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Localized Skein Algebras as Frobenius extensions

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LOCALIZED SKEIN ALGEBRAS AS FROBENIUS EXTENSIONS

by

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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To my grandfather, abuelo Marcial.
I've never met anyone as hardworking, caring and selfless as you. My hope is to someday become at least half the amazing human being you are.

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ABSTRACT

There is an algebra defined on a two dimensional manifold, known as the Skein algebra, which has as elements the simple closed curves of the manifold. Just like with numbers, there are ways to add, subtract and multiply elements. Unfortunately division is not allowed in the Skein algebra, which is why we introduced the notion of the localized Skein algebra, where we define a way to invert elements so that dividing is possible. These algebras have infinitely many elements, may not be commutative and in fact may have torsion, which makes them a hard object to study.

This work is mainly centered in reducing these algebras to something more manageable. We have shown that for any space, its localized Skein algebra is a Frobenius extension of its localized character ring, which means that any element of the algebra can be rewritten as a finite linear combination of a finite subset of basis elements, multiplied by elements that do commute. The importance of this result is that it solves this problem of noncommutativity, by rewriting anything that doesn't commute, as elements of a small set which can be controlled, along with elements that commute and behave nicely, making the Skein algebra far more manageable.

PUBLIC ABSTRACT

Take two of your friends, and if you do not have any friends, take two of your Facebook friends. Give each one of them a piece of string and have them tie as many knots as they want; after they are done, have them glue both ends of the string. The resulting object is what we, mathematicians, call a knot. Does there exists a computational algorithm that can compare whether your friends ended up with the same knot or not? The answer is: Yes, and these are called in mathematics knot invariants. It was previously thought that computing with these algorithms required exponential time. The importance of this research and its results is that shows that there is a rare subset of knots that allows you to compute in linear time instead. We proved this to be the case for any type of two dimensional space.

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CHAPTER 1
INTRODUCTION

1.1 Introduction

In this dissertation we show that the Kauffman bracket skein algebra of a compact surface with nonempty boundary can be localized to give a symmetric Frobenius algebra over the function field of the $SL_2\mathbb{C}$-character variety of the fundamental group of the surface.

A surface F is of finite type if there is a closed oriented surface $\hat{F}$ and a finite set of points $\{p_j\} \in \hat{F}$ so that $F = \hat{F} - \{p_i\}$. For example, a once punctured torus is of finite type, since it is equivalent to removing a point from a torus which is a closed oriented surface. Throughout this dissertation all surfaces are either compact oriented (possibly with boundary) or of finite type. The notation $\Sigma_{g,b}$ is used to denote the compact oriented surface of genus g with b boundary components.

The Kauffman bracket skein algebra is built from the collection of simple diagrams, along with an addition and multiplication operator. The case of the annulus which we cover in chapter 2, is the simplest and possibly most important example in understanding these algebras. Figure 1.1 shows an annulus $\Sigma_{0,2}$, and a blue curve wrapping around. Notice that any nontrivial, simple diagram in $\Sigma_{0,2}$ is just a bunch of copies of the blue curve, hence this blue curve acts as a generator for the elements of this algebra. In this same chapter we also discuss the pair of pants, $\Sigma_{0,3}$. Its corresponding Kauffman bracket skein algebra has three free generators. We explicitly construct the Frobenius extensions required to prove that these are in fact symmetric Frobenius algebra over the function field of the $SL_2\mathbb{C}$-character variety of the fundamental group of the surface.

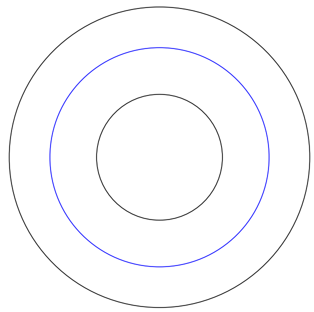


Figure 1.1: Annulus with generator.

Surprisingly, in chapter 3, we find that $K_N(\Sigma_{1,0})$ is not a free module over the universal character ring $\chi(\Sigma_{1,0})$, so although as a vector space over the function field of the character variety, it has dimension N^2, its rank as a module over the coordinate ring is greater than N^2. This is the first surface for which torsion appears and we provide you with an equation of what it looks like. Lucky for us, the localization reduces the rank to the desired N^2. In the process of proving that $S^{-1}K_N(\Sigma_{1,1})$, where S in this case refers to $\chi(\Sigma_{1,1}) - \{0\}$, is Frobenius we identify a basis for $S^{-1}K_N(\Sigma_{1,1})$ over $S^{-1}\chi(\Sigma_{1,1})$ so that multiplication by any simple diagram is easily computable.

In chapter 4, we prove the generalized version that for a surface of finite type F, $K(F)$, the Kauffman bracket skein algebra over $\mathbb{Z}[A, A^{-1}]$ is finitely generated as an algebra by a finite family of simple closed curves S_i. In fact,

$$\{S_{\sigma(1)}^{k_1} * S_{\sigma(2)}^{k_2} * \ldots * S_{\sigma(n)}^{k_n}\},$$

where $k_i \in \mathbb{Z}_{\geq 0}$, spans $K(F)$ for any permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$. This is an extension of a theorem of Bullock [12]. We use this result to prove that $K_N(F)$ is a finitely generated module over $\chi(F)$, the coordinate ring of the $SL_2\mathbb{C}$-character variety of $\pi_1(F)$. Localizing at $S = \chi(F) - \{0\}$, we get $S^{-1}K_N(F)$ is a finite dimensional algebra over $S^{-1}\chi(F)$, the function field of the character variety of $\pi_1(F)$.

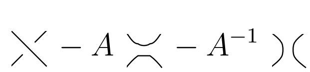
1.2 Preliminaries

1.2.1 Kauffman bracket skein module

Let M be an orientable 3-manifold. A **framed link** in M is an embedding of a disjoint union of annuli into M. Throughout this dissertation $M = F \times [0, 1]$ for an orientable surface F. Diagrammatically we depict framed links by showing the core of the annuli lying parallel to F. Two framed links in M are equivalent if they are isotopic. Let $\mathcal{L}$ denote the set of equivalence classes of framed links in M, including the empty link. By $\mathbb{Z}[A, A^{-1}]$ we mean Laurent polynomials with integral coefficients in the formal variable A. Consider the free module over $\mathbb{Z}[A, A^{-1}]$,

$$\mathbb{Z}[A, A^{-1}]\mathcal{L}$$

with basis $\mathcal{L}$. Let S be the submodule spanned by the Kauffman bracket skein relations,



and

$$\bigcirc \cup L + (A^2 + A^{-2})L.$$

The framed links in each expression are identical outside the balls pictured in the diagrams, and when both arcs in a diagram lie in the same component of the link, the same side of the annulus is up. The Kauffman bracket skein module $K(M)$ is the quotient

$$\mathbb{Z}[A, A^{-1}]\mathcal{L}/S(M).$$

A **skein** is an element of $K(M)$. Let F be a compact orientable surface and let $I = [0, 1]$. There is an algebra structure on $K(F \times I)$ that comes from laying one framed link over the other. The resulting algebra is denoted $K(F)$ to emphasize that it comes from the particular structure as a cylinder over F. Denote the stacking product with a $*$, so $\alpha * \beta$ means α stacked over β. If it is known the two skeins commute the $*$ will be omitted.

A **simple diagram** D on the surface F is a system of disjoint simple closed curves so that none of the curves bounds a disk. A simple diagram D is **primitive** if no two curves in the diagram cobound an annulus. A simple diagram can be made into a framed link by choosing a system of disjoint annuli in F so that each annulus has a single curve in the diagram as its core. This is sometimes called the **blackboard framing**. The set of isotopy classes of blackboard framed simple diagrams form a basis for $K(F)$, [4, 21, 28].

1.2.2 Specializing A

If R is a commutative ring , and $\zeta \in R$ is a unit, then R a $\mathbb{Z}[A, A^{-1}]$-module, where the action

$$\mathbb{Z}[A, A^{-1}] \otimes R \to R$$

is given by letting $p \in \mathbb{Z}[A, A^{-1}]$ act by multiplication by the result of evaluating p at ζ. The skein module specialized at $\zeta \in R$ is,

$$K_R(M) = K(M) \otimes_{\mathbb{Z}[A,A^{-1}]} R.$$

You can think of the specialization as setting A equal to ζ in the Kauffman bracket skein relations.

This is much too general a setting to get nice structure theorems for $K_R(M)$, so we restrict our attention to when the ring R is an integral domain. To emphasize that we are working with an integral domain we denote the ring $\mathfrak{D}$. Since $\mathbb{Z}[A, A^{-1}]$ is an integral domain and A is a unit we can recover $K(M)$ by specialization. For that reason the theorems in this dissertation are all stated in terms of $K_{\mathfrak{D}}(M)$, the skein module specialized at a unit ζ in an integral domain $\mathfrak{D}$.

We are most interested in the case when ζ is a primitive $2N$th root of unity, where $N \in \mathbb{Z}_{\geq 0}$ is odd. The integral domain is $\mathbb{Z}[\frac{1}{2}, \zeta]$ which we think of as embedded in $\mathbb{C}$, so that $\zeta = e^{k\pi \mathbf{i}/N}$, where k is an odd counting number that is relatively prime to N. In this case, the specialized module is denoted $K_N(M)$. This is a little ambiguous because there are in general several primitive $2N$th roots of unity, but the theorems in this dissertation do not depend on the choice of which one. We need 2 to be a unit so that a collection of skeins that are adapted to the computation of the trace will be a basis.

1.2.3 Threading

The Tchebychev Polynomials of the first type T_k are defined recursively by

- $T_0(x) = 2$,
- $T_1(x) = x$ and
- $T_{n+1}(x) = T_1(x) \cdot T_n(x) - T_{n-1}(x)$.

They satisfy some nice properties.

Proposition 1.1. *For $m, n > 0$, $T_m(T_n(x)) = T_{mn}(x)$. Furthermore, for all $m, n \geq 0$, $T_m(x)T_n(x) = T_{m+n}(x) + T_{|m-n|}(x)$.*

Proof. The first formula follows from the observation that $T_m(T_n)$ is a polynomial, and

$$T_m(T_n(q + q^{-1})) = T_n(q^n + q^{-n}) = q^{mn} + q^{-mn}.$$

The second comes from seeing

$$(q^m + q^{-m})(q^n + q^{-n}) = (q^{m+n} + q^{-m-n}) + (q^{|m-n|} + q^{-|m-n|}),$$

and using the characterization of the Chebyshev polynomials of degree k among polynomial functions. □

Let $\Sigma_{0,2} = S^1 \times [0, 1]$. It is easy to see that $K_{\mathfrak{D}}(\Sigma_{0,2})$ is isomorphic to $\mathfrak{D}[x]$ where x is the framed link coming from the blackboard framing of the core of the annulus. Hence $1, x, x^2, \dots x^n, \dots$ is a basis for $K_{\mathfrak{D}}(\Sigma_{0,2})$. Since $T_0(x) = 2$, in order to used the $T_k(x)$ as a basis for $\mathfrak{D}$, 2 must be a unit in $\mathfrak{D}$. If $2 \in \mathfrak{D}$ is a unit then $\{T_k(x)\}$, $k \in \mathbb{Z}_{\geq 0}$ is a basis for $K_{\mathfrak{D}}(\Sigma_{0,2})$.

If the components of the primitive diagram on a finite type surface F are the simple closed curves S_i and $k_i \in \mathbb{Z}_{\geq 0}$ has been chosen for each component, the result of threading the diagram with the T_{k_i} is $\prod_i T_{k_i}(S_i)$. Since the S_i are disjoint from one another they commute so order doesn't matter in the product. For any compact or finite type surface F, the primitive diagrams on F up to isotopy, with their components threaded with $\{T_{k_i}\}$ form a basis for $K_{\mathfrak{D}}(F)$ so long as $2 \in \mathfrak{D}$ is a unit. This basis is becoming more commonly used in the study of skein algebras [18, 26, 23].

The following theorem of Bonahon and Wong is the starting point for this investigation.

Theorem 1.2. *(Bonahon-Wong [17, 22]). If M is a compact oriented three-manifold and we specialize at ζ a $2N$th root of unity with $N \geq 3$ odd, there is a $\mathbb{Z}[\frac{1}{2}, \zeta]$-linear map*

$$\tau : K_1(M) \to K_N(M)$$

given by threading framed links with T_N. Any framed link in the image of τ is central in the sense that if $L' \cup K$ differs from $L \cup K$ by a crossing change of L and L' with K, then $T_N(L) \cup K = T_N(L') \cup K$. In the case that $M = F \times [0,1]$, the map

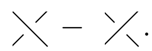
$$\tau : K_1(F) \to K_N(F)$$

is an injective homomorphism of algebras so that the image of τ lies in the center of

$K_N(F)$.

The skein module $K_1(M)$ is a ring under disjoint union. At $A = -1$, the Kauffman bracket skein relation

can be rotated 90 degrees and then subtracted from itself to yield,



This means that in $K_1(M)$ changing crossings does not change the skein. To take the product of two equivalence classes of framed links, choose representatives that are disjoint from one another and take their union. The product is independent of the representatives chosen, since the results differ by isotopy and changing crossings. The product can be extended distributively to give a product on $K_1(M)$. Let $\sqrt{0}$ denote the nilradical of $K_1(M)$. It is a theorem of Bullock, [12], proved independently in [9], that for any oriented compact 3-manifold $K_1(M)/\sqrt{0}$ is canonically isomorphic to the coordinate ring of the $SL_2\mathbb{C}$-character variety of the fundamental group of M. In the case that $M = F \times [0,1]$ the disjoint union product coincides with the stacking product, as stacking is one way to perturb the components of the two links so that they are disjoint. It is a theorem of Przytycki and Sikora [9] that $\sqrt{0} = \{0\}$ when the underlying three-manifold is $F \times [0,1]$. Therefore, $K_1(F)$ is the coordinate ring of the $SL_2\mathbb{C}$-character variety of the fundamental group of F. To alleviate notational ambiguity, and to emphasize the relationship with character varieties, the image of the threading map is denoted by $\chi(M)$.

For any oriented finite type surface F, $\chi(F)$ has as basis the isotopy classes of primitive diagrams threaded with $T_{k_i N}$ for all choices $k_i \in \mathbb{Z}_{\geq 0}$.

1.2.4 Specializing at a place

A place of $\chi(F)$ is a homomorphism $\phi : \chi(F) \to \mathbb{C}$. The places correspond to evaluation at a point on the character variety. A place defines a module structure $\chi(F) \otimes \mathbb{C} \to \mathbb{C}$ by letting $s \in \chi(F)$ act as multiplication by $\phi(s)$ on $\mathbb{C}$. We define the **specialization** of $K_N(F)$ at ϕ to be,

$$K_N(F)_\phi = K_N(F) \otimes_{\chi(F)} \mathbb{C}.$$

The specialization at a place is an algebra over the complex numbers.

1.2.5 Localization

Let $R \to J$ be a central ring extension, where is R is an integral domain, J is an associative ring with unit and the inclusion of R into J is a ring homomorphism. Since R has no zero divisors, $S = R - \{0\}$ is multiplicatively closed. Start with the set of ordered pairs $J \times S$, and place an equivalence relation on $J \times S$ by saying (a, s) is equivalent to (b, t) if $at = bs$. Denote the equivalence class of (a, s) under this relation by $[a, s]$. The set of equivalence classes is denoted $S^{-1}J$, and called the localization of J with respect to S. Denote the set of equivalence classes $[a, s]$ where $a \in R$ by $S^{-1}R$. Define multiplication of equivalence classes by $[a, s][b, t] = [ab, st]$ and addition by $[a, s] + [b, t] = [at + bs, st]$. Under these operations $S^{-1}R$ is a field, and $S^{-1}J$ is an algebra over that field.

In this dissertation, J is a subalgebra of $K_N(F)$ and R is $\chi(F)$. This means that $S^{-1}R$ is the function field of the character variety of $\pi_1(F)$.

1.2.6 Trace and extension of scalars

If $L \in End_F(V)$, we use $tr(L)$ to denote the unormalized trace of L. The linear map L can be represented with respect to a basis $\{v_j\}$, by a matrix (l_i^j). The trace of L is given by

$$tr(L) = \sum_i l_i^i.$$

If W is also a finite dimensional vector space over F and $M : W \to W$ is an F-linear map, then

$$tr(L \otimes_F M) = tr(L)tr(M) \text{ and } tr(L \oplus M) = tr(L) + tr(M).$$

Suppose that $F \leq K$ is a field extension and V is a vector space of dimension n over F, then

$$V \otimes_F K$$

is a vector space of dimension n over K. In fact if $\{v_j\}$ is a basis for V then $\{v_j \otimes 1\}$ is a basis for $V \otimes_F K$ over K.

Under extension of scalars, $L : V \to V$ gets sent to $L \otimes_F 1_K$. The matrix of $L \otimes_F 1_K$ with respect to the basis $\{v_j \otimes 1\}$ is the same as the matrix of L with respect to $\{v_j\}$, so

$$tr(L \otimes_F 1_K) = tr(L),$$

where the trace on the left is taken as a K-linear map, and the trace on the right is taken as an F-linear map, and we are using $F \leq K$ to make the identification.

1.2.7 Geometric intersection numbers

Suppose that X and Z are properly embedded 1-manifolds in the finite type surface F where X is compact. We say that X' is a transverse representative of X, if X' is ambiently isotopic to X via a compactly supported isotopy, and $X' \pitchfork Z$. Define the **geometric intersection number** of X and Z, denoted $i(X, Z)$ to be the minimum cardinality of $X' \cap Z$ over all transverse representatives of X. We could have instead worked with Z up to compactly supported ambient isotopy and taken the minimum over all Z' isotopic to Z and transverse to X and gotten the same number, so $i(X, Z) = i(Z, X)$.

It is a theorem that a transverse representative of X realizes the geometric intersection number $i(X, Z)$ if and only if there are no **bigons**. A bigon is a disk D embedded in F so that the boundary of D consists of the union of two arcs $a \subset X$ and $b \subset Z$, [19]. If there is a bigon, there is always an innermost bigon, whose interior is disjoint from $X \cup Z$.

CHAPTER 2
MOTIVATING EXAMPLES

2.1 The Annulus

The skein algebra of $K_N(\Sigma_{0,2})$ is naturally isomorphic to polynomials in one variable $\mathbb{C}[x]$. Under this isomorphism the variable x is the image of the core of the annulus with the blackboard framing.

Since the polynomials $T_k(x)$ have x^k as their leading terms, we can change bases, so that $K_N(\Sigma_{0,2})$ has basis $T_k(x)$ over the complex numbers, and multiplication obeys the product to sum formula $T_k(x)T_m(x) = T_{k+m}(x) + T_{|k-m|}(x)$.

Proposition 2.1. *The image of the threading map, $\chi(\Sigma_{0,2})$, is the span of all $T_{Na}(x)$ where a ranges over all non-negative integers.*

Proof. This follows from Proposition 1.1, since $\tau(T_k(x)) = T_{Nk}(x)$. □

Proposition 2.2. *$K_N(\Sigma_{0,2})$ is a free module of rank N over $\chi(\Sigma_{0,2})$ with basis $T_k(x)$ where k ranges from 0 to $N-1$.*

Proof. Suppose that a is a non-negative integer and $0 \leq b \leq N-1$. Solving the product to sum formula,

$$T_{aN+b}(x) = T_{aN}(x)T_b(x) - T_{|aN-b|}(x).$$

If $a > 0$ then $aN > 0$ and supposing that $b > 0$, $aN - b < aN + b$. This means, by induction on the largest k so that $T_k(x)$ appears with nonzero coefficient in the skein, every element of $K_N(\Sigma_{0,2})$ can be written as a linear combination of $T_b(X)$ where the b range from 0 to $N-1$. Therefore the proposed basis spans.

Suppose that

$$\sum_{k=0}^{N-1} \chi_k T_k(x) = 0,$$

where the $\chi_k \in \chi(\Sigma_{0,2})$. Rewrite each term using the basis $\{x^k\}$ for $K_N(\Sigma_{0,2})$. The leading term of each summand as polynomials in x must cancel. However the leading term of $\chi_k T_k(x)$ is of the form $\alpha_k x^{aN+k}$, where α_k is a complex number. The highest degree terms $\chi_k T_k(x)$ and $\chi_m T_m(x)$ where $m \neq k$ cannot cancel with each other. Therefore all the leading terms of all $\chi_k T_k(x)$ are all zero. But this means all $\chi_k = 0$. Hence the $T_k(x)$ where k ranges from 0 to $N-1$ are linearly independent over $\chi(\Sigma_{0,2})$. $\square$

If $\alpha \in K_N(\Sigma_{0,2})$ left multiplication by α defines a $\chi(\Sigma_{0,2})$-linear map,

$$L_\alpha : K_N(\Sigma_{0,2}) \to K_N(\Sigma_{0,2}).$$

We can write L_α as an $N \times N$-matrix with coefficients in $\chi(\Sigma_{0,2})$. For instance if $N = 5$, and $\alpha = T_1(x)$ then, the matrix is,

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \frac{1}{2}T_5(x) \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Remark. Notice the determinant of this matrix is $T_5(x)$. In general the determinant of the matrix of $T_k(x)$ acting on $K_N(\Sigma_{0,2})$ as a free module over $\chi(\Sigma_{0,2})$ is $T_{kN}(x)$.

More generally, using Proposition 1.1, the matrix Lk for left multiplication by

$T_k(x)$, where k ranges from 1 to $N-1$ and the indices for the matrix run from 0 to $N-1$, is given by

$$Lk_{i,j} = \begin{cases} 2\delta_i^k & j = 0 \\ \delta_i^{|k-j|} + \delta_i^{k+j} & (k+j \le N-1) \wedge (j \ne 0) \\ \delta_i^{|k-j|} - \delta_i^{2N-k-j} + T_N \delta_i^{k+j-N} & (k+j > N-1) \wedge (j \ne 0) \wedge (k+j-N \ne 0) \\ \delta_i^{|k-j|} - \delta_i^{2N-k-j} + \frac{1}{2} T_N \delta_i^0 & (k+j > N-1) \wedge (j \ne 0) \wedge (k+j-N = 0) \end{cases} \tag{2.1.1}$$

Also $T_k(x)T_0(x) = 2T_k(x)$ from the definition of T_0.

Proposition 2.3. *If $Tr : K_N(F) \to \chi(F)$ is $\frac{1}{N}$ times the trace of an element acting by left translation we find that $Tr(T_k) = 0$ unless $N|k$. Furthermore if $N|k$ then $Tr(T_k) = T_k$.*

Proof. If $1 \le k \le N-1$, There are two ways a diagonal entry can be nonzero, if $|k-j| = j$ and $k+j < N$ or if $2N-k-j = j$ and $k+j > N-1$. In fact, the $j \in \{0, 1, \ldots, N-1\}$ satisfying the first condition can be placed in one-to-one correspondence with the $j \in \{0, \ldots, N-1\}$ that satisfy the second condition. In the first case, since $k > 0$, $k-j = j$, but this implies that $k+N-j > N-1$ and $2N-k-(N-j) = N-j$, so $N-j$ satisfies the second condition. Since j and $N-j$ have the different parity they are not equal. From the formula for $Lk_{i,j}$, j of the first kind gives rise to a $+1$ on the diagonal, and j of the second kind gives rise to a -1 on the diagonal. Hence all diagonal entries are $1, 0$ or -1 and they sum to zero.

If $N|k$ then $T_k \in \chi(\Sigma_{0,2})$ and Lk is a diagonal matrix with each diagonal entry equal to T_k, meaning the normalized trace of Lk is T_k. □

Remark. This gives a simple rule for computing the trace. Write a skein in the form $\sum_i \alpha_i T_i(x)$ where the $\alpha_i \in S^{-1}\chi(\Sigma_{0,2})$, then $Tr(\alpha) = \sum_{N|i} \alpha_i T_i(x)$. That is, throw out all the terms where i is not divisible by N and the sum of the remaining terms is the trace.

We define a pairing $\sigma : K_N(F) \otimes K_N(F) \to \chi(F)$ by $\sigma(\alpha, \beta) = Tr(\alpha\beta)$.

Unfortunately we cannot diagonalize the pairing unless we work over the field of fractions. When $N = 5$ the matrix of the pairing with respect to the basis T_i where i ranges from 0 to 4 is,

$$\begin{pmatrix} 2T_0(x) & 0 & 0 & 0 & 0 \\ 0 & T_0(x) & 0 & 0 & T_5(x) \\ 0 & 0 & T_0(x) & T_5(x) & 0 \\ 0 & 0 & T_5(x) & T_0(x) & 0 \\ 0 & T_5(x) & 0 & 0 & T_0(x) \end{pmatrix}. \tag{2.1.2}$$

This example is general in the sense that the matrix of the pairing consists of a 1×1 block with a 2 and an $N-1 \times N-1$ block that has 1's on the diagonal, $T_N(x)$ on the antidiagonal, and zeroes everywhere else.

Proposition 2.4. *The pairing is nondegenerate. For every $\alpha \neq 0$ there exists $\beta \in K_N(\Sigma_{0,2})$ so that $Tr(\alpha\beta) \neq 0$.*

Proof. At level N the pairing has $\beta(T_0(x)T_0(x)) = 2T_0(x)$, and $\beta(T_k(x)T_l(x)) = T_0(x)$ if $k = l$ and $T_N(x)$ if $k + l = N$.

The corresponding matrix has determinant

$$2T_0(x)(T_0^2(x) - T_N(x)^2)^{\frac{N-1}{2}}.$$

To see this, decompose the matrix of the pairing into blocks. One block has determinant $2T_0(x)$. The other block is easy to make upper triangular with row operations, so that the determinant is evident. □

Theorem 2.5. *$S^{-1}K_N(\Sigma_{0,2})$ is a Frobenius algebra over the function field of the character variety of $\pi_1(\Sigma_{0,2})$.*

□

Proposition 2.6. *$S^{-1}K_N(\Sigma_{0,2})$ is a field.*

Proof. By definition $S^{-1}\chi(\Sigma_{0,2})$ is a field. We obtain $S^{-1}K_N(\Sigma_{0,2})$ as an finite extension of $K_1(\Sigma_{0,2})$ where we are adjoining x that satisfies,

$$x^N = T_N(x) - \sum_{i=1}^{N/2}(-1)^i \frac{N}{N-i}\binom{N-i}{i}x^{N-2i}.$$

Therefore $S^{-1}K_N(\Sigma_{0,2})$ is a finite extension of the field $S^{-1}\chi(\Sigma_{0,2})$, so it is itself a field. □

This gives a second proof that $K_N(\Sigma_{0,2})$ is Frobenius as it has no nontrivial ideals contained in $ker(Tr)$. The computational proof gives more information as knowing the determinant of the pairing tells you the locus of points where the specialized skein algebra is not Frobenius.

Remark. More generally, if F is a compact oriented surface and $S = \chi(F) - \{0\}$, then in $S^{-1}K_N(F)$ any simple diagram is a unit.

Next we explore specializing $K_N(\Sigma_{0,2})$ at a place. The character variety of the annulus can be understood as the complex plane. The places $\phi : \chi(\Sigma_{0,2}) \to \mathbb{C}$ are determined by where $T_N(x)$ is sent. Assume $\phi(T_N(x)) = z$. Denote $K_N(\Sigma_{0,2})/ker(\phi)$ by $K_N(\Sigma_{0,2})_z$. The action of T_1 by left multiplication comes from substituting z for $T_N(x)$ in Equation 2.1.1. In the case $N = 5$ this is,

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \frac{1}{2}z \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The determinant of the matrix for left multiplication by T_1 specialized at $T_N(x) = z$ is z. You can see this by expanding the determinant by the last column. It is easy to see that only one term contributes to the final answer and it's contribution is z. Left multiplication by $T_1(x)$ is an invertible map unless $z = 0$. If $z \neq 0$, then we can find $q \in \mathbb{C}$ so that $q^N + q^{-N} = z$. The roots of $T_N(x) = z$ are exactly the numbers $\zeta q + \zeta^{-1}q^{-1}$ where ζ is a Nth root of unity.

Theorem 2.7. *$K_N(\Sigma_{0,2})_z$ is a Frobenius algebra over $\mathbb{C}$ so long as $z \neq \pm 2$.*

.

Proof. The pairing $\beta : K_N(\Sigma_{0,2})_z \otimes K_N(\Sigma_{0,2})_z \to \mathbb{C}$ comes from the $N \times N$ matrix analogous to the one displayed in Equation 2.1.2 by specializing. It's determinant is $2(T_0(x)^2 - z^2)^{\frac{N-1}{2}}$. Recalling that $T_0(x) = 2$, the pairing is nondegenerate so long as $z \neq \pm 2$. $\square$

2.2 The Pair Of Pants

The skein algebra of the pair of pants is isomorphic to $\mathbb{C}[x, y, z]$, the polynomials with complex coefficients in three variables. The three variables correspond to blackboard framed curves that are parallel to each of the boundary components. Since $\mathbb{C}[x,y,z]$ is the tensor product of three copies of the polynomials in a single variable our analysis of $K_N(\Sigma_{0,2})$ can be used to analyze this case.

In specific, the skeins $T_a(x)T_b(y)T_c(z)$ where $a,b,c \geq 0$ form a basis for $K_N(\Sigma_{0,3})$. The image of the threading map $\chi(\Sigma_{0,3})$ is $\{T_{Na}(x)T_{Nb}(y)T_{Nc}(z)\}$, where the a,b and c range over all non-negative integers. The skeins $T_j(x)T_k(y)T_l(z)$, where j,k,l range over $0\ldots N-1$ form a basis for $K_N(\Sigma_{0,3})$ as a module over $\chi(\Sigma_{0,3})$. With respect to this basis the matrix of left multiplication by $T_j(x)T_k(y)T_l(z)$ is the tensor product of the three matrices coming from their actions on $K_N(\Sigma_{0,3})$. Define $Tr : K_N(\Sigma_{0,3}) \to \chi(\Sigma_{0,3})$ as $\frac{1}{N^3}$ times the trace of this matrix.

Proposition 2.8. *The map $Tr : K_N(\Sigma_{0,3}) \to \chi(\Sigma_{0,3})$ is the identity when restricted to $\chi(\Sigma_{0,3})$ and if one of j,k,l is not divisible by N it sends $T_j(x)T_k(y)T_l(z)$ to zero.*

Proof. The trace is just the tensor product of three copies of the trace on $K_N(\Sigma_{0,2})$.

□

The evaluation of the trace is essentially the same as for the annulus. Write a skein in terms of the basis $T_i(x)T_j(y)T_k(z)$. If one of the i, j or k is not divisible by N throw it out, and the trace is the sum of the remaining terms.

We define the pairing $\sigma : K_N(\Sigma_{0,3}) \otimes K_N(\Sigma_{0,3}) \to \chi(\Sigma_{0,3})$ to be $\sigma(\alpha \otimes \beta) = Tr(\alpha\beta)$.

Proposition 2.9. *As an algebra over the field of fractions of $\chi(\Sigma_{0,3})$, $S^{-1}K_N(\Sigma_{0,3})$ is Frobenius.*

Proof. The tensor product of Frobenius algebras is a Frobenius algebra. □

Proposition 2.10. *$S^{-1}K_N(\Sigma_{0,3})$ is a field.*

Proof. It is a finite extension of the field $S^{-1}\chi(\Sigma_{0,3})$. □

Proposition 2.11. *The character variety of the free group on two generators is $\mathbb{C}^3$. If $\phi : \mathbb{C}[x, y, z] \to \mathbb{C}$ doesn't correspond to evaluating the variables x y or z to ± 2 the specialization of σ is nondegernate and $K_A(\Sigma_{0,3})_\phi$ is a Frobenius algebra.*

CHAPTER 3
THE LOCALIZED SKEIN ALGEBRA OF THE TORUS AND THE ONCE PUNCTURED TORUS

3.1 The Torus

Simple closed curves on $\Sigma_{1,0}$ correspond to $(\pm p, \pm q) \in \mathbb{Z} \times \mathbb{Z}$ where p and q are relatively prime. The correspondence comes from expressing an oriented representative of the curve as an element of the first homology of the torus with respect to a standard basis. Since the curves are not oriented, $(p, q) = (-p, -q)$.

The simple diagrams form a basis for $K_N(\Sigma_{1,0})$. Since two disjoint nontrivial simple closed curves on a torus are parallel, we can identify the simple diagrams with pairs $(\pm p, \pm q) \in \mathbb{Z} \times \mathbb{Z}$. The ordered pair $(0, 0)$ corresponds to the empty skein. The number of components of the simple diagram (p, q) is the greatest common divisor of (p, q) and the diagram consists of d copies of the curve $(p/d/q/d)$.

Once again, the basis of simple diagrams can be replaced by the basis of simple closed curves threaded with Chebyshev polynomials of the first type. For $(p, q) \neq (0, 0)$ with $gcd(p, q) = 1$, we denote (p, q), the (p, q)-curve on the torus. For p, q with $gcd(p, q) = d$, we define

$$(p, q)_T = T_d\big(\big(\frac{p}{d}, \frac{q}{d}\big)\big).$$

Finally, $(0, 0)_T$ is twice the empty skein. Since these skeins have exactly the simple diagrams as their "lead" term they form a basis for $K_N(\Sigma_{1,0})$. Multiplication in $K_N(\Sigma_{1,0})$ has a conveniently simple formula with respect to this basis.

Theorem 3.1. *(Product to Sum Formula in $K_N(\Sigma_{1,0})$ [6]). For any $p, q, r, s \in \mathbb{Z}$, one has*

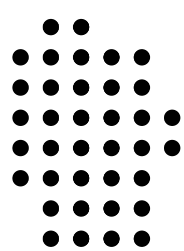
$$(p,q)_T * (r,s)_T = A^{\begin{vmatrix} p & q \\ r & s \end{vmatrix}}(p+r, q+s)_T + A^{-\begin{vmatrix} p & q \\ r & s \end{vmatrix}}(p-r, q-s)_T.$$

Proposition 3.2. *The image of the threading map $\chi(\Sigma_{1,0})$ is exactly the span of $(Np, Nq)_T$ where p, q vary over $\mathbb{Z}$. It is exactly the center of $K_N(\Sigma_{1,0})$.*

Proof. The characterization of the image of the threading map is the same for the annulus. Using the product to sum formula you can show that if some $\alpha_{p,q} \neq 0$ and p and q are not both divisible by N then $\sum_{p,q} \alpha_{p,q}(p,q) \in K_N(\Sigma_{1,0})$ fails to commute with some (r,s). □

Theorem 3.3. *$K_N(\Sigma_{1,0})$ is finitely generated over $\chi(\Sigma_{1,0})$. A spanning set $\mathcal{B}$ consists of $\{(a,b)_T\}$ so that $(a,b) \in (\{0\} \times \{0, 1, ..., N-1\}) \cup (\{1, ..., N-1\} \times \{-\frac{N-1}{2}, .., 0, ..., N-1\}) \cup (\{N\} \times \{1, ..., \frac{N-1}{2}\}) \cup (\{1, ..., \frac{N-1}{2}\} \times \{N\})$.*

The set $\mathcal{B}$ is a little complicated. Below is a diagram that pictorially represents the set when $N = 5$. The dots correpond to points in $\mathbb{Z} \times \mathbb{Z}$ that belong to $\mathcal{B}$. The lowest point in the leftmost column is $(0,0)$.



Proof. Theorem 3.3 The proof is in two steps. Let $\mathcal{A} = \{(p,q)_T\}$ where

$$(p,q) \in \{0\} \times \{0\} \cup \{0, N\} \times \{1, ...N-1\} \cup \{1, ..., N-1\} \times \{-N, ..., 0, ..., N\}.$$

First we show that $\mathcal{A}$ spans $K_N(\Sigma_{1,0})$. Next we show that $\mathcal{A}$ lies in the span of $\mathcal{B}$.

Step 1. We want to show that as a module over $\chi(\Sigma_{1,0})$, $K_N(\Sigma_{1,0})$ is spanned by $(a,b)_T \in \mathcal{A}$. That is to say, any $(p,q)_T$ can be written as a $\chi(\Sigma_{1,0})$ linear combination of elements of $\mathcal{A}$. Let $(p,q)_T \in K_N(\Sigma_{1,0})$. For simplicity we work with pairs (p,q) with $p \geq 0$, which we can do because $(p,q) = (-p,-q)$ as a simple diagram. Suppose $p \neq Nk$ for $k \in \mathbb{N}$ and write p as $p = Np_1 + a_1$ for some $0 < a_1 \leq N-1$. Assume that $p_1 > 0$. By the product to sum formula

$$(p,q)_T = A^{-Np_1q}(Np_1,0)_T * (a_1,q)_T - A^{-2Np_1q}(Np_1 - a_1, -q).$$

Notice that $|Np_1 - a_1| < |p|$, hence recursively applying this identity yields a linear combination of $(r,s)_T$ with coefficients in $\chi(\Sigma_{1,0})$ so that the r lie in $\{0, 1, \ldots N-1\}$. To simplify the case $(p,q)_T$ where $p = Nk$ for some $k \geq 2$, use the product to sum formula with

$$(p,q)_T = (-1)^{-(k-1)q}(N(k-1),0)_T * (N,q)_T - (-1)^{-2(k-1)q}(N(k-2),-q)_T$$

to reduce to a linear combination of $(r,s)_T$ where $r \in \{0,N\}$. Thus $K_N(\Sigma_{1,0})$ is spanned over $\chi(\Sigma_{1,0})$ by $(r,s)_T$ with $r \in \{0,1,\ldots,N\}$.

Reduce the second entries in the same way, however now we must deal with negative integers. Suppose $q \neq Nm$ for $m \in \mathbb{Z} - \{0\}$ is non-negative and write q modulo N, $q = Nq_1 + b_1$ for some $0 < b_1 \leq N-1$. If $q < 0$ induct on $|q|$. As in the case of the first entry this process terminates in finitely many steps, leaving a finite linear combination of $(r,s)_T$ where $0 \leq r \leq N$, and $-N \leq s \leq N$. In the case where $p = N$ reduce q, to get a linear combination of $(r,s)_T$ so that $r = \{0,N\}$ and

$1 \leq s \leq N-1$.

At this point it is clear that,

$$\mathcal{A} = \{0\} \times \{0\} \cup \{0, N\} \times \{1, ...N-1\} \cup \{1, ..., N-1\} \times \{-N, ..., 0, ..., N\}$$

spans $K_N(\Sigma_{1,0})$.

Step 2. We finish the proof by showing $\mathcal{A}$ lies in the span of $\mathcal{B}$. Notice $\mathcal{B}$ doesn't contain $(k, -N)$ for $1 \leq k \leq N-1$ but $\mathcal{A}$ does. By the product to sum formula these can be rewritten as elements in $\mathcal{B}$ in the following way

$$(k, -N)_T = A^{kN}(0, N)_T * (k, 0)_T - A^{-2kN}(k, N).$$

This same argument gets rid of $(0, -k)$ and $(N, -k)$. To rewrite $(a, -b)_T \in \{1, ...N-1\} \times \{\frac{N-1}{2}, ..., N-1\}$ we use the following relation

$$\begin{aligned}(a, -b)_T = &\frac{1}{2}((-1)^{a+b+1}(N, N)_T * (N-a, b-N)_T) \\ &+ \frac{1}{2}((-1)^a(0, N)_T * (a, N-b)_T + (-1)^b(N, 0)_T * (N-a, b)_T).\end{aligned}$$

We can also rewrite $(a, N)_T$ for $a \in \{\frac{N+1}{2}, ..., N-1\}$ as

$$\begin{aligned}(a, N)_T = &\frac{1}{2}((-1)^{a+1}(N, N)_T * (N-a, 0)_T) \\ &+ \frac{1}{2}((-1)^a(0, N)_T * (a, 0)_T + (N, 0)_T * (N-a, N)_T).\end{aligned}$$

And the last case, $(N, b)_T$ for $b \in \{\frac{N+1}{2}, ..., N-1\}$ can be rewritten as

$$\begin{aligned}(N, b)_T = &\frac{1}{2}((-1)^{b+1}(N, N)_T * (0, N-b)_T) \\ &+ \frac{1}{2}((0, N)_T * (N, N-b)_T + (-1)^b(N, 0)_T * (0, b)_T).\end{aligned}$$

This proves that any element of $\mathcal{A}$ not in $\mathcal{B}$ can be obtained by a linear combination of elements of $\mathcal{B}$, hence $\mathcal{B}$ is the spanning set of $\mathcal{A}$ and thus of $K_N(\Sigma_{1,0})$.

□

Lemma 3.4. *Let $\mathcal{B}' = \{(a,b)_T : 0 \leq a \leq N-1, 0 \leq b \leq N-1\}$. The elements in $\mathcal{B}'$ are linearly independent.*

Proof. Suppose that $\sum_{(a,b)_T \in \mathcal{B}'} \chi_{a,b}(a,b)_T = 0$, where $\chi_{a,b} \in \chi(\Sigma_{1,0})$. Using the characterization of $\chi(\Sigma_{1,0})$ as the span of $\{(Np, Nq)_T\}$ where $p, q \in \mathbb{Z}$,

$$\sum_{a,b}\left(\sum_i \alpha_{a,b,i}(Np_i, Nq_i)_T\right)(a,b)_T = 0$$

where $a, b \in \{0, 1, ..., N-1\}$, only finitely many $\alpha_{a,b,i} \in \mathbb{C}$ are nonzero, and $p_i \geq 0$ for all i and if $p_i = 0$ then $q_i \geq 0$. Using the product to sum formula,

$$\sum_{a,b}\left(\sum_i (-1)^{p_i b + q_i a}\alpha_{a,b,i}((Np_i + a, Nq_i + b)_T + (Np_i - a, Nq_i - b)_T)\right) = 0.$$

Let

$$\mathcal{I} = \{(p,q)_T | (p,q)_T = (Np_i \pm a, Nq_i \pm b) \text{ and } \alpha_{a,b,i} \neq 0\}.$$

Assume $\mathcal{I} \neq \emptyset$. Choose $(p,q)_T \in \mathcal{I}$ so that p is maximal, and among all pairs with p maximal, q is maximal. Notice $p \geq 0$.

Case 1. Suppose $p = Np_i + a$ for some $(Np_i + a, Nq_i + b)_T \in \mathcal{I}$ and $a > 0$. Note that $p \neq Np_j - a'$ for some $(Np_j - a', Nq_j - b')_T$ because p was maximal and $Np_j + a' \geq Np_j - a'$, hence $a' = a = 0$. This means that the only elements of $\mathcal{I}$ with first coordinate p are of the form $(Np_j + a', Nq_j + b')_T$. Among these the maximal second coordinate is of the form $Nq_i + b'$. There is a homomorphism $r : \mathbb{Z} \times \mathbb{Z} \to$

$\mathbb{Z}_N \times \mathbb{Z}_N$ obtained by taking residue classes modulo N. If $r(Np_i + a, Nq_i + b) = r(Np_j + a', Nq_j + b')$ where $a, b, a', b' \in \{0, \ldots, N-1\}$ then $a = a'$ and $b = b'$. From this we see that there is a unique $(p_i N + a, q_i + b)_T$ corresponding to the choice of $(p, q)_T$. Therefore $\alpha_{a,b,i} = 0$.

Case 2. Suppose that $p = Np_i$, let $q = Nq_i + b$ be the maximal second coordinate among the pairs $(Np_i, q)_T \in \mathcal{I}$. If $b = 0$ we are done, there is no other way of producing another $(Np_i, Nq_i)_T$ in the sum above, so assume $b \neq 0$. Moreover suppose there is another $\alpha_{a',b',j} \neq 0$ so that $(p, q)_T$ was equal to $(Np_j + a', Nq_j + b')_T$ or $(Np_j - a', Nq_j - b')_T$. Notice that if $(p, q)_T = (Np_j + a', Nq_j + b')_T$ then $a' = 0$ and $(0, b)_T = (0, b')_T$, contradicting our assumption that there was more than one. Hence $(p, q)_T = (Np_j - a', Nq_j - b')_T$. Once again $a' = 0$. Since $Nq_j + b' > Nq_j - b'$ and $\alpha_{a',b',j} \neq 0$ we have contradicted the maximality of q. □

Hence the elements in $\mathcal{B}'$ are linearly independent.

□

The following equation relates a multiple of every element in $\mathcal{B} - \mathcal{B}'$ to two elements in $\mathcal{B}'$.

$$\begin{aligned} 0 = {} & (-1)^p[(2N, N)_T - (0, N)_T] * (p, q)_T \\ & + (-1)^q[-(N, 2N)_T + (N, 0)_T] * (N - p, N - q)_T \\ & + [-(2N, 2N)_T + 2] * (p, q - N)_T. \end{aligned} \tag{3.1.1}$$

For example if $(a, b)_T \in \mathcal{B} - \mathcal{B}'$ with $b < 0$, set $a = p$ and $b = q - N$ in the previous formula, if $(a, N)_T \in \mathcal{B} - \mathcal{B}'$, set $a = p$ and $N = q$ and last if $(N, b)_T \in \mathcal{B} - \mathcal{B}'$, set $N = N - p$ and $b = N - q$.

Let $S = \chi(\Sigma_{1,0}) - \{0\}$. Based on the construction presented earlier, the field of fractions $S^{-1}\chi(\Sigma_{1,0})$, is defined by $\chi(\Sigma_{1,0}) \times S$ with the following equivalence relation, $(z, s) \sim (z', s')$ if $zs' = z's$, where z and $z' \in \chi(\Sigma_{1,0})$ and s and $s' \in S$. Similarly we construct $S^{-1}K_N(\Sigma_{1,0})$.

Equation 5.1 under the image of τ then says that

$$(p, q - N)_T = \frac{(-1)^p((2N, N)_T - (0, N)_T) * (p, q)_T}{-(-(2N, 2N)_T + 2)} + \frac{(-1)^q(-(N, 2N)_T + (N, 0)_T) * (N - p, N - q)_T}{-(-(2N, 2N)_T + 2)}.$$

This relation is what makes the following proposition hold, expressing the image of elements of $\mathcal{B} - \mathcal{B}'$ linear combinations of $\mathcal{B}'$.

Proposition 3.5. *$S^{-1}K_N(\Sigma_{1,0})$ is vector space of dimension N^2 over $S^{-1}\chi(\Sigma_{1,0})$.*

Proof. Let $\mathcal{D}' = \{[(a, b)_T, 1] : 0 \leq a \leq N - 1, 0 \leq b \leq N - 1\}$. Let $[L, s] \in S^{-1}K_N(\Sigma_{1,0})$. By theorem 5.3 $L \in K_A(\Sigma_{1,0})$ can be written as

$$\sum_{\beta \in \mathcal{B}} \chi_\beta \beta$$

where $\chi_\beta \in \chi(\Sigma_{1,0})$. Because of equation 5.1 and the construction of $S^{-1}K_N(\Sigma_{1,0})$,

$$[L, s] = [\sum_{k=0}^{N^2} \chi_k (a_k, b_k)_T, s] = \sum_{k=0}^{N^2} [\chi_k (a_k, b_k)_T, s] = \sum_{k=0}^{N^2} [\chi_k, s] * [(a_k, b_k)_T, 1]$$

where (a_k, b_k) are the elements of $\mathcal{B}'$ ordered lexicographically. This shows that $S^{-1}K_N(\Sigma_{1,0})$ is spanned by $\mathcal{D}'$ as an $S^{-1}\chi(\Sigma_{1,0})$-module. We showed in lemma 5.4

that B' is linearly independent from which it follows that $\mathcal{D}'$ is linearly independent proving the proposition.

□

Now that we have the previous proposition we can state the following.

Proposition 3.6. *$K_N(\Sigma_{1,0})$ is not a free module over $\chi(\Sigma_{1,0})$ and has rank greater than N^2.*

Proof. This simply follows from the fact that the coefficients in the previous proposition are unique and the linear combinations that yield the elements of $\mathcal{B} - \mathcal{B}'$ are not in the image of elements of $\mathcal{B}'$. Hence the rank of $K_N(\Sigma_{1,0})$ over $\chi(\Sigma_{1,0})$ is greater than dimension of $\mathcal{B}'$ and because of equation 5.1 it is not free. □

It will be helpful to have a collection of bases for $S^{-1}K_N(\Sigma_{1,0})$.

Proposition 3.7. *Let α, β be framed links in $\Sigma_{1,0} \times [0,1]$ coming from a pair of simple closed curves a and b on $\Sigma_{1,0}$ by using the blackboard framing, where a and b intersect one another transversely in a single point. The set of skeins $\mathcal{C} = \{T_p(\alpha)T_q(\beta)\}$ where p and q range from 0 to $N-1$, form a basis for $S^{-1}K_N(\Sigma_{1,0})$ as a vector space over $S^{-1}\chi(\Sigma_{1,0})$.*

Proof. There is a diffeomorphism h of $\Sigma_{1,0}$ that takes a to the $(1,0)$ curve and b to the $(0,1)$ curve. If h is orientation preserving it extends to an orientation preserving diffeomorphism $h \times Id : \Sigma_{1,0} \times [0,1] \to \Sigma_{1,0} \times [0,1]$. Let $s : [0,1] \to [0,1]$ be the diffeomorphism $s(t) = 1 - t$. If h is orientation reversing then it extends to an orientation preserving diffeomorphism $h \times s : \Sigma_{1,0} \times [0,1] \to \Sigma_{1,0} \times [0,1]$. An orientation preserving homeomorphism of a three-manifold M induces an automorphism of $K_N(M)$. This means we only need to prove that $\{(p,0)_T * (0,q)_T\}$ and $\{(0,q)_T * (p,0)_T\}$where $p,q \in \{0,\ldots,N-1\}$ form a basis of $S^{-1}K_N(\Sigma_{1,0})$. The two proofs are similar so we restrict our attention to $\mathcal{C} = \{(p,0)_T * (0,q)_T\}$. Since the set has N^2 elements, to prove that it is a basis we only need to prove that $\mathcal{C}$ is linearly independent over the function field of the character variety of $\Sigma_{1,0}$.

As a set we realize the character variety of the fundamental group of the torus $X(\Sigma_{1,0})$ to be the set of *trace equivalence classes* of representations $\rho : \pi_1(\Sigma_{1,0}) \to SL_2\mathbb{C}$. Denote the trace equivalence class of ρ by $[\rho]$. The coordinate ring $C[X(\Sigma_{1,0})]$ of $X(\Sigma_{1,0})$ is the ring generated by functions $\eta_g : X(\Sigma_{1,0}) \to \mathbb{C}$, given by $\eta_g([\rho]) = -tr(\rho(g))$. If $g \in \pi_1(\Sigma_{1,0})$ is homotopic to a simple closed curve, then the isomorphism [3],

$$\Phi : K_1(\Sigma_{1,0}) \to C[X(\Sigma_{1,0})]$$

sends the simple diagram corresponding to g to η_g.

To facilitate the computation we will use a specific 2-fold branched cover of the character variety of the fundamental group of the torus. If $\lambda, \mu \in \mathbb{C}^*$, let $\rho_{\lambda,\mu} : \pi_1(\Sigma_{1,0}) \to SL_2\mathbb{C}$ be the homomorphism with

$$\rho_{\lambda,\mu}((1,0)) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \; \rho_{\lambda,\mu}((0,1)) = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}.$$

We then define

$$\Theta : \mathbb{C}^* \times \mathbb{C}^* \to X(\Sigma_{1,0}),$$

by $\Theta(\lambda,\mu) = [\rho_{\lambda,\mu}]$. The pullback $\Theta^* : C[X(\Sigma_{1,0})] \to C[\lambda^{\pm 1}, \mu^{\pm 1}]$, is given by

$$\Theta^* \circ \Phi((p,q)_T) = (-1)^d(\lambda^p\mu^q + \lambda^{-p}\mu^{-q}),$$

where d is the greatest commond divisor of p and q.

Partition $\mathcal{C}$ into two subsets. Let $\mathcal{C}'$ be the subset where $p = 0$ or $q = 0$, and let $\mathcal{C}''$ be the subset where p, q are both nonzero. Partition $\mathcal{C}''$ further, into sets of four basis elements $(a,b)_T$ where for some fixed $p, q \in \{1, \ldots, N-1\}$, $a \in \{p, N-p\}$ and $b \in \{q, N-Q\}$. For instance when $N = 3$,

$$\mathcal{C}' = \{2(0,0)_T, 2(1,0)_T, 2(2,0)_T, 2(0,1)_T, 2(0,2)_T\}$$

and $\mathcal{C}''$ consists of one set of four,

$$\{(1,2)_T, (2,1)_T, (1,1)_T, (2,2)_T\}.$$

In general $\mathcal{C}'$ has $2N-1$ elements, and $\mathcal{C}''$ decomposes into $\frac{(N-1)^2}{4}$ sets of four elements. The spans of each one of these sets are independent from one another.

Given $p, q \in \{1, \ldots N-1\}$. We need to show that

$$\{(p,0)_T * (0,q)_T, (N-p,0)_T * (0,q)_T, (p,0)_T * (0,N-q)_T, (N-p,0)_T * (0,N-q)_T\}$$

are linearly independent. To this end we rewrite these in terms of the basis $\mathcal{B}$ using relation 3.1.1.

The change of basis matrix, where the columns correspond in order to $(p,q)_T, (p,N-q)_T, (N-p,q)_T$ and $(N-p,N-q)_T$, and the rows correspond in order to $(p,0)_T * (0,q)_T, (p,0)_T * (0,N-q)_T, (N-p,0)_T * (0,q)_T$ and $(N-p,0)_T * (0,N-q)_T$, is given by

$$\begin{pmatrix} A^{pq} & \frac{(-1)^{p+1}A^{-pq}((2N,N)_T-(0,N)_T)}{2-(2N,2N)_T} & \frac{(-1)^{q}A^{-pq}((N,2N)_T-(N,0)_T)}{2-(2N,2N)_T} & 0 \\ \frac{(-1)^{p+1}A^{-(N-p)q}((2N,N)_T-(0,N)_T)}{2-(2N,2N)_T} & A^{p(N-q)} & 0 & \frac{(-1)^{N-q}A^{-p(N-q)}((N,2N)_T-(N,0)_T)}{2-(2N,2N)_T} \\ \frac{(-1)^{q}A^{-(N-p)q}((N,2N)_T-(N,0)_T)}{2-(2N,2N)_T} & 0 & A^{p(N-q)q} & \frac{(-1)^{N-p+1}A^{-(N-p)q}((2N,N)_T-(0,N)_T)}{2-(2N,2N)_T} \\ 0 & \frac{(-1)^{N-q}A^{-(N-p)(N-q)}((N,2N)_T-(N,0)_T)}{2-(2N,2N)_T} & \frac{(-1)^{N-p+1}A^{-(N-p)(N-q)}((2N,N)_T-(0,N)_T)}{2-(2N,2N)_T} & A^{(N-p)(N-q)} \end{pmatrix}.$$

To see this matrix is nonsingular, pull it back to a matrix with coefficients from the field of quotients of $\mathbb{C}[\lambda^{\pm 1}, \mu^{\pm 1}]$ by using $\Theta^* \circ \Phi$. Next multiply by $2\lambda^2\mu^2 - \lambda^4\mu^4 - 1$ to clear the denominator so that the result is a matrix with coefficients in $\mathbb{C}[\lambda, \mu]$. Simple inspection reveals that highest power of λ and μ that appears is 4, taking the coefficient of $\lambda^4\mu^4$ yields,

$$\begin{pmatrix} A^{pq} & 0 & 0 & 0 \\ 0 & A^{p(N-q)} & 0 & 0 \\ 0 & 0 & A^{(N-p)q} & 0 \\ 0 & 0 & 0 & A^{(N-p)(N-q)} \end{pmatrix},$$

which has nontrivial determinant.

The change of basis matrix between the basis $\mathcal{B}'$ and $\mathcal{C}$ decomposes into a block which is the $2N-1$ dimensional identity matrix times 2 corresponding to $(p,0)_T * (0,0)_T$ and $(0,0)_T * (0,q)_T$, and $(\frac{(N-1)^2}{2})^2$ blocks coming from changing basis between $\{(p,q)_T, (p,N-q)_T, (N-p,q)_T, (N-p,N-q)_t\}$ and $\{(p,0)_T * (0,q)_T, (p,0)_T * (0,N-q)_T, (N-p,0)_T * (0,q)_T, (N-p,0)_T * (0,N-q)_T\}$ where p and q range over $1, \dots, \frac{N-1}{2}$. The above computation shows that each one of these blocks is nonsingular. Therefore $\mathcal{C}$ is a basis. □

The next step is to compute the trace. By this we mean let $S^{-1}K_N(\Sigma_{1,0})$ act on itself on the left, and take $\frac{1}{N^2}$ times the trace of the resulting matrix.

Theorem 3.8. *The trace of* $\sum_{p,q} \alpha_{p,q}(p,q)_T$ *where* $\alpha_{p,q} \in S^{-1}\chi(\Sigma_{1,0})$ *is*

$$\sum_{p,q \text{ where } N|p \text{ and} N|q} \alpha_{p,q}(p,q)_T.$$

Proof. The linearity of the trace means we only need to compute the trace of $(p,q)_T$ and see that it is zero unless $N|p$ and $N|q$, in which case it is $(p,q)_T$. Since orientation preserving homeomorphisms of $\Sigma_{1,0}$ induce automorphisms of $S^{-1}\chi(\Sigma_{1,0})$ and $S^{-1}K_N(\Sigma_{1,0})$ that are coherent with one another, and there is an orientation homeomorphism taking and simple closed curve to the $(1,0)$ curve, we only need to compute the trace of $(k,0)$.

It doesn't make any difference what basis we choose for $K_N(\Sigma_{1,0})$ so we choose $\{(p,0)_T * (0,q)_T\}$ where $p,q \in \{0,\ldots,N-1\}$.. From the product to sum formula,

$$(k,0)_T * (p,0)_T * (0,q)_T = (p+k,0)_T * (0,q)_T + (p-k,0)_T * (0,q)_T.$$

Computations similar to the ones for computing the trace on the annulus complete the proof. □

Here is a construction that will come in handy. Suppose that C is a simple closed curve on the surface F and A_0 and A_1 are annular neighborhoods of $C \times \{0\}$ and $C \times \{1\}$ in $F \times \{0,1\}$. By gluing an annulus cross an interval into A_i we can make $K_N(F)$ into a left-module (gluing into A_1) or a right-module over $K_N(\Sigma_{0,2})$. Since the inclusion map sends $\chi(\Sigma_{0,2})$ into $\chi(F)$ the each action extends to make $S^{-1}K_N(F)$ into a vector space over $S^{-1}K_N(\Sigma_{0,2})$.

For instance $S^{-1}K_N(\Sigma_{1,0})$ is a vector space over $S^{-1}K_N(\Sigma_{0,2})$ where the action comes from gluing the annulus into an annulus in $\Sigma_{1,0} \times \{1\}$ corresponding to the $(1,0)$ curve. The basis $(p,0)_T * (0,q)_T$ where $p,q \in \{0,1,\ldots,N-1\}$ corresponds to a splitting of $K_N(\Sigma_{1,0})$ as a direct sum of vector spaces over $S^{-1}(\Sigma_{0,2})$ spanned by

$\chi(\Sigma_{1,0})(0,q)_T$.

$$S^{-1}K_N(\Sigma_{1,0}) = \bigoplus_q (S^{-1}K_N(\Sigma_{0,2}))\chi(\Sigma_{1,0})(0,q)_T.$$

If $T_k(x) \in S^{-1}K_N(\Sigma_{0,2})$ it acts as $(k,0)_T$ on the left.

A similar construction, coming from gluing the annulus into an annulus in $\Sigma_{1,0}$ coming from the $(0,1)$ curve allows us to write $K_N(\Sigma_{1,0})$ as a direct sum, over $S^{-1}K_N(\Sigma_{0,2})$ acting on the right,

$$S^{-1}K_N(\Sigma_{1,0}) = \bigoplus_p \chi(\Sigma_{1,0})(p,0)_T(S^{-1}K_N(\Sigma_{0,2})).$$

Here $T_k(x) \in S^{-1}K_N(\Sigma_{0,2})$ it acts as $(0,k)_T$ on the left.

Theorem 3.9. *$S^{-1}K_N(\Sigma_{1,0})$ is a Frobenius algebra over $S^{-1}\chi(\Sigma_{1,0})$ with the normalized trace $Tr : S^{-1}K_N(\Sigma_{1,0}) \to S^{-1}\chi(\Sigma_{1,0})$.*

Proof. We prove that there are no nontrivial principal ideals in the kernel of Tr. Suppose that $I = (\sum_{p,q} \alpha_{p,q}(p,0)_T * (0,q)_T)$ is a principal ideal where $\alpha_{p,q} \in S^{-1}\chi(\Sigma_{1,0})$, $p,q \in \{0,\ldots,N-1\}$ and some $\alpha_{a,b} \neq 0$. Since $S^{-1}K_N(\Sigma_{0,2})$ is a field there exist $T_a(x)^{-1}$ and $T_b(x)^{-1}$. Using the actions above to include these into $\Sigma_{1,0} \times [0,1]$ there are skeins $(a,0)_T^{-1}$ and $(0,b)_T^{-1}$ in $S^{-1}K_N(\Sigma_{1,0})$. That is, the skeins $(p,q)_T$ are units. Multiplying the generator of the principle ideal on the left by $(a,0)_T^{-1}$ and on the right by $(0,b)_T^{-1}$ we get a skein of the form,

$$(0,0)_T + \sum_{(p,q)\neq(0,0)} \beta_{p,q}(p,0)_T * (0,q)_T$$

with $\beta_{p,q} \in S^{-1}\chi(\Sigma_{1,0})$ is in I. However

$$Tr((0,0)_T + \sum_{(p,q)\neq(0,0)} \beta_{p,q}(p,0)_T * (0,q)_T) = (0,0)_T \neq 0.$$

□

Next we compute the determinant of the pairing $<\alpha,\beta>= Tr(\alpha\beta)$. From the product to sum formula and the linearity of trace,

$$Tr((p,q)_T*(r,s)_T) = A^{\begin{vmatrix} p & q \\ r & s\end{vmatrix}} Tr((p+r,q+s)_T) + A^{-\begin{vmatrix} p & q \\ r & s\end{vmatrix}} Tr((p-r,q-s)_T).$$

This means that $<(p,q)_T,(r,s)_T>= 0$ unless N divides both $p+r$ and $q+s$, or N divides both $p-r$ and $q-s$. As N is odd, both case don't occur at the same time unless one of $(p,q)_T$ or $(r,s)_T$ is $(0,0)_T$. In the first case, $<(p,q)_T,(r,s)_T>= (-1)^{ps+qr}(p+r,q+s)_T$ and in the second case $<(p,q)_T,(r,s)_T>= (-1)^{ps+qr}(p-r,q-s)_T$. Finally $<(0,0)_T,(r,s)_T>$ is zero unless r and s are both divisible by N, in which case it is $2(r,s)_T$. This makes the computation of the matrix of the pairing with respect to the basis $\mathcal{B}'$ straightforward.

Here is the matrix of the pairing for $N=3$ with respect to the basis $(p,q)_T$, where $p,q\in\{0,1,2\}$ ordered lexicographically.

$$\begin{pmatrix}
2(0,0)_T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (0,0)_T & (0,3)_T & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (0,3)_T & (0,0)_T & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (0,0)_T & 0 & 0 & (3,0)_T & 0 & 0 \\
0 & 0 & 0 & 0 & (0,0)_T & 0 & 0 & 0 & (3,3)_T \\
0 & 0 & 0 & 0 & 0 & (0,0)_T & 0 & -(3,3)_T & 0 \\
0 & 0 & 0 & (3,0)_T & 0 & 0 & (0,0)_T & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -(3,3)_T & 0 & (0,0)_T & 0 \\
0 & 0 & 0 & 0 & (3,3)_T & 0 & 0 & 0 & (0,0)_T
\end{pmatrix}$$

The determinant of the pairing can be computed similarly to the case of the annulus.

Theorem 3.10. *The determinant of the pairing* $<\ ,\ >$ *on* $S^{-1}K_N(\Sigma_{1,0})$ *with respect* *to the basis $\mathcal{B}'$ is*

$$2(0,0)_T((0,0)_T^2-(N,0)_T^2)^{\frac{N-1}{2}}((0,0)_T^2-(0,N)_T^2)^{\frac{N-1}{2}}((0,0)_T^2-(N,N)_T^2)^{\left(\frac{N-1}{2}\right)^2}.$$

Hence, $S^{-1}K_N(\Sigma_{1,0})$ is a Frobenius algebra over $S^{-1}\chi(\Sigma_{1,0})$.

Proof. The matrix decomposes into blocks, coming from partitioning the basis $\{(p,q)_T\}$ into subsets, where pairing basis elements in different subsets is always zero.

- The first subset is $\{(0,0)_T\}$. $Tr((0,0)_T^2) = 2(0,0)_T$.
- The second subset is $\{(p,0)_T\}$ where $p \in \{1,\dots,N-1\}$. As a matrix the pairing agrees with the matrix for the pairing of the annulus restricted to the subspace $\{T_k(x)\}$ where $k \in \{1,\dots,N-1\}$ via the obvious correspondence. The determinant of this block is $((0,0)_T^2-(N,0)^2)^{\frac{N-1}{2}}$.
- The third subset is $\{(0,p)_T\}$ where $p \in \{1,\dots,N-1\}$. The determinant of this block is $((0,0)_T^2-(0,N)_T^2)^{\frac{N-1}{2}}$.
- There are $(\frac{N-1}{2})^2$ subsets of four that come from a choice of $p,q \in \{1,\dots,\frac{N-1}{2}\}$ consisting of $\{(p,q)_T,(p,N-q)_T,(N-p,q)_T,(N-p,N-q)_T\}$. With respect to this basis the pairing is,

$$\begin{pmatrix} (0,0)_T & 0 & 0 & (-1)^{p+q}(N,N)_T \\ 0 & (0,0)_T & (-1)^{p+q+1}(N,N)_T & 0 \\ 0 & (-1)^{p+q+1}(N,N)_T & (0,0)_T & 0 \\ (-1)^{p+q}(N,N)_T & 0 & 0 & (0,0)_T \end{pmatrix}.$$

  The determinant of this matrix is $((0,0)_T^2-(N,N)_T^2)$.

The determinant of the pairing is the product of the determinants of the blocks,

$$2(0,0_T)((0,0)_T^2-(N,0)^2)^{\frac{N-1}{2}}((0,0)_T^2-(0,N)_T^2)^{\frac{N-1}{2}}((0,0)_T^2-(N,N)_T^2)^{\left(\frac{N-1}{2}\right)^2}.$$

□

The character variety of the torus is describe as follows. Define $t : Rep(\pi_1(SL_2\mathbb{C})) \to \mathbb{C}^3$ be given by,

$$t(\rho) = (x, y, z),$$

where $x = -tr(\rho((1,0)), y = -tr(\rho(0,1))$ and $z = -tr(\rho((1,1))$. The image of t is in one to one correspondence with the $SL_2\mathbb{C}$-character variety of $\pi_1(\Sigma_{1,0})$. The image of t is the locus,

$$\{(x, y, z) \in \mathbb{C}^3 | x^2 + y^2 + z^2 + xyz - 4 = 0\}.$$

As long as we stay away from the places where x, y, or z is ± 2, the determinant of the pairing specialized at that point is nonzero.

Theorem 3.11. *Let $\phi_{x,y,z} : \chi(\Sigma_{1,0}) \to \mathbb{C}$ be the place corresponding to a representation where $tr(\rho((1,0) = -x$, $tr(\rho(0,1)) = -y$ and $tr(\rho((1,1)) = -z$. As long as none of x, y, or z takes on the values ± 2, $K_N(\Sigma_{1,0})_{\phi_{(x,y,z)}}$ is a Frobenius algebra.*

3.2 The Once Punctured Torus

Let δ denote the blackboard framed link based on a simple closed curve that is parallel to the boundary of $\Sigma_{1,1}$. The skein $\eta = \delta + (A^2 + A^{-2})$ was introduced by Bullock and Przytycki in [5]. The inclusion of $\Sigma_{1,1}$ into $\Sigma_{1,0}$ obtained by gluing in a disk, induces a homomorphism of skein algebras. If (η) denotes the principal ideal of $K_N(\Sigma_{1,1})$ generated by η then the inclusions form a short exact sequence,

$$0 \to (\eta) \to K_N(\Sigma_{1,1}) \to K_N(\Sigma_{1,0}) \to 0.$$

The sequence splits as any two simple closed curves in $\Sigma_{1,1}$ that get mapped by inclusion to isotopic, nontrivial simple closed curves in $\Sigma_{1,0}$ are in fact isotopic in $\Sigma_{1,1}$. Hence it makes sense to talk about the skein $(p,q)_T$ in $\Sigma_{1,1}$. The skeins $\eta^k(p,q)_T$ where k ranges over the non-negative integers and $(p,q)_T$ ranges overs pairs of integers so that that $p \geq 0$ and if $p = 0$, $q \geq 0$ is a basis for $\Sigma_{1,1}$ over the complex numbers.

As a consequence of the splitting of the sequence, any skein in $K_N(\Sigma_{1,1})$ can be written as $\sum_{i,j} \alpha_{i,j}(p_i, q_i)_T + \epsilon$ where $\epsilon \in (\eta)$. Similar to the torus case, define the weight of $\eta^k(p,q)_T$ to be $|p| + |q|$. Define the **weight** of a skein to be the maximum weight of an $\eta^k(p,q)$ appearing with nonzero coefficient in the expansion of the skein in terms of the basis $\eta^k(p,q)$.

Theorem 3.12 (Product to Sum Formula). *[1] If $(p,q),(r,s) \in K_N(\Sigma_{1,1})$ then*

$$(p,q)_T * (r,s)_T = A^{\begin{vmatrix} p & q \\ r & s \end{vmatrix}}(p+r, q+s)_T + A^{-\begin{vmatrix} p & q \\ r & s \end{vmatrix}}(p-r, q-s)_T + \epsilon$$

where $\epsilon \in (\eta)$ and the weight of ϵ is less than or equal to $|p| + |q| + |r| + |s| - 4$.

Lemma 3.13. *For ϵ as in the previous theorem the highest power of η appearing in ϵ is less than or equal to $\lfloor \frac{min\{|p|+|r|,|q|+|s|\}}{2} \rfloor$.*

Proof. Visualize the punctured torus as the square with all four corners removed and this simply follows from the fact that when multiplying, you will have $|p|+|r|$ strands through the top and bottom of your square and $|q|+|s|$ through the sides. To produce an element of (η) you need to borrow two strands from every side of your square and hence this process will be stopped when or before we reach $min\{|p| + |r|, |q| + |s|\}$.

Since we need two strands from every side of our square to produce an element of (η) then the degree must be less than or equal to $\lfloor \frac{min\{|p|+|r|,|q|+|s|\}}{2} \rfloor$. □

The center $Z(K_N(\Sigma_{1,1}))$ of $K_N(\Sigma_{1,1})$ is spanned by all skeins of the form $\eta^k(Na, Nb)_T$. If $\phi : \chi(\Sigma_{1,1}) \to \mathbb{C}$ is a place, supposing that $\phi(T_N(\delta)) = z + z^{-1}$ then choosing any nth root w of z extends to a place $\tilde{\phi} : Z(K_N(\Sigma_{1,1})) \to \mathbb{C}$ that sends δ to w. This fits in with the concept of a **classical shadow** of a representation of the skein algebra due to Bonahon and Wong [2].

The skein algebra $K_N(\Sigma_{1,1})$ is the first skein algebra of a surface with boundary that is noncommutative that we have considered. We begin by treating $K_N(\Sigma_{1,1})$ as a ring extension of its center.

Proposition 3.14. *The algebra $K_N(\Sigma_{1,1})$ as a module over $Z(K_N(\Sigma_{1,1}))$ is spanned by a lift of the spanning set $\mathcal{B}$ of $K_N(\Sigma_{1,0})$ over $\chi(\Sigma_{1,0})$.*

Proof. By the exact sequence from [5] every skein can be written $\sum_{a,b,k} \alpha_{a,b,k}\eta^k(a,b)_T$ where the $\alpha_{a,b,k} \in \mathbb{C}$ and only finitely many are nonzero. The complexity of a skein is the ordered triple consisting of its weight, the highest power of η appearing in a term that realizes it's weight and the number of terms of highest weight with η raised to the highest power. We order the complexity lexicographically.

Suppose that everything with complexity less than (r, n) can be written as a $Z(K_N(\Sigma_{1,1}))$ linear combination of the (a, b) as in the statement of the proposition, and suppose that $\sum_{k,a,b} \alpha_{k,a,b}\eta^k(a,b)_T$ has complexity (r, n, m). Choose a term $\alpha_{k,a,b}\eta^n(a,b)_T$ of weight r. If $|b| > N - 1$ then either we can write $b = c + N$ where

$|c| < |b|$ and either $|c - N| < |c|$ or $|c| < |N|$, or we can write $b = c - N$ where $|c| < |b|$ and either $|c + N| < |c|$ or $|c + N| < N$.

Using the product to sum formula

$$\eta^n(0, dN)_T * (a, c)_T - \eta^n A^{-dNa}(a - Nd, c - Nd)_T = A^{dNc}\eta^k(a, b)_T + \epsilon.$$

Notice the weight of ϵ less than or equal to $|a| + |c| + |dN| - 4$.

Substituting in we have reduced the complexity by lowering the number of terms of weight r with η^n. If b is between $-N + 1$ and $N - 1$ and $|a| > N - 1$ perform the analogous rewriting to reduce the complexity. □

Recall that if $S \subset R$ is a subring of the integral domain R and R is an integral extension of S then if S is a field then R is a field. The center of $K_N(\Sigma_{1,1})$ is also spanned by all $\delta^k(Np, Nq)_T$ where $p, q \in \mathbb{Z}$. The reason is that δ is in the center, and η^k is a polynomial in δ with lead coefficient δ^k. Next, $\chi(\Sigma_{1,1})$ is spanned by all $T_{aN}(\delta)(Np, Nq)_T$. From this we see that $Z(K_N(\Sigma_{1,1}))$ is an integral extension of degree N of $\chi(\Sigma_{1,1})$, where the added element δ satisfies a monic degree N polynomial with coefficients in $\chi(\Sigma_{1,1})$. Specificially

$$\delta^N = T_N(\delta) - \sum_{i=1}^{N/2}(-1)^i \frac{N}{N-i}\binom{N-i}{i}\delta^{N-2i}$$

This means that the ring of fractions coming from inverting all nonzero elements of $\chi(F)$ is isomorphic to the ring of fractions coming from inverting all elements of $Z(K_N(\Sigma_{1,1}))$.

Theorem 3.15. *Letting $S = \chi(\Sigma_{1,1}) - \{0\}$, the algebra $S^{-1}K_N(\Sigma_{1,1})$ has dimension N^3 over $S^{-1}\chi(\Sigma_{1,1})$, the function field of the $SL_2\mathbb{C}$-character variety of $\pi_1(\Sigma_{1,1})$. A* *basis is given by* $T_k(\delta)(p,q)_T$ *where* $k,p,q \in \{0,1,\ldots,N-1\}$.

Proof. The proof is in two steps. First as a vector space over field of fractions of the center, $K_N(\Sigma_{1,1})$ has dimension N^2 with basis $(p,q)_T$ where $p,q \in \{0,1,\ldots,N-1\}$. The product to sum formula for $\Sigma_{1,1}$ says that the elements of $\mathcal{B}$ satisfy relations whose leading term is the same as in $\Sigma_{1,0}$ plus lower complexity terms that are multiplied by powers of η. By continually eliminating the lead term, we terminate with an expression for every element of $\mathcal{B}'$ as a linear combination with coefficients in $S^{-1}Z(K_N(\Sigma_{1,1})$.

Since $S^{-1}Z(K_N(\Sigma_{1,1}))$ is a vector space over $S^{-1}\chi(\Sigma_{1,1})$ with basis $T_k(\delta)$ where $k \in \{0,1,\ldots,N-1\}$ we get the desired result by expanding the coefficients. □

Next we need to compute the trace of the matrices corresponding to left multiplicaiton by $T_k(\delta)(p,q)_T$ where k,p,q range over $\{0,1,\ldots,N-1\}$ Since any simple curve $(p/d,q/d)$ can be taken to $(1,0)$ by an orientation preserving homeomorphism (and that homeomorphism induces an automorphism of $K_N(\Sigma_{1,1})$. We only need to compute the trace of $T_k(\delta) * (d,0)_T)$. This is pretty easy as we can use the basis $T_i(\delta)(p,0)_T * (0,q)_T$ where i,q and q range from $\{0,\ldots,N-1\}$. From the product to sum formula,

$$
\begin{aligned}
(T_k(\delta) * (d,0)_T)(T_i(\delta)(p,0)_T * (0,q)_T) &= T_{k+i}(\delta)(p+d,0)_T * (0,q)_T \\
&+ T_{k+i}(\delta)(|p-d|,0)_T * (0,q)_T \\
&+ T_{|k-i|}(\delta)(p+d,0)_T * (0,q)_T
\end{aligned}
$$

$$+ T_{|k-i|}(\delta)(|p-d|,0)_T * (0,q)_T.$$

This needs to be rewritten when $p+d \geq N$ or $k+i \geq N$, but it works just in the case of the annulus.

Theorem 3.16. *The normalized trace of* $\sum \alpha_{k,p,q} T_k(\delta)(p,q)_T$ *where the* $\alpha_{k,p,q} \in S^{-1}\chi(\Sigma_{1,1})$*, can be computed by deleting all terms where* N *does not divide all of* k*,* p*, and* q*.*

□

Theorem 3.17. *The algebra* $S^{-1}K_N(\Sigma_{1,1})$ *with the normalized trace* $Tr : S^{-1}K_N(\Sigma) \to S^{-1}\chi(\Sigma_{1,1})$ *is Frobenius.*

Proof. Just as in the case of $S^{-1}K_N(\Sigma_{1,0})$ except work with basis $T_k * (\delta)(p,0)_T * (0,q)_T$. Show that any nontrivial principal ideal contains $T_0(\delta)*(0,0)_T + \sum_{k\neq 0,p,q} \beta_{k,p,q} T_k(\delta)*(p,0)_T * (0,q)_T$ whose trace is obviously nonzero. □

The fundamental group of $\Sigma_{1,1}$ is the free group on two generators, it's character variety is $\mathbb{C}^3$ where the coordinates can be taken as the negative of the traces of the matrices that $(1,0)_T$, $(0,1)_T$ and $(1,1)_T$ are sent to. We have not computed the determinant of the trace pairing, so we can't say exactly what the locus is where the algebras fail to be Frobenius is, but the determinant of the pairing is a nonzero element of the function field of the character variety and the divisor of zeroes and poles is a proper subvariety away from which $K_N(\Sigma_{1,1})_\phi$ is Frobenius. The zeroes correspond to points where the pairing is degenerate and the poles correspond to points where the global basis doesn't actually span the specialization of the algebra.

Theorem 3.18. *Away from a proper subvariety C of the character variety, $K_N(\Sigma_{1,1})_\phi$ is Frobenius.*

□

CHAPTER 4
THE LOCALIZED SKEIN ALGEBRA OF A SURFACE OF FINITE TYPE IS FROBENIUS

4.1 $K_{\mathfrak{D}}(F)$ Is Finitely Generated.

4.1.1 Parametrizing the simple diagrams

An ideal triangle is a triangle with its vertices removed. An ideal triangulation of a finite type surface F consists of finitely many ideal triangles Δ_i with their edges identified pairwise, along with a homeomorphism from the resulting quotient space to F. Alternatively an ideal triangulation is defined by a family C of properly embedded lines that cuts F into finitely many ideal triangles. More precisely, place a complete Riemannian metric on F. Define a metric on the components D of $F - C$ by

$$d(p,q) = \inf\{length(\alpha)|\alpha : [0,1] \to D \text{ is smooth}, \alpha(0) = p, \alpha(1) = q\}.$$

Let Δ be the metric space completion of D. If Δ is homeomorphic to a disk with three points removed from its boundary, then D **completes to an ideal triangle**. If all the components of $F - C$ complete to ideal triangles then C **cuts** F **into ideal triangles.**

If Δ is an ideal triangle in an ideal triangulation then $\partial\Delta = \{a, b, c\}$ where a, b and c are homeomorphic to $\mathbb{R}$. The lines a, b, and c are the **sides** of Δ. There is a map of Δ to the closure of a component D of the complement of C into F. If this map is an embedding, then Δ is an **embedded ideal triangle**. It could be that two sides c_1, c_2 of the ideal triangle Δ get mapped to the same line c, in this case Δ is a **folded ideal triangle**. Figure 4.1 is a picture of a folded ideal triangle. There are two punctures in the picture, and the mapping is $2-1$ along the vertical line joining them. The edge that is covered twice by the mapping has **multiplicity** 2.

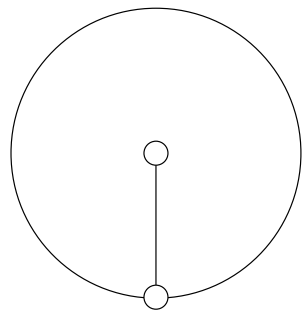


Figure 4.1: A folded triangle

The surface F needs to have at least one puncture, and negative Euler characteristic or it doesn't admit an ideal triangulation. If the Euler characteristic of the surface F is $-e(F)$ then any ideal triangulation of F consists of $2e(F)$ ideal triangles. The cardinality of a set of lines C defining an ideal triangulation is $3e(F)$.

Let C denote a disjoint family of properly embedded lines that defines an ideal triangulation of F, and suppose the triangles are the set $\{\Delta_j\}$. An **admissible coloring**, $f : C \to \mathbb{Z}_{\geq 0}$ is an assignment of a nonnegative integer $f(c)$ to each $c \in C$ so that the following conditions hold;

- If $\{a, b, c\}$ form the boundary of an embedded ideal triangle Δ_j then $f(a)+f(b)+f(c)$ is even and the triple $\{f(a), f(b), f(c)\}$ satisfies the triangle inequality,

$$f(a) \leq f(b) + f(c),\ f(b) \leq f(a) + f(c),\ \text{and}\ f(c) \leq f(a) + f(b).$$

- If $\{a, b\}$ are the image of the boundary of a folded ideal triangle Δ_j where b has multiplicity 2 we require that $f(a) + 2f(b)$ be even and $f(a) \leq 2f(b)$.

If $S \subset F$ is a simple diagram then $f_S : C \to \mathbb{Z}_{\geq 0}$ given by $f_S(c) = i(S, c)$ is an admissible coloring.

Proposition 4.1. *The admissible colorings $f : C \to \mathbb{N}$ are in one-to-one correspondence with isotopy classes of simple diagrams on F.*

□

The admissible colorings form a **pointed integral polyhedral cone**. If C is the set of edges of the ideal triangulation then there is a map,

$$\{f : C \to \mathbb{Z}_{\geq 0} | admissible\} \to \mathbb{Z}^C$$

that sends each f to its tuple of values. The image is defined by linear equations and inequalities, so it is polyhedral. The image is closed under addition, so it is a cone, and the tuple of all zeroes is an admissible coloring so it is pointed.

It is a classical result [20] that any pointed integral polyhedral cone admits an **integral basis**. That is, there are finitely many admissible colorings f_{S_i} so that every admissible coloring is a nonnegative integral linear combination of the f_{S_i}, and the set $\{f_{S_i}\}$ has minimal cardinality with respect to this condition. The integral basis is unique. If P is a pointed integral polyhedral cone, $p \in P$ is **indivisible** if whenever $s, t \in P$ and $s + t = p$ then $s = 0$ or $p = 0$. The set of indivisible elements of P is the integral basis [27]. In the case of the cone of admissible colorings, the diagrams corresponding to indivisible colorings are simple closed curves.

Forgetting positivity, and the triangle inequality, the admissible colorings generate a free module over $\mathbb{Z}$. It makes sense to ask whether a collection $f_{S_i} : C \to \mathbb{Z}_{\geq 0}$ are linearly independent. Oddly, the integral basis need not be linearly independent.

4.1.2 An axample

Decompose the punctured torus $\Sigma_{1,1}$ into two ideal triangles. This requires three edges, that form the boundary of both triangles. In the diagram below we identify the left and right hand sides of the rectangle, and the top and bottom of the rectangle with the vertices deleted to obtain a once punctured torus. The lines defining the triangulation come from the sides of the rectangle and the diagonal shown.

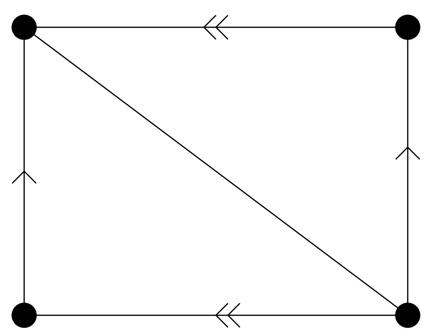


Figure 4.2: An ideal triangulation of $\Sigma_{1,1}$

The admissible colorings can be seen as triples of counting numbers (m, n, p) whose sum is even and satisfy the triangle inequality. The nonzero indecomposable admissible colorings are $(1, 1, 0), (1, 0, 1)$ and $(0, 1, 1)$. This set is an integral basis.

Notice that if (a, b, c) is an admissible coloring and one of the triangle inequalities is strict, say $a < b+c$ we can subtract the corresponding indecomposable $(0, 1, 1)$ to get a triple $(a, b-1, c-1)$ that still satisfies the triangle inequality and the sum of the colors $a+b+c-2 < a+b+c$. If all three triangle inequalities are equalities, $a = b+c$, $b = a+c$ and $c = a+b$ then $(a, b, c) = (0, 0, 0)$. The three curves corresponding to $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ are the generators that Bullock and Przytycki [16] got for $K(\Sigma_{1,1})$. There are infinitely many ideal triangulations of $\Sigma_{1,1}$ but Euler characteristic forces them all to be two triangles that share all their edges. The argument above goes through, even though the curves on the torus will be different. Since the integral basis is unique, any set of skeins that generates $K_{\mathfrak{D}}(\Sigma_{1,1})$ must have at least three elements.

4.1.3 The algebra $K_{\mathfrak{D}}(F)$ is finitely generated over $\mathfrak{D}$

If f_S and $f_{S'}$ are admissible colorings, choose simple diagrams S and S' that realize the colorings as the cardinality of their intersections with the $c_i \in C$ and so that S and S' realize their geometric intersection number, $S \cap S'$ is disjoint from all c_i. Up to isotopy there is a unique simple diagram whose associated coloring is $f_S + f_{S'}$, called the **geometric sum** of S and S'. Since addition of admissible colorings is associative, so is the geometric sum.

More is true. Define the **weight** of a simple diagram S to be

$$weight(S) = \sum_{c \in C} i(S, c) = \sum_{c \in C} f_S(c).$$

Suppose that S and S' transversely represent $i(S, S')$. Furthermore assume that

$S \cap S' \cap C = \emptyset$. If there are n points of intersection in $S \cap S'$ there are 2^n ways of smoothing all the crossings of S and S' to get a system of simple closed curves. We call a system of simple closed curves obtained by smoothing all crossings s a **state**. A state might not be a simple diagram as it may contain some trivial simple closed curves. There is a process for writing the product $S * S'$ as a linear combination of simple diagrams. First expand the product as a sum of states using the Kauffman bracket skein relation for crossings, then delete the trivial components of each state, and for each trivial component deleted from a state multiply the coefficient of the state by $-\zeta^2 - \zeta^{-2}$. Order the crossings of $S * S'$. Based on the ordering there is a rooted tree, where the root is the diagram $S * S'$, the vertices are partial smoothings (resolvents) of the diagram, and the directed edges correspond to smoothing the crossings in order. The states are the leaves of this tree. If the shortest path from the root to a state s passes through a resolvent r we say that s is a **descendent** of r.

Theorem 4.2. *Let S and S' be simple diagrams associated to admissible colorings $f_S, f_{S'} : C \to \mathbb{Z}_{\geq 0}$. Assume the product $S * S' \in K_{\mathfrak{D}}(F)$ has been written as $\sum_D \alpha_D D$ where the D are simple diagrams that are distinct up to isotopy and the $\alpha_D \neq 0 \in \mathfrak{D}$. There exists a unique simple diagram E in this sum, so that $weight(E) = weight(S) + weight(S')$, and all the other simple diagrams appearing with nonzero coefficient in the sum have strictly lower weight. Furthermore, the coefficient α_E is a power of ζ.*

Adam Sikora informs us that working with Jozef Przytycki they proved a similar result based on a filtration coming from Dehn coordinates for the simple closed curves on the surface. The proof is based on the following lemma.

Lemma 4.3. *Let G be a four valent graph with at least one vertex, embedded in a disk D^2. Assume that G is the union of two families of properly embedded arcs $A_1 \cup A_2$ and that there are three special points p, q, r in ∂D^2, so that*

- *the endpoints of the A_1 and A_2 are disjoint from one another and $\{p, q, r\}$ in ∂D^2, and*
- *if $a_1 \in A_1$ and $a_2 \in A_2$ then a_1 and a_2 intersect transversely, and realize their geometric intersection number relative to their boundaries, and*
- *if $a, b \in A_i$ then $a \cap b = \emptyset$, and*
- *for any arc $a \in A_1 \cup A_2$, the endpoints of a are separated by $\{p, q, r\}$,*

Then there is an embedded triangle Δ whose sides consist of an arc of ∂D^2 that is disjoint from $\{p, q, r\}$, an arc contained in some $a \in A_1$ that only intersects A_2 in a single point which is one of its endpoints, and an arc in some $b \in A_2$ that only intersects A_1 in a single point which is one of its endpoints.

We call this an **outermost triangle***.*

Proof. The graph dissects the disk into vertices, edges and faces. The alternating sum of the numbers of vertices, edges and faces is 1 as that is the Euler characteristic of the disk. A face f has two kinds of sides, sides in ∂D^2 and sides in the interior of D^2. Let $e_\partial(f)$ denote the number of sides of f lying in ∂D^2 and $e_i(f)$ the number of sides of f in the interior. Similarly, let $v_\partial(f)$ be the number of vertices of the face that lie in ∂D^2, and $v_i(f)$ be the number of vertices of f that lie in the interior of

D^2. The contribution of the face f to the Euler characteristic of the disk is,

$$c(f) = 1 - \frac{e_i(f)}{2} - e_\partial(f) + \frac{v_i(f)}{4} + \frac{v_{\partial(f)}}{2}.$$

We have that $\sum_f c(f) = 1$. The faces that are contained in the interior of the disk have an even number of sides, as their edges are partitioned into arcs of A_1 and arcs of A_2. Since the arcs of A_1 and A_2 realize their geometric intersection number the interior faces have at least four sides. Hence the largest contribution of an interior face is 0. A face touching the boundary can have two sides, but these faces are cut off by a single component of A_1 or A_2, and contain a point of $\{p, q, r\}$ in their boundary face by the last condition. These components can be discarded and the hypotheses of the theorem hold true for the graph formed by the smaller collection of arcs. The only remaining faces that contribute positively to the Euler characteristic of the disk are triangles with one edge on the boundary. These contribute $\frac{1}{4}$ to the Euler characteristic. There must be at least 4 such triangles. That means one of those triangles does not contain a point from $\{p, q, r\}$, so it is an outermost triangle. □

Proof. Theorem 4.2 Let S and S' be two simple diagrams, with associated colorings $f_S, f_{S'} : C \to \mathbb{Z}_{\geq 0}$ where C is the system of proper lines defining and ideal triangulation with ideal triangles Δ_j. We do not need to distinguish between embedded and folded triangles for this proof, because the combinatorial lemma above is applied in the completed components of the complement of C. Isotope S and S' so that they are transverse to one another, and the lines in C, and realize all geometric intersection numbers $i(S, S')$, $i(S, c)$ and $i(S', c)$ for $c \in C$. Also make sure that $S \cap S' \cap C = \emptyset$.

We resolve $S * S'$ one ideal triangle at a time. The four valent graph $(S \cup S') \cap \Delta_j$ for each Δ_j satisfies the hypotheses of the lemma. To start with, A_1 is made up of the components of $S \cap \Delta_j$ and A_2 is made up of the components of $S' \cap \Delta_j$. Therefore we can find an outermost triangle $\Delta \subset \Delta_j$. If we resolve the crossing of $S * S'$ at the apex of the triangle there are two resolvents. One resolvent forms a bigon with the edge of the triangle, and hence any simple diagram descendent from this resolvent has strictly lower weight than $weight(S) + weight(S')$.

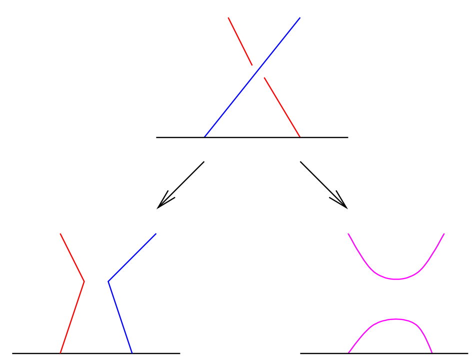


Figure 4.3: Resolving at an outermost triangle

The other resolvent doesn't have a bigon. Any state resulting in a simple diagram of weight $weight(S) + weight(S')$ is a descendent of this resolvent. The triangle Δ has a face $p \subset S$ and a face $q \subset S'$. Assume that p lies in the component a of the family A_1 and q lies in the component b of A_2. We smooth by forming arcs $a - p \cup q$ and $b - q \cup p$ and then perturb them slightly so that they are disjoint. To continue on inductively, we declare that the perturbed version of $a - p \cup q$ is in A_1, whilst removing a, and the perturbate of $b - q \cup p$ is in A_2 and discard b. Notice that the assignment of A_1 and A_2 is now just local to the ideal triangle instead of corresponding to the diagrams S and S'. However we work ideal triangle by ideal triangle, so this isn't a problem. If the new graph has a crossing it still satisfies the hypotheses of the lemma, so we can continue resolving crossings at the apex of an outermost triangle and there is a unique resolvent that can have a descendent of weight $weight(S) + weight(S')$. Continue until there are no crossings in Δ_j. There is a single resolvent with no bigons in Δ_j so that all the crossings in Δ_j have been resolved. All the other resolvents with no crossings in Δ_j have bigons in Δ_j and will lead to simple diagrams of strictly lower weight. Do this for each triangle. In the end, there is a single state of maximum weight which is a simple diagram E. Since there are no bigons between E and the edges of the triangulation, the admissible coloring associated to E is $f_S + f_{S'}$, so E is the geometric sum of S and S'. The coefficient of E is $\zeta^{p(E)-n(E)}$ where $p(E)$ is the number of positive smoothings and $n(E)$ is the number of negative smoothings that gave rise to the state E. The rest of the expansion is a linear combination of simple diagrams having strictly lower weight. □

If $\beta = \sum \alpha_i S_i \in K_{\mathfrak{D}}(F)$ where $\alpha_i \in \mathfrak{D}$, and the S_i are simple diagrams that are not isotopic to one another, the **weight** of β is the maximum weight of the simple diagrams S_i with $\alpha_i \neq 0$. Let

$$\mathcal{F}_i = \{\beta \in K_{\mathfrak{D}}(F) | weight(\beta) \leq i\}.$$

It is clear that $\mathcal{F}_i * \mathcal{F}_j \leq \mathcal{F}_{i+j}$. Filtrations like this has been used by Marché [24], Lê [22] and Muller [25].

The associated graded object $\bigoplus_i \mathcal{F}_i/\mathcal{F}_{i-1}$ is a ring extension of $\mathfrak{D}$. The skeins associated to $f_S : C \to \mathbb{Z}_{\geq 0}$ of weight i form a basis for $\mathcal{G}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$. We denote the element of $\mathcal{G}_i$ associated to S by $[f_S]$ where $f_S : C \to \mathbb{Z}_{\geq 0}$ is the admissible coloring corresponding to S.

If S has weight i, and S' has weight j then the simple diagram of $f_S + f_{S'}$ has weight $i+j$. Let $p(S,S')$ be the number of positive smoothings, and $n(S,S')$ be the number of negative smoothings associated to the state of $S * S'$ with weight $i+j$. We define,

$$e(S,S') = p(S,S') - n(S,S').$$

The product in $K_{\mathfrak{D}}(F)$ descends to a bilinear map,

$$* : \mathcal{G}_i \otimes \mathcal{G}_j \to \mathcal{G}_{i+j}.$$

By Theorem 4.2, in the graded object,

$$[f_S] * [f_{S'}] = \zeta^{e(S,S')}[f_S + f_{S'}].$$

Remark. The filtration, and hence the graded object depends on the choice of ideal triangulation.

If $\beta \in K_{\mathfrak{D}}(F)$, then it has a unique expression as a finite sum $\beta = \sum_S \alpha_S S$ where S are skeins corresponding to simple diagrams, with no two isotopic to one another, and $\alpha_S \in \mathfrak{D}$. If $S_1, \ldots, S_n$ are the simple diagrams appearing with nonzero coefficient in the sum so that $weight(S_i) = weight(\beta)$, then the **symbol** of β is

$$\sigma(\beta) = \sum_i \alpha_{S_i}[f_{S_i}] \in \mathcal{G}_{weight(\beta)}.$$

Proposition 4.4. *The map,*

$$\sigma : K_{\mathfrak{D}}(F) \to \bigoplus_i \mathcal{G}_i,$$

satisfies,

$$\sigma(S * (\sum_i \alpha_i S_i)) = \sigma(S) * \sigma(\sum_i \alpha_i S_i)),$$

for all simple diagrams S and S_i, so that the collection $\{S_i\}$ is linearly independent. Furthermore,

$$weight(\beta + \gamma) \leq \max\{weight(\beta), weight(\gamma)\}.$$

If $weight(\beta) \neq weight(\gamma)$, then

$$weight(\beta + \gamma) = \max\{weight(\beta), weight(\gamma)\}.$$

Finally , if $weight(\beta) = weight(\gamma)$, and $\sigma(\beta) + \sigma(\gamma) \neq 0$, then

$$\sigma(\beta + \gamma) = \sigma(\beta) + \sigma(\gamma).$$

□

In [25], Muller defines a subtropical form that has similar properties.

Lemma 4.5. *Suppose that S_i is a collection of simple diagrams with associated admissible colorings $f_{S_i} : C \to \mathbb{Z}_{\geq 0}$ all having the same weight. Assume further that if $f_{S_i} + f_{S_j} = f_{S_k} + f_{S_l}$ then $\{i, j\} = \{k, l\}$. Let $\alpha_i, \beta_i \in \mathfrak{D}$. If,*

$$weight((\sum_i \alpha_i S_i) * (\sum_i \beta_i S_i)) < weight(\sum_i \alpha_i S_i) + weight(\sum_i \beta_i S_i).$$

then either all $\alpha_i = 0$, or all $\beta_i = 0$.

Proof. Suppose that the weight

$$(\sum_i \alpha_i S_i) * (\sum_i \beta_i S_i)$$

is less than $weight(\sum_i \alpha_i S_i) + weight(\sum_i \beta_i S_i)$. This means that the symbols cancel. Since $f_{S_i} + f_{S_j} = f_{S_k} + f_{S_l}$ if and only if $\{i,j\} = \{k,l\}$, the cancellations in the symbol of

$$(\sum_i \alpha_i S_i) * (\sum_i \beta_i S_i)$$

can be collected as $(\zeta^{e(S_i,S_j)}\alpha_i\beta_j + \zeta^{-e(S_i,S_j)}\alpha_j\beta_i)[f_{S_i} + f_{S_j}] = 0$ if $i \neq j$ and $\alpha_i\beta_i[2f_{S_i}] = 0$. If the set $\{S_i\}$ has n elements, then there are n equations

$$\alpha_i\beta_i = 0$$

and $\binom{n}{2}$ equations

$$\zeta^{e(S_i,S_j)}\alpha_i\beta_j + \zeta^{-e(S_i,S_j)}\alpha_j\beta_i = 0,$$

that the $\{\alpha_i\}$ and $\{\beta_i\}$ satisfy.

The first collection of equations implies for all i, either $\alpha_i = 0$ or $\beta_i = 0$, as $\mathfrak{D}$ has no zero divisors. Fix i. Without loss of generality we may assume that $\alpha_i \neq 0$. Thus $\beta_i = 0$. Using this informaton in the second collection of equations, for every $j \neq i$, $\beta_j = 0$, meaning all $\beta_j = 0$. □

Remark. A collection of skeins $\beta \in B$ spans $K_{\mathfrak{D}}(F)$ over $\mathfrak{D}$ if and only if the set $\sigma(\beta)$ where $\beta \in B$ spans the graded object. This is proved by induction on the weight of a skein.

Theorem 4.6. *Suppose that $\mathfrak{D}$ is an integral domain and $\zeta \in \mathfrak{D}$ is a unit and $2 \in \mathfrak{D}$ is a unit. Let S_i be a family of simple diagrams corresponding to the integral basis of* *the admissible colorings of an ideal triangulation. The skeins* $\{\prod_i T_{k_i}(S_i)\}$ *where the* k_i *range over all nonnegative integers, spans* $K_{\mathfrak{D}}(F)$ *over* $\mathfrak{D}$.

Proof. The symbol of $T_{k_1}(S_1) * T_{k_2}(S_2) * \ldots * T_{k_n}(S_n)$ is a power of ζ times a simple diagram corresponding to the admissible coloring $\sum_i k_i f_{S_i}$ where f_{S_i} is the admissible coloring corresponding to S_i. Since the symbols of these skeins correspond to all simple diagrams we can inductively rewrite any skein as a linear combination of these by starting at the terms of highest weight. □

This extends a theorem of Bullock [15]. In that paper it is proved that the arbitrary products of a finite collection of curves S_i spans. Our theorem is stronger because we can specify the order of the product of the S_i, as no matter what order we work in, the leading terms are the same, though maybe with different powers of ζ as the lead coefficient. It could be that the integral basis of the space of admissible colorings is not linearly independent over $\mathbb{Z}$, so we don't have that the products form a basis.

4.1.4 The case when ζ is a primitive $2N$th root of unity

Now we go on to study $K_N(F)$, meaning the coefficients are $\mathbb{Z}[\frac{1}{2}, \zeta]$, where ζ is a primitive $2N$th root of unity for some odd $N \geq 3$, and A is set equal to ζ. Recall, $\chi(F)$ is the image of the threading map

$$\tau : K_1(F) \to K_N(F).$$

The map τ threads every component of a framed link corresponding to a simple diagram with $T_N(x)$. Since

$$T_k(x) = \sum_{i=0}^{k/2} (-1)^i \frac{k}{k-i} \binom{k-i}{i} x^{k-2i},$$

the symbol of $\tau(S)$ of the simple diagram corresponding to $f_S : C \to \mathbb{Z}_{\geq 0}$ of weight i is $[Nf_S] \in \mathcal{G}_{Ni}$.

Let S be a simple diagram with associated coloring $f_S : C \to \mathbb{Z}_{\geq 0}$. Assume that f_S is not identically zero. The integers $\{f_S(c)\}_{c \in C}$ generate a subgroup of $\mathbb{Z}$, which being cyclic has a smallest positive generator, denoted $gcd(f_S)$.

Lemma 4.7. *If $n > 0$ is odd and $n|gcd(f_S)$ then $\frac{f_S}{n} : C \to \mathbb{Z}_{\geq 0}$ is an admissible coloring with associated simple diagram S' and $(S')^n = S \in K_N(F)$.*

Proof. Since F is orientable, the diagram S' is two sided so that we can push it completely off of itself to take the product. This means that the admissible coloring of $(S')^n$ is $nf_{S'} : C \to \mathbb{Z}_{\geq 0}$. If $f_S : C \to \mathbb{Z}_{\geq 0}$ is an admissible coloring, and for all $c \in C$, the odd integer $n|f(c)$, then for any $\{a, b, c\} = \partial\Delta$ of an embedded deal triangle in the triangulation, $\{f_S(a)/n, f_S(b)/n, f_S(c)/n\}$ satisfy all three triangle inequalities as the triangle inequality is linear. The sum $\frac{f_S(a)+f_S(b)+f_S(c)}{n}$ is even, as an even number divided by an odd number is even. Similarly, $\frac{f_S}{n} : C \to \mathbb{Z}_{\geq 0}$ satisfies the conditions to be admissible for folded triangles. □

Proposition 4.8. *Suppose that $s = \sum_j \alpha_j [f_{S_j}] \in \mathcal{G}_i$ where the $\alpha_j \in \mathbb{Z}[\frac{1}{2}, \zeta]$ and the $f_{S_i} : C \to \mathbb{Z}_{\geq 0}$ are distinct admissible colorings of weight i. The symbol s is the symbol of an element of $\chi(F)$ if and only if for all j, $N|gcd(f_{S_j})$.*

Proof. If $f_{S_i} : C \to \mathbb{Z}_{\geq 0}$ is the admissible coloring corresponding to S_i, by Lemma 4.7, so is $f_{S_i}/N : C \to \mathbb{Z}_{\geq 0}$. Denote the corresponding simple diagram by $(S/N)_i$,

$$\sigma(\tau(\sum_i \alpha_i(S/N)_i)) = \sigma(\sum_i \alpha_i T_N((S/N)_i)) = N\sigma(\sum_i \alpha_i(S/N)_i) = \sum_i \alpha_i[f_{S_i}].$$

On the other hand, if S is a simple diagram associated to the admissible coloring

$$f_S : C \to \mathbb{Z}_{\geq 0}$$

then the coloring associated with $\tau(S)$ is Nf_S, and N divides $gcd(Nf_S)$. □

Theorem 4.9. *Let F be a finite type surface. If S_i is any system of simple diagrams corresponding to an integral basis of the admissible colors, then the skeins $\prod_i T_{k_i}(S_i)$, where $k_i \in \{0, 1, \ldots, N-1\}$ span $K_N(F)$ over $\chi(F)$. In specific $K_N(F)$ is a finite ring extension of $\chi(F)$.*

Proof. The proof is by induction on the weight of the skein. Start with a skein written in terms of the basis over $\mathbb{Z}[\frac{1}{2}, \zeta]$ of simple diagrams,

$$\sum_j \alpha_j \prod_i T_{k_{i,j}}(S_{i,j}),$$

with $\alpha_j \in \mathbb{Z}[\frac{1}{2}, \zeta]$. Start with a term j of highest weight. Since

$$T_{N+k}(x) = T_N(x) * T_k(x) - T_{|N-k|}(x)$$

if some $k_{i,j} \geq N$ then as $\chi(F)$ is central, we can factor out an element of $\chi(F)$ from the term to get a simple diagram of lower weight. Continue on inductively till the skein is written as,

$$\sum_j \beta_j \prod_i T_{k_{i,j}}(S_{i,j})$$

where all $k_{i,j} \in \{0, 1, \ldots, N-1\}$, and $\beta_j \in \chi(F)$. □

After choosing an ideal triangulation for $F - \{p\}$, if the admissible colorings associated with S_i form an integral basis then $T_{k_1}(S_1) * \ldots * T_{k_n}(S_n)$ where $k_i \in \{0, \ldots, N-1\}$ span $K_N(F)$ over $\chi(F)$.

In chapter 2 we proved that $K_N(\Sigma_{1,0})$ is not free over $\chi(\Sigma_{1,0})$, hence there are definitely linear dependencies between the elements of the spanning set produced this way.

Theorem 4.10. *For every $\phi : \chi(F) \to \mathbb{C}$, $K_N(F)_\phi$ is a finite dimensional algebra over the complex numbers.*

□

4.2 Computing The Trace

Suppose that C is a properly embedded system of disjoint lines in the finite type surface F. A **monogon** is a component of the complement of C that completes to a closed disk with a single point removed from its boundary.

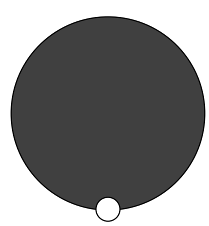


Figure 4.4: A monogon

A **bigon** is a component of the complement of C that completes to a closed disk with two points removed from its boundary.

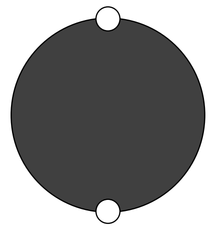


Figure 4.5: A bigon

Proposition 4.11. *Suppose that C is a properly embedded system of disjoint lines in the finite type surface F whose complement has no monogons or bigons. There exists D a collection of properly embedded lines so that $C \cup D$ defines an ideal triangulation of F.*

□

Suppose that $P \subset F$ is a primitive diagram. Place a complete Riemannian metric on F. We say that P **fills** F if the completion of every component of $F - P$ is either a once punctured disk, or a pair of pants. The maps from the completions into the surface can be $2-1$ on boundary components, so the closures of the complements are possibly, a punctured disk, a pair of pants $\Sigma_{0,3}$ or a surface of genus one with one boundary component, $\Sigma_{1,1}$.

The goal of this section is to prove that given a filling diagram P which is a union of components $\{S_k\}$, there is a collection of disjoint properly embedded lines $\{c_j\}$ and $\alpha_{j,m} \in \mathbb{Z}_N$ so that $I_m : \{S_k\} \to \mathbb{Z}_N$ given by $I^m(S_k) = \sum_j \alpha_{j,m} i(c_j, S_k) = \delta_k^m$, where δ_k^m is Kronecker's delta. The functions $\{I^m\}$ are "dual" to the components of the diagram. We are not requiring N to be prime, so $\mathbb{Z}_N$ has zero divisors, so care needs to be taken. For the construction to work, it is essential that N be odd, as there are linear dependencies between the components of P as elements of $H_1(F; \mathbb{Z}_2)$, which implies that the sums of the geometric intersection numbers with components that are linearly dependent must be even. Luckily 2 is a unit in $\mathbb{Z}_N$. The reciprocal of 2 is $\frac{N+1}{2}$, which will be used repeatedly in the construction.

If P fills F, there is a dual 1-dimensional CW-complex, with a 0-cell for every component of the complement of P and a 1-cell for every component of P. The trivalent 0-cells of the CW-complex correspond to components of the complement that complete to pants. The monovalent 0-cells correspond to components of the complement that complete to a punctured disk. If a 1-cell has both its endpoints at the same 0-cell, the corresponding simple closed curve is a nonseparating curve lying in the closure of a component of the complement of P that is homeomorphic to $\Sigma_{1,1}$. The CW-complex minus its valence one vertices can be properly embedded in the surface F, where each edge intersects the corresponding simple closed curve once in a transverse point of intersection and the trivalent vertices embedded in the corresponding components of the complement of P, and the ends of the deleted CW-complex mapped to the ends of the corresponding disk with a point deleted. The edges of the CW-complex are in one to one correspondence with the components of

P, if the edge e and the component S intersect one another we say that they are **dual**. The intersection is neccessarily a single point of transverse intersection.

Below is a twice punctured surface of genus three. The filling diagram is in blue, and the embedded dual graph is red.

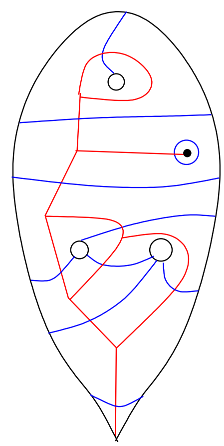


Figure 4.6: A filling diagram and its dual graph

Choose a maximal tree of the CW-complex and a valence one 0-cell. Orient the tree so that it is rooted at the chosen 0-cell. That is, every edge is oriented so that it points towards the root. The monovalent 0-cells of the tree that are sources are the **leaves** of the tree. The rooted tree is in red.

We will build a train track from this tree.

The diagram below is color coded so that each of the following steps is visible. First smooth the vertices ot the tree so that the two edges pointing into each interior 0-cell have the same outward pointing tangent vector. Next for each component of

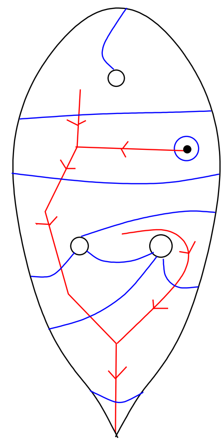


Figure 4.7: A maximal rooted tree

the diagram that doesn't bound a punctured disk, that is dual to an edge of the tree, push it off itself towards the root, and then put a kink in it where it intersects the edge dual to it and smooth the kink to get a switch where both outward normals of the curve at the kink point towards the root. These are in magenta. Next, add the remaining edges of the CW-complex, so that their outward normals, at the switches created, point towards the root. These are in green. If both endpoints of the edge are attached at the same 0-cell, that edge e lies in the closure of a component of the complement of P that is a torus. If S is the dual edge, push it off of itself and add a kink where it intersects e so that the outward tangent vectors point towards the vertex in the torus component. This is in brown. Suppose now that the 0-cells of the tree that e is attached at are distinct. For each one of those 0-cells that is a leaf, add a branch to the track, which is a pushoff of the dual component of P, with a kink in it that makes a switch in the train track pointing at that 0-cell. These are in yellow.

We produce a family of disjoint properly embedded lines by splitting the tree

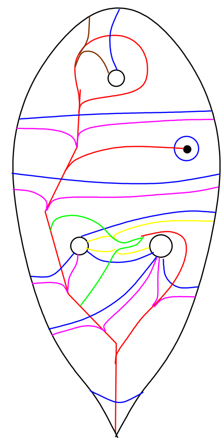


Figure 4.8: The train track

at the switches and cutting open all the way to the root. The switches in the tree point towards the roots, and the switches in the additional edges point towards the tree, so the process of cutting open along switches terminates at the root, and we have produced a family of disjoint properly embedded lines. The train track does not carry any simple closed curves.

Lemma 4.12. *Let F be a noncompact finite surface of negative Euler characteristic, and let C be a collection of lines built as above. There is a subset of these lines so that for any $S \in P$ there is a linear combination*

$$I^S = \sum_i \alpha_S^i i(c_i, \ )$$

where the $\alpha_i \in \mathbb{Z}_N$ so that for any components S, T of P, $I^S(T) \equiv \delta_S^T \mod N$, where δ_S^T is Kronecker's delta. This family can be seen to have no bigons or monogons in it's complement, so it can be built up to an ideal triangulation of F.

Proof. Order the components of P so that S, T are dual to edges in the tree then their relative order is consistent with their distance from the root of the tree, and if they aren't dual to edges of the tree then they come after all the components that are dual to edges of the tree. Working in order we prove that given S a component of P there is a line c_S in our family and $\beta_S \in \mathbb{Z}_N$ so that $\beta_S i(c_S, S) \equiv 1 \mod N$, and if $T > S$ then $c_S \cap T = \emptyset$, or we exchange order so that we can. The family I^S is then produced by taking the appropriate linear combinations of the $\beta_S i(c_S, \ )$. Since the lines c_S are indexed by S, the condition on intersections implies that no line is homotopically trivial (bounds a monogon) and no two lines are parallel (cobound a bigon), so the family c_S can be built up to a triangulation. The complication of the construction is that to construct the line for a given edge in the tree we need to understand what immediately follows the edge in the ordering.

We start at the root. If an edge leaving the root is leaf in the tree, there are three possible cases. The surface could be a once punctured torus, or a thrice punctured sphere, or the terminal points of the edge are at punctures, and the punctured disks containing those punctures abut the same pair of pants. The construction for the punctured torus, and thrice punctured pair of pants can be done by inspection. We focus on the last case, shown in the figure below.

According to our rules either the edge of the tree dual to the blue curve parallel to the outer boundary, or the line joining to the two punctures could come first. You really want the edge dual to the curve parallel to the outer boundary component to be first. If S_1 is the component of P that bounds the punctured disk at the root, let c_{S_1} be the line built from the branch of the track that follows the outer boundary

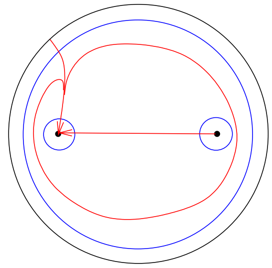
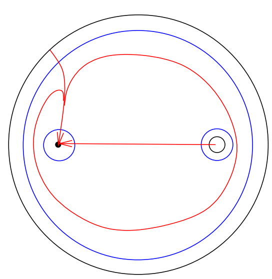


Figure 4.9: An edge that is simultaneously initial and a leaf

component before heading to the puncure. Notice $i(c_{S_1}, S_1) = 2$, so $\beta_{S_1} = \frac{N+1}{2}$ is the correct coefficient so that $\beta_{S_1} i(c_{S_1}, S_1) \equiv 1 \mod N$. The circle S_2 surrounding the other puncture has geometric intersection number one with the line c_{S_2} having one end at each puncture. Since the line c_{S_1} is completely inside the diagram we have that it has geometric intersection number 0 with all later curves.

Now suppose that the edge leaving the root is not a leaf. In the figure below we show the situation. The line c_{S_1} coming from the branch of the track that runs around the outer boundary component has geometric intersection number 2 with S_1 and misses all the other components of the filling diagram, so letting $\beta_{S_1} = \frac{N+1}{2}$, we have that $\beta_{S_1} i(c_{S_1}, \ )$ has $\beta_{S_1} i(c_{S_1}, S_1) \equiv 1 \mod N$ and for all later components it evaluates to 0.



An edge dual to S_i of the tree is intermediate if there is an edge dual to S_{i-1} before it, and an edge dual to S_{i+1} after it in the tree from the ordering. Let c_{S_i} be the line coming from the branch of the track that was built by perturbing S_{i+1}. Since, $i(c_{S_i}, S_i) = 2$, we let $\beta_S = \frac{N+1}{2}$. Do all intermediate edges before doing the leaves.

If an edge is a leaf, then it could end at a puncture, it could have both ends of an edge not part of the tree attached at its terminal 0-cell, or it could have two different edges not in the tree attached at its terminal end. These both occur in Figure 4.9 above. The highest leaf in the diagram is of the first type, and the lower leaf is of the second kind.

In the first case, the component S of P bounds a punctured disk. The line c_S of the train track that emanates from that puncture and ends at the root, has geometric intersection number 1 with S, we let $\beta_S = 1$.

In the second case, the vertex of the edge lies in a torus that is the closure of a component of $F - P$. Call the curve dual to the edge with both its ends attached at that 0-cell S'. The pushoff of S' gives rise to the line c_S that has geometric intersection 2 number with S that is dual to the edge, so $\beta_S = \frac{N+1}{2}$.

In the third case, let S' be the component of P that is dual to one of the edges attached at the leaf. The pushoff of S' towards the leaf gives rise to an embedded line that has geometric intersection number 2 with the curve S dual to the edge. Once again β_S is $\frac{N+1}{2}$.

Throw out any curves that weren't used. Augment to form an ideal triangulation. $\square$

Lemma 4.13. *Let P be a primitive diagram with components $\{S_i\}$. Choose an ideal triangulation as in Lemma 4.12 and let $I^j = \sum_i \alpha_i^j i(c_i, \ )$ be the linear combination of lines so that $I^j(S_k) = \delta_j^k$. Suppose that $\prod_{j \neq m} S_j^{k_j}$ is a monomial not involving m, then*

$$I^m(\prod_{j \neq m} S_j^{k_j} S_m^{k_m}) \equiv k_m \mod N.$$

If $a \in \chi(F)$ is nonzero, then $I^m(\sigma(a \prod_{j \neq m} S_j^{k_j} S_m^{k_m})) \equiv k_m \mod N$.

Proof. Since the curves S_i are disjoint from one another

$$I^m(\prod_{j \neq m} S_j^{k_j} S_m^{k_m}) = \sum_{j \neq m} I^m(S_j^{k_j}) + I^m(S_m^{k_m}) = \sum_{j \neq m} k_j I^m(S_j) + k_m I^m(S_m) \equiv$$

$$\sum_{j \neq m} k_j 0 + k_m 1 \equiv k_m \mod N.$$

For the second part, since $\prod_{j \neq m} S_j^{k_j} S_m^{k_m}$ is a simple diagram, $\sigma(a \prod_{j \neq i} S_j^{k_j} S_m^{k_m}) = \sigma(a)\sigma(\prod_{j \neq m} S_j^{k_j} S_m^{k_m})$. Since $a \neq 0$ its symbol corresponds to an admissible coloring that is divisible by N, $g : C \to N\mathbb{Z}_{\geq 0}$. If $f_{S_j} : C \to \mathbb{Z}_{\geq 0}$ is the coloring corresponding to S_j, the symbol of the product $\sigma(a)\sigma(\prod_{j \neq m} S_j^{k_j} S_m^{k_m})$ is,

$$\sum_j k_j f_{S_j} + g.$$

Since $i(c_m, \sum_j k_j f_{S_j} + g) = \sum_j k_j (f_{S_j}(c_m) + g(c_m))$,

$$I^m(\sigma(a \prod_{j \neq i} S_j^{k_j} S_m^{k_m})) \equiv I^m(\prod_{j \neq m} S_j^{k_j} S_m^{k_m})) \equiv k_i \mod N.$$

□

Proposition 4.14. *Let P be a primitive diagram with n components $\{S_j\}$. Choose S_m and form $\chi(F)[S_1, S_2, \ldots, \hat{S}_i, \ldots, S_n]$ meaning adjoin all S_j with $j \neq m$. If $q(x)$ is* *a nonzero polynomial with coefficients in* $\chi(F)[S_1, S_2, \ldots, \hat{S}_m, \ldots, S_n]$ *and* $q(S_m) = 0$ *then the degree of* q *is at least* N*.*

Proof. Suppose that $p(x) = \sum_l a_l x^l$ where the $a_l \in \chi(F)[S_1, S_2, \ldots \hat{S}_m, \ldots, S_n]$. Suppose further that $p(S_m) = 0$. Finally assume that the degree of p is less than N. Choose an ideal triangulation as in Lemma 4.12, with the I^k so that $I^k(S_j) = \delta_j^k$. Let $\sigma(a_l S_m^l)$ be the symbol of the lth term of $p(S_m)$. Since $p(S_m) = 0$ the symbols cancel. However since $I^m(\sigma(a_l S_m^l)) \equiv l \mod N$ and all the $0 \leq l < N$, no two are isotopic. Meaning they are all zero. If the symbol is zero then so is the polynomial. □

If $S \in F$ is a simple closed curve, there is a polynomial of degree N with coefficients in $\chi(F)$ that S satisfies. Specifically,

$$p(x) = \sum_{i=0}^{N/2} (-1)^i \frac{N}{N-i} \binom{N-i}{i} x^{N-2i} - T_N(S).$$

Before leaping into the final chain of theorems, we take pause to discuss the algebraic themes involved. They are well known, but bear repeating, to give shape to the argument.

Proposition 4.15. *Suppose that* K *is a field, and* $p_m(x)$ *with* $m \in \{1, \ldots n\}$ *are polynomials of positive degree with coefficients in* K*. Suppose further that the least degree nonzero polynomial satisfied by any* S_m *with coefficients in* $K[S_1, \ldots, \hat{S_m}, \ldots S_n]$ *is equal to the degree of* $p_m(x)$*, and that* R *is an commutative algebra over* K *that is generated by* S_m*, with* $p_m(S_m) = 0$*, then if* R *has no zero divisors it is a finite dimensional field extension of* K*, whose degree is the product of the degrees of the* $p_m(x)$*.*

Proof. The proof is by induction on n First the case $n = 1$, the evaluation map $K[x] \to R$ has as it's kernel a principal ideal $(q(x))$ where $q(x)|p_1(x)$. The polynomials over a field have Krull dimension 1 which implies prime ideals are maximal. If R has no zero divisors, $(q(x))$ is prime, so its maximal, and R is a field, and since $p_1(x)$ has minimal degree for all polynomials with coefficients in K that S_1 satisfies $(q(x)) = (p_1(x))$ and the degree of the extension is the degree of $p_1(x)$.

Assume the statement is true in the case of $n-1$ generators, and R has no zero divisors, and is generated by $S_1, \ldots, S_n$. The ring $K[S_1, \ldots, S_{n-1}]$ is a subring of R and has no zero divisors, so by the inductive hypothesis it is a field. If R is the result of adjoining S_n to the field $K[S_1, \ldots, S_{n-1}]$, and having no zero divisors, is a field extension of $K[S_1, \ldots, S_{n-1}]$ of degree equal to $p_n(x)$. Therefore $K[S_1, \ldots, S_n]$ is a finite dimensional field extension of K, whose degree over K is the product of the degrees of the $p_i(x)$.

□

Proposition 4.16. *With the same hypotheses as Proposition 4.15, R is naturally isomorphic to*

$$\otimes_K K[S_i].$$

Proof. Recall that $\otimes_K K[S_i]$ is an algebra over K of dimension equal to the product of the degrees of the $p_i(x)$. The map,

$$\psi : \otimes_K K[S_i] \to R$$

given by $\psi(S_1^{i_1} \otimes \ldots \otimes S_n^{i_n}) = S_1^{i_1} \ldots S_n^{i_n}$ is an onto algebra homomorphism, which is injective because the domain and range have the same dimension. □

The next proposition gives the method by which we will be computing the trace.

Proposition 4.17. *Suppose that $K \leq P$ is a finite dimensional field extension and J is a finite dimensional algebra over P. Thus J is a finite dimensional algebra over K. If $s \in P$, then it defines a K-linear maps $l_s : P \to P$, and $L_s : J \to J$ by left multiplication. If d is the dimension of J over P, then*

$$tr(L_s) = dtr(l_s),$$

where the traces are both taken as linear maps over K.

Proof. Since $K \leq P$ is finite dimensional it has basis $p_1, \ldots p_n$ over K. Since J is a finite dimensional vector space of dimension d over P it has basis $j_1, \ldots j_d$ over P. This implies that $p_a j_c$ is a basis of J over K. Expressing l_s respect to the basis p_a, we get

$$l_s(p_a) = \sum_b l^a_b p_b.$$

Since s acts as scalar multiplicationon J,

$$L_s(p_a j_c) = \sum_b l^a_b p_b j_c.$$

Hence the matrix for L_s decomposes into d blocks that are all copies of the matrix for l_s. Therefore the trace of L_s is equal to d times the trace of l_s. □

Lets get to work.

Proposition 4.18. *Suppose that S_i is a system of disjoint simple closed curves on the surface of finite type, F, no two of which are parallel. That is, $\prod_i S_i$ is a primitive diagram. Then $S^{-1}\chi(F)[S_i]$ has no zero divisors.*

Proof. Choose an ideal triangulation cut out by C, so that there are $I^i = \sum_j \alpha^i_j c_j \in C$ with $I^i(S_k) = \sum_j \alpha^i_j i(c_j, S_k) \equiv \delta^k_i \mod N$, as given by Lemma 4.12, suppose that

$$(\prod_i \alpha_i S_i)(\prod_i \beta_i S_i) = 0,$$

where $\alpha_i, \beta_i \in \chi(F)$. (We can always clear fractions.) Since the symbols of elements of $\chi(F)$ are divisible by N, if $f_{S_i} : C \to \mathbb{Z}_{\geq 0}$ are the admissible colorings associated to S_i, there are admissible colorings $g_i, h_i : C \to \mathbb{Z}_{\geq 0}$ so that

$$\sigma(\alpha_i S_i) = z_i[f_{S_i} + N g_i]$$

and

$$\sigma(\beta_i S_i) = w_i[f_{S_i} + N h_i],$$

where $z_i, w_i \in \mathbb{Z}[\frac{1}{2}, \zeta]$.

Supposing that $f_{S_i} + Ng + f_{S_j} + Nh = f_{S_k} + Nm + f_{S_l} + Np$ for any functions $g, h, m, p : C \to \mathbb{Z}_{\geq 0}$. This means that evaluating both sides on I^i, I^j, I^k, I^l will give the same answer modulo N. The only way this can happen is if $\{i, j\} = \{k, l\}$, so the hypotheses of Lemma 4.5 are satisfied. Therefore,

$$(\prod_i \alpha_i S_i)(\prod_i \beta_i S_i),$$

can only be zero if either the symbols of the α_i are all zero, which means the α_i are all zero or if the symbols of the β_i are all zero which means the β_i are all zero. $\square$

Theorem 4.19. *If S_i, with $i \in \{1, \ldots, n\}$ is a system of simple closed curves on F that forms a primitive diagram then $\chi(F)[S_1, \ldots, S_n]$ is a field of dimension N^n, and*

$$\chi(F)[S_1, \ldots, S_n] \cong \otimes_{\chi(F)}\chi(F)[S_i].$$

Proof. By Proposition 4.18 , we can apply Propositions 4.15 and 4.16. □

Theorem 4.20. *Given S_i a system of disjoint simple closed curves on the surface of finite type, F, no two of which is parallel, $S^{-1}K_N(F)$ is a finite dimensional algebra over $S^{-1}\chi(F)[S_i]$.*

Proof. By the theorem in the last section $K_N(F)$ is a finitely generated module over $\chi(F)$. Localizing this means $S^{-1}K_N(F)$ is a finite dimensional vector space over $S^{-1}\chi(F)$. Since $S^{-1}\chi(F) \leq S^{-1}\chi(F)[S_i] \leq S^{-1}K_N(F)$, we have that $S^{-1}K_N(F)$ is finite dimensional over $S^{-1}\chi(F)[S_i]$. □

If $S \subset F$ is a nontrivial simple closed curve, let $\Sigma_{0,2}(S)$ be an annular neighborhood of S in F. There is a left action of $K_N(\Sigma_{0,2}(S)) \otimes K_N(F) \to K_N(F)$ by gluing a copy of $\Sigma_{0,2}(S) \times [0,1]$ onto the top of $F \times [0,1]$. Notice that it restricts to give and action $\chi(\Sigma_{0,2}(S))$ on $\chi(F)$ making $S^{-1}\chi(\Sigma_{0,2}(S)) \leq S^{-1}\chi(F)$ a field extension.

Remark. It is worth mentioning that

$$S^{-1}\chi(\Sigma_{0,2}(S))[S] = S^{-1}K_N(\Sigma_{0,2}(S)).$$

Theorem 4.21. *$S^{-1}\chi(F)[S]$ is the result of extending the coefficients of $S^{-1}\chi(\Sigma_{0,2}(S))[S]$ as a vector space over $S^{-1}\chi(\Sigma_{0,2}(S))$ to a vector space over $S^{-1}\chi(F)$.*

Proof. The dimension of $S^{-1}\chi(\Sigma_{0,2}(S))[S]$ over $S^{-1}\chi(\Sigma_{0,2}(S))$ is equal to the dimension of $S^{-1}\chi(F)[S]$ over $S^{-1}\chi(F)$, so the map,

$$S^{-1}\chi(\Sigma_{0,2}(S))[S] \otimes_{S^{-1}\chi(\Sigma_{0,2})} S^{-1}\chi(F) \to \chi(F)[S]$$

that sends $S \otimes 1$ to S is a linear isomorphism. □

In Chapter 2 Proposition 2.3, we proved that if $\Sigma_{0,2}$ is an annulus and x is the skein at its core, and

$$tr : K_N(\Sigma_{0,2}) \to \chi(\Sigma_{0,2})$$

is the unormalized trace, $tr(L_{T_k(x)}) = 0$ unless $k|N$ at which point $tr(L_{T_{aN}(x)}) = NT_{aN}(x)$.

This implies the same result for $T_k(S) : S^{-1}K_N(\Sigma_{0,2}(S)) \to S^{-1}K_N(\Sigma_{0,2}(S))$.

Proposition 4.22. *Let $S \subset F$ be a nontrivial simple closed curve. Define $L_{T_k(S)}$: $S^{-1}\chi(F)[S] \to S^{-1}\chi(F)[S]$ by left multiplication, then $tr(L_{T_k(S)}) = 0$ unless $k|N$ at which point $tr(L_{T_{k(S)}}) = NT_k(S)$.*

Proof. The map $L_{T_k(S)} : S^{-1}\chi(F)[S] \to S^{-1}\chi(F)[S]$ comes from

$$L_{T_k(S)} : S^{-1}\chi(\Sigma_{0,2}(S))[S] \to S^{-1}\chi(\Sigma_{0,2}(S))[S]$$

by extension of scalars, and the fact that $\chi(\Sigma_{0,2}(S))[S] = K_N(\Sigma_{0,2}(S))$. □

Proposition 4.23. *Let $\prod_i T_{k_i}(S_i)$ act on $S^{-1}\chi(F)[S_1, \ldots, S_n]$ by multiplication*

$$L_{\prod_i T_{k_i}(S_i)} : S^{-1}\chi(F)[S_1, \ldots, S_n] \to S^{-1}\chi(F)[S_1, \ldots, S_n],$$

the unnormalized trace of $L_{k_1,\ldots,k_n}$ is zero unless $N|k_i$ for all i, in which case it is $N^n \prod_i T_{k_i}(S_i)$.

Proof. ψ is the natural isomorphism, commutes. This means that the trace of $L_{\prod_i T_{k_i}(S_i)}$ is the product of the traces of the

$$L_{T_{k_i}(S_i)} : \chi(\Sigma_{0,2}(S_i))[S_i] \otimes_{\chi(\Sigma_{0,2}(S_i))} \chi(F) \to \chi(\Sigma_{0,2}(S_i))[S_i] \otimes_{\chi(\Sigma_{0,2}(S_i))} \chi(F)$$

which are obtained by extension of scalars from $L_{T_{k_i}(S_i)} : K_N(\Sigma_{0,2}(S_i)) \to K_N(\Sigma_{0,2}(S_i))$.

□

Theorem 4.24. *Suppose that* $d = [S^{-1}\chi(F)[S_1, \ldots, S_n] : S^{-1}K_N(F)]$. *The unnormalized trace of*

$$L_{k_1,\ldots,k_n} : S^{-1}K_N(F) \to S^{-1}K_N(F)$$

is zero unless $N|k_i$ *for all* i *in which case it is* $dN^n \prod_i T_{k_i}(S_i)$.

Proof. By Theorem 4.20 $S^{-1}K_N(F)$ is a finite dimensional vector space over

$$S^{-1}\chi(F) \leq S^{-1}\chi(F)[S_1, \ldots, S_n].$$

so Proposition 4.17 applies. □

We define the normalized trace

$$Tr : S^{-1}K_N(F) \to S^{-1}\chi(F),$$

to be the trace divided by dN^n. The map Tr is $S^{-1}\chi(F)$ linear, cyclic, and $Tr(1) = 1$.

Theorem 4.25. *Suppose that* $s = \sum_i \beta_i P_i$ *where the* $\beta \in S^{-1}\chi(F)$ *and the* P_i *are primitive diagrams whose components have been threaded with* T_k. *Let* J *be those indices* i *so that the components of* P_i *have only been threaded with* T_k *where* $N|k$, *then,*

$$Tr(s) = \sum_{i \in J} \beta_i P_i.$$

Theorem 4.26. *The restriction of* $Tr : S^{-1}K_N(F) \to S^{-1}\chi(F)$ *to* $K_N(F)$*, embedded in* $S^{-1}K_N(F)$ *as fractions having denominator* 1 *yields,*

$$Tr : K_N(F) \to \chi(F).$$

which is a $\chi(F)$*-linear map, so that* $Tr(1) = 1$ *and for every* $\alpha, \beta \in K_N(F)$*,*

$$Tr(\alpha * \beta) = Tr(\beta * \alpha).$$

Proof. From the formula for Tr, the only fractions that appear in the coefficients in the trace come from fractions that are in the coefficients of the skein. □

4.3 The Trace Is Nondegenerate

Lemma 4.27. *Let* F *be a finite type surface with an ideal triangulation cut out by* C*. Suppose that* $\sum_i z_i S_i \in K_N(F)$*, where the* $z_i \in \mathbb{Z}[\frac{1}{2}, \zeta]$ *and the* S_i *are distinct simple diagrams. If for some* $[f_S]$ *appearing in the symbol of* $\sum_i z_i S_i$ *with nonzero coefficient* z*, has* $N|gcd(f_S)$*, then*

$$Tr(\sum_i z_i S_i) \neq 0.$$

Proof. Suppose that the primitive diagram P underlying S is made up of simple closed curves S'_j. The threaded diagram having lead coefficient S is $\prod_j T_{Nk_j}(S'_j)$ for some $k_j \in \mathbb{Z}_{\geq 0}$. Rewriting $\sum_i z_i S_i$ in terms of threaded diagrams, the threaded diagrams appearing in the symbol appear with the same coefficients and are distinct from one another in the sum. Hence $\prod_j T_{Nk_j}(S'_j)$ appears in the trace with coefficient $z \neq 0$. This term can't cancel with other highest weight terms in the trace, as the S_i were distinct nor can it cancel with lower weight terms, as that would violate the filtration of $\chi(F)$, so $Tr(\sum_i z_i S_i) \neq 0$. □

Theorem 4.28. *Let F be a noncompact, finite type surface. There are no nontrivial principal ideals in the kernel of*

$$Tr : K_N(F) \to \chi(F).$$

Proof. Suppose that $(\sum_i \alpha_i P_i)$ is a two-sided principal ideal in $K_N(F)$ where $\alpha_i \in \chi(F)$, and the P_i are primitive diagrams whose components have been threaded with Tchebychev polynomials of the first type where the threadings come from $\{0, \dots, N-1\}$. Also assume that $\sum_i \alpha_i P_i \neq 0$, and the diagrams $\{P_i\}$ are a linearly independent set over $\chi(F)$. Rewrite the sum as $\sum_j z_j S_j$ where the $z_j \in \mathbb{Z}[\frac{1}{2}, \zeta]$, and the S_j are distinct simple diagrams. Choose an ideal triangulation C. As the skein $\sum_j z_j S_j \neq 0$, it's symbol is nonzero. If some $f_S : C \to \mathbb{Z}_{\geq 0}$ appearing in its symbol with nonnegative coefficient has $N | gcd(f_S)$ we are done by the last lemma. If not, choose $f_S : C \to \mathbb{Z}_{\geq 0}$ appearing in the symbol with nonzero coefficient. We can write $S = \prod_m J_m^{a_m N + r_m}$ where the J_m are a disjoint system of simple closed curves, $a_m \in \mathbb{Z}_{\geq 0}$ and $r_m \in \{0, \dots, N-1\}$. Consider the simple diagram,

$$\prod_m J_m^{N-r_m}.$$

Since $(\sum_i \alpha_i P_i)$ is an ideal,

$$\prod_m J^{N-r_m} * \sum_j z_j S_j \in (\sum_i \alpha_i P_i).$$

If $\sum_i z_i [f_{S_i}]$ is the symbol of $\sum_j z_j S_j$, then the symbol of the product is

$$\sum_i z_i \zeta^{e(\prod_m J^{N-r_m}, S_i)} [f_{S_i} + f_{\prod_m J^{N-r_m}}].$$

The coefficient of $[f_S + f_{\prod_m J^{N-r_m}}]$ is nonzero, and $N|gcd(f_S + f_{\prod_m J^{N-r_m}})$ as the corresponding diagram is,

$$\prod_m J_m^{(a_m+1)N}.$$

By the last lemma, the ideal $(\sum_i \alpha_i P_i)$ contains an element with nonzero trace. $\square$

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