fixed in advance, as described in Definition 4.12 and Theorem 4.14.

In Chapter 5, we build a directed system of contractive intertwining pairs by iterating the construction from Chapter 4. The inductive limit of this directed system is a coisometric extension, and we show that this extension is minimal and adapted to the transfer operator (fixed in advance) if each step of the directed system is (see Theorem 5.2). Moreover, we prove that the coisometric extension obtained in this way is unique up to unitary equivalence (Theorem 5.3).

The sixth chapter is devoted to proving a commutant lifting theorem in the context of partially isometric extensions (Theorem 6.4). We begin this chapter by observing that an additional assumption needs to be placed on the underlying contractive intertwining pair in order to develop the commutant lifting theorem. We outline the difficulties in proceeding without this additional assumption, and then we prove the lifting theorem under this assumption.

Finally, in Chapter 7 we extend the commutant lifting theorem from Chapter 6 to the level of coisometric extensions. We proceed in a fashion analogous to the approach taken in Chapter 5: We iterate the construction from Chapter 6, and in doing so, we obtain a directed system; we then take the inductive limit of this directed system to prove the commutant lifting theorem for coisometric extensions (Theorem 7.1). Just as in Theorem 6.4, an additional assumption on the underlying contractive intertwining pair is necessary in order to establish Theorem 7.1.

CHAPTER 2

BACKGROUND AND PRELIMINARIES

2.1 Classical Dilation Theory

The origins of dilation theory can be traced back to at least 1944, when Gaston Julia initiated a study of certain relations between operators [5, 6, 7]. Observe that if S is an isometric dilation of $T \in B(H)$, i.e., $P_H \, S \,|_H = T$, then for any $h \in H$,

$$||Th|| = ||PSh|| \leq ||Sh|| = ||h||.$$

Therefore, an operator T for which there exists an isometric dilation is necessarily a contraction, i.e., $||T|| \leq 1$. Julia's dilation theorem asserts the converse: Every contraction on Hilbert space has an isometric dilation to a larger space.

Theorem 2.1 (Julia's Dilation Theorem)**.** *Given an operator T on a Hilbert space H with $||T|| \leq 1$, there exists a Hilbert space K containing H and an isometric operator V on K such that*

$$P_H V|_H = T.$$

Halmos showed that V can in fact be chosen to be unitary, so that every contraction has a unitary dilation to a larger space. To be clear, a *unitary dilation* of $T \in B(H)$ is a unitary operator U on a space $K \supseteq H$ such that U is an isometric dilation of T and U^* is an isometric dilation of T^*.

Theorem 2.2 (Halmos's Dilation Theorem). *Given a contraction $T \in B(H)$, there exists a Hilbert space $K \supseteq H$ and a unitary operator $U \in B(K)$ such that*

$$P_H U|_H = T \text{ and } P_H U^*|_H = T^*.$$

Since every unitary operator is isometric, once Halmos's dilation theorem is established, Julia's follows immediately. Halmos proved his result via a clever matricial construction, which we now recapitulate. Let $K = H \oplus H$ and, viewing elements of K as column vectors, define $U : K \to K$ by the matrix

$$U = \begin{bmatrix} T & \Delta_* \\ \Delta & -T^* \end{bmatrix},$$

where $\Delta = (I - T^*T)^{1/2}$ and $\Delta_* = (I - TT^*)^{1/2}$ are the *defect operators* for T and T^*, respectively. It is clear that $P_H U|_H = T$ and $P_H U^*|_H = T^*$, and simple computations verify that $U^*U = UU^* = I$ (see [17, Appendix.4] for more details).

Note that we use the indefinite article "a" in the definition of unitary dilation, indicating that such dilations are not unique. However, if $K = H \oplus \mathcal{D}_*$, where $\mathcal{D}_* = [\Delta_* H]$[1] denotes the *defect space* associated to T^*, then K is the smallest reducing subspace for U containing H (identified with $H \oplus \{0\}$), and the dilation is unique up to unitary equivalence.

A natural question to ask is if Halmos's dilation theorem holds for polynomials. That is, is it the case that U, as defined above, satisfies

$$P_H p(U)|_H = p(T),$$

[1]For an arbitrary Hilbert space H and any subset $\mathcal{M} \subseteq H$, we denote by $[\mathcal{M}]$ the closed linear span of $\mathcal{M}$ in H.

for every (complex) polynomial $p(z)$? Unfortunately, the above construction does not in general yield $P_H U^2|_H = T^2$, much less $P_H U^n|_H = T^n$ for all $n \in \mathbb{Z}_+$. However, Eugene Egerváry [3] was able to modify Halmos's construction to accommodate powers of T, up to a fixed natural number. To be precise, Egerváry proved the following via a modification of Halmos's matricial construction (again, see [17, Appendix.4] for the details).

Theorem 2.3 (Egerváry's Dilation Theorem)**.** *Given a contraction $T \in B(H)$ and a natural number k, there exists a Hilbert space $K \supseteq H$ and a unitary operator $U \in B(K)$ such that*

$$P_H U^n|_H = T^n \text{ and } P_H U^{*n}|_H = T^{*n}, \quad 1 \leq n \leq k.$$

Béla Sz.-Nagy generalized Halmos's result even further to allow for *all* powers of T [20, 21].

Theorem 2.4 (Sz.-Nagy's Dilation Theorem)**.** *Given a contraction $T \in B(H)$, there exists a Hilbert space $K \supseteq H$ and a unitary operator $U \in B(K)$ such that*

$$P_H U^n|_H = T^n \text{ and } P_H U^{*n}|_H = T^{*n}, \quad n \geq 1,$$

Moreover, the dilation space K can be chosen in such a way that U is the unique such operator up to unitary equivalence. Namely, one stipulates that the dilation space is minimal in the sense that the smallest subspace of K which contains H and reduces U is K itself.

There are many proofs of Sz.-Nagy's dilation theorem (see [22, I.12] for a discussion of several). One approach is to dilate T to an isometry and then to extend the result to a coisometry. The process of extending an isometry to a coisometry preserves the isometric structure of the dilation, so the result is a dilation U that is both isometric and coisometric, hence unitary. Minimality is imposed by setting

$$K = \bigvee_{n=-\infty}^{\infty} U^n H.$$

Another, more constructive proof was provided by J.J. Schäffer in 1955 [18]. In fact, his proof is a sort of infinite-dimensional generalization of the matricial constructions of Halmos and Egerváry. Set

$$K = \cdots \oplus H \oplus H \oplus H \oplus H \oplus H \oplus \cdots,$$

and define $U : K \to K$ by the *Schäffer matrix* for T:

$$U = \begin{bmatrix} \ddots & \ddots & \ddots & & & & & & & \\ & 0 & I & 0 & & & & & & \\ & & 0 & I & 0 & & & & & \\ & & & 0 & \Delta & -T^* & & & & \\ & & & & \boxed{T} & \Delta_* & 0 & & & \\ & & & & & 0 & I & 0 & & \\ & & & & & & 0 & I & 0 & \\ & & & & & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Here, I is the identity operator on H, and the box around T indicates that it is the $(0,0)$-entry of the matrix. One readily verifies by matrix multiplication that $U^*U = UU^* = I_K$, i.e., that U is unitary. Further, identifying H with the zeroth summand of $K = \bigoplus_{-\infty}^{\infty} H$, we have

$$P_H U^n|_H = T^n \text{ and } P_H U^{*n}|_H = T^{*n}, \quad n \geq 1.$$

Therefore, U is a unitary dilation for T. In fact, if we write $\mathcal{D} = [\Delta H]$ and $\mathcal{D}_* = [\Delta_* H]$ for the *defect spaces* associated to T and T^*, respectively, then the subspace

$$\cdots \oplus \mathcal{D} \oplus \mathcal{D} \oplus H \oplus \mathcal{D}_* \oplus \mathcal{D}_* \oplus \cdots,$$

reduces U, and the restriction of U to this subspace is *the* minimal unitary dilation of T (which is unique up to unitary equivalence).

Another dilation theorem that we shall use repeatedly in the sequel was proven by W.F. Stinespring around the same time as the previous two [19]. In order to state Stinespring's dilation theorem, we require the very important concept of *complete positivity.*

Definition 2.5. *Given a linear map $\phi : A \to B$ between C^*-algebras A and B, we define, for each $n \geq 1$, a linear map $\phi_n : M_n(A) \to M_n(B)$ by*

$$\phi_n((a_{ij})) = (\phi(a_{ij})).$$

(i) We call ϕ completely positive *in case each ϕ_n is a positive map between C^*-algebras, i.e., if $\phi_n((a_{ij})^*(a_{ij})) \geq 0$ whenever $n \geq 1$ and $(a_{ij}) \in M_n(A)$.*

*(ii) We define the cb-*norm *of ϕ by $||\phi||_{cb} = \sup_n ||\phi_n||$, and we say that ϕ is* completely bounded *if*

$$||\phi||_{cb} < \infty.$$

(iii) It is then natural to define ϕ *to be* completely contractive *if*

$$||\phi||_{cb} \leq 1.$$

Stinespring's theorem asserts that any completely positive map of a C^*-algebra into the bounded operators on some Hilbert space can be dilated to a representation of that C^*-algebra on a larger space.

Theorem 2.6 (Stinespring's Dilation Theorem)**.** *Let* A *be a* C^**-algebra, and let* $\psi : A \to B(H)$ *be a completely positive map of* A *into the bounded operators on a Hilbert space* H*. Then there exists a Hilbert space* K*, a representation* $\rho : A \to B(K)$*, and a bounded map* $V : H \to K$ *such that*

$$V^*\rho(a)V = \psi(a), \quad a \in A.$$

Remark 2.7. *If* A *and* ψ *are unital, then* V *is an isometry. In this case, we view* V *as an embedding of* H *into* K*, and we view* V^* *as the projection* P_H *of* K *onto* H*. Hence, we can write*

$$P_H\rho(a)|_H = \psi(a), \quad a \in A.$$

An advantage of this formulation is that it more closely resembles the dilation results of Julia, Halmos, Egervary, and Sz.-Nagy. We will use both formulations in what follows.

Regardless of the formulation of Stinespring's result, the Hilbert space K is called the *Stinespring space* associated to ψ, and the representation ρ is referred to as the *Stinespring representation* associated to ψ. Also, (K, ρ, V) is a *Stinespring triple* associated to ψ. Furthermore, (K, ρ, V) is a *minimal Stinespring triple* if the smallest subspace of K containing (the image under V of) H and reducing ρ is K itself. The next result says that minimal Stinespring triples are unique up to unitary equivalence.

Proposition 2.8. *The Stinespring triple (K, ρ, V) associated to a completely positive map $\psi : A \to B(H)$ is unique up to unitary equivalence if K is minimal. That is, if (K', ρ', V') is another minimal Stinespring triple associated to ψ, then there exists a Hilbert space isomorphism $U : K \to K'$ such that $U\rho(\cdot)U^* = \rho'(\cdot)$ and $UV = V'$.*

One of our primary goals in what follows is to generalize Sz.-Nagy's dilation theorem to the context of completely contractive covariant representations of C^*-correspondences (definitions below). These representations are direct generalizations of operators on Hilbert space, and so our theorem extends that of Sz.-Nagy. We accomplish our goal by constructing an analogue of the Schäffer matrix in the more general setting.

2.2 Hilbert C^*-Modules and C^*-Correspondences

As mentioned in the previous section, one of our primary goals is to investigate the extent to which classical dilation theory can be extended to the more general setting of covariant representations and C^*-correspondences. *Covariant representations* can in a sense be viewed as natural generalizations of operators on Hilbert space, and *C^*-correspondences* are the objects over which we define them. As one might expect,

C^*-correspondences are themselves generalizations of Hilbert spaces (along with some additional structure). We assume the reader is familiar with basic Hilbert space and C^*-algebra theory, and we follow the notation and terminology established in [10].

Let A be a C^*-algebra. An *inner product module over* A is a right A-module E equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$ satisfying the following conditions for each $\xi, \eta, \zeta \in E$, each $\lambda \in \mathbb{C}$, and each $a \in A$:

$$\langle \xi, \eta + \zeta \rangle = \langle \xi, \eta \rangle + \langle \xi, \zeta \rangle,$$

$$\langle \xi, \lambda\eta \rangle = \lambda \langle \xi, \eta \rangle,$$

$$\langle \xi, \eta \cdot a \rangle = \langle \xi, \eta \rangle a,$$

$$\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle,$$

$$\langle \xi, \xi \rangle \geq 0, \text{ and}$$

$$\langle \xi, \xi \rangle = 0 \iff \xi = 0.$$

In brief, the first three conditions can be summarized by saying that $\langle \cdot, \cdot \rangle$ is linear in the second variable. Combined with the fourth condition, this implies that $\langle \cdot, \cdot \rangle$ is conjugate-linear in the first variable. The last two conditions mean that $\langle \cdot, \cdot \rangle$ is positive definite.

Definition 2.9. *Given an inner product module E over A, we define a norm on E by the formula*

$$||\xi||_E = ||\langle \xi, \xi \rangle||_A^{1/2}.$$

If E is complete with respect to this norm, then E is called a Hilbert C^*-module over A, *or more succinctly, a* Hilbert A-module. *If we wish to be explicit about the C^*-algebra over which E is a Hilbert C^*-module, we write $E = E_A$.*

Thus, the notion of a Hilbert C^*-module is a natural generalization of Hilbert space. Indeed, when $A = \mathbb{C}$, a Hilbert A-module *is* a Hilbert space. However, there are some subtle differences, including the fact that not every closed submodule of a Hibert C^*-module is complemented (see [10, p. 7] for an example). Additionally, some care must be taken when talking about "operators" on a Hilbert C^*-module. In particular, a bounded map between Hilbert C^*-modules need not have an adjoint, as is the case for bounded maps between Hilbert spaces. Let $E = E_A$ and $F = F_A$ be Hilbert A-modules, and let $T : E \to F$ be a map from E to F. We say that T is *adjointable* if there exists a map $T^* : F \to E$ such that

$$\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle, \quad \xi \in E, \eta \in F.$$

From this condition, it follows that T is A-linear, i.e., $T(\xi a) = T(\xi)a$, and bounded. We write $\mathcal{L}(E, F)$ for the collection of all adjointable maps from E to F; $\mathcal{L}(E, F)$ is the proper generalization of the collection $B(H, K)$ of all bounded linear transformations between two Hilbert space H and K (in fact, $\mathcal{L}(H, K) = B(H, K)$). It is clear that if $T \in \mathcal{L}(E, F)$, then $T^* \in \mathcal{L}(F, E)$, so $\mathcal{L}(E) := \mathcal{L}(E, E)$ is a $*$-algebra. In fact, $\mathcal{L}(E)$ is a C^*-algebra, a fact that will be very important as we proceed.

Another important class of maps between Hilbert A-modules E and F are the finite-rank operators. For $\xi \in E$ and $\eta \in F$, we define $\xi \otimes \eta^* : F \to E$ by

$$\xi \otimes \eta^*(\zeta) = \xi\langle \eta, \zeta\rangle, \quad \zeta \in F.$$

We denote by $\mathcal{K}(F,E)$ the closed linear subspace of $\mathcal{L}(F,E)$ spanned by $\{\xi \otimes \eta^* \mid \xi \in E, \eta \in F\}$, and we write $\mathcal{K}(E)$ for $\mathcal{K}(E,E)$ (so that the definition of $\mathcal{K}(H,K)$ coincides with the usual notion of compact operators from H to K). As is conventional, we refer to elements of $\mathcal{K}(F,E)$ as "compact operators," even though they need not be compact when viewed as operators between the Banach spaces F and E (see [10, p. 10] for an example).

Thus far, we've considered only right-modules over C^*-algebras, which makes sense in view of the fact that a Hilbert $\mathbb{C}$-module is really just a Hilbert space. Of course, every Hilbert space H may also be endowed with a left-module structure over $\mathbb{C}$, over $\mathcal{K}(H)$, or over $B(H)$. This leads to the concept of a C^*-correspondence.

Definition 2.10. *Let A and B be C^*-algebras. A* C^*-correspondence from B to A*, or a* B-A-correspondence*, is a B-A-bimodule E that is a Hilbert A-module as well as a left-module over B, with the action of B on E given by a homomorphism $\phi : B \to \mathcal{L}(E)$. We write $b \cdot \xi = \phi(b)\xi$, $b \in B$, $\xi \in E$, for this action, and we write $E = {}_BE_A$ to indicate that E is a B-A-correspondence.*

When $B = A$, we refer to $E = {}_AE_A$ as a C^*-correspondence over A*, or simply an* A-correspondence.

Let us consider several examples, many of which we will revisit later. As a general rule, we use Greek letters to denote elements of Hilbert C^*-modules and C^*-correspondences, and Roman letters are used to denote elements of the underlying C^*-algebra.

Example 2.11. *The simplest example of a Hilbert C^*-module is obtained by viewing the C^*-algebra $\mathbb{C}$ of complex numbers as a right-module over itself, i.e., by setting $E = A = \mathbb{C}$. In this case, the module action is given by multiplication, and the inner product is defined by $\langle \xi, \eta \rangle = \overline{\xi}\eta$, for $\xi, \eta \in \mathbb{C}$. It is easy to see that $\mathcal{K}(E)$ and $\mathcal{L}(E)$ can be identified with $\mathbb{C}$ in this case. In fact, the identification $\mathcal{L}(E) = \mathbb{C}$ allows us to view $\mathbb{C}$ as a $\mathbb{C}$-correspondence by defining ϕ by left-multiplication.*

Example 2.12. *Expanding on the previous example,* any *C^*-algebra A can be viewed as a Hilbert C^*-module over itself with module action given by multiplication, and with the following inner product:*

$$\langle \xi, \eta \rangle = \xi^* \eta, \quad \xi, \eta \in E = A.$$

Here, $\mathcal{K}(E) = \mathcal{K}(A)$ can be identified with A via the map $\xi \otimes \eta^* \mapsto \xi\eta^*$. Moreover, if A is unital, as we always assume in what follows, then $\mathcal{L}(E) = \mathcal{L}(A) = \mathcal{K}(A)$, since any $T \in \mathcal{L}(A)$ consists of multiplication by $T(1_A)$. Thus, A is in fact an A-correspondence, with ϕ being given by left-multiplication. More generally, given an endomorphism $\alpha : A \to A$, A can be made into an A-correspondence by letting $\phi = \alpha$. We write $E = {}_\alpha A$ for the C^*-correspondence obtained in this way.

Muhly and Solel's work (see [14]) with this example motivated much of what follows, especially the contents of Chapters 4 and 5.

Example 2.13. *As remarked above, every Hilbert space H is a Hilbert $\mathbb{C}$-module. If, in addition, σ is a representation of a C^*-algebra A on H, then H can be made into a C^*-correspondence from A to $\mathbb{C}$, with left-action map $\phi = \sigma$, i.e., $a \cdot \xi = \sigma(a)\xi$.*

In particular, in addition to being a Hilbert $\mathbb{C}$-module, every Hilbert space is also a C^*-correspondence over $\mathbb{C}$, with $\phi = \sigma$ having the only form it can (left-multiplication).

Example 2.14. *Of particular interest to us is the Hilbert $\mathbb{C}$-module $E = \mathbb{C}^d$, for $1 \leq d \leq \infty$ ($\mathbb{C}^\infty := \ell^2(\mathbb{N})$, but we shall focus on finite d). In this case, we view elements of $\mathbb{C}^d$ as column vectors and define the action of $A = \mathbb{C}$ on $\mathbb{C}^d$ by (matrix) multiplication (for this reason, $\mathbb{C}^d$ is sometimes referred to as column Hilbert space). Moreover, by choosing an orthonormal basis for $\mathbb{C}^d$, it is easy to see that $\mathcal{K}(E) = \mathcal{L}(E) = M_d(\mathbb{C})$, the $d \times d$ matrices over $\mathbb{C}$, when $d < \infty$. Then, $\mathbb{C}^d$ is naturally a C^*-correspondence from $M_d := M_d(\mathbb{C})$ to $\mathbb{C}$, where the left-action map ϕ is given simply by matrix multiplication.*

Given C^*-algebras A and B and a B-A-correspondence $E = {}_BE_A$, there is a natural A-B-correspondence known as the *opposite correspondence*, denoted $\widetilde{E} = {}_A\widetilde{E}_B$. As a set, $\widetilde{E}$ equals E, with elements of $\widetilde{E}$ distinguished from those of E by superscripted asterisks. To be precise, for every $\xi \in E$, there is a $\xi^* \in \widetilde{E}$, and vice versa (with $(\xi^*)^* = \xi$). $\widetilde{E}$ is then made into a C^*-correspondence from A to B by defining the module actions and a B-valued inner product as follows:

$$a \cdot \xi^* = (\xi \cdot a^*)^*, \qquad \xi^* \cdot b = (b^* \cdot \xi)^*, \qquad \langle \xi^*, \zeta^* \rangle = \xi \otimes \zeta^*,$$

where $a \in A$, $b \in B$, and $\xi^*, \zeta^* \in \widetilde{E}$.

Returning to the setting of Example 2.14, the opposite correspondence $\widetilde{\mathbb{C}^d}$ is a $\mathbb{C}$-M_d-correspondence, known as row Hilbert space. Of course, the reasons for the names column (respectively, row) Hilbert space are that elements of $\mathbb{C}^d$ (resp., $\widetilde{\mathbb{C}^d}$) are identified with column (resp., row) vectors. Under these identifications, the actions of M_d and $\mathbb{C}$ are then given by matrix multiplication. Furthermore, the maps $\widetilde{\mathbb{C}^d} \times \mathbb{C}^d \mapsto \mathbb{C}$ and $\mathbb{C}^d \times \widetilde{\mathbb{C}^d} \mapsto M_d$, also given by matrix multiplication, are really just the Hilbert C^*-module inner products on $\mathbb{C}^d$ and $\widetilde{\mathbb{C}^d}$, respectively. This is representative of a more general fact, one that requires the notion of (internal) tensor product in order to be made precise.

Let A, B, and C be C^*-algebras, let E be an A-B-correspondence with left-action map $\phi : A \to \mathcal{L}(E)$, and let F be a B-C-correspondence with left-action map $\psi : B \to \mathcal{L}(F)$. We form the algebraic tensor product $E \otimes_{alg} F$ and define a C-valued sesquilinear form on elementary tensors $\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \in E \otimes_{alg} F$ as follows:

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, \psi(\langle \xi_1, \xi_2 \rangle)\eta_2 \rangle.$$

We extend this form by linearity and take the quotient by the subspace $\mathcal{N}$ spanned by elementary tensors of the form $\xi a \otimes \eta - \xi \otimes \psi(a)\eta$, where $\xi \otimes \eta \in E \otimes_{alg} F$ and $a \in A$. Taking the Hausdorff completion of $(E \otimes_{alg} F)/\mathcal{N}$ with respect to this form, we arrive at the *internal tensor product* of E and F, denoted $E \otimes_B F$ or $E \otimes_\psi F$. $E \otimes_\psi F$ is a C^*-correspondence from A to C with left-action map $\phi(\cdot) \otimes I_F : A \to \mathcal{L}(E \otimes_{alg} F)$. The subscripted "$\psi$" (or "B") is a reminder that $E \otimes_\psi F$ is balanced over B in the sense that $\xi b \otimes \eta = \xi \otimes \psi(b)\eta$ for $b \in B$ and $\xi \otimes \eta \in E \otimes_\psi F$.

Remark 2.15. *Although we have given the internal tensor product construction for C^*-correspondences using the homomorphism that implements the left-action of the* *right tensor factor, all that is really required to form $E \otimes_\psi F$ is a Hilbert B-module $E = E_B$, a B-C-correspondence $F = {}_B F_C$, and a linear, completely positive (not necessarily multiplicative) map $\psi : B \to \mathcal{L}(F)$. In this case, the construction proceeds exactly as above, but now $E \otimes_\psi F$ is not balanced over B. This is relevant because one of our main tools is the notion of a transfer operator, and these maps are, in general, not multiplicative.*

Returning to the discussion above (prior to the definition of the internal tensor product), if $E = {}_B E_A$ is a C^*-correspondence from a C^*-algebra B to a C^*-algebra A, then the tensor product $E \otimes_A \widetilde{E}$ can be identified with $\mathcal{K}(E)$. Also, $\widetilde{E} \otimes_B E$ can be identified with A (via the map $\xi^* \otimes \eta \mapsto \xi^* \eta$), viewing both spaces as A-correspondences. Additionally, given a B-A-correspondence E with left-action map $\phi : B \to \mathcal{L}(E)$, we may make the identifications $B \otimes_B E = E$ (via $b \otimes \xi \mapsto \phi(b)\xi$) and $E \otimes_A A = E$ (via $\xi \otimes a \mapsto \xi a$). These identifications and the internal tensor product construction in general will play an important role throughout the constructions that follow.

There is one final type of C^*-correspondence that we use extensively in what follows, namely, the so-called intertwining space determined by two representations. Given representations σ_1 and σ_2 of a C^*-algebra A on Hilbert spaces H_1 and H_2, respectively, we denote by $\mathcal{I}(\sigma_1, \sigma_2)$ the space consisting of all operators from H_1 into H_2 which *intertwine* σ_1 and σ_2, that is,

$$\mathcal{I}(\sigma_1, \sigma_2) := \{X \in B(H_1, H_2) \mid X\sigma_1(a) = \sigma_2(a)X \text{ for all } a \in A\}.$$

Note that if $X \in \mathcal{I}(\sigma_1, \sigma_2)$, then for any $a \in A$,

$$X^*\sigma_2(a) = (\sigma_2(a)^*X)^* = (\sigma_2(a^*)X)^* = (X\sigma_1(a^*))^* = (X\sigma_1(a)^*)^* = \sigma_1(a)X^*.$$

Therefore, if $Y \in \mathcal{I}(\sigma_1, \sigma_2)$, then $X^*Y\sigma_1(a) = X^*\sigma_2(a)Y = \sigma_1(a)X^*Y$ for all $a \in A$, and so, $X^*Y \in \sigma_1(A)'$. This allows us to define a $\sigma_1(A)'$-valued inner product on $\mathcal{I}(\sigma_1, \sigma_2)$ by $\langle X, Y\rangle = X^*Y$. Furthermore, if $Z_i \in \sigma_i(A)'$, $i = 1, 2$, then it is easy to see that XZ_1 and Z_2X are in $\mathcal{I}(\sigma_1, \sigma_2)$. Thus, $\mathcal{I}(\sigma_1, \sigma_2)$ has the structure of a C^*-correspondence from $\sigma_2(A)'$ to $\sigma_1(A)'$. We call this C^*-correspondence the *intertwining space of* σ_1 *and* σ_2.

Note that in the process of showing that $\mathcal{I}(\sigma_1, \sigma_2)$ is a C^*-correspondence, we proved the following result.

Proposition 2.16. *A map $X : H_1 \to H_2$ is an element of $\mathcal{I}(\sigma_1, \sigma_2)$ if and only if its adjoint $X^* : H_2 \to H_1$ is an element of $\mathcal{I}(\sigma_2, \sigma_1)$, that is,*

$$\mathcal{I}(\sigma_1, \sigma_2)^* = \mathcal{I}(\sigma_2, \sigma_1).$$

Moreover, if $X, Y \in \mathcal{I}(\sigma_1, \sigma_2)$, then $X^*Y \in \sigma_1(A)'$ and $XY^* \in \sigma_2(A)'$. In particular, $X^*X \in \sigma_1(A)'$ and $XX^* \in \sigma_2(A)'$.

We also have following result, which we include for completeness.

Proposition 2.17. *For $i = 1, 2, 3$, let H_i be a Hilbert space and let σ_i be a representation of an arbitrary C^*-algebra A on H_i. Assume there are operators $X : H_2 \to H_3$ and $Y : H_1 \to H_2$ such that the composition $Z = XY$ is in the intertwining space $\mathcal{I}(\sigma_1, \sigma_3)$ and such that Y is a surjective operator in the intertwining space $\mathcal{I}(\sigma_1, \sigma_2)$. Then $X \in \mathcal{I}(\sigma_2, \sigma_3)$.*

Proof. The proof rests essentially on the following string of equalities, which hold for every $a \in A$ since $Z \in \mathcal{I}(\sigma_1, \sigma_3)$ and $Y \in \mathcal{I}(\sigma_1, \sigma_2)$:

$$\sigma_3(a)XY = \sigma_3(a)Z = Z\sigma_1(a) = XY\sigma_1(a) = X\sigma_2(a)Y.$$

Therefore, $\sigma_3(\cdot)X = X\sigma_2(\cdot)$ on $\mathrm{Ran}(Y)$, hence on all of H_2 by the assumption that Y is surjective. That is, $X \in \mathcal{I}(\sigma_2, \sigma_3)$.

$\square$

2.3 Induced Representations and Rieffel's Imprimitivity and Equivalence Theorems

We begin this section by observing that every Hilbert C^*-module can be viewed as a C^*-correspondence in a natural way. Specifically, if $E = E_A$ is a Hilbert C^*-module over a C^*-algebra A, then E is also a C^*-correspondence from $\mathcal{L}(E)$ (alternatively, $\mathcal{K}(E)$) to A, with left-action defined as follows:

$$X \cdot \xi = X\xi, \quad X \in \mathcal{L}(E), \ \xi \in E.$$

For the moment, view E as a $\mathcal{K}(E)$-A-correspondence. Then the opposite correspondence $\widetilde{E}$ associated to E is an $A - \mathcal{K}(E)$-correspondence, and it follows from remarks in the previous section that $E \otimes_A \widetilde{E} = \mathcal{K}(E)$ as $\mathcal{K}(E)$-correspondences. In fact, this identification is what motivates the notation $\xi \otimes \eta^*$ for a finite-rank operator on E.

For the remainder of this section, fix a C^*-algebra A, an A-correspondence E, and a representation $\sigma : A \to B(H)$ of A on a Hilbert space H. Then we can form the (internal) tensor product $E \otimes_\sigma H$ of the A-correspondence E and the $A - \mathbb{C}$-correspondence H. Recall that the inner product on $E \otimes_\sigma H$ is defined on elementary tensors $\xi_1 \otimes h_1, \xi_1 \otimes h_1 \in E \otimes_\sigma H$ by the following formula:

$$\langle \xi_1 \otimes h_1, \xi_2 \otimes h_2 \rangle = \langle h_1, \sigma(\langle \xi_1, \xi_2 \rangle) h_2 \rangle.$$

Also, $E \otimes_\sigma H$ is balanced over A in the sense that $\xi a \otimes h = \xi \otimes \sigma(a) h$ for all $a \in A$ and all $\xi \otimes h \in E \otimes_\sigma H$. Notice that $E \otimes_\sigma H$ is an $A - \mathbb{C}$-correspondence, so in particular, it is a Hilbert space. We call $E \otimes_\sigma H$ the *induced Hilbert space* associated to σ.

There is a natural representation of $\mathcal{L}(E)$ (or, by restriction, of $\mathcal{K}(E)$) on $E \otimes_\sigma H$, called the *representation of* $\mathcal{L}(E)$ *(resp.,* $\mathcal{K}(E)$*) induced from* σ. This representation, denoted $\sigma^E : \mathcal{L}(E) \to B(E \otimes_\sigma H)$, is defined on elementary tensors by the formula

$$\sigma^E(X)(\xi \otimes h) = (X\xi) \otimes h, \quad X \in \mathcal{L}(E),\ \xi \otimes h \in E \otimes_\sigma H.$$

In particular, $\sigma^E \circ \phi : A \to B(E \otimes_\sigma H)$ (which is nondegenerate by [16, Theorem 5.1]) implements the left-A-action that makes $E \otimes_\sigma H$ into a C^*-correspondence from A to $\mathbb{C}$ (note that $B(E \otimes_\sigma H) = \mathcal{L}(E \otimes_\sigma H)$ since $E \otimes_\sigma H$ is a Hilbert space). That is, $\sigma^E(\phi(\cdot)) = \phi(\cdot) \otimes I_H$.

More generally, if $E = {}_B E_A$ is a C^*-correspondence from B to A, and if $\sigma : A \to B(H)$ is a representation of A on a Hilbert space H, then σ^E is a representation of B on $E \otimes_\sigma H$. In this case, we say that σ^E is a representation of B induced from (a representation of) A with respect to E.

Rieffel's equivalence theorem [16, Theorem 6.23] establishes a bijective correspondence between representations of A and representations of $\mathcal{L}(E)$ induced from A with respect to E. In what follows, we give a special case of Rieffel's theorem, which is suitably general for our purposes. Specifically, we only consider the case where $B = \mathcal{L}(E)$.

Theorem 2.18 (Rieffel's Equivalence Theorem)**.** *Let $\mathfrak{A}$ denote the category whose objects are nondegenerate representations of A (written $\sigma_i : A \to B(H_i)$) with morphisms operators from the following intertwining space:*

$$\mathcal{I}(\sigma_1, \sigma_2) = \{Y \in B(H_1, H_2) : Y\sigma_1(a) = \sigma_2(a)Y \text{ for all } a \in A\}.$$

Also, let $\mathfrak{B}$ be the category whose objects are nondegenerate representations of $\mathcal{L}(E)$ (denoted $\rho_i : \mathcal{L}(E) \to B(K_i)$) with the property that the restriction of each object to $\mathcal{K}(E)$ is nondegenerate, and with morphisms operators from the following intertwining space:

$$\mathcal{I}(\rho_1, \rho_2) = \{Z \in B(K_1, K_2) : Z\rho_1(X) = \rho_2(X)Z \text{ for all } X \in \mathcal{K}(E)\}.$$

Then the functor F from $\mathfrak{A}$ to $\mathfrak{B}$ that act on objects by $\sigma \mapsto \sigma^E$ and the functor G from $\mathfrak{B}$ to $\mathfrak{A}$ that acts by $\rho \mapsto \rho^{\widetilde{E}}$ together establish an equivalence between categories, that is, $G \circ F \cong I_{\mathfrak{A}}$ and $F \circ G \cong I_{\mathfrak{B}}$ (where $I_{\mathfrak{A}}$ and $I_{\mathfrak{B}}$ are the identity maps on $\mathfrak{A}$ and $\mathfrak{B}$, respectively).

The functor F also sends a morphism $Y \in \mathcal{I}(\sigma_1, \sigma_2)$ of $\mathfrak{A}$ to the morphism $I_E \otimes Y \in \mathcal{I}(\sigma_1^E, \sigma_2^E)$ of $\mathfrak{B}$, where I_E denotes the identity map on E, while G sends a morphism $Z \in \mathcal{I}(\rho_1, \rho_2)$ of $\mathfrak{B}$ to $I_{\widetilde{E}} \otimes Z$, where $I_{\widetilde{E}}$ denotes the identity map on $\widetilde{E}$.

Remark 2.19. *The last part of Theorem 2.18 means that to every intertwiner $Y \in \mathcal{I}(\sigma_1, \sigma_2)$ there is an associated intertwiner $I_E \otimes Y \in \mathcal{I}(\sigma_1^E, \sigma_2^E)$ of $\mathfrak{B}$, and conversely, every intertwiner $Z \in \mathcal{I}(\sigma_1^E, \sigma_2^E)$ is of the form $I_E \otimes Z_0$ for some $Z_0 \in \mathcal{I}(\sigma_1, \sigma_2)$. This fact will be used repeatedly in what follows.*

Having discussed how a representation of A can be induced to a representation of $\mathcal{K}(E)$ (hence $\mathcal{L}(E)$), and vice versa, it is natural to ask when a given representation of $\mathcal{L}(E)$ is *induced*.

Definition 2.20. *A representation $\rho : \mathcal{L}(E) \to B(K)$ is said to be* induced *(from a representation of A) if ρ is unitarily equivalent to a representation induced from (a representation of) A with respect to E. That is, ρ is induced if there is a Hilbert space H, a representation $\theta : A \to B(H)$, and a Hilbert space isomorphism $U : E \otimes_\theta H \to K$ such that $U\theta^E(\cdot) = \rho(\cdot)U$.*

The question of when a given representation is induced is answered by Rieffel's imprimitivity theorem [16, Theorem 6.29]. We now state a special case of the imprimitivity theorem, which like the equivalence theorem, is suitably general for our purposes.

Theorem 2.21 (Rieffel's Imprimitivity Theorem)**.** *A representation $\rho : \mathcal{L}(E) \to B(H)$ of $\mathcal{L}(E)$ on a Hilbert space H is induced from a representation of A if and only if there is a nondegenerate representation $\psi : \mathcal{K}(E) \to H$ such that*

$$\rho(X)(\psi(k)h) = \psi(Xk)h,$$

for all $X \in \mathcal{L}(E)$, $k \in \mathcal{K}(E)$, $h \in H$.

Note that in particular, a representation $\rho : \mathcal{L}(E) \to B(H)$ is induced if the restriction of ρ to $\mathcal{K}(E)$ is nondegenerate. We will revisit this observation in Section 2.6, as well as in Chapters 3 and 4. In fact, one of the main results from Chapter 3, Theorem 3.4, is simply the observation that the Stinespring representation of a certain type of completely positive map from $\mathcal{L}(E)$ to the bounded operators on Hilbert space is an induced representation (see also Lemma 4.7).

2.4 Representations of the Tensor Algebra, Covariant Representations, and Intertwining Pairs

As mentioned previously, a C^*-correspondence generalizes the notion of a Hilbert space. In the present section, we develop several concepts that generalize properties of operators on Hilbert space. For the purposes of dilation theory, the most important types of operators are contractions, partial isometries, isometries, coisometries, and unitaries, and we give appropriate generalizations of each.

Fix a unital C^*-algebra A and let E be a C^*-correspondence over A, with the left-action of A on E given by a homomorphism $\phi : A \to \mathcal{L}(E)$. Recall that $E \otimes_\phi E$ is an A-correspondence with left-action map $\phi^{(2)}(\cdot) := \phi(\cdot) \otimes I_E$. We set $E^{\otimes 0} = A$, $E^{\otimes 1} = E$, and $E^{\otimes 2} = E \otimes_\phi E$. We then inductively define A-correspondences $E^{\otimes n}$, $n \geq 3$, by setting $E^{\otimes n} = E \otimes_{\phi^{(n-1)}} E^{\otimes n-1}$, where the left-action of A on $E^{\otimes n}$ is given by the homomorphism $\phi^{(n)}(\cdot) := \phi(\cdot) \otimes I_{E^{\otimes n-1}}$. We call $E^{\otimes n}$ the n^{th} tensor power of E.

As a set, the *Fock space* associated to E, $\mathcal{F}(E)$, is just the direct sum of the tensor powers of E:

$$\mathcal{F}(E) = A \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \cdots .$$

$\mathcal{F}(E)$ is endowed with the structure of an A-correspondence by way of the homomorphism $\phi_\infty : A \to \mathcal{L}(\mathcal{F}(E))$ defined by $\phi_\infty(a) = diag(a, \phi(a), \phi^{(2)}(a), \phi^{(3)}(a), \ldots)$.

A very important class of operators on the Fock space are the so-called creation operators $\{T_\xi\}_{\xi \in E}$. For each $n \geq 0$ and each $\xi \in E$, define a map $T_\xi^{(n)} : E^{\otimes n} \to E^{\otimes n+1}$ by $T_\xi^{(n)}(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n$ ($T_\xi^{(0)} a = \xi \cdot a$). Each of the operators $T_\xi^{(n)}$ is adjointable, with adjoint $(T_\xi^{(n)})^* : E^{\otimes n+1} \to E^{\otimes n}$ given by the formula $(T_\xi^{(n)})^*(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_{n+1}) = \phi^{(n)}(\langle \xi, \xi_1 \rangle)(\xi_2 \otimes \cdots \otimes \xi_{n+1})$. The *creation operator* $T_\xi : \mathcal{F}(E) \to \mathcal{F}(E)$ determined by ξ is then given matricially as follows:

$$T_\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ T_\xi^{(0)} & 0 & 0 & 0 & \cdots \\ 0 & T_\xi^{(1)} & 0 & 0 & \ddots \\ 0 & 0 & T_\xi^{(2)} & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

It is not difficult to see that $T_\xi \in \mathcal{L}(\mathcal{F}(E))$ (in fact, $||T_\xi|| \leq ||\xi||$), with the adjoint of T_ξ given by

$$T_\xi^* = \begin{bmatrix} 0 & (T_\xi^{(0)})^* & 0 & 0 & \cdots \\ 0 & 0 & (T_\xi^{(1)})^* & 0 & \cdots \\ 0 & 0 & 0 & (T_\xi^{(2)})^* & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

T_ξ^* is called the *annihilation operator* determined by ξ.

The creation operators and $\phi_\infty(A)$ together generate the so-called tensor algebra associated to E. To be precise, the *tensor algebra* $\mathcal{T}_+(E)$ associated to a C^*-correspondence E is defined to be the norm-closed subalgebra of $\mathcal{L}(\mathcal{F}(E))$ generated by the collection of operators $\{T_\xi\}_{\xi \in E}$ and $\{\phi_\infty(a)\}_{a \in A}$.

We are particularly interested in (building extensions of) completely contractive representations of $\mathcal{T}_+(E)$. Our first objective is to associate to any such representation two other very useful notions, both of which turn out to be in bijective correspondence with the collection of all completely contractive representations of $\mathcal{T}_+(E)$. Toward that end, let $\rho : \mathcal{T}_+(E) \to B(H)$ be a completely contractive representation of the tensor algebra associated to E on a Hilbert space H. The formula

$$\sigma(a) = \rho(\phi_\infty(a)), \quad a \in A, \tag{2.1}$$

defines a representation $\sigma : A \to B(H)$, and the formula

$$\mathfrak{z}(\xi \otimes h) = \rho(T_\xi)h, \quad \xi \otimes h \in E \otimes_\sigma H,$$

determines a map $\mathfrak{z} : E \otimes_\sigma H \to H$. For any $a \in A$ and any $\xi \otimes h \in E \otimes_\sigma H$, we see that

$$\begin{aligned} \mathfrak{z}\sigma^E(\phi(a))(\xi \otimes h) &= \mathfrak{z}((\phi(a)\xi) \otimes h) = \rho(T_{\phi(a)\xi})h \\ &= \rho(\phi_\infty(a)T_\xi)h = \rho(\phi_\infty(a))\rho(T_\xi)h \\ &= \sigma(a)\mathfrak{z}(\xi \otimes h). \end{aligned}$$

It follows that $\mathfrak{z} \in \mathcal{I}(\sigma^E \circ \phi, \sigma)$. Furthermore, $\mathfrak{z}$ is contractive since ρ is completely contractive. This leads to the following definition.

Definition 2.22. *Given a Hilbert space H, an* intertwining pair $(\mathfrak{z}, \sigma)$ associated to (E, A, H) *consists of a representation $\sigma : A \to B(H)$ and a map $\mathfrak{z} : E \otimes_\sigma H \to H$ in the intertwiner space $\mathcal{I}(\sigma^E \circ \phi, \sigma)$.*

In addition, $(\mathfrak{z}, \sigma)$ is called a contractive intertwining pair*, a* partially isometric intertwining pair*, an* isometric intertwining pair*, a* coisometric intertwining pair*, or a* unitary intertwining pair *associated to (E, A, H) whenever $\mathfrak{z}$ has the corresponding property.*

Note that in the simplest case, i.e., the setting of Example 2.11, where $E = A = \mathbb{C}$ and ϕ is the identity map, σ has the only form it can: $\sigma(a)h = ah$. Moreover, in this case $E \otimes_\sigma H = H$ and so, $\mathfrak{z} \in B(H)$. In particular, a contractive intertwining pair $(\mathfrak{z}, \sigma)$ in this context simply amounts to a contraction on Hilbert space. Likewise, an isometric intertwining pair is really just an isometry, a coisometric intertwining pair is a coisometry, etc. In this way, we see that Definition 2.22 really does provide plausible generalizations of familiar operator theory notions, as claimed in this section's opening remarks.

Consider again the completely contractive representation $\rho : \mathcal{T}_+(E) \to B(H)$, and define a representation $\sigma : A \to B(H)$ as in (2.1), i.e., $\sigma = \rho \circ \phi_\infty$. Also, define a map $T : E \to B(H)$ by

$$T(\xi) = \rho(T_\xi), \quad \xi \in E.$$

Then T is completely contractive (by the complete contractivity of ρ), and for $a, b \in A$

and $\xi \in E$,

$$T(\phi(a)\xi b) = \rho(T_{\phi(a)\xi b}) = \rho(\phi_\infty(a)T_\xi\phi_\infty(b)) = \rho(\phi_\infty(a))\rho(T_\xi)\rho(\phi_\infty(b))$$
$$= \sigma(a)T(\xi)\sigma(b).$$

That is, $T : E \to B(H)$ is a bimodule map when $B(H)$ is viewed as a bimodule over A via the representation σ. A pair (T, σ) satisfying these properties is known as a completely contractive covariant representation.

Definition 2.23. *A* covariant representation *of* (E, A) *on a Hilbert space* H *is a pair* (T, σ) *consisting of a representation* $\sigma : A \to B(H)$ *and a linear, completely contractive map* $T : E \to B(H)$*, such that* T *is a bimodule map in the following sense:*

$$T(\phi(a)\xi b) = \sigma(a)T(\xi)\sigma(b), \quad a, b \in A, \, \xi \in E. \tag{2.2}$$

Additionally, (T, σ) *is a* completely contractive covariant representation *of* (E, A) *on* H *if* T *is completely contractive.*

Remark 2.24. *We should clarify that to say* $T : E \to B(H)$ *is completely contractive means that we are viewing* E *with its canonical operator space structure and that* T *is completely contractive with respect to it. This point is developed at length in [12] (see Section 3, in particular). Since it does not play an essential role in this thesis, we simply call attention to [12, Lemma 3.5], where it is proved that* T *is completely contractive if and only if for each choice of vectors* $\eta_1, \eta_2, \ldots, \eta_n \in E$*, the operator* *matrix inequality*

$$(T(\eta_i)^*T(\eta_j)) \leq (\sigma(\langle \eta_i, \eta_j \rangle))$$

holds in $M_n(B(H))$. One may thus take this as the definition of what it means for a covariant representation to be completely contractive.

Given a completely contractive representation $\rho : \mathcal{T}_+(E) \to B(H)$ of the tensor algebra, we have seen how to associate a contractive intertwining pair $(\mathfrak{z}, \sigma)$ and a completely contractive covariant representation (T, σ) to ρ. They are related by the following formulas: $\sigma = \rho \circ \phi_\infty$ and

$$\mathfrak{z}(\xi \otimes h) = \rho(T_\xi)h = T(\xi)h. \tag{2.3}$$

Evidently, this process is reversible. That is, starting from the point-of-view of a contractive intertwining pair $(\mathfrak{z}, \sigma)$ or a completely contractive covariant representation (T, σ), there is a completely contractive representation ρ of the tensor algebra [12, Theorem 3.10]. Therefore, by studying properties of intertwiners and of bimodule maps, we can obtain information about completely contractive representations of $\mathcal{T}_+(E)$. The proof that equation (2.3) sets up an equivalence between completely contractive covariant representations of (E, A) and completely contractive representations of $\mathcal{T}_+(E)$ is long. In fact, much of [12] is devoted to establishing this correspondence. However, for our purposes, intertwining pairs and covariant representations are more manageable, and so, these will be the objects on which we focus our efforts.

We want to highlight some of the details behind equation (2.3). Let $(\mathfrak{z}, \sigma)$ be an intertwining pair associated to (E, A, H). We may define a (linear) map $T_\mathfrak{z} : E \to B(H)$ by the formula

$$T_\mathfrak{z}(\xi)h = \mathfrak{z}(\xi \otimes h), \tag{2.4}$$

for $\xi \in E$ and $h \in H$. Fix $a, b \in A$ and $\xi \in E$. Then for all $h \in H$,

$$\begin{aligned} T_\mathfrak{z}(\phi(a)\xi b)h &= \mathfrak{z}((\phi(a)\xi b) \otimes h) = \mathfrak{z}(\sigma^E(\phi(a))((\xi b) \otimes h)) \\ &= \mathfrak{z}\sigma^E(\phi(a))(\xi \otimes (\sigma(b)h)) = \sigma(a)\mathfrak{z}(\xi \otimes (\sigma(b)h)) \\ &= \sigma(a)T_\mathfrak{z}(\xi)\sigma(b)h. \end{aligned}$$

Therefore, $T_\mathfrak{z}(\phi(a)\xi b) = \sigma(a)T_\mathfrak{z}(\xi)\sigma(b)$. That is, $T_\mathfrak{z}$ is a bimodule map of E into $B(H)$, via σ.

On the other hand, given a covariant representation (T, σ) of (E, A) on a Hilbert space H, we may define a map $\mathfrak{z}_T : E \otimes_\sigma H \to H$ on elementary tensors $\xi \otimes h \in E \otimes_\sigma H$ by the formula

$$\mathfrak{z}_T(\xi \otimes h) = T(\xi)h. \tag{2.5}$$

By (2.2), for any $a \in A$,

$$\begin{aligned} \mathfrak{z}_T\sigma^E(\phi(a))(\xi \otimes h) &= \mathfrak{z}_T((\phi(a)\xi) \otimes h) = T(\phi(a)\xi)h = \sigma(a)T(\xi)h \\ &= \sigma(a)\mathfrak{z}_T(\xi \otimes h). \end{aligned}$$

Since elements of the form $\xi \otimes h$ span a dense subset of $E \otimes_\sigma H$, we conclude that $\mathfrak{z}_T\sigma^E(\phi(\cdot)) = \sigma(\cdot)\mathfrak{z}_T$, that is, $\mathfrak{z}_T \in \mathcal{I}(\sigma^E \circ \phi, \sigma)$. We have thus proved part of the following proposition. For the norm equalities see [12, Lemma 3.5].

Proposition 2.25. *Let σ be a representation of A on a Hilbert space H.*

(i) If $\mathfrak{z} : E \otimes_\sigma H \to H$ is a bounded map in the intertwiner space $\mathcal{I}(\sigma^E \circ \phi, \sigma)$, then the map $T_\mathfrak{z} : E \to B(H)$ given by (2.4) is a linear bimodule map as in (2.2), i.e., $T_\mathfrak{z}(\phi(a)\xi b)h = \sigma(a)T_\mathfrak{z}(\xi)\sigma(b)$ for all $a, b \in A$ and $\xi \in E$. Furthermore, since $\mathfrak{z}$ is bounded, $T_\mathfrak{z}$ is completely bounded, and

$$||\mathfrak{z}|| = ||T_\mathfrak{z}||_{cb}.$$

In particular, $\mathfrak{z}$ contractive if and only if $T_\mathfrak{z}$ is completely contractive.

(ii) If $T : E \to B(H)$ is a completely bounded linear bimodule map as in (2.2)*, then the map $\mathfrak{z}_T : E \otimes_\sigma H \to H$ defined by (2.5) is in the intertwiner space $\mathcal{I}(\sigma^E \circ \phi, \sigma)$. Moreover, since T is completely bounded, $\mathfrak{z}_T$ is bounded, and*

$$||T||_{cb} = ||\mathfrak{z}_T||.$$

Hence, T is completely contractive if and only if $\mathfrak{z}_T$ is contractive.

If there is no chance for confusion, we often drop the subscripts and simply write T for $T_\mathfrak{z}$ and $\mathfrak{z}$ for $\mathfrak{z}_T$. Using this notation, we may summarize the bijective correspondence between bimodule maps and intertwiners established in Proposition 2.25 as follows:

The pair (T, σ) is a completely contractive covariant representation of (E, A) on H if and only if $(\mathfrak{z}, \sigma)$ is a contractive intertwining pair associated to (E, A, H).

The discussion to this point motivates us to define partially isometric covariant representations, isometric covariant representations, coisometric covariant representations, and unitary covariant representations in terms of the already established notions of partially isometric, isometric, coisometric, and unitary intertwining pairs.

Definition 2.26. *(T, σ) is a* partially isometric covariant representation*, an* isometric covariant representation*, a* coisometric covariant representation*, or a* unitary covariant representation *of (E, A) on H if and only if $\mathfrak{z}_T$ is partially isometric, isometric, coisometric, or unitary, respectively.*

Remark 2.27. *There are several things to note about Definition 2.26.*

(i) If $(T, \sigma) : (E, A) \to B(H)$ is an isometric covariant representation, then $\mathfrak{z} = \mathfrak{z}_T$ is an isometry, so for all $\xi, \eta \in E$ and all $h, k \in H$,

$$\begin{aligned}\langle T(\xi)^* T(\eta) k, h\rangle &= \langle T(\eta)k, T(\xi)h\rangle = \langle \mathfrak{z}(\eta \otimes k), \mathfrak{z}(\xi \otimes h)\rangle \\ &= \langle \eta \otimes k, \xi \otimes h\rangle = \langle k, \sigma(\langle \eta, \xi\rangle)h\rangle \\ &= \langle \sigma(\langle \xi, \eta\rangle)k, h\rangle.\end{aligned}$$

Thus, $T(\xi)^* T(\eta) = \sigma(\langle \xi, \eta\rangle)$ for all $\xi, \eta \in E$. In fact, a simple rearrangement of the above string of equalities proves that the condition $T(\xi)^* T(\eta) = \sigma(\langle \xi, \eta\rangle)$ is equivalent to the definition of isometric covariant representation given in 2.26. This gives a way of defining an isometric covariant representation independent of $\mathfrak{z}$.

(ii) Let $(T, \sigma) : (E, A) \to B(H)$ be a covariant representation. Define a map $\Psi : \mathcal{K}(E) \to B(H)$ by $\Psi(\xi \otimes \eta^*) = T(\xi)T(\eta)^*$. Muhly and Solel called Ψ

the completely positive extension of (T, σ) *[12, Lemma 5.1], and they showed that* Ψ *can be extended to* $\mathcal{L}(E)$*, with*

$$\Psi(X) = \mathfrak{z}_T \sigma^E(X) \mathfrak{z}_T^*, \quad X \in \mathcal{L}(E).$$

From this latter formula for Ψ*, it is easy to see that* Ψ *is unital (i.e.,* $\Psi(I_E) = I_H$*) if and only if* (T, σ) *a coisometric covariant representation of* (E, A) *on* H*. Even though* $\mathfrak{z}_T$ *was used, the conclusion is still equivalent to the definition of coisometric covariant representation and is independent of* $\mathfrak{z}_T$*. That is,* (T, σ) *is a coisometric covariant representation of* (E, A) *on* H *if and only if the unique completely positive extension of the map* $\Psi : \mathcal{K}(E) \to B(H)$*,* $\xi \otimes \eta^* \mapsto T(\xi)T(\eta)^*$*, is unital.*

This also gives a way of classifying partially isometric and unitary covariant representations independent of $\mathfrak{z}$*.* $(T, \sigma) : (E, A) \to B(H)$ *is a partially isometric covariant representation if and only if* $\Psi(I_E)$ *is a projection, and* (T, σ) *is a unitary covariant representation if* (T, σ) *is isometric and coisometric.*

(iii) What we call a coisometric covariant representation Muhly and Solel called fully *coisometric covariant representations in order to distinguish them from their J-*coisometric covariant representations *(which occur when* $\Psi(I_E)$ *is a projection; see [12, Definition 5.3]). We only consider the former type of covariant representations, so there should be no confusion if we drop the word "fully". Note that ours is the "natural" definition when viewed from the context of intertwining pairs.*

(iv) We shall have more to say about coisometric covariant representations in Chapter 5. For now, we remark that in [12], the authors observe that (fully) coisometric extensions need not always exist, and even when they do, they need not be unique. However, as we will show, with one simple assumption on the left-action map ϕ – namely, that there exists a (smooth, unital) generalized transfer operator *for ϕ – both existence and uniqueness of a coisometric extension is guaranteed.*

2.5 Directed Systems and Inductive Limits

In this section we review the basics of directed systems and inductive limits, important ingredients in the proof that coisometric extensions exist. Specifically, we outline the existence and uniqueness of inductive limits of Hilbert spaces, C^*-algebras, operators, and representations (see [8, p. 885-887]).

Throughout this section, we let $(I, \leq)$ denote a fixed directed set. Let $\{\mathcal{H}_i\}_{i \in I}$ be a family of Hilbert spaces with the property that whenever $i \leq j$, there exists an isometry $V_{ji} : \mathcal{H}_i \to \mathcal{H}_j$. Assume also that if $i \leq j \leq k$, then $V_{ki} = V_{kj}V_{ji}$, i.e., the following diagram commutes:

$$\begin{tikzcd} \mathcal{H}_i \arrow[r, "V_{j,i}"] \arrow[rr, bend right, "V_{k,i}"'] & \mathcal{H}_j \arrow[r, "V_{k,j}"] & \mathcal{H}_k \end{tikzcd} \tag{2.6}$$

Note that if $i \leq j$, then $V_{ji}V_{ii} = V_{ji}$ and so,

$$V_{ii} = (V_{ji}^* V_{ji})V_{ii} = V_{ji}^*(V_{ji}V_{ii}) = V_{ji}^* V_{ji} = I_{H_i}. \tag{2.7}$$

This shows that V_{ii} is the identity operator on H_i for each $i \in I$. In a general category, a collection of objects and morphisms satisfying properties like (2.6) and (2.7) is called a *directed system*.

The *inductive limit* of the directed system of Hilbert spaces $\{H_i\}_{i\in I}$ and isometries $\{V_{ji}\}_{i\leq j}$ consists of a Hilbert space $\mathcal{H}$ and, for each $i \in I$, an isometry $V_i : H_i \to \mathcal{H}$, such that $V_j V_{ji} = V_i$ for $i \leq j$.

The inductive limit of the directed system $\{\{H_i\}_{i\in I}, \{V_{ji}\}_{i\leq j}\}$ can be summarized by the following diagram, in which every path commutes.

$$\begin{tikzcd}
\mathcal{H}_i \arrow[r, "V_{j,i}"] \arrow[rr, bend right, "V_{k,i}"'] \arrow[rrrr, bend left, "V_i"] & \mathcal{H}_j \arrow[r, "V_{k,j}"] \arrow[rrr, bend left, "V_j"] & \mathcal{H}_k \arrow[r] \arrow[rr, bend left, "V_k"] & \cdots \arrow[r] & \mathcal{H}
\end{tikzcd}$$

Furthermore, the span of the ranges of the V_i's is dense in $\mathcal{H}$, i.e.,

$$\bigvee_{i\in I} V_i(H_i) = \mathcal{H}.$$

The inductive limit is unique in the following sense. If $\mathcal{K}$ is a Hilbert space, if $\{W_i : \mathcal{H}_i \to \mathcal{K}\}_{i\in I}$ is a collection of isometries such that $W_j V_{ji} = W_i$ for $i \leq j$, and if $\bigvee_i W_i(\mathcal{H}_i) = \mathcal{K}$, then there is a Hilbert space isomorphism $U : \mathcal{H} \to \mathcal{K}$ such that $W_i = UV_i$ for all $i \in I$.

The notation $\mathcal{H} = \varinjlim H_i$ is used to denote the inductive limit of the directed system $\{\{H_i\}_{i\in I}, \{V_{ji}\}_{i\leq j}\}$. It is important to keep in mind that $\varinjlim H_i$ always comes equipped with isometries $V_i : H_i \to \varinjlim H_i$ satisfying the stated properties.

Assume now that $\{H_i\}_{i\in I}$ and $\{K_i\}_{i\in I}$ are two directed systems of Hilbert spaces, with isometries $V_{ji} : H_i \to H_j$ and $W_{ji} : K_i \to K_j$. Write $\mathcal{H} = \varinjlim H_i$ and $\mathcal{K} = \varinjlim K_i$ for the inductive limits of these directed systems, and denote by $V_i : H_i \to \mathcal{H}$

and $W_i : K_i \to \mathcal{K}$ the isometries associated to these inductive limits. Further, suppose that $\{T_i : H_i \to K_i\}_{i\in I}$ is a collection operators satisfying $\sup_{i\in I} ||T_i|| < \infty$ and such that there exists $i_0 \in I$ so that $T_j V_{ji} = W_{ji} T_i$ whenever $i_0 \leq i \leq j$. Then there is a unique element $T \in B(\mathcal{H}, \mathcal{K})$, the *inductive limit* of $\{T_i\}_{i\in I}$, such that $TV_i = W_i T_i$ whenever $i_0 \leq i$. Indeed, one simply defines T on $\bigvee_i V_i(\mathcal{H}_i)$ by $TV_i = W_i T_i$ and then extends to an operator of $\mathcal{H}$ into $\mathcal{K}$. The equalities $T_j V_{ji} = W_{ji} T_i$ and the density of $\bigvee_i V_i(\mathcal{H}_i)$ and $\bigvee_i W_i(\mathcal{K}_i)$ in $\mathcal{H}$ and $\mathcal{K}$, respectively, ensure that T is unique.

The final type of inductive limit we need is the inductive limit of a directed system of C^*-algebras and C^*-representations. Let $\{\{H_i\}_{i\in I}, \{V_{ji}\}_{i\leq j}\}$ be a directed system of Hilbert spaces and isometries between those spaces. For each $i \in I$, let A_i be a C^*-algebra and let π_i be a representation of A_i on H_i. Also, assume there is a homomorphism $\phi_{ji} : A_i \to A_j$ for each $i \leq j$, and suppose that

$$\pi_j(\phi_{ji}(a))V_{ji} = V_{ji}\pi_i(a), \quad a \in A_i,\ i \leq j.$$

Denote by A the inductive limit of the directed system $\{A_i\}$ of C^*-algebras, written $A = \varinjlim A_i$, and let $H = \varinjlim H_i$. For each $i \in I$ there exists a $*$-isomorphism $\phi_i : A_i \to A$, and there is a unique representation $\pi : A \to B(H)$ such that

$$\pi(\phi_i(a))V_i = V_i\pi_i(a), \quad a \in A.$$

It is a non-trivial fact that A even exists as a C^*-algebra, but if every $A_i = B$ for some C^*-algebra B and if each ϕ_{ji} is the identity, then it is easy to see that the inductive limit of $\{A_i\}_{i\in I}$ exists and equals B (this is the setting in which we are most interested). In case $A_i = B$ and ϕ_i is the identity for all $i \in I$, the intertwining condition above simplifies to $\pi_j(\cdot)V_{ji} = V_{ji}\pi_i(\cdot)$, i.e., $V_{ji} \in \mathcal{I}(\pi_i, \pi_j)$. Also, the condition on π becomes $\pi(\cdot)V_i = V_i\pi_i(\cdot)$, i.e., $V_i \in \mathcal{I}(\pi_i, \pi)$. We mention in passing that the situation for W^*-algebras is more complicated, as the inductive limit of a directed system of W^*-algebras need not exist. In this setting, the C^*-inductive limit exists, but it need not be a W^*-algebra.

2.6 Transfer Operators

One of the key tools in the analysis that follows is the notion of a transfer operator. Let $\alpha : A \to A$ be an endomorphism of a C^*-algebra A. A *transfer operator* for α is a linear, completely positive map $\tau : A \to A$ satisfying the so-called conditional expectation property on Ran α:

$$\tau(\alpha(a)b) = a\tau(b), \quad a, b \in A.$$

This definition is due to Exel [4], who borrowed the term from classical dynamics. Our motivation for incorporating transfer operators into the study of coisometric extensions comes from Muhly and Solel, who in [14] suggested a way of building a coisometric extension for any contractive intertwining pair associated to a C^*-correspondence of the form ${}_\alpha A$, where A is a C^*-algebra and $\alpha \in \mathrm{End}(A)$. In fact, two of our main results, Theorems 5.2 and 5.3, are generalizations of the main result (Theorem 1.2) from [14]. In the more general case, where E is an arbitrary C^*-correspondence over a C^*-algebra A, the endomorphism α is replaced by the left-action map $\phi : A \to \mathcal{L}(E)$. Accordingly, we need a more general notion of transfer operator in order for the construction in [14] to carry through.

Definition 2.28. *Let $\psi : A \to B$ be a homomorphism between two C^*-algebras A and B. A* generalized transfer operator *for ψ is a linear, completely positive map $\tau : B \to A$ such that*

$$\tau(\psi(a)b) = a\tau(b)$$

for all $a \in A$ and all $b \in B$.

We call τ smooth *if for any $b \in B$,*

$$\tau((e_\lambda b - b)^*(e_\lambda b - b)) \to 0$$

for some, and hence any, approximate identity $\{e_\lambda\}$ for B.

Assume that B is unital. Then by positivity and linearity, τ is self-adjoint, i.e., $\tau(b) = \tau^*(b) := \tau(b^*)^*$ for all $b \in B$ (to see this, use the fact that b can be written as a sum of self-adjoint elements and that every self-adjoint element is the difference of two positive elements). Thus, for $a \in A$ and $b \in B$,

$$\tau(b\psi(a)) = \tau((b\psi(a))^*)^* = (\tau(\psi(a^*)b^*))^* = (a^*\tau(b^*))^* = \tau(b^*)^*a = \tau(b)a,$$

That is, $\tau(b\psi(a)) = \tau(b)a$ for all $a, b \in A$.

Notice that if A is also unital, and if $\tau(1_B) = 1_A$, then τ is surjective. Indeed, for any $a \in A$,

$$\tau(\psi(a)) = \tau(\psi(a)1_B) = a\tau(1_B) = a1_A = a.$$

The definition of smoothness for a generalized transfer operator is adapted from Definition 4.12 of [16] and applies to completely positive maps in general, not just transfer operators. The reason we want to impose smoothness is that the Stinespring representation associated to a smooth map of $\mathcal{L}(E)$ into $B(H)$ is nondegenerate when restricted to $\mathcal{K}(E)$, and as we observed in Section 2.3, such a representation is induced (see the discussion leading up to and following Theorem 3.4).

It is easy to see that the definition of generalized transfer operator extends naturally the notion introduced by Exel. It is also clear that the zero operator serves as a transfer operator for *any* homomorphism $\psi : A \to B$. However, even in the setting in which $B = A$ and $\psi \in \mathrm{End}(A)$, nonzero transfer operators need not exist. To complicate matters further, even if a nonzero transfer operator exists, it is in general not unique. However, there are many interesting settings in which generalized transfer operators do exist, some of which we explore in the examples below. Also, we remark that existence is guaranteed under the conditions that ψ is faithful and there is a conditional expectation onto Ran ψ (see [9, Theorem 1.6]).

Example 2.29. *We saw above that a generalized transfer operator generalizes Exel's notion of a transfer operator (hence the terminology). Indeed, the two definitions agree in the setting of Example 2.12, where A is a unital C^*-algebra and $E = {}_\alpha A$ for a homomorphism $\alpha : A \to \mathcal{L}(E) \simeq A$ (i.e., an endomorphism of A).*

Example 2.30. *Recall the setting of Example 2.13, where $A = \mathbb{C}$, $H = {}_{\mathbb{C}}H_{\mathbb{C}}$ is a Hilbert space, and $\phi : \mathbb{C} \to B(H)$ is given by left-multiplication. Then a generalized transfer operator τ for ϕ is a linear, completely positive map of $B(H)$ into $\mathbb{C}$. If, in addition, τ is unital, then it is a positive linear functional of norm one, that is, a state on $B(H)$. Further, if τ is smooth (and unital), then it is a* normal *state on*

$B(H)$. And conversely, every normal state on $B(H)$ is a smooth transfer operator for ϕ.

These two examples show that the idea of a unital generalized transfer operator simultaneously generalizes the classical definition of a transfer operator and the notion of a state on the C^*-algebra of bounded operators on a Hilbert space. It also shows that the definition of generalized transfer operator is not vacuous, and in fact, is a very important concept in many settings. After all, there are lots of states on $B(H)$ for a fixed Hilbert space H, and in certain cases it is even known what the most general state looks like. Indeed, if H is finite-dimensional, say $\dim H = d$, then we may identify $B(H)$ with M_d, the $d \times d$ matrices over $\mathbb{C}$, and a state $\tau : M_d \to \mathbb{C}$ is necessarily of the form:

$$\tau(X) = tr(DX), \quad X \in M_d,$$

where $D \in M_d$ is a (fixed) *density matrix*, i.e., a positive $d \times d$ matrix with trace one. In fact, all normal states have this form in $B(H)$.

Although it is not known in general what conditions on a homomorphism $\psi : A \to B$ guarantee the existence of a generalized transfer operator, it *is* known when a transfer operator exists if $B = A = C(X)$ for a compact Hausdorff space X and ψ is given by a local homeomorphism. It is also known when transfer operators exist in the setting $B = A = B(H)$ for a (separable) Hilbert space H and where ψ is an endomorphism of $B(H)$.

Example 2.31. *If $A = C(X)$ for a compact Hausdorff space X, if θ is a local* *homeomorphism of X, and if $\psi : C(X) \to C(X)$ is the endomorphism given by composition with θ, i.e., $\psi(f) = f \circ \theta$, then every transfer operator $\tau : C(X) \to C(X)$ for ψ has the following form:*

$$\tau(f)(x) = \sum_{\substack{y \in X \\ \theta(y)=x}} W(y) f(y), \quad f \in C(X), \, x \in X,$$

where W is a positive continuous function on X such that $\sum_{\theta(y)=x} W(y) = 1$ for all $x \in X$. If τ has this form, then for $f, g \in C(X)$ and $x \in X$,

$$\begin{aligned}
\tau(\psi(f)g)(x) &= \sum_{\theta(y)=x} W(y)\, \psi(f)(y)\, g(y) = \sum_{\theta(y)=x} W(y)\, f(\theta(y))\, g(y) \\
&= \sum_{\theta(y)=x} W(y)\, f(x)\, g(y) = f(x) \sum_{\theta(y)=x} W(y)\, g(y) \\
&= (f\tau(g))(x).
\end{aligned}$$

Therefore, $\tau(\psi(f)g) = f\tau(g)$ for all $f, g \in C(X)$ and so, τ is a transfer operator for ψ (the other requirements for a transfer operator are readily verified). The converse is proved in [9, Proposition 2.6].

Example 2.32. *Let H be a separable Hilbert space and let ψ be a normal endomorphism of $B(H)$. The normality of ψ implies that there is a row-contraction $S = (S_1, S_2, \ldots, S_d)$ (i.e., $SS^* = \sum_i S_i S_i^* \leq I_H$) such that*

$$\psi(X) = \sum_{i=1}^{d} S_i X S_i^*, \quad X \in B(H).$$

B.K. Kwaśniewski [9, Theorem 3.3] showed that $\tau : B(H) \to B(H)$ is a transfer operator for ψ if and only if

$$\tau(X) = \sum_{i,j=1}^{d} D(i,j) S_i^* X S_j, \quad X \in B(H),$$

where $(D(i,j)) \in M_d$ *is a matrix of a positive trace-class operator* $D \in K(H)$, *i.e.,* $(D(i,j))$ *is a* $d \times d$ *density matrix. Furthermore, in this case* $||\tau|| = tr(D) = \sum_i D(i,i)$.

CHAPTER 3

STINESPRING AND INDUCED REPRESENTATIONS

Fix a C^*-correspondence E over a unital C^*-algebra A, and let ϕ denote the homomorphism of A into $\mathcal{L}(E)$ that gives the left-action of A on E. Also, assume a unital, smooth generalized transfer operator for ϕ exists.

Remark 3.1 (A Note on Notation). *We work with many different representations of the C^*-algebras A and $\mathcal{L}(E)$ in this and subsequent chapters. As a result, it will be convenient to subscript Hilbert spaces associated to representations with the name of the representation, using "H" for representations of A and "K" for representations of $\mathcal{L}(E)$. One advantage of this notation is that just by seeing the Hilbert space, one can immediately deduce the name of the representation and the C^*-algebra that is being represented. For example, in the result that follows, there are Hilbert spaces H_σ and K_ρ – from this information alone we know that there is a representation σ of A on H_σ and a representation ρ of $\mathcal{L}(E)$ on K_ρ.*

As a preliminary step toward our goal of constructing a (unique) coisometric extension for any given contractive intertwining pair $(\mathfrak{z}, \sigma)$, we prove several results that follow from Stinespring's dilation theorem. Taken together, the results in this chapter show that the Stinespring representation associated to a representation composed with a (smooth) transfer operator is induced (see, in particular, Corollary 3.6). We shall invoke these results repeatedly in what follows.

Theorem 3.2. *Let σ be a representation of A on a Hilbert space H_σ, and let $\tau : \mathcal{L}(E) \to A$ be a unital, generalized transfer operator for the homomorphism $\phi : A \to \mathcal{L}(E)$. Then there is a Hilbert space K_ρ, a representation $\rho : \mathcal{L}(E) \to B(K_\rho)$, and an isometry $V : H_\sigma \to K_\rho$, such that*

$$V^*\rho(X)V = \sigma \circ \tau(X), \quad X \in \mathcal{L}(E). \tag{3.1}$$

In particular,

$$\rho(\phi(a))V = V\sigma(a), \quad a \in A, \tag{3.2}$$

i.e., $V \in \mathcal{I}(\sigma, \rho \circ \phi)$, and

$$V^*\rho(\phi(a))V = \sigma(a), \quad a \in A. \tag{3.3}$$

Moreover, we may assume (without a loss of generality) that K_ρ is minimal in the sense that the smallest subspace of K_ρ that contains $[VH_\sigma]$ and reduces ρ is all of K_ρ. Then $K_\rho = [\rho(\mathcal{L}(E))VH_\sigma]$, and the triple (K_ρ, ρ, V) is unique up to unitary equivalence. More precisely, if $(K_{\rho'}, \rho', V')$ is another triple satisfying (3.1), and if $K_{\rho'}$ is minimal in the sense described, then $K_{\rho'} = [\rho'(\mathcal{L}(E))V'H_\sigma]$, and there is a Hilbert space isomorphism $U : K_\rho \to K_{\rho'}$ such that $U\rho(\cdot)U^* = \rho'(\cdot)$ and such that $UV = V'$.

It is worth noting that (3.2) and (3.3) are not automatic consequences of (3.1). Rather, they follow from (3.1) *and* the fact that τ is a (unital) transfer operator.

Proof. We apply Stinespring's dilation theorem (Theorem 2.6) to the completely positive map $\sigma \circ \tau : \mathcal{L}(E) \to B(H_\sigma)$ to obtain a Hilbert space K_ρ, a representation

$\rho : \mathcal{L}(E) \to B(K_\rho)$, and a bounded operator $V : H_\sigma \to K_\rho$, such that

$$V^*\rho(X)V = \sigma(\tau(X)), \quad X \in \mathcal{L}(E).$$

Recall that $K_\rho = \mathcal{L}(E) \otimes_{\sigma\circ\tau} H_\sigma$ and that the Stinespring representation $\rho : \mathcal{L}(E) \to B(K_\rho)$ has the following form:

$$\rho(Y)(X \otimes h) = (YX) \otimes h, \quad Y \in \mathcal{L}(E), \ X \otimes h \in K_\rho.$$

Recall also that the bounded operator $V : H_\sigma \to K_\rho$ is given by the formula $Vh = I_E \otimes h$, where I_E denotes the identity operator on E. Note that (3.1) and the unitality of $\sigma \circ \tau$ imply that V is an isometry. Further, for any $a \in A$ and any elementary tensor $X \otimes h \in K_\rho$, we have

$$\begin{aligned} V^*\rho(\phi(a))(X \otimes h) &= V^*\rho(\phi(a))(\rho(X)Vh) = V^*\rho(\phi(a)X)Vh \\ &= \sigma(\tau(\phi(a)X))h = \sigma(a\tau(X))h \\ &= \sigma(a)\sigma(\tau(X))h = \sigma(a)V^*\rho(X)Vh \\ &= \sigma(a)V^*(X \otimes h). \end{aligned}$$

It follows that $V^*\rho(\phi(a)) = \sigma(a)V^*$ for all $a \in A$. That is, $V^* \in \mathcal{I}(\rho \circ \phi, \sigma)$, and therefore, $V \in \mathcal{I}(\sigma, \rho\circ\phi)$. Combining this with our observation that V is an isometry, we see that

$$V^*\rho(\phi(a))V = V^*V\sigma(a) = \sigma(a), \quad a \in A.$$

If K_ρ is minimal in the sense described in the statement of the lemma, then $K_\rho = [\rho(\mathcal{L}(E))VH_\sigma]$, and the triple (K_ρ, ρ, V) is the minimal Stinespring representation for $\sigma \circ \tau$. Assume that $(K_{\rho'}, \rho', V')$ is another minimal triple satisfying (3.1). Minimality implies that $K_{\rho'} = [\rho'(\mathcal{L}(E))V'H_\sigma]$, and evidently the map

$$\rho(X)Vh \mapsto \rho'(X)V'h, \quad X \in \mathcal{L}(E), h \in H_\sigma,$$

extends to a Hilbert space isomorphism $U : K_\rho \to K_{\rho'}$ such that $UV = V'$ and $U\rho U^* = \rho'$ (see for example, [15, Proposition 4.2] for the full details).

□

Definition 3.3. *We call the triple (K_ρ, ρ, V) the* extension of σ adapted to τ. *Note that we are justified in using the definite article by uniqueness (since we identify unitarily equivalent extensions).*

We observed in Section 2.3 that if σ is a representation of A, then there is an induced representation σ^E of $\mathcal{L}(E)$ that is nondegenerate on $\mathcal{K}(E)$, and conversely, given a representation of $\mathcal{L}(E)$ that restricts to $\mathcal{K}(E)$ nondegenerately, we can recover a (nondegenerate) representation of A. The next result is simply the observation that the Stinespring representation associated to a smooth map is nondegenerate on $\mathcal{K}(E)$, and therefore, is necessarily an induced representation. We mention in passing that Muhly, Skeide, and Solel proved a much more general version of Theorem 3.4 [11, Theorems 1.4 & 1.8], but the following is sufficiently general for our purposes.

Theorem 3.4. *Let $\psi : \mathcal{L}(E) \to B(H)$ be a completely positive map, and let (K_ρ, ρ, V) be the Stinespring triple associated to ψ. If ψ is smooth, then ρ is induced uniquely from a representation of A with respect to E. That is, there is a Hilbert space H_θ, a representation θ of A on H_θ, and a Hilbert space isomorphism of $E \otimes_\theta H_\theta$ onto*

K_ρ that implements an equivalence between the representations ρ and θ^E. Further, (H_θ, θ) is the unique pair satisfying these properties, up to unitary equivalence.

Proof. As observed in the discussion after Theorem 2.21, we can conclude that ρ is induced if we know that $\rho(\cdot)|_{\mathcal{K}(E)}$ is nondegenerate. Further, by Proposition 2.5 of [10], $\rho(\cdot)|_{\mathcal{K}(E)}$ is nondegenerate if and only if $\rho(k_\lambda)$ converges to the identity $I_K \in B(K)$ in the strong-$*$ topology for some, hence any, approximate identity $\{k_\lambda\}$ of $\mathcal{K}(E)$. To see that this is the case, fix a simple tensor $X \otimes h \in K_\rho = \mathcal{L}(E) \otimes_\psi H$, and observe that

$$\begin{aligned} ||\rho(k_\lambda)(X \otimes h) - X \otimes h||^2 = ||(k_\lambda X) \otimes h - X \otimes h||^2 &= ||(k_\lambda X - X) \otimes h||^2 \\ &= \langle (k_\lambda X - X) \otimes h, (k_\lambda X - X) \otimes h \rangle \\ &= \langle h, \psi((k_\lambda X - X)^*(k_\lambda X - X))h \rangle. \end{aligned}$$

This last expression converges to zero by the assumption that ψ is smooth. It follows that $\rho(k_\lambda)x \to x$ for every $x \in K_\rho$. Since each k_λ is self-adjoint, we conclude that $\rho(k_\lambda) \to I_K$. Therefore, ρ is induced from A with respect to E. That is, there is a Hilbert space H_θ, a representation θ of A on H_θ, and a Hilbert space isomorphism $U : E \otimes_\theta H_\theta \to K_\rho$ such that $U\theta^E(\cdot) = \rho(\cdot)U$. In particular, $U\theta^E(\cdot)U^* = \rho(\cdot)$, showing that θ^E and ρ are unitarily equivalent representations.

The fact that (H_θ, θ) is unique up to unitary equivalence follows from Rieffel's equivalence theorem (Theorem 2.18). Assume that $H_{\theta'}$ is a Hilbert space and $\theta' : A \to B(H_{\theta'})$ is a representation such that there exists a Hilbert space isomorphism

$U' : E \otimes_{\theta'} H_{\theta'} \to K_\rho$ with $U'\theta'^E(\cdot) = \rho(\cdot)U'$. Then the composition $U^*U' : E \otimes_{\theta'} H_{\theta'} \to E \otimes_\theta H_\theta$ is a Hilbert space isomorphism that intertwines the representations θ'^E and θ^E. In particular, U^*U' is a morphism between the two objects θ'^E and θ^E in the category of (nondegenerate) representations of $\mathcal{L}(E)$, with morphisms intertwining operators between representations. Hence, by Theorem 2.18, U^*U' has the form $I_E \otimes W$, where $W : H_{\theta'} \to H_\theta$ is a Hilbert space isomorphism such that $W\theta'(\cdot) = \theta(\cdot)W$, i.e., $W\theta'(\cdot)W^* = \theta(\cdot)$. Therefore, θ' is unitarily equivalent to θ, and the proof is complete.

□

Remark 3.5. *We claim that in fact, $H_\theta = \widetilde{E} \otimes_\rho K_\rho$ and $\theta = \rho^{\widetilde{E}}$. We will prove this by showing that $\widetilde{E} \otimes_\rho K_\rho$ and $\rho^{\widetilde{E}}$ satisfy the conditions laid out in Theorem 3.4. Then necessarily $H_\theta = \widetilde{E} \otimes_\rho K_\rho$ and $\theta = \rho^{\widetilde{E}}$ (up to unitary equivalence) by the uniqueness of (H_θ, θ).*

Set $H_\theta = \widetilde{E} \otimes_\rho K_\rho$ and define a representation $\theta : A \to B(H_\theta)$ by $\theta = \rho^{\widetilde{E}}$, that is,

$$\theta(a)(\eta^* \otimes x) = (a \cdot \eta^*) \otimes x = (\eta \otimes a^*)^* \otimes x, \quad a \in A, \ \eta^* \otimes x \in H_\theta.$$

Note that these definitions make sense since $\widetilde{E}$ can be viewed as an $A - \mathcal{L}(E)$-correspondence. Consider the mapping from $E \otimes_\theta H_\theta$ to K_ρ given by $\xi \otimes (\eta^* \otimes x) \mapsto$

$\rho(\xi \otimes \eta^*)x$. *We have*

$$\begin{aligned}
\langle \xi_1 \otimes (\eta_1^* \otimes x_1), \xi_2 \otimes (\eta_2^* \otimes x_2)\rangle &= \langle \eta_1^* \otimes x_1, \theta(\langle \xi_1, \xi_2\rangle)(\eta_2^* \otimes x_2)\rangle \\
&= \langle \eta_1^* \otimes x_1, (\eta_2 \cdot \langle \xi_1, \xi_2\rangle^*)^* \otimes x_2\rangle \\
&= \langle x_1, \rho(\langle \eta_1^*, (\eta_2 \cdot \langle \xi_2, \xi_1\rangle)^*\rangle)x_2\rangle \\
&= \langle x_1, \rho(\eta_1 \otimes (\eta_2 \cdot \langle \xi_2, \xi_1\rangle)^* x_2\rangle \\
&= \langle x_1, \rho((\eta_1 \cdot \langle \xi_1, \xi_2\rangle) \otimes \eta_2^*)x_2\rangle \\
&= \langle x_1, \rho((\eta_1 \otimes \xi_1^*)(\xi_2 \otimes \eta_2^*))x_2\rangle \\
&= \langle x_1, \rho((\xi_1 \otimes \eta_1^*)^*(\xi_2 \otimes \eta_2^*))x_2\rangle \\
&= \langle \rho(\xi_1 \otimes \eta_1^*)x_1, \rho(\xi_2 \otimes \eta_2^*)x_2\rangle.
\end{aligned}$$

Therefore, the map $\xi \otimes (\eta^* \otimes x) \mapsto \rho(\xi \otimes \eta^*)x$ *is an isometry of* $E \otimes_\theta H_\theta$ *into* K_ρ. *Furthermore, if* $\{k_\lambda\} \subseteq \mathcal{K}(E)$ *is an approximate identity and if* x *is any element of* K_ρ, *then* $\rho(k_\lambda)x \to x$. *It follows that the map* $\xi \otimes (\eta^* \otimes x) \mapsto \rho(\xi \otimes \eta^*)x$ *can be extended uniquely to a Hilbert space isomorphism* $U : E \otimes_\theta H_\theta \to K_\rho$. *Then for any* $X \in \mathcal{L}(E)$ *and any* $\xi \otimes (\eta^* \otimes x) \in E \otimes_\theta H_\theta$, *we see that*

$$\begin{aligned}
U\theta^E(X)(\xi \otimes (\eta^* \otimes x)) &= U((X\xi) \otimes (\eta^* \otimes x)) = \rho((X\xi) \otimes \eta^*)x \\
&= \rho(X \cdot (\xi \otimes \eta^*))x = \rho(X)\rho(\xi \otimes \eta^*)x \\
&= \rho(X)U(\xi \otimes (\eta^* \otimes x)).
\end{aligned}$$

Therefore, the Hilbert space isomorphism U, *which identifies* K_ρ *and* $E \otimes_\theta H_\theta$, *inter-*

twines ρ *and* θ^E *and so, these representations are unitarily equivalent.*

Theorem 3.4 has the following immediate corollary, which highlights the importance of our assumption that transfer operators are smooth.

Corollary 3.6. *Let* $\sigma : A \to B(H_\sigma)$ *be a representation, let* $\tau : \mathcal{L}(E) \to A$ *be a unital generalized transfer operator for the homomorphism* $\phi : A \to \mathcal{L}(E)$*, and let* (K_ρ, ρ, V) *be the extension of* σ *adapted to* τ*. If* τ *is smooth, then the representation* ρ *is induced uniquely from a representation of* A *with respect to* E*.*

CHAPTER 4

PARTIALLY ISOMETRIC EXTENSIONS

4.1 The Classical Case

In this section, we review some useful facts about partially isometric extensions for contractive operators on Hilbert space. Although the results of this section are well-known, we include proofs for the sake of completeness. Additionally, the proofs in this section serve as models for the more general setting (e.g., compare Theorem 4.5 with Theorems 4.13 and 4.14). The reader who is familiar with dilating and extending operators on Hilbert space may wish to proceed directly to Section 4.2.

To begin, fix a Hilbert space H and a contraction $T \in B(H)$. Denote by Δ_* the defect operator for T^*, i.e., $\Delta_* = (I_H - TT^*)^{1/2}$, and let $\mathcal{D}_* = [\Delta_* H]$. Then the operator $\mathcal{W} : H \oplus \mathcal{D}_* \to H \oplus \mathcal{D}_*$ defined by $\mathcal{W}(h, x) = (Th + \Delta_* x, 0)$, $(h, x) \in H \oplus \mathcal{D}_*$, is a minimal partial isometric extension of T. That is, $\mathcal{W}$ is a partial isometry (with final space H) such that H is invariant for $\mathcal{W}$ with $\mathcal{W}|_H = T$, and such that $\mathcal{W}$ is minimal in the sense that the smallest reducing subspace for $\mathcal{W}$ that contains H is $H \oplus \mathcal{D}_*$. In fact, we claim that $\mathcal{W}$ is *the* minimal partial isometric extension of T, i.e., that $\mathcal{W}$ is unique up to unitary equivalence.

In order to prove that $\mathcal{W}$ is the unique minimal partially isometric extension of T, we exploit the fact that an operator on the direct sum of n Hilbert spaces (vector spaces, for that matter) can be written as an $n \times n$ operator matrix by viewing elements of the direct sum as $(n \times 1)$ column vectors. For example, viewing elements of $H \oplus \mathcal{D}_*$ as 2×1 column vectors, we may write

$$\mathcal{W} = \begin{bmatrix} T & \Delta_* \\ 0 & 0 \end{bmatrix}.$$

More generally, assume that W is an operator on a Hilbert space K containing H. Then we can write $K = H \oplus H^\perp$, and we can express W as follows:

$$W = \begin{bmatrix} B & D \\ C & F \end{bmatrix},$$

for operators $B : H \to H$, $D : H^\perp \to H$, $C : H \to H^\perp$, and $F : H^\perp \to H^\perp$.

Lemma 4.1. *W extends T, i.e., $WH \subseteq H$ and $W|_H = T$, if and only if $B = T$ and $C = 0$.*

Proof. Since $B \in B(H)$, $D \in B(H^\perp, H)$, $C \in B(H, H^\perp)$, and $F \in B(H^\perp)$, it is easy to see that $WH \subseteq H$ if and only if $C = 0$. Moreover, simple computations show that $W|_H = T$ if and only if $C = 0$ *and* $B = T$.

□

Assume now that W extends T. By the previous lemma we can write

$$W = \begin{bmatrix} T & D \\ 0 & F \end{bmatrix}.$$

Lemma 4.2. *W is a partial isometry if and only if $TT^* + DD^*$ is a projection, $DF^* = 0$, and FF^* is a projection (i.e., F is a partial isometry).*

Proof. Recall that W is a partial isometry if and only if WW^* is a projection. We compute

$$WW^* = \begin{bmatrix} T & D \\ 0 & F \end{bmatrix} \begin{bmatrix} T^* & 0 \\ D^* & F^* \end{bmatrix} = \begin{bmatrix} TT^* + DD^* & DF^* \\ FD^* & FF^* \end{bmatrix}.$$

Write $P = TT^* + DD^*$ and $Q = FF^*$, and observe that WW^*, P, and Q are all self-adjoint. Furthermore,

$$WW^* = (WW^*)^2 \iff \begin{bmatrix} P & DF^* \\ FD^* & Q \end{bmatrix} = \begin{bmatrix} P^2 + DF^*FD^* & PDF^* + DF^*Q \\ FD^*P + QFD^* & FD^*DF^* + Q^2 \end{bmatrix}$$

$$\iff P = P^2,\ DF^* = 0,\ \text{and } Q = Q^2.$$

□

Corollary 4.3. *(i) Ran* $W \supseteq H$ *if and only if* $TT^* + DD^* = I_H$*, and*

(ii) Ran $W = H$ *if and only if* $TT^* + DD^* = I_H$ *and* $F = 0$*.*

Proof. By the computations in the last proposition, $H \subseteq \text{Ran } W = WW^*K$ if and only if $I_H \leq WW^*$ if and only if $TT^* + DD^* = P = I_H$. Further, Ran $W = H$ if and only if $TT^* + DD^* = P = I_H$ and $FF^* = Q = 0$. This latter condition holds if and only if $F = 0$.

□

Theorem 4.4. *$W \in B(K)$ $(K \supseteq H)$ is a partially isometric extension of T (with final space H) if and only if W can be written*

$$W = \begin{bmatrix} T & \Delta_* U \\ 0 & 0 \end{bmatrix},$$

where $U : H^\perp \to \mathcal{D}_*$ is a surjective partial isometry.

Proof. If W is a partial isometry with final space H and W extends T, we have already established that

$$W = \begin{bmatrix} T & D \\ 0 & 0 \end{bmatrix},$$

where $D \in B(H^\perp, H)$. By the assumption that the final space of W is H, we have

$$\begin{bmatrix} I_H & 0 \\ 0 & 0 \end{bmatrix} = WW^* = \begin{bmatrix} T & D \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ D^* & 0 \end{bmatrix} = \begin{bmatrix} TT^* + DD^* & 0 \\ 0 & 0 \end{bmatrix}.$$

That is, $TT^* + DD^* = I_H$, which in turn implies that

$$DD^* = I_H - TT^* = \Delta_*^2.$$

Note also that $\Delta_*^2 = \Delta_*\Delta_*^*$ since Δ_* is self-adjoint. Therefore, by Douglas's range inclusion, majorization, and factorization theorem [1, Theorem 1], there exists a unique partial isometry $U : H^\perp \to H$ such that $D = \Delta_* U$, $\ker B = \ker U$, and Ran $U \subseteq [\text{Ran } \Delta_*] = \mathcal{D}_*$. It remains to show that U maps onto $\mathcal{D}_*$. Toward that end, note that

$$\Delta_*^2 = DD^* = \Delta_* UU^* \Delta_*,$$

i.e., $\Delta_*(I_{\mathcal{D}_*} - UU^*)\Delta_* = 0$. Writing P for the projection $I_{\mathcal{D}_*} - UU^*$, the equality $\Delta_*(I_{\mathcal{D}_*} - UU^*)\Delta_* = 0$ is equivalent to the condition $(\Delta_* P)(\Delta_* P)^* = 0$, which is true if and only if $\Delta_* P = 0$, and this latter condition is true if and only if $P = 0$ on Inn $\Delta_* = (\ker \Delta_*)^\perp$. However, since Δ_* is self-adjoint, Inn $\Delta_* = [\text{Ran } \Delta_*] = \mathcal{D}_*$, and so, $UU^* = I_{\mathcal{D}_*}$. Therefore, Ran $U = \mathcal{D}_*$, and this completes the proof.

□

Theorem 4.5. *Every* minimal *partially isometric extension of* T *(with final space* H*) is of the form*

$$W = \begin{bmatrix} T & \Delta_* U \\ 0 & 0 \end{bmatrix}$$

for a Hilbert space isomorphism $U : H^\perp \to \mathcal{D}_*$.

Proof. By the previous theorem, W is a partially isometric extension of T (with final space H) if and only if $D = \Delta_* U$ for a surjective partial isometry $U : H^\perp \to \mathcal{D}_*$. Let P denote the projection $U^*U \in B(H^\perp)$. We have

$$\begin{aligned} W(I_H \oplus P) &= \begin{bmatrix} T & \Delta_* U \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_H & 0 \\ 0 & P \end{bmatrix} = \begin{bmatrix} T & \Delta_* U P \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} T & \Delta_* U U^* U \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T & \Delta_* U \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I_H & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} T & \Delta_* U \\ 0 & 0 \end{bmatrix} = (I_H \oplus P)W. \end{aligned}$$

But since $I_H \oplus P$ is a projection, $W(I_H \oplus P) = (I_H \oplus P)W$ if and only if $H \oplus PH^\perp$ reduces W. Therefore, W is minimal if and only if $H \oplus PH^\perp = H \oplus H^\perp$. This latter condition holds if and only if $PH^\perp = H^\perp$, which is equivalent to $U^*U = P = I_{H^\perp}$. In conclusion, W is minimal if and only if $U : H^\perp \to \mathcal{D}_*$ is a Hilbert space isomorphism.

□

Finally, putting all the pieces together, we conclude that W is a minimal partially isometric extension of T (with final space H) if and only if there exists a Hilbert space isomorphism of the form $I_H \oplus U : H \oplus \mathcal{D}_* \to K$ such that $(I_H \oplus U)\mathcal{W} = W(I_H \oplus U)$. In other words,

$$\mathcal{W} = \begin{bmatrix} T & \Delta_* \\ 0 & 0 \end{bmatrix}$$

is the *unique* minimal partially isometric extension of T (with final space H), up to unitary equivalence.

Remark 4.6. *We may equivalently reformulate Theorem 4.5 as follows: If $W_1 \in B(K_1)$ and $W_2 \in B(K_2)$ are minimal partially isometric extensions of $T \in B(H)$, each with final space H, then W_1 and W_2 are unitarily equivalent. Indeed, Theorem 4.5 says that there exist Hilbert space isomorphisms $U_i : \mathcal{D}_* \to H_i^\perp$, $i = 1, 2$, such that*

$$W_i = \begin{bmatrix} T & \Delta_* U_i \\ 0 & 0 \end{bmatrix},$$

where $H_i^\perp$ denotes the subspace $K_i \ominus H$, $i = 1, 2$. Then $U := I_H \oplus U_2^* U_1$ is a Hilbert space isomorphism from K_1 onto K_2 such that $UW_1U^* = W_2$.

4.2 The General Case

Let A be a unital C^*-algebra, and let E be an A-correspondence with left-action map $\phi : A \to \mathcal{L}(E)$. Assume there exists – and fix – a unital, smooth generalized transfer operator $\tau : \mathcal{L}(E) \to A$ for ϕ. Given a Hilbert space H_σ, recall that a contractive intertwining pair associated to (E, A, H_σ) consists of a representation

$\sigma : A \to B(H_\sigma)$ and a contractive operator $\mathfrak{z} : E \otimes_\sigma H_\sigma \to H_\sigma$ such that $\mathfrak{z}$ is in the intertwining space $\mathcal{I}(\sigma^E \circ \phi, \sigma)$, i.e., such that $\mathfrak{z}\sigma^E(\phi(\cdot)) = \sigma(\cdot)\mathfrak{z}$. Our first result in this section follows from the fact that a representation of $\mathcal{L}(E)$ that restricts to $\mathcal{K}(E)$ nondegenerately is an induced representation (see Section 2.3). Note that Lemma 4.7 does not necessarily follow from Corollary 3.6, since we first restrict to $\mathcal{D}_*$; 3.6 implies that ρ is an induced representation, while 4.7 implies that the *restriction* of ρ to $\mathcal{D}_*$ is induced. However, just as in 3.6, the key to the proof of 4.7 is still the smoothness of τ.

Lemma 4.7. *Fix a Hilbert space H_σ, let $(\mathfrak{z}, \sigma)$ be a contractive intertwining pair associated to (E, A, H_σ), and let $\Delta_* \in B(H_\sigma)$ denote the defect operator for $\mathfrak{z}^*$, i.e., $\Delta_* = (I - \mathfrak{z}\mathfrak{z}^*)^{1/2}$. Additionally, let (K_ρ, ρ, V) denote the extension of σ adapted to τ, and let $\mathcal{D}_* \subseteq K_\rho$ denote smallest reducing subspace for ρ containing $[V\Delta_* H_\sigma]$.*

Then $\mathcal{D}_* = [\rho(\mathcal{L}(E))V\Delta_* H_\sigma]$, and the map $\hat{\rho} : \mathcal{L}(E) \to B(\mathcal{D}_*)$ defined by $\hat{\rho}(\cdot) = \rho(\cdot)|_{\mathcal{D}_*}$ is a representation of $\mathcal{L}(E)$ on $\mathcal{D}_*$. Further, $\hat{\rho}$ is induced from a representation of A with respect to E, so that there is a Hilbert space H_θ, a representation $\theta : A \to B(H_\theta)$, and a Hilbert space isomorphism $\mathcal{U} : E \otimes_\theta H_\theta \to \mathcal{D}_*$ such that $\mathcal{U}\theta^E(\cdot) = \hat{\rho}(\cdot)\mathcal{U}$. Still further, $(H_\theta, \theta, \mathcal{U})$ is the unique triple satisfying these properties, up to unitary equivalence.

Note that although the proof of Theorem 3.4 is not directly applicable, the same construction works here: Set $H_\theta = \widetilde{E} \otimes_{\hat{\rho}} \mathcal{D}_*$, $\theta = \hat{\rho}^{\widetilde{E}}$, and $\mathcal{U}(\xi \otimes (\eta^* \otimes x)) = \hat{\rho}(\xi \otimes \eta^*)x$. Then $\mathcal{U}$ is a Hilbert space isomorphism of $E \otimes_\theta H_\theta$ onto $\mathcal{D}_*$ such that $\mathcal{U}\theta^E(\cdot)\mathcal{U}^* = \hat{\rho}(\cdot)$. Uniqueness of $(H_\theta, \theta, \mathcal{U})$ follows from Theorem 2.18 (Rieffel's equivalence theorem).

Proof. The equivalence of the minimality condition and the equality $\mathcal{D}_* = [\rho(\mathcal{L}(E))V\Delta_* H_\sigma]$ is straightforward. The fact that $\mathcal{D}_*$ reduces ρ implies that the restriction $\hat{\rho}(\cdot) = \rho(\cdot)|_{\mathcal{D}_*}$ is a representation of $\mathcal{L}(E)$ on $\mathcal{D}_*$. The remainder of the proof follows from simple and obvious modifications to the proof of Theorem 3.4, as remarked above.

□

Define an operator $D_* : E \otimes_\theta H_\theta \to H_\sigma$ by $D_* = \Delta_* V^* \mathcal{U}$. The following result summarizes some useful intertwining properties of the operators Δ_* and D_*.

Lemma 4.8. *With the notation established in Lemma 4.7 and in the preceding remarks, we have the following intertwining relations.*

(i) $\Delta_*\sigma(a) = \sigma(a)\Delta_*$ *for all* $a \in A$*, i.e.,* $\Delta_* \in \sigma(A)' = \sigma(\tau(\mathcal{L}(E)))'$ *(the commutants are equal by the fact that* τ *is surjective).*

(ii) $D_*\theta^E(X) = \sigma(\tau(X))D_*$ *for all* $X \in \mathcal{L}(E)$*, i.e.,* $D_* \in \mathcal{I}(\theta^E, \sigma \circ \tau)$*. In particular,* $D_* \in \mathcal{I}(\theta^E \circ \phi, \sigma)$*,* $D_*^* D_* \in \theta^E(\mathcal{L}(E))'$*, and* $D_* D_*^* \in \sigma(A)'$*.*

Proof. From the fact that $\mathfrak{z} \in \mathcal{I}(\sigma^E \circ \phi, \sigma)$, we get $\mathfrak{z}^* \in \mathcal{I}(\sigma, \sigma^E \circ \phi)$. Thus, for any $a \in A$,

$$\begin{aligned}\Delta_*^2\sigma(a) &= (I - \mathfrak{z}\mathfrak{z}^*)\sigma(a) = \sigma(a) - \mathfrak{z}\mathfrak{z}^*\sigma(a) = \sigma(a) - \mathfrak{z}\sigma^E(\phi(a))\mathfrak{z}^* = \sigma(a) - \sigma(a)\mathfrak{z}\mathfrak{z}^* \\ &= \sigma(a)\Delta_*^2\end{aligned}$$

It follows that $\Delta_*^{2n}\sigma(a) = \sigma(a)\Delta_*^{2n}$ for all $n \geq 1$, and hence, $p(\Delta_*^2)\sigma(a) = \sigma(a)p(\Delta_*^2)$ for all polynomials $p(x)$. Since Δ_* is the positive square root of Δ_*^2, there is a sequence of polynomials $\{p_n(x)\}_{n=1}^{\infty}$ such that $p_n(\Delta_*^2) \to \Delta_*$. Therefore,

$$\Delta_*\sigma(a) = \lim_{n\to\infty} p_n(\Delta_*^2)\sigma(a) = \lim_{n\to\infty} \sigma(a)p_n(\Delta_*^2) = \sigma(a)\Delta_*.$$

As $a \in A$ was arbitrary, we conclude that $\Delta_* \in \sigma(A)' = \sigma(\tau(\mathcal{L}(E)))'$. (See the appendix of [17] for the full details.)

In order to show that $D_* \in \mathcal{I}(\theta^E, \sigma \circ \tau)$, we note that $VV^*\mathcal{U} = \mathcal{U}$ (since VV^* is the projection of K_ρ onto the subspace $\mathcal{D}_*$). Then, using the fact that $\mathcal{U} \in \mathcal{I}(\theta^E, \hat{\rho})$ and formula (3.1), we see that

$$\begin{aligned} D_*\theta^E(X) &= \Delta_*V^*\mathcal{U}\theta^E(X) = \Delta_*V^*\hat{\rho}(X)\mathcal{U} = \Delta_*V^*\rho(X)VV^*\mathcal{U} \\ &= \Delta_*\sigma(\tau(X))V^*\mathcal{U} = \sigma(\tau(X))\Delta_*V^*\mathcal{U} = \sigma(\tau(X))D_*, \end{aligned}$$

for all $X \in \mathcal{L}(E)$. This proves that $D_* \in \mathcal{I}(\theta^E, \sigma\circ\tau)$, and it follows that $D_*\theta^E(\phi(a)) = \sigma(\tau(\phi(a))D_* = \sigma(a)D_*$, i.e., $D_* \in \mathcal{I}(\theta^E \circ \phi, \sigma)$. Finally, by Lemma 2.16, $D_*^*D_* \in \theta^E(\mathcal{L}(E))'$ and $D_*D_*^* \in \sigma(\tau(X))' = \sigma(A)'$.

$\square$

Our objective (in this chapter and the next) is to show that there is a *minimal coisometric extension* of a given contractive intertwining pair $(\mathfrak{z}, \sigma)$, subject to the fixing of a (unital, smooth generalized) transfer operator $\tau : \mathcal{L}(E) \to A$. We show that once $(\mathfrak{z}, \sigma)$ and τ are fixed, a minimal coisometric extension adapted to τ is unique up to unitary equivalence. In order to accomplish these goals, we build an inductive system of Hilbert spaces and operators between those spaces, the inductive limit of which is the desired minimal coisometric extension. At each stage of the construction, we obtain a formula analogous to (3.1), making the role of τ apparent. The first step is to build a *minimal partially isometric extension of* $(\mathfrak{z}, \sigma)$ *adapted to* τ. We need a result and some related terminology before explaining what is meant by a partially isometric extension.

Recall the close relation, established in Section 2.4, among contractive intertwining pairs $(\mathfrak{z}, \sigma)$ associated to (E, A, H_σ), completely contractive covariant representations (T, σ) of (E, A) on H_σ, and completely contractive representations ρ of the tensor algebra $\mathcal{T}_+(E)$ on H_σ. Recall that this correspondence is given by the formulas $\sigma(a) = \rho(\phi_\infty(a))$, $a \in A$, and

$$\mathfrak{z}(\xi \otimes h) = \rho(T_\xi)h = T(\xi)h, \quad \xi \in E, \; h \in H_\sigma. \tag{4.1}$$

Starting from the point-of-view of a contractive intertwining pair $(\mathfrak{z}, \sigma)$ (respectively, a completely contractive covariant representation (T, σ)) we call the completely contractive representation ρ of $\mathcal{T}_+(E)$ in (4.1) the *integrated form* of $(\mathfrak{z}, \sigma)$ (respectively, of (T, σ)), and we write $\rho = \mathfrak{z} \times \sigma$ (respectively, $\rho = T \times \sigma$).

Proposition 4.9. *Let $(\mathfrak{z}, \sigma)$ be a contractive intertwining pair associated to (E, A, H_σ), and let $T = T_\mathfrak{z} : E \to B(H_\sigma)$ be the completely contractive bimodule map associated to $(\mathfrak{z}, \sigma)$. For $X \in B(H_\sigma)$, the following are equivalent:*

(i) X is in the commutant of the integrated form of $(\mathfrak{z}, \sigma)$, i.e., $X \in (\mathfrak{z} \times \sigma)(\mathcal{T}_+(E))'$.

(ii) $X \in T(E)'$ *and* $X \in \sigma(A)'$*, that is,* X *commutes with the collection of operators* $\{T(\xi)\}_{\xi \in E} \cup \{\sigma(a)\}_{a \in A}$.

(iii) $X\mathfrak{z} = \mathfrak{z}(I_E \otimes X)$ *and* $X \in \sigma(A)'$*, where* I_E *denotes the identity operator on* E.

Moreover, if $\mathfrak{z}$ *is coisometric, then we can replace* (*ii*) *and* (*iii*) *with following two simpler conditions, respectively:*

(ii') $X \in T(E)'$.

(iii') $X\mathfrak{z} = \mathfrak{z}(I_E \otimes X)$.

Proof. Recall that $\mathcal{T}_+(E)$ is generated by the image of the homomorphism $\phi_\infty : A \to \mathcal{L}(\mathcal{F}(E))$ and the collection of creation operators $\{T_\xi\}_{\xi \in E}$. Fix $a \in A$ and $\xi \in E$. If $X \in (\mathfrak{z} \times \sigma)(\mathcal{T}_+(E))'$, then for all $h \in H_\sigma$,

$$\sigma(a)Xh = (\mathfrak{z} \times \sigma)(\phi_\infty(a))Xh = X(\mathfrak{z} \times \sigma)(\phi_\infty(a))h = X\sigma(a)h,$$

while

$$XT(\xi)h = X(\mathfrak{z} \times \sigma)(T_\xi)h = (\mathfrak{z} \times \sigma)(T_\xi)Xh = T(\xi)Xh.$$

Therefore, $X \in \sigma(A)'$ and $X \in T(E)'$, so (i) $\Longrightarrow$ (ii). The reverse implication follows from simply rearranging these two strings of equalities.

The equivalence of the conditions $X \in T(E)'$ and $X\mathfrak{z} = \mathfrak{z}(I_E \otimes X)$ is a consequence of the following two equalities:

$$XT(\xi)h = X\mathfrak{z}(\xi \otimes h)$$

and

$$T(\xi)Xh = \mathfrak{z}(\xi \otimes Xh) = \mathfrak{z}(I_E \otimes X)(\xi \otimes h),$$

where $\xi \in E$ and $h \in H_\sigma$, alternatively, $\xi \otimes h \in E \otimes_\sigma H_\sigma$ (depending on which direction of the equivalence one is trying to prove). Thus, (ii) $\Longleftrightarrow$ (iii) and (ii') $\Longleftrightarrow$ (iii'). In particular, (i), (ii), and (iii) are equivalent.

Now, assume that $\mathfrak{z}$ is coisometric. If $X \in T(E)'$, then for $a \in A$ and $h \in H_\sigma$,

$$\begin{aligned} X\sigma(a)T(\xi)h &= XT(\phi(a)\xi)h = T(\phi(a)\xi)Xh = \sigma(a)T(\xi)Xh \\ &= \sigma(a)XT(\xi)h. \end{aligned}$$

Thus, $X\sigma(a) = \sigma(a)X$ on $T(E)H_\sigma = \mathfrak{z}(E \otimes_\sigma H_\sigma)$, which is dense in H_σ as a result of the assumption that $\mathfrak{z}$ is coisometric. It follows that $X \in \sigma(A)'$. Therefore (ii') $\Longrightarrow$ (ii), and the other direction is trivial, so (ii') and (ii) are equivalent when $\mathfrak{z}$ is coisometric.

Finally, (ii') $\Longleftrightarrow$ (iii'), as we showed above, so (iii') $\Longrightarrow$ (ii') $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) when $\mathfrak{z}$ is coisometric. Since (iii) $\Longrightarrow$ (iii') trivially, the proof is complete.

$\square$

Definition 4.10. *For a contractive intertwining pair* $(\mathfrak{z}, \sigma)$ *associated to* (E, A, H_σ)*, we say that* $X \in B(H_\sigma)$ commutes *with* $(\mathfrak{z}, \sigma)$ *if any of the equivalent conditions of Proposition 4.9 hold.*

Let H be a subspace of H_σ, and let P denote the projection of H_σ onto H. Then it is not difficult to see that P commutes with $(\mathfrak{z}, \sigma)$, i.e., $P\mathfrak{z} = \mathfrak{z}(I_E \otimes P)$ and

$P \in \sigma(A)'$, if and only if $\mathfrak{z}(E \otimes_{\sigma|_H} H) \subseteq H$, $\mathfrak{z}^* H \subseteq E \otimes_{\sigma|_H} H$, and $\sigma(a)H \subseteq H$. This motivates the following definitions.

Definition 4.11. *Let H_σ be a Hilbert space, and let $(\mathfrak{z}, \sigma)$ be a contractive intertwining pair associated to (E, A, H_σ). We say that a subspace $H \subseteq H_\sigma$ is an* invariant subspace *for $(\mathfrak{z}, \sigma)$ if $\mathfrak{z}(E \otimes_{\sigma|_H} H) \subseteq H$ and $\sigma(a)H \in H$ for all $a \in A$. Additionally, $H \subseteq H_\sigma$ is a* reducing subspace *for $(\mathfrak{z}, \sigma)$ in case H is an invariant subspace for $(\mathfrak{z}, \sigma)$ and $\mathfrak{z}^* H \subseteq E \otimes_{\sigma|_H} H$.*

Straightforward calculations show that H is an invariant subspace for $(\mathfrak{z}, \sigma)$ if and only if H is invariant under each $T(\xi)$ and each $\sigma(a)$, if and only if H is invariant under $\rho(\mathcal{T}_+(E))$. What it means for H to be a reducing subspace for $(\mathfrak{z}, \sigma)$ is more difficult to formulate in terms of (T, σ) and ρ; this is one of the primary reasons we focus on contractive intertwining pairs in our analysis rather than on completely contractive covariant representations or completely contractive representations of the tensor algebra. In any case, we now have everything we need to define a partially isometric extension, which is the primary object of interest in the current chapter.

Definition 4.12 (Partially Isometric Extensions)**.**

(i) A partially isometric extension *of $(\mathfrak{z}, \sigma)$ acting on a Hilbert space $H_\pi \supseteq H_\sigma$ is a partially isometric intertwining pair $(\mathfrak{w}, \pi)$ associated to (E, A, H_π), that is, a representation $\pi : A \to B(H_\pi)$ and a partially isometric operator $\mathfrak{w} : E \otimes_\pi H_\pi \to H_\pi$ in the space $\mathcal{I}(\pi^E \circ \phi, \pi)$, such that H_σ is an invariant subspace for $(\mathfrak{w}, \pi)$ with $\mathfrak{w}|_{E \otimes_\sigma H_\sigma} = \mathfrak{z}$ and $\pi(\cdot)|_{H_\sigma} = \sigma(\cdot)$.*

(ii) A partially isometric extension $(\mathfrak{w}, \pi)$ *of* $(\mathfrak{z}, \sigma)$ *is* minimal *if the smallest reducing subspace* $H \subseteq H_\pi$ *for* $(\mathfrak{w}, \pi)$ *containing* H_σ *is* $H = H_\pi$.

(iii) Let $\widehat{H}$ *denote the subspace* $H_\pi \ominus H_\sigma$*, and write* $\hat{\pi}$ *for the restriction of* π *to* $\widehat{H}$*, so that* $H_\pi = H_\sigma \oplus \widehat{H}$ *and* $\pi = \sigma \oplus \hat{\pi}$*. Further, let* $D : E \otimes_{\hat{\pi}} \widehat{H} \to H_\sigma$ *denote the operator given by*

$$D = P_{H_\sigma} \mathfrak{w} \mid_{E \otimes_{\hat{\pi}} \widehat{H}} .$$

We say that $(\mathfrak{w}, \pi)$ *is* adapted to τ *if* $D \in \mathcal{I}(\hat{\pi}^E, \sigma \circ \tau)$.

Note that when $\mathfrak{w}$ is viewed as an operator from $(E \otimes_\sigma H_\sigma) \oplus (E \otimes_{\hat{\pi}} \widehat{H})$ to $H_\sigma \oplus \widehat{H}$, it has a 2×2 matrix characterization, and D is the $(1, 2)$ entry of this matrix. It may seem somewhat arbitrary to require that $D \in \mathcal{I}(\hat{\pi}^E, \sigma \circ \tau)$, but as we will see, this is exactly the condition needed to conclude that a minimal partially isometric extension adapted to τ is unique up to unitary equivalence. For now, we note that $D \in \mathcal{I}(\hat{\pi}^E, \sigma \circ \tau)$ is a stronger condition than just requiring $D \in \mathcal{I}(\hat{\pi}^E \circ \phi, \sigma)$. In fact, the latter is automatic if $(\mathfrak{w}, \pi)$ is a partially isometric extension of $(\mathfrak{z}, \sigma)$, by the fact that $\mathfrak{w} \in \mathcal{I}(\pi^E \circ \phi, \pi)$. Also, if it is the case that $D \in \mathcal{I}(\hat{\pi}^E, \sigma \circ \tau)$, then $D\hat{\pi}^E(\phi(a)) = \sigma(\tau(\phi(a)))D = \sigma(a)D$ for all $a \in A$, i.e., $D \in \mathcal{I}(\hat{\pi}^E \circ \phi, \sigma)$; in other words, $\mathcal{I}(\hat{\pi}^E, \sigma \circ \tau) \subseteq \mathcal{I}(\hat{\pi}^E \circ \phi, \sigma)$.

In order to simplify the statements and proofs of results in subsequent chapters, we modify the notation and definitions established in Lemma 4.7 and in the remark preceding Lemma 4.8, as follows: Let $(K_{\rho_1}, \rho_1, V_1)$ denote the extension of σ adapted to τ, and set $\hat{\rho}_1 = \hat{\rho}$, $H_{\theta_1} = H_\theta$, $\theta_1 = \theta$, $\mathcal{U}_1 = \mathcal{U}$, and $D_{1*} = D_* = \Delta_* V_1^* \mathcal{U}_1$.

Theorem 4.13 (Existence of Minimal Partially Isometric Extensions Adapted to τ)**.** *Let H_{π_1} be the Hilbert space given by the direct sum $H_\sigma \oplus H_{\theta_1}$, and let π_1 be the representation of A on H_{π_1} defined by $\pi_1 = \sigma \oplus \theta_1$. Under the identification $E \otimes_{\pi_1} H_{\pi_1} = (E \otimes_\sigma H_\sigma) \oplus (E \otimes_{\theta_1} H_{\theta_1})$, define an operator $\mathfrak{w}_1 : E \otimes_{\pi_1} H_{\pi_1} \to H_{\pi_1}$ matricially by*

$$\mathfrak{w}_1 = \begin{bmatrix} \mathfrak{z} & D_{1*} \\ 0 & 0 \end{bmatrix}.$$

Then $(\mathfrak{w}_1, \pi_1)$ is a minimal partially isometric extension of $(\mathfrak{z}, \sigma)$ adapted to τ.

Proof. To begin, we have

$$\mathfrak{z}\mathfrak{z}^* + D_{1*}D_{1*}^* = \mathfrak{z}\mathfrak{z}^* + (\Delta_* V_1^* \mathcal{U}_1)(\mathcal{U}_1^* V_1 \Delta_*) = \mathfrak{z}\mathfrak{z}^* + \Delta_* V_1^* V_1 \Delta_*$$
$$= \mathfrak{z}\mathfrak{z}^* + \Delta_*^2 = \mathfrak{z}\mathfrak{z}^* + (I_{H_\sigma} - \mathfrak{z}\mathfrak{z}^*) = I_{H_\sigma}.$$

Thus,

$$\mathfrak{w}_1\mathfrak{w}_1^* = \begin{bmatrix} \mathfrak{z} & D_{1*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathfrak{z}^* & 0 \\ D_{1*}^* & 0 \end{bmatrix} = \begin{bmatrix} \mathfrak{z}\mathfrak{z}^* + D_{1*}D_{1*}^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_{H_\sigma} & 0 \\ 0 & 0 \end{bmatrix}.$$

This shows that $\mathfrak{w}_1$ is a partial isometry with final space H_σ.

The intertwining condition $\mathfrak{w}_1 \in \mathcal{I}(\pi_1^E \circ \phi, \pi_1)$ follows from another matrix computation. Recall that $D_{1*} \in \mathcal{I}(\theta_1^E \circ \phi, \sigma)$, as we showed in Lemma 4.8. Hence, for any $a \in A$,

$$\begin{aligned}
\mathfrak{w}_1\, \pi_1^E(\phi(a)) &= \begin{bmatrix} \mathfrak{z} & D_{1*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma^E(\phi(a)) & 0 \\ 0 & \theta_1^E(\phi(a)) \end{bmatrix} \\
&= \begin{bmatrix} \mathfrak{z}\sigma^E(\phi(a)) & D_{1*}\theta_1^E(\phi(a)) \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sigma(a)\mathfrak{z} & \sigma(a)D_{1*} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma(a) & 0 \\ 0 & \theta_1(a) \end{bmatrix} \begin{bmatrix} \mathfrak{z} & D_{1*} \\ 0 & 0 \end{bmatrix} \\
&= \pi_1(a)\mathfrak{w}_1.
\end{aligned}$$

It is clear from the definitions of $\mathfrak{w}_1$ and π_1 that H_σ is an invariant subspace for $(\mathfrak{w}_1, \pi_1)$. Make the natural identification $H_\sigma = H_\sigma \oplus \{0\} \subseteq H_{\pi_1}$. Then for $a \in A$ and $h \in H_\sigma$, we have

$$\pi_1(a)h = \begin{bmatrix} \sigma(a) & 0 \\ 0 & \theta_1(a) \end{bmatrix} \begin{bmatrix} h \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma(a)h \\ 0 \end{bmatrix} = \sigma(a)h.$$

Therefore, $\pi_1(\cdot)|_{H_\sigma} = \sigma(\cdot)$. To see that $\mathfrak{w}_1|_{E\otimes_\sigma H_\sigma} = \mathfrak{z}$, fix $\xi \otimes h \in E \otimes_\sigma H_\sigma$, and simply observe that

$$\mathfrak{w}_1(\xi \otimes h) = \begin{bmatrix} \mathfrak{z} & D_{1*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \otimes h \\ 0 \end{bmatrix} = \begin{bmatrix} \mathfrak{z}(\xi \otimes h) \\ 0 \end{bmatrix} = \mathfrak{z}(\xi \otimes h).$$

This shows that $(\mathfrak{w}_1, \pi_1)$ is a partially isometric extension of $(\mathfrak{z}, \sigma)$. Notice that $\widehat{H} = H_{\theta_1}$, $\hat{\pi} = \theta_1$, and $D = D_{1*}$ (comparing the notation from Definition 4.12 and the notation in the statement of the lemma). By Lemma 4.8, $D_{1*} \in \mathcal{I}(\theta_1^E, \sigma \circ \tau)$. Therefore, $(\mathfrak{w}_1, \pi_1)$ is a partially isometric extension of $(\mathfrak{z}, \sigma)$ adapted to τ.

It only remains to show that $(\mathfrak{w}_1, \pi_1)$ is minimal in the sense of part (ii) of Definition 4.12. For this, let $H \subseteq H_{\pi_1}$ be the smallest reducing subspace for $(\mathfrak{w}_1, \pi_1)$ that contains H_σ. Write H' for the complement of H_σ in H, i.e., $H' = H \ominus H_\sigma$. Note that $H = H_\sigma \oplus H'$ and that $H' \subseteq H_{\pi_1} \ominus H_\sigma = H_{\theta_1}$. Let Q denote the projection of H_{θ_1} onto H'; then $I_{H_\sigma} \oplus Q$ is the projection of H_{π_1} onto H. By the remarks prior to Definition 4.11 and the fact that H reduces $(\mathfrak{w}_1, \pi_1)$, we have $\mathfrak{w}_1(I_E \otimes (I_{H_\sigma} \oplus Q)) = (I_{H_\sigma} \oplus Q)\mathfrak{w}_1$ and $(I_{H_\sigma} \oplus Q) \in \pi_1(A)'$. Note that $I_E \otimes (I_{H_\sigma} \oplus Q) = I_{E \otimes_\sigma H_\sigma} \oplus (I_E \otimes Q)$. Moreover,

$$\mathfrak{w}_1(I_{E\otimes_\sigma H_\sigma} \oplus (I_E \otimes Q)) = \begin{bmatrix} \mathfrak{z} & D_{1*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{E\otimes_\sigma H_\sigma} & 0 \\ 0 & I_E \otimes Q \end{bmatrix}$$
$$= \begin{bmatrix} \mathfrak{z} & D_{1*}(I_E \otimes Q) \\ 0 & 0 \end{bmatrix},$$

while

$$(I_{H_\sigma} \oplus Q)\mathfrak{w}_1 = \begin{bmatrix} I_{H_\sigma} & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} \mathfrak{z} & D_{1*} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathfrak{z} & D_{1*} \\ 0 & 0 \end{bmatrix}.$$

It follows that $\mathfrak{w}_1(I_E \otimes (I_{H_\sigma} \oplus Q)) = (I_{H_\sigma} \oplus Q)\mathfrak{w}_1$ if and only if $D_{1*}(I_E \otimes Q) = D_{1*}$. Also, $(I_{H_\sigma} \oplus Q) \in \pi_1(A)'$ if and only if $Q \in \theta_1(A)'$ if and only if $I_E \otimes Q \in \theta_1^E(\mathcal{L}(E))'$ (see Theorem 2.18).

Write P for the projection $\mathcal{U}_1(I_E \otimes Q)\mathcal{U}_1^* \in B(\mathcal{D}_*)$, and consider the subspace $P\mathcal{D}_* \subseteq \mathcal{D}_*$. We claim that $P\mathcal{D}_*$ reduces ρ_1 and contains $[V_1\Delta_* H_\sigma]$. For the former, recall that $\mathcal{U}_1\hat{\rho}_1(\cdot) = \theta_1^E(\cdot)\mathcal{U}_1$. Then since $I_E \otimes Q \in \theta_1^E(\mathcal{L}(E))'$, for any $X \in \mathcal{L}(E)$ we have

$$\hat{\rho}_1(X)P = \hat{\rho}_1(X)\mathcal{U}_1(I_E \otimes Q)\mathcal{U}_1^* = \mathcal{U}_1\theta_1^E(X)(I_E \otimes Q)\mathcal{U}_1^*$$
$$= \mathcal{U}_1(I_E \otimes Q)\theta_1^E(X)\mathcal{U}_1^* = \mathcal{U}_1(I_E \otimes Q)\mathcal{U}_1^*\hat{\rho}_1(X)$$
$$= P\hat{\rho}_1(X).$$

Since $P\mathcal{D}_*$ and $\hat{\rho}_1(\mathcal{L}(E))\mathcal{D}_*$ are contained in $\mathcal{D}_*$, it follows that $\rho_1(X)Px = \hat{\rho}_1(X)Px = P\hat{\rho}_1(X)x \in P\mathcal{D}_*$ for all $x \in \mathcal{D}_*$. That is, $P\mathcal{D}_*$ reduces ρ_1. Additionally, since $D_{1*}^* = \mathcal{U}_1^*V_1\Delta_*$ and $D_{1*}(I_E \otimes Q) = D_{1*}$,

$$PV_1\Delta_* = \mathcal{U}_1(I_E \otimes Q)\mathcal{U}_1^*V_1\Delta_* = \mathcal{U}_1(I_E \otimes Q)D_{1*}^*$$
$$= \mathcal{U}_1 D_{1*}^* = \mathcal{U}_1\mathcal{U}_1^*V_1\Delta_* = V_1\Delta_*,$$

from which it follows that $[V_1\Delta_*H_\sigma] = P[V_1\Delta_*H_\sigma] \subseteq P\mathcal{D}_*$.

To summarize, we have shows that $P\mathcal{D}_* \subseteq \mathcal{D}_*$ is a reducing subspace for ρ_1 containing $[V_1\Delta_*H_\sigma]$. By the minimality of $\mathcal{D}_*$, $P = I_{\mathcal{D}_*}$ and so, $I_E \otimes Q = \mathcal{U}_1^*\mathcal{U}_1 = I_{E\otimes_\sigma H_\sigma}$. But this is only possible if $Q = I_{H_{\theta_1}}$, and we conclude that $H' = H_{\theta_1}$. Finally,

$$H = H_\sigma \oplus H' = H_\sigma \oplus H_{\theta_1} = H_{\pi_1}.$$

Therefore, H_{π_1} is the smallest reducing subspace for $(\mathfrak{w}_1, \pi_1)$ containing H_σ, and so, $(\mathfrak{w}_1, \pi_1)$ is a minimal partially isometric extension of $(\mathfrak{z}, \sigma)$ adapted to τ.

□

Theorem 4.14 (Uniqueness of Minimal Partially Isometric Extensions Adapted to τ). *The minimal partially isometric extension $(\mathfrak{w}_1, \pi_1)$ of $(\mathfrak{z}, \sigma)$ adapted to τ, defined in Theorem 4.13, is unique up to unitary equivalence. That is, if $(\mathfrak{w}, \pi)$, acting on a*

Hilbert space H_π, is another minimal partially isometric extension of $(\mathfrak{z},\sigma)$ adapted to τ, then there is a Hilbert space isomorphism $\Phi : H_\pi \to H_{\pi_1}$ such that $\Phi\pi(\cdot) = \pi_1(\cdot)\Phi$ and $\Phi\mathfrak{w} = \mathfrak{w}_1(I_E \otimes \Phi)$.

Proof. Assume that $(\mathfrak{w},\pi)$ is another minimal partially isometric extension of $(\mathfrak{z},\sigma)$ adapted to τ, acting on a Hilbert space H_π. That is, π is a nondegenerate representation of A on H_π, and $\mathfrak{w} : E \otimes_\pi H_\pi \to H_\pi$ is a partially isometric operator in the intertwiner space $\mathcal{I}(\pi^E \circ \phi, \pi)$. Write H_θ for the subspace $H_\pi \ominus H_\sigma$, so that $H_\pi = H_\sigma \oplus H_\theta$, and let $\theta : A \to H_\theta$ be the representation of A obtained from restricting π to H_θ: $\theta(\cdot) = \pi(\cdot)|_{H_\theta}$. In this way, we may view $\mathfrak{w}$ as a map from the direct sum $(E \otimes_\sigma H_\sigma) \oplus (E \otimes_\theta H_\theta)$ to $H_\sigma \oplus H_\theta$, and so, $\mathfrak{w}$ can be expressed as a 2×2 matrix as follows:

$$\mathfrak{w} = \begin{bmatrix} B & D \\ C & F \end{bmatrix},$$

for operators $B : E \otimes_\sigma H_\sigma \to H_\sigma$, $C : E \otimes_\sigma H_\sigma \to H_\theta$, $C \in E \otimes_\theta H_\theta \to H_\theta$, and $F : E \otimes_\theta H_\theta \to H_\sigma$. In fact, straightforward modifications of the proofs in Section 4.1 allow us to conclude that $B = \mathfrak{z}$, $C = 0$, $F = 0$, and $D = D_{1*}U$ for a Hilbert space isomorphism $U : E \otimes_\theta H_\theta \to E \otimes_{\theta_1} H_{\theta_1}$. That is,

$$\mathfrak{w} = \begin{bmatrix} \mathfrak{z} & D_{1*}U \\ 0 & 0 \end{bmatrix}$$

Moreover, $D_{1*}U = D \in \mathcal{I}(\theta^E, \sigma \circ \tau)$ by the assumption that $(\mathfrak{w},\pi)$ is adapted to τ. Thus,

$$D_{1*}U\theta^E(\cdot) = D\theta^E(\cdot) = \sigma(\tau(\cdot))D = \sigma(\tau(\cdot))D_{1*}U = D_{1*}\theta_1^E(\cdot)U.$$

Therefore, $U\theta^E(\cdot) = \theta_1^E(\cdot)U$ on Inn D_{1*}. But Inn $D_{1*} = E \otimes_{\theta_1} H_{\theta_1}$ by the fact that $D_{1*} = \Delta_* V_1 \mathcal{U}_1$ (and $\mathcal{U}_1 : E \otimes_{\theta_1} H_{\theta_1} \to \mathcal{D}_*$ is a Hilbert space isomorphism). By Theorem 2.18 (Rieffel's equivalence theorem), there exists a Hilbert space isomorphism $U_0 : H_\theta \to H_{\theta_1}$ such that $U = I_E \otimes U_0$ and such that $U_0\theta(\cdot) = \theta_1(\cdot)U_0$. Further, we see that

$$\begin{aligned}
\mathfrak{w}_1(I_E \otimes (I_{H_\sigma} \oplus U_0)) &= \mathfrak{w}_1(I_{E\otimes_\sigma H_\sigma} \oplus (I_E \otimes U_0)) = \mathfrak{w}_1(I_{E\otimes_\sigma H_\sigma} \oplus U) \\
&= \begin{bmatrix} \mathfrak{z} & D_{1*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{E\otimes_\sigma H_\sigma} & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} \mathfrak{z} & D_{1*}U \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} I_{H_\sigma} & 0 \\ 0 & U_0 \end{bmatrix} \begin{bmatrix} \mathfrak{z} & D_{1*}U \\ 0 & 0 \end{bmatrix} = (I_{H_\sigma} \oplus U_0)\mathfrak{w}.
\end{aligned}$$

Therefore, the map $\Phi = I_{H_\sigma} \oplus U_0$ is a Hilbert space isomorphism of $H_\sigma \oplus H_\theta = H_\pi$ onto $H_\sigma \oplus H_{\theta_1} = H_{\pi_1}$ such that $\Phi\pi(\cdot) = \pi_1(\cdot)\Phi$ and $\Phi\mathfrak{w} = \mathfrak{w}_1(I_E \otimes \Phi)$. That is, Φ implements an equivalence between $(\mathfrak{w}, \pi)$ and $(\mathfrak{w}_1, \pi_1)$, and this completes the proof.

$\square$

CHAPTER 5

COISOMETRIC EXTENSIONS

5.1 An Inductive Limit Approach

As usual, let E denote a C^*-correspondence over a unital C^*-algebra A, let $\phi : A \to \mathcal{L}(E)$ be the homomorphism that gives the left-action of A on E, and assume there exists a nonzero, smooth, unital generalized transfer operator $\tau : \mathcal{L}(E) \to A$ for ϕ. In this chapter we build a *coisometric extension* for a given contractive intertwining pair $(\mathfrak{z}, \sigma)$. As in the last chapter, the extension we construct is unique subject to a minimality condition and to the construction respecting τ in a sense that is made precise in the following definition.

Definition 5.1 (Coisometric Extension)**.** *Let H_σ be a Hilbert space, and let $(\mathfrak{z}, \sigma)$ be a contractive intertwining pair associated to (E, A, H_σ).*

(i) A coisometric extension *of $(\mathfrak{z}, \sigma)$ acting on a Hilbert space $H_\pi \supseteq H_\sigma$ is a coisometric intertwining pair $(\mathfrak{w}, \pi)$ associated to (E, A, H_π) (i.e., a representation $\pi : A \to B(H_\pi)$ and a coisometry $\mathfrak{w} : E \otimes_\pi H_\pi \to H_\pi$ in the space $\mathcal{I}(\pi^E \circ \phi, \pi)$), such that H_σ is an invariant subspace for $(\mathfrak{w}, \pi)$ with $\mathfrak{w}|_{E \otimes_\sigma H_\sigma} = \mathfrak{z}$ and $\pi(\cdot)|_{H_\sigma} = \sigma(\cdot)$.*

(ii) A coisometric extension $(\mathfrak{w}, \pi)$ of $(\mathfrak{z}, \sigma)$ is minimal *if the smallest reducing subspace $H \subseteq H_\pi$ for $(\mathfrak{w}, \pi)$ containing H_σ is $H = H_\pi$*

(iii) Let $\widehat{H}$ denote the subspace $H_\pi \ominus H_\sigma$, and write $\hat{\pi}$ for the restriction of π to $\widehat{H}$,

so that $H_\pi = H_\sigma \oplus \widehat{H}$ *and* $\pi = \sigma \oplus \hat{\pi}$*. Further, let* $D : E \otimes_{\hat{\pi}} \widehat{H} \to H_\sigma$ *denote the operator given by*

$$D = P_{H_\sigma} \mathfrak{w} \mid_{E \otimes_{\hat{\pi}} \widehat{H}},$$

and let $S : E \otimes_{\hat{\pi}} \widehat{H} \to \widehat{H}$ *denote the operator*

$$S = P_{H_{\widehat{H}}} \mathfrak{w} \mid_{E \otimes_{\hat{\pi}} \widehat{H}} .$$

We say that $(\mathfrak{w}, \pi)$ *is* adapted to τ *if* $D \in \mathcal{I}(\hat{\pi}^E, \sigma \circ \tau)$ *and* $S \in \mathcal{I}(\hat{\pi}^E, \hat{\pi} \circ \tau)$*.*

The next two results show that minimal coisometric extensions adapted to τ exist and are unique. Theorems 4.13 and 4.14 comprise the first step in an inductive process, the limit of which is the desired coisometric extension.

Theorem 5.2 (Existence of Minimal Coisometric Extensions Adapted to τ)**.** *Fix a Hilbert space* H_σ*, and let* $(\mathfrak{z}, \sigma)$ *be a contractive intertwining pair associated to* (E, A, H_σ)*. Then there exists a* minimal coisometric extension *of* $(\mathfrak{z}, \sigma)$ *adapted to* τ*.*

Proof. It will be convenient to write $\mathfrak{w}_0 = \mathfrak{z}$, $\pi_0 = \sigma$, and $H_{\pi_0} = H_\sigma$. Let $(\mathfrak{w}_1, \pi_1)$ denote the (unique) minimal partially isometric extension of $(\mathfrak{w}_0, \pi_0)$ adapted to τ, and let $H_{\pi_1} \supseteq H_{\pi_0}$ denote the Hilbert space on which $(\mathfrak{w}_1, \pi_1)$ acts, i.e., $\mathfrak{w}_1 \in B(E \otimes_{\pi_1} H_{\pi_1}, H_{\pi_1})$ and $\pi_1 : A \to B(H_{\pi_1})$. The existence and uniqueness of $(\mathfrak{w}_1, \pi_1)$ is guaranteed by Theorems 4.13 and 4.14.

Now we simply iterate this process. To be precise, for each $n \geq 0$, we denote by $(\mathfrak{w}_{n+1}, \pi_{n+1})$ the (unique) minimal partially isometric extension of $(\mathfrak{w}_n, \pi_n)$ adapted to τ, acting on $H_{\pi_{n+1}} \supseteq H_{\pi_n}$. This process yields two nested sequences of Hilbert spaces $\{H_{\pi_n}\}_{n\geq 0}$ and $\{E \otimes_{\pi_n} H_{\pi_n}\}_{n\geq 0}$, as well as a sequence of partially isometric operators $\{\mathfrak{w}_n : E \otimes_{\pi_n} H_{\pi_n} \to H_{\pi_n}\}_{n\geq 0}$ and a sequence of representations $\{\pi_n : A \to B(H_{\pi_n})\}_{n\geq 0}$.

For $0 \leq m \leq n < \infty$, denote by $\iota_{n,m}$ the inclusion map of H_{π_m} into H_{π_n} ($\iota_{n,n}$ is the identity on H_{π_n} for all $n \geq 0$). Using this notation, to say that $(\mathfrak{w}_{n+1}, \pi_{n+1})$ is a partially isometric extension of $(\mathfrak{w}_n, \pi_n)$ is equivalent to the conditions $\mathfrak{w}_{n+1}(I_E \otimes \iota_{n+1,n}) = \iota_{n+1,n}\mathfrak{w}_n$ and $\pi_{n+1}(\cdot)\iota_{n+1,n} = \iota_{n+1,n}\pi_n(\cdot)$ for all $n \geq 0$. In fact, note that $\iota_{n,m} \circ \iota_{m,k} = \iota_{n,k}$ for all $0 \leq k \leq m \leq n < \infty$, since $H_{\pi_k} \subseteq H_{\pi_m} \subseteq H_{\pi_n}$. It follows that $\pi_n(\cdot)\iota_{n,m} = \iota_{n,m}\pi_m(\cdot)$ and $\mathfrak{w}_n(I_E \otimes \iota_{n,m}) = \iota_{n,m}\mathfrak{w}_m$ for $0 \leq m \leq n < \infty$. That is, $(\mathfrak{w}_n, \pi_n)$ is a partially isometric extension of $(\mathfrak{w}_m, \pi_m)$ (with final space $H_{\pi_{n-1}}$) whenever $0 \leq m < n < \infty$.

In addition to satisfying $\iota_{n,m} \circ \iota_{m,k} = \iota_{n,k}$ for $0 \leq k \leq m \leq n < \infty$, each of the maps $\iota_{n,m}$ is an isometry. Therefore, the collection $\{\{H_{\pi_n}\}_{n=0}^{\infty}, \{\iota_{n,m}\}_{m\leq n}\}$ is a directed system of Hilbert spaces and isometries between those spaces, so there is a unique Hilbert space $H_{\pi_\infty} = \varinjlim H_{\pi_n}$ and a family of isometries $\{\iota_{\infty,n} : H_{\pi_n} \to H_{\pi_\infty}\}_{n=0}^{\infty}$ such that $H_{\pi_\infty} = \bigvee_n \iota_{\infty,n}H_{\pi_n}$ and such that

$$\iota_{\infty,n} \circ \iota_{n,m} = \iota_{\infty,m}, \quad 0 \leq m \leq n < \infty,$$

Additionally, since the collection of representations $\{\pi_n : A \to B(H_{\pi_n})\}_{n\geq 0}$ satisfies $\pi_n(\cdot)\iota_{n,m} = \iota_{n,m}\pi_m(\cdot)$ for $0 \leq m \leq n < \infty$, there is a unique representation $\pi_\infty : A \to B(H_{\pi_\infty})$ such that

$$\pi_\infty(\cdot)\iota_{\infty,n} = \iota_{\infty,n}\pi_n(\cdot), \quad n \geq 0.$$

We have another inductive system of Hilbert spaces and isometries between those spaces, namely, $\{\{E \otimes_{\pi_n} H_{\pi_n}\}_{n=0}^{\infty}, \{I_E \otimes \iota_{n,m}\}_{m \leq n}\}$. Moreover, by the uniqueness of the inductive limit H_{π_∞}, we have $\varinjlim(E \otimes_{\pi_n} H_{\pi_n}) = E \otimes_{\pi_\infty} H_{\pi_\infty}$ and $\{I_E \otimes \iota_{\infty,n} : H_{\pi_n} \to H_{\pi_\infty}\}_{n=0}^{\infty}$ is a family of isometries satisfying:

$$(I_E \otimes \iota_{\infty,n}) \circ (I_E \otimes \iota_{n,m}) = I_E \otimes \iota_{\infty,m}, \quad 0 \leq m \leq n < \infty.$$

Recall that $\mathfrak{w}_n : E \otimes_{\pi_n} H_{\pi_n} \to H_{\pi_n}$ is a partial isometry with range $H_{\pi_{n-1}}$, $n \in \mathbb{N}$, and that $\mathfrak{w}_n(I_E \otimes \iota_{n,m}) = \iota_{n,m}\mathfrak{w}_m$ for $0 \leq m \leq n < \infty$. Therefore, there is a partial isometric operator $\mathfrak{w}_\infty : E \otimes_{\pi_\infty} H_{\pi_\infty} \to H_{\pi_\infty}$ – the inductive limit of the operators $\{\mathfrak{w}_n\}_{n \geq 0}$ – determined by the following family of equalities:

$$\mathfrak{w}_\infty(I_E \otimes \iota_{\infty,n}) = \iota_{\infty,n}\mathfrak{w}_n, \quad n \geq 0.$$

We claim that $\mathfrak{w}_\infty$ is coisometric. Since $\operatorname{Ran} \mathfrak{w}_n = H_{\pi_{n-1}}$ for $n \geq 1$, it follows that $\mathfrak{w}_\infty$ maps onto $\bigcup_n H_{\pi_n}$. But $\bigcup_n H_{\pi_n}$ is dense in H_{π_∞} and so, $\mathfrak{w}_\infty : E \otimes_{\pi_\infty} H_{\pi_\infty} \to H_{\pi_\infty}$ is surjective. Therefore, $\mathfrak{w}_\infty \mathfrak{w}_\infty^* = I_{H_{\pi_\infty}}$, that is, $\mathfrak{w}_\infty$ is a coisometry.

Moreover, for any $a \in A$ and any elementary tensor $\xi \otimes \iota_{\infty,n} h \in E \otimes_{\pi_\infty} H_{\pi_\infty}$, we have

$$\begin{aligned}\mathfrak{w}_\infty \pi_\infty^E(\phi(a))(\xi \otimes \iota_{\infty,n} h) &= \mathfrak{w}_\infty((\phi(a)\xi) \otimes \iota_{\infty,n} h) \\ &= \mathfrak{w}_\infty(I_E \otimes \iota_{\infty,n})((\phi(a)\xi) \otimes h) \\ &= \iota_{\infty,n}\mathfrak{w}_n((\phi(a)\xi) \otimes h) \\ &= \iota_{\infty,n}\mathfrak{w}_n \pi_n^E(\phi(a))(\xi \otimes h) \\ &= \iota_{\infty,n}\pi_n(a)\mathfrak{w}_n(\xi \otimes h) \\ &= \pi_\infty(a)\iota_{\infty,n}\mathfrak{w}_n(\xi \otimes h) \\ &= \pi_\infty(a)\mathfrak{w}_\infty(I_E \otimes \iota_{\infty,n})(\xi \otimes h) \\ &= \pi_\infty(a)\mathfrak{w}_\infty(\xi \otimes \iota_{\infty,n} h).\end{aligned}$$

It follows that $\mathfrak{w}_\infty \in \mathcal{I}(\pi_\infty^E \circ \phi, \pi_\infty)$, and so, $(\mathfrak{w}_\infty, \pi_\infty)$ is a coisometric intertwining pair. Furthermore, $(\mathfrak{w}_\infty, \pi_\infty)$ extends $(\mathfrak{w}_0, \pi_0) = (\mathfrak{z}, \sigma)$ by the manner in which $\mathfrak{w}_\infty$ and π_∞ were defined. To be specific, the equality $\pi_\infty(\cdot)\iota_{\infty,0} = \iota_{\infty,0}\sigma(\cdot)$ means that

$$\pi_\infty(\cdot)|_{H_\sigma} = \sigma(\cdot),$$

and the equality $\mathfrak{w}_\infty(I_E \otimes \iota_{\infty,0}) = \iota_{\infty,0}\mathfrak{z}$ is equivalent to

$$\mathfrak{w}_\infty|_{E\otimes_\sigma H_\sigma} = \mathfrak{z}.$$

In order to show that H_{π_∞} is the minimal reducing subspace for $(\mathfrak{w}_\infty, \pi_\infty)$ containing H_σ, we assume that H is a proper subspace of H_{π_∞} that reduces $(\mathfrak{w}_\infty, \pi_\infty)$ and contains H_σ. Then since $H_{\pi_\infty} = \bigvee_{n=0}^\infty H_{\pi_n}$, there must exist some minimal integer n such that $\iota_{\infty,n} H_{\pi_n} \nsubseteq H$. But then $H \cap H_{\pi_n}$ is a reducing subspace for

$(\mathfrak{w}_n, \pi_n)$ such that $H_{\pi_{n-1}} \subsetneq H \cap H_{\pi_n} \subsetneq H_{\pi_n}$, contradicting the minimality of H_{π_n}. Therefore, $H = H_{\pi_n}$ is the minimal reducing subspace for $(\mathfrak{w}_\infty, \pi_\infty)$ containing H_σ, that is, $(\mathfrak{w}_\infty, \pi_\infty)$ is a minimal coisometric extension of $(\mathfrak{z}, \sigma)$. (Incidentally, this also shows that $(\mathfrak{w}_\infty, \pi_\infty)$ is a minimal coisometric extension of each $(\mathfrak{w}_n, \pi_n)$.)

To see that $(\mathfrak{w}_\infty, \pi_\infty)$ is adapted to τ, set $\widehat{H}_n = H_{\pi_n} \ominus \iota_{n,0} H_{\pi_0}$ and define $\hat{\pi}_n : A \to B(\widehat{H}_n)$ by $\hat{\pi}_n(\cdot) = \pi_n(\cdot)|_{\widehat{H}_n}$, $n \geq 1$. Since $(\mathfrak{w}_n, \pi_n)$ is adapted to τ when viewed as a partial isometric extension of $(\mathfrak{z}, \sigma)$, if we define a sequence of operators $\{D_n : E \otimes_{\hat{\pi}_n} \widehat{H}_n \to H_\sigma\}_{n \geq 1}$ by $D_n = P_{H_\sigma} \mathfrak{w}_n|_{E \otimes_{\hat{\pi}_n} \widehat{H}_n}$, then $D_n \in \mathcal{I}(\hat{\pi}_n^E, \sigma \circ \tau)$, $n \geq 1$. Similarly, if we define a sequence of operators $S_n : E \otimes_{\hat{\pi}_n} \widehat{H}_n \to \widehat{H}_n$ by $S_n = P_{\widehat{H}_n} \mathfrak{w}_n|_{E \otimes_{\hat{\pi}_n} \widehat{H}_n}$, then $S_n \in \mathcal{I}(\hat{\pi}_n^E, \hat{\pi}_n \circ \tau)$, $n \geq 1$. By identifying $E \otimes_{\pi_n} H_{\pi_n}$ with $(E \otimes_\sigma H_\sigma) \oplus (E \otimes_{\hat{\pi}_n} \widehat{H}_n)$, $\mathfrak{w}_n$ can then be represented matricially as follows:

$$\mathfrak{w}_n = \begin{bmatrix} \mathfrak{z} & D_n \\ 0 & S_n \end{bmatrix}.$$

If we set $\widehat{H}_\infty = \varinjlim \widehat{H}_n$, then by uniqueness of inductive limits, $\widehat{H}_\infty = H_{\pi_\infty} \ominus \iota_{\infty,0} H_\sigma$. Moreover, there exist operators $D_\infty : E \otimes_{\hat{\pi}_\infty} \widehat{H}_\infty \to H_\sigma$ and $S_\infty : E \otimes_{\hat{\pi}_\infty} \widehat{H}_\infty \to \widehat{H}_\infty$ such that $D_\infty \in \mathcal{I}(\hat{\pi}_\infty^E, \sigma \circ \tau)$ and $S_\infty \in \mathcal{I}(\hat{\pi}_\infty^E, \hat{\pi}_\infty \circ \tau)$, where $\hat{\pi}_\infty : A \to B(\widehat{H}_\infty)$ is the representation that satisfies $\pi_\infty = \sigma \oplus \hat{\pi}_\infty$. Since $D_\infty|_{E \otimes_{\hat{\pi}_n} \widehat{H}_n} = D_n$ for all n, it follows that $D_\infty = P_{H_\sigma} \mathfrak{w}_\infty|_{E \otimes_{\hat{\pi}_\infty} \widehat{H}_\infty}$. In much the same way, $S_\infty = P_{\widehat{H}_\infty} \mathfrak{w}_\infty|_{E \otimes_{\hat{\pi}_\infty} \widehat{H}_\infty}$ since $S_\infty|_{E \otimes_{\hat{\pi}_n} \widehat{H}_n} = S_n$ for all n. This shows that $(\mathfrak{w}_\infty, \pi_\infty)$ is adapted to τ, which completes the proof.

□

Theorem 5.3 (Uniqueness of Minimal Coisometric Extensions Adapted to τ)**.** *The* *minimal coisometric extension* $(\mathfrak{w}_\infty, \pi_\infty)$ *of* $(\mathfrak{z}, \sigma)$ *adapted to* τ*, built in Theorem 5.2, is unique up to unitary equivalence. In particular, if* $(\mathfrak{v}, \gamma)$ *is a minimal coisometric extension of* $(\mathfrak{z}, \sigma)$ *adapted to* τ*, acting on* H_γ*, then there is a Hilbert space isomorphism* $\Phi_\infty : H_\gamma \to H_{\pi_\infty}$ *such that* $\Phi_\infty \gamma(\cdot) = \pi_\infty(\cdot)\Phi_\infty$ *and* $\Phi_\infty \mathfrak{v} = \mathfrak{w}_\infty(I_E \otimes \Phi_\infty)$*.*

Proof. By virtue of $(\mathfrak{v}, \gamma)$ being an extension of $(\mathfrak{z}, \sigma)$, $\mathfrak{v}(E \otimes_\sigma H_\sigma) \subseteq H_\sigma$ with $\mathfrak{v}|_{E\otimes_\sigma H_\sigma} = \mathfrak{z}$, and $\gamma(\cdot)H_\sigma \subseteq H_\sigma$ with $\gamma(\cdot)|_{H_\sigma} = \sigma(\cdot)$. Write $K_1 = H_\gamma \ominus H_\sigma$ and let $\omega_1(\cdot) = \gamma(\cdot)|_{K_1}$, so that $H_\gamma = H_\sigma \oplus K_1$ and $\gamma = \sigma \oplus \omega_1$. Under the identification $E \otimes_\gamma H_\gamma = (E \otimes_\sigma H_\sigma) \oplus (E \otimes_{\omega_1} K_1)$, we can write

$$\mathfrak{v} = \begin{bmatrix} \mathfrak{z} & D \\ 0 & S \end{bmatrix}.$$

The following simple computation reveals the structure of $\mathfrak{v}$:

$$\mathfrak{v}\mathfrak{v}^* = \begin{bmatrix} \mathfrak{z} & D \\ 0 & S \end{bmatrix} \begin{bmatrix} \mathfrak{z}^* & 0 \\ D^* & S^* \end{bmatrix} = \begin{bmatrix} \mathfrak{z}\mathfrak{z}^* + DD^* & DS^* \\ SD^* & SS^* \end{bmatrix}.$$

This shows that $\mathfrak{v}$ is coisometric if and only if $DD^* = (I_{H_\sigma} - \mathfrak{z}\mathfrak{z}^*) = \Delta_*^2$, $SD^* = 0$, and $SS^* = I_{K_1}$. Also, the assumption that $(\mathfrak{v}, \gamma)$ is adapted to τ means that $D \in \mathcal{I}(\omega_1^E, \sigma \circ \tau)$ and $S \in \mathcal{I}(\omega_1^E, \omega_1 \circ \tau)$. The latter condition implies that the projection S^*S commutes with ω_1^E, and is therefore of the form $I_E \otimes P$ for a projection $P \in \gamma(A)' \subseteq B(K_1)$. Define subspaces $K_2 = PK_1$ and $H_{\psi_1} = K_1 \ominus K_2$, so that $K_1 = H_{\psi_1} \oplus K_2$. Also, define representations $\psi_1 : A \to B(H_{\psi_1})$ and $\omega_2 : A \to B(K_2)$ of γ by $\psi(\cdot) = \gamma(\cdot)|_{H_{\psi_1}}$ and $\omega_2(\cdot) = \gamma(\cdot)|_{K_2}$. Then $\omega_1 = \psi_1 \oplus \omega_2$, and by the fact that $S^*S = I_E \otimes P$, we may make the identifications $E \otimes_{\omega_2} K_2 = \operatorname{Ran} S^*$ and $E \otimes_{\psi_1} H_{\psi_1} = \ker S$.

Define a new Hilbert space H_{γ_1} by the direct sum $H_\sigma \oplus H_{\psi_1}$. It is easy to see that H_{γ_1} reduces γ. Let γ_1 denote the representation of A on H_{γ_1} given by $\gamma_1 = \gamma|_{H_{\gamma_1}} = \sigma \oplus \psi_1$. We claim that H_{γ_1} is invariant under $\mathfrak{v}$, i.e., $\mathfrak{v}(E \otimes_{\gamma_1} H_{\gamma_1}) \subseteq H_{\gamma_1}$. To see this, we view $\mathfrak{v}$ as a map from $E \otimes_\gamma H_\gamma = (E \otimes_\sigma H_\sigma) \oplus (E \otimes_{\psi_1} H_{\psi_1}) \oplus (E \otimes_{\omega_2} K_2)$ to $H_\gamma = H_\sigma \oplus H_{\psi_1} \oplus K_2$, and we note that $S\,|_{E \otimes_{\psi_1} H_{\psi_1}} = 0$ since $E \otimes_{\psi_1} H_{\psi_1} = \ker S$. Additionally, $SD^* = 0$ implies that Inn $D =$ Ran $D^* = \ker S = E \otimes_{\psi_1} H_{\psi_1}$ and so, $D\,|_{E \otimes_{\omega_2} K_2} = 0$. From the intertwining condition $D \in \mathcal{I}(\omega_1^E, \sigma \circ \tau)$, it follows that $D\,|_{E \otimes_{\psi_1} H_{\psi_1}} \in \mathcal{I}(\psi_1^E, \sigma \circ \tau)$. Set $D_1 = D\,|_{E \otimes_{\psi_1} H_{\psi_1}}$, $D_2 = P_{H_{\psi_1}} S\,|_{E \otimes_{\omega_2} K_2}$, and $S' = P_{K_2} S\,|E \otimes_{\omega_2} K_2$. Notice that $D_1 D_1^* = DD^*$ since $D|_{E \otimes_{\psi_1} H_{\psi_1}} = 0$. Further, we can write

$$\mathfrak{v} = \begin{bmatrix} \mathfrak{z} & D_1 & 0 \\ 0 & 0 & D_2 \\ 0 & 0 & S' \end{bmatrix},$$

From this matricial representation, it is easy to see that $\mathfrak{v}(E \otimes_{\gamma_1} H_{\gamma_1}) \subseteq H_{\gamma_1}$. By identifying $E \otimes_{\gamma_1} H_{\gamma_1}$ with $(E \otimes_\sigma H_\sigma) \oplus (E \otimes_{\psi_1} H_{\psi_1})$, we define $\mathfrak{v}_1 : E \otimes_{\gamma_1} H_{\gamma_1} \to H_{\gamma_1}$ by

$$\mathfrak{v}_1 = \mathfrak{v}|_{E \otimes_{\gamma_1} H_{\gamma_1}} = \begin{bmatrix} \mathfrak{z} & D_1 \\ 0 & 0 \end{bmatrix},$$

We claim that $(\mathfrak{v}_1, \gamma_1)$ is the minimal partially isometric extension of $(\mathfrak{z}, \sigma)$ adapted to τ. It is clear that H_{γ_1} reduces $(\mathfrak{v}_1, \gamma_1)$, while $\mathfrak{v}_1|_{E \otimes_\sigma H_\sigma} = \mathfrak{z}$ and $\gamma_1(\cdot)|_{H_\sigma} = \sigma(\cdot)$. The fact that $\mathfrak{z}\mathfrak{z}^* + D_1 D_1^* = \mathfrak{z}\mathfrak{z}^* + DD^* = I_{H_\sigma}$ implies that $\mathfrak{v}_1$ is a partial isometry with final space H_σ. Furthermore, since $D_1 \in \mathcal{I}(\psi_1^E, \sigma \circ \tau)$ and $\mathfrak{z} \in \mathcal{I}(\sigma^E \circ \phi, \sigma)$, for any $a \in A$, we have

$$\begin{aligned}
\mathfrak{v}_1\gamma_1^E(\phi(a)) &= \begin{bmatrix} \mathfrak{z} & D_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma^E(\phi(a)) & 0 \\ 0 & \psi_1^E(\phi(a)) \end{bmatrix} \\
&= \begin{bmatrix} \mathfrak{z}\sigma^E(\phi(a)) & D_1\psi_1^E(\phi(a)) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma(a)\mathfrak{z} & \sigma(\tau(\phi(a)))D_1 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sigma(a)\mathfrak{z} & \sigma(a)D_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma(a) & 0 \\ 0 & \psi_1(a) \end{bmatrix} \begin{bmatrix} \mathfrak{z} & D_1 \\ 0 & 0 \end{bmatrix} \\
&= \gamma_1(a)\mathfrak{v}_1.
\end{aligned}$$

Therefore, $(\mathfrak{v}_1, \gamma_1)$ is a partially isometric extension of $(\mathfrak{z}, \sigma)$ adapted to τ. We want to show that $(\mathfrak{v}_1, \gamma_1)$ is minimal. Toward this end, consider the subspace $[\psi_1^E(\mathcal{L}(E))D_1^* H_\sigma]$ of $E \otimes_{\omega_1} K_1$; by definition it is the smallest reducing subspace for ψ_1^E containing the range of D_1^*. Further, if $X \in \mathcal{L}(E)$ and $h \in H_\sigma$, then since $D_1^* \in \mathcal{I}(\sigma \circ \tau, \psi_1^E)$, we have $\psi_1^E(X)D_1^*h = D_1^*\sigma(\tau(X))h \in \text{Ran } D_1^*$. We conclude that Ran D_1^* reduces ψ_1^E, and therefore, the smallest reducing subspace for ψ_1^E containing $[D_1^* H_\sigma]$ is Ran D_1^*. But Ran $D_1^* = \text{Inn } D_1 = \text{Inn } D = \ker S = E \otimes_{\psi_1} H_{\psi_1}$, so in fact, $E \otimes_{\psi_1} H_{\psi_1}$ is the smallest reducing subspace for ψ_1^E containing the range of D_1^*, that is, $(\mathfrak{v}_1, \gamma_1)$ is minimal. Therefore, $(\mathfrak{v}_1, \gamma_1)$ is (unitarily equivalent to) the unique minimal partially isometric extension of $(\mathfrak{z}, \sigma)$ adapted to τ.

By repeating the process just outlined, we obtain the minimal partially isometric extension of $(\mathfrak{v}_1, \gamma_1)$ adapted to τ. We denote this extension by $(\mathfrak{v}_2, \gamma_2)$. In a similar way, we obtain the minimal partially isometric extension $(\mathfrak{v}_{n+1}, \gamma_{n+1})$ of

$(\mathfrak{v}_n, \gamma_n)$ adapted to τ, for all $n \geq 1$. Just as in the proof of Theorem 5.2 we then take (inductive) limits to get a minimal coisometric extension $(\mathfrak{v}_\infty, \gamma_\infty)$ of $(\mathfrak{z}, \sigma)$ adapted to τ. By the uniqueness of the inductive limits involved, $(\mathfrak{v}_\infty, \gamma_\infty)$ is unitarily equivalent to $(\mathfrak{w}_\infty, \pi_\infty)$. It remains to show that $(\mathfrak{v}_\infty, \gamma_\infty)$ is equivalent to $(\mathfrak{v}, \gamma)$. This will be accomplished by showing that $\bigvee_n \ker S_n = K_1$, where $S_n = P_{K_n}\mathfrak{v}|_{E\otimes_{\omega_n} K_n}$ (and $K_n = H_\gamma \ominus H_{\gamma_{n-1}}$). Toward that end, define the following subspaces of K_1:

$$K_P = \bigvee_{n=1}^{\infty} \ker S_n, \quad K_U = K_1 \ominus K_P.$$

(The subscripts on K_P and K_U stand for "pure coisometry" and "unitary," respectively, adjectives that describe properties of $\mathfrak{v}$ on those spaces.) It follows that H_γ can be written $H_\sigma \oplus K_P \oplus K_U$. Under this identification, simple computations reveal that $\mathfrak{v}$ has the following form:

$$\mathfrak{v} = \begin{bmatrix} \mathfrak{z} & D & 0 \\ 0 & S_P & 0 \\ 0 & 0 & S_U \end{bmatrix}.$$

It is now easy to see that $H_\sigma \oplus K_P$ reduces $(\mathfrak{v}, \gamma)$. If $K_U \neq \{0\}$, then $H_\sigma \oplus K_P$ is a *proper* subspace of H_γ that reduces $(\mathfrak{v}, \gamma)$ and contains H_σ, contradicting minimality of H_γ. Therefore, $K_U = \{0\}$ and $H_\gamma = H_\sigma \oplus K_P$. We conclude that $(\mathfrak{v}_\infty, \gamma_\infty)$ is unitarily equivalent to $(\mathfrak{v}, \gamma)$, and this completes the proof (to get Φ_∞, simply compose the isomorphism identifying H_{γ_∞} and H_{π_∞} with the isomorphism between H_γ and H_{γ_∞}).

$\square$

5.2 A Matricial Approach

Given a Hilbert space H_σ and a contractive intertwining pair $(\mathfrak{z}, \sigma)$ associated to (E, A, H_σ), the unique minimal coisometric extension $(\mathfrak{w}_\infty, \pi_\infty)$ adapted to τ, guaranteed by Theorems 5.2 and 5.3, has a straightforward matrix representation. For emphasis and to highlight some features of the extension that we exploit later, we give an outline of this matricial form of $(\mathfrak{w}_\infty, \pi_\infty)$. Recall that we have a fixed transfer operator $\tau : \mathcal{L}(E) \to A$ for the homomorphism $\phi : A \to \mathcal{L}(E)$. To keep things manageable, we "reset" the notation from the proofs in this chapter; however, we still write $(\mathfrak{w}_n, \pi_n)$ for the sequence of minimal partially isometric extensions adapted to τ, and we write $(\mathfrak{w}_\infty, \pi_\infty)$ for the minimal coisometric extension of $(\mathfrak{z}, \sigma)$ adapted to τ.

The first step of the construction proceeds just as in Section 5.1. We apply Theorem 3.2 to σ, we obtain a unique triple $(K_{\rho_1}, \rho_1, V_1)$, where K_{ρ_1} is a Hilbert space, $\rho_1 : \mathcal{L}(E) \to B(K_{\rho_1})$ is a representation, and $V_1 : H_\sigma \to K_{\rho_1}$ is an isometry, such that

$$V_1^* \rho_1(\cdot) V_1 = \sigma(\tau(\cdot)) \implies V_1 \in \mathcal{I}(\sigma, \rho_1 \circ \phi) \implies V_1^* \rho_1(\phi(\cdot)) V_1 = \sigma(\cdot).$$

$(K_{\rho_1}, \rho_1, V_1)$ is the extension of σ adapted to τ.

We (still) denote by Δ_* the defect operator for $\mathfrak{z}^*$, and we let $\mathcal{D}_{1*}$ denote the subspace $[\rho_1(\mathcal{L}(E)) V_1 \Delta_* H_\sigma] \subseteq K_{\rho_1}$. As noted in Lemma 4.7 , $\mathcal{D}_{1*}$ is the smallest subspace of K_{ρ_1} that reduces ρ_1 and contains $[V_1 \Delta_* H_\sigma]$. Thus, we can define a representation $\hat{\rho}_1 : \mathcal{L}(E) \to B(\mathcal{D}_{1*})$ by $\hat{\rho}_1(\cdot) = \rho_1(\cdot)|_{\mathcal{D}_{1*}}$. By Lemma 4.7, $\hat{\rho}_1$ is induced from a representation of A with respect to E, so there is a Hilbert space H_{θ_1}, a representation $\theta_1 : A \to B(H_{\theta_1})$, and a Hilbert space isomorphism $\mathcal{U}_1 : E \otimes_{\theta_1} H_{\theta_1} \to \mathcal{D}_{1*}$ such that $\hat{\rho}_1 = \mathcal{U}_1 \theta_1^E \mathcal{U}_1^*$. Further, (H_{θ_1}, θ_1) is the unique pair satisfying these properties, up to unitary equivalence. By Lemma 4.8, the operator $D_{1*} : E \otimes_{\theta_1} H_{\theta_1} \to H_\sigma$ given by $D_{1*} = \Delta_* V_1^* \mathcal{U}_1$ intertwines θ_1^E and $\sigma \circ \tau$, i.e., $D_{1*} \in \mathcal{I}(\theta_1^E, \sigma \circ \tau) \subseteq \mathcal{I}(\theta_1^E \circ \phi, \sigma)$.

The Hilbert space H_{π_1} is defined to be the direct sum $H_\sigma \oplus H_{\theta_1}$, and the representation $\pi_1 : A \to B(H_{\pi_1})$ is given by $\pi_1 = \sigma \oplus \theta_1$. Under the identification $(E \otimes_\sigma H_\sigma) \oplus (E \otimes_{\theta_1} H_{\theta_1}) = E \otimes_{\pi_1} H_{\pi_1}$, the intertwining map $\mathfrak{w}_1 : E \otimes_{\pi_1} H_{\pi_1} \to H_{\pi_1}$ can be written matricially as follows:

$$\mathfrak{w}_1 = \begin{bmatrix} \mathfrak{z} & D_{1*} \\ 0 & 0 \end{bmatrix}.$$

By Theorem 4.14, $(\mathfrak{w}_1, \pi_1)$ is *the* minimal partially isometric extension of $(\mathfrak{z}, \sigma)$ adapted to τ. In particular, $\mathfrak{w}_1$ is a partial isometry in the intertwiner space $\mathcal{I}(\pi_1^E \circ \phi, \pi_1)$. In fact, $\mathfrak{w}_1 \mathfrak{w}_1^*$ is the projection of H_{π_1} onto H_σ. It follows that the defect operator Δ_{1*} for $\mathfrak{w}_1^*$ is also a projection, so that

$$\Delta_{1*} = (I - \mathfrak{w}_1 \mathfrak{w}_1^*)^{1/2} = I - \mathfrak{w}_1 \mathfrak{w}_1^*.$$

Therefore, Ran $\Delta_{1*} = H_{\theta_1}$, i.e., Δ_{1*} is the projection of H_{π_1} onto H_{θ_1}. In fact, this is the key observation that allows us to get a matricial picture of $(\mathfrak{w}_\infty, \pi_\infty)$.

Until this point, the construction has proceeded exactly as in the previous section. There, the next step was to apply Theorem 3.2 to the completely positive map π_1 to obtain the extension of π_1 adapted to τ. However, the fact that Δ_{1*} is a projection allows us to take a different (but in the end, equivalent) approach, by applying 3.2 to the subrepresentation θ_1 of π_1.

Applying Theorem 3.2 to $\theta_1 : A \to B(H_{\theta_1})$, we obtain a unique triple $(K_{\gamma_2}, \gamma_2, S_2)$ (the extension of θ_1 adapted to τ). Recall that K_{γ_2} is a Hilbert space, γ_2 is a nondegenerate representation of $\mathcal{L}(E)$ on K_{γ_2}, and $S_2 : H_{\theta_1} \to K_{\gamma_2}$ is an isometry, together satisfying

$$S_2^*\gamma_2(X)S_2 = \theta_1(\tau(X)), \quad X \in \mathcal{L}(E).$$

In particular, $S_2 \in \mathcal{I}(\theta_1, \gamma_2 \circ \phi)$, and

$$S_2^*\gamma_2(\phi(a))S_2 = \theta_1(a), \quad a \in A.$$

Moreover, by Corollary 3.6 there is a (unique) representation θ_2 of A on a Hilbert space H_{θ_2}, and there exists a Hilbert space isomorphism of K_{γ_2} onto $E \otimes_{\theta_2} H_{\theta_2}$ which implements an equivalence between γ_2 and the induced representation $\theta_2^E : \mathcal{L}(E) \to B(E \otimes_{\theta_2} H_{\theta_2})$.

We are justified in using the symbol θ_2, even though it has been used previously to denote another representation, because as we will see, the two representations are equivalent. For this, recall that K_{γ_2} is the Stinespring space associated to θ_1 and is defined by $K_{\gamma_2} = \mathcal{L}(E) \otimes_{\theta_1 \circ \tau} H_{\theta_1}$. As in Section 5.1, let $(K_{\rho_2}, \rho_2, V_2)$ denote the extension of π_1 adapted to τ, let $\mathcal{D}_{2*} = [\rho_2(\mathcal{L}(E))V_2\Delta_{1*}H_{\pi_1}]$, and let $\hat{\rho}_2 : \mathcal{L}(E) \to B(\mathcal{D}_{2*})$ be the restriction of ρ_2 to $\mathcal{D}_{2*}$, i.e., $\hat{\rho}_2(\cdot) = \rho_2(\cdot)|_{\mathcal{D}_{2*}}$. Recall that $\mathcal{D}_{2*}$ is the smallest reducing subspace for ρ containing $[V_2\Delta_{1*}H_{\pi_1}] = [V_2 H_{\theta_1}]$. Under this identification, we can define a map $W : K_{\gamma_2} \to \mathcal{D}_{2*}$ on elementary tensors by $X \otimes h \mapsto$

$\rho_2(X)V_2h$. For $X \otimes h, Y \otimes k \in K_{\gamma_2}$, we see that

$$\begin{aligned}\langle X \otimes h, Y \otimes k\rangle &= \langle h, \theta_1(\tau(X^*Y))k\rangle = \langle h, \pi_1(\tau(X^*Y))k\rangle \\ &= \langle h, V_2^*\rho_2(X^*Y)V_2k\rangle = \langle \rho_2(X)V_2h, \rho_2(Y)V_2k\rangle \\ &= \langle W(X \otimes h), W(Y \otimes k)\rangle.\end{aligned}$$

This shows that W is a well-defined isometry, and it is clear the W is surjective. Thus, W is a Hilbert space isomorphism of K_{γ_2} onto $\mathcal{D}_{2*}$. Further, for $X \in \mathcal{L}(E)$ and $Y \otimes h \in K_{\gamma_2}$,

$$\begin{aligned}\hat{\rho}_2(X)W(Y \otimes h) &= \hat{\rho}_2(X)\rho_2(Y)V_2h = \rho_2(XY)V_2h = W(XY \otimes h) \\ &= W\gamma_2(X)(Y \otimes h).\end{aligned}$$

Therefore, $W \in \mathcal{I}(\gamma_2, \hat{\rho}_2)$, showing that γ_2 and $\hat{\rho}_2$ are unitarily equivalent. Finally, we claim that $WS_2 = V_2|_{H_{\theta_1}}$. Indeed, recalling that $S_2 : H_{\theta_1} \to K_{\gamma_2}$ is given by $S_2h = I_E \otimes h$, $h \in H_{\theta_1}$, we have $WS_2h = W(I_E \otimes h) = \rho_2(I_E)V_2h = V_2h$.

We conclude that the Stinespring triple $(K_{\gamma_2}, \gamma_2, S_2)$ associated to $\theta_1 \circ \tau$ is unitarily equivalent to the triple $(\mathcal{D}_{2*}, \hat{\rho}_2, V_2|_{H_{\theta_1}})$. It follows that θ_2^E is unitarily equivalent to both $\hat{\rho}_2$ and γ_2 (via Hilbert space isomorphisms $\mathcal{U}_2$ and $W^*\mathcal{U}_2$ of $E \otimes_{\theta_2} H_{\theta_2}$ onto $\mathcal{D}_{2*}$ and K_{γ_2}, respectively). Therefore, we are justified in "reusing" the symbols H_{θ_2} and θ_2. We make the identifications $K_{\gamma_2} = \mathcal{D}_{2*} = E \otimes_{\theta_2} H_{\theta_2}$, $\gamma_2 = \hat{\rho}_2 = \theta_2^E$, and $S_2 = V_2|_{H_{\theta_1}}$.

The minimal partial isometric extension $(\mathfrak{w}_2, \pi_2)$ of $(\mathfrak{w}_1, \pi_1)$ adapted to τ acts on the Hilbert space $H_{\pi_2} = H_{\pi_1} \oplus H_{\theta_2} = H_\sigma \oplus H_{\theta_1} \oplus H_{\theta_2}$, and is comprised of the representation $\pi_2 = \pi_1 \oplus \theta_2 = \sigma \oplus \theta_1 \oplus \theta_2$ of A on $B(H_{\pi_2})$ and the map $\mathfrak{w}_2 : E \otimes_{\pi_2} H_{\pi_2} \to$

H_{π_2} which, after identifying $E \otimes_{\pi_2} H_{\pi_2}$ with $(E \otimes_\sigma H_\sigma) \oplus (E \otimes_{\theta_1} H_{\theta_1}) \oplus (E \otimes_{\theta_2} H_{\theta_2})$, is defined matricially by

$$\mathfrak{w}_2 = \begin{bmatrix} \mathfrak{z} & D_{1*} & 0 \\ 0 & 0 & S_2^* \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that for $X \in \mathcal{L}(E)$, $S_2^* \theta_2^E(X) = S_2^* \gamma_2(X) S_2 S_2^* = \theta_1(\tau(X)) S_2^*$; that is, $S_2^* \in \mathcal{I}(\theta_2^E, \theta_1 \circ \tau)$ and hence, $S_2 \in \mathcal{I}(\theta_1 \circ \tau, \theta_2^E)$. Verifying that $(\mathfrak{w}_2, \pi_2)$ is a minimal partially isometric extension of $(\mathfrak{w}_1, \pi_1)$ is now a matter of straightforward matrix computations, highlighting one of the primary advantages of this (matricial) approach.

Continuing in the fashion just described, we obtain inductively a sequence of triples $\{(K_{\gamma_n}, \gamma_n, S_n)\}_{n=1}^{\infty}$, where, for $n \geq 1$, $(K_{\gamma_n}, \gamma_n, S_n)$ is the extension of θ_{n-1} adapted to τ. To be precise, K_{γ_n} is a Hilbert space, $\gamma_n : \mathcal{L}(E) \to B(K_{\gamma_n})$ is a (nondegenerate) representation of $\mathcal{L}(E)$ on K_{γ_n}, and $S_n : H_{\theta_{n-1}} \to K_{\gamma_n}$ is an isometry, such that

$$S_n^* \gamma_n(X) S_n = \theta_{n-1}(\tau(X)), \quad X \in \mathcal{L}(E), \tag{5.1}$$

and such that

$$S_n^* \gamma_n(\phi(a)) S_n = \theta_{n-1}(a), \quad a \in A. \tag{5.2}$$

In fact, $S_n \in \mathcal{I}(\theta_{n-1} \circ \tau, \theta_n^E)$, from which (5.1) and (5.2) follow immediately.

We know from Corollary 3.6 that γ_n is induced from a representation θ_n of A with respect to E. That is, $\theta_n : A \to B(H_{\theta_n})$ is a nondegenerate representation of A on a Hilbert space H_{θ_n}, and there is a Hilbert space isomorphism $\mathcal{U}_n : E \otimes_{\theta_n} H_{\theta_n} \to K_{\gamma_n}$

such that $\mathcal{U}_n \theta_n^E(\cdot) = \gamma_n(\cdot)\mathcal{U}_n$. Just as in the $n = 2$ case, we make the identifications $K_{\gamma_n} = \mathcal{D}_{n*} = E \otimes_{\theta_n} H_{\theta_n}$, $\gamma_n = \hat{\rho}_n = \theta_n^E$, and $S_n = V_n|_{H_{\theta_n}}$, whenever $n \geq 3$.

We get a nested sequence of Hilbert spaces H_{π_n} by defining $H_{\pi_n} = H_{\pi_{n-1}} \oplus H_{\theta_n} = H_\sigma \oplus H_{\theta_1} \oplus \cdots \oplus H_{\theta_n}$, and for each $n \geq 1$, we define a representation $\pi_n : A \to B(H_{\pi_n})$ by $\pi_n = \pi_{n-1} \oplus \theta_n = \sigma \oplus \theta_1 \oplus \cdots \oplus \theta_n$. We identify $E \otimes_{\pi_n} H_{\pi_n} = (E \otimes_\sigma H_\sigma) \oplus (E \otimes_{\theta_1} H_{\theta_1}) \oplus \cdots \oplus (E \otimes_{\theta_n} H_{\theta_n})$, and under this identification, $\mathfrak{w}_n : E \otimes_{\pi_n} H_{\pi_n} \to H_{\pi_n}$ is given by

$$\mathfrak{w}_n = \begin{bmatrix} \mathfrak{z} & D_{1*} & & & & \\ & 0 & S_2^* & & & \\ & & 0 & S_3^* & & \\ & & & 0 & \ddots & \\ & & & & \ddots & S_n^* \\ & & & & & 0 \end{bmatrix}.$$

Again, simple matrix computations verify that $(\mathfrak{w}_n, \pi_n)$ is the minimal partially isometric extension of $(\mathfrak{w}_{n-1}, \pi_{n-1})$ for $n \geq 2$.

The Hilbert space H_{π_∞} on which the minimal coisometric extension of $(\mathfrak{z}, \sigma)$ adapted to τ acts has as a dense subspace $H_\sigma \oplus H_{\theta_1} \oplus H_{\theta_2} \oplus H_{\theta_3} \oplus \cdots$. The representation $\pi_\infty : A \to B(H_{\pi_\infty})$ is given by the direct sum

$$\pi_\infty = \sigma \oplus \theta_1 \oplus \theta_2 \oplus \theta_3 \oplus \cdots$$

Finally, under the identification

$$E \otimes_{\pi_\infty} H_{\pi_\infty} = (E \otimes_\sigma H_\sigma) \oplus (E \otimes_{\theta_1} H_{\theta_1}) \oplus (E \otimes_{\theta_2} H_{\theta_2}) \oplus \cdots ,$$

the coisometry $\mathfrak{w}_\infty : E \otimes_{\pi_\infty} H_{\pi_\infty} \to H_{\pi_\infty}$ is defined matricially as follows:

$$\mathfrak{w}_\infty = \begin{bmatrix} \mathfrak{z} & D_{1*} & & & & & \\ & 0 & S_2^* & & & & \\ & & 0 & S_3^* & & & \\ & & & 0 & S_4^* & & \\ & & & & 0 & \ddots & \\ & & & & & \ddots & \end{bmatrix}.$$

Once more, straightforward matrix multiplication shows that $\mathfrak{w}_\infty$ is a coisometry that intertwines $\pi_\infty^E \circ \phi$ and π_∞, i.e., $\mathfrak{w}_\infty \in \mathcal{I}(\pi_\infty^E \circ \phi, \pi_\infty)$. Furthermore, it is clear that H_σ (identified with the subspace $H_\sigma \oplus \{0\} \oplus \{0\} \oplus \cdots \subseteq H_{\pi_\infty}$) is invariant for $(\mathfrak{w}_\infty, \pi_\infty)$ with $\mathfrak{w}_\infty|_{E\otimes_\sigma H_\sigma} = \mathfrak{z}$ and $\pi_\infty(\cdot)|_{H_\sigma} = \sigma(\cdot)$. Thus, $(\mathfrak{w}_\infty, \pi_\infty)$ is a minimal coisometric extension of $(\mathfrak{z}, \sigma)$ (minimality is immediate from Theorem 5.2 and the definition of H_{π_∞}).

Define $\widehat{H} = H_{\pi_\infty} \ominus H_\sigma$ and $\hat{\pi}(\cdot) = \pi_\infty(\cdot)|_{\widehat{H}}$, and let $D = P_{H_\sigma}\mathfrak{w}_\infty|_{E\otimes_{\hat{\pi}}\widehat{H}}$ and $S = P_{\widehat{H}}\mathfrak{w}_\infty|_{E\otimes_{\hat{\pi}}\widehat{H}}$. In other words, $\widehat{H} = H_{\theta_1} \oplus H_{\theta_2} \oplus \cdots$, $\hat{\pi} = \theta_1 \oplus \theta_2 \oplus \cdots$, $D = [D_{1*}\ 0\ \ 0\ \ \cdots]$ and

$$S = \begin{bmatrix} 0 & S_2^* & & & \\ & 0 & S_3^* & & \\ & & 0 & S_4^* & \\ & & & \ddots & \ddots \end{bmatrix}.$$

Recall that $D_{1*} \in \mathcal{I}(\theta_1^E, \sigma \circ \tau)$ and $S_n \in \mathcal{I}(\theta_{n-1} \circ \tau, \theta_n^E)$ for each $n \geq 2$, so that $S_n^* \in \mathcal{I}(\theta_n^E, \theta_{n-1} \circ \tau)$, $n \geq 2$. Evidently, $D \in \mathcal{I}(\hat{\pi}^E, \sigma \circ \tau)$ and $S \in \mathcal{I}(\hat{\pi}^E, \hat{\pi} \circ \tau)$ (this again amounts simply to matrix multiplication). Therefore, $(\mathfrak{w}_\infty, \pi_\infty)$ is the minimal coisometric extension of $(\mathfrak{z}, \sigma)$ adapted to τ.

CHAPTER 6

COMMUTANT LIFTING FOR PARTIALLY ISOMETRIC EXTENSIONS

6.1 Introduction

Fix a unital C^*-algebra A, and let E be an A-correspondence, with $\phi : A \to \mathcal{L}(E)$ denoting the homomorphism that gives the left-action of A on E. Assume that ϕ has a smooth, unital generalized transfer operator $\tau : \mathcal{L}(E) \to A$. We use the notation and terminology established in Chapters 4 and 5.

Our goal in this chapter is to prove a commutant lifting theorem for partially isometric extensions of contractive intertwining pairs. Ideally, we would like to show that if $(\mathfrak{z}, \sigma)$ is a contractive intertwining pair associated to (E, A, H_σ) and if X commutes with $(\mathfrak{z}, \sigma)$, then there exists an operator X_1 that commutes with $(\mathfrak{w}_1, \pi_1)$, extends X, and such that $||X_1|| = ||X||$. Unfortunately, there is strong evidence suggesting that this is an unattainable goal unless some additional assumptions are placed on $(\mathfrak{z}, \sigma)$. In what follows, we first outline the difficulties in lifting the commutant to the level of a partially isometric extension. In outlining these difficulties, it will become clear what sort of additional assumption on $(\mathfrak{z}, \sigma)$ is sufficient to guarantee that the commutant can be lifted. We then impose this condition on $(\mathfrak{z}, \sigma)$ and prove our version of the commutant lifting theorem (for partially isometric extensions).

Recall that the minimal partially isometric extension $(\mathfrak{w}_1, \pi_1)$ of $(\mathfrak{z}, \sigma)$ adapted to τ acts on the Hilbert space $H_{\pi_1} = H_\sigma \oplus H_{\theta_1}$, that $\pi_1 = \sigma \oplus \theta_1$, and that

$$\mathfrak{w}_1 = \begin{bmatrix} \mathfrak{z} & \mathcal{D}_{1*} \\ 0 & 0 \end{bmatrix},$$

where $D_{1*} : E \otimes_{\theta_1} H_{\theta_1} \to H_\sigma$ is defined by $D_{1*} = \Delta_* V_1^* \mathcal{U}_1$.

Assume $X \in B(H_\sigma)$ commutes with $(\mathfrak{z}, \sigma)$, i.e., $X\mathfrak{z} = \mathfrak{z}(I_E \otimes X)$ and $X \in \sigma(A)'$. We would like to construct an operator $X_1 \in B(H_{\pi_1})$ such that $X_1 H_\sigma \subseteq H_\sigma$, $X_1|_{H_\sigma} = X$, $||X_1|| = ||X||$, and X_1 commutes with $(\mathfrak{w}_1, \pi_1)$. Note, first of all, that since X_1 acts on $H_{\pi_1} = H_\sigma \oplus H_{\theta_1}$, there exist operators $P : H_\sigma \to H_\sigma$, $Q : H_\sigma \to H_{\theta_1}$, $R : H_{\theta_1} \to H_\sigma$, and $S : H_{\theta_1} \to H_{\theta_1}$ such that

$$X_1 = \begin{bmatrix} P & R \\ Q & S \end{bmatrix}.$$

Appealing to this matricial form of X_1, it is easy to see that X_1 extends X (i.e., $X_1 H_\sigma \subseteq H_\sigma$ and $X_1|_{H_\sigma} = X$) if and only if $P = X$ and $Q = 0$, that is, if and only if

$$X_1 = \begin{bmatrix} X & R \\ 0 & S \end{bmatrix}.$$

Note that for a fixed $a \in A$,

$$X_1 \pi_1(a) = \begin{bmatrix} X & R \\ 0 & S \end{bmatrix} \begin{bmatrix} \sigma(a) & 0 \\ 0 & \theta_1(a) \end{bmatrix} = \begin{bmatrix} X\sigma(a) & R\theta_1(a) \\ 0 & S\theta_1(a) \end{bmatrix},$$

while

$$\pi_1(a) X_1 = \begin{bmatrix} \sigma(a) & 0 \\ 0 & \theta_1(a) \end{bmatrix} \begin{bmatrix} X & R \\ 0 & S \end{bmatrix} = \begin{bmatrix} \sigma(a)X & \sigma(a)R \\ 0 & \theta_1(a)S \end{bmatrix}.$$

Thus, $X_1 \in \pi_1(A)'$ if and only if $R \in \mathcal{I}(\theta_1, \sigma)$ and $S \in \theta_1(A)'$. Similarly,

$$X_1 \mathfrak{w}_1 = \begin{bmatrix} X & R \\ 0 & S \end{bmatrix} \begin{bmatrix} \mathfrak{z} & D_{1*} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X\mathfrak{z} & XD_{1*} \\ 0 & 0 \end{bmatrix},$$

while

$$\begin{aligned} \mathfrak{w}_1(I_E \otimes X_1) &= \begin{bmatrix} \mathfrak{z} & D_{1*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_E \otimes X & I_E \otimes R \\ 0 & I_E \otimes S \end{bmatrix} \\ &= \begin{bmatrix} \mathfrak{z}(I_E \otimes X) & \mathfrak{z}(I_E \otimes R) + D_{1*}(I_E \otimes S) \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

showing that $X_1 \mathfrak{w}_1 = \mathfrak{w}_1(I_E \otimes X_1)$ if and only if $XD_{1*} = \mathfrak{z}(I_E \otimes R) + D_{1*}(I_E \otimes S)$ (of course, $X\mathfrak{z} = \mathfrak{z}(I_E \otimes X)$ is automatic from the assumption that X commutes with $(\mathfrak{z}, \sigma)$).

We may assume without a loss of generality that $||X|| = 1$. By a theorem of Sz.-Nagy and Foiaş [22, Théorème 1] (see also [2, Proposition 2.2]), $||X_1|| = 1$ as well if and only if $||S|| \leq 1$ and $R = D_{X^*}CD_S$, where $D_{X^*} = (I_{H_\sigma} - XX^*)^{1/2}$ and $D_S = (I_{H_{\theta_1}} - S^*S)^{1/2}$ are the defect operators for X^* and S, respectively, and where $C : H_{\theta_1} \to H_\sigma$ satisfies $||C|| \leq 1$. Note that since $X \in \sigma(A)'$ and $S \in \theta_1(A)'$, $D_{X^*} \in \sigma(A)'$ and $D_S \in \theta_1(A)'$. Since $R \in \mathcal{I}(\theta_1, \sigma)$, we must also have that $C \in \mathcal{I}(\theta_1, \sigma)$.

In summary, X_1 extends X, preserves the norm of X, and commutes with $(\mathfrak{w}_1, \pi_1)$ if and only if

$$X_1 = \begin{bmatrix} X & D_{X^*}CD_S \\ 0 & S \end{bmatrix},$$

where $C : H_{\theta_1} \to H_\sigma$ is a contraction in the intertwining space $\mathcal{I}(\theta_1, \sigma)$ and $S \in \theta_1(A)'$. We now set about trying to find such operators C and S. A key tool throughout the analysis that follows is Douglas's range inclusion, majorization, and factorization theorem [1, Theorem 1].

Theorem 6.1 (Douglas's Range Inclusion, Majorization, and Factorization Theorem)**.** *Given operators A and B on a Hilbert space H, the following are equivalent.*

1. *Ran $A \subseteq$ Ran B,*

2. *$AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0$, and*

3. *there exists a bounded operator $C \in B(H)$ such that $A = BC$.*

Moreover, if any of the equivalent conditions 1., 2., or 3. hold, then there is a unique operator C such that $||C||^2 = \inf\{\mu \mid AA^* \leq \mu BB^*\}$, $\ker A = \ker C$, and Ran $C \subseteq$ (Ran $B^*)^\perp$.

Since X is contractive, we see that

$$\begin{aligned}
(XD_{1*})(XD_{1*})^* &= X\Delta_* V_1^* \mathcal{U}_1 \mathcal{U}_1^* V_1 \Delta_* X^* \\
&= X\Delta_*^2 X^* \\
&= X(I_{H_\sigma} - \mathfrak{z}\mathfrak{z}^*)X^* \\
&= XX^* - X\mathfrak{z}\mathfrak{z}^* X^* \\
&\leq I_{H_\sigma} - X\mathfrak{z}\mathfrak{z}^* X^*.
\end{aligned}$$

Furthermore, since $X\mathfrak{z} = \mathfrak{z}(I_E \otimes X)$,

$$\begin{aligned} I_{H_\sigma} - X\mathfrak{z}\mathfrak{z}^* X^* &= I_{H_\sigma} - \mathfrak{z}(I_E \otimes X)(I_E \otimes X)^* \mathfrak{z}^* \\ &= I_{H_\sigma} - \mathfrak{z}(I_E \otimes XX^*)\mathfrak{z}^* \\ &= \mathfrak{z}\mathfrak{z}^* - \mathfrak{z}(I_E \otimes XX^*)\mathfrak{z}^* + (I_{H_\sigma} - \mathfrak{z}\mathfrak{z}^*) \\ &= \mathfrak{z}(I_{E\otimes_\sigma H_\sigma} - (I_E \otimes XX^*))\mathfrak{z}^* + \Delta_*^2 \\ &= \mathfrak{z}(I_E \otimes (I_{H_\sigma} - XX^*))\mathfrak{z}^* + \Delta_* V_1^* \mathcal{U}_1 \mathcal{U}_1^* V_1 \Delta_* \\ &= \mathfrak{z}(I_E \otimes D_{X*}^2)\mathfrak{z}^* + (\Delta_* V_1^* \mathcal{U}_1)(\Delta_* V_1^* \mathcal{U}_1)^* \\ &= \mathfrak{z}(I_E \otimes D_{X*})^2 \mathfrak{z}^* + D_{1*} D_{1*}^* \end{aligned}$$

Therefore, $(XD_{1*})(XD_{1*})^* \leq BB^*$, where $B : E \otimes_{\pi_1} H_{\pi_1} \to H_\sigma$ is the operator given by the 1×2 block operator matrix $\begin{bmatrix} \mathfrak{z}(I_E \otimes D_{X*}) & D_{1*} \end{bmatrix}$. Applying Theorem 6.1 to the inequality $(XD_{1*})(XD_{1*})^* \leq BB^*$, we obtain a contraction $\hat{Z} : E \otimes_{\theta_1} H_{\theta_1} \to E \otimes_{\pi_1} H_{\pi_1}$ such that $B\hat{Z} = XD_{1*}$.

We claim that $B^* \in \mathcal{I}(\sigma, \pi_1^E \circ \phi)$, i.e., $B^*\sigma(a) = \pi_1^E(\phi(a))B^*$ for all $a \in A$. For this, recall that $\mathfrak{z}^* \in \mathcal{I}(\sigma, \sigma^E \circ \phi)$ (so that $\Delta_* \in \sigma(A)'$) and that $D_{1*}^* \in \mathcal{I}(\sigma \circ \tau, \theta_1^E) \subseteq \mathcal{I}(\sigma, \theta_1^E \circ \phi)$. In addition, note that $I_E \otimes D_{X*} \in \sigma^E(\phi(A))'$ since $D_{X^*} \in \sigma(A)'$. Thus, for any $a \in A$,

$$\begin{aligned}
\pi_1^E(\phi(a))B^* &= \begin{bmatrix} \sigma^E(\phi(a)) & 0 \\ 0 & \theta_1^E(\phi(a)) \end{bmatrix} \begin{bmatrix} (I_E \otimes D_{X*})\mathfrak{z}^* \\ D_{1*}^* \end{bmatrix} \\
&= \begin{bmatrix} \sigma^E(\phi(a))(I_E \otimes D_{X*})\mathfrak{z}^* \\ \theta_1^E(\phi(a))D_{1*}^* \end{bmatrix} \\
&= \begin{bmatrix} (I_E \otimes D_{X*})\mathfrak{z}^*\sigma(a) \\ D_{1*}^*\sigma(a) \end{bmatrix} \\
&= \begin{bmatrix} (I_E \otimes D_{X*})\mathfrak{z}^* \\ D_{1*}^* \end{bmatrix} \sigma(a) \\
&= B^*\sigma(a).
\end{aligned}$$

This shows that $B^* \in \mathcal{I}(\sigma, \pi_1^E \circ \phi)$, as claimed. Therefore,

$$\begin{aligned}
\hat{Z}^*\pi_1^E(\phi(a))B^* &= \hat{Z}^*B^*\sigma(a) = (B\hat{Z})^*\sigma(a) = (XD_{1*})^*\sigma(a) \\
&= D_{1*}^*X^*\sigma(a) = D_{1*}^*\sigma(a)X^* = \theta_1^E(\phi(a))D_{1*}^*X^* \\
&= \theta_1^E(\phi(a))\hat{Z}^*B^*.
\end{aligned}$$

This proves that $\hat{Z}^*$ intertwines the representations $\pi_1^E \circ \phi$ and $\theta_1^E \circ \phi$ on the range of B^*. If we set $\hat{Z}^* = 0$ on the orthogonal complement of $\mathrm{Ran}(B^*)$, then

$$\mathrm{Ran}\,\hat{Z} = (\ker \hat{Z}^*)^\perp \subseteq \mathrm{Ran}\,B^* = \mathrm{Inn}\,B.$$

In fact, if one requires the condition $\mathrm{Ran}\,\hat{Z} \subseteq \mathrm{Inn}\,B$, then $\hat{Z}$ is the unique contraction satisfying $B\hat{Z} = XD_{1*}$ by Theorem 6.1; fix this choice of $\hat{Z}$. Since $B^* \in \mathcal{I}(\sigma, \pi_1^E \circ$

ϕ), it follows that Ran B^* is invariant under $\pi_1^E(\phi(A))$. We conclude that $\hat{Z}^*$ is in $\mathcal{I}(\pi_1^E \circ \phi, \theta_1^E \circ \phi)$, and hence, that $\hat{Z} \in \mathcal{I}(\theta_1^E \circ \phi, \pi_1^E \circ \phi)$.

We have reached an impasse. Note that there have been no degrees of freedom in the construction up to this point (we took care of any ambiguities in the choice of $\hat{Z}$ by assuming Ran $\hat{Z} \subseteq$ Inn B). We would like to apply Rieffel's equivalence theorem to conclude that $\hat{Z} = I_E \otimes Z$ for an operator $Z \in \mathcal{I}(\theta_1, \pi_1)$, but this is possible if and only if $\hat{Z} \in \mathcal{I}(\theta_1^E, \pi_1^E)$. Retracing our steps, we see that $\hat{Z} \in \mathcal{I}(\theta_1^E, \pi_1^E)$ if and only if $B^* \in \mathcal{I}(\sigma \circ \tau, \pi_1^E)$. However, the latter intertwining condition is only true if $\mathfrak{z} \in \mathcal{I}(\sigma^E, \sigma \circ \tau)$; our original assumption that $\mathfrak{z} \in \mathcal{I}(\sigma^E \circ \phi, \sigma)$ is not sufficient. This is indicative of the difficulty in lifting the commutant of a contractive intertwining pair $(\mathfrak{z}, \sigma)$ to the level of its minimal partially isometric extension adapted to τ. However, it also suggests that if we assume $\mathfrak{z} \in \mathcal{I}(\sigma^E, \sigma \circ \tau)$, then there is hope for lifting the commutant. Of course, a different choice of $\hat{Z}$ might allow us to conclude that $\hat{Z} \in \mathcal{I}(\theta_1^E, \pi_1^E)$ without assuming $\mathfrak{z} \in \mathcal{I}(\sigma^E, \sigma \circ \tau)$, but then we lose the uniqueness condition on $\hat{Z}$ and the construction becomes ad hoc. While imposing the additional condition that $\mathfrak{z} \in \mathcal{I}(\sigma^E, \sigma \circ \tau)$ eliminates the bijective correspondence among contractive intertwining pairs associated to (E, A, H_σ), completely contractive covariant representations of (E, A) on H_σ, and completely contractive representations of $\mathcal{T}_+(E)$ on H_σ, our primary focus throughout has been on contractive intertwining pairs. Hence, a commutant lifting theorem in the context of a *strong contractive intertwining pair* is still of interest.

6.2 Strong Intertwining Pairs

Definition 6.2. *A* strong intertwining pair with respect to τ*, associated to* (E, A, H_σ)*, is a pair* $(\mathfrak{z}, \sigma)$ *consisting of a representation* $\sigma : A \to B(H_\sigma)$ *and an operator* $\mathfrak{z} : E \otimes_\sigma H_\sigma \to H_\sigma$ *in the intertwining space* $\mathcal{I}(\sigma^E, \sigma \circ \tau)$*.*

Generally we work only with a single, fixed transfer operator τ*, in which case there should be no cause for confusion if we just call* $(\mathfrak{z}, \sigma)$ *a* strong intertwining pair*.*

Note that the terminology is sensible since every strong intertwining pair is necessarily an intertwining pair. Indeed, if $(\mathfrak{z}, \sigma)$ is a strong intertwining pair (with respect to τ), then

$$\mathfrak{z}\sigma^E(\phi(a)) = \sigma(\tau(\phi(a))\mathfrak{z} = \sigma(a)\mathfrak{z}.$$

Thus, if $\mathfrak{z}$ is also contractive, we can build the minimal partially isometric extension $(\mathfrak{w}_1, \pi_1)$ and the minimal coisometric extension $(\mathfrak{w}_\infty, \pi_\infty)$ of $(\mathfrak{z}, \sigma)$ adapted to τ, as in Theorems 4.13 and 5.2, respectively. Moreover, we observe that being a strong intertwining pair carries over to the level of partially isometric and coisometric extensions. More precisely, we have the following corollaries to Theorems 4.13 and 5.2.

Theorem 6.3. *Let* $(\mathfrak{z}, \sigma)$ *be a contractive intertwining pair associated to* (E, A, H_σ)*. Also, let* $(\mathfrak{w}_1, \pi_1)$ *and* $(\mathfrak{w}_\infty, \pi_\infty)$ *denote the minimal partially isometric and the minimal coisometric extensions of* $(\mathfrak{z}, \sigma)$ *adapted to* τ*, respectively. Then* $(\mathfrak{z}, \sigma)$ *is a strong intertwining pair if and only if* $(\mathfrak{w}_1, \pi_1)$ *is a strong intertwining pair if and only if* $(\mathfrak{w}_\infty, \pi_\infty)$ *is a strong intertwining pair.*

Proof. First, note that if $(\mathfrak{w}, \pi)$, acting on H_π, is an extension of $(\mathfrak{z}, \sigma)$, i.e., if H_σ is a reducing subspace for $(\mathfrak{w}, \pi)$ with $\mathfrak{w}|_{E\otimes_\sigma H_\sigma} = \mathfrak{z}$ and $\pi(\cdot)|_{H_\sigma} = \sigma(\cdot)$, then $\pi^E(\cdot)|_{E\otimes_\sigma H_\sigma} = \sigma^E(\cdot)$. Therefore, if $(\mathfrak{w}, \pi)$ is a strong intertwining pair, then for any $X \in \mathcal{L}(E)$,

$$\mathfrak{z}\sigma^E(X) = \mathfrak{w}\pi^E(X)|_{E\otimes_\sigma H_\sigma} = \pi(\tau(X))\mathfrak{w}|_{E\otimes_\sigma H_\sigma} = \sigma(\tau(X))\mathfrak{z}.$$

That is, $(\mathfrak{z}, \sigma)$ is a strong intertwining pair as well.

On the other hand, assume $(\mathfrak{z}, \sigma)$ is a strong intertwining pair. Let $(\mathfrak{w}_1, \pi_1)$, acting on H_{π_1}, denote the minimal partially isometric extension of $(\mathfrak{z}, \sigma)$ adapted to τ. Recall that $H_{\pi_1} = H_\sigma \oplus H_{\theta_1}$, and that with respect to this direct sum decomposition of H_{π_1}, $\pi_1 = \sigma \oplus \theta_1$ and

$$\mathfrak{w}_1 = \begin{bmatrix} \mathfrak{z} & D_{1*} \\ 0 & 0 \end{bmatrix},$$

where $D_{1*} \in \mathcal{I}(\theta_1^E, \sigma \circ \tau)$. Therefore, for any $X \in \mathcal{L}(E)$,

$$\begin{aligned}
\mathfrak{w}_1\pi_1^E(X) &= \begin{bmatrix} \mathfrak{z} & D_{1*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma^E(X) & 0 \\ 0 & \theta_1^E(X) \end{bmatrix} = \begin{bmatrix} \mathfrak{z}\sigma^E(X) & D_{1*}\theta_1^E(X) \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sigma(\tau(X))\mathfrak{z} & \sigma(\tau(X))D_{1*} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma(\tau(X)) & 0 \\ 0 & \theta_1(\tau(X)) \end{bmatrix} \begin{bmatrix} \mathfrak{z} & D_{1*} \\ 0 & 0 \end{bmatrix} \\
&= \pi_1(\tau(X))\mathfrak{w}_1,
\end{aligned}$$

showing that $\mathfrak{w}_1 \in \mathcal{I}(\pi_1^E, \pi_1 \circ \tau)$. In other words, $(\mathfrak{w}_1, \pi_1)$ is a strong intertwining pair with respect to τ.

Finally, let $(\mathfrak{w}_\infty, \pi_\infty)$, acting on H_{π_∞}, be the minimal coisometric extension of $(\mathfrak{z}, \sigma)$ adapted to τ. By virtue of $(\mathfrak{w}_\infty, \pi_\infty)$ being an extension of $(\mathfrak{z}, \sigma)$, H_{π_∞} has a direct sum decomposition $H_\sigma \oplus \widehat{H}_\infty$. Further, under this decomposition, we can write $\pi_\infty = \sigma \oplus \hat{\pi}_\infty$ and

$$\mathfrak{w}_\infty = \begin{bmatrix} \mathfrak{z} & D_\infty \\ 0 & S_\infty \end{bmatrix},$$

where $D_\infty \in \mathcal{I}(\hat{\pi}^E_\infty, \sigma \circ \tau)$ and $S_\infty \in \mathcal{I}(\hat{\pi}^E_\infty, \hat{\pi}_\infty \circ \tau)$. As a result, for any $X \in \mathcal{L}(E)$, we see that

$$\begin{aligned}
\mathfrak{w}_\infty \pi^E_\infty(X) &= \begin{bmatrix} \mathfrak{z} & D_\infty \\ 0 & S_\infty \end{bmatrix} \begin{bmatrix} \sigma^E(X) & 0 \\ 0 & \hat{\pi}^E_\infty(X) \end{bmatrix} = \begin{bmatrix} \mathfrak{z}\sigma^E(X) & D_\infty \hat{\pi}^E_\infty(X) \\ 0 & S_\infty \hat{\pi}^E_\infty(X) \end{bmatrix} \\
&= \begin{bmatrix} \sigma(\tau(X))\mathfrak{z} & \sigma(\tau(X))D_\infty \\ 0 & \hat{\pi}_\infty(\tau(X))S_\infty \end{bmatrix} = \begin{bmatrix} \sigma(\tau(X)) & 0 \\ 0 & \hat{\pi}_\infty(\tau(X)) \end{bmatrix} \begin{bmatrix} \mathfrak{z} & D_\infty \\ 0 & S_\infty \end{bmatrix} \\
&= \pi_\infty(\tau(X))\mathfrak{w}_\infty.
\end{aligned}$$

Therefore, $\mathfrak{w}_\infty \in \mathcal{I}(\pi^E_\infty . \pi_\infty \circ \tau)$, i.e., $(\mathfrak{w}_\infty, \pi_\infty)$ is a strong intertwining pair with respect to τ.

$\square$

We are now in a position to prove our commutant lifting theorem for partially isometric extensions. As a reminder, we assume that $\tau : \mathcal{L}(E) \to A$ is a fixed transfer operator throughout.

Theorem 6.4 (Commutant Lifting for Partially Isometric Extensions). *Let $(\mathfrak{z}, \sigma)$ be a strong contractive intertwining pair associated to (E, A, H_σ), and let $(\mathfrak{w}_1, \pi_1)$ denote the minimal partially isometric extension of $(\mathfrak{z}, \sigma)$ adapted to τ. If $X \in B(H_\sigma)$ commutes with $(\mathfrak{z}, \sigma)$, then there is an operator $X_1 \in B(H_{\pi_1})$ such that*

1. $X_1 H_\sigma \subseteq H_\sigma$ *and* $X_1|_{H_\sigma} = X$,

2. $||X_1|| = ||X||$, *and*

3. X_1 *commutes with* $(\mathfrak{w}_1, \pi_1)$.

Proof. We have shown that X_1 extends X, preserves the norm of X, and commutes with $(\mathfrak{w}_1, \pi_1)$ if and only if

$$X_1 = \begin{bmatrix} X & D_{X^*} C D_S \\ 0 & S \end{bmatrix},$$

where $C : H_{\theta_1} \to H_\sigma$ is a contraction in the intertwining space $\mathcal{I}(\theta_1, \sigma)$ and $S \in \theta_1(A)'$.

As before, $(XD_{1*})(XD_{1*})^* \leq BB^*$, where $B = \begin{bmatrix} \mathfrak{z}(I_E \otimes D_{X*}) & D_{1*} \end{bmatrix}$. Hence, Theorem 6.1 yields a contraction $\hat{Z} : E \otimes_{\theta_1} H_{\theta_1} \to E \otimes_{\pi_1} H_{\pi_1}$ such that $B\hat{Z} = XD_{1*}$. Previously, we showed that $B^* \in \mathcal{I}(\sigma, \pi_1^E \circ \phi)$. However, the fact that $(\mathfrak{z}, \sigma)$ is a strong intertwining pair allows us to conclude more, namely, $B^* \in \mathcal{I}(\sigma \circ \tau, \pi_1^E)$. Indeed, for any $Y \in \mathcal{L}(E)$,

$$\begin{aligned}
\pi_1^E(Y)B^* &= \begin{bmatrix} \sigma^E(Y) & 0 \\ 0 & \theta_1^E(Y) \end{bmatrix} \begin{bmatrix} (I_E \otimes D_{X*})\mathfrak{z}^* \\ D_{1*}^* \end{bmatrix} \\
&= \begin{bmatrix} \sigma^E(Y)(I_E \otimes D_{X*})\mathfrak{z}^* \\ \theta_1^E(Y) D_{1*}^* \end{bmatrix} \\
&= \begin{bmatrix} (I_E \otimes D_{X*})\mathfrak{z}^* \sigma(a) \\ D_{1*}^* \sigma(a) \end{bmatrix} \\
&= \begin{bmatrix} (I_E \otimes D_{X*})\mathfrak{z}^* \\ D_{1*}^* \end{bmatrix} \sigma(a) \\
&= B^* \sigma(a).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{Z}^* \pi_1^E(Y) B^* &= \hat{Z}^* B^* \sigma(a) = (B\hat{Z})^* \sigma(a) = (X D_{1*})^* \sigma(a) \\
&= D_{1*}^* X^* \sigma(a) = D_{1*}^* \sigma(a) X^* = \theta_1^E(Y) D_{1*}^* X^* \\
&= \theta_1^E(Y) \hat{Z}^* B^*.
\end{aligned}$$

As we did above, we set $\hat{Z}^* = 0$ on $(\text{Ran } B^*)^\perp$, and Theorem 6.1 then ensures that $\hat{Z}$ is the unique operator satisfying $B\hat{Z} = XD_{1*}$ and $\operatorname{Ran} \hat{Z} \subseteq \operatorname{Inn} B$. Since $B^* \in \mathcal{I}(\sigma \circ \tau, \pi_1^E)$, Ran B^* is invariant under $\pi_1^E(\mathcal{L}(E))$, and we conclude that $\hat{Z}^*$ is in $\mathcal{I}(\pi_1^E, \theta_1^E)$, so that $\hat{Z} \in \mathcal{I}(\theta_1^E, \pi_1^E)$.

Rieffel's equivalence theorem [16, Theorem 6.23] then implies that there is a unique operator $Z : H_{\theta_1} \to H_{\pi_1}$ such that $||Z|| \leq 1$, $Z \in \mathcal{I}(\theta_1, \pi_1)$, and $\hat{Z} = I_E \otimes Z$.

The fact that Z is a map of H_{θ_1} into $H_{\pi_1} = H_\sigma \oplus H_{\theta_1}$ means that we can write

$$Z = \begin{bmatrix} R \\ S \end{bmatrix},$$

for operators $R : H_{\theta_1} \to H_\sigma$ and $S : H_{\theta_1} \to H_{\theta_1}$. Furthermore, we claim that the intertwining condition $Z \in \mathcal{I}(\theta_1, \pi_1)$ implies that $R \in \mathcal{I}(\theta_1, \sigma)$ and $S \in \theta_1(A)'$. Indeed, on the one hand, for any $a \in A$,

$$Z\theta_1(a) = \begin{bmatrix} R \\ S \end{bmatrix} \theta_1(a) = \begin{bmatrix} R\theta_1(a) \\ S\theta_1(a) \end{bmatrix}$$

while on the other,

$$\pi_1(a)Z = \begin{bmatrix} \sigma(a) & 0 \\ 0 & \theta_1(a) \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix} = \begin{bmatrix} \sigma(a)R \\ \theta_1(a)S \end{bmatrix},$$

showing that $R \in \mathcal{I}(\theta_1, \sigma)$ and $S \in \theta_1(A)'$. Since Z is contractive we have

$$R^*R + S^*S = Z^*Z \leq I_{H_{\theta_1}}.$$

Therefore, $||S|| \leq 1$ and $R^*R \leq I_{H_{\theta_1}} - S^*S = D_S^* D_S$ (recall that D_S denotes the defect operator for S). Another application of Theorem 6.1 (this time to the operators R^* and $D_S^* = D_S$) yields a contraction $C^* : H_\sigma \to H_{\theta_1}$ such that $R^* = D_S C^*$, and hence, such that $R = CD_S$.

Finally, the operator $X_1 : H_{\pi_1} \to H_{\pi_1}$ that we are seeking is defined by the operator matrix

$$X_1 = \begin{bmatrix} X & D_{X*}CD_S \\ 0 & S \end{bmatrix}.$$

We have already seen that this formula guarantees that H_σ is invariant under X_1 with $X_1|_{H_\sigma} = X$ and that $||X_1|| = ||X||$. Recalling that $CD_S = R \in \mathcal{I}(\theta_1, \sigma)$ and that $S \in \theta_1(A)'$, we have

$$\begin{aligned} X_1\pi_1(a) &= \begin{bmatrix} X & D_{X*}CD_S \\ 0 & S \end{bmatrix} \begin{bmatrix} \sigma(a) & 0 \\ 0 & \theta_1(a) \end{bmatrix} = \begin{bmatrix} X\sigma(a) & D_{X*}CD_S\,\theta_1(a) \\ 0 & S\theta_1(a) \end{bmatrix} \\ &= \begin{bmatrix} \sigma(a)X & D_{X*}\sigma(a)(CD_S) \\ 0 & \theta_1(a)S \end{bmatrix} = \begin{bmatrix} \sigma(a)X & \sigma(a)D_{X*}CD_S \\ 0 & \theta_1(a)S \end{bmatrix} \\ &= \begin{bmatrix} \sigma(a) & 0 \\ 0 & \theta_1(a) \end{bmatrix} \begin{bmatrix} X & D_{X*}CD_S \\ 0 & S \end{bmatrix} \\ &= \pi_1(a)X_1. \end{aligned}$$

Hence, $X_1 \in \pi_1(A)'$, and it only remains to show that $X_1\mathfrak{w}_1 = \mathfrak{w}_1(I_E \otimes X_1)$. To that end, recall that $B\hat{Z} = XD_{1*}$, from which it follows that

$$\begin{aligned} XD_{1*} = B(I_E \otimes Z) &= \begin{bmatrix} \mathfrak{z}(I_E \otimes D_{X*}) & D_{1*} \end{bmatrix} \begin{bmatrix} I_E \otimes R \\ I_E \otimes S \end{bmatrix} \\ &= \mathfrak{z}(I_E \otimes D_{X*})(I_E \otimes CD_S) + D_{1*}(I_E \otimes S) \\ &= \mathfrak{z}(I_E \otimes D_{X*}CD_S) + D_{1*}(I_E \otimes S). \end{aligned}$$

We conclude that

$$
\begin{aligned}
X_1\mathfrak{w}_1 &= \begin{bmatrix} X & D_{X*}CD_S \\ 0 & S \end{bmatrix} \begin{bmatrix} \mathfrak{z} & D_{1*} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X\mathfrak{z} & XD_{1*} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathfrak{z}(I_E \otimes X) & \mathfrak{z}(I_E \otimes D_{X*}CD_S) + D_{1*}(I_E \otimes S) \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathfrak{z} & D_{1*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_E \otimes X & I_E \otimes D_{X*}CD_S \\ 0 & I_E \otimes S \end{bmatrix} \\
&= \mathfrak{w}_1(I_E \otimes X_1).
\end{aligned}
$$

Therefore $X_1\mathfrak{w}_1 = \mathfrak{w}_1(I_E \otimes X)$, and the proof is complete.

$\square$

CHAPTER 7

COMMUTANT LIFTING FOR COISOMETRIC EXTENSIONS

In this chapter we extend the commutant lifting theorem from Chapter 6 to the level of coisometric extensions of contractive intertwining pairs. Toward that end, let E be a C^*-correspondence over a C^*-algebra A, let $\phi : A \to \mathcal{L}(E)$ denote the left-action map, and assume that ϕ has a smooth unital generalized transfer operator $\tau : \mathcal{L}(E) \to A$. As in Chapter 6, the starting point for our analysis is a *strong* contractive intertwining pair $(\mathfrak{z}, \sigma)$ with respect to τ. Recall that this means that $\mathfrak{z}$ is a contractive operator in the intertwining space $\mathcal{I}(\sigma^E, \sigma \circ \tau) \subseteq \mathcal{I}(\sigma^E \circ \phi, \sigma)$. In much the same way as in the construction of the minimal coisometric extension $(\mathfrak{w}_\infty, \pi_\infty)$ (of $(\mathfrak{z}, \sigma)$) adapted to τ, we proceed inductively, with Theorem 6.4 being the first step in an iterative process.

Theorem 7.1 (Commutant Lifting for Coisometric Extensions)**.** *Let $(\mathfrak{z}, \sigma)$ be a strong contractive intertwining pair associated to (E, A, H_σ), and let $(\mathfrak{w}_\infty, \pi_\infty)$, acting on H_{π_∞}, denote the minimal coisometric extension of $(\mathfrak{z}, \sigma)$ adapted to τ (the existence and uniqueness of which is guaranteed by Theorems 5.2 and 5.3).*

Given any $X \in B(H_\sigma)$ that commutes with $(\mathfrak{z}, \sigma)$, there is an operator $X_\infty \in B(H_{\pi_\infty})$ satisfying the following conditions,

1. $X_\infty H_\sigma \subseteq H_\sigma$ *and* $X_\infty|_{H_\sigma} = X$,

2. $||X_\infty|| = ||X||$, *and*

3. X_∞ *commutes with* $(\mathfrak{w}_\infty, \pi_\infty)$.

Proof. Recall that $(\mathfrak{w}_n, \pi_n)$ is the minimal partially isometric extension of $(\mathfrak{w}_{n-1}, \pi_{n-1})$ adapted to τ, for each $n \geq 1$, where $H_{\pi_0} = H_\sigma$, $\pi_0 = \sigma$, and $\mathfrak{w}_0 = \mathfrak{z}$. Therefore for each $n \geq 1$, Theorem 6.4 guarantees the existence of an operator $X_n \in B(H_{\pi_n})$ satisfying the following properties,

$$X_n H_\sigma \subseteq H_\sigma, \quad X_n|_{H_\sigma} = X, \quad ||X_n|| = ||X||, \quad \text{and} \quad X_n \text{ commutes with } (\mathfrak{w}_n, \pi_n).$$

Recall that $\{H_{\pi_n}\}_{n\geq 0}$ and $\{E \otimes_{\pi_n} H_{\pi_n}\}_{n\geq 0}$ are nested sequences of Hilbert spaces, and that $\bigcup_{n\geq 0} H_{\pi_n}$ is a dense subspace of H_{π_∞}. Set $X_0 = X$, and view $\{X_n\}_{n\geq 0}$ as a sequence of operators on H_{π_∞} by setting $X_n h = 0$ for $h \in H_{\pi_\infty} \ominus H_{\pi_n}$.

We claim that the sequence $\{X_n h\}_{n\geq 0}$ is strongly Cauchy for every element $h \in \bigcup_{n=0}^\infty H_{\pi_n}$. Indeed, $X_n|_{H_{\pi_m}} = X_m$ for $m \leq n$, so for any $h \in \bigcup_{n=0}^\infty H_{\pi_n}$, we can always find sufficiently large N so that $X_n h - X_m h = 0$ for $n, m \geq N$. In conjunction with the fact that $||X_n|| = ||X||$ for all $n \geq 1$, this implies that the sequence $\{X_n\}_{n\geq 0}$ converges strongly to an operator $X_\infty \in B(H_{\pi_\infty})$ (the inductive limit of the sequence of operators $\{X_n\}$). From the properties of each X_n, it follows that $X_\infty H_\sigma \subseteq H_\sigma$, $X_\infty|_{H_\sigma} = X$, and $||X_\infty|| = ||X||$.

Viewing $\mathfrak{w}_n(\xi)$ and $\pi_n(a)$, $n \geq 0$, $\xi \in E$, $a \in A$, as operators on H_{π_∞} in the natural way, we see that X_∞ commutes with each $(\mathfrak{w}_n, \pi_n)$, since $X_\infty|_{H_{\pi_n}} = X_n$ for all $n \geq 0$. Therefore, since $\mathfrak{w}_\infty(\xi)$ and $\pi_\infty(a)$ are the SOT-limits of the sequences $\{\mathfrak{w}_n(\xi)\}_{n\geq 0}$ and $\{\pi_n(a)\}_{n\geq 0}$, respectively, for all $\xi \in E$ and all $a \in A$, we conclude that X_∞ commutes with $(\mathfrak{w}_\infty, \pi_\infty)$.

$\square$

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