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Modules and orbits of the regular action, and deformations of incidence algebras

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MODULES AND ORBITS OF THE REGULAR ACTION, AND DEFORMATIONS OF INCIDENCE ALGEBRAS

by

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CERTIFICATE OF APPROVAL

PH.D. THESIS

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ABSTRACT

The property of having a finite number of orbits under the regular action has been used to study properties of rings and algebras. For example, in ring theory, Yasuyuki Hirano was able to use this property to show that rings with finitely many orbits under the regular action can be decomposed as direct sum of uniserial rings and a finite ring. In this thesis, we study modules under the regular action. More precisely, if R is a unital ring and M is a left(right) R-module, we describe all modules M that have finitely many orbits under the regular action. Along the way, we give a (new) module theoretical proof to the theorem of Yasuyuki Hirano on the classification of rings with finitely many orbits under the regular action which was proven using using methods from ring theory. Our charaterization of modules with finitely many orbits under the regular action shows a connection between algebras with finitely many submodules and distributive modules. A particular algebra that is of interest to us is the incidence algebra of a finite poset.

Incidence algebras were originally introduced in the 1960's by Gian-Carlo Rota as a way to study combinatorial problems but it became apparent later on that such algebras were an interesting object to study in their own right. They include ring theoretical examples such as the product of copies of a ring R and the upper triangular matrices over R. Robert B. Feinberg in his work on incidence algebras developed an internal characterization of incidence algebras of lower finite quasi-ordered sets. For example, he showed that an associative unital complete topological algebra Λ over a field k, where k has the discrete topology, is isomorphic to an incidence algebra if and only if

1. Λ has a faithful unital left module M with a distributive lattice of submodules. Fur-

ther, every finitely generated submodule of M is finite dimensional and Λ has the coarset topology such that its action on M is continuous in Λ, when M has the discrete topology.

2. For every maximal closed ideal J, Λ/J is isomorphic to $M_n(k)$ for some integer n.
3. For every closed ideal J, the center of Λ/J is isomorphic to the direct product of copies of k.

This thesis investigates the deformations of incidence algebras and how such deformations relate to cohomology. We show that distributivity of projective indecomposable modules of algebras largely characterizes precisely those algebras which are deformations of incidence algebras.

PUBLIC ABSTRACT

This thesis studies mathematical objects called modules under certain algebraic action. We characterize those objects for nice underline algebraic object called rings. This generalizes the work done by Yasuyuli Hirano when studying the same algebraic action for the underline algebraic objects.

In the second part of the thesis, we investigate the deformations of an algebraic object called incidence algebra and characterize all algebraic objects that are deformations of incidence algebra.

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CHAPTER 1
INTRODUCTION

1.1 Motivation

The motivation of this work is twofold. First, we use the action of a group, namely, the action of the group of units of a ring R on a left R-module M to study some properties of M. Modules serve as a generalization of vector spaces. They are the representation objects for a ring, and are used in representation theory to study the structure of the rings that act on them. Second, we study the deformations of incidence algebras to classify certain types of finite-dimensional basic algebras. More precisely, let k be an infinite field and Λ a finite-dimensional basic k-algebra. We show that the characterization of Λ as a deformation of an incidence algebra consists of two axioms concerning the Ext-quiver of Λ and projective Λ-modules and another axiom stipulating the existence of distributive Λ-modules.

1.2 Overview

In Chapter 2, some necessary definitions and results are given which will be used in the remainder of the thesis. We define the regular action, orbits, distributive modules, incidence algebras, basic algebras, quivers, and cohomology.

In chapter 3, we consider all R-modules M under the regular action. Each module M is a left module. In section 3.1, we give an introduction and in section 3.2, we state the main theorem of this chapter and prove several results using this theorem. In section 3.3, we prove the main theorem. We classify all R-modules M

with finitely many orbits under the regular action for semilocal rings R. In section 3.4, we consider the case where M is the regular module ${}_RR$ and give a new proof, a module theoretical proof of the structure theorem for rings with finitely many orbits under the regular action.

In chapter 4, we investigate the deformations of incidence algebras $I(P, k)$ for a finite poset P. In section 4.1, we provide many definitions and present many nice theorems which illustrate important properties of the deformations of incidence algebras. In section 4.2, we relate the deformations of incidence algebras to cohomology. In the case where the finite poset P has no non-trivial automorphism, we show that the isomorphism classes of deformations of incidence algebras are in one to one correspondence with a second cohomology group with coefficients in the abelian group $(k^*, \cdot)$ of a simplicial set(topological space) canonically associated to the finite poset P, and classify all basic algebras which are isomorphic to deformations of incidence algebras.

CHAPTER 2
DEFINITIONS AND BACKGROUND

Let R be a ring with 1 and k be an infinite field. We will use M to denote a module, $L(M)$ the lattice of submodules of M and Λ to denote an algebra. All modules will be left modules and all algebras will be finite dimensional k-algebras. Denote the category of R-modules by R-mod.

2.1 Ring Theory

We assume familiarity with some basic notions of ring theory and refer the reader to [19].

Definition 2.1. The regular action(or left regular action) of the group $G = U(R)$ of units of the ring R on the module M is the action inherited by multiplication with elements of R on M.

In this thesis, we will only consider left modules and the induced left action of $G = U(R)$, and the results obviously apply to the right modules as well by going to the opposite ring R^{op}.

Definition 2.2. Let G be a group acting on a set X. To each $x \in X$, the set

$$Gx = \{y \in X : y = g \cdot x, g \in G\},$$

is called the orbit of x under the action of G.

Definition 2.3. A ring R is said to be semilocal if $R/\text{rad } R$ is a left artinian ring, or, equivalently, if $R/\text{rad } R$ is a semisimple ring.

The class of rings which are semilocal rings is a big class which includes, for instance, all local rings(rings with a unique maximal ideal) and all left(resp. right) artinian rings(in particualar, all finite-dimensional algebras over fields). Many rings which arise naturally in ring theory are semilocal rings. An alternate definition of a semilocal ring is given by the following proposition.

Proposition 2.1. *For a ring R, consider the following two conditions:*

1. *R is semilocal.*
2. *R has finitely many maximal left ideals.*

We have, in general, $(2) \Rightarrow (1)$. The converse holds if $R/rad\ R$ is commutative.

Proof. See [19, Proposition 20.2] □

Theorem 2.2. *(Bass) Let R be a semilocal ring, $a \in R$, and $\mathfrak{B}$ be a left ideal of R. If $R \cdot a + \mathfrak{B} = R$, then the coset $a + \mathfrak{B}$ contains a unit of R.*

Proof. See [19, Theorem 20.9] □

The notion of semilocal rings has a bearing on the general structure theory of rings through the use of idempotents. By definition, an element e in a ring is called an idempotent if $e^2 = e$. Two idempotents e and f in a ring are called orthogonal if $ef = fe = 0$. An idempotent e in a ring is said to be primitive if e has no decomposition into a sum of nonzero orthogonal idempotents $e = e_1 + e_2$.

Lemma 2.3. *Let R be a ring with 1. Let $e_1 + \cdots + e_n = f_1 + \cdots + f_n$ be two decompositions of 1 into sums of primitive orthogonal idempotents. Then, after some permutation σ of $\{1, 2, \ldots, n\}$,* *there exists an invertible element $a \in R$ such that $f_{\sigma(i)} = a^{-1}e_i a$.*

Proof. See [14, Lemma 10.3.6] □

2.2 Module Theory

Definition 2.4. Let M be an R-module.

1. The socle of M, denoted by $soc(M)$, is the submodule of M generated by all the semisimple submodules of M.

2. The radical of M, denoted by $rad(M)$, is the intersection of all the maximal submodules of M.

3. The top of M, denoted by $top(M)$, is the quotient module $M/rad(M)$.

Definition 2.5. A module M is distributive if for any submodules A, B, C we have $A \cap (B + C) = (A \cap B) + (A \cap C)$. Equivalently, a module is distributive if the lattice of its submodules is a distributive lattice.

Theorem 2.4. *Let M be a module. Then M is a distributive module if and only if for every submodule N, M/N has square-free socle.*

Proof. See [2, Theorem 1] □

Remark. Over a semilocal ring R, the above theorem is very easy to prove. It is done via localization to primitive idempotents. In particular, we have the following proposition.

Proposition 2.5. *Let R be a semilocal ring. A left R-module M is distributive if and only if for each primitive idempotent $e \in R$, the left eRe-module eM is uniserial,* *hence distributive.*

Proof. See [10, Lemma 4] □

Definition 2.6. 1. An epimorphism $f : A \to B$ in R-mod is called an essential epimorphism if $\ker(f) \subseteq \mathrm{rad}(A)$.

2. Let M be a module. A projective cover of M is a projective module $P(M)$ together with an essential epimorphism $f : P(M) \to M$, denoted by $(P(M), f)$.

Remark. 1. Sometimes we shall refer to $P(M)$ as a projective cover of M, suppressing the role of f.

2. A module M does not in general have a projective cover. However, if R is an artinian ring, then M is guaranteed to have a projective cover. In particular, if M is a simple module, then M is the top of its projective cover, if it exists.

2.3 Homological Algebra

Definition 2.7. Let $\mathcal{A}$ be an abelian category.

1. A chain complex of objects in $\mathcal{A}$ consists of a collection $\mathbf{C} = \{C_n \mid n \in \mathbb{Z}\}$ of objects C_n of $\mathcal{A}$ indexed by the integers, together with maps $\partial_n : C_n \to C_{n-1}$ satisfying $\partial_n \circ \partial_{n+1} = 0$.

2. A cochain complex of objects in $\mathcal{A}$ consists of a collection $\mathbf{C} = \{C^n \mid n \in \mathbb{Z}\}$ of objects C^n of $\mathcal{A}$ indexed by the integers, together with maps $\delta^n : C^n \to C^{n+1}$ satisfying $\delta^n \circ \delta^{n-1} = 0$.

Definition 2.8. 1. The homology of a chain complex $\mathbf{C}$ is given by

$$\mathrm{H}_n(\mathbf{C}) = \frac{\ker(\partial_n)}{\mathrm{im}(\partial_{n+1})}.$$

2. The cohomology of a cochain complex $\mathbf{C}$ is given by

$$\mathrm{H}^n(\mathbf{C}) = \frac{\ker(\delta^n)}{\mathrm{im}(\delta^{n-1})}.$$

Definition 2.9. An augmented algebra Λ over a communitative ring of coefficients R is an algebra together with a surjective map $\epsilon : \Lambda \to R$ of R-algebras called an augmentation map.

Definition 2.10. Let M and N be modules, and let

$$\cdots \to P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\epsilon} M \to 0$$

be a projective resolution of M.

1. Applying the contravariant functor $\mathrm{Hom}_R(-, N)$ to the projective resolution of M, we obtain a sequence of $\mathbb{Z}$-modules(abelian groups)

$$0 \to \mathrm{Hom}_R(P_0, N) \xrightarrow{\delta_1^*} \mathrm{Hom}_R(P_1, N) \xrightarrow{\delta_2^*} \mathrm{Hom}_R(P_2, N) \xrightarrow{\delta_3^*} \cdots$$

For all $n \geq 0$, define

$$\mathrm{Ext}_R^n(M, N) = \ker(\delta_{n+1}^*)/\mathrm{im}(\delta_n^*).$$

2. If N is a right module, then applying the covariant functor $N \otimes_R -$ to the projective resolution of M, we obtain a sequence of $\mathbb{Z}$-modules(abelian groups)

$$\cdots N \otimes_R P_2 \xrightarrow{1\otimes\delta_2} N \otimes_R P_1 \xrightarrow{1\otimes\delta_1} N \otimes_R P_0 \to 0.$$

For all $n \geq 0$, define

$$\mathrm{Tor}_n^R(N, M) = \ker(1 \otimes \delta_n)/\mathrm{im}(1 \otimes \delta_{n+1}).$$

Note that $\mathrm{Tor}_0^R(N, M) = N \otimes_R M$ and $\mathrm{Ext}_R^0(M, N) = \mathrm{Hom}_R(M, N)$.

Definition 2.11. Suppose Λ is an augmented R-algebra. Then

1. The homology groups of Λ with coefficients in a right Λ-module M are defined to be
$$\mathrm{H}_n(\Lambda, M) = \mathrm{Tor}_n^\Lambda(M, R).$$

2. The cohomology groups of Λ with coefficients in a left Λ-module M are defined to be
$$\mathrm{H}^n(\Lambda, M) = \mathrm{Ext}_\Lambda^n(R, M).$$

2.4 Incidence Algebras and Quivers

Definition 2.12. A partially ordered set P is said to be locally finite if the subset $[y, z] = \{x \in P : y \leq x \leq z\}$ is finite for all $y, z \in P$ such that $y \leq z$.

Definition 2.13. The incidence algebra $I(P, R)$ of a locally finite partially ordered set P, over the commutative ring R with identity is

$$I(P, R) = \{f : P \times P \to R | f(x, y) = 0 \text{ if } x \not\leq y\}$$

with the operations defined by

$$(f+g)(x,y) = f(x,y) + g(x,y),$$

$$(f \cdot g)(x,y) = \sum_{x \le z \le y} f(x,z) \cdot g(z,y) \text{ and}$$

$$(r \cdot f)(x,y) = r \cdot f(x,y),$$

for $f, g \in I(P,R)$ with $r \in R$ and $x, y, z \in P$.

The identity element of $I(P,R)$ is

$$\delta(x,y) = \begin{cases} 1 & \text{if} \quad x = y \\ 0 & \text{otherwise.} \end{cases}$$

For all $x, y \in P$ such that $x \le y$, let $e_{xy} \in I(P,R)$ denote the function defined by

$$e_{xy}(a,b) = \begin{cases} 1 & \text{if } a = x \text{ and } b = y, \\ 0 & \text{otherwise.} \end{cases}$$

When $x = y$, we will denote the function e_{xx} by e_x. If $R = k$ is a field and P is finite, the set

$$\{e_{xy} \in I(P,k) | \text{ for all } x, y \in P \text{ such that } x \le y\}$$

is a k-basis for $I(P,k)$. In particular, if $g \in I(P,k)$, we can write

$$g = \sum_{x \le y} a_{xy} e_{xy}$$

where $a_{xy} = g(x,y)$.

Definition 2.14. A quiver $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two sets: Q_0, whose elements are called points or vertices, and Q_1, whose elements are called arrows, and two maps $s, t : Q_1 \to Q_0$ which associate to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$, respectively.

Definition 2.15. Let $Q = (Q_0, Q_1, s, t)$ be a quiver.

1. A path of length $l \geq 1$ in Q from vertex a to vertex b is a sequence $(\alpha_1, \cdots, \alpha_{l-1}, \alpha_l)$ where $\alpha_j \in Q_1$ for all $1 \leq j \leq l$, and where $s(\alpha_1) = a$, $t(\alpha_l) = b$, and $t(\alpha_j) = s(\alpha_{j+1})$ for all $1 \leq j \leq l-1$. We denote this path by $\alpha_1 \cdots \alpha_{l-1}\alpha_l$. We also associate a path of length $l = 0$ to each vertex $i \in Q_0$, which we call the trivial path at i and which we denote by e_i.

2. The path algebra kQ of Q is defined to be the k-vector space whose k-basis is the set of all paths in Q. The product of two paths $\alpha_1\alpha_2 \cdots \alpha_l$ and $\beta_1\beta_2 \cdots \beta_k$ in kQ is $\alpha_1\alpha_2 \cdots \alpha_l\beta_1\beta_2 \cdots \beta_k$ if $t(\alpha_l) = s(\beta_1)$ and 0 otherwise. This defines a k-algebra structure on kQ.

Remark. When P is finite, an alternate definition of an incidence algebra $I(P, k)$ (where $R = k$ is a field) is the following.

1. Let $(P, \leq)$ be a finite partially ordered set. In a natural way, we associate with $(P, \leq)$ its quiver Q as follows: the set Q_0 of vertices of Q coincides with P and there is an arrow from x to y, $x, y \in Q_0$ if and only if $x < y$ and there is no $u \in Q_0$ such that $x < u < y$.

2. Let I be the two-sided ideal in kQ generated by all differences $p - g$ where p, g are pairs of paths which share the same starting-point and the same end-point, respectively. The quotient algebra $\Lambda = kQ/I$ by the ideal I is isomorphic to the incidence algebra of the poset P.

Definition 2.16. 1. A labelled graph is an undirected graph together with a pair of positive integers $({}_xd_y, {}_yd_x)$ for each edge $x - y$.

2. An orientation of a labelled graph assigns a direction $x \to y$ or $x \leftarrow y$ to each edge $x - y$.

3. A valued graph is a labelled graph with the property that there exist positive integers f_x, one for each vertex with ${}_xd_y f_y = {}_yd_x f_x$ for each edge $x - y$.

4. A modulation of a valued graph consists of an assignment of a division ring Δ_x to each vertex x, and a $\Delta_x - \Delta_y$-bimodule ${}_xM_y$ for each edge $x - y$ satisfying

   (a) ${}_yM_x \cong \mathrm{Hom}_{\Delta_x}({}_yM_x, \Delta_x) \cong \mathrm{Hom}_{\Delta_y}({}_xM_y, \Delta_y)$

   (b) $\dim_{\Delta_y}({}_xM_y) = {}_xd_y$.

5. A modulated quiver consists of a valued graph together with an orientation and a modulation.

Definition 2.17. If Λ is a finite dimensional algebra over a field k, which is not necessarily algebraically closed, its Ext-quiver is defined as a modulated quiver as follows. The vertices x_i correspond to the isomorphism classes of simple modules S_i, with $\Delta_i = \mathrm{End}_\Lambda(S_i)^{\mathrm{op}}$. There is an arrow $x_i \to x_j$ if and only if $\mathrm{Ext}^1_\Lambda(S_i, S_j) \neq 0$, and

1. ${}_jM_i = \mathrm{Ext}^1_\Lambda(S_i, S_j)$ as a $\Delta_j - \Delta_i$-bimodule,

2. ${}_iM_j = \mathrm{Hom}_k({}_jM_i, k) \cong \mathrm{Hom}_{\Delta_i}({}_jM_i, \Delta_i) \cong \mathrm{Hom}_{\Delta_j}({}_jM_i, \Delta_j)$,

3. ${}_jd_i = \dim_{\Delta_i}({}_jM_i)$,

4. ${}_id_j = \dim_{\Delta_j}({}_iM_j)$ and $f_i = \dim_k(\Delta_i)$.

Remark. If k is algebraically closed, then the above construction reduces to the following. Suppose Λ is a finite dimensional algebra over an algebraically closed field k. Let $S_1, \ldots, S_r$ be the isomorphism classes of simple Λ-modules with projective covers Λe_i. The Ext-quiver $Q(\Lambda)$ has vertices $x_1, \ldots, x_r$ corresponding to these simple modules, and the number of arrows from x_i to x_j is $\dim_k \mathrm{Ext}^1_\Lambda(S_i, S_j)$.

Proposition 2.6. *Suppose Λ is a finite dimensional basic algebra over an algebraically closed field k, and let $Q = Q(\Lambda)$ be its Ext-quiver. Then there is a surjective map $\phi : kQ \twoheadrightarrow \Lambda$ such that the kernel of ϕ is contained in the ideal of paths of length at least two. In particular, if kQ is finite dimensional, this latter ideal is equal to the square of the radical, and ϕ induces a bijection between the simple Λ-modules and the simple kQ-modules, and between the blocks of Λ and the blocks of kQ.*

Proof. See [1, Proposition 4.1.7] □

Let R be an artinian ring. Then the following lemma says when there is a path in the Ext-quiver of R between two simple R-modules S and T.

Lemma 2.7. *Let S and T be two simples R-modules. If $[rad(P(S)) : T] \neq 0$ where $[rad(P(S)) : T]$ is the multiplicity of T in a composition series of $rad(P(S))$, then there exists a path in the Ext-quiver of R of length greater than or equal to one that starts at S and ends at T.*

Proof. See [7, Lemma 1] □

2.5 Morita Equivalence

Definition 2.18. Let Λ_1 and Λ_2 be two finite-dimensional k-algebras. We say that Λ_1 and Λ_2 are Morita equivalent, written $\Lambda_1 \sim_M \Lambda_2$, if Λ_1-mod and Λ_2-mod are equivalent categories.

Definition 2.19. An algebra Λ is said to be a basic algebra if $\Lambda/J(\Lambda)$ is a finite product of division algebras, where $J(\Lambda)$ is the Jacobson radical of Λ.

Remark. Let Λ be a k-algebra where k is not necessary algebraically closed. Then, in this thesis, we say that Λ is basic if every simple Λ-module S is 1-dimensional over k or $\text{End}_\Lambda(S) = k$. Obviously, if k is algebraically closed, then this corresponds to the above definition of a basic algebra.

Proposition 2.8. *If Λ is a left Artinian R-algebra, then there exists a left ideal P of Λ such that:*

1. *P is a direct summand of ${}_\Lambda\Lambda$;*
2. *$P\Lambda = \Lambda$;*
3. *$End_\Lambda(P)$ is a basic algebra.*

An ideal P that satisfies (i), (ii), and (iii) is unique up to isomorphism and $End_\Lambda(P)$ is the basic algebra of Λ.

Proof. See [20, Proposition a. Section 6.6]. □

Proposition 2.9. *Let Λ be a left artinian algebra, and suppose B is its basic algebra. Then $\Lambda \sim_M B$.*

Proof. See [20, Proposition. Section 9.6]. □

CHAPTER 3
MODULES WITH FINITELY MANY ORBITS

The main purpose of this chapter is to prove a structure theorem for modules with finitely many orbits under the regular action over particularly nice rings, namely semilocal rings. This theorem is a generalization of the structure theorem for rings with finitely many orbits under the regular action(see [16]). Applying this result to the case where M is the regular module ${}_RR$, we recover Hirano's theorem and obtain a new proof to his theorem. Other new results are proved about modules using the structure theorem. In particular, we show that there is a connection between a left R-module M being distributive and the finiteness of the number of orbits of the action of $G = U(R)$ on M.

3.1 A structure theorem

In this section, we state a structure theorem for modules with finitely many orbits under the regular action and prove several results that follow from the use of the structure theorem.

Theorem 3.1. *Let R be a ring. Consider the following conditions for a module M.*

1. *M has finitely many orbits under the regular action,*
2. *M has finitely many submodules,*
3. *M has finite length and has no sub-factor isomorphic to $T \oplus T$ where T is a simple left R-module and $End_R(T)$ is infinite.*

Then the following implications hold: (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3). *Moreover, if R is a semilocal ring, then* (3) $\Rightarrow$ (1).

The following corollaries are immediate consequences of the above Theorem.

Corollary 3.2. *Let Λ be a k-algebra where k is an infinite field. Then a left Λ-module M has finitely submodules if and only if it is a distributive module and has finite length.*

Proof. ($\Rightarrow$). Suppose a left Λ-module M has finitely many submodules. Let T be a simple left Λ-submodule of M. Then by hypothesis, $\mathrm{End}_\Lambda(T)$ is a k-vector space. Since k is infinite, we have that $\mathrm{End}_\Lambda(T)$ is infinite as well. Hence, by Theorem 3.1, $T \oplus T$ cannot appear has a sub-factor of M. This shows that every quotient of M has square-free socle. It follows then from theorem 2.9 that M is a distributive module. Since M has finitely many submodules, it satisfies both the ascending and descending chain conditions, so it is of finite length as it is noetherian and artinian.

($\Leftarrow$). Suppose M is a distributive module and has finite length. Then every submodule and quotient module of M have square-free socle. Hence, by Theorem 3.1 again, we have that M has finitely many submodules. □

Corollary 3.3. *Let R be an artinian ring with* 1 *and M a left R-module with no nonzero finite factor. Then, the following are equivalent:*

1. *M has finitely many orbits under the regular action.*
2. *M is left artinian and distributive.*

Proof. ($\Rightarrow$) We first show that M has no finite simple sub-factor. By way of contradiction, suppose S is a finite simple sub-factor of M. Let

$$\oplus_{i=1}^{t} P_i \xrightarrow{\varphi} M \to 0$$

where the P_i's are projective indecomposable R-modules and $P_i \nsubseteq \ker \varphi$ for all $1 \leq i \leq t$. Then the multiplicity of S in the Jordan-Holder series of M is greater than or equal to 1. This implies that the multiplicity of S in the Jordan-Holder series of $\oplus_{i=1}^{t} P_i$ is greater than or equal to 1, which in turn implies that the multiplicity of S in the Jordan-Holder series of P_i is greater than or equal to 1 for some i. Let $T = \text{top}(P_i)$. Since M has no finite factor module, we know that S cannot appear on top of P_i, otherwise it would be easy to see that S would appear as a quotient of M. It follows then that $[rad(P_i) : S] \geq 1$, and so, by Lemma 2.24, we see that there is a non-trivial path from T to S, say

$$T = T_1 \to T_2 \to T_3 \to \cdots \to T_n = S.$$

If T is infinite, then the above path must contain an arrow $T_j \to T_{j+1}$ where T_j is infinite and T_{j+1} is finite. But then $\text{Ext}^1_R(T_j, T_{j+1}) \neq 0$, which contradicts Proposition 2.1 in [18]. Thus, M has no finite simple sub-factor.

Now, if M has finitely many orbits, then it is artinian since it has finitely many submodules by Theorem 3.1. Let M/N be an arbitrary factor module for M. Then M/N is infinite. Suppose $T \oplus T$ is a submodule of M/N(and thus a sub-factor of M) with T simple. Then T is infinite by the above arugment. Since T is a finite vector space over $\text{End}_R(T)$, it follows that $T = (\text{End}_R(T))^n$ where n is the dimension of T

as a $\text{End}_R(T)$-vector space. Since T is infinite, it must be that $\text{End}_R(T)$ is infinite as well. But that contradicts Theorem 3.1. Hence any factor module of M has square free socle. Therefore, by Theorem 2.9, M is distributive. Since M has finitely many orbits under the regular action, we have by Theorem 3.1 that is has finitely many submodules, so it is left artinian.

($\Leftarrow$) Since M is left artinian over the artinian ring R, M has finite length. Since M is also distributive, by Theorem 2.9, it follows that it satisfies the conditions of (3) in Theorem 3.1. Hence, M has finitely many submodules and also finitely many orbits under the regular action. □

Definition 3.1. Let M be a left R-module and $N_1, N_2, \ldots, N_n$ be submodules of M. We say that $M = N_1 \cup N_2 \cup \cdots \cup N_n$ is an efficient union if none of the N_i can be omitted from the union.

The following lemma shows that left R-modules with finitely many orbits under the regular action are efficient unions of submodules.

Lemma 3.4. *Let M be a left R- module with finitely many orbits under the regular action. Then M is an efficient union of submodules.*

Proof. If M has n orbits, let $x_1, x_2, \ldots, x_n \in M$ such that $M = \bigcup_{i=1}^n Gx_i$, where the Gx_i's are the distinct orbits. Then $M = \bigcup_{i=1}^n Rx_i$ and this union can be reduced to an efficient one by eliminating some of the Rx_i's. □

The efficiency of the union above puts a lower bound on the number of orbits when the action of G on M is not transitive.

Proposition 3.5. *Let M be a left R- module with finitely many orbits. Suppose the action of G on M is not transitive. Then M has at least three orbits.*

Proof. Suppose M has n orbits. Since the G-action is not transitive, we have that $n \neq 1$. We show that $n \neq 2$ as well. By way of contradiction, suppose $n = 2$. Then by Lemma 3.4, we have that $M = Rx_1 \cup Rx_2$ with $x_1, x_2 \in M$ and this union is efficient. Thus, by Lemma 1 in [11], we must have $M \subseteq Rx_1$ or $M \subseteq Rx_2$. This contradicts the efficiency of the union. Hence, $n \geq 3$. □

It is known that not every module has a composition series. For example, $\mathbb{Z}$ viewed as a $\mathbb{Z}$-module does not have a composition series. However, when a left R-module M has finitely many orbits under the regular action, it does have a composition series.

Proposition 3.6. *Let M be a module with finitely many orbits under the regular action. Then every proper submodule of M is a union of orbits. A fortiori, M has finitely many submodules.*

Proof. A submodule N is obviously fixed by the action of G, and thus is a union of orbits. Since there are only finitely many orbits, there are only finitely many possible combinations to form unions. Thus, every proper submodule of M is a union of orbits. If follows that M has finitely many submodules. □

Corollary 3.7. *Let M be a left R-module with finitely many orbits under the regular action. Then M has finite length. In particular, M is left artinian and has a composition series.*

Proof. Since M has finitely many orbits, and each submodule is a union of some subset of these, by Proposition 3.6, it follows that M has finitely many submodules. Hence, M has finite length. The rest of the proposition follows. □

Corollary 3.8. *Let M be a non zero left R-module with finitely many orbits. Then M has a finite indecomposable decomposition.*

Proof. Since M has finite length, then by the Krull-Schmidt Theorem, M has the required decomposition. □

The finiteness of the number of orbits of a left R-module under the regular action carries over to its submodules and quotient modules.

Proposition 3.9. *Let M be a left R-module with finitely many orbits. Then for every submodule N, both N and M/N have finitely many orbits.*

Proof. By Proposition 3.6, it is clear that N has finitely many orbits. Now, by hypothesis, $M = \cup_{i=1}^{n} Gx_i$, with $x_i \in M$. Let $S = \{i | x_i \notin N\}$. Then $M/N = \cup_{i \in S} G\overline{x_i}$. Since S is finite, we have that M/N has finitely many orbits. □

3.2 Proof of the structure theorem

Proposition 3.10. *Let R be a ring with 1 and T be a simple left R-module. If $End_R(T)$ is infinite, then $T \oplus T$ has infinitely many submodules.*

Proof. Let $\sigma, \tau \in \mathrm{End}_R(T)$. If the submodules $\{(x, \sigma(x)) | x \in T\}$ and $\{(x, \tau(x)) | x \in T\}$ of $T \oplus T$ are equal, then there exist $(x_0, \sigma(x_0)) = (y_0, \tau(y_0)) \neq (0, 0)$, so that $x_0 = y_0$ and $\sigma(x_0) = \tau(y_0) = \tau(x_0)$. Now, for every $y \in T$, there exists $a \in R$ such that $y = ax_0$. It follows then that $\sigma(y) = \sigma(ax_0) = a\sigma(x_0) = a\tau(x_0) = \tau(ax_0) = \tau(y)$. Hence, $\sigma = \tau$. This shows that $T \oplus T$ has infinitely many submodules. □

Corollary 3.11. *Let R be a ring and T be a simple module. If $End_R(T)$ is infinite, then T^n has infinitely many submodules for any $n \geq 2$.*

Proof. Since $T \oplus T$ is a submodule of T^n, it follows from the previous proposition that T^n has infinitely many submodules for any $n \geq 2$. □

The idea and insight for the following key proposition came from Victor Camillo, to whom I am very grateful.

Proposition 3.12. *Let R be a ring and A and B be two left R-modules. If $Hom_R(A, B)$ is finite, then B has finitely many complements in $A \oplus B$.*

Proof. Let X be a complement of B in $A \oplus B$. Let $\sigma_A : A \hookrightarrow A \oplus B$ be the natural injection and $\pi_X : X \oplus B \twoheadrightarrow B$ be the natural surjection which is the projection onto B with kernel X. Define $f : A \to B$ by $f = \pi_X \sigma_A$. Then f can be extended to $\overline{f} \in \mathrm{End}_R(A \oplus B)$ such that $\overline{f}|_A = f|_A$ and $f = 0$ in B. We claim that $X = (\mathrm{Id} - \overline{f})(A)$. Indeed, if $a \in A$, then $a = x + b$ for $x \in X$ and $b \in B$. Thus $\overline{f}(a) = f(a) = \pi(x+b) = b$, so that $x = (\mathrm{Id} - \overline{f})(a)$. On the other hand, if $x \in X$, then $x = a' + b'$ where $a' \in A$ and $b' \in B$. Thus, $a' = x - b'$. This implies that $\pi(a') = -b' = f(a') = \overline{f}(a')$. Hence, $x = a' - \overline{f}(a') = (\mathrm{Id} - \overline{f})(a')$. This proves the claim. Now, since each complement is obtained uniquely from some $f \in \mathrm{Hom}_R(A, B)$, there are only finitely many such complements X. □

Proof of Theorem 3.1

Proof. (1) $\Rightarrow$ (2): Suppose M has finitely many orbits under the regular action. Then by Proposition 3.5, we have that M has finitely many submodules.

(2) $\Rightarrow$ (3): Suppose M has finitely many submodules. Then by Proposition 3.5, M has finite length. We show that if N is a submodule of M, then M/N has finitely submodules. Indeed, by the lattice isomorphism Theorem, the submodules of M/N are in one-to-one correspondence with the submodules of M containing N. But since M has finitely many submodules, we must have that M/N has finitely many submodules as well. Now, If M has a sub-quotient isomorphic to $T \oplus T$ with T a simple R-module and $\text{End}_R(T)$ infinite, then by Proposition 3.7, $T \oplus T$ has infinitely many submodules. But this contradicts the previous argument. Thus M has no sub-quotient isomorphic to $T \oplus T$ with T simple and $\text{End}_R(T)$ infinite.

(3) $\Rightarrow$ (2): Suppose that M has no sub-quotient isomorphic to $T \oplus T$ with T simple and $\text{End}_R(T)$ infinite. We show that M has finitely many submodules by induction on the length $l(M)$ of M. If $l(M) = 1$, then M is simple, so it has finitely many submodules. Suppose that this is true for modules of length less than $l(M)$. Let Σ be the socle of M. Then, there are two cases to look at.

Case 1: M properly contains Σ. For a fixed simple submodule S of M, let $\mathfrak{L}(S) = \{N \subseteq M | N \supseteq S\}$. Then the lattice $L(M)$ of submodules of M is given by $L(M) = \cup_S \mathfrak{L}(S)$. This is a finite union since $l(\Sigma) < l(M)$ and by the induction hypothesis Σ has finitely many submodules. We claim that each $\mathfrak{L}(S)$ is finite. Indeed, by the lattice isomorphism theorem, we have $\mathfrak{L}(S) = L(M/S)$ where $L(M/S)$ is the lattice of submodules of M/S. By Proposition 3.6 and the induction hypothesis, we have that $L(M/S)$ is finite. Hence, the claim. Therefore, $L(M)$ is finite.

Case 2: $M = \Sigma$. Then $M = S_1 \oplus S_2 \oplus \cdots S_n \oplus T_1 \oplus \cdots \oplus T_p$ where the S_i's are non-isomorphic simple R-modules with infinite endomorphism rings and the T_j's are simple R-modules with finite endomorphism rings. Since $S_1 \oplus \cdots \oplus S_n$ has finitely many submodules, to show that M has finitely many submodules, we need to show that $\oplus_{j=1}^{p} T_j$ has finitely many submodules. Submodules of $\oplus_{j=1}^{p} T_j$ are of two types. Those containing T_1 and those not containing T_1. We show by induction on p that there are finitely submodules of $\oplus_{j=1}^{p} T_j$ containing T_1. If $p = 1$, we are done since T_1 is simple.

Suppose that $\oplus_{j=1}^{n} T_j$ has finitely many submodules for all $n < p$. By the lattice isomorphism theorem, the submodules of $\oplus_{j=1}^{p} T_j$ containing T_1

are in one-to-one correspondence with the submodules of $\oplus_{j=2}^{p} T_j$. By the induction hypothesis, $\oplus_{j=2}^{p} T_j$ has finitely many submodules. Hence, there are finitely many submodules of $\oplus_{j=1}^{p} T_j$ containing T_1. This completes the induction. Now, we consider the submodules of $\oplus_{j=1}^{n} T_j$ which do not contain T_1. If X is such a submodule, it is contained in a complement of T_1, say Y and $T_1 \oplus Y = \oplus_{j=1}^{p} T_j$. Thus, the submodules of $\oplus_{j=1}^{n} T_j$ not containing T_1 are in one-to-one correspondence with the complements of T_1 in $\oplus_{j=1}^{p} T_j$. But by Proposition 3.6, we have that the complements of T_1 in $\oplus_{j=1}^{p} T_j$ are in one-to-one correspondence with the elements of $\mathrm{Hom}_R(\oplus_{j=2}^{p} T_j, T_1) = \oplus_{j=2}^{p} \mathrm{Hom}_R(T_j, T_1)$. For $j \neq 1$, we have that $\mathrm{Hom}_R(T_j, T_1) =$

0 and for $j = 1$, $\mathrm{Hom}_R(T_1, T_1)$ is finite. Hence, we have that there are finitely many submodules of $\oplus_{j=1}^{n} T_j$ not containing T_1. Therefore, $\oplus_{j=1}^{p} T_j$ has finitely many submodules. Consequently, M has finitely many submodules.

(2) $\Rightarrow$ (1): Suppose M has finitely many submodules. Moreover, suppose R is a semilocal ring. We show that $L(M)$ being finite implies that M has finitely many orbits. Indeed, if $x, y \in M$ such that $Rx = Ry$, then there exist $a, b \in R$ such that $y = ax$ and $x = by$. Thus $(1 - ab) \in \mathrm{ann}(y)$ where $\mathrm{ann}(y)$ is the annihilator of y. This implies $1 = ab + c$ for some $c \in \mathrm{ann}(y)$. So $Rb + \mathrm{ann}(y) = R$. By Theorem 2.5, we have that $(b + \mathrm{ann}(y)) \cap G \neq \emptyset$. Let $u \in (b + \mathrm{ann}(y)) \cap G$. Then $(b - u)y = 0$, so that $x = by = uy$. This implies that $Gx = Gy$. Conversely, if $Gx = Gy$, obviously $Rx = Ry$, and this shows that x and y are in the same G-orbit if and only if they generate the same cyclic submodule. Since $L(M)$ is finite, it follows that M has finitely many orbits. $\square$

Remark. In the last part of the proof of Theorem 3.1, we use the fact that the ring R was semilocal. We give below an example to show that if the ring R is not semilocal, then having finitely many submodules does not necessary imply that M has finitely many orbits under the regular action.

Example 3.13. *Consider $R = \mathbb{C}\langle x, y\rangle$. Then R is not semilocal. Now, let V be a 2-dimensional $\mathbb{C}$-module where x and y act like the matrices*

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

respectively. Then V is simple since A and B do not have a common eigenvector.

The group of units of R is $\mathbb{C}$. Thus, for every non-zero $v \in V$, we have that $\mathbb{C}v$ is an orbit of V under the regular action. Hence, V has infinitely many orbits.

3.3 Rings with finitely many orbits

In this section we consider the case where our module is the regular module, that is, $M = {}_RR$. If we apply the theorem of the previous section to the left regular module then we obtain a new module theoretical proof of the structure theorem for rings with finitely many orbits under the left regular action.

Proposition 3.14. *Let R be a ring with finitely many left ideals. Suppose $T \ncong S$ are simple R-modules and S is infinite. Then, $Ext^1_R(T, S) = 0$.*

Proof. Suppose $\mathrm{Ext}^1_R(T, S) \neq 0$. Then there exists an indecomposable R-module V such that the short exact sequence $0 \to S \to V \to T \to 0$ does not split. Let $P(T)$ be the projective cover of T. Then there is a surjection $\pi : P(T) \to V$. Note that $P(T)$ has top T and S shows up at the top of $\mathrm{rad}(P(T))$ in the composition series of $P(T)$. Let $P(S)$ be the projective cover of S. Since $T \ncong S$, by Corollary 3.4, we have that R is isomorphic to $P(T) \oplus P(S) \oplus X$ for some module X. Let $N = \ker \pi \oplus \mathrm{rad}(P(S)) \oplus X$. Then

$$R/N = \begin{bmatrix} T \\ S \\ \vdots \end{bmatrix} \oplus \begin{bmatrix} S \\ \vdots \end{bmatrix}.$$

This implies that the regular module ${}_RR$ has sub-factor $S \oplus S$. Now, Since R is left artinian, $R/\mathrm{Rad}(R)$ is semisimple artinian, so S is a finite dimensional vector space over the division ring $\mathrm{End}_R(S)$. Hence, since S is infinite, it follows that $\mathrm{End}_R(S)$ is infinite as well. But this contradicts the statement of Theorem 3.1 because ${}_RR$

has finitely many left submodules (left ideals) by hypothesis. Hence, we must have $\mathrm{Ext}^1_R(T,S)=0$. □

Proposition 3.15. *Let R be a ring with finitely many left ideals. Suppose F and S are simple submodules of R with F finite and S infinite. Then*

1. *$Ext^1_R(F,S)=0$, and*

2. *$Ext^1_R(S,F)=0$.*

Proof. (1) follows from the above Proposition 3.9. For (2), suppose $\mathrm{Ext}^1_R(S,F)\neq 0$. Since R is also left artinian(R has finitely many left ideals by hypothesis) by Proposition 2.1 in [18], we must have that $|S|\leq|F|$. This contradicts that S is infinite and F is finite. Hence, $\mathrm{Ext}^1_R(S,F)=0$. □

Lemma 3.16. *Let $R=\prod_{k=1}^n R_k$ be a product of rings R_k with groups of units G_k. If each R_k has finitely many orbits under the regular action of G_k, then R has finitely many orbits under its group of units.*

Proof. Let O be an orbit of R under the regular action. Then, $O=\prod_{k=1}^n O_k$ where O_k is an orbit of R_k. Since each R_k has finitely many orbits under the action of their groups of units, we have that R has finitely many orbits under its group of units. □

We now have all the ingredients to give a new proof of theorem 2.4 in [16].

Theorem 3.17. *(Hirano) The following statements are equivalent for a ring R:*

1. *R has only finitely many orbits under the regular action.*

2. *R has only finitely many left ideals.*

3. R is the direct sum of finitely many left uniserial rings and a finite ring.

Proof. (3) $\Rightarrow$ (1) By hypothesis, we can write R as $R = F \oplus R_1 \oplus \cdots \oplus R_n$ where F is a finite ring and each R_i is a left uniserial ring (hence local)and has an infinite simple S_i. Hence, each R_i satisfies condition (3) in Theorem 3.1, and so has finitely many orbits from the group of units of R and of R_i as well. Thus, from Lemma 3.11, R has finitely many orbits under the regular action. Hence, (1) follows.

(1) $\Rightarrow$ (2) This also follows directly from Theorem 3.1.

(2) $\Rightarrow$ (3). Since infinite-dimensional and finite-dimensional simples are Ext-orthogonal, we can use Proposition 3.10 and block or Pierce decomposition to show that $R = R_f \oplus R_I$, where R_f is a finite ring and R_I is a ring with only infinite simple modules. Now, by Proposition 3.9, every two infinite simples S and T are Ext- orthogonal, i.e., $\mathrm{Ext}^1_R(S,T) = 0$. Hence, by block decomposition, R_I is a product of local rings R_i. Since R is artinian, so is R_I and also has only finitely many R or R_I-submodules (hence ideals) since this property is preserved by submodules and quotients. So, J_i^k/J_i^{k+1} in each of the R_i must be simple as a left module (cannot contain $S \oplus S$) where J_i is the Jacobson radical of R_i. This shows that each R_i is left uniserial; they are obviously artinian as they have only finitely many left submodules.

□

Let Λ be a k-algebra with each simple module being 1-dimensional. Then, for S and T simple Λ-modules, we use Theorem 3.1 to obtain strong information on $\mathrm{Ext}^1_\Lambda(S,T)$ under the weaker assumption that only finitely many two sided ideals exist in Λ. We have

Proposition 3.18. *Let Λ be a basic k-algebra with k infinite and suppose Λ has only finitely many two sided ideals. Let S and T be two simple Λ-modules. Then $dim_k Ext^1_A(T, S) \leq 1$.*

Proof. There exist two sided ideals I and J such that $S = \Lambda/I$ and $T = \Lambda/J$. Then, it is known that there is an identification of $\mathrm{Ext}^1_\Lambda(S, T) = \frac{I \cap J}{JI}$ as a Λ-bimodule. Now, since $\frac{I \cap J}{JI}$ is a subquotient of ${}_\Lambda\Lambda_\Lambda$, we have that $\frac{I \cap J}{JI}$ is isomorphic to $(S \otimes_\Lambda T)^n$, for some $n \geq 1$. Since ${}_\Lambda\Lambda_\Lambda$ has finitely many sub-bimodules, we have by Theorem 3.1 that $n = 1$. This implies that $\dim_k(\mathrm{Ext}^1_\Lambda(S, T)) \leq 1$ as required. □

CHAPTER 4
DEFORMATIONS OF INCIDENCE ALGEBRAS

4.1 Introduction

Let k be a field and P be a poset. Feinberg in [8] provided intrinsic characterizations of all incidence algebras $I(P,k)$ where P is a lower finite quasi-order set. In this chapter we study the deformations of incidence algebras $I(P,k)$ where P is a finite poset and k is a field. We show that the deformations of an incidence algebra $I(P,k)$ are unital associative algebras and describe when two deformations of the same incidence algebra are isomorphic. These considerations can be made for arbitrary lower finite posets as well, but for clarity we restrict ourselves to the finite case, as it is also of interest to representation theory.

4.2 Deformation of Incidence Algebras

Let $S = \{(x,y,z) \in P \times P \times P | x \leq y \leq z \in P\}$. Then for each function

$$\lambda : S \to k^*,$$

we will denote $\lambda(x,y,z)$ by λ_{xz}^{y}.

Definition 4.1. For each λ, we introduce a new multiplication $*_\lambda$ on $I(P,k)$ as follows.

1. $e_{xy} *_\lambda e_{yz} = \lambda_{xz}^{y} \cdot e_{xz}$, and
2. $e_{xy} *_\lambda e_{tz} = 0$ when $y \neq t$.

We denote the incidence algebra $I(P,k)$ with the new multiplication $*_\lambda$ by $I_\lambda(P,k)$.

Theorem 4.1. *$I_\lambda(P,k)$ is an associative algebra if and only if*

$$\lambda_{xz}^{y} \cdot \lambda_{xt}^{z} = \lambda_{xt}^{y} \cdot \lambda_{yt}^{z}$$

for any $x \leq y \leq z \leq t \in P$.

Proof. For any $x \leq y \leq z \leq t \in P$, we have that

$$(e_{xy} *_\lambda e_{yz}) *_\lambda e_{zt} = e_{xy} *_\lambda (e_{yz} *_\lambda e_{zt}). \tag{4.1}$$

This is equivalent by definition to

$$\lambda_{xz}^{y} \cdot \lambda_{xt}^{z} \cdot e_{xt} = \lambda_{xt}^{y} \cdot \lambda_{yt}^{z} \cdot e_{xt}$$

which in turn is equivalent to

$$\lambda_{xz}^{y} \cdot \lambda_{xt}^{z} = \lambda_{xt}^{y} \cdot \lambda_{yt}^{z},$$

as desired. ∎

Remark.

1. The associativity law on other types of triplets always trivially works, since both sides of (4.1) would multiply to zero.
2. If the function $\lambda : S \to k^*$ satisfies the identity

   $$\lambda_{xz}^{y} \cdot \lambda_{xt}^{z} = \lambda_{xt}^{y} \cdot \lambda_{yt}^{z}, \tag{4.2}$$

   then

   (a) $\lambda_{xy}^{y} = \lambda_{yy}^{y}$ for $x \leq y$, and

   (b) $\lambda_{xx}^{x} = \lambda_{xt}^{x}$ for $x \leq t$.

Identities (1) and (2) follow from (4.2) by setting $y = z = t$ and $x = y = z$, respectively.

Proposition 4.2. *Let $\lambda : S \to k^*$ and $\mu : S \to k^*$ be two functions. If there exists $\alpha_{xy} \in k^*$ for all $x, y \in P$ with $x \leq y$ such that*

$$\lambda_{xz}^{y} = \frac{\alpha_{xy} \cdot \alpha_{yz}}{\alpha_{xz}} \cdot \mu_{xz}^{y}, \tag{4.3}$$

for all $x, y, z \in P$ with $x \leq y \leq z$, then the algebras $I_\lambda(P, k)$ and $I_\mu(P, k)$ are isomorphic.

Proof. Let $\phi : I_\lambda(P, k) \to I_\mu(P, k)$ be given by $\phi(e_{xy}) = \alpha_{xy} e_{xy}$ where the e_{xy}'s form a basis for both $I_\lambda(P, k)$ and $I_\mu(P, k)$, and extend ϕ by linearity to obtain a linear map of k-vector spaces. We show that ϕ is an isomorphism of k-algebras. Suppose $\phi(g) = 0$. Then, since

$$g = \sum_{x \leq y} a_{xy} e_{xy},$$

we have that $\phi(g) = 0$ implies that

$$\sum_{x \leq y} a_{xy} \alpha_{xy} e_{xy} = 0.$$

Since the e_{xy}'s are linearly independent, it follows that $a_{xy} = 0$. Thus, $g = 0$. Hence, ϕ is injective. Since $\dim_k I_\lambda(P, k) = \dim_k I_\mu(P, k) = \dim_k I(P, k)$ as a k-vector space, we have that ϕ is a bijection. Now, if

$$g = \sum_{x \leq y} a_{xy} e_{xy} \quad \text{and} \quad h = \sum_{x \leq y} b_{xy} e_{xy}$$

are two elements in $I_\lambda(P, k)$, then

$$g *_\lambda h = \sum_{x \leq y} \left(\sum_{x \leq t \leq y} a_{xt} b_{ty} \lambda_{xy}^{t} \right) e_{xy}.$$

This implies that

$$\phi(g *_\lambda h) = \sum_{x \le y} \left(\sum_{x \le t \le y} a_{xt} b_{ty} \alpha_{xy} \lambda_{xy}^t \right) e_{xy}.$$

Thus, by (4.3), we have that

$$\phi(g *_\lambda h) = \sum_{x \le y} \left(\sum_{x \le t \le y} a_{xt} b_{ty} \alpha_{xy} \cdot \frac{\alpha_{xt} \cdot \alpha_{ty}}{\alpha_{xy}} \cdot \mu_{xy}^t \right) e_{xy}$$

$$= \sum_{x \le y} \left(\sum_{x \le t \le y} a_{xt} b_{ty} \alpha_{xt} \alpha_{ty} \mu_{xy}^t \right) e_{xy}.$$

On the other hand, since

$$\phi(g) *_\mu \phi(h) = \left(\sum_{x \le y} a_{xy} \alpha_{xy} e_{xy} \right) *_\mu \left(\sum_{x \le y} b_{xy} \alpha_{xy} e_{xy} \right)$$

$$= \sum_{x \le y} \left(\sum_{x \le t \le y} a_{xt} b_{ty} \alpha_{xt} \alpha_{ty} \mu_{xy}^t \right) e_{xy},$$

we have that $\phi(g *_\lambda h) = \phi(g) *_\mu \phi(h)$. Hence, ϕ is an isomorphism of k-algebras.

□

Proposition 4.3. *For each $\lambda : S \to k^*$, there exists $\beta : S \to k^*$ such that $I_\beta(P, k)$ is a unital associative algebra isomorphic to $I_\lambda(P, k)$. Moreover, the corresponding identities for β in (4.2) are all equal to* 1.

Proof. For the function $\lambda : S \to k^*$, define for all $x, y \in P$ such that $x \le y$,

$$\alpha_{xy} = \frac{1}{\lambda_{xy}^x}.$$

Then, we consider the function $\beta : S \to k^*$ defined by

$$\beta^y_{xz} = \frac{\alpha_{xy} \cdot \alpha_{yz}}{\alpha_{xz}} \cdot \lambda^y_{xz} \tag{4.4}$$

for all $x, y, z \in P$ such that $x \leq y \leq z$. Now, for all $x, y, z, t \in P$ such that $x \leq y \leq z \leq t$, since

$$\beta^y_{xz} \cdot \beta^z_{xt} = \frac{\alpha_{xy} \cdot \alpha_{yz}}{\alpha_{xz}} \lambda^y_{xz} \cdot \frac{\alpha_{xz} \cdot \alpha_{zt}}{\alpha_{xt}} \lambda^z_{xt}$$

$$= \frac{(\lambda^x_{xy})^{-1} \cdot (\lambda^y_{yz})^{-1}}{(\lambda^x_{xz})^{-1}} \lambda^y_{xz} \cdot \frac{(\lambda^x_{xz})^{-1} \cdot (\lambda^z_{zt})^{-1}}{(\lambda^x_{xt})^{-1}} \lambda^z_{xt}$$

$$= \left(\frac{(\lambda^x_{xy})^{-1} \cdot (\lambda^y_{yz})^{-1} \cdot (\lambda^z_{zt})^{-1}}{(\lambda^x_{xt})^{-1}} \right) \cdot \lambda^y_{xz} \cdot \lambda^z_{xt}$$

and

$$\beta^y_{xt} \cdot \beta^z_{yt} = \frac{\alpha_{xy} \cdot \alpha_{yt}}{\alpha_{xt}} \lambda^y_{xt} \cdot \frac{\alpha_{yz} \cdot \alpha_{zt}}{\alpha_{yt}} \lambda^z_{yt}$$

$$= \frac{(\lambda^x_{xy})^{-1} \cdot (\lambda^y_{yt})^{-1}}{(\lambda^x_{xt})^{-1}} \lambda^y_{xt} \cdot \frac{(\lambda^y_{yz})^{-1} \cdot (\lambda^z_{zt})^{-1}}{(\lambda^y_{yt})^{-1}} \lambda^z_{yt}$$

$$= \left(\frac{(\lambda^x_{xy})^{-1} \cdot (\lambda^y_{yz})^{-1} \cdot (\lambda^z_{zt})^{-1}}{(\lambda^x_{xt})^{-1}} \right) \cdot \lambda^y_{xt} \cdot \lambda^z_{yt},$$

It follows that

$$\beta^y_{xz} \cdot \beta^z_{xt} = \beta^y_{xt} \cdot \beta^z_{yt}.$$

Thus, by Theorem 4.2, we have that $I_\beta(P, k)$ is an associative algebra. Hence, by (4.3) and Proposition 4.3, it follows that $I_\beta(P, k)$ is isomorphic to $I_\lambda(P, k)$. From (4.3) and using also the relations in Remark 4.1, we see that for $x \leq y$ and $x \leq t$,

$$\beta^x_{xx} = \beta^y_{yy} = \beta^y_{xy} = \beta^x_{xt} = 1. \tag{4.5}$$

Let $g \in I_\beta(P, k)$. We have that

$$(\delta *_\beta g)(x, y) = \sum_{x \le z \le y} \beta^z_{xy} \delta(x, z) \cdot g(z, y)$$

$$= \beta^x_{xy} \delta(x, x) \cdot g(x, y)$$

$$= g(x, y).$$

Similarly, we have that $g *_\beta \delta = g$. Thus, $I_\beta(P, k)$ is unital. □

Remark. If $\lambda^y_{xz} = 1$ for any $x \le y \le z \in P$, we have that $I_\lambda(P, k)$ is the incidence algebra $I(P, k)$ itself.

Remark. We may assume now for the rest of thesis that every function $\lambda : S \to k^*$ satisfies

$$\lambda^y_{xy} = \lambda^y_{yy} = \lambda^x_{xx} = \lambda^x_{xt} = 1,$$

for $x \le y$ and $x \le t$. In this case, it follows that the functions e_x for all $x \in P$ are orthogonal primitive idempotents for the algebra $I_\lambda(P, k)$.

Definition 4.2. We call the unital associative algebra $I_\lambda(P, k)$ a deformation of the incidence algebra $I(P, k)$.

The following theorem shows when two deformations of the incidence algebra $I(P, k)$ are isomorphic.

Theorem 4.4. *Suppose P has no non-trivial automorphism. Then $I_\lambda(P,k)$ is isomorphic to $I_\mu(P,k)$ as a k-algebra if and only if there exist $\alpha_{xy} \in k^*$ for every $x, y \in P$ with $x \leq y$ such that*

$$\lambda_{xz}^{y} = \frac{\alpha_{xy} \cdot \alpha_{yz}}{\alpha_{xz}} \mu_{xz}^{y} \tag{4.6}$$

for all $x \leq y \leq z \in P$.

Proof. ($\Rightarrow$) Suppose $\phi : I_\lambda(P,k) \to I_\mu(P,k)$ is an isomorphism of algebras. To avoid confusion, we will use different letters for the basis e_{xy} for the second algebra $I_\mu(P,k)$. So, let e_{xy} and f_{xy} be the bases of $I_\lambda(P,k)$ and $I_\mu(P,k)$, respectively. Then by the above remark, we have that $\{e_x | x \in P\}$ and $\{f_x | x \in P\}$ are systems of primitive orthogonal idempotents for $I_\lambda(P,k)$ and $I_\mu(P,k)$, respectively. Now, let $h_{xy} = \phi(e_{xy})$. Then since ϕ is an isomorphism, $\{h_{xy} | x \leq y \in P\}$ and $\{h_x | x \in P\}$ are a basis and a complete system of primitive orthogonal idempotents, respectively, for $I_\mu(P,k)$. By Lemma 2.6, we have that there exists an invertible element $a \in I_\mu(P,k)$ such that after some permutation σ of the elements of P,

$$a^{-1} h_x a = f_{\sigma(x)}. \tag{4.7}$$

Since $\phi(e_x) = h_x$, (4.7) becomes

$$a^{-1} \phi(e_x) a = f_{\sigma(x)}.$$

Since $a^{-1}\phi(I_\lambda(P,k))a$ is isomorphic to $I_\mu(P,k)$(in fact, they are equal as sets), we have

$$f_{\sigma(x)}I_\mu(P,k)f_{\sigma(y)} = a^{-1}\phi(e_x)a \cdot a^{-1}\phi(I_\lambda(P,k))a \cdot a^{-1}\phi(e_y)a.$$

$$= a^{-1}\phi(e_x)\phi(I_\lambda(P,k))\phi(e_y)a$$

$$= a^{-1}\phi(e_x I_\lambda(P,k)e_y)a.$$

This implies that

$$f_{\sigma(x)}I_\mu(P,k)f_{\sigma(y)} \neq 0 \quad \text{if and only if} \quad e_x I_\lambda(P,k)e_y \neq 0.$$

Hence,

$$\sigma(x) \leq \sigma(y) \quad \text{if and only if} \quad x \leq y.$$

Therefore, σ preserves the order of P. If P has no trivial automorphism, then σ is the identity. So, we have in $I_\mu(P,k)$ that

$$f_x = a^{-1}h_x a.$$

Now, if $x \leq y$, we have that

$$f_x *_\mu (a^{-1}h_{xy}a) *_\mu f_y = (a^{-1}(af_xa^{-1})a) *_\mu (a^{-1}h_{xy}a) *_\mu (a^{-1}(af_ya^{-1})a)$$

$$= (a^{-1}h_xa) *_\mu (a^{-1}h_{xy}a) *_\mu (a^{-1}h_ya)$$

$$= a^{-1}((h_x *_\mu h_{xy}) *_\mu h_y)a$$

$$= a^{-1}((\mu^x_{xy} h_{xy}) *_\mu h_y)a$$

$$= a^{-1}\mu^x_{xy}(h_{xy}) *_\mu h_y)a$$

$$= a^{-1}\mu^y_{xy}\mu^x_{xy}h_{xy}a$$

$$= a^{-1}h_{xy}a \qquad \text{(since by Remark 4.3, we have } \mu^x_{xy} = \mu^y_{xy} = 1\text{).}$$

The above computation shows that $a^{-1}h_{xy}a$ is a non-zero multiple of f_{xy}, that is $a^{-1}h_{xy}a = \alpha_{xy}f_{xy}$ for some $\alpha_{xy} \in k^*$. Now, since

$$a\alpha_{xz}f_{xz}a^{-1} = h_{xz}$$

$$= \phi(e_{xz})$$

$$= \phi\left(\frac{e_{xy} *_\lambda e_{yz}}{\lambda^y_{xz}}\right)$$

$$= \frac{\phi(e_{xy}) *_\mu \phi(e_{yz})}{\lambda^y_{xz}}$$

$$= \frac{h_{xy} *_\mu h_{yz}}{\lambda^y_{xz}}$$

$$= \frac{(a\alpha_{xy}f_{xy}a^{-1}) *_\mu (a\alpha_{yz}f_{yz}a^{-1})}{\lambda_{xz}^y}$$

$$= \frac{\alpha_{xy}\alpha_{yz}a(f_{xy} *_\mu f_{yz})a^{-1}}{\lambda_{xz}^y}$$

$$= \frac{a(\alpha_{xy}\alpha_{yz}\mu_{xz}^y f_{xz})a^{-1}}{\lambda_{xz}^y},$$

we have that

$$\alpha_{xz} = \frac{\alpha_{xy}\alpha_{yz}\mu_{xz}^y}{\lambda_{xz}^y}.$$

This implies that

$$\lambda_{xz}^y = \frac{\alpha_{xy}\alpha_{yz}}{\alpha_{xz}}\mu_{xz}^y,$$

as desired.

($\Leftarrow$) Suppose there exist $\alpha_{xy} \in k^*$ for every $x \leq y \in P$ such that (4.6) is satisfied. Then, by Proposition 4.3, $I_\lambda(P,k)$ is isomorphic to $I_\mu(P,k)$. □

Corollary 4.5. *Suppose P has no non-trivial automorphism. Then, $I_\lambda(P,k) \simeq I(P,k)$ if and only if there exist $\alpha_{xy} \in k^*$ for every $x \leq y \in P$ such that*

$$\lambda_{xz}^y = \frac{\alpha_{xy} \cdot \alpha_{yz}}{\alpha_{xz}} \tag{4.8}$$

for all $x \leq y \leq z \in P$.

Proof. This follows from the above Theorem by taking $\mu_{xz}^{y} = 1$ for all $x, y, z \in P$ such that $x \leq y \leq z$. □

Remark. We define a relation $\sim$ on the set of functions from S into k^* by $\lambda \sim \mu$ if there exists $\alpha : P \to k^*$ such that

$$\lambda_{xz}^{y} = \frac{\alpha_{xy} \cdot \alpha_{yz}}{\alpha_{xz}} \mu_{xz}^{y}$$

for all $x, y, z \in P$ such that $x \leq y \leq z$. It is easy to see that this is an equivalence relation. The reason is, at least in the case when P has no non-trivial automorphism, one has that $\lambda \sim \mu$ if and only if the corresponding deformations of the incidence algebra are isomorphic. We will see in the next section that $\lambda \sim \mu$ if and only if they are cohomologous.

4.3 Cohomology of the deformations of incidence algebras

In this section we investigate the cohomology of the deformations of incidence algebras. We use the method of the geometric realization of a poset to achieve our goal. This method shows that to every finite poset, we can associate a simplicial complex. We begin this section with a definition.

Definition 4.3. An element $f \in I(P, k)$ is a multiplicative function if

- $f(x, y) \neq 0$ for all $x \leq y \in P$, *and*
- $f(x, y) \cdot f(y, z) = f(x, z)$ for all $x \leq y \leq z \in P$.

Remark. The set of all multiplicative functions form an abelian group under point-wise multiplication. We denote this group by $Z(P, k)$. For any $\alpha : P \to k \setminus \{0\}$, we define

$d_\alpha \in Z(P,k)$ by $d_\alpha(x,y) = \alpha(x)\alpha(y)^{-1}$ and $B(P,k) = \{d_\alpha \in Z(P,k) | \alpha : P \to k\backslash\{0\}\}$. Then $B(P,k)$ is a subgroup of $Z(P,k)$. Let $H(P,k) = Z(P,k)/B(P,k)$. Then it is shown in [9] that $H(P,k)$ is in bijection with the isomorphism classes of faithful distributive left $I(P,k)$-modules. In this section, we show that the elements of $H(P,k)$ are exactly the elements of a cohomology group.

To every poset P, we associate an abstract simplicial complex $\Delta(P)$ called the geometric realization of P as follows. The vertices of $\Delta(P)$ are the elements of P and the faces of $\Delta(P)$ are the totally ordered subsets of P(i.e., the chains of P). Let

$$\mathbb{Z}C^n = \mathbb{Z} - \text{span}\{(s_0, s_1, s_2, \cdots, s_n) : s_0 < s_1 < s_2 < \cdots < s_n, s_i \in P\}.$$

Then we get a chain complex

$$\mathbb{Z}C_n \xrightarrow{\delta_n} \mathbb{Z}C_{n-1} \tag{4.9}$$

realizing singular homology with boundary map given by

$$\partial_n(s_0, s_1, s_2, \cdots, s_n) = \sum_{i=0}^{n} (-1)^i (s_0, s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_n).$$

Dualizing (4.9) with respect to the abelian group $(k^*, \cdot)$, we obtain the cochain complex

$$E^{n-1} \xrightarrow{(\delta)^*} E^n \tag{4.10}$$

where $E^n = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}C^n, k^*)$ and

$$(\partial_n)^*(f)(s_0, s_1, s_2, \cdots, s_n) = \prod_{i=0}^{n} f(s_0, s_1, s_2, \cdots, s_{i-1}, s_{i+1}, \cdots, s_n)^{(-1)^i}.$$

Remark. Given $\lambda : S \to k^*$, we can define $\lambda_{xz}^y = f(x,y,z)$ for some $f \in E^2$.

Proposition 4.6. *If $I_\lambda(P,k)$ is a deformation of an incidence algebra, then* $(\partial_3)^*(f) = 1$.

Proof. By Theorem 4.2, we have that

$$f(x,y,z)\cdot f(x,z,t) = f(x,y,t)\cdot f(y,z,t).$$

This implies that

$$f(y,z,t)\cdot f(x,z,t)^{-1}\cdot f(x,y,t)\cdot f(x,y,z)^{-1} = 1.$$

If we denote

$$s_0 = x, s_1 = y, s_2 = z \text{ and } s_3 = t,$$

it follows from the definition of $(\partial_3)^*$ that $(\partial_3)^*(f) = 1$. □

Proposition 4.7. *Given the functions $\lambda : S \to k^*$ and $\mu : S \to k^*$, we have that $\lambda \sim \mu$ if and only if $(\lambda\cdot\mu^{-1}) \in im(\partial_2)^*$, i.e., $\lambda : S \to k^*$ and $\mu : S \to k^*$ represent the same element in the cohomology group $H^2(P,k^*)$.*

Proof. By Theorem 4.6, $\lambda \sim \mu$ if and only if there exists $\alpha_{xy} \in k^*$ for every $x \leq y \in P$ such that

$$\lambda_{xz}^y = \frac{\alpha_{xy}\cdot\alpha_{yz}}{\alpha_{xz}}\cdot\mu_{xz}^y. \tag{4.11}$$

Now, if we let $\lambda_{xz}^y = f(x,y,z), \mu_{xz}^y = g(x,y,z)$ and $\alpha_{xy} = h(x,y)$ for some $f,g \in E^2$ and $h \in E^1$, then equation (4.11) is equivalent to $f(x,y,z)\cdot g(x,y,z)^{-1} = h(y,z)\cdot h(x,z)^{-1}\cdot h(x,y)$. Thus, we have that $(\partial_2)^*(h)(x,y,z) = f(x,y,z)\cdot g(x,y,z)^{-1}$. This is equivalent to $\lambda\cdot\mu^{-1} \in \text{im}(\partial_2)^*$. □

The following corollary is a restatement of Theorem 1.1 in [9] in terms of cohomology in the case where the poset P is finite.

Corollary 4.8. *Let P be a finite poset and $I(P,k)$ be its incidence algebra. Then the isomorphism classes of faithful distributive left $I(P,k)$-modules are exactly the elements of $H^1(P,k^*)$.*

Proof. By Theorem 1.1 in [9], the isomorphism classes of faithful distributive left $I(P,k)$-modules are in bijection with $H(P,k)$. Now, if $f \in \ker(\partial_1)^*$, we have that

$$f(y,z) \cdot f(x,z)^{-1} \cdot f(x,y) = 1.$$

This implies that $\ker(\partial_1)^*$ is the set of all multiplicative functions of $I(P,k)$. If $g \in \mathrm{im}(\partial_0)^*$ we have that there exists $h \in E^0$ such that

$$(\partial_0)^*(g)(x,y) = h(x)h(y)^{-1}.$$

This implies that the elements of $\mathrm{im}(\partial_0)^*$ are exactly the elements of $B(P,k)$ as described at the beginning of the section. Hence, we have that $H(P,k) = \ker(\partial_1)^*/\mathrm{im}(\partial_0)^* = H^1(P,k^*)$. □

Corollary 4.9. *Suppose P has no non-trivial automorphism. Then the cohomology group $H^2(P,k^*)$ parametrizes the isomorphism classes of deformations of the incidence algebra $I(P,k)$.*

Remark. If P has non-trivial automorphisms, then the isomorphisms of deformations of the incidence algebra $I(P,k)$ are parametrized by the orbits of the automorphisms of P on $H^2(P,k^*)$ under a natural action.

Proof. This follows from Theorems 4.2 and 4.6. □

Definition 4.4. A finite dimensional k-algebra Λ is locally hereditary if every local Λ-submodule of an indecomposable projective Λ-module is projective.

Remark. By Lemma 2.8 in [6], the above definition of locally hereditary is equivalent to the condition that every non-zero morphism between two indecomposable projective modules is injective.

Before we state and prove the next theorem, we need the following propositions. As usual, we will take k to be an infinite field.

Proposition 4.10. *Let Λ be a finite-dimensional basic k-algebra with no oriented cycles and suppose all the projective indecomposable Λ-modules are distributive. Then for any simple modules S and T in the Ext-quiver of Λ, we have that $dim_k Hom_\Lambda(P(S), P(T)) \leq 1$ where $P(S)$ and $P(T)$ are the projective covers of the simple modules S and T, respectively.*

Proof. Suppose $[P(S) : T] \geq 2$ where $[P(S) : T]$ denotes the multiplicity of T in the Jordan-Holder series of $P(S)$. Then there exist $x, y \in \text{rad}(P(S))$ such that $\Lambda x \neq \Lambda y$ and T is on top of both Λx and Λy in their Jordan-Holder series. Without loss of generality, we may assume that both Λx and Λy are local left Λ-modules. Let J be the Jacobson radical of Λ. Now, we have two cases.

Case 1: $\Lambda x + Jy = Jy + Jx$(or $\Lambda y + Jx = Jy + Jx$). Then, $\Lambda x \subseteq \Lambda y + Jx$. This implies that $x = ty + lx$ for some $l, t \in J$. Since $1 - l$ is invertible, we have that $x = (1 - l)^{-1}ty$. Thus, $\Lambda x \subset \Lambda y$, and so Λy has the module T in its Jordan-Holder series, other than the top, which is also isomorphic to T. Hence, by Lemma 2.24, there is a non-trivial path in the Ext-quiver from S to T. This contradicts that the Ext-quiver of Λ had no oriented cycles.

Case 2: The quotients $\frac{\Lambda x+Jy}{Jx+Jy}$ and $\frac{\Lambda y+Jx}{Jx+Jy}$ are non-zero. These quotients are distinct since otherwise, we get case 1. Now, consider the quotient $\frac{P(S)}{Jx+Jy}$. Then, $\frac{\Lambda x}{Jx}$ is isomorphic to $\frac{\Lambda y}{Jy}$ and they both inject into $\frac{P(S)}{Jx+Jy}$. Further, since they are both isomorphic to T, we get that $T \oplus T$ is contained in $\frac{P(S)}{Jx+Jy}$. This contradicts the assumption that all projective indecomposable Λ-modules are distributive. From the above arguments, it follows that $[P(S) : T] \leq 1$. Therefore, since

$\dim_k \mathrm{Hom}_\Lambda(P(S), P(T)) = [P(S) : T]$, we have that $\dim_k \mathrm{Hom}_\Lambda(P(S), P(T)) \leq 1$. □

Proposition 4.11. *Let Λ be a finite dimensional basic k-algebra where all the projective indecomposable Λ-modules are distributive. Then Λ has finitely many two sided-ideals.*

Proof. Let I be a two sided ideal of Λ. Then $I = \oplus_{j=1}^{n} Ie_j$ where the e_j's are the primitive idempotents of Λ. Now, let P_j be the projective indecomposable Λ-modules corresponding to the e_j's. Since $Ie_j \subseteq P_j$ and the P_j's are distributive, we get that each Ie_j can be only one of finitely many submodules of P_j. Then, since $I = \oplus_{j=1}^{n} Ie_j$, I can be only one of finitely many choices. Hence, Λ has finitely many two-sided-ideals. □

Now we are ready to describe all basic k-algebras that are deformations of incidence algebras.

Theorem 4.12. *Let Λ be a finite-dimensional basic k-algebra. Then the following are equivalent.*

1. *Λ is a deformation of an incidence algebra*
2. (a) *The Ext-quiver of Λ has no oriented cycle.*

   (b) *$dim_k Hom_\Lambda(S,T) \leq 1$ where $P(S)$ and $P(T)$ are the projective covers of the simple Λ-modules S and T, respectively, and Λ is locally hereditary.*
3. (a) *The Ext-quiver of Λ has no oriented cycle*

   (b) *Λ is locally hereditary and its projective indecomposable modules are distributive.*

Proof. (1) $\Rightarrow$ (2) Suppose Λ is a deformation of an incidence algebra. Then $\Lambda = I_\lambda(P,k)$ for some finite poset P and $\lambda : S \to k^*$. By Remark 4.3, we have that if $x \leq y$, $e_x \Lambda e_y = k e_{xy}$. Thus, the projectives of Λ are given by $\Lambda e_y = \sum_x e_x \Lambda e_y = \bigoplus_{x \leq y} k e_{xy}$. Hence, $\dim_k \mathrm{Hom}_\Lambda(\Lambda e_x, \Lambda e_y) = \dim_k(e_x \Lambda e_y) = \dim_k(k e_{xy}) = 1$ if $x <= y$ and zero otherwise. Now, let

$$\varphi : \Lambda e_x = \bigoplus_{u \leq x} k e_{ux} \to \Lambda e_y = \bigoplus_{u \leq y} k e_{uy}$$

be a non-zero morphism defined by $\varphi(e_{ux}) = \lambda^x_{uy} \cdot e_{uy}$ and extend it by linearity to obtain a morphism of k-vector spaces. We show that φ is a morphism of Λ-modules. Indeed, for $z \leq u \leq x \leq y$, since

$$\varphi(e_{zu} *_\lambda e_{ux}) = \varphi(\lambda^u_{zx} e_{zx})$$

$$= \lambda^u_{zx} \varphi(e_{zx})$$

$$= \lambda^u_{zx}\lambda^x_{zy}e_{zy}$$

$$= \lambda^x_{uy}\lambda^u_{zy}e_{zy} \qquad \text{by (4.1)}$$

$$= \lambda^x_{uy}e_{zu} *_\lambda e_{uy}$$

$$= e_{zu}\varphi(e_{ux}),$$

φ is a morphism of λ-modules. Now, since $x \leq y$, we have that ϕ is injective. Then, since $\dim_k \mathrm{Hom}_\Lambda(\Lambda e_x, \Lambda e_y) \leq 1$, every other non-zero morphism must be a multiple of φ, thus injective. Hence, by the above remark Λ is locally hereditary. Finally, suppose the Ext-quiver of Λ has a non-trivial oriented cycle, say $e_{x_1} \to e_{x_2} \to \cdots \to e_{x_n} \to e_{x_1}$. Then we would have that $e_{x_1}\Lambda e_{x_2}$ is not equal to zero, and $e_{x_2}\Lambda e_{x_3}$ is not zero, etc. But this implies that $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_1$, so that $x_1 = x_2 = \cdots = x_n = x_1$. But this contradicts that $e_{x_1} \to e_{x_2} \to \cdots \to e_{x_n} \to e_{x_1}$ is a non-trivial cycle. Hence, the Ext-quiver of Λ has no oriented cycles.

(2) $\Rightarrow$ (1) We define a relation on the set of vertices of the Ext-quiver of Λ as follows: $x \leq y$ if and only if $e_x\Lambda e_y \neq 0$. By hypothesis 2b, we have that $e_x\Lambda e_x \neq 0$(note that e_x is a set of primitive orthogonal idempotents, indexed by the vertices of the Ext quiver). Thus, $x \leq x$ and the relation is reflexive. Arguing as in the above implication (1) $\Rightarrow$ (2), since the Ext quiver has no oriented cycles we have that if $x \leq y$ but x is

not y, then there is an injective morphism from Λx into Λy which is not surjective. Similarly, if $y \leq x$ but x is not y, there is an injective morphism from Λy into Λx which is not surjective. This gives us a contradiction. Thus, the relation is antisymmetric. If $x \leq y$ and $y \leq z$, then $e_x \Lambda e_y \neq 0$ and $e_y \Lambda e_z \neq 0$. But by the second part of 2b, this means that we have a non-zero injective morphism $\Lambda x \to \Lambda y$ and a non-zero injective morphism $\Lambda y \to \Lambda z$, and hence non-zero morphisms $\Lambda x \to \Lambda z$, meaning that $e_x \Lambda e_z \neq 0$. Thus, $x \leq z$ and the relation is transitive. Therefore, the relation is a partial order. Now, by Pierce decomposition, we have that $\Lambda = \bigoplus_{x \leq y} e_x \Lambda e_y$. Take a basis of Λ as follows. For $e_x \Lambda e_x$, denote that basis element by g_{xx} and for $e_x \Lambda e_y$, denote the basis element by g_{xy}; this is possible since these spaces have dimension 1. Then $\{g_{xy} | x \leq y\}$ is a k-basis for Λ. Since every nonzero morphism between projective indecomposable Λ-modules is injective, we have that $g_{xy} \cdot g_{tz} = \delta_{yt} \lambda^y_{xz} g_{xz}$ for some $\lambda^y_{xz} \in k$. Since we have that $e_x \Lambda e_y \cdot e_y \Lambda e_z \subseteq e_x \Lambda e_z$, the composition between the injective nonzero morphism $\Lambda x \to \Lambda y$ and the nonzero injective morphism $\Lambda y \to \Lambda z$ is, obviously, nonzero. It follows that λ^y_{xz} is a non-zero multiple of g_{xz}. Hence, Λ is a deformation of an incidence algebra.

(2) $\Leftrightarrow$ (3) It is clear from the above remark and the definition that (2) and (3) are equivalent. $\square$

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