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Generalized factorization in commutative rings with zero-divisors

Christopher Park Mooney
University of Iowa

GENERALIZED FACTORIZATION IN COMMUTATIVE RINGS WITH ZERO-DIVISORS

by

Christopher Park Mooney

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Thesis Supervisor: Professor Daniel D. Anderson

Graduate College
The University of Iowa
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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Christopher Park Mooney

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the August 2013 graduation.

Thesis Committee: ______________________
Daniel Anderson, Thesis Supervisor

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Frauke Bleher

____
Victor Camillo

____
Charles Frohman

____
Paul Muhly

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ABSTRACT

The study of factorization in integral domains has a long history. Unique factorization domains, like the integers, have been studied extensively for many years. More recently, mathematicians have turned their attention to generalizations of this such as Dedekind domains or other domains which have weaker factorization properties. Many authors have sought to generalize the notion of factorization in domains. One particular method which has encapsulated many of the generalizations into a single study is that of τ-factorization, studied extensively by A. Frazier and D.D. Anderson.

Another generalization comes in the form of studying factorization in rings with zero-divisors. Factorization gets quite complicated when zero-divisors are present due to the existence of several types of associate relations as well as several choices about what to consider the irreducible elements.

In this thesis, we investigate several methods for extending the theory of τ-factorization into rings with zero-divisors. We investigate several methods including: 1) the approach used by A.G. Ağargün and D.D. Anderson, S. Chun and S. Valdes-Leon in several papers; 2) the method of U-factorization developed by C.R. Fletcher and extended by M. Axtell, J. Stickles, and N. Baeth and 3) the method of regular factorizations and 4) the method of complete factorizations.

This thesis synthesizes the work done in the theory of generalized factorization and factorization in rings with zero-divisors. Along the way, we encounter several nice applications of the factorization theory. Using τ_z-factorizations, we discover a nice

relationship with zero-divisor graphs studied by I. Beck as well as D.D. Anderson, D.F. Anderson, A. Frazier, A. Lauve, and P. Livingston. Using τ-U-factorization, we are able to answer many questions that arise when discussing direct products of rings.

There are several benefits to the regular factorization factorization approach due to the various notions of associate and irreducible coinciding on regular elements greatly simplifying many of the finite factorization property relationships. Complete factorization is a very natural and effective approach taken to studying factorization in rings with zero-divisors. There are several nice results stemming from extending τ-factorization in this way. Lastly, an appendix is provided in which several examples of rings satisfying the various finite factorization properties studied throughout the thesis are given.

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CHAPTER 1
INTRODUCTION AND BACKGROUND

Factorization has played an important part in the theory of algebra for quite some time. Unique factorization domains (UFDs) have played a central role in the development of algebra. They have been studied extensively by many authors including but not limited to P.M. Cohn, [23], R.M. Fossum, [26], and P. Samuel, [36, 37, 38]. Since then many have turned their attention towards generalizations of UFDs either in the form of Dedekind domains or else by weakening the factorization properties. This has given rise to the notion of half factorization domains studied extensively by S. Chapman and J. Coykendall in [21]. D.D. Anderson, D.F. Anderson, M. Zafrullah and others have looked at further generalizations coming in the form of bounded factorization domains, finite factorization domains, irreducible divisor finite domains, atomic domains, domains which satisfy the ascending chain condition on principal ideals, see the survey article [3]. These will be defined more formally in Section 1.1.

In more recent times, attention has been turned to several generalizations of irreducible factorizations. One that was especially successful is the co-maximal factorization studied by S. McAdam and R. Swan in [33]. There has been much work at studying elasticity of domains and variations in lengths of irreducible factorizations, especially by A. Geroldinger and F. Halter-Koch which has been surveyed in [28]. A recent survey article in 2011 by D.D. Anderson and A. Frazier, [6], has developed the notion of τ-factorization which collects much of this theory into a single study.

Another direction to take the study of factorization which has been pursued by many authors is to look outside of domains and towards rings with zero-divisors. There are many approaches that authors have taken to study this problem. We have, for instance, various definitions of unique factorization ring given by A. Bouvier, D.D. Anderson and Markanda [9, 10], M. Billis [16], C.R. Fletcher [24, 25] and Galovich [27]. There is a more general approach by D.D. Anderson and S. Valdes-Leon in [8, 1] and S. Chun in [5] which subsumes much of the previous theory into a single study. More recently, the method of U-factorization originated by C.R. Fletcher has been revisited and extended out of unique factorization rings, to a more general class of rings by M. Axtell, S. Forman, N. Roersma, and J. Stickles in [13, 14].

The purpose of this dissertation has been to study how one can effectively use the τ-factorization techniques in commutative rings with zero-divisors. There are several approaches investigated here, and much of the thesis draws from results in [34, 35]. We begin here in Chapter 1 with much of the preliminary definitions and results from the aforementioned research. This will set us up to proceed into many new factorization results concerning generalized factorization theory in commutative rings with zero-divisors.

In Chapter 2, we are ready to study τ-factorization in rings with zero-divisors. In this chapter, we provide many definitions: we define various τ-irreducible elements as well as rings satisfying various τ-finite factorization properties. Here we present many nice theorems which illustrate relationships between the τ-irreducible elements and rings satisfying the τ-finite factorization properties. In Chapter 3, we investigate a particular choice of a relation τ, namely τ_z and τ_z^{Δ}. Of particular interest is the close relationship between τ_z and τ_z^Δ-factorizations and zero-divisor graphs. By using many of the results from the theory of zero-divisor graphs, we get several nice theorems about rings which satisfy τ_z and τ_z^Δ-finite factorization properties. Much of the work here in these two chapters stems from the research done in [34] to appear in the *Houston Journal of Mathematics.*

In Chapter 4, we investigate how to extend τ-factorization by way of U-factorization developed by C.R. Fletcher in [24, 25]. This will come in the form of what has been termed τ-U-factorization. We begin with many definitions of various τ-U-irreducible factorizations as well as rings which satisfy τ-U-finite factorization properties. We see this is an excellent way to deal with idempotent elements and find that a much larger class of rings will satisfy the τ-U-finite factorization properties using this approach. In Chapter 5, we look at an application of the τ-U-factorizations. They are extremely helpful when it comes to studying direct products of rings. We show that we are able to get similar results as in the usual case of factorization in rings with zero-divisors across direct products to go through with τ-U-factorizations. Much of the work here stems from the research coming from [35] to appear in the *Rocky Mountain Journal of Mathematics.*

In Chapter 6, we investigate τ-regular factorizations. This provides another possible approach to extending τ-factorization to rings with zero-divisors. We see that focusing on regular elements greatly simplifies many of the diagrams given previously because all the associate relations and notions of τ-irreducible coincide on regular elements. We provide several definitions and prove many nice theorems about the relationship between rings satisfying various τ-regular finite factorization conditions. In Chapter 7, we focus on yet another possible way to study τ-factorization in rings with zero-divisors. This is the method of τ-complete factorizations which was introduced in D.D. Anderson and A. Frazier [6]. The extension is quite natural when discussing rings with zero-divisors. We provide many definitions of τ-complete finite factorization properties rings may have. We then proceed to prove several theorems illustrating the relationship between rings satisfying the τ-complete factorization properties as well as comparing with the τ-finite factorization properties laid forth previously in Chapter 2.

Appendix A and Appendix B serve as a collection of examples from the literature. These examples demonstrate that many of the arrows from various theorems are not reversible. This list is by no means exhaustive, but I have found it useful to have some rings in mind when thinking about these theorems. Many of the nice examples that I have come across have been included here, while I do not claim to always indicate the *first* person to have presented the example, I have tried my best to give credit where it is due.

1.1 Preliminary Definitions and Notation

We will let R be a commutative ring with 1 such that $1 \neq 0$. We will use D to denote an integral domain, a commutative ring such that $ab = 0$ implies $a = 0$ or $b = 0$. We will often refer to integral domains by simply domains. We use $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ for the natural numbers, integers, rationals, real numbers, and complex numbers respectively. Let $R^* = R - \{0\}$, the non-zero elements of the ring R. We will use $U(R)$ to denote the set of invertible elements of the ring R, that is $U(R) = \{a \in R \mid \exists b \in R \text{ with } ab = 1\}$. We will use $R^\# = R^* - U(R)$ for the non-zero, non-units of R.

Let D be a domain. For $a, b \in D$, we say a and b are *associate* if $(a) = (b)$, written $a \sim b$. An element $a \in D^\#$ is *irreducible* or *an atom* if $a = bc$ implies that $a \sim b$ or $a \sim c$. The classical situation is when every non-zero, non-unit element of D has a factorization into irreducible elements and this factorization is unique up to associates and rearrangement. This is a *unique factorization domain (UFD).*

We take a moment to define some of the weaker factorization conditions which may be placed on a domain and are summarized nicely in a survey article by D.D. Anderson, D.F. Anderson, and M. Zafrullah [3]. Following Cohn [22], we say that D is *atomic* if each non-zero non-unit of D is a product of a finite number of irreducible elements (atoms) of D. We say that D satisfies the *ascending chain condition on principal ideals (ACCP)* if there does not exist an infinite strictly ascending chain of principal ideals of D. This is a weaker form of the Noetherian condition on all ideals. The domain D is a *bounded factorization domain (BFD)* if D is atomic and for each non-zero non-unit of D there is a bound on the length of factorizations into products of irreducible elements. We say that D is a *half-factorial domain (HFD)* if D is atomic and each factorization of a non-zero non-unit of D into a product of irreducible elements has the same length. This concept was introduced by Zaks in [39]. The domain D is an *idf-domain* (for irreducible-divisor-finite) if each non-zero non-unit element of D has at most a finite number of non-associate irreducible divisors. They were introduced by Grams and Warner in [30]. It has been shown that atomic idf-domains are precisely the domains in which each non-zero non-unit has only a finite number of non-associate divisors (and hence, only a finite number of factorizations up to order and associates). We will call such a domain a *finite factorization domain (FFD)*.

In general, we have the following diagram from [3] which illustrates the relative strengths of the various properties.

$$\begin{tikzcd}
 & \text{HFD} \arrow[dr, Rightarrow] & & & \\
\text{UFD} \arrow[ur, Rightarrow] \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{FFD} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{BFD} \arrow[r, Rightarrow] & \text{ACCP} \arrow[r, Rightarrow] & \text{atomic} \\
 & \text{idf-domain} & & &
\end{tikzcd}$$

Examples are given in [3] which show no more arrows can be drawn. Many of these examples can also be found in Appendix A and B.

1.2 τ-Factorization in Domains

We follow the notation of D.D. Anderson and A. Frazier [6]. Let $a \in D^{\#}$, then by a factorization we mean $a = \lambda a_1 \cdots a_n$ where $\lambda \in U(D)$ and $a_i \in D^{\#}$ for $1 \leq i \leq n$. Let τ be a relation on $D^{\#}$. For our purposes in this thesis, we will typically assume τ is symmetric, but this is not necessary and we will provide several examples of interesting relations τ which are not symmetric. We give several examples of possible relations τ in Section 2.1. For $a \in D^{\#}$, a τ-*factorization* is a factorization

$a = \lambda a_1 \cdots a_n$ such that $a_i \tau a_j$ for all $i \neq j$. We refer to each a_i as a *τ-factor of a* and say that a_i *τ-divides* a, written $a_i \mid_\tau a$. We call $a \in D^\#$ *τ-irreducible* or a *τ-atom* if $a = \lambda(\lambda^{-1}a)$ are the only τ-factorizations of a. We will say D is *τ-atomic* if each $a \in D^\#$ has a τ-factorization into τ-irreducibles.

They proceed to define τ-factorization properties for integral domains.

Definition 1.1. Given a domain D and a relation τ.

1. D is said to be a τ-UFD if (1) D is τ-atomic and (2) given $a \in D^\#$, given two τ-atomic factorizations of $a = \lambda a_1 \cdots a_n = \mu b_1 \cdots b_m$, $m = n$ and there is a rearrangement of the τ-factors so that $a_i \sim b_i$ for each $1 \leq i \leq n$.

2. D is said to be a τ-HFD if (1) D is τ-atomic and (2) given $a \in D^\#$, any two τ-atomic factorizations of $a = \lambda a_1 \cdots a_n = \mu b_1 \cdots b_m$, $m = n$.

3. D is said to be a τ-FFD if every $a \in D^\#$ has a finite number of τ-factorizations up to rearrangement and associate.

4. D is said to be a τ-idf-domain if every $a \in D^\#$ has a finite number of τ-atomic τ-divisors up to associate.

5. D is said to be a τ-BFD if for every $a \in D^\#$, there is a natural number $N(a)$ such that if $a = \lambda a_1 \cdots a_n$ is a τ-factorization, then $n \leq N(a)$.

6. D is said to satisfy τ-ACCP if for each infinite sequence $\{a_i\}_{n=1}^\infty$ of elements of $D^\#$ with $a_{i+1} \mid_\tau a_i$ for each $i \geq 1$, there is an N (depending on the sequence) such that $a_{k+1} \sim a_k$ for each $k \geq N$.

They then proceed to prove the following diagram [6, Figure 2]

$$\begin{tikzcd}
& \tau\text{-HFD} \arrow[dr, Rightarrow] & & & \\
\tau\text{-UFD} \arrow[ur, Rightarrow] \arrow[r, Rightarrow] & \tau\text{-FFD} \arrow[r, Rightarrow] & \tau\text{-BFD} \arrow[r, Rightarrow, "\star"] & \tau\text{-ACCP} \arrow[r, Rightarrow, "\star"] & \tau\text{-atomic} \\
\text{UFD} \arrow[u, Rightarrow, "\star"] & \text{FFD} \arrow[u, Rightarrow, "\star"] & \text{BFD} \arrow[u, Rightarrow, "\star"] & \text{ACCP} \arrow[u, Rightarrow] &
\end{tikzcd}$$

Where $\star$ represents τ being divisive.

1.3 Factorization in Rings with Zero-Divisors

We now turn our attention towards some of the work that has been done on factorization in commutative rings with zero-divisors. We let R be a commutative ring with identity. Let $R^* = R - \{0\}$, $U(R)$ be the units of R, and $R^\# = R^* - U(R)$, the non-zero, non-units of R. As in [8], we let $a \sim b$ if $(a) = (b)$, $a \approx b$ if there exists $\lambda \in U(R)$ such that $a = \lambda b$, and $a \cong b$ if (1) $a \sim b$ and (2) $a = b = 0$ or if $a = rb$ for some $r \in R$ then $r \in U(R)$. As in [4], a ring R is said to be *strongly associate* (resp. *very strongly associate*) ring if for any $a, b \in R$, $a \sim b$ implies $a \approx b$ (resp. $a \cong b$). Each of these associate relations which were equivalent in the case of an integral domain can be different with zero-divisors present. This leads us several choices for different types of irreducible, all of which in a domain would coincide, yet in rings with zero-divisors are distinct.

Definition 1.2. Let R be a commutative ring with $1 \neq 0$. Given a non-zero, non-unit a,

1. If given any factorization, $a = \lambda a_1 \cdots a_n$ implies $a \sim a_i$ for some i, then a is said to be *irreducible* or an *atom*.

2. If given any factorization, $a = \lambda a_1 \cdots a_n$ implies $a \approx a_i$ for some i, then a is said to be *strongly irreducible* or a *strong atom*.

3. If $a = rb$ implies $r \in U(R)$, then a is said to be *m-irreducible* or *m-atomic*. Note: this condition is equivalent to a being maximal among principal ideals, explaining the name.

4. If given any factorization, $a = \lambda a_1 \cdots a_n$ implies $a \cong a_i$ for some i, then a is said to be *very strongly irreducible* or *very strongly atomic*.

Example 1.1. Examples to show that each type of irreducible is distinct.

- $(1, 0) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is m-irreducible, but not very strongly irreducible.

  Notice that $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}/((1, 0)) \cong \mathbb{Z}/2\mathbb{Z}$, a field, so $((1, 0))$ is actually a maximal ideal, not just maximal among principal ideals. On the other hand, $(1, 0) = (1, 0)(1, 0)$ shows that $(1, 0)$ is *not* very strongly irreducible since $(1, 0)$ is not a unit.

- $(1, 0) \in \mathbb{Z} \times \mathbb{Z}$ is strongly irreducible, but not m-irreducible.

  We consider $(1, 0) \subsetneq (1, 2)$, so it is not maximal among principal ideals. On the other hand, it is easy to see why it must be strongly irreducible.

- $3 \in \mathbb{Z}[\sqrt{-5}]$ is irreducible, but not prime.

  $3^2 = (2+\sqrt{-5})(2-\sqrt{-5})$, $3 \mid 9$; however 3 does not divide either factor, showing 3 is not prime. One can use norms to check that 3 is indeed irreducible.

- $x \in \frac{F[X,Y,Z]}{(X-XYZ)}$ (where x denotes the image of X under the canonical surjection) is prime and therefore irreducible, but not strongly irreducible.

Consider $\frac{F[X,Y,Z]}{(X-XYZ)}/\frac{X}{(X-XYZ)} \cong F[Y,Z]$ which is a domain, making x prime as claimed. On the other hand, $x = yz$ yet one can see that neither y nor z is strongly associate to x.

$\boxplus$

This leads to several analogous definitions for various finite factorization properties in commutative rings with zero-divisors with various choices of associate and irreducible.

Definition 1.3. Given a ring R, let $\alpha \in \{$very strongly atomic, m-atomic, strongly atomic, atomic$\}$ and $\beta \in \{$ very strongly associate, strongly associate, associate $\}$.

1. R is said to be α if every $a \in R^{\#}$ has a factorization $a = \lambda a_1 \cdots a_n$ such that each a_i is α.

2. R is said to satisfy ACCP if for each infinite sequence $\{a_i\}_{n=1}^{\infty}$ of elements of $R^{\#}$ with $a_{i+1} \mid_\tau a_i$ for each $i \geq 1$, there is an N (depending on the sequence) such that $a_{k+1} \sim a_k$ for each $k \geq N$.

3. R is said to be a α-β-UFR if (1) R is α and (2) given $a \in R^{\#}$, given two α-factorizations of $a = \lambda a_1 \cdots a_n = \mu b_1 \cdots b_m$, $m = n$ and there is a rearrangement of the τ-factors so that a_i and b_i are β for each $1 \leq i \leq n$.

4. R is said to be a α-HFR if (1) R is α and (2) given $a \in R^{\#}$, any two α-factorizations of $a = \lambda a_1 \cdots a_n = \mu b_1 \cdots b_m$, $m = n$.

5. R is said to be a β-FFR if every $a \in R^{\#}$ has a finite number of factorizations up to rearrangement and up to β.

6. R is said to be a β-WFFR if every $a \in R^{\#}$ has a finite number of divisors up to β.

7. R is said to be a β-α-df-ring if every $a \in R^{\#}$ has a finite number of α divisors up to β.

8. R is said to be a BFR if for every $a \in R^{\#}$, there is a natural number $N(a)$ such that if $a = \lambda a_1 \cdots a_n$ is a factorization, then $n \leq N(a)$.

This results in culminating in the following diagram in [8]:

$$\begin{tikzcd}
 & \alpha\text{-HFR} \arrow[dr, Rightarrow] & & & \\
\alpha\text{-}\beta\text{-UFR} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \text{BFR} \arrow[r, Rightarrow] & \text{ACCP} \arrow[r, Rightarrow] & \alpha \\
 & \beta\text{-FFR} \arrow[ur, Rightarrow] \arrow[r, Rightarrow] & \beta\text{-WFFR} \arrow[r, Rightarrow] & \alpha\text{-}\beta\text{-idf ring} &
\end{tikzcd}$$

$$\alpha \in \{\text{very strongly atomic, m-atomic, strongly atomic, atomic}\}$$

$$\beta \in \{\text{very strongly associate, strongly associate, associate}\}.$$

CHAPTER 2
τ-FACTORIZATION IN RINGS WITH ZERO-DIVISORS

In this chapter we will define τ-factorization in rings with zero-divisors. We define our new τ-irreducible elements as well as rings satisfying τ-finite factorization properties. We proceed to prove several theorems describing the relationships between the various irreducible elements as well as rings with τ-finite factorization properties. Many of the main results from Chapter 2 and Chapter 3 stem from my article to appear in the *Houston Journal of Mathematics* [34].

2.1 τ-Factorization Definitions

For a non-unit $a \in R$, we define $a = \lambda a_1 \cdots a_n$, $\lambda \in U(R)$, $a_i \in R^{\#}$ to be a *τ-factorization of* a if $a_i \tau a_j$ for each $i \neq j$. We call $a = \lambda(\lambda^{-1}a)$ a *trivial τ-factorization of* a. We say that a is a *τ-product* of the a_i's and that a_i is a *τ-factor* or a *τ-divisor* of a. We do not allow 0 to occur as a τ-factor of a non-trivial τ-factorization; however, we do allow the trivial factorization, $0 = \lambda 0$ for $\lambda \in U(R)$. For $a, b \in R^{\#}$ we say that a *τ-divides* b, written $a \mid_\tau b$, if a occurs as a τ-factor in some τ-factorization of b. Note that if $a = \lambda a_1 \cdots a_n$ is a τ-factorization, then for $\sigma \in S_n$, the symmetric group on n letters, so is each rearrangement of $a = \lambda a_{\sigma(1)} \cdots a_{\sigma(n)}$ because τ is assumed to be symmetric.

A *τ-refinement* of a τ-factorization $\lambda a_1 \cdots a_n$ is a τ-factorization of the form

$$\lambda \cdot b_{1_1} \cdots b_{1_{m_1}} \cdot b_{2_1} \cdots b_{2_{m_2}} \cdots b_{n_1} \cdots b_{n_{m_n}}$$

where $a_i = b_{i_1} \cdots b_{i_{m_i}}$ is a τ-factorization for each i. We say that τ is *refinable* if every τ-refinement of a τ-factorization is a τ-factorization. We say τ is *combinable* if whenever $\lambda a_1 \cdots a_n$ is a τ-factorization, then so is each $\lambda a_1 \cdots a_{i-1}(a_i a_{i+1})a_{i+2} \cdots a_n$.

We pause briefly to give some examples of particular relations τ.

Example 2.1. Let R be a commutative ring with 1.

1. $\tau = R^{\#} \times R^{\#}$. This yields the usual factorizations in R and $|_\tau$ is the same as the usual divides. τ is multiplicative and divisive (hence associate preserving as we shall soon see).

2. $\tau = \emptyset$. For every $a \in R^{\#}$, there is only the trivial factorization and $a \mid_\tau b \Leftrightarrow a = \lambda b$ for $\lambda \in U(R) \Leftrightarrow a \approx b$. Again τ is both multiplicative and divisive (vacuously).

3. Let S be a nonempty subset of $R^{\#}$ and let $\tau = S \times S$, $a \tau b \Leftrightarrow a, b \in S$. So τ is multiplicative (resp. divisive) if and only if S is multiplicatively closed (resp. closed under non-unit factors). A non-trivial τ-factorization is up to unit factors a factorization into elements from S.

4. Let I be an ideal of R and define $a \tau b$ if and only if $a - b \in I$. This relation is certainly symmetric, but need not be multiplicative or divisive. Let $R = \mathbb{Z}$ and $I = (5)$. Consider $7 \tau 2$ and $7 \tau 7$, but $7 \not\tau 14$, and $9 \tau 4$, but $2 \mid 4$ yet $9 \not\tau 2$.

5. Let $\star$ be a star-operation on R and define $a \tau b \Leftrightarrow (a, b)^\star = R$, that is a and b are $\star$-coprime or $\star$-comaximal. This particular operation has been studied more in depth by J. Juett in [31].

6. Suppose that $\leq$ is a transitive order on $R^{\#}$. Define $a\tau b \Leftrightarrow a \leq b$. Then an ordered τ-factorization $a = \lambda a_1 \cdot \ldots \cdot a_n$ is just a factorization where $\lambda \in U(R)$ and $a_i \in R^{\#}$ with $a_1 \leq \cdot \ldots \cdot \leq a_n$. For example for $R[x]$ we can define $f \leq g \Leftrightarrow deg(f) \leq deg(g)$ and so an ordered τ-factorization is simply a factorization into polynomials of ascending degree.

7. Let $a\tau_n b \Leftrightarrow ab \neq 0$. Notice in a domain, this is precisely $\tau = D^{\#} \times D^{\#}$, so this example is different only for rings with zero-divisors. τ is divisive, but not multiplicative. Let $a\tau b$ and $a' \mid a$ and $b' \mid b$, say $a's = a$ and $b't = b$. Then $a'sb't = ab \neq 0$, so certainly $a'b' \neq 0$ so $a'\tau b'$ as desired. On the other hand, in $\mathbb{Z}/12\mathbb{Z}$ we have $2\tau 2$ and $2\tau 3$, but $2 \not\tau 6$.

8. Let $a\tau_z b \Leftrightarrow ab = 0$. Then every $a \in R^{\#}$ is a τ-atom. The only nontrivial τ-factorizations are $0 = \lambda a_1 \cdot \ldots \cdot a_n$ where $a_i \cdot a_j = 0$ for all $i \neq j$. This example will be studied more in depth in Chapter 3.

9. Let $a\tau b \Leftrightarrow a, b \in \text{Reg}(R)$. Then this gives us the regular factorization studied in [1]. This too will be studied more in depth Chapter 6.

10. Let $\tau \subseteq R^{\#} \times R^{\#}$, then we define $\tau_{reg} := \tau \cap (Reg(R) \times Reg(R))$. Then this makes every zero-divisor have only trivial τ_{reg}-factorizations. We then focus on factoring regular elements so the various associate relations coincide. This type of factorization will have much in common with the generalization of τ-factorization using regular factorizations as in [1]. This will be studied more in depth in Chapter 6.

$\boxplus$

From this point on, we will assume that τ is symmetric unless specifically mentioned otherwise. In the case in which *tau* is not symmetric, we will refer to these τ-factorizations, in accordance with [6], as τ_{ord}-factorizations.

Lemma 2.1. *Let R be a commutative ring and let $a, b \in R^{\#}$.*

(1) $a \cong b \Rightarrow a \approx b \Rightarrow a \sim b$.

(2) $\sim$ and $\approx$ are equivalence relations.

(3) $\cong$ need only be transitive and symmetric.

(4) For $a \in R$, the following are equivalent.

(a) $a \sim b$ for some $b \in R$ implies $a \cong b$.

(b) $a \cong a$.

(c) $a = 0$ or $ann(a) \subseteq J(R)$.

If a satisfies one of the above conditions, then for $\lambda \in U(R)$, $a \cong b \Leftrightarrow a \cong \lambda b$.

(5) The following conditions are equivalent.

(a) R is very strongly associate.

(b) R is présimplifiable (for all $x, y \in R$, $xy = x$ implies $x = 0$ or $y \in U(R)$).

(c) $\cong$ is reflexive on R.

(d) $\cong$ is an equivalence relation on R.

(e) $\sim$, $\approx$, and $\cong$ all coincide on R.

In particular, domains and quasi-local rings all satisfy the above conditions.

Proof. See [8, Theorem 2.2] and the discussion preceding the theorem. $\square$

The following theorem is a slight generalization of [6, Proposition 2.2].

Theorem 2.2. *Let R be a commutative ring and τ a relation on $R^{\#}$. Let $a, b, b' \in R^{\#}$, $\lambda \in U(R)$.*

(1) If τ is divisive, then τ is associate (resp. strongly associate, very strongly associate) preserving.

(2) If τ is divisive, then τ is refinable.

(3) If τ is multiplicative, then τ is combinable.

Proof. (1) Let $a\tau b$. Now $b \sim b' \Rightarrow b \mid b'$ and $b' \mid b$. So by the definition of divisive, $a\tau b \Rightarrow a\tau b'$ and $a\tau b' \Rightarrow a\tau b$. As $b \cong b'$ and $b \approx b'$ each imply $b \sim b'$ by Lemma 2.1, the result follows. Proofs of (2) and (3) can be found in [6]. $\square$

2.2 Types of τ-Irreducible Elements

We would like to define what it means for an element to be τ-irreducible in a ring with zero-divisors. This definition needs to be consistent with the definitions of τ-irreducible in domains as well as the various types of irreducible elements when zero-divisors are present. These definitions are generalizations of those given in [8].

Proposition 2.3. *Let R be a commutative ring and τ a relation on $R^{\#}$ with $a \in R$ a non-unit and $\lambda \in U(R)$. Consider the following statements.*

(1) $a = \lambda a_1 \cdots a_n$, $n \in \mathbb{N}$, is a τ-factorization implies $a \sim a_i$ for some $1 \leq i \leq n$.

(2) $(a) = (a_1) \cdots (a_n)$, $n \in \mathbb{N}$, with $a_i \tau a_j$ for all $i \neq j$ implies $(a) = (a_i)$ for some $1 \leq i \leq n$.

(3) $a \sim a_1 \cdots a_n$, $n \in \mathbb{N}$, with $a_i \tau a_j$ for all $i \neq j$ implies $a \sim a_i$ for some $1 \leq i \leq n$.

We have (2) $\Leftrightarrow$ (3) $\Rightarrow$ (1). If R is strongly associate, we also have (1) $\Rightarrow$ (2).

Proof. (2) $\Leftrightarrow$ (3) are seen to be equivalent after noting that

$$(a_1)(a_2)\cdots(a_n) = (a_1 \cdot a_2 \cdots a_n)$$

.

(2) $\Rightarrow$ (1) If $a = \lambda a_1 \cdots a_n$ is a τ-factorization, then

$$(a) = (a_1 \cdots a_n) = (a_1)\cdots(a_n)$$

with $a_i \tau a_j$ for all $i \neq j$, so by (2) we have $(a) = (a_i)$ for some $1 \leq i \leq n$.

We now assume R is strongly associate and show (1) $\Rightarrow$ (2). Let $(a) = (a_1)\cdots(a_n)$ with $a_i \tau a_j$ for $i \neq j$, then $a \sim a_1 \cdots a_n$ implies there exists a $\lambda \in U(R)$ with $a = \lambda a_1 \cdots a_n$ a τ-factorization, so by (1) we have $a \sim a_i$ for some i. □

We will call a non-unit $a \in R$ *τ-irreducible* or *τ-atomic* if it satisfies condition (1) of Proposition 2.3.

Theorem 2.4. *A strong associate of a τ-irreducible element is τ-irreducible.*

Proof. Let a be a τ-irreducible element. Suppose $a = \lambda a'$ for $\lambda \in U(R)$. Let $a' = \mu b_1 \cdots b_n$ be a τ-factorization. Then $a = \lambda a' = (\lambda\mu) b_1 \cdots b_n$ is a τ-factorization and a is τ-irreducible, so $a \sim b_i$ for some $1 \leq i \leq n$. We have $a \approx a' \Rightarrow a \sim a'$, so $a' \sim a \sim b_i$ which shows a' is τ-irreducible. □

Proposition 2.5. *Let R be a commutative ring and τ a relation on $R^{\#}$. For $a \in R$, a non-unit and $\lambda \in U(R)$, the following are equivalent.*

(1) $a = \lambda a_1 \cdots a_n$, $n \in \mathbb{N}$, *with* $a_i \tau a_j$ *for* $i \neq j$ *implies* $a \approx a_i$ *for some* i.

(2) $a \approx a_1 \cdots a_n$, $n \in \mathbb{N}$, *with* $a_i \tau a_j$ *for* $i \neq j$ *implies* $a \approx a_i$ *for some* i.

Proof. This is immediate from definitions. □

We will call a non-unit element $a \in R$ *τ-strongly irreducible* or *τ-strongly atomic* if a satisfies one of the conditions of Proposition 2.5.

Theorem 2.6. *A strong associate of a τ-strongly irreducible element is τ-strongly irreducible.*

Proof. Let $a' \approx a$ with a τ-strongly irreducible. Suppose $a' \approx a_1 \cdots a_n$, $n \in \mathbb{N}$, with $a_i \tau a_j$ for all $i \neq j$. Then we have $a \approx a' \approx a_1 \cdots a_n$ which implies $a \approx a_i$ for some $1 \leq i \leq n$. Hence $a' \approx a \approx a_i$, showing a' to be τ-strongly irreducible. □

Proposition 2.7. *Let R be a commutative ring and τ a relation on $R^{\#}$. For a non-unit $a \in R$, $\lambda \in U(R)$ we consider the following statements.*

(1) (a) *is maximal in the set* $S' := \{(b) \mid b \in R$, *a non-unit and* $b \mid_\tau a\}$.

(2) $a = \lambda a_1 \cdots a_n$, *a τ-factorization implies* $a \sim a_i$ *for all* i.

(3) $a = \lambda a_1 \cdots a_n$, *a τ-factorization implies* $a \approx a_i$ *for all* i.

Then (3) $\Rightarrow$ (1) $\Leftrightarrow$ (2) and for R strongly associate, (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) Let a satisfy (1) and suppose $a = \lambda a_1 \cdots a_n$ is a τ-factorization. Then $(a) \subseteq (a_i)$, each $a_i \mid_\tau a$, so we have $(a_i) \in S'$ for all i. Hence by maximality of (a) in S', we have $(a) = (a_i)$ as desired. (2) $\Rightarrow$ (1) Suppose a satisfies (2), and we have $(a) \subseteq (b) \in S'$. We have $b \mid_\tau a$. Say $a = \lambda b a_1 \cdots a_n$ is a τ-factorization. By (2) we have $a \sim b$, thus proving (a) is maximal in S' as desired.

(3) $\Rightarrow$ (2) Clear. Furthermore, given R strongly associate it is clear that the converse will also hold since $a \sim a_i \Rightarrow a \approx a_i$. $\square$

We say a non-unit element $a \in R$ is *τ-m-irreducible* or *τ-m-atomic* if a satisfies conditions (1) or (2) of Proposition 2.7.

Theorem 2.8. *A strong associate of a τ-m-irreducible element is τ-m-irreducible.*

Proof. Let a be a τ-m-irreducible element. Suppose $a' \approx a$. Say there is a unit μ in R with $a = \mu a'$. We suppose $a' = \lambda a_1 \cdots a_n$, $n \in \mathbb{N}$, with $a_i \tau a_j$ for all $i \neq j$. Then $a = \mu a' = (\mu\lambda)a_1 \cdots a_n$ remains a τ-factorization. So by (2), $a \sim a_i$ for all $1 \leq i \leq n$. But then we have $a' \sim a \sim a_i$ for all non-units $a_i \in R$, showing a is τ-m-irreducible as desired. $\square$

Proposition 2.9. *Let R be a commutative ring and τ a relation on $R^{\#}$. For a non-unit $a \in R$, $\lambda \in U(R)$, with $a \cong a$, the following are equivalent.*

(1) $a = \lambda a_1 \cdots a_n$, $n \in \mathbb{N}$, with $a_i \tau a_j$ for all $i \neq j$ implies $a \cong a_i$ for some i.

(2) $a \cong a_1 \cdots a_n$, $n \in \mathbb{N}$, with $a_i \tau a_j$ for all $i \neq j$ implies $a \cong a_i$ for some i.

(3) $a \sim a_1 \cdots a_n$, $n \in \mathbb{N}$, with $a_i \tau a_j$ for $i \neq j$ implies $a \sim a_i$ for some i .

(4) a has no non-trivial τ-factorizations.

Proof. (1) $\Rightarrow$ (2) Suppose $a \cong a_1 \cdots a_n$ with $a_i \tau a_j$ for all $i \neq j$. We have $a = \lambda a_1 \cdots a_n$ for some $\lambda \in U(R)$. Thus by (1) $a \cong a_i$ for some i. (2) $\Rightarrow$ (3) Suppose $a \sim a_1 \cdots a_n$ with $a_i \tau a_j$ for all $i \neq j$. We have $a \cong a$, so $a \cong a_1 \cdots a_n$. By (2) $a \cong a_i$ for some i. (3) $\Rightarrow$ (1) Suppose $a = \lambda a_1 \cdots a_n$ with $a_i \tau a_j$ for all $i \neq j$. Then $a \sim a_1 \cdots a_n$. By (3) we have $a \sim a_i$ for some i. Thus we have $a \cong a_i$ for some i, proving the equivalence of (1)-(3).

(1) $\Rightarrow$ (4) Suppose $a = \lambda a_1 \cdots a_n$. By assumption $a \cong a_i$ for some i, say $a_i = \mu a$ for $\mu \in U(R)$. This factorization can be written as $a = \lambda a_1 \cdots \hat{a_i} \cdots a_n \cdot (\mu a)$. But $a \cong a$, which means $a_1 \cdots \hat{a_i} \cdots a_n = \lambda' \in U(R)$, so $n = 1$ and we have the trivial factorization $a = \lambda'(\mu a)$ after all. (4) $\Rightarrow$ (1) The only types of τ-factorizations are the trivial ones, $a = \lambda(\lambda^{-1}a)$ and we have by assumption $a \cong a$, and by Lemma 2.1, $a \cong \lambda^{-1}a$. $\square$

We shall call a non-unit $a \in R$ with $a \cong a$ τ*-very strongly irreducible* or τ*-very strongly atomic* if it satisfies one of the equivalent conditions (1)-(4) of Proposition 2.9.

Theorem 2.10. *A strong associate of a τ-very strongly irreducible element is τ-very strongly irreducible.*

Proof. Let a be τ-very strongly irreducible. Let $a \approx a'$, say $a = \mu a'$ for some $\mu \in U(R)$. Then $a \cong a$ if and only if $a' \cong a'$ by Lemma 2.1. We suppose $a' = \lambda a_1 \cdots a_n$, $n \in \mathbb{N}$, with $a_i \tau a_j$ for all $i \neq j$ and $\lambda \in U(R)$. So we have $a = \mu a' = (\mu\lambda)a_1 \cdots a_n$ which remains a τ-factorization. Since a is τ-very strongly irreducible, we have $a \cong a_i$ for some i. So $a' \cong a \cong a_i$ showing a' is τ-very strongly irreducible. $\square$

Theorem 2.11. *Let R be a commutative ring and τ a relation on $R^\#$. Let $a \in R$ be a non-unit.*

(1) a is τ-very strongly irreducible implies a is τ-m-irreducible.

(2) For R strongly associate, a is τ-m-irreducible implies a is τ-strongly irreducible.

(3) a τ-strongly irreducible implies a is τ-irreducible.

(4) a τ-very strongly irreducible implies a is τ-strongly irreducible.

(5) a τ-m-irreducible implies a is τ-irreducible.

The following diagram summarizes our result († represents a strongly associate ring):

$$\begin{tikzcd}
\tau\textit{-very strongly irred.} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \tau\textit{-strongly irred.} \arrow[r, Rightarrow] & \tau\textit{-irred.} \\
& \tau\textit{-m-irred.} \arrow[u, Rightarrow, "\dagger"'] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Proof. (1) Let a be τ-very strong irreducible, and suppose $(a) \subseteq (a_i) \in S'$. The only τ-factorizations of a are trivial ones. We must have $a = \lambda(\lambda^{-1}a) = \lambda a_i$, that is $a \approx a_i$ and thus $(a) = (a_i)$, proving a is τ-m-irreducible.

(2) Let R be a strongly associate ring, with a, a τ-m-irreducible element. We suppose $a = \lambda a_1 \cdots a_n$ is a τ-factorization. Then $a_i \mid_\tau a$ for each i. But a is τ-m-irreducible, so we have $a \sim a_i$ and hence R strongly associate implies $a \approx a_i$ as desired.

(3) Let a be a τ-strongly irreducible element. Suppose $a = \lambda a_1 \cdots a_n$ is a τ-factorization. Since a is τ-strongly irreducible, $a \approx a_i \Rightarrow a \sim a_i$ for some i, showing a is τ-irreducible as desired.

The proofs of (4) and (5) are immediate from definitions. $\square$

Theorem 2.12. *Let R be a présimplifiable commutative ring and τ a relation on $R^\#$. Then τ-irreducible, τ-strongly irreducible, τ-m-irreducible and τ-very strongly irreducible are equivalent.*

Proof. Let $a \in R$ be a non-unit with a τ-irreducible. If R is présimplifiable, then $a \cong a$ for all $a \in R$. Let $a \cong a_1 \cdots a_n$ with $a_i \tau a_j$ for all $i \neq j$, then $a = \lambda a_1 \cdots a_n$ for some $\lambda \in U(R)$ is a τ-factorization of a. Because a is τ-irreducible, we know $a \sim a_i$ for some i. Therefore $a \cong a_i$ for some i, proving a is τ-very strongly irreducible as desired. □

When R is a domain, all the types of irreducibles coincide and for non-zero elements, our definitions match the τ-irreducible elements defined in [6]. Furthermore, when we set $\tau = R^{\#} \times R^{\#}$, we get the usual factorization in integral domains for non-zero elements. In domains, 0 has no non-trivial factorizations anyway, so this is not much of an impediment.

2.3 τ-Finite Factorization Conditions

Let $\alpha \in$ {atomic, strongly atomic, m-atomic, very strongly atomic}, $\beta \in$ {associate, strong associate, very strong associate} and τ a symmetric relation on $R^{\#}$. Then R is said to be τ-α if every non-unit $a \in R$ has a τ-factorization $a = \lambda a_1 \cdots a_n$ with a_i being τ-α for all $1 \leq i \leq n$. We will call such a factorization a *τ-α-factorization*. We say R satisfies *τ-ACCP* if for every chain $(a_0) \subseteq (a_1) \subseteq \cdots \subseteq (a_i) \subseteq \cdots$ with $a_{i+1} \mid_\tau a_i$, there exists an $N \in \mathbb{N}$ such that $(a_i) = (a_N)$ for all $i > N$.

A ring R is said to be a *τ-α-β-UFR* if (1) R is τ-α and (2) for every non-unit $a \in R$ any two τ-α factorizations $a = \lambda_1 a_1 \cdots a_n = \lambda_2 b_1 \cdots b_m$ have $m = n$ and there is a rearrangement so that a_i and b_i are β. A ring R is said to be a *τ-α-HFR* if (1) R is τ-α and (2) for every non-unit $a \in R$ any two τ-α-factorizations have the same length. A ring R is said to be a τ-*BFR* if for every non-unit $a \in R$, there exists a natural number $N(a)$ such that for any τ-factorization $a = \lambda a_1 \cdots a_n$, $n \leq N(a)$. A ring R is said to be a τ-β-*FFR* if for every non-unit $a \in R$ there are only a finite number of non-trivial τ-factorizations up to rearrangement and β. A ring R is said to be a τ-β-*WFFR* if for every non-unit $a \in R$, there are only finitely many $b \in R$ such that b is a non-trivial τ-divisor of a up to β. A ring R is said to be a τ-α-β-*divisor finite (df)* if for every non-unit $a \in R$, there are only finitely many τ-α τ-divisors of a up to β.

Theorem 2.13. *Let R be a commutative ring and τ a relation on $R^{\#}$. We have the following.*

(1) R is a τ-α-β-UFR implies R is a τ-α-HFR.

(2) For τ refinable and associate preserving R is a τ-α-HFR implies R is a τ-BFR.

(3) For τ refinable and associate preserving, R is a τ-α-β-UFR implies R is a τ-β-FFR .

(4) R is a τ-β-FFR implies R is a τ-BFR.

(5) R is a τ-β-FFR implies R is a τ-β-WFFR and R is a τ-β-WFFR implies R is a τ-α-β-df ring.

(6) For τ-refinable and associate preserving, R is a τ-α-WFFR implies R is a τ-α τ-α-β-df ring.

(7) R is a τ-α τ-α-β-df ring implies R is a τ-α-β-df ring.

(8) For τ refinable and associate preserving, R is a τ-BFR implies R satisfies τ-ACCP.

(9) For τ refinable and associate preserving, R satisfies τ-ACCP implies R is τ-α.

(10) R satisfying ACCP implies R satisfies τ-ACCP.

We have the following diagram ($\star$ represents τ being refinable and associate preserving).

$$\begin{tikzcd}
 & \tau\text{-}\alpha\text{-}HFR \arrow[dr, Rightarrow, "\star"] & & & \\
\tau\text{-}\alpha\text{-}\beta\text{-}UFR \arrow[ur, Rightarrow] \arrow[r, Rightarrow, "\star"] & \tau\text{-}\beta\text{-}FFR \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \tau\text{-}BFR \arrow[r, Rightarrow, "\star"] & \tau\text{-}ACCP \arrow[r, Rightarrow, "\star"] & \tau\text{-}\alpha \\
 & \tau\text{-}\beta\text{-}WFFR \arrow[dl, Rightarrow, "\star"'] \arrow[d, Rightarrow] & & ACCP \arrow[u, Rightarrow] & \\
\tau\text{-}\alpha\ \tau\text{-}\alpha\text{-}\beta\text{-}df\ ring \arrow[r, Rightarrow] & \tau\text{-}\alpha\text{-}\beta\text{-}df\ ring & & &
\end{tikzcd}$$

Proof. (1) Let R be a τ-α-β-UFR. Then we have R is τ-α and every τ-α-factorization of any non-unit $a \in R$ has the same length, so R is a τ-α-HFR.

(2) Let τ be refinable and associate preserving, with R a τ-α-HFR. Let $a \in R$ be a non-unit, and $a = \lambda a_1 \cdots a_n$ be a τ-α-factorization of a. Set $N(a) = n$. Suppose there were a τ-factorization of a, of length $m > n$, $a = \mu b_1 \cdots b_m$. This can be τ-refined into a τ-α-factorization since R is τ-α and τ is refinable and associate preserving. This would lead to a strictly longer τ-α-factorization of a contradicting the fact that R is a τ-α-HFR.

(3) Let R be a τ-α-β-UFR, with τ refinable and associate preserving. Let $a \in R$ be a non-unit. Say $a = \lambda a_1 \cdots a_n$ is the unique τ-α factorization up to rearrangement and β. For any τ-factorization $a = \mu b_1 \cdots b_m$, take the unique τ-α-factorization of each b_i, say $b_i = \mu_i c_{i_1} \cdots c_{i_{mi}}$. We may now refine our τ-factorization to be

$$a = \mu(\mu_1 c_{1_1} \cdots c_{1_{m1}})(\mu_2 c_{2_1} \cdots c_{2_{m2}}) \cdots (\mu_m c_{m_1} \cdots c_{m_{mm}})$$

$$= (\mu\mu_1\mu_2 \cdots \mu_m) c_{1_1} \cdots c_{1_{m1}} c_{2_1} \cdots c_{2_{m2}} \cdots c_{m_1} \cdots c_{m_{mm}}$$

This is a τ-α-factorization of a, so there is a rearrangement such that c_i and a_i are β. This means any τ-factorization of a is simply some grouping of β of the a_i in the original τ-α-factorization of a. There are only 2^n possible ways to do this up to β, so R is a τ-β-FFR.

(4) Let R be a τ-β-FFR, with $a \in R$ a non-unit. There are only finitely many τ-factorizations of a up to β. Simply set $N(a)$ to the maximum length of any of these τ-factorizations.

(5) Let R be a τ-α-FFR with a a non-unit $a \in R$. We collect each of the τ-factors in the finite number of τ-factorizations up to β. This is a complete list of non-trivial τ-divisors of a up to β. Moreover, it is a finite union of finite sets, hence is finite. This proves R is a τ-β-WFFR. Every τ-α-divisor is certainly a τ-divisor, so the second implication is immediate.

(6) Let R be a τ-β-WFFR with τ refinable and associate preserving. We have just seen that R is a τ-α-β-df ring, so we need only show R is τ-α. In light of (9), it suffices to show that R satisfies τ-ACCP. We suppose for a moment there is an infinite ascending chain of properly contained principal ideals $(a_0) \subsetneq (a_1) \subsetneq \cdots$ with $a_{i+1} \mid_\tau a_i$. Say $a_i = \lambda_i a_{i+1} b_{i_1} \cdots b_{i_{n_i}}$ for each i. We must have $n_i \geq 1$ for all i otherwise $(a_i) = (a_{i+1})$. Using the fact that τ is refinable and associate preserving, we know

that we have the following τ-factorizations of a:

$$a_0 = \lambda_0 a_1 b_{0_1} \cdots b_{0_{n_0}} = \lambda_0(\lambda_1 a_2 b_{1_1} \cdots b_{1_{n1}}) b_{01} \cdots b_{0n_0} =$$

$$(\lambda_0\lambda_1\lambda_2) a_3 b_{2_1} \cdots b_{2_{ni}} b_{1_1} \cdots b_{1_{n1}} b_{0_1} \cdots b_{0_{n0}} = \ldots$$

So in particular, for $i > 0$, each a_i is a τ-divisor of a_0. Furthermore, none are even associate, so certainly none are β. Hence a_0 has an infinite number of τ-divisors up to β. This contradicts the hypothesis that R is a τ-α-WFFR.

(7) This is immediate from definitions.

(8) Let τ be refinable and associate preserving and R a τ-BFR. Suppose $(a_0) \subsetneq (a_1) \subsetneq \cdots \subsetneq (a_i) \subsetneq \cdots$ is an infinite chain of properly ascending principal ideals such that $a_{i+1} \mid_\tau a_i$ for each i. Then we use the same factorization as in (6) to see that we get arbitrarily long τ-factorizations of a_0 contradicting the hypothesis.

(9) Suppose R satisfies τ-ACCP, and τ is refinable and associate preserving. Let $a \in R$ be a non-unit. We show a has a τ-α factorization. If a is τ-α, we are done, so we may assume $a = \lambda_1 a_{1_1} \cdots a_{1_{n_1}}$ is a non-trivial τ-factorization with a and a_{1_i} not β for all $1 \leq i \leq n_1$. If all of the a_{i_1} are τ-α, we are done as we have found a τ-α factorization of a. So at least one must not be τ-α, say it is a_{1_1}, so suppose $a_{1_1} = \lambda_2 a_{2_1} \cdots a_{2_{n_2}}$ is a non-trivial τ-factorization with a_{1_1} and a_{2_i} not β for all $1 \leq i \leq n_2$. Then we have $a = (\lambda_1\lambda_2) a_{2_1} \cdots a_{2_{n_2}} a_{1_1} \cdots a_{1_{n_1}}$ is a τ-factorization. We could continue in the fashion picking out one factor that is not τ-α, always just saying it is a_{i_1} after reordering if necessary. This yields an infinite chain of principal ideals $(a) \subsetneq (a_{1_1}) \subsetneq (a_{2_1}) \subsetneq \cdots$ with $a_{i+1_1} \mid_\tau a_{i_1}$ which contradicts R satisfying τ-ACCP.

(10) This is clear by noting that if $a \mid_\tau b$, then $a \mid b$. If R failed to satisfy

τ-ACCP, there would be a properly ascending infinite chain of principal ideals $(a_0) \subsetneq (a_1) \subsetneq \cdots$ with $a_{i+1} \mid_\tau a_i$ also satisfies $a_{i+1} \mid a_i$. Hence we would have an infinite chain of properly ascending principal ideals which contradicts ACCP. $\square$

CHAPTER 3
τ_Z-FACTORIZATIONS AND ZERO-DIVISOR GRAPHS

In this chapter we take an in depth look at a particular relation τ which arises naturally when there are zero-divisors present. We see that studying the τ-factorization properties of rings has an extremely close relationship with the zero-divisor graphs first studied by I. Beck in [15]. There has been much research since then including, but not limited to D.D. Anderson and Naseer in [7], D.F. Anderson and P.S. Livingston in [12] and D.F. A. Frazier, A. Lauve, and P.S. Livingston in [11]. There are many nice results that have been published in the world of zero-divisor graphs, and they have strong impacts when it comes to studying factorizations of this particular type. This chapter seeks to illustrate a more concrete example of τ-factorization in rings with zero-divisors by picking a particular τ relation and classifying many rings which satisfy the various τ-finite factorization properties from Section 2.3.

3.1 The Relation $a\tau_z b \Leftrightarrow ab = 0$

Let $a, b \in R^{\#}$. We will consider the relation τ_z defined by $a\tau_z b$ if and only if $ab = 0$. We will analyze the relation τ_z and investigate rings satisfying the τ_z-finite factorization properties described in Section 2.3.

We observe that with the exception of nilpotent elements, we have a strong correspondence between τ_z-factorizations and the zero-divisor graphs studied first by Beck in [15] and then by several more authors in particular in [7, 11, 12]. The zero-divisor graph, denoted $\Gamma(R)$, is defined to be the graph with vertex set $Z(R) - \{0\}$.

Edges given by the relationship $a, b \in Z(R) - \{0\}$ are adjacent if $ab = 0$. So we see $a\tau_z b \Leftrightarrow ab = 0 \Leftrightarrow a$ and b are adjacent in $\Gamma(R)$ or $a = b$ with $a^2 = 0$. We would like to be able to say $a\tau_z b$ if and only if a and b are adjacent in $\Gamma(R)$.

There are two approaches to ensuring this can be said: (1) insist that our ring R is reduced so there are no non-trivial nilpotent elements or (2) define a modification of τ_z to be $\tau_z^\Delta := \tau_z - \Delta \cap (\mathrm{Nil}(R) \times \mathrm{Nil}(R))$, that is $a\tau_z^\Delta b \Leftrightarrow ab = 0$ and $a \neq b$. Both of these choices result in having no repeated factors in any given τ_z^Δ (τ_z)-factorization (in a reduced ring) which will be useful in several of the proofs.

Theorem 3.1. *Let R be a commutative ring and τ_z and τ_z^Δ be as defined above.*

(1) For $a \in R^\#$, a has only trivial $\tau_z^\Delta(\tau_z)$-factorizations and therefore is a $\tau_z^\Delta(\tau_z)$-atom.

(2) $\tau_z^\Delta(\tau_z)$ is symmetric, but not combinable, and therefore not multiplicative. Furthermore, τ_z^Δ (τ_z) is refinable, but is not divisive.

(3) R satisfies $\tau_z^\Delta(\tau_z)$-ACCP.

(4) R is $\tau_z^\Delta(\tau_z)$-atomic.

(5) If R is an integral domain, then R is a $\tau_z^\Delta(\tau_z)$-atomic-associate-UFR.

(6) τ_z is associate (resp. strongly associate, resp. very strongly associate) preserving, while τ_z^Δ is not.

Proof. (1) Let $a \in R^\#$. Suppose $a = \lambda a_1 \cdots a_n$ is a τ_z^Δ (τ_z)-factorization. If $n \geq 2$, then $a_1 \cdot a_2 = 0$, so $a = 0$, a contradiction. Hence, $n = 1$ and there are only trivial $\tau_z^\Delta(\tau_z)$-factorizations.

(2) $\tau_z^\Delta(\tau_z)$ is clearly symmetric. Let $R = \mathbb{Z}/30\mathbb{Z}$ and consider $0 = 6 \cdot 10 \cdot 15$

is a $\tau_z^{\Delta}(\tau_z)$-factorization, but $0 = 6 \cdot 150 = 6 \cdot 0$ is not a τ_z^{Δ} (τ_z)-factorization. This shows $\tau_z^{\Delta}(\tau_z)$ is not combinable, and hence not multiplicative. Now let $R = \mathbb{Z}/12\mathbb{Z}$, we have $2\tau_z^{\Delta}(\tau_z)6$, but $2 \not\tau_z^{\Delta}(\not\tau_z)3$, so $\tau_z^{\Delta}(\tau_z)$ is not divisive. Every non-trivial τ_z^{Δ} (τ_z)-factor is non-zero and in light of (1) has no non-trivial τ_z^{Δ} (τ_z)-factorizations so τ_z^{Δ} (τ_z) is vacuously refinable.

(3) Let $(a) \subseteq (b)$ with $b \mid_{\tau_z^{\Delta}} a$ $(b \mid_{\tau_z} a)$, say $a = \lambda b b_1 \cdots b_n$ is a τ_z^{Δ} (τ_z)-factorization. If $n \geq 1$, $b\tau_z^{\Delta}(\tau_z)b_1 \Rightarrow a = 0$. If $n = 0$, then $(a) = (b)$. Hence, the longest $\tau_z^{\Delta}(\tau_z)$-ascending chain has length 1.

(4) We have already seen that all non-zero, non-units are τ_z^{Δ} (τ_z)-atoms from (1). If $Z(R) = 0$, then 0 has only trivial factorizations, making it a τ_z^{Δ} (τ_z)-atom. Suppose R is not a domain. Choose an $x \in Z(R)$ such that there is a $y \in R$ such that $xy = 0$ for $x, y \neq 0$ and $x \neq y$. Then $0 = xy$ is a τ_z^{Δ} (τ_z)-atomic factorization of 0. If it is not possible to choose such an x, then $x^2 = 0$ for every $0 \neq x \in Z(R)$. This means 0 itself is a τ_z^{Δ}-atom ($0 = x \cdot x$ is a τ_z-atomic factorization).

(5) If R is a domain, then $Z(R) = 0$ and we have $\tau_z^{\Delta}(\tau_z) = \emptyset$. There are only trivial τ_z^{Δ} (τ_z)-factorizations, so every non-unit is a τ_z^{Δ} (τ_z)-atom, and so R is a τ_z^{Δ} (τ_z)-atomic-associate-UFR.

(6) Suppose $a\tau_z b$, with $a \sim a'$ (resp. $a \approx a'$, $a \cong a'$). Then in all cases, we have $(a) = (a')$ and therefore $a' = ra$ for some $r \in R$. We have $ab = 0$, but by substitution, we have $a'b = (ra)b = r(ab) = 0$, so $a'\tau_z b$ as well. This shows τ_z is associate (resp. strongly associate, very strongly associate) preserving. On the other hand, let $R = \mathbb{Z}/9\mathbb{Z}$. $3 \sim 6$ (resp. $3 \approx 6$, $3 \cong 6$) and $3\tau_z^{\Delta}6$; however, $3 \not\tau_z^{\Delta}3$. Thus τ_z^{Δ}

is not associate (resp. strongly associate, very strongly associate) preserving. □

3.2 Zero-Divisor Graph Results

We begin by stating a theorem which summarizes some results about zero-divisor graphs. We denote the complete graph on r vertices with K^r, and define $\omega(\Gamma(R))$ to be the clique number of $\Gamma(R)$. This is the largest integer $r \geq 1$ with $K^r \subseteq \Gamma(R)$. If $K^r \subseteq \Gamma(R)$ for all $r \geq 1$, then we say $\omega(\Gamma(R)) = \infty$. We use $\min(R)$ to denote the set of minimal prime ideals of R.

Theorem 3.2. *(Zero-divisor graph results) Let R be a commutative ring.*

(1) $\Gamma(R)$ is connected and has diameter less than or equal to 3.

(2) $\Gamma(R)$ is finite if and only if R is a domain or R is finite.

(3) $\omega(\Gamma(R)) = \infty$ if and only if $\Gamma(R)$ has an infinite clique (a complete subgraph).

(4) $\omega(\Gamma(R)) < \infty$ if and only if $|Nil(R)| < \infty$ and $Nil(R)$ is a finite intersection of primes, that is $|min(R)| < \infty$.

(5) All rings with $|\Gamma(R)| \leq 4$ have been classified up to isomorphism.

(6) All finite rings with $|\omega(\Gamma(R))| \leq 3$ have been classified up to isomorphism.

(7) If $R = R_1 \times \cdots \times R_n$ with R_i domains, $n \geq 2$, $\omega(\Gamma(R)) = n$.

Proof. (1) [12, Theorem 2.3]. (2) [12, Theorem 2.2]. (3) and (4) [15, Theorem 3.7]. (5) [12, Example 2.1]. (6) [15, Page 226] and [7, Theorem 4.4]. (7) [11, Theorem 3.7]. □

3.3 τ_z and τ_z^{Δ} Factorization Results

Theorem 3.3. *Let R be a commutative ring. The following are equivalent.*

(1) $|\Gamma(R)| < \infty$.

(2) R is a domain or R is finite.

(3) R is a strong-τ_z^{Δ}-FFR (for every non-unit $a \in R$, there are only a finite number of non-trivial τ_z^{Δ}-factorizations of a).

(4) R is a strong-$\tau_z^{\Delta}(\tau_z)$-WFFR (for every non-unit $a \in R$, there are only a finite number of non-trivial $\tau_z^{\Delta}(\tau_z)$-divisors of a).

(5) R is a strong-$\tau_z^{\Delta}(\tau_z)$-atomic-divisor finite-ring (for every non-unit $a \in R$, there are only a finite number $\tau_z^{\Delta}(\tau_z)$-divisors which are $\tau_z^{\Delta}(\tau_z)$-atoms).

(6) $\Gamma(R)$ has a finite number of complete subgraphs K^r for $r \geq 2$.

Proof. (1) $\Leftrightarrow$ (2) This is given by Theorem 3.2 (2).

(2) $\Rightarrow$ [(3) and (4)] For R a domain, the result is trivial as every non-unit is a $\tau_z^{\Delta}(\tau_z)$-atom. For R finite, say $|R| = n$ we see that (4) clearly holds as there are only n possible $\tau_z^{\Delta}(\tau_z)$-divisors. Furthermore, because no factor can be repeated in a τ_z^{Δ}-factorization, there are at most 2^n possible τ_z^{Δ}-factorizations.

[(3) or (4)] $\Rightarrow$ (1) Suppose $|\Gamma(R)|$ is infinite. The $\Gamma(R)$ has an infinite number of distinct vertices; say $\{x_i\}_{i=1}^{\infty}$. Recall, $\Gamma(R)$ is connected, so every vertex is adjacent to another distinct vertex, say y_i for each i. Then $\{x_i y_i\}$ is an infinite collection of $\tau_z^{\Delta}(\tau_z)$-factorizations of 0 up to reordering. This contradicts (3). Each x_i is a distinct non-trivial $\tau_z^{\Delta}(\tau_z)$-divisor of 0 which contradicts (4).

(4) $\Leftrightarrow$ (5) Every $\tau_z^{\Delta}(\tau_z)$-divisor is non-zero and hence is $\tau_z^{\Delta}(\tau_z)$-atomic. Every

$\tau_z^{\Delta}(\tau_z)$-atomic $\tau_z^{\Delta}(\tau_z)$-divisor is certainly a $\tau_z^{\Delta}(\tau_z)$-divisor.

(3) $\Rightarrow$ (6) Suppose there were an infinite number of distinct complete subgraphs in $\Gamma(R)$ of size at least 2. Each subgraph corresponds to a distinct non-trivial $\tau_z^{\Delta}(\tau_z)$-factorization of 0 by taking the product of the vertices in the given complete subgraph, contradicting (3).

(6) $\Rightarrow$ (1) Suppose for a moment $|\Gamma(R)|$ were infinite. Let $\{x_i\}_{i=1}^{\infty}$ be an infinite set of distinct vertices. Recall, $\Gamma(R)$ is connected, so every vertex x_i must be adjacent to another vertex, say y_i. Then x_i and y_i form a complete subgraph of size 2, and this generates an infinite collection, contradicting (6). $\square$

Theorem 3.4. *Let R be a commutative ring. The following are equivalent.*

(1) Nil$(R) = 0$ and $\mid \Gamma(R) \mid < \infty$.

(2) R is a strong-τ_z-FFR (for every non-unit $a \in R$, there are only a finite number of non-trivial τ_z-factorizations).

(3) R is a domain or a finite reduced ring.

(4) R is a domain or $R \cong K_1 \times \cdots \times K_n$ with K_i a finite field for $1 \leq i \leq n$ with $n \geq 2$.

(5) Nil$(R) = 0$ and $\Gamma(R)$ has a finite number of complete subgraphs K^r with $r \geq 2$.

Proof. (1) $\Rightarrow$ (2) Suppose there are an infinite number of non-trivial τ_z-factorizations of 0. All τ_z-factors are distinct. If a were a repeated τ_z factor of 0, then $a\tau_z a \Rightarrow a^2 = 0$ implies $0 \neq a \in \mathrm{Nil}(R)$, a contradiction. Hence, there can only be at most $2^{|\Gamma(R)|}$ non-trivial τ_z-factorizations, a contradiction.

(2) $\Rightarrow$ (5) Suppose there were an infinite number of distinct complete subgraphs

in $\Gamma(R)$ of size at least 2. Each such complete subgraph corresponds to a distinct non-trivial τ_z-factorization of 0 contradicting (2). Suppose R were not reduced. Suppose $0 \neq x \in \mathrm{Nil}(R)$, with $x^k = 0$ with k minimal. Then $0 = (x^{k-1})^i$ is a τ_z factorization of 0 of length i for any $i \geq 2$. Hence R must be reduced.

(5) $\Rightarrow$ (1) Suppose there were an infinite number of vertices in $\Gamma(R)$. We recall that $\Gamma(R)$ is connected. We could find paths connecting all the vertices. This would certainly require an infinite number of edges. This yields an infinite number of K^2 subgraphs, contradicting (5).

(1) $\Leftrightarrow$ (3) We have now added the hypothesis that $\mathrm{Nil}(R) = 0$ to both (1) and (2) of Theorem 3.3, so the equivalence remains.

(3) $\Leftrightarrow$ (4) This is well known. $\square$

We now introduce the notion of the associated zero-divisor graph, $\Gamma(R/\sim)$. The vertices are now represented by a zero-divisor up to associate, and an edge between two zero-divisor representatives a and b if $ab = 0$. Recall $\sim$ is an equivalence relation, and one can check the edge relation is well defined. We record two analogous theorems, but omit the proofs. The proofs are nearly identical to those of Theorems 3.3 and 3.4 except now uniqueness is only up to associate and reordering.

Theorem 3.5. *Let R be a commutative ring. The following are equivalent.*

(1) $|\Gamma(R/\sim)| < \infty$ (There are a finite number of zero-divisors up to associate).

(2) R is a τ_z^{Δ}-associate-FFR.

(3) R is a $\tau_z^{\Delta}(\tau_z)$-associate-WFFR.

(4) R is a $\tau_z^{\Delta}(\tau_z)$-atomic-associate-divisor finite ring.

(5) $\Gamma(R/\sim)$ *has a finite number of complete subgraphs* K^r *for* $r \geq 2$.

Theorem 3.6. *Let* R *be a commutative ring. The following are equivalent.*

(1) R *is a* τ_z*-associate-FFR.*

(2) $Nil(R) = 0$ *and* $|\ \Gamma(R/\sim)\ | < \infty$.

(3) $Nil(R) = 0$ *and* $\Gamma(R/\sim)$ *has a finite number of complete subgraphs* K^r *for* $r \geq 2$.

Example 3.1. Consider $R = \mathbb{Z}/4\mathbb{Z}$**.**

We have $\tau_z = \{(2,2)\}$, while $\tau_z^{\Delta} = \emptyset$. $0 = 2^i$ is a τ_z-factorization for all $i \geq 2$, so R is not a (strong-)τ_z-associate-FFR; however, R is certainly a (strong)-τ_z^{Δ}-associate-FFR since there are only trivial factorizations. Hence the items given in Theorems 3.3 and 3.4 (resp. 3.6 and 3.5) cannot be combined and still maintain equivalence. $\boxplus$

Theorem 3.7. *Let* R *be a commutative ring. Then* R *is a* $\tau_z^{\Delta}(\tau_z)$*-BFR if and only if (*R *is reduced and)* $\omega(\Gamma(R))$ *is finite.*

Proof. All $\tau_z^{\Delta}(\tau_z)$-factorizations of non-zero elements are trivial, and hence length 1. Let $0 = \lambda a_1 \cdots a_n$ be a $\tau_z^{\Delta}(\tau_z)$-factorization. Now $a_i a_j = 0$ for all $i \neq j$, (R being reduced tells us) $a_i \neq a_j$ for all $i \neq j$. Hence, every non-trivial $\tau_z^{\Delta}(\tau_z)$-factorization corresponds precisely with a complete subgraph of $\Gamma(R)$.

($\Rightarrow$)If R is a $\tau_z^{\Delta}(\tau_z)$-BFR, then there is a bound on the maximum length of any $\tau_z^{\Delta}(\tau_z)$-factorization of 0, say n. There can be no complete subgraph of size larger than K^n. (Furthermore, suppose $0 \neq x \in \text{Nil}(R)$ with $x^k = 0$, the smallest such integer k, then $0 = (x^{k-1})^i$ for $i \geq 2$ yields arbitrarily long τ_z-factorizations, a contradiction, so R is reduced.)

($\Leftarrow$) Conversely, if we assume $\omega(\Gamma(R)) = n$ (, with R reduced). Then all of the $\tau_z^\Delta(\tau_z)$-factorizations of 0 are bounded by n. All τ_z^Δ (τ_z) factorizations of non-zero elements are of length 1. Hence R is a $\tau_z^\Delta(\tau_z)$-BFR as desired. $\square$

Example 3.2. We can construct a $\tau_z^\Delta(\tau_z)$-BFR that has a factorization of length n and no longer for any $n \geq 1$.

Consider the ring $R = K_1 \times \cdots \times K_n$ with K_i a field for $1 \leq i \leq n$. By (7) of Theorem 3.2, we have $\omega(\Gamma(R)) = n$. This gives us a complete subgraph of size n with vertices $\{x_1, \ldots, x_n\}$ which corresponds to a $\tau_z^\Delta(\tau_z)$-factorization $0 = x_1 \cdots x_n$. There can be no longer factorizations or else there would be a complete subgraph of size larger than n. One can simply take the standard basis $x_i = e_i := (0_{K_1}, \cdots, 1_{K_i}, \cdots, 0_{K_n})$ where the 1 occurs in the i^{th} coordinate for $1 \leq i \leq n$. $\boxplus$

Corollary 3.1. *Let R be a commutative ring. R is a $\tau_z^\Delta(\tau_z)$-BFR if and only if (R is reduced and) Nil(R) is finite and Nil(R) is a finite intersection of prime ideals, i.e. min(R) is finite.*

Proof. This is a consequence of (4) from Theorem 3.2 and the above theorem. (For a reduced ring Nil(R) $= 0$ which is certainly finite.) If min(R) is finite, then $\omega(\Gamma(R))$ is finite, and hence by the above theorem, R is a $\tau_z^\Delta(\tau_z)$-BFR. Conversely, if R is a $\tau_z^\Delta(\tau_z)$-BFR, from the theorem, we know (R must be reduced and) that $\omega(\Gamma(R))$ is finite. Therefore, min(R) is finite proving the claim. $\square$

Corollary 3.2. *Any (reduced) Noetherian ring with Nil(R) finite, or more generally* *any (reduced) ring with Nil(R) finite that satisfies the ascending chain condition on radical ideals is a $\tau_z^{\Delta}(\tau_z)$-BFR.*

Proof. This is a consequence of [32, Theorem 87] and the fact that (R being reduced yields $\sqrt{0} = \mathrm{Nil}(R) = 0$) $\mathrm{Nil}(R)$ is a radical ideal and hence a finite intersection of primes, with $\mathrm{Nil}(R)$ finite. $\square$

Theorem 3.8. *Let R be a commutative ring. Then R is a $\tau_z^{\Delta}(\tau_z)$-atomic-HFR if and only if (R is reduced and) $|\omega(\Gamma(R))| \leq 2$.*

Proof. ($\Rightarrow$) Let R be a $\tau_z^{\Delta}(\tau_z)$-atomic-HFR. (If R is not reduced, it is not even a τ_z-BFR, so this is impossible.) Suppose $|\omega(\Gamma(R))| > 2$, then there is a $K^3 \subset \Gamma(R)$, say $ab = 0$, $ac = 0$ and $bc = 0$ with $a, b, c \in Z(R)$ all distinct. The $\tau_z^{\Delta}(\tau_z)$-factorizations $0 = ab = abc$ show that R cannot be a $\tau_z^{\Delta}(\tau_z)$-atomic-HFR, a contradiction.

($\Leftarrow$) Let $|\omega(\Gamma(R))| \leq 2$. Recall R is always $\tau_z^{\Delta}(\tau_z)$-atomic. All non-zero elements have only trivial $\tau_z^{\Delta}(\tau_z)$-factorizations, and hence have the same length, 1. If $|\omega(\Gamma(R))| = 0$, then R is a domain and R is even a $\tau_z^{\Delta}(\tau_z)$-atomic-associate-UFR. If $|\omega(\Gamma(R))| = 1$, then there is only one possible non-trivial $\tau_z^{\Delta}(\tau_z)$-factor. Our hypothesis implies that any $\tau_z^{\Delta}(\tau_z)$-factorization cannot have a repeated factor. Hence 0 has only trivial $\tau_z^{\Delta}(\tau_z)$-factorizations. If $\omega(\Gamma(R)) = 2$, then 0 is not a $\tau_z^{\Delta}(\tau_z)$-atom, so $\tau_z^{\Delta}(\tau_z)$-atomic-factorizations of 0 must be at least length 2. Furthermore, $|\omega(\Gamma(R))| \leq 2$ implies $\tau_z^{\Delta}(\tau_z)$-factorizations of 0 have length at most 2, proving R is a $\tau_z^{\Delta}(\tau_z)$-atomic-HFR. $\square$

The following lemma is well known, so we omit the proof.

Lemma 3.9. *Let R be a commutative ring. Suppose $(a) = (a^2)$, then there exists $e \in R$ such that $e^2 = e$ and $e \approx a$. Furthermore, R is decomposable, i.e. $R = eR \times (1-e)R = R_1 \times R_2$.*

Theorem 3.10. *Let R be a commutative ring. Then R is a τ_z-atomic-associate-UFR if and only if R is a domain or a direct product of two fields.*

Proof. ($\Rightarrow$) R being a τ_z-atomic-associate-UFR implies R is a τ_z-atomic-HFR, and therefore by the previous result $\omega(\Gamma(R)) \leq 2$. Now $\omega(\Gamma(R)) = 0 \Leftrightarrow R$ is a domain. If $\omega(\Gamma(R)) = 1$, then there is only one non-zero zero-divisor. As before, this is forced to be nilpotent, making R not even a τ_z-atomic-BFR. The only possibility remaining is for $\omega(\Gamma(R)) = 2$. Hence there is a τ_z-factorization of 0 of length 2, say $xy = 0$. R is a τ_z-atomic-associate-UFR, so x and y are the only two τ_z-factors up to associate.

We wish to show that R is decomposable. We cannot have $x^2 = 0$ or $y^2 = 0$ since R must be reduced. All the same, x^2 and y^2 are certainly still zero-divisors. They must be associate to either x or y. If $x^2 \sim x$ we have a non-trivial idempotent element and R is decomposable and we are done. Thus we may assume $x^2 \sim y$ and $y^2 \sim x$. This means $x^2 \mid y$, so certainly $x^2 \mid y^2$, and $y^2 \mid x$. Hence we have $x^2 \mid x$ and therefore $x^2 \sim x$.

In all cases there is a non-trivial idempotent element e and we can write $R = R_1 \times R_2$. As in Lemma 3.9, $x \approx e$, where e is identified with $(1, 0)$. Furthermore, x can be identified with $(\lambda_x, 0)$ where $\lambda_x \in U(R_1)$ and y can be identified with $(0, \lambda_y)$ where $\lambda_y \in U(R_2)$. Let $0 \neq a \in R_1$, and $0 \neq b \in R_2$. We show they must be units. Every element of $R = R_1 \times R_2$ of the form $(a, 0)$ or $(0, b)$ with a, b non-zero is a zero-divisor. They must be associate to either $(\lambda_x, 0)$ or $(\lambda_y, 0)$. This forces $(a) = (\lambda_x) = R_1$ and $(b) = (\lambda_y) = R_2$. Hence we must have $a \in U(R_1)$ and $b \in U(R_2)$ which means R_1 and R_2 are fields as desired.

($\Leftarrow$) For domains this is immediate. If $R = K_1 \times K_2$ for fields K_1, K_2, then the only non-units are of the form $(a, 0)$ and $(0, b)$. So 0 is not a τ_z-atom. The only non-trivial τ_z-factorizations are of the form $(0, 0) = (a, 0)(0, b)$ for $0 \neq a \in K_1$, $0 \neq b \in K_2$. This is the only factorization up to rearrangement and associate, so R is a τ_z-atomic-associate-UFR. □

Theorem 3.11. *Let R be a finite reduced commutative ring. Then R is a τ_z-atomic-associate-UFR if and only if τ_z-atomic-HFR.*

Proof. ($\Rightarrow$) This is always true. ($\Leftarrow$) τ_z-atomic-HFR implies $\omega(\Gamma(R)) \leq 2$. Any finite, reduced ring is of the form $R = K_1 \times \cdots \times K_n$ with K_i finite fields. We recall from Theorem 3.2 (7) that $\omega(\Gamma(K_1 \times \cdots \times K_n)) = n$. So in fact, we must have $R = K_1$ or $R \cong K_1 \times K_2$ for some finite fields K_1, K_2. Both cases are covered by the previous theorem, so R is a τ_z-atomic-associate-UFR. □

Example 3.3. R is a τ_z^{Δ}-HFR does not imply that R is a τ_z^{Δ}-UFR.

Consider the ring $R = K_1 \times \mathbb{Z}/4\mathbb{Z}$ (with K_1 a finite field). Now $\omega(\Gamma(R)) = 2$ by [11, Theorem 3.2], so R is a τ_z^{Δ}-atomic-HFR by the above theorem. However, $(0, 0) = (1, 0)(0, 1) = (1, 2)(0, 2)$ but $(0, 2) \not\sim (1, 0)$ and $(0, 2) \not\sim (0, 1)$, showing there exist non-unique τ_z^{Δ}-factorizations of 0 in this ring. ⊞

CHAPTER 4
τ-U-FACTORIZATION

In this chapter we investigate another method of studying factorization in rings with zero-divisors. The motivation here was to study how to pass τ-factorization properties through direct products of rings to achieve many of the nice theorems in the literature of this type. Of course the main problem is that as soon as there is a decomposable ring, there is a non-trivial idempotent element e with $e^2 = e$. But this yields arbitrarily long factorizations $e = e^n$ for any $n \geq 1$, causing R to not be even a BFR.

There is a nice method which deals with the problems caused by idempotent elements, called U-factorization. This is the method was first developed by C.R. Fletcher in [24, 25]. This method of factorization has been studied extensively by A.G. Ağargün, D.D. Anderson, M. Axtell, N. S. Forman, N. Roersma , J. Stickles, S. Valdez-Leon in [13, 14, 1] and others. In this chapter, we extend τ-factorization to rings with zero-divisors using the method of U-factorization and in so doing synthesize this work done into a single study of what we will call τ-U-factorization.

This will set up the opportunity to study τ-factorization across direct products of rings in Chapter 5 using the methods developed here in Chapter 4. Many of the results in Chapter 4 and Chapter 5 stem from the article [35] to appear in the *Rocky Mountain Journal of Mathematics.*

4.1 U-Factorization Definitions and Background

As in [14], we define U-factorization as follows. Let $a \in R$ be a non-unit. If $a = \lambda a_1 \cdots a_n b_1 \cdots b_m$ is a factorization with $\lambda \in U(R)$, $a_i, b_i \in R^{\#}$, then we will call $a = \lambda a_1 a_2 \cdots a_n \lceil b_1 b_2 \cdots b_m \rceil$ a *U-factorization* of a if (1) $a_i(b_1 \cdots b_m) = (b_1 \cdots b_m)$ for all $1 \leq i \leq n$ and (2) $b_j(b_1 \cdots \widehat{b_j} \cdots b_m) \neq (b_1 \cdots \widehat{b_j} \cdots b_m)$ for $1 \leq j \leq m$ where $\widehat{b_j}$ means b_j is omitted from the product. Here $(b_1 \cdots b_m)$ is the principal ideal generated by $b_1 \cdots b_m$. The b_i's in this particular U-factorization above will be referred to as *essential divisors*. The a_i's in this particular U-factorization above will be referred to as *inessential divisors*. A U-factorization is said to be *trivial* if there is only one essential divisor.

Note: we have added a single unit factor in front with the inessential divisors which was not in M. Axtell's original paper. This is added for consistency with the τ-factorization definitions and it is evident that a unit is always inessential. We allow only one unit factor, so it will not affect any of the finite factorization properties.

Remark 4.1. If $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ is a U-factorization, then for any $1 \leq i_0 \leq m$, we have $(a) = (b_1 \cdots b_m) \subsetneq (b_1 \cdots \widehat{b_{i_0}} \cdots b_m)$. This is immediate from the definition of U-factorization. ⊞

In [13], M. Axtell defines a non-unit a and b to be associate if $(a) = (b)$ and a non-zero non-unit a said to be irreducible if $a = bc$ implies a is associate to b or c. R is commutative ring R to be *U-atomic* if every non-zero non-unit has a U-factorization in which every essential divisor is irreducible. R is said to be a *U-finite factorization ring* if every non-zero non-unit has a finite number of distinct U-factorizations. R

is said to be a *U-bounded factorization ring* if every non-zero non-unit has a bound on the number of essential divisors in any U-factorization. R is said to be a *U-weak finite factorization ring* if every non-zero non-unit has a finite number of non-associate essential divisors. R is said to be a *U-atomic idf-ring* if every non-zero non-unit has a finite number of non-associate irreducible essential divisors. R is said to be a *U-half factorization ring* if R is U-atomic and every U-atomic factorization has the same number of irreducible essential divisors. R is said to be a *U-unique factorization ring* if it is a U-HFR and in addition each U-atomic factorization can be arranged so the essential divisors correspond up to associate. In [14, Theorem 2.1], it is shown this definition of U-UFR is equivalent to the one given by C.R. Fletcher in [24, 25].

4.2 τ-U-Irreducible Elements

A τ-*U-factorization* of a non-unit $a \in R$ is a U-factorization

$$a = \lambda a_1 a_2 \cdots a_n \lceil b_1 b_2 \cdots b_m \rceil$$

for which $\lambda a_1 \cdots a_n b_1 \cdots b_m$ is also a τ-factorization.

Given a symmetric relation τ on $R^{\#}$, we say R is τ-*U-refinable* if for every τ-U-factorization of any non-unit $a \in U(R)$, $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$, any τ-U-factorization of an essential divisors, $b_i = \lambda' c_1 \cdots c_{n'} \lceil d_1 \cdots d_{m'} \rceil$ satisfies

$$a = \lambda\lambda' a_1 \cdots a_n c_1 \cdots c_{n'} \lceil b_1 \cdots b_{i-1} d_1 \cdots d_{m'} b_{i+1} \cdots \ b_m \rceil$$

is a τ-U-factorization.

Example 4.1. Let $R = \mathbb{Z}/20\mathbb{Z}$, and let $\tau = R^{\#} \times R^{\#}$.

Certainly $0 = \lceil 10 \cdot 10 \rceil$ is a τ-U-factorization. But $10 = \lceil 2 \cdot 5 \rceil$ is a τ-U-factorization; however, $0 = \lceil 2 \cdot 5 \cdot 2 \cdot 6 \rceil$ is not a U-factorization since 5 becomes inessential after a τ-U-refinement. It will sometimes be important to ensure the essential divisors of a τ-U-refinement of a τ-U-factorization's essential divisors remain essential. We will see that in a présimplifiable ring, there are no inessential divisors, so for τ-refinable, R will be τ-U-refinable. ⊞

As stated in [13], the primary benefit of looking at U-factorizations is the elimination of troublesome idempotent elements that ruin many of the finite factorization properties. For instance, even $\mathbb{Z}_6$ is not a BFR (a ring in which every non-unit has a bound on the number of non-unit factors in any factorization) because we have $3 = 3^2$. Thus, 3 is an idempotent, so $3 = 3^n$ for all $n \geq 1$ which yields arbitrarily long factorizations. When we use U-factorization, we see any of these factorizations can be rearranged to $3 = 3^{n-1} \lceil 3 \rceil$, which has only one essential divisor.

Let $\alpha \in \{$ irreducible, strongly irreducible, m-irreducible, very strongly irreducible $\}$. Let a be a non-unit. If $a = \lambda a_1 a_2 \cdots a_n \lceil b_1 b_2 \cdots b_m \rceil$ is a τ-U-factorization, then this factorization is said to be a *τ-U-α-factorization* if it is a τ-U-factorization and the essential divisors b_i are τ-α for $1 \leq i \leq m$.

One must be somewhat more careful with U-factorizations as there is a loss of uniqueness in the factorizations. For instance, if we let $R = \mathbb{Z}_6 \times \mathbb{Z}_8$, then we can factor $(3,4)$ as $(3,1) \lceil (3,3)(1,4) \rceil$ or $(3,3) \lceil (3,1)(1,4) \rceil$. On the bright side, we have [1, Proposition 4.1].

Theorem 4.1. *Every factorization can be rearranged into a U-factorization.*

Corollary 4.1. *Let R be a commutative ring and τ a symmetric relation on $R^{\#}$. Let $\alpha \in \{$ irreducible, strongly irreducible, m-irreducible, very strongly irreducible $\}$. For every τ-α factorization of a non-unit $a \in R$, $a = \lambda a_1 \cdots a_n$, we can rearrange this factorization into a τ-U-α-factorization.*

Proof. Let $a = \lambda a_1 \cdots a_n$ be a τ-α-factorization. By Theorem 4.1 we can rearrange this to form a U-factorization. This remains a τ-factorization since τ is assumed to be symmetric. Lastly each a_i is τ-α, so the essential divisors are τ-α. $\square$

This leads us to another equivalent definition of τ-irreducible.

Theorem 4.2. *Let $a \in R$ be a non-unit. Then a is τ-irreducible if and only if any τ-U-factorization of a has only one essential divisor.*

Proof. ($\Rightarrow$) Let a be τ-irreducible. Let $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ be a τ-U-factorization. Suppose $m \geq 2$. Then $a = \lambda a_1 \cdots a_n b_1 \cdots b_m$ is certainly a τ-factorization. This implies $a \sim a_{i_0}$ for some $1 \leq i_0 \leq n$ or $a \sim b_{i_0}$ for some $1 \leq i_0 \leq m$. We have either

$$(a) = (a_1 \cdots a_n b_1 \cdots b_m) \subsetneq (a_1 \cdots a_n \widehat{b_1} b_2 \cdots b_m) \subseteq (a_{i_0}) = (a)$$

or

$$(a) = (a_1 \cdots a_n b_1 \cdots b_m) = (b_1 \cdots b_m) \subsetneq (\widehat{b_1} \cdots \widehat{b_{i_0-1}} \cdot b_{i_0} \cdot \widehat{b_{i_0+1}} \cdots \widehat{b_m}) \subseteq (b_{i_0}) = (a)$$

a contradiction.

($\Leftarrow$) Suppose $a = \lambda a_1 \cdots a_n$. Then this can be rearranged into a U-factorization, and hence a τ-U-factorization. By hypothesis, there can only be one essential divisor.

Suppose it is a_n. We have $a = \lambda a_1 \cdots a_{n-1} \lceil a_n \rceil$ is a τ-U-factorization and $a \sim a_n$ as desired. $\square$

We now define the finite factorization properties using the τ-U-factorization approach. Let $\alpha \in \{$ irreducible, strongly irreducible, m-irreducible, very strongly irreducible $\}$ and let $\beta \in \{$ associate, strongly associate, very strongly associate$\}$. R is said to be *τ-U-α* if for all non-units $a \in R$, there is a τ-U-α-factorization of a. R is said to satisfy *τ-U-ACCP* (ascending chain condition on principal ideals) if every properly ascending chain of principal ideals $(a_1) \subsetneq (a_2) \subsetneq \cdots$ such that a_{i+1} is an essential divisor in some τ-U-factorization of a_i, for each i terminates after finitely many principal ideals. R is said to be a *τ-U-BFR* if for all non-units $a \in R$, there is a bound on the number of essential divisors in any τ-U-factorization of a.

R is said to be a *τ-U-β-FFR* if for all non-units $a \in R$, there are only finitely many τ-U-factorizations up to rearrangement of the essential divisors and β. R is said to be a *τ-U-β-WFFR* if for all non-units $a \in R$, there are only finitely many essential divisors among all τ-U-factorizations of a up to β. R is said to be a *τ-U-α-β-divisor finite (df) ring* if for all non-units $a \in R$, there are only finitely many essential τ-α divisors up to β in the τ-U-factorizations of a.

R is said to be a *τ-U-α-HFR* if R is τ-U-α and for all non-units $a \in R$, the number of essential divisors in any τ-U-α-factorization of a is the same. R is said to be a *τ-U-α-β-UFR* if R is a τ-U-α-HFR and the essential divisors of any two τ-U-α-factorizations can be rearranged to match up to β.

R is said to be *présimplifiable* if for every $x \in R$, $x = xy$ implies $x = 0$ or

$y \in U(R)$. This is a condition which has been well studied and is satisfied by any domain or local ring. We introduce two slight modifications of this. R is said to be *τ-présimplifiable* if for every $x \in R$, the only τ-factorizations of x which contain x as a τ-factor are of the form $x = \lambda x$ for a unit λ. R is said to be *τ-U-présimplifiable* if for every non-zero non-unit $x \in R$, all τ-U-factorizations have no non-unit inessential divisors.

Theorem 4.3. *Let R be a commutative ring with 1 and let τ be a symmetric relation on $R^{\#}$. We have the following.*

(1) If R is présimplifiable, then R is τ-U-présimplifiable.

(2) If R is τ-U-présimplifiable, then R is R is τ-présimplifiable.

That is présimplifiable $\Rightarrow$ τ-U-présimplifiable $\Rightarrow$ τ-présimplifiable. If $\tau = R^{\#} \times R^{\#}$, then all are equivalent.

Proof. (1) Let R be présimplifiable, and $x \in R^{\#}$. Suppose $x = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ is a τ-U-factorization. Then $(x) = (b_1 \cdots b_m)$. R présimplifiable implies that all the associate relations coincide, so in fact $x \cong b_1 \cdots b_m$ implies that $\lambda a_1 \cdots a_n \in U(R)$ and hence all inessential divisors are units.

(2) Let R be τ-U-présimplifiable, and $x \in R$ such that $x = \lambda x a_1 \cdots a_n$ is a τ-factorization. We claim that $x = \lambda a_1 \cdots a_n \lceil x \rceil$ is a τ-U-factorization. For any $1 \leq i \leq n$, $x \mid a_i x$ and $(a_i x)(\lambda a_1 \cdots \widehat{a_i} \cdots a_n) = x$ shows $a_i x \mid x$, proving the claim. This implies $\lambda a_1 \cdots a_n \in U(R)$ as desired.

Let $\tau = R^{\#} \times R^{\#}$ and suppose R is τ-présimplifiable. Suppose $x = xy$, for $x \neq 0$, we show $y \in U(R)$. If $x \in U(R)$, then multiplying through by x^{-1} yields

$1 = x^{-1}x = x^{-1}xy = y$ and $y \in U(R)$ as desired. We may now assume $x \in R^{\#}$. If $y = 0$, then $x = 0$, a contradiction. If $y \in U(R)$ we are already done, so we may assume $y \in R^{\#}$. Thus $x\tau y$, and $x = xy$ is a τ-factorization, so $y \in U(R)$ as desired. □

4.3 τ-U-Finite Factorization Relations

We now would like to show the relationship between rings with various τ-U-α-finite factorization properties as well as compare these rings with the τ-α-finite factorization properties defined in Section 2.3.

Theorem 4.4. *Let R be a commutative ring with 1 and let τ be a symmetric relation on $R^{\#}$. Consider the following statements.*

(1) R is a τ-BFR.

(2) R is τ-présimplifiable and for every non-unit $a_1 \in R$, there is a fixed bound on the length of chains of principal ideals (a_i) ascending from a_1 such that at each stage $a_{i+1} \mid_\tau a_i$.

(3) R is τ-présimplifiable and a τ-U-BFR.

(4) For every non-unit $a \in R$, there are natural numbers $N_1(a)$ and $N_2(a)$ such that if $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ is a τ-U-factorization, then $n \leq N_1(a)$ and $m \leq N_2(a)$.

Then (4) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3). For τ refinable, (1) $\Rightarrow$ (2) and for R τ-U-présimplifiable, (3) $\Rightarrow$ (4). Thus all are equivalent if R is τ-U-présimplifiable and τ is refinable.

Let $\star$ represent τ being refinable, and $\dagger$ represent R being τ-U-présimplifiable, *then the following diagram summarizes the theorem.*

$$\begin{tikzcd}
(1) \arrow[r, Rightarrow, "\star"] & (2) \arrow[d, Rightarrow] \\
(4) \arrow[u, Rightarrow] & (3) \arrow[l, Rightarrow, "\dagger"']
\end{tikzcd}$$

Proof. (1) $\Rightarrow$ (2) Let τ be refinable. Suppose there were a non-trivial τ-factorization $x = \lambda x a_1 \cdots a_n$ with $n \geq 1$. Since τ is assumed to be refinable we can continue to replace the τ-factor x with this factorization.

$$x = \lambda x a_1 \cdots a_n = (\lambda\lambda) x a_1 \cdots a_n a_1 \cdots a_n = \cdots = (\lambda\lambda\lambda) x a_1 \cdots a_n a_1 \cdots a_n a_1 \cdots a_n = \cdots$$

yields an unbounded series of τ-factorizations of increasing length.

Let a_1 be a non-unit in R. Suppose N is the bound on the length of any τ-factorization of a_1. We claim that N satisfies the requirement of (2). Let $(a_1) \subsetneq (a_2) \subsetneq \cdots$ be an ascending chain of principal ideals generated by elements which satisfy $a_{i+1} \mid_\tau a_i$ for each i. Say $a_i = \lambda_i a_{i+1} a_{i1} \cdots a_{in_i}$ for each i. Furthermore, we can assume $n_i \geq 1$ for each i or else the containment would not be proper. Then we can write

$$a_1 = \lambda_1 a_2 a_{11} \cdots a_{1n_1} = \lambda_1 \lambda_2 a_3 a_{21} \cdots a_{2n_2} a_{11} \cdots a_{1n_1} = \cdots .$$

Each remains a τ-factorization since τ is refinable and we have added at least one factor at each step. If the chain were greater than length N we would contradict R being a τ-BFR.

(2) $\Rightarrow$ (3) Let $a \in R$ be a non-unit. Let N be the bound on the length of any properly ascending chain of principle ideals ascending from a such that $a_{i+1} \mid_\tau a_i$. If

$a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ is a τ-U-factorization, then we get an ascending chain with $b_1 \cdots b_{i-1} \mid_\tau b_1 \cdots b_i$ for each i:

$$(a) = (b_1 \cdots b_m) \subsetneq (b_1 \cdots b_{m-1}) \subsetneq (b_1 \cdots b_{m-2}) \subsetneq \cdots \subsetneq (b_1 b_2) \subsetneq (b_1).$$

Hence, $m \leq N$ and we have found a bound on the number of essential divisors in any τ-U-factorization of a, making R a τ-U-BFR.

(3) $\Rightarrow$ (4) Let $a \in R$ be a non-unit. Let $N_e(a)$ be the bound on the number of essential divisors in any τ-U-factorization of a. Since R is τ-U-présimplifiable, there are no inessential τ-U-divisors of a. We can set $N_1(a) = 0$, and $N_2(a) = N_e(a)$ and see that this satisfies the requirements of the theorem.

(4) $\Rightarrow$ (1) Let $a \in R$ be a non-unit. Then any τ-factorization $a = \lambda a_1 \cdots a_n$ can be rearranged into a τ-U-factorization, say $a = \lambda a_{s_1} \cdots a_{s_i} \lceil a_{s_{i+1}} \cdots a_{s_n} \rceil$. But then $n = i + (n - i) \leq N_1(a) + N_2(a)$. Hence the length of any τ-factorization must be less than $N_1(a) + N_2(a)$ proving R is a τ-BFR as desired. $\square$

The way we have defined our finite factorization properties on only the essential divisors causes a slight problem. Given a τ-U-factorization $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$, we only know that $a \sim b_1 \cdots b_m$. This may no longer be a τ-factorization of a, but rather only some associate of a. This is easily remedied by insisting that our rings are strongly associate.

Lemma 4.5. *Let R be a strongly associate ring with τ a symmetric relation on $R^\#$, and let $\alpha \in \{$ irreducible, strongly irreducible, m-irreducible, very strongly irreducible $\}$. Let $a \in R$, a non-unit. If $a = \lambda a_1 a_2 \cdots a_n \lceil b_1 b_2 \cdots b_m \rceil$ is a τ-U-α-factorization,* *then there is a unit $\mu \in U(R)$ such that $a = \mu b_1 \cdots b_m$ is a τ-α-factorization.*

Proof. Let $a = \lambda a_1 a_2 \cdots a_n \lceil b_1 b_2 \cdots b_m \rceil$ be a τ-U-α-factorization. By definition, $(a) = (b_1 \cdots b_m)$, and R strongly associate implies that $a \approx b_1 \cdots b_m$. Let $\mu \in U(R)$ be such that $a = \mu b_1 \cdots b_m$. We still have $b_i \tau b_j$ for all $i \neq j$, and b_i is τ-α for every i. Hence $a = \mu b_1 \cdots b_m$ is the desired τ-factorization, proving the lemma. □

4.4 τ-U-Finite Factorization Property Diagrams

Theorem 4.6. *Let R be a commutative ring with 1, and let τ be a symmetric relation on $R^{\#}$. Let $\alpha \in \{$ irreducible, strongly irreducible, m-irreducible, very strongly irreducible $\}$ and $\beta \in \{$ associate, strongly associate, very strongly associate $\}$. We have the following.*

(1) If R is τ-α, then R is τ-U-α.

(2) If R satisfies τ-ACCP, then R satisfies τ-U-ACCP.

(3) If R is a τ-BFR, then R is a τ-U-BFR.

(4) If R is a τ-β-FFR, then R is a τ-U-β-FFR.

(5) Let R be a τ-β-WFFR, then R is a τ-U-β-WFFR.

(6) Let R be a τ-α-β-divisor finite ring, then R is τ-U-α-β-divisor finite ring.

(7) Let R be a strongly associate τ-α-HFR (resp. τ-α-β-UFR), then R is τ-U-α-HFR (resp. τ-U-α-β-UFR).

Proof. (1) This is immediate from Corollary 4.1.

(2) Suppose there were a infinite properly ascending chain of principal ideals $(a_1) \subsetneq (a_2) \subsetneq \cdots$ such that a_{i+1} is an essential divisor in some τ-U-factorization of a_i,

for each i. Every essential τ-U-divisor is certainly a τ-divisor. This would contradict the fact that R satisfies τ-ACCP.

(3) We suppose that there is a non-unit $a \in R$ with τ-U-factorizations having arbitrarily large numbers of essential τ-U-divisors. Each is certainly a τ-factorization, having at least as many τ-factors as there are essential τ-divisors, so this would contradict the hypothesis.

(4) Every τ-U-factorization is certainly among the τ-factorizations. If the latter is finite, then so is the former.

(5) For any given non-unit $a \in R$, every essential τ-U-divisor of a is certainly a τ-factor of a which has only finitely many up to β. Hence there can be only finitely many essential τ-U-factors up to β.

(6) Let $a \in R$ be a non-unit. Every essential τ-U-α-divisor of a is a τ-α-factor of a. There are only finitely many τ-α-divisors up to β, so then there can be only finitely many τ-U-α-divisors of a up to β.

(7) We have already seen that R being τ-α implies R is τ-U-α. Let $a \in R$ be a non-unit. We suppose for a moment there are two τ-α-U-factorizations:

$$a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil = \lambda' a'_1 \cdots a'_{n'} \lceil b'_1 \cdots b'_{m'} \rceil$$

such that $m \neq m'$ (resp. $m \neq m'$ or there is no rearrangement such that b_i and b'_i are β for each i). Lemma 4.5 implies $\exists \mu, \mu' \in U(R)$ with $a = \mu b_1 \cdots b_m = \mu' b'_1 \cdots b'_{m'}$ are two τ-α-factorizations of a, so $m = m'$ (resp. $m = m'$ and there is a rearrangement so that b_i and b'_i are β for each $1 \leq i \leq m$), a contradiction, proving R is indeed a τ-U-α-HFR (resp. -β-UFR) as desired. $\square$

Theorem 4.7. *Let R be a commutative ring with 1 and τ a symmetric relation on $R^{\#}$. Let $\alpha \in \{$irreducible, strongly irreducible, m-irreducible, very strongly irreducible$\}$, and let $\beta \in \{$associate, strongly associate, very strongly associate$\}$.*

(1) If R is a τ-U-α-β-UFR, then R is a τ-α-U-HFR.

(2) If R is τ-U-refinable and R is a τ-U-α-β-UFR, then R is a τ-U-β-FFR.

(3) If R is τ-U-refinable and R is a τ-U-α-HFR, then R is a τ-U-BFR.

(4) If R is a τ-U-β-FFR, then R is a τ-U-BFR.

(5) If R is a τ-U-β-FFR, then R is a τ-U-β-WFFR.

(6) If R is a τ-U-β-WFFR, then R is a τ-U-α-β-divisor finite ring.

(7) If R is τ-U-refinable and R is a τ-U-α-BFR, then R satisfies τ-U-ACCP.

(8) If R is τ-U-refinable and R satisfies τ-U-ACCP, then R is τ-U-α.

Proof. (1) This is immediate from definitions.

(2) Let $a \in R$ be a non-unit. Let $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ be the unique τ-α-U-factorization up to rearrangement and β. Given any other τ-U-factorization, we can τ-U-refine each essential τ-U-divisor into a τ-U-α-factorization of a. There is a rearrangement of the essential divisors to match up to β with b_i for each $1 \leq i \leq m$. Thus the essential divisors in any τ-U-factorization come from some combination of products of β of the m τ-U-α essential factors in our original factorization. Hence there are at most 2^m possible distinct τ-U-factorizations up to β, making this a τ-U-β-FFR as desired.

(3) For a given non-unit $a \in R$, the number of essential divisors in any τ-U-α-factorization is the same, say N. We claim this is a bound on the number

of essential divisors of any τ-U-factorization. Suppose there were a τ-U-factorization $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ with $m > N$. For every i, b_i has a τ-U-α-factorization with at least one essential divisor. Since R is τ-U-refinable, we can τ-U-refine the factorization yielding a τ-U-α-factorization of a with at least m τ-U-α essential factors. This contradicts the assumption that R is a τ-U-α-HFR.

(4) Let R be a τ-U-β-FFR. Let $a \in R$ be a non-unit. There are only finitely many τ-U-factorizations of a up to rearrangement and β of the essential divisors. We can simply take the maximum of the number of essential divisors among all of these factorizations. This is an upper bound for the number of essential divisors in any τ-U-factorization.

(5) Let R be a τ-U-β-FFR, then for any non-unit $a \in R$. Let S be the collection of essential divisors in the finite number of representative τ-U-factorizations of a up to β. This gives us a finite collection of elements up to β. Every essential divisor up to β in a τ-U-factorization of a must be among these, so this collection is finite as desired.

(6) If every non-unit $a \in R$ has a finite number of proper essential τ-U divisors, then certainly there are a finite number of essential τ-α-U-divisors.

(7) Suppose R is a τ-U-BFR, but $(a_1) \subsetneq (a_2) \subsetneq \cdots$ is a properly ascending chain of principal ideals such that a_{i+1} is an essential factor in some τ-U-factorization of a_i, say

$$a_i = \lambda_i a_{i1} \cdots a_{in_i} \lceil a_{i+1} b_{i1} \cdots b_{im_i} \rceil$$

for each i. Furthermore, $m_i \geq 1$, for each i otherwise we would have $(a_{i+1}) = (a_i)$

contrary to our assumption that our chain is properly increasing. Our assumption that R is τ-U refinable allows us to factor a_1 as follows:

$$a_1 = \lambda_1 a_{11} \cdots a_{1n_1} \lceil a_2 b_{11} \cdots b_{1m_1} \rceil =$$

$$\lambda_1 \lambda_2 a_{11} \cdots a_{1n_1} a_{21} \cdots a_{2n_2} \lceil a_3 b_{21} \cdots b_{2m_2} b_{11} \cdots b_{1m_1} \rceil$$

and so on. At each iteration i we have at least $i+1$ essential factors in our τ-U-factorization. This contradicts the assumption that a_1 should have a bound on the number of essential divisors in any τ-U-factorization.

(8) Let $a_1 \in R$ be a non-unit. If a_1 is τ-U-α we are already done, so there must be a non-trivial τ-U factorization of a_1, say:

$$a_1 = \lambda_1 a_{11} \cdots a_{1n_1} \lceil a_2 b_{11} \cdots b_{1m_1} \rceil .$$

Now if all of the essential divisors are τ-U-α we are done as we have found a τ-U-α-factorization. After rearranging if necessary, we suppose that a_2 is not τ-U-α. Therefore a_2 has a non-trivial τ-U-factorization, say:

$$a_2 = \lambda_2 a_{21} \cdots a_{2n_1} \lceil a_3 b_{21} \cdots b_{2m_2} \rceil .$$

Because R is τ-U-refinable, this gives us a τ-U-factorization:

$$a_1 = \lambda_1 \lambda_2 a_{11} \cdots a_{1n_1} a_{21} \cdots a_{2n_2} \lceil a_3 b_{21} \cdots b_{2m_2} b_{1_1} \cdots b_{1m_1} \rceil$$

which cannot be τ-U-α or else we would be done. We can continue in this fashion and get an ascending chain of principal ideals

$$(a_1) \subseteq (a_2) \subseteq \cdots$$

such that a_{i+1} is an essential τ-U-divisor of a_i for each i.

Claim: This chain must be properly ascending. Suppose $(a_i) = (a_{i+1})$ for some i. When we look at $a_i = \lambda_i a_{i1} \cdots a_{in_i} \lceil a_{i+1} b_{i1} \cdots b_{im_i} \rceil$, we see that $(a_i) = (a_{i+1} b_{i1} \cdots b_{im_i})$. But then we could remove any of the b_{ij} for any $1 \leq j \leq m_i$ and still have $(a_i) = (a_{i+1} b_{i1} \cdots \widehat{b_{ij}} \cdots b_{im_i})$ contradicting the fact that the factorization was a τ-U-factorization since b_{ij} is inessential.

We certainly have $(a_i) \subseteq (a_{i+1} b_{i1} \cdots \widehat{b_{ij}} \cdots b_{im_i})$. To see the other containment holds, $(a_i) = (a_{i+1}) \Rightarrow a_{i+1} = a_i r$ for some $r \in R$, and we can simply multiply by $b_{i1} \cdots \widehat{b_{ij}} \cdots b_{im_i}$ on both sides to see that

$$a_{i+1} b_{i1} \cdots \widehat{b_{ij}} \cdots b_{im_i} = a_i (r b_{i1} \cdots \widehat{b_{ij}} \cdots b_{im_i})$$

showing the other containment. Proving the claim.

This is a contradiction to the fact that R satisfies τ-U-ACCP, proving we must in finitely many steps arrive at a τ-U-α-factorization of a_1, proving R is indeed τ-U-α as desired. $\square$

The following diagram summarizes our results from the Theorems 4.6 and 4.7 where $\star$ represents R being strongly associate, and $\dagger$ represents R is τ-U-refinable:

$$
\begin{tikzcd}
\tau\text{-}\alpha\text{-}\beta\text{-UFR} \arrow[d, Rightarrow, "\star"] & \tau\text{-U-}\alpha\text{-HFR} \arrow[dr, Rightarrow, "\dagger"] & \tau\text{-}\alpha\text{-HFR} \arrow[l, Rightarrow, "\star"'] & & \\
\tau\text{-U-}\alpha\text{-}\beta\text{-UFR} \arrow[r, Rightarrow, "\dagger"] \arrow[ur, Rightarrow] & \tau\text{-U-}\beta\text{-FFR} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \tau\text{-U-BFR} \arrow[r, Rightarrow, "\dagger"] & \tau\text{-U-ACCP} \arrow[r, Rightarrow, "\dagger"] & \tau\text{-U-}\alpha \\
\tau\text{-}\beta\text{-WFFR} \arrow[r, Rightarrow] & \tau\text{-U-}\beta\text{-WFFR} \arrow[d, Rightarrow] & \tau\text{-BFR} \arrow[u, Rightarrow] & \tau-\text{ACCP} \arrow[u, Rightarrow] & \tau\text{-}\alpha \arrow[u, Rightarrow] \\
\tau\text{-}\alpha\text{-}\beta \text{ df ring} \arrow[r, Rightarrow] & \tau\text{-U-}\alpha\text{-}\beta \text{ df ring} & \tau\text{-}\beta\text{-FFR} \arrow[uul, Rightarrow] & &
\end{tikzcd}
$$

We have left off the relations which were proven in Chapter 2, and focused instead on the rings satisfying the U-finite factorization properties. Examples given in [13, 14, 6, 3] show that arrows can neither be reversed nor added to the diagram with a few exceptions.

Question 4.1. *Does U-atomic imply atomic?*

D.D. Anderson and S. Valdez-Leon show in [8, Theorem 3.13] that if R has a finite number of non-associate irreducibles, then U-atomic and atomic are equivalent. This remains open in general.

Question 4.2. *Does U-ACCP imply ACCP?*

We can modify M. Axtell's proof of [13, Theorem 2.9] to add a partial converse to Theorem 4.7 (5) if τ is combinable and associate preserving. The idea is the same, but slight adjustments are required to adapt it to τ-factorizations and to allow uniqueness up to any type of associate.

Theorem 4.8. *Let $\beta \in \{$associate, strongly associate, very strongly associate$\}$. Let R be a commutative ring with 1 and let τ be a symmetric relation on $R^{\#}$ which is both combinable and associate preserving. R is a τ-U-β-FFR if and only if R is a τ-U-β-WFFR.*

Proof. ($\Rightarrow$) was already shown, so we need only prove the converse. ($\Leftarrow$) Suppose R is not a τ-U-β-FFR. Let $a \in R$ be a non-unit which has infinitely many τ-U-factorizations up to β. Let $b_1, b_2, \ldots, b_m$ be a complete list of essential τ-U-divisors of a up to β. Let

$$a = a_1 \cdots a_n \lceil c_1 \cdots c_k \rceil = a'_1 \cdots a'_{n'} \lceil d_1 \cdots d_n \rceil$$

be two τ-U-factorizations of a and assume we have re-ordered the essential divisors in both factorizations above so that the β of b_1 appear first, followed by β of b_2, etc. Let $A = \langle (c_1), (c_2), \ldots, (c_k) \rangle$ and $B = \langle (d_1), (d_2), \ldots, (d_n) \rangle$ be sequences of ideals. We call the factorizations *comparable* if A is a subsequence of B or vice versa.

Suppose A is a proper subsequence of B

$$B = \langle (d_1), \ldots, (d_{i_1}) = (c_1), \ldots, (d_{i_2}) = (c_2), \ldots, (d_{i_k}) = (c_k), \ldots, (d_n) \rangle$$

with $n > k$. Because τ is combinable and symmetric,

$$a = a'_1 \cdots a'_{n'} \left\lceil d_{i_1} d_{i_2} \cdots d_{i_k} (d_1 \cdots \widehat{d_{i_1}} \widehat{d_{i_2}} \cdots \widehat{d_{i_k}} \cdots d_n) \right\rceil$$

remains a τ-factorizations and [13, Lemma 1.3] ensures this remains a U-factorization.

This yields

$$(a) = (d_1 \cdots \widehat{d_{i_1}} \widehat{d_{i_2}} \cdots \widehat{d_{i_k}} \cdots d_n)(d_{i_1} d_{i_2} \cdots d_{i_k}) = (d_1 \cdots d_n) = (c_1 \cdots c_k)$$

$$= (c_1) \cdots (c_k) = (d_{i_1}) \cdots (d_{i_k}) = (d_{i_1} \cdots d_{i_k}).$$

But then, $(d_1 \cdots \widehat{d_{i_1}} \widehat{d_{i_2}} \cdots \widehat{d_{i_k}} \cdots d_n)$ cannot be an essential divisor, a contradiction, unless $n = k$.

If $n = k$, then the sequences of ideals are identical, and we seek to prove this means the τ-U-factorizations are the same up to β. It is certainly true for $\beta =$ associate as demonstrated in [13, Theorem 2.9]. So we have a pairing of the c_i and d_i such that $c_i \sim b_j \sim d_i$ for one of the essential τ-U-divisors b_j. We know further that c_i and b_j (resp. d_i and b_j) are β since R is by assumption a τ-U-β-WFFR.

It is well established that β is transitive, so we can conclude that this same pairing demonstrates that c_i and d_i are β, not just associate. Thus the number of distinct τ-U-factorizations up to β is less than or equal to the number of non-comparable finite sequences of elements from the set $\{(b_1), (b_2), \ldots, (b_m)\}$.

From here we direct the reader to the proof of the second claim in [13, Theorem 2.9] where it is shown that this set is finite. $\square$

CHAPTER 5
τ-U-FACTORIZATION ON DIRECT PRODUCTS

In this chapter we look at the primary motivation for developing the theory of τ-U-factorization. There are many nice theorems that demonstrate how finite factorization properties extend through direct products of rings. The main issue with direct products is that there are many idempotent elements. These are elements, $0 \neq e \neq 1$, such that $e^2 = e$. These are particularly problematic prior to using U-factorization techniques because $e = e^2 = e^3 = \cdots$ provides arbitrarily long factorizations. With direct products of rings, the element $(1, 0, \ldots, 0)$ or $(1, 0, \ldots)$ is always idempotent, so it simply must be dealt with. By using U-factorization, we see that we are able to get many of the analogues of the nice theorems to go through with τ-factorization as well.

5.1 Direct Products and the Relation $\tau_\times$

For each i, $1 \leq i \leq N$, let R_i be commutative rings with τ_i a symmetric relation on $R_i^\#$. We define a relation $\tau_\times$ on $R = R_1 \times \cdots \times R_N$ which preserves many of the theorems about direct products from [1] for τ-factorizations. Let $(a_i), (b_i) \in R^\#$, then $(a_i)\tau_\times(b_i)$ if and only if whenever a_i and b_i are both non-units in R_i, then $a_i \tau_i b_i$.

For convenience we will adopt the following notation: Suppose $x \in R_i$, then $x^{(i)} = (1_{R_1}, \cdots, 1_{R_{i-1}}, x, 1_{R_{i+1}}, \cdots 1_{R_N})$. so x appears in the i^{th} coordinate, and all other entries are the identity. Thus for any $(a_i) \in R$, we have $(a_i) = a_1^{(1)} a_2^{(2)} \cdots a_n^{(n)}$ is a $\tau_\times$-factorization. We will always move any $\tau_\times$-factors which may become units in this process to the front and collect them there.

Lemma 5.1. *Let $R = R_1 \times \cdots \times R_N$ for $N \in \mathbb{N}$. Then $(a_i) \sim (b_i)$ (resp. $(a_i) \approx (b_i)$) if and only if $a_i \sim b_i$ (resp. $a_i \approx b_i$) for every i. Furthermore, $(a_i) \cong (b_i)$ implies $a_i \cong b_i$ for all i, and for a_i, b_i all non-zero, $a_i \cong b_i$ for all $i \Rightarrow (a_i) \cong (b_i)$.*

Proof. See [8, Theorem 2.15]. □

Example 5.1. If $a_{i_0} = 0$ for even one index $1 \leq i_0 \leq N$, then $a_i \cong b_i$ for all i need not imply $(a_i) \cong (b_i)$.

Consider the ring $R = \mathbb{Z} \times \mathbb{Z}$, with $\tau_i = \mathbb{Z}^{\#} \times \mathbb{Z}^{\#}$ for $i = 1, 2$, the usual factorization. We have $1 \cong 1$ and $0 \cong 0$ since $\mathbb{Z}$ is a domain; however $(0, 1) = (0, 1)(0, 1)$ shows $(0, 1) \ncong (0, 1)$. ⊞

Lemma 5.2. *Let $R = R_1 \times \cdots \times R_N$ for $N \in \mathbb{N}$ with τ_i a symmetric relation on $R_i^{\#}$ for each i. Let $\alpha \in \{$ irreducible, strongly irreducible, m-irreducible, very strongly irreducible $\}$. If $(a_i) \in R$ is τ-α, then precisely one coordinate is not a unit.*

Proof. Let $a = (a_i) \in R$ be a non-unit which is $\tau_\times$-α. Certainly not all coordinates can be units, or else $a \in U(R)$. Suppose for a moment there were at least two coordinates for which a_i is not a unit in R_i. After reordering, we may assume a_1 and a_2 are not units. Then $a = a_1^{(1)}(1_{R_1}, a_2, \cdots, a_N)$ is a $\tau_\times$-factorization. But a is not even associate to either $\tau_\times$-factor, a contradiction. □

Theorem 5.3. *Let $R = R_1 \times \cdots \times R_N$ for $N \in \mathbb{N}$ with τ_i a symmetric relation on $R_i^{\#}$ for each i.*

(1) A non-unit $(a_i) \in R$ *is* $\tau_\times$*-atomic (resp. strongly atomic) if and only if* a_{i_0} *is* τ_{i_0}*-atomic (resp. strongly atomic) for some* $1 \leq i_0 \leq n$ *and* $a_i \in U(R_i)$ *for all* $i \neq i_0$.

(2) A non-unit $(a_i) \in R$ *is* $\tau_\times$*-m-atomic if and only if* a_{i_0} *is* τ_{i_0}*-m-atomic for some* $1 \leq i_0 \leq n$ *and* $a_i \in U(R_i)$ *for all* $i \neq i_0$.

(3) A non-unit $(a_i) \in R$ *is* $\tau_\times$*-very strongly atomic if and only if* a_{i_0} *is* τ_{i_0}*-very strongly atomic and non-zero for some* $1 \leq i_0 \leq n$ *and* $a_i \in U(R_i)$ *for all* $i \neq i_0$.

Proof. (1) ($\Rightarrow$) Let $a = (a_i) \in R$ be a non-unit which is $\tau_\times$-atomic (resp. strongly atomic). By Lemma 5.2, there is only one non-unit coordinate. Suppose after re-ordering if necessary that a_1 is the non-unit. If a_1 were not τ_1-atomic (resp. strongly atomic), then there is a τ_1-factorization, $\lambda_{1_1} a_{1_1} a_{1_2} \cdots a_{1_k}$ for which $a_1 \not\sim a_{1_j}$ (resp. $a_1 \not\approx a_{1_j}$) for any $1 \leq j \leq k$. But then

$$(a_i) = (\lambda_{1_1}, a_2, \ldots, a_n) a_{1_1}^{(1)} a_{1_2}^{(1)} \cdots a_{1_k}^{(1)}$$

is a $\tau_\times$-factorization. Furthermore, by Lemma 5.1 $(a_i) \not\sim a_{1_j}^{(1)}$ (resp. $(a_i) \not\sim a_{1_j}^{(1)}$) for all $1 \leq j \leq k$. This would contradict the assumption that a was $\tau_\times$-atomic (resp. strongly atomic).

($\Leftarrow$) Let $a_1 \in R_1$, a non-unit with a_1 being τ_1-atomic (resp. strongly atomic). Let $\mu_i \in U(R_i)$ for $2 \leq i \leq N$. We show $a = (a_1, \mu_2, \cdots \mu_N)$ is $\tau_\times$-atomic (resp. strongly atomic). Suppose $a = (\lambda_1, \ldots, \lambda_N)(a_{1_1}, \ldots, a_{1_N}) \cdots (a_{k_1}, \ldots, a_{k_N})$ is a $\tau_\times$-factorization of a. We first note $a_{i_j} \in U(R_j)$ for all $j \geq 2$. Furthermore, this means a_{i_1} is not a unit in R_1 for $1 \leq i \leq k$, otherwise we would have units as factors in a $\tau_\times$ factorization. This means $a_1 = \lambda_1 a_{1_1} \cdots a_{k_1}$ is a τ_1 factorization of a τ_1-atomic (resp.

strongly atomic) element. Thus, we must have $a_1 \sim a_{j_1}$ (resp. $a_1 \approx a_{j_1}$) for some $1 \leq j \leq k$. Hence by Lemma 5.1, we have $a \sim (a_{j_1}, \dots, a_{j_N})$ (resp. $a \approx (a_{j_1}, \dots, a_{j_N})$ for some $1 \leq j \leq k$ and a is $\tau_\times$ atomic (resp. strongly atomic) as desired.

(2) ($\Rightarrow$) Let $a = (a_i) \in R$ be a non-unit which is $\tau_\times$-m-atomic. By Lemma 5.2, there is only one non-unit coordinate, say a_1 after reordering if necessary. Let $a_1 = \lambda_{1_1} a_{1_1} a_{1_2} \cdots a_{1_k}$ be a τ_1 factorization for which $a_1 \not\sim a_{1_{j_0}}$ for at least one $1 \leq j_0 \leq k$. But then

$$(a_i) = (\lambda_{1_1}, a_2, \dots, a_n) a_{1_1}^{(1)} a_{1_2}^{(1)} \cdots a_{1_k}^{(1)}$$

is a $\tau_\times$-factorization of a for which (by Lemma 5.1) $a = (a_i) \not\sim a_{1_{j_0}}^{(1)}$. This contradicts the hypothesis that a is $\tau_\times$-m-atomic.

($\Leftarrow$) Let $a_1 \in R_1$, a non-unit with a_1 being τ_1-m-atomic. Let $\mu_i \in U(R_i)$ for $2 \leq i \leq N$. We show $a = (a_1, \mu_2, \cdots \mu_N)$ is $\tau_\times$-m-atomic. Suppose

$$a = (\lambda_1, \dots, \lambda_N)(a_{1_1}, \dots, a_{1_N}) \cdots (a_{k_1}, \dots, a_{k_N})$$

is a $\tau_\times$-factorization of a. We first note $a_{i_j} \in U(R_j)$ for all $j \geq 2$. As before, this means $a_1 = \lambda_1 a_{1_1} \cdots a_{k_1}$ is a τ_1 factorization of a τ_1-m-atomic element. Hence $a_1 \sim a_{j_1}$ for each $1 \leq j \leq k$. By Lemma 5.1 we have $a \sim (a_{j_1}, \dots, a_{j_N})$ for all $1 \leq j \leq k$ and thus a is $\tau_\times$-m-atomic as desired.

(3) ($\Rightarrow$) Let $a = (a_1, \dots a_N)$ be a non-unit which is $\tau_\times$-very strongly atomic. By Lemma 5.2, we may assume a_1 is the non-unit, and a_j is a unit for $j \geq 2$. We suppose for a moment that $a_1 = 0_1$. But then $(0, a_2, \dots a_N) = (0, 1, \dots 1) \cdot (0, a_2, \dots a_N)$ shows that $a \ncong a$, a contradiction. Lemma 5.1 shows that if $a \cong a$, then $a_i \cong a_i$

for each $1 \leq i \leq N$. Hence, if a_1 were not τ_1-very strongly atomic, then there is a τ_1-factorization, $\lambda_{1_1} a_{1_1} a_{1_2} \cdots a_{1_k}$ for which $a_1 \not\cong a_{1_j}$ for any $1 \leq j \leq k$. But then

$$(a_i) = (\lambda_{1_1}, a_2, \ldots, a_n) a_{1_1}^{(1)} a_{1_2}^{(1)} \cdots a_{1_k}^{(1)}$$

is a $\tau_\times$-factorization. Furthermore, since every coordinate is non-zero, by Lemma 5.1 $(a_i) \not\cong a_{1_j}^{(1)}$ for all $1 \leq j \leq k$. This would contradict the assumption that a was $\tau_\times$-very strongly atomic.

($\Leftarrow$) Let $a_1 \in R_1^\#$ be τ_1-very strongly atomic. Let $\mu_i \in U(R_i)$ for $2 \leq i \leq N$. We show $a = (a_1, \mu_2, \cdots \mu_N)$ is $\tau_\times$-very strongly atomic. We first check $a \cong a$. By definition of τ_1-very strongly atomic, $a_1 \cong a_1$. Certainly as units, we have $\mu_i \cong \mu_i$ for each $i \geq 2$. Lastly, all of these are non-zero, so we may apply Lemma 5.1 to see that $a \cong a$. Suppose $a = (\lambda_1, \ldots, \lambda_N)(a_{1_1}, \ldots, a_{1_N}) \cdots (a_{k_1}, \ldots, a_{k_N})$ is a $\tau_\times$-factorization of a. We first note $a_{i_j} \in U(R_j)$ for all $j \geq 2$. As before, this means $a_1 = \lambda_1 a_{1_1} \cdots a_{k_1}$ is a τ_1 factorization of a τ_1-very strongly atomic element. Hence $a_1 \cong a_{j_1}$ for some $1 \leq j \leq k$. By Lemma 5.1 we have $a \cong (a_{j_1}, \ldots, a_{j_N})$ and thus a is $\tau_\times$-very strongly atomic as desired. $\square$

Lemma 5.4. *Let $R = R_1 \times \cdots \times R_N$ for $N \in \mathbb{N}$ with τ_i a symmetric relation on $R_i^\#$. Let $\alpha \in \{$ irreducible, strongly irreducible, m-irreducible, very strongly irreducible $\}$. Then we have the following.*

(1) If $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ is a τ_i-U-α-factorization of some non-unit $a \in R_i$, then

$$a^{(i)} = \lambda^{(i)} a_1^{(i)} \cdots a_n^{(i)} \left\lceil b_1^{(i)} \cdots b_m^{(i)} \right\rceil$$

is a $\tau_\times$-U-α-factorization.

(2) Conversely, let $a_{i_0} \in R_{i_0}$ be a non-unit and $\mu_i \in U(R_i)$ for all $i \neq i_0$. Let

$$(\mu_1, \mu_2, \ldots, \mu_{i_0-1}, a_{i_0}, \mu_{i_0+1} \ldots, \mu_N) = (\lambda_i)(a_{1_i})(a_{2_i}) \cdots (a_{n_i}) \left\lceil (b_{1_i})(b_{2_i}) \cdots (b_{m_i}) \right\rceil$$

be a $\tau_\times$-U-α-factorization. Then

$$a_{i_0} = \lambda_{i_0} a_{1_{i_0}} \cdots a_{n_{i_0}} \left\lceil b_{1_{i_0}} \cdots b_{_{i_0}} \right\rceil$$

is a τ_{i_0}-U-α-factorization.

Proof. (1) Let $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ be a τ_i-U-α-factorization of some non-unit $a \in R_i$. It is easy to see that

$$a^{(i)} = \lambda^{(i)} a_1^{(i)} \cdots a_n^{(i)} \left\lceil b_1^{(i)} \cdots b_m^{(i)} \right\rceil$$

is a $\tau_\times$-factorization. Furthermore, $b_j \neq 0$ for all $1 \leq j \leq$ or else it would not be a τ_i-factorization. Hence by Theorem 5.3 $b_j^{(i)}$ is $\tau_\times$-α for each $1 \leq j \leq m$. Thus it suffices to show that we actually have a U-factorization.

Since $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ is a U-factorization, we know $a_k(b_1 \cdots b_m) = (b_1 \cdots b_m)$ for all $1 \leq k \leq n$. In the other coordinates, we have $(1_{R_j}) = (1_{R_j})$ for all $j \neq i$. Hence, we apply Lemma 5.1 and see that this implies that $a_k^{(i)}(b_1^{(i)} \cdots b_m^{(i)}) = (b_1^{(i)} \cdots b_m^{(i)})$ for all $1 \leq k \leq n$. Similarly we have $b_j(b_1 \cdots \widehat{b_j} \cdots b_m) \neq (b_1 \cdots \widehat{b_j} \cdots b_m)$ which implies

$$b_j^{(i)}(b_1^{(i)} \cdots \widehat{b_j^{(i)}} \cdots b_m^{(i)}) \neq (b_1^{(i)} \cdots \widehat{b_j^{(i)}} \cdots b_m^{(i)}).$$

So this is indeed a U-factorization.

(2) Let

$$(\mu_1, \mu_2, \ldots, \mu_{i_0-1}, a_{i_0}, \mu_{i_0+1} \ldots, \mu_N) = (\lambda_i)(a_{1_i})(a_{2_i}) \cdots (a_{n_i}) \left\lceil (b_{1_i})(b_{2_i}) \cdots (b_{m_i}) \right\rceil$$

be a $\tau_\times$-U-α-factorization. We note that $a_{j_i} \in U(R_i)$ for all $i \neq i_0$ and all $1 \leq j \leq n$ and $b_{j_i} \in U(R_i)$ for all $i \neq i_0$ and all $1 \leq j \leq m$ since they divide the unit μ_i. Next, every coordinate in the i_0 place must be a non-unit in R_{i_0} or else this factor would be a unit in R and therefore could not occur as a factor in a $\tau_\times$-factorization. This tells us that

$$a_{i_0} = \lambda_{i_0} a_{1_{i_0}} \cdots a_{n_{i_0}} \left\lceil b_{1_{i_0}} \cdots b_{_{i_0}} \right\rceil$$

is a τ_{i_0}-factorization. Furthermore, (b_{k_i}) is assumed to be $\tau_\times$-α for all $1 \leq k \leq m$, and the other coordinates are units, so $b_{k_{i_0}}$ is τ_{i_0}-α for all $1 \leq k \leq m$ by Theorem 5.3. Again, we need only show that

$$a_{i_0} = \lambda_{i_0} a_{1_{i_0}} a_{2_{i_0}} \cdots a_{n_{i_0}} \left\lceil b_{1_{i_0}} b_{2_{i_0}} \cdots b_{m_{i_0}} \right\rceil$$

is a U-factorization. Since all the coordinates other than i_0 are units, we simply apply Lemma 5.1 and see that we indeed maintain a U-factorization. □

5.2 Direct Products of Rings Results

Theorem 5.5. *Let $R = R_1 \times \cdots \times R_N$ for $N \in \mathbb{N}$ with τ_i a symmetric relation on $R_i^\#$. Let $\alpha \in \{$ irreducible, strongly irreducible, m-irreducible, very strongly irreducible $\}$. Then R is $\tau_\times$-U-α if and only if R_i is τ_i-U-α for each $1 \leq i \leq N$.*

Proof. ($\Rightarrow$) Let $a \in R_{i_0}$ be a non-unit. Then $a^{(i_0)}$ is a non-unit in R and therefore has a $\tau_\times$-U-α-factorization. Furthermore, the only possible non-unit factors in this factorization must occur in the i_0^{th} coordinate. Thus as in Lemma 5.4 (2), we have found a τ_{i_0}-U-α-factorization of a by taking the product of the i_0^{th} entries. This shows R_{i_0} is τ_{i_0}-U-α as desired.

($\Leftarrow$) Let $a = (a_i) \in R$ be a non-unit. For each non-unit $a_i \in R_i$, there is a τ_i-U-α-factorization of a_i, say

$$a_i = \lambda_i a_{i_1} \cdots a_{i_{n_i}} \left\lceil b_{i_1} \cdots b_{i_{m_i}} \right\rceil .$$

If $a_i \in U(R_i)$, then $a_i^{(i)} \in U(R)$ and we can simply collect these unit factors in the front, so we need not worry about these factors. This yields a $\tau_\times$-U-α-factorization

$$a = (a_i) = \prod_{i=1}^{n} \lambda_i^{(i)} a_{i_1}^{(i)} \cdots a_{i_{n_i}}^{(i)} \left\lceil \prod_{i=0}^{m} b_{i_1}^{(i)} \cdots b_{i_{m_i}}^{(i)} \right\rceil .$$

It is certainly a $\tau_\times$-factorization. Furthermore, $b_{j_k} \neq 0_j$ for $1 \leq j \leq m$ and $1 \leq k \leq m_j$, so $b_{j_k}^{(j)}$ is $\tau_\times$-α by Theorem 5.3. It is also clear from Lemma 5.4 that this is a U-factorization, showing every non-unit in R has a $\tau_\times$-U-α-factorization. □

Theorem 5.6. *Let $R = R_1 \times \cdots \times R_N$ for $N \in \mathbb{N}$ with τ_i a symmetric relation on $R_i^{\#}$. Let $\alpha \in \{$ irreducible, strongly irreducible, m-irreducible, very strongly irreducible $\}$ and let $\beta \in \{$ associate, strongly associate, very strongly associate $\}$. Then R is a $\tau_\times$-U-α-β-df ring if and only if R_i is τ_i-U-α-df ring for each $1 \leq i \leq N$.*

Proof. ($\Rightarrow$) Let $a \in R_{i_0}$ be a non-unit. Suppose there were an infinite number of τ_{i_0}-U-α essential divisors of a, say $\{b_j\}_{j=1}^{\infty}$ none of which are β. But then $\{b_j^{(i_0)}\}_{j=1}^{\infty}$ yields an infinite set of $\tau_\times$-U-α-divisors of $a^{(i_0)}$ by Lemma 5.4. Furthermore, none of them are β by Lemma 5.1.

($\Leftarrow$) Let $(a_i) \in R$ be a non-unit. We look at the collection of $\tau_\times$-U-α essential divisors of (a_i). Each must be of the form $(\lambda_1, \cdots, b_{i_0}, \cdots \lambda_N)$ with $\lambda_i \in U(R_i)$ for each i and with b_{i_0} τ_{i_0}-α for some $1 \le i_0 \le N$. But then b_{i_0} is a τ_{i_0}-α essential divisor of a_{i_0}. For each i between 1 and N, R_i is a τ_i-U-α-β-df ring, so there can be only finitely many τ_i-α essential divisors of a_i up to β, say $N(a_i)$. If $a_i \in R_i$, then we can simply set $N(a_i) = 0$ since it is a unit and has no non-trivial τ_i-U-factorizations. Hence there can be only

$$N((a_i)) := N(a_1) + N(a_2) + \cdots + N(a_N) = \sum_{i=1}^{N} N(a_i)$$

$\tau_\times$-α essential divisors of (a_i) up to β. This proves the claim. □

Corollary 5.1. *Let α and β be as in the theorem. Let $R = R_1 \times \cdots \times R_N$ for $N \in \mathbb{N}$ with τ_i a symmetric relation on $R_i^{\#}$. Then R is a $\tau_\times$-U-α $\tau_\times$-U-α-β-df ring if and only if R_i is τ_i-U-α τ_i-U-α-df ring for each $1 \le i \le N$.*

Proof. This is immediate from Theorem 5.6 and Theorem 5.5. □

Theorem 5.7. *Let $R = R_1 \times \cdots \times R_N$ for $N \in \mathbb{N}$ with τ_i a symmetric relation on $R_i^{\#}$. Then R is a $\tau_\times$-U-BFR if and only if R_i is a τ_i-U-BFR for every i.*

Proof. ($\Rightarrow$) Let $a \in R_{i_0}$ be a non-unit. Then $a^{(i_0)}$ is a non-unit in R, and hence has a bound on the number of essential divisors in any $\tau_\times$-U-factorization, say $N_e(a^{(i_0)})$. We claim this also bounds the number of essential divisors in any τ_{i_0}-U-factorization of a. Suppose for a moment $a = a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ were a τ_{i_0}-U-factorization with $m > N_e(a^{(i)})$. But then

$$a = \lambda^{(i_0)} a_1^{(i_0)} \cdots a_n^{(i_0)} \left\lceil b_1^{(i_0)} \cdots b_m^{(i_0)} \right\rceil$$

is a $\tau_\times$-U-factorization with more essential divisors than is allowed, a contradiction.

($\Leftarrow$) Let $a = (a_i) \in R$ be a non-unit. Let $B(a) = \max\{N_e(a_i)\}_{i=1}^N$. Where $N_e(a_i)$ is the number of essential divisors in any τ_i-U-factorization of a_i, and will say for $a_i \in U(R_i)$, $N_e(a_i) = 0$. We claim that $B(a)N$ is a bound on the number of essential divisors in any $\tau_\times$-U-factorization of a. Let

$$(a_i) = (\lambda_i)(a_{1_i}) \cdots (a_{n_i}) \lceil (b_{1_i}) \cdots (b_{m_i}) \rceil$$

be a $\tau_\times$-U-factorization. We can decompose this factorization so that each factor has at most one non-unit entry as follows:

$$(a_i) = \prod_{i=1}^N \lambda_i^{(i)} a_{1_i}^{(i)} \cdots \prod_{i=1}^N a_{n_i}^{(i)} \prod_{i=1}^N b_{1_i}^{(i)} \cdots \prod_{i=1}^N b_{m_i}^{(i)}.$$

Some of these factors may indeed be units; however, by allowing a unit factor in the front of every τ-U-factorization, we simply combine all the units into one at the front, and maintain a $\tau_\times$-factorization. We can always rearrange this to be a $\tau_\times$-U-factorization. Furthermore, since a_{j_i} is inessential, by Lemma 5.1 $a_{j_i}^{(i)}$ is inessential. Only some of the components of the essential divisors could become inessential, for instance if one coordinate were a unit. At worst when we decompose, $b_{j_i}^{(i)}$ remains an essential divisor for all $1 \le j \le m$ and for all $1 \le i \le N$. But then the product of each of the i^{th} coordinates gives a τ_i-U-factorization of a_i and thus is bounded by $N_e(a_i)$, so we have $m \le N_e(a_i) \le B(a)$ and therefore there are no more than $B(a)N$ essential divisors. Certainly the original factorization is no longer than the one we constructed through the decomposition, proving the claim and completing the proof. $\square$

Theorem 5.8. *Let $R = R_1 \times \cdots \times R_N$ for $N \in \mathbb{N}$ with τ_i a symmetric relation on $R_i^{\#}$. Let $\alpha \in \{$ irreducible, strongly irreducible, m-irreducible, very strongly irreducible $\}$. Then R is $\tau_\times$-U-α-HFR if and only if R_i is a τ_i-U-α-HFR for each i.*

Proof. ($\Rightarrow$) Let $a \in R_{i_0}$ be a non-unit. We know by Theorem 5.5 Then $a^{(i_0)}$ is a non-unit in R, and has an $\tau_\times$-U-α-factorization. Suppose there were τ_{i_0}-U-α-factorizations of a with different numbers of essential divisors, say:

$$a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil = \mu c_1 \cdots c_{n'} \lceil d_1 \cdots d_{m'} \rceil$$

where $m \neq m'$. By Lemma 5.4 this yields two $\tau_\times$-U-α-factorizations:

$$a^{(i_0)} = \lambda^{(i_0)} a_1^{(i_0)} \cdots a_n^{(i_0)} \left\lceil b_1^{(i_0)} \cdots b_n^{(i_0)} \right\rceil = \mu^{(i_0)} c_1^{(i_0)} \cdots c_{n'}^{(i_0)} \left\lceil d_1^{(i_0)} \cdots d_{n'}^{(i_0)} \right\rceil.$$

This contradicts the hypothesis that R is a R is $\tau_\times$-U-α-HFR.

($\Leftarrow$) Let $(a_i) \in R$ be a non-unit. Suppose we had two $\tau_\times$-U-α factorizations

$$(a_i) = (\lambda_i)(a_{1_i})(a_{2_i}) \cdots (a_{n_i}) \lceil (b_{1_i})(b_{2_i}) \cdots (b_{m_i}) \rceil =$$

$$(\mu_i)(a'_{1_i})(a'_{2_i}) \cdots (a'_{n'_i}) \left\lceil (b'_{1_i})(b'_{2_i}) \cdots (b'_{m'_i}) \right\rceil.$$

For each i_0, if a_{i_0} is a non-unit in R_{i_0}, then since each $\tau_\times$-α element can only have one coordinate which is not a unit, we can simply collect all the $\tau_\times$-divisors which have the i_0 coordinate a non-unit. This product forms a τ_{i_0}-U-α-factorization of a_{i_0} and therefore the number of essential $\tau_\times$-factors with coordinate i_0 a non-unit must be the same in the two factorizations. This is true for each coordinate i_0, hence $m = m'$ as desired. $\square$

Theorem 5.9. *Let $R = R_1 \times \cdots \times R_N$ for $N \in \mathbb{N}$ with τ_i a symmetric relation on $R_i^{\#}$. Let $\alpha \in \{$ irreducible, strongly irreducible, m-irreducible, very strongly irreducible $\}$ and let $\beta \in \{$ associate, strongly associate $\}$. Then R is $\tau_\times$-U-α-β-UFR if and only if R_i is a τ_i-U-α-β-UFR for each i.*

Proof. We simply apply Lemma 5.1 to the proof of Theorem 5.8, to see that the factors can always be rearranged to match associates of the correct type. □

CHAPTER 6
τ-REGULAR FACTORIZATION

In this chapter, we look at another way to extend τ-factorization to rings with zero-divisors. In Section 6.1, we develop many of the definitions of τ-regular-factorization, τ-regular irreducible elements as well as τ-regular finite factorization properties that rings may have. This is done by restricting our study of τ-factorization to the regular elements of a commutative ring with 1. In Section 6.2, we prove several theorems which describe the relationships between the various τ-regular finite factorization properties that rings may possess. In Section 6.3, we compare this new method of extending τ-factorization with the previous work in Chapter 2.3 and the relation $\tau_r := \tau \cap \mathrm{Reg}(R) \times \mathrm{Reg}(R)$. In Section 6.4, we demonstrate how these τ-regular finite factorization properties are related to other finite factorization properties defined in other works.

The primary benefit of looking at the factorization of the regular elements is that for regular elements, all the associate relations coincide. That is, let $a, b \in \mathrm{Reg}(R)$, then $a \sim b$ implies $a \cong b$. Suppose $a = rb$. Neither a nor b can be zero, or else they would not be regular elements. But $a \sim b$ implies there is an $s \in R$ such that $b = sa$. Thus $a = rb = r(sa) = (rs)a$, but a is regular, so $a(1 - rs) = 0$ implies $rs - 1 = 0$ or $rs = 1$, so $r \in U(R)$ as desired. Another important consequence is that for a regular element, we always have $a \cong a$. This also means that for a regular, non-unit element $a \in \mathrm{Reg}(R)$, if a is irreducible, then a is very strongly irreducible. As a consequence, for a regular, non-unit $a \in R$ we can simply refer to it as *irreducible* without any ambiguity. We will soon see that this simplifies matters considerably.

6.1 τ-Regular Factorizations Definitions

Let τ be a symmetric relation on $R^{\#}$. A τ-factorization, $a = \lambda a_1 \cdots a_n$ with $\lambda \in U(R)$, and $a_i \tau a_j$ for all $i \neq j$ is said to be a *τ-regular-factorization* or *τ-r-factorization* if $a \in \text{Reg}(R)$. Note that a is regular if and only if a_i is regular for each $1 \leq i \leq n$.

Proposition 6.1. *Let R be a commutative ring with 1 and let τ be a symmetric relation on $R^{\#}$. Given $a \in Reg(R)$, the following are equivalent.*

(1) For any τ-regular-factorization, $a = \lambda a_1 \cdots a_n$, we have $a \sim a_i$ for some $1 \leq i \leq n$.

(2) For any τ-regular-factorization, $a = \lambda a_1 \cdots a_n$, we have $a \approx a_i$ for some $1 \leq i \leq n$.

(3) For any τ-regular-factorization, $a = \lambda a_1 \cdots a_n$, we have $a \sim a_i$ for all $1 \leq i \leq n$.

(4) The only τ-regular factorizations of a are of the form $a = \lambda(\lambda^{-1} a)$.

(5) $a \cong a$ and for any τ-regular-factorization, $a = \lambda a_1 \cdots a_n$, we have $a \cong a_i$ for some $1 \leq i \leq n$.

Proof. (5) $\Rightarrow$ (4) Suppose $a = \lambda a_1 \cdots a_n$ is a τ-regular factorization with $n \geq 2$. Then by hypothesis $a \cong a_i$ for some $1 \leq i \leq n$. Then

$$a = (\lambda a_1 \cdots a_{i-1} \widehat{a_i} a_{i+1} \cdots a_n) a_i$$

implies that $(\lambda a_1 \cdots a_{i-1} \widehat{a_i} a_{i+1} \cdots a_n)$ is a unit. Hence the factorization was a trivial factorization to begin with.

(4) $\Rightarrow$ (3) is immediate. After noting that any divisor of a regular element must be regular and hence $\sim, \approx$ and $\cong$ coincide, it is clear that (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (5) Since a is regular by hypothesis, $a \cong a$ and again $\sim, \approx$ and $\cong$ coincide on any divisors of a regular element, completing the proof. $\square$

We say $a \in \text{Reg}(R)$ is *τ-r-irreducible* or *a τ-r-atom* if a satisfies any of the above equivalent conditions. We say R is *τ-r-atomic* if for all $a \in \text{Reg}(R)^{\#}$, there is a τ-r-factorization into τ-r-irreducible elements. R satisfies *τ-r-ACCP* if for every chain of principal ideals generated by regular elements $(a_1) \subsetneq (a_2) \subsetneq \cdots (a_i) \subsetneq \cdots$ with a_{i+1} occurring as a τ-divisor in some τ-r-factorization of a_i for all i becomes stationary.

R is a *τ-r-half factorization ring (HFR)* if (1) R is τ-r-atomic and (2) if $\lambda a_1 \cdots a_m = \mu b_1 \cdots b_n$ are two τ-r-atomic τ-factorizations implies that $m = n$. R is said to be a *τ-r-unique factorization ring (UFR)* if R is a τ-r-HFR and there is a rearrangement of any two τ-r-atomic factorizations as above such that $a_i \sim b_i$ for all $1 \leq i \leq n = m$. We define the *τ-regular-elasticity* as τ-r-$\rho(R) = \sup\{\rho(a) \mid a \in \text{Reg}(R)^{\#}\}$ where $\rho(a) = \sup\{\frac{m}{n} \mid \lambda a = a_1 \cdots a_m = \mu b_1 \cdots b_n$ are τ-atomic-factorizations $\}$. Then it is clear that R is a τ-r-HFR if and only if R is τ-atomic and τ-r-$\rho(R)$=1.

R is a *τ-r-bounded factorization domain (BFR)* if for every $a \in \text{Reg}(R)$ there exists a natural number $N_r(a)$ such that for all τ-r-factorizations $a = \lambda a_1 \cdots a_n$, we have $n \leq N_r(a)$. R is said to be a *τ-r-irreducible-divisor-finite ring (idf ring)* if each $a \in \text{Reg}(R)^{\#}$ has at most a finite number of non-associate τ-irreducible τ-divisors. R is said to be a *τ-r-finite factorization ring (FFR)* if for every $a \in \text{Reg}(R)^{\#}$, a has only a finite number (up to order and associates) of τ-factorizations. R is said to be a *τ-r-weak finite factorization ring (WFFR)* if for every $a \in \text{Reg}(R)^{\#}$ there are only a finite number of non-associate τ-divisors.

6.2 τ-Regular Factorization Results

Proposition 6.2. *Let R be a commutative ring with 1. Let τ be a symmetric relation on $R^{\#}$ with τ refinable, then the following are equivalent.*

(1) R is a τ-r-FFR.

(2) R is a τ-r-WFFR.

(3) R is a τ-r-atomic τ-r-idf ring.

(4) R is τ-r-atomic and each $a \in Reg(R)^{\#}$, a has only finitely many τ-r-atomic τ-factorizations up to order and associates.

(5) For all $a \in Reg(R)^{\#}$, there are only finitely many $b \in Reg(R)^{\#}$ up to associate such that b occurs as a τ-factor in a τ-r-factorization of a.

(6) For all $a \in Reg(R)^{\#}$, (a) is contained in only finitely many principal ideals (b) where $b \in Reg(R)^{\#}$ such that b occurs as a τ-factor in a τ-r-factorization of a.

(7) For all $a \in Reg(R)^{\#}$, there are only finitely many $b \in Reg(R)^{\#}$ up to associate such that $b \mid_\tau a$.

(8) For all $a \in Reg(R)^{\#}$, (a) is contained in only finitely many principal ideals (b) where $b \in Reg(R)^{\#}$ such that $b \mid_\tau a$.

Proof. (1) $\Rightarrow$ (2) Let R be a τ-r-FFR and $a \in \text{Reg}(R)^{\#}$, then there are only a finite number of τ-factorizations (up to order and associate), each of these is of finite length. Hence, since every τ-divisor of a must be among these up to associate, R is a τ-r-WFFR.

(2) $\Rightarrow$ (3) Let R be a τ-r-WFFR and $a \in$Reg$(R)^{\#}$. If a has a finite number of τ-divisors, then certainly it has a finite number of irreducible τ-divisors, so it suffices to show a has a τ-r-atomic factorization. We instead show the stronger condition, that R satisfies τ-r-ACCP, that is any chain of principal ideals generated by regular elements $(a_0) \subsetneq (a_1) \subsetneq \cdots \subset (a_i) \subsetneq \cdots$ with a_{i+1} occurring as a τ-factor in a τ-r-factorization of a_i and $a_i \in$Reg$(R)^{\#}$ for all i comes to a halt. Suppose there is an infinite chain, but then each a_i is a τ-divisor of a_0 and none of them are associate since each containment is proper, so we would have an infinite number of non-associate τ-r-divisors contradicting the fact that R is a τ-r-WFFR (note: we use strongly here that τ is refinable to ensure that at each step we retain a τ-factorization).

(3) $\Rightarrow$ (1) This proof is similar to [3, Thm 5.1]. Let R be a τ-r-atomic τ-r-idf ring and $x \in$ Reg$(R)^{\#}$. Let $x_1, \cdots, x_n$ be the τ-r-irreducible τ-factors of x, in particular they are all regular elements of R. Suppose that in a τ-factorization of x, $x = \lambda x_1^{s_1} \cdots x_n^{s_n}$, we always have $0 \le s_i \le N_i$ for each $1 \le i \le n$. Then there is a bound on the number of non-associate factors of x. So we suppose that this is not the case. There must then be some s_i which is not bounded, we assume it is the first one s_1. Hence for each $k \ge 1$, we can write $x = \lambda_k x_1^{s_{k_1}} \cdots x_n^{s_{k_n}}$, where $\lambda_k \in U(R)$ and $s_{1_1} < s_{2_1} < s_{3_1} < \cdots$. Suppose that in this set of factorizations $\{s_{k_i}\}$ is bounded for each i with $1 < i \le n$. Then since there are only finitely many choices for $s_{k_2}, \cdots, s_{k_n}$ we must have $s_{k_2} = s_{j_2}, \cdots, s_{k_n} = s_{j_n}$ for some $j > k$. But then $\lambda_j x_1^{s_{j_1}} \cdots x_n^{s_{j_n}} = x = \lambda_k x_1^{s_{k_1}} \cdots x_n^{s_{k_n}}$, but since each x_i is regular, we can cancel to get $\lambda_j x_1^{s_{j_1}} = \lambda_k x_1^{s_{k_1}}$, where $s_{j_1} > s_{k_1}$, but then x_1 would be a unit, a

contradiction. Thus we must have some set $\{s_{k_i}\}$ for a fixed i with $1 < i \leq n$ is unbounded, say for $i = 2$. By taking sub-sequences at each stage, we may assume that $s_{1_1} < s_{2_1} < s_{3_1} < \cdots$ and $s_{1_2} < s_{2_2} < s_{3_2} < \cdots$. Continuing in this manner, we may assume for each $1 \leq i \leq n$ that $s_{1_i} < s_{2_i} < s_{3_i} < \cdots$. But then we would have $\lambda_1 x_1^{s_{1_1}} \cdots x_n^{s_{1_n}} = x = \lambda_2 x_1^{s_{2_1}} \cdots x_n^{s_{2_n}}$ where $s_{1_i} < s_{2_i}$, a contradiction as again, we would have x_i must be units after cancellation, which is impossible.

(1) $\Rightarrow$ (4) This is clear as we have already seen that a τ-r-FFR is τ-r-atomic and a τ-r-atomic factorization is certainly a τ-r-factorization, so there must be a finite number of τ-r-atomic factorizations up to order and associate for every $a \in \text{Reg}(R)^{\#}$.

(4) $\Rightarrow$ (3) Let $a \in \text{Reg}(R)^{\#}$, then there are a finite number of τ-r-atomic factorizations, each has a finite number of τ-r-atomic factors, so the collection of τ-r-atomic divisors is finite, so R is a τ-r-atomic τ-r-idf ring.

(5), (6) are restatements of (2) and their equivalence is immediate. Furthermore, (5) and (7) (resp. (6) and (8)) are seen to be equivalent after noting that for $b \in \text{Reg}(R)$, $a \mid_\tau b$ implies there is some τ-factorization $b = \lambda a a_1 \cdots a_n$, but since b is regular and the set of regular elements is saturated, every τ-factor must be regular so this is really a τ-factorization. $\square$

Theorem 6.3. *Let R be a commutative ring with 1, with τ a symmetric relation on $R^{\#}$. We have the following.*

(1) R is a τ-r-UFR implies R is a τ-r-HFR.

(2) For τ refinable, R is a τ-r-HFR implies R is a τ-r-BFR.

(3) For τ refinable, R is a τ-r-UFR implies R is a τ-r-FFR.

(4) R is a τ-r-FFR implies R is a τ-r-BFR.

(5) For τ refinable, R is a τ-r-BFR implies R satisfies τ-r-ACCP.

(6) For τ refinable, R satisfies τ-r-ACCP implies R is τ-r-atomic.

The following diagram summarizes our result ($\star$ represents τ being refinable):

$$\begin{tikzcd}
& \tau\text{-r-HFR} \arrow[dr, Rightarrow, "\star"] & & \tau\text{-ACCP} \arrow[d, Rightarrow] & \\
\tau\text{-r-UFR} \arrow[ur, Rightarrow] \arrow[r, Rightarrow, "\star"] & \tau\text{-r-FFR} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow, "\star"] & \tau\text{-r-BFR} \arrow[r, Rightarrow, "\star"] & \tau\text{-}r\text{-ACCP} \arrow[r, Rightarrow, "\star"] & \tau\text{-r-atomic} \\
& \tau\text{-r-WFFR} \arrow[dr, Rightarrow] \arrow[d, Leftrightarrow, "\star"] & & & \\
& \tau\text{-atomic } \tau\text{-r-idf} \arrow[r, Rightarrow] & \tau\text{-r-idf} & &
\end{tikzcd}$$

Proof. (1) This is immediate from the definition.

(2) Let R be a τ-r-HFR. Suppose $a = \lambda a_1 \cdots a_n$ is a τ-r-atomic factorization. We claim $N_r(a) = n$. Let $a = \mu b_1 \cdots b_m$ be a τ-r-factorization of a. Since R is τ-r-atomic, we can find τ-r-atomic factorizations for b_i for each $1 \leq i \leq m$. We have assumed τ to be refinable, so we can replace each b_i with the corresponding τ-r-atomic factorization and collect the units in the front of the factorization and retain a τ-r-factorization which is τ-atomic and thus must have length n. The refinement process can only increase the length of the factorization, so the length of the original factorization is no longer than n, proving the claim.

(3) We show for τ-refinable, R a τ-r-UFR, R is a τ-r-atomic τ-r-idf-ring which has been shown in Theorem 6.2 to be equivalent to being a τ-r-FFR. R being τ-r-factorial gives us τ-r-atomic for free. Furthermore, any τ-atomic factorization of $a \in$Reg$(R)^{\#}$ has the same length, say n and can be reordered so that the associates

match up. This tells us there are precisely n τ-irreducible divisors of a up to associate, hence R is a τ-r-idf-ring.

(4) Suppose R is a τ-r-FFR, by definition, we know R is τ-r-atomic. Now, let $a \in$Reg$(R)^{\#}$, let S be the finite set of all τ-atomic factors of a. Set $N(a) = |S|$. Let $a = \lambda a_1 \cdots a_n$ be a τ-atomic factorization of a, then $a_i \in S$ for all i, but then $\{a_i\}_{i=1}^n \subseteq S$ and hence is finite and $n \leq N(a) = |S|$ as desired, so R is a τ-r-BFR.

(5) Let R be a τ-r-BFR, and we suppose for a moment that R does not satisfy τ-r-ACCP. There must exist and infinite sequence $\{a_i\}_{i=1}^{\infty} \subseteq$Reg$(R)^{\#}$ such that $a_{n+1} \mid_\tau a_n$, but $a_{n+1} \not\sim a_n$ for all $n \geq 1$. Let $a_n = \lambda_{n+1} r_{n+11} \cdots r_{n+1s_{n+1}} a_{n+1}$ be a τ factorization of a_n for all $n \geq 1$. But then we have

$$a_1 = \lambda_2 r_{2_1} \cdots r_{2_{s_2}} a_2 = \lambda_2 r_{2_1} \cdots r_{2_{s_2}} \lambda_3 r_{3_1} \cdots r_{3_{s_3}} a_3 = \cdots$$

is a τ factorization (note we use τ refinable here). Furthermore, each of these factorizations can be refined into τ-atomic elements, and it will still be a τ-factorization the length of which $L_\tau(a_1) \geq s_2 + s_3 + \cdots s_n + 1 \geq n$ which shows we can find arbitrarily large τ-atomic factorizations of a_1 which contradicts the fact the R is a τ-r-BFR.

(6) Let R satisfy τ-r-ACCP, but suppose that R is not τ-r-atomic. Then there exists $a \in$Reg$(R)^{\#}$ with no τ-factorization into τ-atoms. a itself cannot be a τ-atom, so say $a = \lambda a_1 \cdots a_n$ is a τ-factorization with $n > 1$. Now again some a_i must not be a product of τ-atoms, or with τ refinable, we could find a τ-atomic factorization, say it is a_1. Then $a_1 \mid_\tau a$ and $a_1 \not\sim a$ put $b_1 = a_1$. Then a_1 must have a τ-factorization $a_1 = \lambda_2 a_{2_1} \cdots a_{2_{n_2}}$ where $n_2 > 1$. Again, one of the τ-factors, say a_{2_1} cannot be a τ-product of τ-atoms. Here $a_{2_1} \mid_\tau a_1 = b_1$ and $a_{21} \not\sim a_1$. Put $b_2 = a_{2_1}$. Continuing in

this fashion, we obtain a sequence $\{b_i\}_{i=1}^{\infty}$ of elements of $\mathrm{Reg}(R)^{\#}$ such that $b_{n+1} \mid_\tau b_n$ but $b_{n+1} \not\sim b_n$ for every $n \geq 1$. This contradicts R satisfying τ-r-ACCP. $\square$

6.3 τ_{reg}-Factorizations

In this section, we study another approach which could have be used to extend τ-factorization to commutative rings with zero-divisors using regular factorizations. In Section 6.1, we decided to only consider factorizations of the regular elements. We chose to simply restrict the elements we attempt to factor to the regular elements of a commutative ring R. We could have instead chosen to restrict the relation τ itself. This will be the motivation of this section.

Let R be a commutative ring with 1 and τ a symmetric relation on $R^{\#}$. Then we define a new relation

$$\tau_{\mathrm{reg}} := \tau \cap (\mathrm{Reg}(R) \times \mathrm{Reg}(R))\,.$$

We may now pursue the τ-factorizations using the approach from Chapter 2 and look at factoring all the non-units in R instead of just the regular elements. There is certainly a very close relationship between τ_{reg}-factorizations and τ-regular factorizations; however, there are a few subtle differences that cause some problems, especially with the definition of τ_{reg}-very strongly atomic elements since the author insisted that part of a being τ-very strongly atomic was that $a \cong a$. The fact that the very strongly associate relation need not be reflexive is the main reason there is not a perfect correspondence between the two approaches. We will see that τ_{reg}-factorizations are simply very poorly behaved when it comes to τ_{reg}-very strong atoms and rearrangement up to very strong associates.

Of course any non-trivial idempotent element, e, is a zero-divisor since $e(e-1)=0$. Furthermore, since $e=e^2$, we see that $e \not\approx e$. This means that e is not very strongly atomic for any non-trivial idempotent element. On the other hand, since every non-trivial τ_{reg}-factorization consists of a product of regular elements, we can have no non-trivial τ_{reg}-factorizations of e. This means the only τ_{reg}-factorizations of any zero-divisor, in particular e, are the trivial factorizations. Unfortunately, in the case of a non-trivial idempotent, e, this means e is not a τ-very strong atom, and will never have a τ_{reg}-very strongly atomic factorization. We demonstrate this in the following example.

Example 6.1. Let K be an infinite field. $R = K \times K$ with $\tau = R^{\#} \times R^{\#}$.

We consider the element $(1,0) \in Z(R)$. This ring has only idempotent elements and units. So the set of non-unit regular elements is empty and our ring is vacuously a τ-r-UFR. On the other hand, for any unit $\mu \in K^*$, we have $(1,0) = (\mu^{-1},1)(\mu,0)$ is the only type of τ_{reg}-factorization of $(1,0)$, yet none of these are τ_{reg}-very strongly atomic factorizations. The problem is that $(\mu,0) \not\approx (\mu,0)$ since we have $(\mu,0) = (1,0)(\mu,0)$ and $(1,0)$ is not a unit. This shows we can have a τ-r-UFR which is not even τ_{reg}-atomic. It gets worse. Each of these factorizations is non-very strongly associate. Let $\mu, \lambda \in K^*$. Then $(1,0) = (\mu^{-1},1)(\mu,0) = (\lambda^{-1},1)(\lambda,0)$ are two τ_{reg}-factorizations of $(1,0)$, but $(\mu,0) = (\mu\lambda^{-1},0)(\lambda,0)$ with $(\mu\lambda^{-1},0)$ not a unit shows $(\mu,0) \not\approx (\lambda,0)$. Since K is infinite, there are infinitely many τ_{reg}-factorizations of $(1,0)$, none of which are very strongly associate. $\boxplus$

This problem is more an indication of the problem with the very strongly associate relation not being reflexive and the requirement in Section 2.2 that τ-very strong atoms be self very strongly associate rather than an issue with regular factorizations. We could have perhaps added yet another type of τ-irreducible which is not quite as powerful as being τ-very strongly atomic, but stronger than being τ-m-atomic or τ-strongly atomic. We call it τ-unrefinable. A non-unit $a \in R$ is said to be *τ-unrefinable* if a admits only trivial τ-factorizations. A ring R is said to be *τ-unrefinably atomic* if for every non-unit $a \in R$, there is a τ-factorization into τ-unrefinable elements.

Theorem 6.4. *Let R be a commutative ring with 1 and τ be a symmetric relation on $R^{\#}$. Let $a \in R$ be a non-unit. The following diagram illustrates the relationship between the various types of τ-irreducibles a might satisfy where $\dagger$ represents the implication requires a strongly associate ring:*

$$
\begin{tikzcd}
\tau\textit{-very strongly irred.} \arrow[r, Rightarrow] & \tau\textit{-unrefinable} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \tau\textit{-strongly irred.} \arrow[r, Rightarrow] & \tau\textit{-irred.} \\
& & \tau\textit{-m-irred.} \arrow[u, Rightarrow, "\dagger"'] \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

Proof. If a is τ-very strongly irreducible, then it is immediate that a is also τ-unrefinable. We have simply removed that $a \cong a$ condition. If a is τ-unrefinable, then the only τ-factorizations of a are of the form $a = \lambda b$ for some $b \in R$, but this shows $a \approx b$ and therefore $a \sim b$ proving a is both τ-m-atomic and τ-strongly atomic. $\square$

This leads us to the following results.

Lemma 6.5. *Let R be a commutative ring with 1 and let τ be a symmetric relation on $R^{\#}$. Let $\tau_{reg} := \tau \cap (Reg(R) \times Reg(R))$. The collection of non-trivial τ-regular-factorizations and non-trivial τ_{reg}-factorizations coincide.*

Proof. Let $a = \lambda a_1 \cdots a_n$ be a non-trivial τ-regular factorization. Then $a \in \mathrm{Reg}(R)$ by definition of τ-regular factorization, and $a_i \tau a_j$ for all $i \neq j$. Since a is regular, and the set of regular elements is saturated, we have $a_i \mid a \in \mathrm{Reg}(R)$ for each $1 \leq i \leq n$, we know that $a_i \in \mathrm{Reg}(R)$ for each $1 \leq i \leq n$. This means $a_i \tau_{\mathrm{reg}} a_j$ for each $i \neq j$. Thus $a = \lambda a_1 \cdots a_n$ is a τ_{reg}-factorization.

Conversely, suppose $a = \lambda a_1 \cdots a_n$ is a non-trivial τ_{reg}-factorization. Then $a_i \tau_{\mathrm{reg}} a_j$ for each $i \neq j$. This means $a_i \tau a_j$ and $a_i, a_j \in \mathrm{Reg}(R)$. In particular, since $n \geq 2$, we can conclude that $a_1 a_2 \cdots a_n$ is a product of regular elements, so $a \in \mathrm{Reg}(R)$. This means $a = \lambda a_1 \cdots a_n$ is a τ-regular-factorization. □

Theorem 6.6. *Let R be a commutative ring with 1 and let τ be a symmetric relation on $R^{\#}$. Let $\tau_{reg} := \tau \cap (Reg(R) \times Reg(R))$. For $a \in Reg(R)$, the following are equivalent.*

(1) a is a τ-regular-atom.

(2) a is a τ_{reg}-atom.

(3) a is a τ_{reg}-strong atom.

(4) a is a τ_{reg}-m-atom.

(5) a is τ_{reg}-unrefinable.

(6) a is a τ_{reg}-very strong atom.

Proof. When we consider Theorem 6.4, it suffices to show that (2) $\Rightarrow$ (6) and then we show that (1) $\Leftrightarrow$ (5). Let $a \in \text{Reg}(R)$, be a τ_{reg}-atom. Since $a \in \text{Reg}(R)$, we have $a \cong a$ since $a = ra$ implies $r = 1$. Furthermore, if $a = \lambda a_1 \cdots a_n$ is a τ_{reg}-factorization of a, then $a \sim a_i$ for some $1 \leq i \leq n$. Since $a \in \text{Reg}(R)$, $a \cong a_i$ and we have shown that a is a τ_{reg}-very strongly atom.

(1) $\Leftrightarrow$ (5) In light of Lemma 6.5, a has a non-trivial τ-regular factorization if and only if a has a non-trivial τ_{reg}-factorization. $\square$

Corollary 6.1. *Let R be a commutative ring with 1 and let τ be a symmetric relation on $R^{\#}$. Let $\tau_{reg} := \tau \cap (Reg(R) \times Reg(R))$. Let $\alpha \in \{$ atomic, strongly atomic, m-atomic, unrefinably atomic $\}$. Let $a \in Reg(R)$ be a non-unit, then $a = \lambda a_1 \cdots a_n$ is a τ_{reg}-α-factorization if and only if $a = \lambda a_1 \cdots a_n$ is a τ-regular-factorization.*

Proof. This is immediate from what we have shown in Theorem 6.6. $\square$

Theorem 6.7. *Let R be a commutative ring with 1 and let τ be a symmetric relation on $R^{\#}$. Let $\tau_{reg} := \tau \cap (Reg(R) \times Reg(R))$. If $a \in Z(R)$, then following hold.*

(1) a is a τ_{reg}-atom.

(2) a is a τ_{reg}-strong atom.

(3) a is a τ_{reg}-m-atom.

(4) a is τ_{reg}-unrefinable.

Proof. By Theorem 6.4, it suffices to show, for $a \in Z(R)$, (1) $\Rightarrow$ (4). Let a be a τ_{reg}-atom, and suppose $a = \lambda a_1 \cdots a_n$ is a non-trivial τ_{reg}-factorization. This implies $n \geq 2$, and therefore $a_i \tau_{\text{reg}} a_j$ for each $i \neq j$. In particular, $a_i \in \text{Reg}(R)$ for all $1 \leq i \leq n$. This means a is a product of regular elements and is therefore regular, a contradiction. □

Theorem 6.8. *Let R be a commutative ring with 1 and let τ be a symmetric relation on $R^{\#}$. Let $\tau_{reg} := \tau \cap (Reg(R) \times Reg(R))$. The following are equivalent.*

(1) R is τ-regular-atomic.

(2) R is a τ_{reg}-atomic.

(3) R is a τ_{reg}-strongly atomic.

(4) R is a τ_{reg}-m-atomic.

(5) R is τ_{reg}-unrefinably atomic.

Proof. Let a be a non-unit in R. Then $a \in Z(R)$ or $a \in \mathrm{Reg}(R)$. If $a \in Z(R)$, we apply Theorem 6.7 to see that a itself is τ_{reg}-atomic, τ_{reg}-strongly atomic, τ_{reg}-m-atomic, and τ_{reg}-unrefinably atomic. For R to be a τ-regular-atomic ring, we need only check the regular elements for τ-regular atomic factorizations. If $a \in \mathrm{Reg}(R)$, we apply Corollary 6.1 to see the equivalence. □

Lemma 6.9. *Let R be a commutative ring with 1 and let τ be a symmetric relation on $R^{\#}$. Let $a = \lambda(\lambda^{-1}a) = \mu(\mu^{-1}a)$ be two trivial factorizations of a. Then we have the following*

(1) $\lambda^{-1}a$ and $\mu^{-1}a$ are associates.

(2) $\lambda^{-1}a$ and $\mu^{-1}a$ are strong associates.

Proof. (2) $(\mu^{-1}\lambda)(\lambda^{-1}a) = \mu^{-1}a$ with $(\mu^{-1}\lambda) \in U(R)$ proves $\lambda^{-1}a \approx \mu^{-1}a$. If $\lambda^{-1}a \approx \mu^{-1}a$, then $\lambda^{-1}a \sim \mu^{-1}a$. □

Remark 6.1. Given the above situation, $\lambda^{-1}a$ and $\mu^{-1}a$ need not be very strong associates. For instance $R = \mathbb{R} \times \mathbb{R}$,

$$(1,0) = (1,1)(1,0) = (-1,-1)(-1,0)$$

yet $(1,0) \not\cong (-1,0)$. $\boxplus$

Theorem 6.10. *Let R be a commutative ring with 1 and let τ be a symmetric relation on $R^{\#}$. Let $\tau_{reg} := \tau \cap (Reg(R) \times Reg(R))$. Let $\alpha \in \{$ atomic, strongly atomic, m-atomic, unrefinably atomic $\}$ and $\beta \in \{$ associate, strongly associate $\}$. Then we have the following.*

(1) R satisfies τ-regular-ACCP if and only if R satisfies τ_{reg}-ACCP.

(2) R is a τ-regular-UFR if and only if R is a τ_{reg}-α-β-UFR.

(3) R is a τ-regular-HFR if and only if R is a τ_{reg}-α-HFR.

(4) R is a τ-regular-BFR if and only if R is a τ_{reg}-BFR.

(5) R is a τ-regular-idf ring if and only if R is a τ_{reg}-α-β-df ring.

(6) R is a τ-regular-atomic τ-regular-idf ring if and only if R is a τ_{reg}-α, τ_{reg}-α-β-df ring.

(7) R is a τ-regular-WFFR if and only if R is a τ_{reg}-β-WFFR.

(8) R is a τ-regular-FFR if and only if R is a τ_{reg}-β-FFR.

If τ is refinable, then $(6) \Leftrightarrow (7) \Leftrightarrow (8)$.

Proof. (1) The statement that $(a) \subsetneq (a_1)$ with $a_1 \mid_\tau a$ implies that $a = \lambda a_1 a_2 \cdots a_n$. We notice here that $n \geq 2$ or else we would have $a = \lambda a_1$ or $a \approx a_1$ which implies $(a) = (a_1)$, a contradiction. So these properly ascending chains yield non-trivial factorizations at each step. Thus any properly ascending chain of principal ideals

$$(a_1) \subsetneq (a_2) \subsetneq (a_3) \subsetneq \cdots \tag{6.1}$$

such that $a_{i+1} \mid_{\tau_{\text{reg}}} a_i$ yields a τ-regular factorization of a_i with a_{i+1} as a τ-regular factor. Conversely, any ascending chain as in (6.1) with a_i regular for all i and a_{i+1} occurring as a τ-factor in some τ-regular factorization of a_i yields a τ_{reg}-factorization of a_i as well. Hence R fails to satisfy τ-regular ACCP if and only if R fails to satisfy τ_{reg}-ACCP, and the proof is complete.

(2) We know from Theorem 6.8 that R is τ-regular-α if and only if R is τ_{reg}-α. Let $a \in R$ be a non-unit. If $a \in Z(R)$, we know from Theorem 6.7 that a is τ_{reg}-α. Furthermore, any trivial τ_{reg}-factorization of a is unique up to β by Lemma 6.9. For R to be a τ-regular UFR, we need only check the regular elements. Let $a \in \text{Reg}(R)$. We know from Corollary 6.1, for regular elements, τ-atomic and τ_{reg}-α-factorizations of a coincide, so the uniqueness up to rearrangement and β is immediate.

(3) By Theorem 6.8, R is τ-regular-α if and only if R is τ_{reg}-α. If $a \in Z(R)$, then a is τ_{reg}-α and has only trivial τ_{reg}-factorizations each of which has length 1. For $a \in \text{Reg}(R)$, τ-atomic and τ_{reg}-α-factorizations of a coincide by Corollary 6.1, and the equivalence is clear.

(4) For $a \in Z(R)$, all τ_{reg}-factorizations are trivial and have length 1. By Lemma 6.5, the set of non-trivial τ-regular factorizations and τ_{reg}-factorizations coincide and the equivalence is apparent.

(5) If $a \in Z(R)$, a itself is τ_{reg}-α and there is precisely one unique τ_{reg}-α-divisor of a up to β since all trivial τ_{reg}-factorizations are β from Lemma 6.9. If $a \in \text{Reg}(R)$,

then the set of τ-regular atomic divisors and τ_{reg}-α-divisors of a are all regular and hence coincide by Theorem 6.6 so the equivalence is clear.

(6) This is simply (5) plus Theorem 6.8.

(7) For $a \in Z(R)$, the only τ_{reg}-divisors of a are unit multiples of a, so there is only one τ_{reg}-divisor of a up to β. For $a \in \text{Reg}(R)$, since the set of τ-regular factorizations and the set of τ_{reg}-factorizations of a are the same, the set of τ_{reg}-divisors and τ-regular divisors coincide and are regular, so the associate relations also coincide. Thus the equivalence follows.

(8) For $a \in Z(R)$, the only τ_{reg}-factorizations of a are of the form $a = \lambda(\lambda^{-1}a)$, so there is only one τ_{reg}-factorization of a up to β. For $a \in \text{Reg}(R)$, since the set of τ-regular factorizations and the set of τ_{reg}-factorizations of a are the same. Moreover, the set of τ_{reg}-factors and τ-regular factors coincide and are regular, hence the associate relations also coincide. Thus the equivalence follows. □

6.4 Relation with Other Factorization Properties

We now compare the τ-r-finite factorization properties with the τ-finite factorization properties defined originally in Section 2.3.

Theorem 6.11. *Let R be a commutative ring with 1 and let τ be a symmetric relation on $R^{\#}$. Let $\alpha \in$ {atomic, strongly atomic, m-atomic, very strongly atomic}, $\beta \in$ {associate, strong associate, very strong associate}. Then we have the following:*

(1) If R is a τ-α-β-UFR, then R is a τ-r-UFR.

(2) If R is a τ-α-HFR, then R is a τ-r-HFR.

(3) If R is a τ-β-FFR, then R is a τ-r-FFR.

(4) If R is a τ-β-WFFR, then R is a τ-r-WFFR.

(5) If R is a τ-β-α df ring, then R is a τ-r idf ring.

(6) If R is a τ-BFR, then R is a τ-r-BFR.

(7) If R satisfies τ-ACCP, then R satisfies τ-r-ACCP.

(8) If R is τ-α, then R is τ-r-atomic.

This yields the following diagram where $\star$ represents τ is refinable.

$$\begin{tikzcd}
\tau\text{-}\alpha\text{-}\beta\text{-}UFR \arrow[d, Rightarrow] & \tau\text{-}r\text{-}HFR \arrow[dr, Rightarrow, "\star"] & \tau\text{-}\alpha\text{-}HFR \arrow[l, Rightarrow] & \tau-ACCP \arrow[d, Rightarrow] & \tau\text{-}\alpha \arrow[d, Rightarrow] \\
\tau\text{-}r\text{-}UFR \arrow[ur, Rightarrow] \arrow[r, Rightarrow, "\star"] & \tau\text{-}r\text{-}FFR \arrow[r, Rightarrow] \arrow[d, Leftrightarrow, "\star"] & \tau\text{-}r\text{-}BFR \arrow[r, Rightarrow, "\star"] & \tau\text{-}r\text{-}ACCP \arrow[r, Rightarrow, "\star"] & \tau\text{-}r\text{-}atomic \\
\tau\text{-}\beta\text{-}WFFR \arrow[r, Rightarrow] & \tau\text{-}r\text{-}WFFR \arrow[d, Leftrightarrow, "\star"] & \tau\text{-}BFR \arrow[u, Rightarrow] & & \\
\tau\text{-}\alpha\text{-}\beta \text{ df ring} \arrow[r, Rightarrow] & \tau\text{-}r\text{-}idf \text{ ring} & \tau\text{-}\beta\text{-}FFR \arrow[uul, Rightarrow] & &
\end{tikzcd}$$

Proof. (8) Let $a \in \text{Reg}(R)$. Since R is a τ-α, there is a τ-α-factorization of the form $a = \lambda a_1 \cdots a_n$. Since $a \in \text{Reg}(R)$, $a_i \in \text{Reg}(R)$ for all i, by Proposition 6.1, each of these factorizations is a τ-r-atomic factorization of a, showing R is τ-r-atomic.

(2) (resp. (1)) Let a be a regular non-unit element. We have just seen that R is τ-r-atomic. Given two τ-r-atomic factorizations, $a = \lambda a_1 \cdots a_n = \mu b_1 \cdots b_m$, this is also two τ-α-factorizations. By assumption we have $m = n$ (resp. and there is a rearrangement so that $a_i \sim b_i$ for each $1 \leq i \leq n$.) This proves R is a τ-r-HFR (resp. τ-r-UFR).

[(3)-(6)] Let $a \in \text{Reg}(R)$. For a regular element a, the set of τ-r-factorizations and τ-factorizations are identical, proving (3) and (6). Similarly, since every divisor of a regular element is regular, the set of regular τ-divisors is the same as the set of τ-divisors, proving (4). As in 6.1, we know that the set of τ-α-divisors is the same as the set of τ-r-atoms, proving (5).

(7) Suppose $(a_1) \subsetneq (a_2) \subsetneq \cdots$ is an chain of regular principal ideals such that $a_{i+1} \mid_\tau a_i$, then since R satisfies τ-ACCP, it must become stationary, proving (7). $\square$

Theorem 6.12. *Let R be a commutative ring with 1 and $\tau \subset$ Reg(R)$^\#$ $\times$ Reg(R)$^\#$. Then we have the following.*

(1) R a r-BFR implies R is a τ-r-BFR.

(2) R a r-FFR implies R is a τ-r-FFR.

(3) R a r-WFFR implies R is a τ-r-WFFR.

(4) R satisfies r-ACCP implies R satisfies τ-r-ACCP.

Proof. (1) Let R be a r-BFR, but suppose R is not a τ-r-BFR, then there exists a regular element $a \in$ Reg(R)$^\#$ with τ-factorizations of arbitrarily long length, but any τ-factorization is certainly a factorization into regular elements, so this would contradict the fact that R is a r-BFR.

(2) Let R be a r-FFR, but suppose that R is not a τ-r-FFR. We then have a regular element $a \in$ Reg(R)$^\#$ that has an infinite number of τ-r-factorizations up to rearrangement and associate, but again each of these is also r-factorization and are still unique up to rearrangement and associates which contradicts the fact that R is a r-FFR.

(3) Let $a \in$ Reg(R). Every τ-r-divisor divisor is a regular divisor of a, so there can be only finitely many up to associate.

(4) Suppose we have an infinite sequence $\{a_i\}_{i=1}^{\infty}$, $a_k \in$Reg$(R)^{\#}$ for all k with $a_{n+1} \mid_\tau a_n$ but $a_{n+1} \not\sim a_n$ for all $n \geq 1$. But then we still have $a_{n+1} \mid_\tau a_n$, $a_k \in$Reg$(R)^{\#}$ for all k but $a_{n+1} \not\sim a_n$ so we contradict r-ACCP. Concluding the proof. □

Corollary 6.2. *The r-UFRs, r-FFRs, r-HFRs, r-BFRs as defined in [1, Section 5] satisfy r-ACCP, and therefore τ-r-ACCP. Hence for τ refinable, each is τ-r-atomic by Theorem 6.12 and Theorem 6.3.*

The following diagram summarizes our results ($\star$ represents τ being refinable):

$$\begin{tikzcd}
 & \tau\text{-r-HFR} \arrow[dr, Rightarrow, "\star"] & \text{r-BFR} \arrow[d, Rightarrow] & r\text{-ACCP} \arrow[d, Rightarrow] & \\
\tau\text{-r-UFR} \arrow[ur, Rightarrow] \arrow[r, Rightarrow, "\star"] & \tau\text{-r-FFR} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow, "\star"] & \tau\text{-r-BFR} \arrow[r, Rightarrow, "\star"] & \tau\text{-}r\text{-ACCP} \arrow[r, Rightarrow, "\star"] & \tau\text{-r-atomic} \\
\text{r-FFR} \arrow[ur, Rightarrow] & \tau\text{-r-WFFR} \arrow[dr, Rightarrow] \arrow[d, Leftrightarrow, "\star"] & & \tau\text{-ACCP} \arrow[u, Rightarrow] & \\
\text{r-WFFR} \arrow[ur, Rightarrow] & \tau\text{-atomic } \tau\text{-r-idf} \arrow[r, Rightarrow] & \tau\text{-r-idf} & &
\end{tikzcd}$$

CHAPTER 7
τ-COMPLETE FACTORIZATIONS

In this chapter, we investigate another particularly effective way to extend τ-factorization to rings with zero-divisors. This is the method of τ-complete factorization. This method was originally defined in the integral domain case in [6]. In Section Two, we provide some necessary background definitions and theorems from the theory of τ-factorization in domains as well as the theory of factorization in rings with zero-divisors. In Section Three, we define what we refer to as τ-complete factorizations in rings with zero-divisors. These are τ-factorizations in which the factorizations cannot be refined to create any strictly longer τ-factorization. We proceed to define several τ-complete finite factorization properties rings may possess. In Section Four, we investigate the relationship between these new τ-complete factorizations and the the previous τ-irreducible factorizations studied in Chapter 2.

7.1 τ-Complete Factorizations Definitions

Another approach to factorization studied in the domain case is that of τ-complete factorization. In some ways, this notion is more natural. The idea behind complete factorization is simply to factor and element as far as possible. One says a factorization is complete when it is no longer possible to replace one of the factors with a strictly longer factorization. In the τ-factorization case, we see τ-complete factorizations have several nice consequences. Many of the properties such as τ being divisive, multiplicative, refinable, combinable, associate preserving are no longer necessary for many of the major desirable theorems to hold.

We begin with some definitions. Recall that a τ-*refinement* of a τ-factorization $\lambda a_1 \cdots a_n$ is a τ-factorization of the form

$$(\lambda\lambda_1 \cdots \lambda_n) b_{11} \cdots b_{1m_1} \cdot b_{21} \cdots b_{2m_2} \cdots b_{n1} \cdots b_{nm_n}$$

where $a_i = \lambda_i b_{i_1} \cdots b_{i_{m_i}}$ is a τ-factorization for each i. A τ-*complete factorization* is a τ-factorization that cannot be τ-refined into a longer τ-factorization. R is said to be τ-complete if every non-unit has a τ-complete factorization. R is said to be τ-*completable* (resp. τ-*atomicable*, τ-*strongly-atomicable*, τ-*m-atomicable*, τ-*very strongly atomicable*) if every τ-factorization can be τ-refined to a τ-complete (resp. τ-atomic, τ-strongly atomic, τ-m-atomic, τ-very strongly atomic) factorization.

Let $\alpha \in$ {completable, atomicable, strongly atomicable, m-atomicable, very strongly atomicable} and $\beta \in$ {associate, strong associate, very strong associate}. If $\alpha =$ completable, set $\alpha' =$ complete. If $\alpha =$ atomicable (resp. strongly atomicable, m-atomicable, very strongly atomicable), set $\alpha' =$ atomic (resp. strongly atomic, m-atomic, very strongly atomic).

We then say R is a τ-α-β-*UFR* if (1) R is τ-α and (2) if $a = \lambda \cdot a_1 \cdots a_n = \mu b_1 \cdots$ are two τ-α' factorizations of a non-unit $a \in R$, then $n = m$ and after re-ordering, if necessary, a_i and b_i are β for all $i \in \{1, \ldots, n\}$. R is a τ-α-β-*HFR* if (1) R is τ-α and (2) if $a = \lambda \cdot a_1 \cdots a_n = \mu b_1 \cdots$ are two τ-α' factorizations of a non-unit $a \in R$, then $n = m$. We say that R is a τ-*complete*-β-*FFR* (resp. τ-*complete-BFR*) if for each non-unit $a \in R$, there are only a finite number of τ-complete factorizations of a up to reordering and β (resp. there is a natural number $N(a)$ so that for each τ-

complete factorization $a = \lambda a_1 \cdots a_n$, $n \leq N(a)$). We say R is a *τ-complete-β-divisor finite ring or τ-β-cdf ring* if for every non-unit $a \in R$ there are a finite number of divisors up to β, which appear in a τ-complete factorization of β.

Theorem 7.1. *Let R be a commutative ring with 1, τ a symmetric relation on $R^{\#}$. Let $a = \lambda a_1 \cdots a_n$ be a τ-factorization. We consider the following statements.*

(1) This is a τ-very strongly atomic factorization.

(2) This is a τ-complete factorization.

(3) This is a τ-strongly atomic factorization.

(4) This is a τ-m-atomic factorization.

(5) This is a τ-atomic factorization.

Let $\star$ represent τ-refinable and $\dagger$ represent R is strongly associate. Then we have the following relationship between the different factorizations.

$$\begin{tikzcd}
\tau\text{-very strongly irred.} \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \tau\text{-strongly irred.} \arrow[r, Rightarrow] & \tau\text{-irred.} \\
\tau\text{-complete} \arrow[r, Rightarrow, "\star"] \arrow[ur, Rightarrow, "\star"] & \tau\text{-m-irred.} \arrow[u, Rightarrow, "\dagger"'] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Proof. Many of these implications were shown in Chapter 2. We need only prove the arrows entering and exiting from the τ-complete factorizations. We begin with (1) $\Rightarrow$ (2). If $a = \lambda a_1 \cdots a_n$ is a τ-very strongly atomic factorization, then a_i is τ-very strongly atomic and hence has only trivial τ-factorizations. This means there simply are no refinements of a_i which can possibly increase the length of the factorization making the factorization τ-complete.

If τ is refinable, then (2) $\Rightarrow$ (3). Let $a = \lambda a_1 \cdots a_n$ be a τ-complete factorization. We show that a_i is τ-strongly atomic for all $1 \leq i \leq n$. Suppose there a_i is not

τ-strongly atomic. Then there is a τ-factorization $a_i = \mu b_1 \cdots b_m$ such that $a_i \not\approx b_j$ for any $1 \leq j \leq m$. In particular, $m \geq 2$, or else we have $a_i = \mu b_1$ and $a_i \approx b_1$, a contradiction. Because τ is refinable, we can refine the factorization into

$$a = (\lambda\mu)a_1 \cdots a_{i-1}b_1 \cdots b_m a_{i+1} \cdots a_n.$$

This is a τ-factorization of strictly longer length contradicting the assumption that the factorization was τ-complete.

If τ is refinable, then (2) $\Rightarrow$ (4). Let $a = \lambda a_1 \cdots a_n$ be a τ-complete factorization. We show that a_i is τ-m-atomic for all $1 \leq i \leq n$. Suppose there a_i is not τ-m-atomic. Then there is a principal ideal generated by some $b_1 \in R$ such that $b \mid_\tau a_i$ and $(a_i) \subsetneq (b_1)$. Because $b_1 \mid_\tau a_i$, there exists a τ-factorization of the form $a_i = \mu b_1 \cdots b_m$. In particular, $m \geq 2$, or else we have $a_i = \mu b_1$ and $a_i \sim b_1$, a contradiction. Because τ is refinable, we can refine the factorization into

$$a = (\lambda\mu)a_1 \cdots a_{i-1}b_1 \cdots b_m a_{i+1} \cdots a_n.$$

This is a τ-factorization of strictly longer length contradicting the assumption that the factorization was τ-complete. $\square$

We now provide examples to show τ-complete factorizations are indeed distinct.

Example 7.1. Let $R = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with $\tau = \{((1,0),(1,0))\}$.

Consider the τ-factorization $(1,0) = (1,0)(1,0)$. This is a τ-m-atomic and τ-strongly atomic factorization, but not a τ-complete factorization. To see this, $(1,0)R$

is a maximal ideal since $R/(1,0)R \cong \mathbb{Z}/2\mathbb{Z}$, a field. If an ideal is maximal, it is certainly maximal among principal ideals, so it is m-atomic and therefor τ-m-atomic. R is strongly associate, so we know that the factorization is also τ-strongly atomic. On the other hand,

$$(1,0) = (1,0)(1,0) = (1,0)\left((1,0)\cdot(1,0)\right) = (1,0)(1,0)(1,0)$$

gives us a τ-refinement of the factorization which is properly longer, showing it is not a τ-complete factorization. ⊞

Example 7.2. Let $R = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with $\tau = \{((1,0),(0,1)),((0,1),(1,0))\}$.

Consider the τ-factorization $(0,0) = (1,0)(0,1)$. There are no non-trivial τ-factorizations of $(1,0)$ or $(0,1)$, this makes the factorization τ-complete since it cannot be refined into any longer τ-factorization. On the other hand $(1,0)$ is not τ-very strongly atomic because it fails the $(1,0) \cong (1,0)$ part of the definition. The factorization $(1,0) = (1,0)(1,0)$ and noting that $(1,0)$ is not a unit in R shows this. Hence we have a τ-complete factorization which is not τ-very strongly atomic. ⊞

Examples given in [8] show that the other arrows are not reversible even when $\tau = R^{\#} \times R^{\#}$.

As in a series of papers by A. Bouvier, [20, 18, 19, 17], a commutative ring is said to be *présimplifiable* if $x = xy$ for some $x, y \in R$ implies $x = 0$ or $y \in U(R)$. A nice property of the various τ-irreducibles defined in Section 2.2 is that they all coincide when a ring is présimplifiable. When R is présimplifiable and τ is refinable, we can add τ-complete factorizations to the list of equivalent τ-irreducible factorizations. We summarize this in the following proposition which follows relatively easily from Theorem 7.1.

Proposition 7.2. *Let R be présimplifiable and τ be a symmetric, refinable relation on $R^{\#}$. For a τ-factorization $a = \lambda a_1 \cdots a_n$, the following are equivalent.*

(1) This is a τ-very strongly atomic factorization.

(2) This is a τ-complete factorization.

(3) This is a τ-strongly atomic factorization.

(4) This is a τ-m-atomic factorization.

(5) This is a τ-atomic factorization.

Proof. In a présimplifiable ring, all the associate relations coincide. We have $x \sim y \Rightarrow x \cong y$, in particular $a_i \sim a_i \Rightarrow a_i \cong a_i$, so we have (5) $\Rightarrow$ (1). This coupled with what was shown in Theorem 7.1 completes the proof. □

Proposition 7.3. *Let R be a commutative ring with 1 and let τ be a refinable, symmetric relation on $R^{\#}$. If $a = \lambda a_1 \cdots a_n$ is a τ-factorization and $a_i = \lambda_i b_{i1} \cdots b_{im_i}$ are τ-complete factorizations for $1 \leq i \leq n$, then*

$$a = (\lambda\lambda_1 \cdots \lambda_n) b_{11} \cdots b_{1m_1} b_{21} \cdots b_{2m_2} \cdots b_{n1} \cdots b_{nm_n} \tag{7.1}$$

is a τ-complete factorization.

Proof. Because τ is refinable, we know that the factorization given in 7.1 is certainly a τ-factorization. It remains to be seen that this factorization is τ-complete. Taking the notation from the statement of the proposition, we suppose there is a τ-refinement of b_{ij} for some $1 \leq i \leq n$ and $1 \leq j \leq m_i$ of the form $b_{ij} = \mu c_1 \cdots c_k$ which makes the factorization in equation (7.1) properly longer. This also yields a τ-refinement of the τ-complete factorization $a_i = \lambda_i b_{i1} \cdots b_{im_i}$ into

$$a_i = (\lambda_i \mu) b_{i1} \cdots b_{i(j-1)} c_1 \cdots c_k b_{i(j+1)} \cdots b_{im_i}$$

into a τ-factorization which is properly longer, a contradiction completing the proof.

$\square$

7.2 τ-Complete Factorization Relationships

In this section, we look at the relationship between the τ-complete (completable) factorizations defined in Section 7.1 and the τ-atomic (atomicable) (resp. strongly atomic (atomicable), m-atomic (atomicable), very strongly atomic (atomicable)) factorizations defined in Section 2.2.

Let $\alpha \in \{$completable, atomicable, strongly atomicable, m-atomicable, very strongly atomicable$\}$. If $\alpha =$ completable, set $\alpha' =$ complete. If $\alpha =$ atomicable (resp. strongly atomicable, m-atomicable, very strongly atomicable), set $\alpha' =$ atomic (resp. strongly atomic, m-atomic very strongly atomic).

Theorem 7.4. *Let R be a commutative ring with 1. Let τ-be a symmetric relation on $R^{\#}$. We have the following.*

(1) If R is τ-very strongly atomic, then R is τ-complete.

(2) If τ is refinable and R is τ-complete, then R is τ-strongly atomic.

(3) If τ is refinable and R is τ-complete, then R is τ-m-atomic.

(4) If R is τ-m-atomic and strongly associate, then R is τ-strongly atomic.

(5) If R is τ-m-atomic, then R is τ-atomic.

(6) If R is τ-strongly atomic, then R is τ-atomic.

(7) If R is τ-very strongly atomicable, then R is τ-completable.

(8) If τ is refinable and R is τ-completable, then R is τ-strongly atomicable.

(9) If τ is refinable and R is τ-completable, then R is τ-m-atomicable.

(10) If R is τ-m-atomicable and strongly associate, then R is τ-strongly atomicable.

(11) If R is τ-m-atomicable, then R is τ-atomicable.

(12) If R is τ-strongly atomicable, then R is τ-atomicable.

(13) If R is τ-α, then R is τ-α'.

(14) If τ is refinable and R is τ-α', then R is τ-α.

If R is présimplifiable, then (1)-(6) are equivalent and (7)-(12) are equivalent.

Proof. (1) (resp. (2), (3), (4), (5), (6)) Let $a \in R^{\#}$. Because R is τ-very strongly atomic (resp. τ-complete, τ-complete, τ-m-atomic, τ-m-atomic, τ-strongly atomic) we have a τ-factorization $a = \lambda a_1 \cdots a_n$ which is τ-very strongly atomic (resp. τ-complete, τ-complete, τ-m-atomic, τ-m-atomic, τ-strongly atomic). We now apply Theorem 7.1 and the hypothesis to conclude that this factorization is τ-complete (resp. τ-strongly atomic, τ-m-atomic, τ-strongly atomic, τ-atomic, τ-atomic), proving the claim.

(7) (resp. (8), (9), (10), (11), 12)) Let $a = \lambda a_1 \cdots a_n$ be a τ-factorization. Because R is τ-very strongly atomicable (resp. τ-completable, τ-completable, τ-m-atomicable, τ-m-atomicable, τ-strongly atomicable) we can τ-refine this factorization into a τ-factorization $a = \lambda b_1 \cdots b_m$ which is τ-very strongly atomic (resp. τ-complete,

τ-complete, τ-m-atomic, τ-m-atomic, τ-strongly atomic). By Theorem 7.1, this factorization is τ-complete (resp. τ-strongly atomic, τ-m-atomic). This proves any τ-factorization can be refined to a τ-very strongly atomic (resp. τ-complete, τ-complete, τ-strongly atomic, τ-atomic, τ-atomic) factorization as desired.

(13) Let R be τ-α. Let $a \in R^{\#}$. Then $a = 1 \cdot a$ is a τ-factorization and thus it can be τ-refined into a τ-α'-factorization. Thus for any non-zero, non-unit, we can find a τ-α'-factorization proving R is α' as desired.

(14) Let R be α' with τ-refinable. Let $a = \lambda a_1 \cdots a_n$ be a τ-factorization. R is α', so let

$$a_i = \lambda_i b_{1i} \cdots b_{m_i i}$$

be a τ-α' factorization of a_i for $1 \leq i \leq n$. By hypothesis τ-is refinable, so

$$a = (\lambda \lambda_1 \cdots \lambda_n) b_{11} \cdots b_{m_1 1} \cdots b_{12} \cdots b_{m_2 2} \cdots b_{1n} \cdots b_{m_n n} \tag{7.2}$$

is a τ-factorization. Furthermore, for $\alpha' \in \{$ atomic, strongly atomic, m-atomic, very strongly atomic $\}$, we can immediately conclude this factorization is τ-α', proving the claim.

For $\alpha' =$ complete, we apply Proposition 7.3 to Equation (7.2) to see that for a refinable τ, a τ-factorization comprised of τ-complete parts remains τ-complete.

The final sentence follows from an application of Proposition 7.2 since all these factorization types coincide. $\square$

The following diagrams help summarize the relationship between the above properties. Where $\star$ indicates τ is refinable, and $\dagger$ indicates R is a strongly associate ring.

$$\begin{tikzcd}
R \text{ is } \tau\text{-v.s. atomic} \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & R \text{ is } \tau\text{-strongly atomic} \arrow[r, Rightarrow] & R \text{ is } \tau\text{-atomic} \\
R \text{ is } \tau\text{-complete} \arrow[r, Rightarrow, "\star"] \arrow[ur, Rightarrow, "\star"] & R \text{ is } \tau\text{-m-atomic} \arrow[u, Rightarrow, "\dagger"'] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

$$\begin{tikzcd}
R \text{ is } \tau\text{-v.s. atomicable} \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & R \text{ is } \tau\text{-strongly atomicable} \arrow[r, Rightarrow] & R \text{ is } \tau\text{-atomicable} \\
R \text{ is } \tau\text{-completable} \arrow[r, Rightarrow, "\star"] \arrow[ur, Rightarrow, "\star"] & R \text{ is } \tau\text{-m-atomicable} \arrow[u, Rightarrow, "\dagger"'] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Let $\alpha \in \{$ completable, atomicable, strongly atomicable, m-atomicable, very strongly atomicable $\}$. If α = completable, set α' = complete. If α = atomicable (resp. strongly atomicable, m-atomicable, very strongly atomicable), set α' = atomic (resp. strongly atomic, m-atomic very strongly atomic). Let $\star$ indicate τ-refinable.

$$\begin{tikzcd}
R \text{ is } \tau\text{-}\alpha \arrow[d, Rightarrow] \\
R \text{ is } \tau\text{-}\alpha' \arrow[u, bend right, "\star"']
\end{tikzcd}$$

The following corollary is an immediate consequence of definitions and parts (13) and (14) of Theorem 7.4. The proof is clear and thus has been omitted.

Corollary 7.1. *Let R be a commutative ring with 1 and τ be a symmetric relation on $R^{\#}$. Let $\beta \in \{$ associate, strongly associate, very strongly associate $\}$. We have the following.*

(1) R is a τ-completable-β-UFR (resp. τ-completable-HFR) implies R is a τ-complete-β-UFR (resp. τ-complete-HFR).

(2) R is a τ-atomicable-β-UFR (resp. τ-atomicable-HFR) implies R is a τ-atomic-β-UFR (resp. τ-atomic-HFR).

(3) R is a τ-strongly atomicable-β-UFR (resp. τ-strongly atomicable-HFR) implies R is a τ-strongly atomic-β-UFR (resp. τ-strongly atomic-HFR).

(4) R is a τ-m-atomicable-β-UFR (resp. τ-m-atomicable-HFR) implies R is a τ-m-atomic-β-UFR (resp. τ-m-atomic-HFR).

(5) R is a τ-very strongly atomicable-β-UFR (resp. τ-very strongly atomicable-HFR) implies R is a τ-very strongly atomic-β-UFR (resp. τ-very strongly atomic-HFR).

If τ is refinable, then the converses also hold.

Theorem 7.5. *Let R be a commutative ring with 1 and τ be a refinable, associate preserving, symmetric relation on $R^{\#}$. If R satisfies τ-ACCP, then*

(1) R is τ-complete and τ-completable.

(2) R is τ-very strongly atomic and τ-very strongly atomicable.

(3) R is τ-m-atomic and τ-m-atomicable.

(4) R is τ-strongly atomic and τ-strongly atomicable.

(5) R is τ-atomic and τ-atomicable.

Proof. It was shown in Chapter 2, that for τ refinable and associate preserving if R satisfies τ-ACCP, then R is τ-very strongly atomic. Hence for each non-unit $a \in R$, there is a τ-very strongly atomic factorization of a. By Theorem 7.1, this factorization is also τ-complete, τ-m-atomic, τ-strongly atomic, and τ-atomic. This shows R is a τ-complete, τ-very strongly atomic, τ-m-atomic, τ-strongly atomic, and τ-atomic ring. Lastly, using part (14) of Theorem 7.4 and the fact that τ is refinable, we see that R is τ-completable, τ-very strongly atomicable, τ-m-atomicable, τ-strongly atomicable, and τ-atomicable. $\square$

Theorem 7.6. *Let R be a commutative ring with 1 and τ be a symmetric relation on $R^{\#}$. Let $\beta \in \{$ associate, strongly associate, very strongly associate $\}$.*

(1) R is a τ-complete (resp. completable)-β-UFR implies R is a τ-complete (resp. completable)-β-HFR.

(2) R is a τ-complete (resp. completable)-β-UFR implies R is a τ-complete-β-FFR.

(3) R is a τ-complete (resp. completable)-HFR implies R is a τ-complete-BFR.

(4) R is a τ-complete-β-FFR implies R is a τ-complete-BFR.

(5) R is a τ-complete-β-FFR implies R is a τ-β-cdf ring.

(6) For R τ-complete and τ refinable (resp. For R τ-completable), R is τ-complete-BFR implies R satisfies τ-ACCP.

Proof. [(1) and (2)] Let R be a τ-complete (resp. completable)-β-UFR. Then R is τ-complete (resp. completable) by definition. Furthermore, for a non-unit $a \in R$, if there is precisely one complete τ-complete-factorization up to rearrangement and β, so certainly the length is unique, proving R is a τ-complete (resp. completable)-β-HFR. This also shows R is a τ-complete-β-FFR since there is only one τ-complete factorization up to rearrangement and β.

(3) Let R be a τ-complete (resp. completable)-HFR. Let $a \in R$ be a non-unit. Then a has a τ-complete factorization, say $a = \lambda a_1 \cdots a_n$. We then set $N(a) = n$. Given any τ-complete factorization, we know it has length n, so R is a τ-complete-BFR.

(4) Let R be a τ-complete-β-FFR. Let $a \in R$ be a non-unit. Then there are a finite number of τ-complete factorizations of a up to rearrangement and β. Set

$N(a)$ equal to the length of the largest such τ-complete factorization. Given any τ-complete factorization of a, it is either among the given factorizations, or there is a rearrangement and switching of β. In any case, the τ-factorization has length less than $N(a)$, proving R is a τ-complete-BFR.

(5) Let R be a τ-complete-β-FFR, and let $a \in R$ be a non-unit. There are a finite number of τ-complete factorizations up to rearrangement and β. Each of these τ-complete factorizations has a finite length. Thus the set of all τ-factors which occur as a divisor in some τ-complete factorization of a must be finite.

(6) Let R be a τ-complete-BFR. We suppose there is an ascending chain $(a_1) \subsetneq (a_2) \subsetneq \cdots \subsetneq (a_i) \subsetneq \cdots$ of principal ideals such that $a_{i+1} \mid_\tau a_i$. Let $N(a_1)$ be the bound on the length of the τ-complete factorizations of a_1. We have τ-factorizations $a_i = \lambda_i a_{i+1} b_{i1} \cdots b_{in_i}$ for each i. We note here that $n_i \geq 1$ or else we would have $(a_i) = (a_{i+1})$. Because τ is refinable, we can create τ-factorizations as follows:

$$a_1 = \lambda_1 a_2 b_{11} \cdots b_{1n_1} = \lambda_1 \lambda_2 a_3 b_{21} \cdots b_{2n_2} b_{11} \cdots b_{1n_1} = \cdots .$$

After $N(a_1)$ iterations, we will arrive at a τ-factorization, †, of length at least $N(a_1)$ since at each stage the length increases by at least 1. Now τ is refinable and R is τ-complete, so we apply Proposition 7.3 to τ-refine the τ-factorization, †, of length $N(a_1)$ into a τ-complete factorization. (resp. Because R is τ-completable, we can τ-refine the factorization, †, into a τ-complete factorization.) This can only increase the length of the factorization which contradicts the fact that $N(a_1)$ is the bound on the length of τ-complete factorizations of a_1. $\square$

Theorem 7.7. *Let R be a commutative ring with 1 and τ be a symmetric relation on $R^{\#}$. Let $\alpha \in \{$atomicable, strongly atomicable, m-atomicable, very strongly atomicable $\}$. If $\alpha =$ atomicable (resp. strongly atomicable, m-atomicable, very strongly atomicable), set $\alpha' =$ atomic (resp. strongly atomic, m-atomic very strongly atomic). Let $\beta \in \{$associate, strongly associate, very strongly associate $\}$. If R is a τ-α-β-UFR, then R is a τ-α-HFR.*

Proof. Suppose R is a τ-α-β-UFR. By hypothesis R is τ-α. Let a be a non-unit. Suppose $a = \lambda a_1 \cdots a_n = \mu b_1 \cdots b_m$ were two τ-α' factorizations with different lengths. Then this contradicts the fact that R is a τ-α-β-UFR, and proves the theorem. $\square$

Theorem 7.8. *Let R be a commutative ring with 1 and τ be a symmetric relation on $R^{\#}$. We have the following.*

(1) If R is a BFR, then R is a τ-BFR.

(2) If R is a τ-BFR, then R is a τ-complete-BFR.

(3) Let R be τ-complete and τ refinable. Then R is a τ-complete-BFR implies R is a τ-BFR.

(4) Let R be τ-completable. Then R is a τ-complete-BFR implies R is a τ-BFR.

Proof. (1) Let R be a BFR, and a be a non-unit. Suppose $N(a)$ is the bound on the length of any factorization of a. Any τ-factorization $a = \lambda a_1 \cdots a_n$ is certainly a factorization, so $n \leq N(a)$, proving R is a τ-BFR.

(2) Let R be a τ-BFR, and a be a non-unit. Suppose $N(a)$ is the bound on the length of any τ-factorization of a. Any τ-complete-factorization $a = \lambda a_1 \cdots a_n$ is certainly a τ-factorization, so $n \leq N(a)$, proving R is a τ-complete-BFR.

(3) Let R be τ-complete and τ refinable. Suppose R is a τ-complete-BFR. Let a be a non-unit. Let $N(a)$ be the bound on the length of any τ-complete factorization. We claim this also serves as a bound on the length of any τ-factorization. Let $a = \lambda a_1 \cdots a_n$ be any τ-factorization of a. Because R is τ-complete, each a_i has a τ-complete factorization and τ is refinable, we can τ-refine this factorization into a τ-complete factorization, say $a = \lambda' b_1 \cdots b_m$. We have $n \leq m \leq N(a)$ as desired.

(4) Let R be τ-completable. Suppose R is a τ-complete-BFR. Let a be a non-unit. Let $N(a)$ be the bound on the length of any τ-complete factorization. This also serves as a bound on the length of any τ-factorization. Let $a = \lambda a_1 \cdots a_n$ be any τ-factorization of a. By hypothesis, we can τ-refine this factorization into a τ-complete factorization, say $a = \lambda' b_1 \cdots b_m$. We have $n \leq m \leq N(a)$ as desired. □

Theorem 7.9. *Let R be a commutative ring with 1 and τ be a symmetric relation on $R^{\#}$. Let $\beta \in \{$associate, strongly associate, very strongly associate $\}$. We have the following.*

(1) If R is a β-FFR, then R is a τ-β-FFR.

(2) If R is a τ-β-FFR, then R is a τ-complete-β-FFR.

(3) Let R be τ-complete and τ refinable. Then R is a τ-complete-FFR implies R is a τ-β-FFR.

(4) Let R be τ-completable. Then R is a τ-complete-FFR implies R is a τ-β-FFR.

Proof. (1) Let a be a non-unit. The set of τ-factorizations of a up to β is among the set of factorizations of a. By hypothesis the latter is finite, so certainly the former is.

(2) Let a be a non-unit. The set of τ-complete-factorizations of a up to β is

among the set of τ-factorizations of a. By hypothesis, the latter is finite, so certainly the former is.

(3) Let a be a non-unit. We claim every τ-factorization of a, up to β can be realized as coming from grouping of factors of a τ-complete-factorization up to β. Since there are a finite number of τ-factorizations up to β, each with a finite number of factors, there is a finite number of ways of grouping the factors to generate different τ-factorizations up to β. Given a τ-factorization $a = \lambda a_1 \cdots a_n$, there is a τ-complete factorization for each a_i with $1 \leq i \leq n$. Suppose $a_i = \lambda_i b_{i1} \cdots im_i$ is the τ-complete factorization of a_i for $1 \leq i \leq n$. By hypothesis, τ is refinable, so

$$a = (\lambda\lambda_1 \cdots \lambda_n) b_{11} \cdots b_{1m_1} \cdot b_{21} \cdots b_{2m_2} \cdots b_{n1} \cdots b_{nm_n} \tag{7.3}$$

is a τ-factorization. By Proposition 7.3, this is a τ-complete factorization and hence was among the finite number of τ-complete factorizations of a up to β.

(4) This proof is nearly identical to the proof of (3). The only modification is that we can use the fact that since R is τ-completable to automatically conclude that any factorization $a = \lambda a_1 \cdots a_n$ can be τ-refined into a τ-complete factorization of the form of Equation 7.3. $\square$

Theorem 7.10. *Let R be a commutative ring with 1 and τ be a symmetric relation on $R^{\#}$. Let $\beta \in \{$associate, strongly associate, very strongly associate $\}$. We have the following.*

(1) If R is a β-WFFR, then R is a τ-β-WFFR.

(2) If R is a τ-β-WFFR, then R is a τ-complete-β-divisor finite ring.

(3) If R is a τ-β-atomic divisor finite ring, then R is a τ-complete-β-divisor finite ring.

(4) If R is a τ-β-strongly atomic divisor finite ring, then R is a τ-complete-β-divisor finite ring.

(5) If τ is refinable and R is a τ-β-m-atomic divisor finite ring, then R is a τ-complete-β-divisor finite ring.

Proof. (1) Let $a \in R$ be a non-unit. If a has a finite number of divisors up to β, then it certainly has a finite number τ-divisors up to β.

(2) Let $a \in R$ be a non-unit. If there are a finite number of τ-divisors of a up to β, then certainly there are a finite number of τ-divisors which occur as a τ-factor in some τ-complete-factorization of a up to β.

(3) (resp. (4)) Let $a \in R$ be a non-unit. Suppose $\{a_i\}_{i=1}^{\infty}$ is a infinite collection of non-β τ-divisors which occur in some τ-complete factorization of a. Say $a = \lambda_i a_i b_{i1} \cdots b_{in_i}$ is one such τ-complete factorization. By Theorem 7.1, this τ-complete factorization is τ-atomic (resp. strongly atomic). This provides an infinite number of non-β τ-atomic (resp. τ-strongly atomic) divisors of a, a contradiction.

(5) Let $a \in R$ be a non-unit. Suppose $\{a_i\}_{i=1}^{\infty}$ is a infinite collection of non-β τ-divisors which occur in some τ-complete factorization of a. Say $a = \lambda_i a_i b_{i1} \cdots b_{in_i}$ is one such τ-complete factorization. We have a τ which is refinable, so by Theorem 7.1, this τ-complete factorization is τ-m-atomic. This provides an infinite number of non-β τ-m-atomic divisors of a, a contradiction. $\square$

We notice at this point that many of the τ-finite factorization and τ-complete finite factorization properties result in R having the property that for a given non-unit $a \in R$, there is a finite number of divisors of a which occur as a τ-factor of some τ-complete factorization. We summarize these in the form of the following corollary.

Corollary 7.2. *Let R be a commutative ring with 1 and τ be a symmetric relation on $R^{\#}$. Let $\beta \in \{$associate, strongly associate, very strongly associate $\}$. If R satisfies any of the following conditions, then R is a τ-complete-β-divisor finite ring.*

(1) R is a β-FFR.

(2) R is a τ-β-FFR.

(3) R is a τ-complete-β-FFR.

(4) R is a τ-complete (completable)-β-UFR.

(5) R is a β-WFFR.

(6) R is a τ-β-WFFR.

(7) R is a τ-β-irreducible (resp. strongly irreducible) divisor finite ring.

(8) R is a strongly associate ring and R is a τ-β-m-irreducible divisor finite ring.

Proof. We have seen in Theorem 7.6 part (2) that (4) $\Rightarrow$ (1). By Theorem 7.9, we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and Theorem 7.6 part (5) proves that (3) implies R is a τ-β-cdf ring.

We know from Theorem 7.10 part (1) and (2), that (5) $\Rightarrow$ (6) and that (6) implies R is a τ-β-cdf ring. (7) and (8) are restatements of 7.10 parts (3), (4) and (5). This completes the proof. □

The following diagram serves as an illustration of several of the previous results. Let ∇ represent a strongly associate ring, $\star$ represent τ is refinable and let $\dagger$ represent R is both τ-complete and τ-is refinable. Let $\gamma \in \{$ complete, completable, atomic, atomicable, strongly atomic, strongly atomicable, m-atomic, m-atomicable, very strongly atomic, very strongly atomicable $\}$.

$$
\begin{tikzcd}
\tau\text{-}\beta \text{ irr. df ring} \arrow[d, Rightarrow] & \beta\text{-WFFR} \arrow[d, Rightarrow] & & \\
\tau\text{-}\beta \text{ s. irr. df ring} \arrow[d, Rightarrow, "\nabla"] \arrow[dr, Rightarrow] & \tau\text{-}\beta\text{-WFFR} \arrow[l, Rightarrow] \arrow[ul, Rightarrow] \arrow[d, Rightarrow] \arrow[dl, Rightarrow] & \beta\text{-FFR} \arrow[ul, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{BFR} \arrow[d, Rightarrow] \\
\tau\text{-}\beta \text{ m-irr. df ring} \arrow[r, Rightarrow, "\nabla"] \arrow[d, Rightarrow] & \tau\text{-}\beta\text{-cdf ring} & \tau\text{-}\beta\text{-FFR} \arrow[ul, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \tau\text{-BFR} \arrow[d, Rightarrow] \\
\tau\text{-}\beta \text{ v.s. irr. df ring} & \tau\text{-cplt-}\beta\text{-UFR} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] \arrow[d, bend left, "\star"] & \tau\text{-cplt-}\beta\text{-FFR} \arrow[ul, Rightarrow] \arrow[r, Rightarrow] \arrow[u, bend right, "\dagger"'] & \tau\text{-cplt-BFR} \arrow[d, Rightarrow] \arrow[u, bend right, "\dagger"'] \\
& \tau\text{-cpltable-}\beta\text{-UFR} \arrow[u, Rightarrow] \arrow[r, Rightarrow] \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \tau\text{-cplt-HFR} \arrow[ur, Rightarrow] \arrow[d, bend right, "\star"'] & \tau\text{-ACCP} \arrow[d, Rightarrow] \\
& & \tau\text{-cpltable-HFR} \arrow[u, Rightarrow] \arrow[uur, Rightarrow] & \tau\text{-}\gamma
\end{tikzcd}
$$

APPENDIX A
FACTORIZATION PROPERTY EXAMPLES

Here we would like to provide a collection of interesting examples of rings which satisfy certain finite factorization properties, but not others. We first note that when R is a domain, all types of associate and irreducible coincide. Furthermore, all divisors are essential even when R is présimplifiable. Thus when $\tau = R^{\#} \times R^{\#}$ the τ-U-factorization is exactly the same as the usual factorizations in domains. In particular, we get several counter-examples from D.D. Anderson, D.F. Anderson, and M. Zafrullah in [3]. These show the arrows between rings satisfying τ-U-finite factorization properties cannot be reversed in our case in the diagram following Theorem 4.7.

Example A.1. An HFD and a FFD which is not a UFD.

Consider the example given following [3, Proposition 5.3]. Let $F_1 \subset F_2$ be finite fields. Then $R = F_1 + XF_2[X]$ and $R = F_1 + XF_2[[X]]$ are both FFDs and a HFDs, but neither is a UFD. In general, A. Zaks shows that any Krull domain D with $\mathrm{Cl}(D) = \mathbb{Z}/2\mathbb{Z}$ is an HFD, but not a UFD in [39]. $\boxplus$

Example A.2. An HFD and a BFD which is not an idf-Domain and therefore, not a FFD or a WFFD.

Consider [3, Example 4.1 (a)] Let $R = \mathbb{R} + X\mathbb{C}[X]$. Then R is Noetherian, but R is not an irreducible divisor finite domain. We see $\{(r + i)X \mid r \in \mathbb{R}\}$ is an infinite family of non-associate irreducible divisors of X^2. R is a HFD and hence a BFD, but not an idf-domain and therefore not a FFD or a WFFD. Note: For domains WFFD and FFD are equivalent. $\boxplus$

Example A.3. A (W)FFD and an idf-Domain which is neither an HFD nor a UFD.

D.D. Anderson et al provide the following example in [3, Example 5.4] Let k be a field and $T = \{n + \frac{i}{n!} \mid 0 \leq i \leq n! - 1,\ n = 0, 1, \ldots\}$ an additive submonoid of Q^+. Then the monoid domain $R = k[X, T]$ is a one-dimensional domain which is a FFD, but not a HFD. R is a BFD since each nonconstant $f \in R$ has $deg(f) \geq 1$. R is an atomic idf-domain, and hence a FFD and a WFFD, since any factorization of an $f \in R$ takes place in some polynomial ring $k[X^{\frac{1}{n!}}]$. However, R is not a HFD since $X^5 = X^{\frac{5}{2}} X^{\frac{5}{2}}$, and X and $X^{\frac{5}{2}}$ are each irreducible. R is not an HFD, so it is not a UFD. $\boxplus$

Example A.4. A BFD which is not an HFD.

D.D. Anderson et al provide the following example of a BFD which is not an HFD. Consider the Krull domain $R = k[X^3, XY, Y^3]$, where k is a field. Anderson proves in [3, Proposition 2.2] that Noetherian and Krull domains are BFDs; however, R is not a HFD since XY, X^3, and Y^3 are each irreducible in R and $(XY)^3 = X^3Y^3$ (note that $\mathrm{Cl}(R) = \mathbb{Z}/3\mathbb{Z}$). $\boxplus$

Example A.5. A domain satisfying ACCP, but is not a BFD.

In [3, Example 2.1], D.D. Anderson et al give the following example of a domain which satisfies the ascending chain condition on principal ideals, but is not a BFD. Let k be a a field and T the additive submonoid of $\mathbb{Q}^+$ generated by $\{\frac{1}{3}, \frac{1}{5} \ldots \frac{1}{p_j}, \ldots\}$, where p_j is the j^{th} prime. Let R be the monoid domain $k[X;T]$ is a one-dimensional domain which satisfies ACCP, but is not a BFD. $\boxplus$

Example A.6. An atomic domain which does not satisfy ACCP.

A. Grams provided the following example of a ring which is atomic but fails to satisfy the ascending chain condition on principal ideals in [29]. Let F be a a field and T the additive submonoid of $\mathbb{Q}^+$ generated by $\{\frac{1}{3}, \frac{1}{2\cdot 5} \cdots \frac{1}{2^j p_j}, \ldots\}$, where $p_0 = 3, p_1 = 5, \ldots$ is the sequence of odd primes. Let R be the monoid domain $F[X;T]$ and $N = \{f \in R \mid f \text{has non-zero constant term}\}$. Then $A = F[X;T]_N$, is an atomic domain which does not satisfy ACCP. $\boxplus$

APPENDIX B
U-FACTORIZATION EXAMPLES

We would now like to move to show examples which show the U-factorization definitions properly extends the class of rings with the various finite factorization properties. Many of these examples come from M. Axtell in [13, 14]. We set $\tau = R^{\#} \times R^{\#}$ and note that these examples will yield counter-examples to show that none of the converses can possibly hold when we add τ-factorizations.

Example B.1. A U-UFR and a U-HFR which is neither a UFR nor a HFR.

Let $R = \mathbb{Z}/6\mathbb{Z}$. As in [14], R is a Fletcher unique factorization ring and hence a U-UFR and a U-HFR. However, $3 = 3^n$ for all $n \geq 1$, which shows R is not even a BFR, so it cannot be a UFR or an HFR. ⊞

Example B.2. A U-BFR which is not a BFR.

Consider [13, Example 2.4]. Let $R = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then $(0,1) = (0,1)^i$ for all $i \geq 1$ shows R is not a BFR; however, there are only two possible essential divisors in the entire ring, $(0,1)$ and $(1,0)$. This means the largest number of essential divisors in any factorization is two, which is certainly bounded. ⊞

Example B.3. A U-(W)FFR which is not a (W)FFR.

Consider [13, Example 2.6]. Let $R = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then $(0,1) = (p,1)(0,1)$ for every prime $p \in \mathbb{Z}$ shows R is not a WFFR; however, M. Axtell shows it to be a U-WFFR. M. Axtell proves in [13, Theorem 2.9] that U-WFFR and U-FFR are equivalent, so this is also an example of a U-FFR which is not an FFR since

$(0,1) = (p,1)(0,1)$ for each prime $p \in \mathbb{Z}$ provides an infinite collection of non-associate factorizations. $\boxplus$

Example B.4. A U-idf-Ring which is not an idf-Ring.

Again take $R = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then $(0,1) = (p,1)(0,1)$ for every prime $p \in \mathbb{Z}$. As above R is a U-WFFR and hence is a U-idf ring. On the other hand, $(p,1)$ is prime and therefore irreducible for every prime p. This yields and infinite number of irreducible divisors of $(0,1)$, showing R is not an idf-ring. Furthermore, $(p,1)$ is regular, so all the types of irreducible coincide and R is strongly associate. $\boxplus$

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