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Resonance sums for Rankin-Selberg products

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RESONANCE SUMS FOR RANKIN-SELBERG PRODUCTS

by

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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To my wife, Sarah, and my son, Nikolai, for their love and support.

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ABSTRACT

Consider either (i) $f = f_1 \boxtimes f_2$ for two Maass cusp forms for $SL_m(\mathbb{Z})$ and $SL_{m'}(\mathbb{Z})$, respectively, with $2 \leq m \leq m'$, or (ii) $f = f_1 \boxtimes f_2 \boxtimes f_3$ for three weight $2k$ holomorphic cusp forms for $SL_2(\mathbb{Z})$. Let $\lambda_f(n)$ be the normalized coefficients of the associated L-function $L(s, f)$, which is either (i) the Rankin-Selberg L-function $L(s, f_1 \times f_2)$, or (ii) the Rankin triple product L-function $L(s, f_1 \times f_2 \times f_3)$. First, we derive a Voronoi-type summation formula for $\lambda_f(n)$ involving the Meijer G-function. As an application we obtain the asymptotics for the smoothly weighted average of $\lambda_f(n)$ against $e(\alpha n^\beta)$, i.e. the asymptotics for the associated resonance sums. Let ℓ be the degree of $L(s, f)$. When $\beta = \frac{1}{\ell}$ and α is close or equal to $\pm \ell q^{\frac{1}{\ell}}$ for a positive integer q, the average has a main term of size $|\lambda_f(q)| X^{\frac{1}{2\ell}+\frac{1}{2}}$. Otherwise, when α is fixed and $0 < \beta < \frac{1}{\ell}$ it is shown that this average decays rapidly. Similar results have been established for individual $SL_m(\mathbb{Z})$ automorphic cusp forms and are due to the oscillatory nature of the coefficients $\lambda_f(n)$.

PUBLIC ABSTRACT

Automorphic forms can be thought of as a generalization of classical trigonometric and elliptic functions. The latter are periodic functions defined on the complex numbers, whereas the former are invariant functions defined on more general topological groups. Examples of automorphic forms include modular forms and Maass forms, both of which are discussed in this thesis. These automorphic forms all admit a Fourier expansion because they are well-behaved and periodic. Moreover, attached to each one of these forms is a special function, called the L-function, which is defined as a series involving the Fourier coefficients. These L-functions are generalizations of important fundamental number theoretic objects like the Riemann zeta function and the Dirichlet L-series.

It is possible to combine automorphic forms in a variety of ways to construct new forms. One of the most straightforward constructions is to take the product of two or more automorphic forms. Unfortunately, it is not known whether or not the resulting form is automorphic, although this is suspected to be true and is known in a few select cases. This problem is referred to as the Langlands Functoriality Conjecture and remains one of the most difficult unsolved problems in modern number theory. This thesis establishes a collection of summation formulas involving the L-series coefficients attached to the product of several automorphic forms. As an application it is shown that these L-series coefficients have the exact properties that one would expect if the products were indeed automorphic.

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CHAPTER 1
INTRODUCTION

1.1 Motivation

Consider the following weighted exponential sum

$$\sum_{n=1}^{\infty} \lambda_f(n)\phi\left(\frac{n}{X}\right) e(\alpha n^{\beta}) \tag{1.1}$$

where the $\lambda_f(n)$ are coefficients attached to an object of interest, $\phi(x)$ is a compactly supported function, $e(x) = e^{2\pi i x}$, and $\alpha, \beta, X \in \mathbb{R}$ are parameters to be specified later. The goal is to examine the oscillatory behavior of the coefficients $\lambda_f(n)$. One way this is accomplished is through the study of weighted sums of these coefficients against various exponential functions whose oscillatory behavior is well known. The constructive and destructive interference of these oscillations give rise to resonance and decay, respectively. Consequently, we refer to the summation in equation (1.1) as a *resonance sum.*

Much has been done recently in the case where f is an automorphic form or representation. The following cases are currently known:

- When f is an automorphic representation of $GL_m(\mathbb{A}_{\mathbb{Q}})$ and $\lambda_f(n)$ are the coefficients of the associated L-function, Booker [3] showed that (1.1) has rapid decay for $\alpha = 0$.
- When f is a $SL_2(\mathbb{Z})$ cusp form and $\lambda_f(n)$ are the coefficients of the associated L-function (or, equivalently, the Fourier-Whittaker coefficients), Iwaniec, Luo, and Sarnak [16] and later Ren and Ye [27] gave asymptotics and upper bounds

of (1.1) for $\alpha \neq 0$.

- When f is a $SL_3(\mathbb{Z})$ Maass (cusp) form and $\lambda_f(n)$ are the coefficients of the associated L-function (or, equivalently, the Fourier-Whittaker coefficients), Ernvall-Hytönen [7] and later Ren and Ye [29, 30] gave asymptotics and upper bounds of (1.1) for $\beta = 1$ and $\alpha, \beta \geq 0$, respectively.
- When f is a $SL_m(\mathbb{Z})$ Maass cusp form and $\lambda_f(n)$ are the coefficients of the associated L-function (or, equivalently, the Fourier-Whittaker coefficients), Ren and Ye [28] gave asymptotics and upper bounds of (1.1) for $\alpha, \beta \geq 0$. Concurrently and independently, Ernvall-Hytönen et al. [8] gave asymptotics and Ω-results of (1.1) for $\beta = 1$.
- When f is an element of degree d of the extended Selberg class of functions and $\lambda_f(n)$ are the coefficients of the associated L-function, Kaczorowski and Perelli [18] gave asymptotics of (1.1) for $\alpha > 0$ and $\beta = \frac{1}{d}$ under various hypotheses.

This thesis is a continuation of these investigations for the following two cases:

- When $f = f_1 \boxtimes f_2$ is the product of two Maass cusp forms for $SL_m(\mathbb{Z})$ and $SL_{m'}(\mathbb{Z})$, respectively, and $\lambda_f(n)$ are the coefficients of the Rankin-Selberg L-function.
- When $f = f_1 \boxtimes f_2 \boxtimes f_3$ is the product of three holomorphic cusp forms for $SL_2(\mathbb{Z})$ and $\lambda_f(n)$ are the coefficients of the Rankin triple product L-function.

In both of these cases we first derive the following Voronoi-type summation formula.

Theorem 1.1. *Assume one of the following cases:*

(I) f is a Maass cusp form for $SL_m(\mathbb{Z})$.

(II) $f = f_1 \boxtimes f_2$ is the product of two Maass cusp forms for $SL_m(\mathbb{Z})$ and $SL_{m'}(\mathbb{Z})$, respectively.

(III) $f = f_1 \boxtimes f_2 \boxtimes f_3$ is the product of three holomorphic cusp forms for $SL_2(\mathbb{Z})$.

Let $\tilde{f}$ denote the dual of f. Let $\lambda_f(n)$ be the coefficients of the associated L-function, and assume further that this L-function is entire. Let $\psi(x) \in C^\infty(0,\infty)$ with compact support, then

$$\sum_{n=1}^{\infty} \lambda_f(n)\psi(n) = \kappa_\ell \sum_{n=1}^{\infty} \lambda_{\tilde{f}}(n) \int_0^\infty \psi(x) G_{\tilde{f}}(\pi^{2\ell} x^2 n^2) dx \tag{1.2}$$

where ℓ is the degree of the L-function, κ_ℓ is a constant depending on ℓ, and

$$G_{\tilde{f}}(x) = G^{\ell,0}_{0,2\ell}\left(\begin{matrix} - \\ \boldsymbol{b}_j \end{matrix} \,\middle|\, x \right).$$

Here $G^{\ell,0}_{0,2\ell}$ is the Meijer G-function, and $\boldsymbol{b}_j$ are parameters depending on f.

The proof as well as the explicit expressions for κ_ℓ and $\mathbf{b}_j$ in each case will be given later in their respective chapters.

The particular choice of $\psi(x) = e(\alpha x^\beta)\phi\left(\frac{x}{X}\right)$ in equation (1.2) gives an exact formula for the resonance sum under consideration. With this choice of ψ we refer to equation (1.2) as a Voronoi-type summation formula because it is similar to the full Voronoi summation formulas of Goldfeld and Li [11, 12] and of Miller and Schmid [25, 26] discussed in the Appendix. This Voronoi-type summation formula, together with the known asymptotics of the Meijer G-function, yield the following theorem.

Theorem 1.2. *Assume one of the following cases:*

(I) f is a Maass cusp form for $SL_m(\mathbb{Z})$.

(II) $f = f_1 \boxtimes f_2$ is the product of two Maass cusp forms for $SL_m(\mathbb{Z})$ and $SL_{m'}(\mathbb{Z})$, respectively.

(III) $f = f_1 \boxtimes f_2 \boxtimes f_3$ is the product of three holomorphic cusp forms for $SL_2(\mathbb{Z})$.

Let $\tilde{f}$ denote the dual of f. Let $\lambda_f(n)$ be the coefficients of the associated L-function, and assume further that this L-function is entire. Let $\phi(x) \in C^\infty(0,\infty)$ with compact support in $[1,2]$ and $\phi^{(j)}(x) \ll 1$ for $j \geq 1$. Moreover, let ℓ be the degree of the associated L-function, $X > 1$, and $\alpha, \beta \geq 0$.

(i) If $2\max\{1, 2^{\beta-\frac{1}{\ell}}\}(\alpha\beta)^\ell \leq X^{1-\beta\ell}$, then

$$\sum_{n=1}^{\infty} \lambda_f(n)\phi\left(\frac{n}{X}\right) e(\pm\alpha n^\beta) \ll_{\ell,\beta,M} X^{-M}$$

holds for any $M > 0$.

(ii) If $2\max\{1, 2^{\beta-\frac{1}{\ell}}\}(\alpha\beta)^\ell > X^{1-\beta\ell}$, then

$$\sum_{n=1}^{\infty} \lambda_f(n)\phi\left(\frac{n}{X}\right) e(\pm\alpha n^\beta) \ll_{\ell,\beta} (\alpha X^\beta)^{\frac{\ell}{2}}$$

holds for $\beta \neq \frac{1}{\ell}$, and

$$\sum_{n=1}^{\infty} \lambda_f(n)\phi\left(\frac{n}{X}\right) e(\pm\alpha n^\beta) \ll_{\ell,\beta} (\alpha X^\beta)^{\frac{1+\ell}{2}}$$

holds for $\beta = \frac{1}{\ell}$.

(iii) *If* $X > \alpha^{\frac{\ell(\ell-1)}{1-\ell\varepsilon}}$ *with* $0 < \varepsilon < \frac{1}{\ell}$, *then*

$$\sum_{n=1}^{\infty} \lambda_f(n)\phi\left(\frac{n}{X}\right) e(\pm\alpha n^{\frac{1}{\ell}}) = \frac{\lambda_{\tilde{f}}(n_\alpha)}{n_\alpha} \times \sum_{k=0}^{r} c_{k,\ell}^{-} I_k(n_\alpha; -)(n_\alpha X)^{\frac{1}{2\ell}+\frac{1}{2}-\frac{k}{\ell}} + O_{\ell,r,\varepsilon}\left(X^{\frac{1}{2\ell}+\frac{1}{2}-\frac{r+1}{\ell}}\right)$$

for any $r > \frac{\ell-1}{2}$. *Here* n_α *is the unique positive integer satisfying* $\left(\frac{\alpha}{\ell}\right)^\ell - n_\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right]$,

$$I_k(n_\alpha; -) = \int_0^\infty \phi(t^\ell) e\left(\left(\alpha - n_\alpha^{\frac{1}{\ell}}\ell\right) X^{\frac{1}{\ell}} t\right) t^{\frac{\ell}{2}-\frac{1}{2}-k} dt,$$

and $c_{k,\ell}^{-}$ *are constants depending on* f, k, *and* ℓ.

(iv) *In particular, if* q *is a positive integer and* $0 < \varepsilon < \frac{1}{\ell}$, *then for* $X > \left(\ell^\ell q\right)^{\frac{\ell-1}{1-\ell\varepsilon}}$ *we have*

$$\sum_{n=1}^{\infty} \lambda_f(n)\phi\left(\frac{n}{X}\right) e(\pm\ell(qn)^{\frac{1}{\ell}}) = \frac{\lambda_{\tilde{f}}(q)}{q} \times \sum_{k=0}^{r} c_{k,\ell}^{-} I_k(q; -)(qX)^{\frac{1}{2\ell}+\frac{1}{2}-\frac{k}{\ell}} + O_{\ell,r,\varepsilon}\left(X^{\frac{1}{2\ell}+\frac{1}{2}-\frac{r+1}{\ell}}\right)$$

for any $r > \frac{\ell-1}{2}$ *where*

$$I_k(q; -) = \frac{1}{\ell}\int_0^\infty \phi(x) x^{\frac{1}{2\ell}-\frac{1}{2}-\frac{k}{\ell}} dx$$

and $c_{k,\ell}^{-}$ *are constants depending on* f, k, *and* ℓ.

Note that for fixed α and $0 < \beta < \frac{1}{\ell}$ case (i) holds and we have rapid decay. We conclude this section with a few remarks about these results.

Remark 1.1. *The condition that the associated* L*-function be entire may be dropped in both Theorems 1.1 and 1.2 with appropriate modifications. Indeed, the only complication of this case is the possible existence of poles in the completed* L*-function* *which will lead to a residual term, say $\mathcal{R}$, in equation (1.2). It can then be shown that $\mathcal{R} \ll_{\ell,f} \int_0^\infty e(\alpha x^\beta)\phi\left(\frac{x}{X}\right)dx$ which can then be made negligible via repeated integration by parts.*

Remark 1.2. *Theorems 1.1 and 1.2 demonstrate that the L-function coefficients (and therefore the Fourier-Whittaker coefficients) of the Rankin-Selberg product of a $SL_m(\mathbb{Z})$ form and a $SL_{m'}(\mathbb{Z})$ form exhibit the same behavior as those of a $SL_{mm'}(\mathbb{Z})$ form. This provides indirect evidence for functoriality in that these theorems precisely are what one would expect given the known resonance of cusp forms.*

1.2 Overview

In Chapter 2 we fix our notation and recall some of the necessary background material. First, we discuss $SL_2(\mathbb{Z})$ modular forms in Section 2.1 and then $SL_m(\mathbb{Z})$ Maass forms in Section 2.2. In both of these sections we give the Fourier-Whittaker expansion and define the associated L-function. We conclude Chapter 2 with the definition and asymptotic expansion of the Meijer-G function in Section 2.3. Chapter 3 begins with the definition and properties of the Rankin-Selberg L-function of two Maass forms in Section 3.1. Next, we give the proofs of Theorem 1.1 and Theorem 1.2 for the Rankin-Selberg product case in Section 3.2 and Section 3.3, respectively. Chapter 4 is structured similarly and is devoted to the case of the Rankin triple product of three $SL_2(\mathbb{Z})$ holomorphic cusp forms. Finally, in the Appendix we recall the full Voronoi summation formula and discuss the case of an individual Maass cusp form.

CHAPTER 2
BACKGROUND

2.1 Modular Forms for $SL_2(\mathbb{Z})$

In this section we fix our notation and recall a few basic results in the theory of $SL_2(\mathbb{Z})$ modular forms which will be needed for Chapter 4. For more details we refer the reader to one of the many great references such as Bump [5], Bump et al. [4], Diamond & Shurman [6], and Iwaniec [14].

Definition 2.1. *A modular form of weight $k \in \mathbb{Z}^+$ for $SL_2(\mathbb{Z})$ is a holomorphic function f on the complex upper half-plane $\mathfrak{h}^2$ that satisfies the following two conditions:*

(i) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ *for all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

(ii) $f(z)$ *is holomorphic at the cusp at infinity.*

Note that the first condition implies that any odd weight modular form is identically zero. The second condition is interpreted in the following way. First, observe that $f(z+1) = f(z)$ since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$; consequently, f has a Fourier expansion of the form

$$f(z) = \sum_{n \in \mathbb{Z}} A(n) e(nz) = \sum_{n \in \mathbb{Z}} A(n) q^n \tag{2.1}$$

where $e(z) = e^{2\pi i z} = q$. If $A(n) = 0$ for $n < 0$ then we say that f is holomorphic at infinity. If, in addition, $A(0) = 0$ then we say f is cuspidal at infinity and call f a cusp form.

The standard example of a weight k modular form is the holomorphic Eisenstein series $E_k(z)$ defined for even integers $k \geq 4$ as

$$E_k(z) := \frac{1}{2} \sum_{\substack{n,m \in \mathbb{Z} \\ (n,m) \neq (0,0)}} \frac{1}{(mz+n)^k}.$$

The first condition of Definition 2.1 is satisfied because

$$E_k\left(\frac{az+b}{cz+d}\right) = (cz+d)^k \cdot \frac{1}{2} \sum_{\substack{n,m \in \mathbb{Z} \\ (n,m) \neq (0,0)}} \frac{1}{((ma+nc)z + (mb+nd))^k}$$

and this sum is just $E_k(z)$ after reindexing (which is possible since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible). The second condition of Definition 2.1 follows form the Fourier expansion

$$E_k(z) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where $\zeta(k)$ is the Riemann zeta function and $\sigma_k(n) = \sum_{d|n} d^k$ is the divisor sum. We can normalize $E_k(z)$ by defining $G_k(z) = \zeta(k)^{-1} E_k(z)$.

It is possible to construct modular cusp forms as well by using these Eisenstein series. For example,

$$\Delta(z) := \frac{G_4^3(z) - G_6^2(z)}{1728} = \sum_{n=1}^{\infty} \tau(n) q^n$$

is a weight 12 cusp form known as the modular discriminant. The Fourier coefficients $\tau(n)$ define an arithmetic function known as Ramanujan's tau function. Ramanujan conjectured that $\tau(n)$ satisfied the following properties:

- $\tau(nm) = \tau(n)\tau(m)$ whenever $(n, m) = 1$.
- $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$ for primes p and $r \in \mathbb{Z}^+$.
- $\tau(p) \leq 2p^{\frac{11}{2}}$ for primes p.

These properties are characteristic of Fourier-Whittaker coefficients for automorphic forms and are not unique to the Ramanujan tau function. The first two properties are the multiplicativity relations; they were first proved for $\tau(n)$ by Mordell and later by Hecke in more generality. The third property and its generalization, known as the Ramanujan-Petersson Conjecture, is considerably more difficult. Although the Ramanujan-Petersson Conjecture is suspected to be true, it has only been proved for $SL_2(\mathbb{Z})$ modular forms by Deligne.

Given a modular form f there is an associated L-function $L(s, f)$ constructed from the coefficients $A(n)$ in the Fourier expansion in equation (2.1). Before giving the definition of the L-function associated to a modular form, we first recall a few facts about the Hecke operators on $SL_2(\mathbb{Z})$.

Let $\mathcal{L}^2\left(SL_2(\mathbb{Z}) \setminus \mathfrak{h}^2\right)$ be the completed subspace of all smooth, square integrable functions $f : SL_2(\mathbb{Z}) \setminus \mathfrak{h}^2 \to \mathbb{C}$. The space $\mathcal{L}^2\left(SL_2(\mathbb{Z}) \setminus \mathfrak{h}^2\right)$ is a Hilbert space with the Petersson inner product

$$\langle f, g \rangle = \int_{SL_2(\mathbb{Z}) \setminus \mathfrak{h}^2} f(z)\overline{g(z)}dz.$$

For every integer $N > 1$ we may define a Hecke operator T_N acting on $\mathcal{L}^2\left(SL_2(\mathbb{Z}) \setminus \mathfrak{h}^2\right)$ via

$$T_N f(z) = \frac{1}{\sqrt{N}} \sum_{\substack{ad=N \\ 0 \le b < d}} f\left(\frac{az+b}{d}\right).$$

It is known that these Hecke operators are normal operators that commute with one another and with the Laplacian $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ (i.e. the $GL_2(\mathbb{R})$-invariant differential operator). Consequently, the space $\mathcal{L}^2\left(SL_2(\mathbb{Z}) \setminus \mathfrak{h}^2\right)$ may be simultaneously diagonalized by these operators. Furthermore, we have the following connection between the Hecke operators and the Fourier coefficients.

Proposition 2.1. *Let f be a modular form as in (2.1). Suppose f is an eigenfunction of the full Hecke ring. If the Fourier coefficients are normalized so that $A(1) = 1$ then*

$$T_n f(z) = A(n) f(z) \text{ for all } n \geq 1, \text{ and}$$

$$A(n)A(m) = \sum_{d|(n,m)} A\left(\frac{nm}{d^2}\right).$$

In particular, $A(n)A(m) = A(nm)$ whenever $(n, m) = 1$, and

$$A(p^{r+1}) = A(p)A(p^r) - A(p^{r-1})$$

for primes p and $r \in \mathbb{Z}^+$.

Proof. See Theorem 3.12.8 in [10] and Section 1.4 in [5]. □

For convenience we will assume that the Fourier coefficients $A(n)$ are normalized from this point forward. We are now in a position to define the standard L-function associated to f.

Definition 2.2. *Let f be a modular form for $SL_2(\mathbb{Z})$ which is also an eigenfunction of the full Hecke ring. Then the L-function $L(s, f)$ associated to f is defined as*

$$L(s, f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} \tag{2.2}$$

where $\lambda_f(n) = A(n)$ for $s \in \mathbb{C}$ with $\Re(s) > 1$.

If we define

$$L_p(s,f) := \sum_{k=0}^{\infty} \frac{\lambda_f(p^k)}{p^{ks}}$$

then using the multiplicativity relations in Proposition 2.1 we have

$$L(s,f) = \prod_p L_p(s,f).$$

Furthermore, one can use these multiplicativity relations to derive the Euler product expansion

$$L(s,f) = \prod_p (1 - A(p)p^{-s} + p^{-2s})^{-1} = \prod_p \prod_{j=1}^{2} (1 - \alpha_{p,j} p^{-s})^{-1}.$$

In this setting the Ramanujan-Petersson Conjecture takes the form $|\alpha_{p,j}| = 1$ for $1 \leq j \leq 2$ and all primes p.

It is possible to analytically continue $L(s,f)$ to the rest of the complex plane via the completed L-function

$$\Lambda(s,f) := \pi^{-s}\Gamma\left(\frac{s}{2} + \frac{k-1}{4}\right)\Gamma\left(\frac{s}{2} + \frac{k+1}{4}\right)L(s,f)$$

which satisfies the functional equation $\Lambda(s,f) = \Lambda(1-s,f)$. The completed L-function is entire when f is cuspidal, otherwise it has simple poles at $s = 0$ and $s = 1$.

2.2 Maass Forms for $SL_m(\mathbb{Z})$

In this section we recall the necessary and essential information regarding Maass forms. For further information we refer the reader to Bump [5], Bump et al. [4], Goldfeld [10], and Iwaniec & Kowalski [15]. A Maass form is a smooth function from the generalized upper half plane to the complex numbers that is invariant under $SL_m(\mathbb{Z})$, or certain subgroups thereof, and that is also an eigenfunction for every differential operator arising from the center of the universal enveloping algebra. Therefore, we begin with the generalized upper half plane

$$\mathfrak{h}^m = GL_m(\mathbb{R}) / \left(O_m(\mathbb{R}) \cdot \mathbb{R}^\times \right).$$

An arbitrary element $z \in \mathfrak{h}^m$ may be written as $z = x \cdot y$ with

$$x = \begin{pmatrix} 1 & x_{m-1} & * & \dots & * \\ & 1 & x_{m-2} & \dots & * \\ & & \ddots & \ddots & \vdots \\ & & & 1 & x_1 \\ & & & & 1 \end{pmatrix}, y = \begin{pmatrix} y_1 y_2 \cdots y_{m-1} & & & & \\ & y_2 y_2 \cdots y_{m-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

where $x_i, y_i \in \mathbb{R}$ and $y_i > 0$. An important function on $\mathfrak{h}^m$ is

$$I_\nu(z) = \prod_{i,j=1}^{m-1} y_i^{b_{i,j}\nu_j}$$

where

$$b_{i,j} = \begin{cases} ij & \text{if } i+j \leq n, \\ (m-i)(m-j) & \text{if } i+j \geq n, \end{cases}$$

and $\nu = (\nu_1, \nu_2, \dots, \nu_{m-1}) \in \mathbb{C}^{m-1}$. The function $I_\nu(z)$ is an eigenfunction of every differential operator $D \in \mathfrak{D}^m$, the center of the universal enveloping algebra of $\mathfrak{gl}_m(\mathbb{R})$ (the Lie algebra of $GL_m(\mathbb{R})$), and can be thought of as a generalization of the $\Im(z)$ on $\mathfrak{h}^2$. As we shall see $I_\nu(z)$ plays a crucial role in the development of Maass forms.

Definition 2.3. *Let $m \geq 2$ and $\nu = (\nu_1, \nu_2, \dots, \nu_{m-1}) \in \mathbb{C}^{m-1}$. A Maass form for $SL_m(\mathbb{Z})$ of type ν is a smooth function f on $\mathfrak{h}^m$ that satisfies*

(i) $f(\gamma z) = f(z)$ for all $\gamma \in SL_m(\mathbb{Z})$ and $z \in \mathfrak{h}^m$,

(ii) $Df(z) = \lambda_D f(z)$ for all $D \in \mathfrak{D}^m$.

Additionally, if f satisfies

(iii) $\int_{(SL_m(\mathbb{Z})\cap U)\backslash U} f(uz)du = 0,$

for all upper triangluar groups U of the form

$$U = \left\{ \begin{pmatrix} I_{r_1} & * & * & * \\ & I_{r_2} & * & * \\ & & \ddots & * \\ & & & I_{r_k} \end{pmatrix} \right\}$$

with $r_1 + r_2 + \cdots + r_k = m$, then f is called a Maass cusp form.

Note that some authors require that f be square integrable as well.

It is known (see Chapter 4 of [10]) that an infinite number of Maass cusp forms exist for $SL_m(\mathbb{Z})$. For a simple example of a $SL_m(\mathbb{Z})$ Maass form it is possible to create one by averaging the function $I_\nu(z)$ over the group. When $m = 2$ the real analytic Eisenstein series is defined as

$$E(z,s) := \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \frac{I_s(\gamma z)}{2} = \frac{1}{2} \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}}$$

for $z \in \mathfrak{h}^2$ and $\Re(s) > 1$, where

$$\gamma \in \Gamma_\infty := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| \, n \in \mathbb{Z} \right\}.$$

Note that it is necessary to factor out by Γ_∞ since $I_\nu(z)$ is left-invariant under this group. While the Eisenstein series $E(z,s)$ is not square integrable, it does satisfy properties (i) and (ii) of Definition 2.3. The former is immediate since $\Gamma_\infty \setminus SL_2(\mathbb{Z})$ is left-invariant under $SL_2(\mathbb{Z})$, while the latter follows from the fact that

$$\Delta I_s(gz) = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) I_s(gz) = s(1-s) I_s(gz)$$

for any $g \in GL_2^+(\mathbb{Z})$.

Next, we recall the Fourier-Whittaker expansion of a Maass cusp form.

Proposition 2.2. *Let f be a $SL_m(\mathbb{Z})$ Maass cusp form. Then for all $z \in \mathfrak{h}^m$*

$$f(z) = \sum_{\gamma \in U_{m-1}(\mathbb{Z}) \backslash SL_{m-1}(\mathbb{Z})} \sum_{n_1=1}^{\infty} \cdots \sum_{n_{m-2}=1}^{\infty} \sum_{n_{m-1} \neq 0} \hat{f}_{(n_1,\ldots,n_{m-1})} \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z \right), \qquad (2.3)$$

where $U_m(\mathbb{Z})$ is the group of $m \times m$ unipotent upper triangluar matricies with 1's on the diagonal, and

$$\hat{f}_{(n_1,\ldots,n_{m-1})}(z) := \int_0^1 \cdots \int_0^1 f(u \cdot z) e(-(n_1 u_{1,2} + n_2 u_{2,3} + \cdots + n_{m-1} u_{m-1,m})) d^* u$$

with $u \in U_m(\mathbb{Z})$, $e(z) = e^{2\pi i z}$, and $d^* u = \prod_{1 \le i < j \le m} d_{i,j}$.

Proof. See Theorem 5.3.2 in [10]. $\square$

It is a consequence of the Multiplicity One Theorem that the Fourier coefficients $\hat{f}_{(n_1,\ldots,n_{m-1})}(z)$ are precisely constant multiples of Jacquet's Whittaker function

$$W_{\text{Jacquet}}(z, \nu, \psi_n) := \int_{U_m(\mathbb{R})} I_\nu(w_m \cdot u \cdot z) \overline{\psi_n(u)} du$$

where ψ_n is a character on $U_m(\mathbb{R})$ given by

$$\psi_n(u) = e(n_1 u_1 + \cdots n_{m-1} u_{m-1})$$

for $n = (n_1, \cdots, n_{m-1})$, and

$$w_m = \begin{pmatrix} & & & (-1)^{\lfloor m/2 \rfloor} \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} \in SL_m(\mathbb{Z})$$

is the long element of the Weyl group. Thus, every Maass cusp form f of type ν has the Fourier-Whittaker expansion

$$
\begin{aligned}
f(z) \quad = & \sum_{\gamma \in U_{m-1}(\mathbb{Z})\backslash SL_{m-1}(\mathbb{Z})} \sum_{n_1=1}^{\infty} \cdots \sum_{n_{m-2}=1}^{\infty} \sum_{n_{m-1}\neq 0} \frac{A(n_1, \cdots, n_{m-1})}{\prod_{k=1}^{m-1} |n_k|^{k(m-k)/2}} \\
& \times \; W_{\text{Jacquet}}\left(N \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, \nu, \psi_{1,\ldots,1,\frac{n_{m-1}}{|n_{m-1}|}} \right), \qquad (2.4)
\end{aligned}
$$

where $U_m(\mathbb{Z})$ is the group of $m \times m$ unipotent upper triangluar matricies with 1's on the diagonal,

$$
N = \begin{pmatrix} n_1 \cdots n_{m-2}|n_{m-1}| & & & & \\ & \ddots & & & \\ & & n_1 n_2 & & \\ & & & n_1 & \\ & & & & 1 \end{pmatrix},
$$

the $A(n_1, \cdots, n_{m-1}) \in \mathbb{C}$ are the Fourier-Whittaker coefficients, and

$$
\psi_{1,\ldots,1,\epsilon}\left(\begin{pmatrix} 1 & u_{m-1} & * & * & * \\ & 1 & u_{m-2} & * & * \\ & & \ddots & \ddots & * \\ & & & 1 & u_1 \\ & & & & 1 \end{pmatrix} \right) = e(u_1 + \cdots + u_{m-2} + \epsilon u_{m-1}).
$$

To every Maass form f we can define another form $\tilde{f}$ called the dual of f.

Definition 2.4. *Let f be a Maass form for $SL_m(\mathbb{Z})$. Then*

$$
\tilde{f}(z) := f\left(w_m (z^{-1})^t w_m \right)
$$

is called the dual Maass form. Here w_m is the long element of the Weyl group as given above.

It is known that the dual $\tilde{f}$ is again a $SL_m(\mathbb{Z})$ Maass form, however it is of type $(\nu_{m-1}, \ldots, \nu_1)$. Moreover, if $A(n_1, \ldots, n_{m-1})$ is the $(n_1, \ldots, n_{m-1})$ Fourier-Whittaker coefficient of f, then $A(n_{m-1}, \ldots, n_1)$ is the corresponding coefficient for $\tilde{f}$.

Given a Maass form f there is an associated L-function $L(s,f)$ constructed from the coefficients $A(n_1,\dots,n_{m-1})$ in the Fourier-Whittaker expansion in equation (2.4). Before giving the defintion of the L-function associated to a Maass form, we first recall a few facts about the Hecke operators on $SL_m(\mathbb{Z})$.

Let $\mathcal{L}^2\left(SL_m(\mathbb{Z})\setminus\mathfrak{h}^m\right)$ be the completed subspace of all smooth, square integrable functions $f: SL_m(\mathbb{Z})\setminus\mathfrak{h}^m \to \mathbb{C}$. The space $\mathcal{L}^2\left(SL_m(\mathbb{Z})\setminus\mathfrak{h}^m\right)$ is a Hilbert space with the Petersson inner product

$$\langle f,g\rangle = \int_{SL_m(\mathbb{Z})\setminus\mathfrak{h}^m} f(z)\overline{g(z)}dz.$$

For every integer $N>1$ we may define a Hecke operator T_N acting on $\mathcal{L}^2\left(SL_m(\mathbb{Z})\setminus\mathfrak{h}^m\right)$ via

$$T_N f(z) = \frac{1}{N^{m-\frac{1}{2}}} \sum_{\substack{\prod_{\ell=1}^m c_\ell = N \\ 0\le c_{i,\ell}<c_\ell (1\le i<\ell\le m)}} f\left(\begin{pmatrix} c_1 & c_{1,2} & \cdots & c_{1,m} \\ & c_2 & \cdots & c_{2,m} \\ & & \ddots & \vdots \\ & & & c_m \end{pmatrix}\cdot z\right).$$

It is known that these Hecke operators are normal operators that commute with one another and with the $GL_m(\mathbb{R})$-invariant differential operators. Consequently, the space $\mathcal{L}^2\left(SL_m(\mathbb{Z})\setminus\mathfrak{h}^m\right)$ may be simultaneously diagonalized by these operators. Furthermore, we have the following connection between the Hecke operators and the Fourier-Whittaker coefficients.

Proposition 2.3. *Let f be a Maass form as in (2.4). Suppose f is an eigenfunction of the full Hecke ring. If $A(1,\dots,1)=0$, then $f(z)$ is identically zero. If the Fourier-Whittaker coefficients are normalized so that $A(1,\dots,1)=1$ then*

$$T_n f(z) = A(n,1,\dots,1)f(z) \text{ for all } n\ge 1,$$

$$A(n_1n_1',\ldots,n_{m-1}n_{m-1}') = A(n_1,\ldots,n_{m-1})A(n_1',\ldots,n_{m-1}'),$$

whenever $(n_1\cdots n_{m-1}, n_1'\cdots n_{m-1}') = 1$, *and*

$$A(n,1,\ldots,1)A(n_1,\ldots,n_{m-1}) = \sum_{\substack{\prod_{\ell=1}^{m} c_\ell = n \\ c_i | n_i\ (1\le i<m)}} A\left(\frac{n_1c_m}{c_1}, \frac{n_2c_1}{c_2}, \ldots, \frac{n_{m-1}c_{m-2}}{c_{m-1}}\right).$$

Proof. See Theorem 9.3.11 in [10]. □

For convenience we will assume that the Fourier-Whittaker coefficients $A(n_1,\ldots,n_{m-1})$ are normalized from this point forward. We are now in a position to define the standard L-function associated to f (i.e. the Godement-Jacquet L-function).

Definition 2.5. *Let f be a Maass form for $SL_m(\mathbb{Z})$ which is also an eigenfunction of the full Hecke ring. Then the L-function $L(s,f)$ associated to f is defined as*

$$L(s,f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} \tag{2.5}$$

where $\lambda_f(n) = A(n,1,\ldots,1)$ for $s\in\mathbb{C}$ with $\Re(s) > \frac{m+1}{2}$.

Initially the L-function converges absolutely for $\Re(s) > \frac{m+1}{2}$, since the Fourier-Whittaker coefficients satisfy the so-called trivial bound

$$\frac{A(n_1,\ldots,n_{m-1})}{\prod_{k=1}^{m-1}|n_k|^{k(m-k)/2}} = O(1),$$

but ultimately Jacquet and Shalika [17] showed absolute convergence for $\Re(s) > 1$. If we define

$$L_p(s,f) := \sum_{k=0}^{\infty} \frac{A(p^k,1,\ldots,1)}{p^{ks}} = \sum_{k=0}^{\infty} \frac{\lambda_f(p^k)}{p^{ks}}$$

then using the multiplicativity relations in Proposition 2.3 we have

$$L(s,f) = \prod_p L_p(s,f).$$

Furthermore, one can use these multiplicativity relations to derive the Euler product expansion

$$\begin{aligned} L(s,f) &= \prod_p (1 - A(p,1,\ldots,1)p^{-s} + A(1,p,1,\ldots,1)p^{-2s} - \ldots \\ &\quad + (-1)^{m-1}A(1,\ldots,1,p)p^{-(m-1)s} + (-1)^m p^{-ms})^{-1} \\ &= \prod_p \prod_{j=1}^{m} (1 - \alpha_{p,j}p^{-s})^{-1}. \end{aligned}$$

Observe here that the Fourier-Whittaker coefficients can be given in terms of the elementary symmetric polynomials in $\alpha_{p,j}$. For instance, $A(p,1,\ldots,1) = \sum_{j=1}^{m} \alpha_{p,j}$. A very important unsolved problem involves the bound $|\alpha_{p,j}| = p^{\theta}$. The generalized Ramanujan Conjecture states that $\theta = 0$ and is known to be true for holomorphic $SL_2(\mathbb{Z})$ modular forms. In general, it can be shown that $\theta = \frac{1}{2}$; however, at present the best bound is $\theta = \frac{1}{2} - \frac{1}{m^2+1}$ due to Luo, Rudnick, and Sarnak [24].

It is possible to analytically continue $L(s,f)$ to the rest of the complex plane via the completed L-function

$$\Lambda(s,f) := \pi^{-\frac{ms}{2}} \prod_{j=1}^{m} \Gamma\left(\frac{s - \mu_f(j)}{2}\right) L(s,f)$$

which satisfies the functional equation $\Lambda(s,f) = \Lambda(1-s,\tilde{f})$. Here the $\mu_f(j) \in \mathbb{C}$ are the parameters at the infinite place. The completed L-function is entire when f is cuspidal except when $m = 1$ and $L(s,f)$ is the Riemann zeta function.

2.3 The Meijer G-function

The main references for this section are Luke [22, 23] and Bateman [2]. The Meijer G-function is a generalized function which encompasses many special functions.

It was originally defined through a series representation, but it is now more commonly given by the following Mellin-Barnes integral.

Definition 2.6. *The Meijer G-function is defined as*

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\middle| z\right) := \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)} z^s ds \quad (2.6)$$

where $0 \le m \le q$ *and* $0 \le n \le p$ *are integers, the complex parameters* a_k *and* b_j *are such that no pole of* $\Gamma(b_j - s), j = 1, \ldots, m$*, coincides with any pole of* $\Gamma(1-a_k+s), k = 1, \ldots, n$*, and* $z \neq 0$*. The path of integration may be chosen as follows:*

(i) L *goes from* $\sigma - i\infty$ *to* $\sigma + i\infty$ *so that all the poles of* $\Gamma(b_j - s), j = 1, \ldots, m$*, lie to the right of the path, and all the poles of* $\Gamma(1 - a_k + s), k = 1, \ldots, n$*, lie to the left of the path. The integral converges when* $|\arg(z)| < \delta\pi$ *where* $\delta = m+n-\frac{1}{2}(p+q) > 0$*. If* $|\arg(z)| = \delta\pi$ *when* $\delta \ge 0$*, then the integral converges absolutely when* $p = q$ *if* $\Re(\nu) < -1$*; or when* $p = q$ *if, given* $s = \sigma + i\tau, \sigma$ *is chosen so that for* $\tau \to \pm\infty$*,*

$$(q-p)\sigma > \Re(\nu) + 1 - \frac{1}{2}(q-p),$$

$$\nu = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j.$$

(ii) L *is a loop beginning and ending at* $+\infty$ *and encircling all poles of* $\Gamma(b_j - s), j = 1, \ldots, m$*, once in the negative direction, but none of the poles of* $\Gamma(1-a_k+s), k = 1, \ldots, n$*. The integral converges if* $q \ge 1$ *and either* $p < q$ *or* $p = q$ *and* $|z| < 1$*.*

(iii) L *is a loop beginning and ending at* $-\infty$ *and encircling all poles of* $\Gamma(1 - a_k + s), k = 1, \ldots, n$*, once in the positive direction, but none of the poles of* $\Gamma(b_j -$

$s), j = 1, \ldots, m$. The integral converges if $p \geq 1$ and either $p > q$ or $p = q$ and $|z| > 1$.

When the parameters a_k and b_j are clear the Meijer G-function is usually denoted as $G_{p,q}^{m,n}\left(\begin{matrix} a_k \\ b_j \end{matrix} \middle| z\right)$ or $G_{p,q}^{m,n}(z)$. We note that in our applications we will have $\arg(z) = 0$ and consequently will take the first contour described above. The Meijer G-function satisfies many interesting properties including multiplication formulas and differential equations. For our purposes, however, we will only require the following asymptotic expansion.

Lemma 2.1. *For integers $m \geq 1$ and $r \geq 0$ we have*

$$G_{0,2m}^{m,0}\left(\begin{matrix} - \\ b_j \end{matrix} \middle| x\right) = {A^{m,0}}_{2m} H_{0,2m}\left(xe^{i\pi m}; r\right) + {\bar{A}^{m,0}}_{2m} H_{0,2m}\left(xe^{-i\pi m}; r\right) \tag{2.7}$$

as $x \to +\infty$ where

$${A^{m,0}}_{2m} = \left(-\frac{1}{2\pi i}\right)^m \exp\left(-i\pi \sum_{j=m+1}^{2m} b_j\right),$$

$$H_{0,2m}(x; r) = \frac{(2\pi)^m}{2\sqrt{\pi m}} \exp\left(-2m x^{\frac{1}{2m}}\right) x^\theta \sum_{k=0}^{r} M_k x^{-\frac{k}{2m}} + O\left(x^{\theta - \frac{r+1}{2m}}\right),$$

$$\theta = \frac{1}{4m}\left(1 - 2m + 2\sum_{j=1}^{2m} b_j\right),$$

and the M_k's are constants, independent of x, depending on m and b_j.

Proof. The asymptotic expansions of $G_{p,q}^{m,n}(z)$ for various m, n, p, and q are given in [22, 23], but we repeat the relevant formulas here for completeness. From equation (4) in Theorem 2 of section 5.9.2 in [23] we have

$$G_{p,q}^{m,0}(z) = {A^{m,0}}_q H_{p,q}\left(ze^{i\pi(q-m)}; r\right) + {\bar{A}^{m,0}}_q H_{p,q}\left(ze^{-i\pi(q-m)}; r\right)$$

as $z \to +\infty$ for $\arg z = 0$ and $1 \le p+1 \le m \le q-1$. There is a lot of notation to unpack here. First, $A^{m,n}_{\ q}$ is given in equation (2) of section 5.8.2. in [23] as

$$A^{m,n}_{\ q} = \left(-\frac{1}{2\pi i}\right)^{q-n-m} \exp\left[i\pi\left(\sum_{j=1}^{n} a_j - \sum_{j=m+1}^{q} b_j\right)\right].$$

Second, $H_{p,q}$ is defined via Theorem 5 of section 5.7 of [22] where

$$H_{p,q}(z;r) = \frac{(2\pi)^{(\sigma-1)/2}}{\sigma^{1/2}} \exp\left(-\sigma z^{1/\sigma}\right) z^{\theta} \sum_{k=0}^{r} M_k z^{-k/\sigma} + O\left(z^{\theta-\frac{r+1}{\sigma}}\right).$$

Finally, $\sigma = q - p$,

$$\theta = \frac{1}{2\sigma}\left(1 - \sigma + 2\sum_{j=1}^{q} b_j - 2\sum_{j=1}^{p} a_j\right),$$

and the M_k are constants, independent of z, depending on σ, a_j, and b_j. Taking $n = p = 0$ and $q = 2m$ gives the result of the lemma. $\square$

CHAPTER 3
THE $SL_m(\mathbb{Z}) \times SL_{m'}(\mathbb{Z})$ CASE

3.1 The Rankin-Selberg L-function

The Rankin-Selberg product was originally established for two $SL_2(\mathbb{Z})$ forms as a means to study the properties of L-functions and the Fourier-Whittaker coefficients. The technique has been generalized to $SL_m(\mathbb{Z}) \times SL_{m'}(\mathbb{Z})$ and beyond, although the construction of the convolution varies slightly depending on whether or not $m = m'$. The main idea is to express the completed L-function as a integral involving either the Eisenstein series when $m = m'$, or a certain type of projection operator when $m \neq m'$. For an overview of this construction we refer the reader to Goldfeld [10]. We first recall the definition of the Rankin-Selberg L-function in the case when $m = m'$.

Definition 3.1. *Let f and g be two $SL_m(\mathbb{Z})$ Maass forms whose Fourier-Whittaker coefficients are $A(n_1, \ldots, n_{m-1})$ and $B(n_1, \ldots, n_{m-1})$, respectively. Then the Rankin-Selberg L-function is defined as*

$$L(s, f \times \tilde{g}) := \zeta(ms) \sum_{n_1=1}^{\infty} \cdots \sum_{n_{m-1}=1}^{\infty} \frac{A(n_1, \ldots, n_{m-1})\overline{B(n_1, \ldots, n_{m-1})}}{(n_1^{m-1} n_2^{m-2} \cdots n_{m-1})^s}, \tag{3.1}$$

or equivalently as

$$L(s, f \times \tilde{g}) := \sum_{n=1}^{\infty} \frac{\lambda_{f \times \tilde{g}}(n)}{n^s} \tag{3.2}$$

where the coefficients $\lambda_{f \times \tilde{g}}(n)$ are given by

$$\lambda_{f \times \tilde{g}}(n) = \sum_{n = n_0^m n_1^{m-1} \cdots n_{m-1}} A(n_1, \ldots, n_{m-1})\overline{B(n_1, \ldots, n_{m-1})}.$$

The definition in the case when $m \neq m'$ is similar.

Definition 3.2. *Let f and g be Maass forms for $SL_m(\mathbb{Z})$ and $SL_{m'}(\mathbb{Z})$ with $2 \leq m < m'$ whose Fourier-Whittaker coefficients are $A(n_1, \ldots, n_{m-1})$ and $B(n_1, \ldots, n_{m'-1})$, respectively. Then the Rankin-Selberg L-function is defined as*

$$L(s, f \times \tilde{g}) := \sum_{n_1=1}^{\infty} \cdots \sum_{n_m=1}^{\infty} \frac{A(n_2, \ldots, n_m)\overline{B(n_1, \ldots, n_m, 1, \ldots, 1)}}{(n_1^m n_2^{m-1} \cdots n_m)^s}, \tag{3.3}$$

or equivalently as

$$L(s, f \times \tilde{g}) := \sum_{n=1}^{\infty} \frac{\lambda_{f \times \tilde{g}}(n)}{n^s} \tag{3.4}$$

where the coefficients $\lambda_{f \times \tilde{g}}(n)$ are given by

$$\lambda_{f \times \tilde{g}}(n) = \sum_{n = n_1^m n_2^{m-1} \cdots n_m} A(n_2, \ldots, n_m)\overline{B(n_1, \ldots, n_m, 1, \ldots, 1)}.$$

The Rankin-Selberg L-function initially converges absolutely for $\Re(s) \gg 1$, but Jacquet & Shalika [17] proved that ultimately we obtain $\Re(s) > 1$. Regardless of whether or not $m = m'$ the Euler product of the Rankin-Selberg L-function has the particularly simple form

$$L(s, f \times \tilde{g}) = \prod_p \prod_{j=1}^{m} \prod_{k=1}^{m'} (1 - \alpha_{p,j}\overline{\beta_{p,k}}p^{-s})^{-1}$$

provided that the Euler products associated to f and g are given by

$$L(s, f) = \prod_p \prod_{j=1}^{m} (1 - \alpha_{p,j}p^{-s})^{-1} \quad \text{and} \quad L(s, g) = \prod_p \prod_{j=1}^{m'} (1 - \beta_{p,j}p^{-s})^{-1}.$$

The completed L-function

$$\Lambda(s, f \times \tilde{g}) := \pi^{-\frac{mm's}{2}} \prod_{j=1}^{m} \prod_{k=1}^{m'} \Gamma\left(\frac{s - \mu_f(j) - \overline{\mu_g(k)}}{2}\right) L(s, f \times \tilde{g})$$

is an entire function when f and $\tilde{g}$ are not twist equivalent and satisfies the functional equation $\Lambda(s, f\times\tilde{g}) = \Lambda(1-s, \tilde{f}\times g)$. In particular, we have the equality

$$L(s, f\times\tilde{g}) = \pi^{mm's - \frac{mm'}{2}} \prod_{j=1}^{m}\prod_{k=1}^{m'} \frac{\Gamma\left(\frac{1-s-\overline{\mu_f(j)}-\mu_g(k)}{2}\right)}{\Gamma\left(\frac{s-\mu_f(j)-\overline{\mu_g(k)}}{2}\right)} L(1-s, \tilde{f}\times g) \tag{3.5}$$

from the functional equation.

3.2 The Summation Formula

In this section we prove Theorem 1.1 for case (II) where f is the product of two Maass cusp forms.

Theorem 3.1. *Let f and g be Maass cusp forms for $SL_m(\mathbb{Z})$ and $SL_{m'}(\mathbb{Z})$, respectively, with $2 \le m \le m'$. Let $\lambda_{f\times\tilde{g}}(n)$ be the coefficients of the Rankin-Selberg L-function $L(s, f\times\tilde{g})$. If $m = m'$ then suppose that f and $\tilde{g}$ are not twist equivalent so that $L(s, f\times\tilde{g})$ is entire. Let $\psi(x) \in C^\infty(0,\infty)$ with compact support, then*

$$\sum_{n=1}^{\infty} \lambda_{f\times\tilde{g}}(n)\psi(n) = 2\pi^{\frac{\ell}{2}} \sum_{n=1}^{\infty} \lambda_{\tilde{f}\times g}(n) \int_0^\infty \psi(x) G_{\tilde{f}\times g}(\pi^{2\ell}x^2n^2)dx \tag{3.6}$$

where $\ell = mm'$ and

$$G_{\tilde{f}\times g}(x) = G^{\ell,0}_{0,2\ell}\left(\begin{matrix} - \\ -\frac{\overline{\mu_f(j)}+\mu_g(k)}{2}, \frac{1+\mu_f(j)+\overline{\mu_g(k)}}{2} \end{matrix} \,\middle|\, x \right).$$

Here $G^{\ell,0}_{0,2\ell}$ is the Meijer G-function, μ are the parameters at ∞, and it is understood that the indicies j and k take all possible values.

We proceed as in Goldfeld & Li [11], Iwaniec & Kowalski [15], Kowalski et al. [20], etc. to deduce the associated summation formula. Let

$$\mathcal{M}(A)(s) := \int_0^\infty A(x)x^{s-1}dx \quad \text{and} \quad \mathcal{M}^{-1}(B)(x) := \frac{1}{2\pi i}\int_{(\sigma)} B(s)x^{-s}ds, \tag{3.7}$$

denote the Mellin and inverse Mellin transforms, respectively, and let $\psi(x)$ be a test function satisfying the hypotheses of Theorem 3.1. Multiplying both sides of (3.5) by $\frac{1}{2\pi i}\mathcal{M}(\psi)(s)$ and integrating along the line $\Re(s) = 1+\varepsilon$ for some $\varepsilon > 0$ we have

$$\frac{1}{2\pi i}\int_{(1+\varepsilon)} L(s, f\times\tilde{g})\mathcal{M}(\psi)(s)ds = \frac{1}{2\pi i}\int_{(1+\varepsilon)} \pi^{mm's-\frac{mm'}{2}}\mathcal{M}(\psi)(s) \times \prod_{j=1}^{m}\prod_{k=1}^{m'} \frac{\Gamma\left(\frac{1-s-\overline{\mu_f(j)}-\mu_g(k)}{2}\right)}{\Gamma\left(\frac{s-\mu_f(j)-\overline{\mu_g(k)}}{2}\right)} L(1-s, \tilde{f}\times g)ds. \quad (3.8)$$

Note that the integration is well defined since the L-function has at most polynomial growth in vertical strips of $\Re(s) > 0$ and $\mathcal{M}(\psi)(s)$ is of rapid decay as $\Im(s) \to \pm\infty$. We evaluate the left hand side of (3.8) by interchanging the summation and integration, which is valid by Lemma 3.1 below, to obtain the sum

$$\sum_{n=1}^{\infty}\lambda_{f\times\tilde{g}}(n)\cdot\frac{1}{2\pi i}\int_{(1+\varepsilon)}\mathcal{M}(\psi)(s)n^{-s}ds = \sum_{n=1}^{\infty}\lambda_{f\times\tilde{g}}(n)\psi(n).$$

To evaluate the right hand side of (3.8) we shift the contour left to $\Re(s) = -1$ and make the change of variable $s \mapsto 1-s$ to get

$$\frac{1}{2\pi i}\int_{(2)} \pi^{-mm's+\frac{mm'}{2}}\mathcal{M}(\psi)(1-s)\prod_{j=1}^{m}\prod_{k=1}^{m'}\frac{\Gamma\left(\frac{s-\overline{\mu_f(j)}-\mu_g(k)}{2}\right)}{\Gamma\left(\frac{1-s-\mu_f(j)-\overline{\mu_g(k)}}{2}\right)}L(s, \tilde{f}\times g)ds.$$

Observe that the integrand has no poles because, by the functional equation, it is equal to $L(1-s, f\times\tilde{g})\mathcal{M}(\psi)(1-s)$ which is entire by hypothesis. Furthermore, the residual horizontal integrals vanish since $\mathcal{M}(\psi)$ has rapid decay. Putting in the series for $L(s, \tilde{f}\times g)$ and making the change of variables $s \mapsto 2s$ we have

$$2\pi^{\frac{mm'}{2}}\cdot\frac{1}{2\pi i}\int_{(1)}\sum_{n=1}^{\infty}\lambda_{\tilde{f}\times g}(n)\left(\pi^{2mm'}n^2\right)^{-s} \times \mathcal{M}(\psi)(1-2s)\prod_{j=1}^{m}\prod_{k=1}^{m'}\frac{\Gamma\left(s-\frac{\overline{\mu_f(j)}+\mu_g(k)}{2}\right)}{\Gamma\left(1-s-\frac{1+\mu_f(j)+\overline{\mu_g(k)}}{2}\right)}ds. \quad (3.9)$$

To interchange the summation and integration we need the following two lemmas.

Lemma 3.1. *Suppose $\{D(n,s)\}$ is a sequence of complex measurable functions defined almost everywhere on $\Re(s)=\sigma_0$ such that*

$$\sum_{n=1}^{\infty}\int_{(\sigma_0)}|D(n,s)|ds<\infty.$$

Then we have

$$\int_{(\sigma_0)}\sum_{n=1}^{\infty}D(n,s)ds=\sum_{n=1}^{\infty}\int_{(\sigma_0)}D(n,s)ds.$$

Proof. Theorem 1.38 in Rudin [31]. □

Lemma 3.2. *Let $D(n,s)$ be given by*

$$D(n,s):=\lambda_{\tilde{f}\times g}(n)\left(\pi^{2mm'}n^2\right)^{-s}\mathcal{M}(\psi)(1-2s)\prod_{j=1}^{m}\prod_{k=1}^{m'}\frac{\Gamma\left(s-\frac{\overline{\mu_f(j)}+\mu_g(k)}{2}\right)}{\Gamma\left(1-s-\frac{1+\mu_f(j)+\overline{\mu_g(k)}}{2}\right)},$$

then it follows that

$$\int_{(\sigma_0)}\sum_{n=1}^{\infty}D(n,s)ds=\sum_{n=1}^{\infty}\int_{(\sigma_0)}D(n,s)ds$$

whenever $L(2s,\tilde{f}\times g)$ converges absolutely.

Proof. By Lemma 3.1 it suffices to show that

$$\sum_{n=1}^{\infty}\int_{(\sigma_0)}|D(n,s)|ds<\infty$$

whenever $L(2s,\tilde{f}\times g)$ converges absolutely. For $s=\sigma_0+it$ we have

$$|D(n,s)|=|\lambda_{\tilde{f}\times g}(n)|\left(\pi^{2mm'}n^2\right)^{-\sigma_0}$$
$$\times|\mathcal{M}(\psi)(1-2\sigma_0-2it)|\prod_{j=1}^{m}\prod_{k=1}^{m'}\frac{\left|\Gamma\left(\sigma_0+it-\frac{\overline{\mu_f(j)}+\mu_g(k)}{2}\right)\right|}{\left|\Gamma\left(1-\sigma_0-it-\frac{1+\mu_f(j)+\overline{\mu_g(k)}}{2}\right)\right|}.$$

First, we claim that

$$\prod_{j=1}^{m}\prod_{k=1}^{m'}\frac{\left|\Gamma\left(\sigma_0+it-\frac{\overline{\mu_f(j)}+\mu_g(k)}{2}\right)\right|}{\left|\Gamma\left(1-\sigma_0-it-\frac{1+\mu_f(j)+\overline{\mu_g(k)}}{2}\right)\right|}=O\left(|t|^{mm'(2\sigma_0-\frac{1}{2})}\right).$$

By Stirling's Formula (equation (6.1.45) in [1]) we know

$$|\Gamma(\sigma+it)|\sim\sqrt{2\pi}e^{-\frac{\pi|t|}{2}}|t|^{\sigma-\frac{1}{2}}$$

as $t\to\pm\infty$. Letting $\mu_f(j)=\rho_f(j)+i\iota_f(j)$, and similarly for $\mu_g(k)$, we have

$$\begin{aligned}\left|\Gamma\left(\sigma_0+it-\frac{\overline{\mu_f(j)}+\mu_g(k)}{2}\right)\right|&\\ &=\left|\Gamma\left(\left(\sigma_0-\frac{\rho_f(j)+\rho_g(k)}{2}\right)+i\left(t+\frac{\iota_f(j)-\iota_g(k)}{2}\right)\right)\right|\\ &\sim\sqrt{2\pi}e^{-\frac{\pi\left|t+\frac{\iota_f(j)-\iota_g(k)}{2}\right|}{2}}\left|t+\frac{\iota_f(j)-\iota_g(k)}{2}\right|^{\sigma_0-\frac{\rho_f(j)+\rho_g(k)}{2}-\frac{1}{2}}\end{aligned}$$

and similarly

$$\begin{aligned}\left|\Gamma\left(\frac{1}{2}-\sigma_0-it-\frac{\mu_f(j)+\overline{\mu_g(k)}}{2}\right)\right|&\\ &\sim\sqrt{2\pi}e^{-\frac{\pi\left|t+\frac{\iota_f(j)-\iota_g(k)}{2}\right|}{2}}\left|t+\frac{\iota_f(j)-\iota_g(k)}{2}\right|^{-\sigma_0-\frac{\rho_f(j)+\rho_g(k)}{2}}\end{aligned}$$

which implies

$$\frac{\left|\Gamma\left(\sigma_0+it-\frac{\overline{\mu_f(j)}+\mu_g(k)}{2}\right)\right|}{\left|\Gamma\left(1-\sigma_0-it-\frac{1+\mu_f(j)+\overline{\mu_g(k)}}{2}\right)\right|}\sim\left|t+\frac{\iota_f(j)-\iota_g(k)}{2}\right|^{2\sigma_0-\frac{1}{2}}.$$

Consequently,

$$\frac{\left|\Gamma\left(\sigma_0+it-\frac{\overline{\mu_f(j)}+\mu_g(k)}{2}\right)\right|}{\left|\Gamma\left(1-\sigma_0-it-\frac{1+\mu_f(j)+\overline{\mu_g(k)}}{2}\right)\right|}=O\left(|t|^{2\sigma_0-\frac{1}{2}}\right)$$

and the claim follows.

Next, we show that

$$\mathcal{I} = \int_{-\infty}^{\infty} |\mathcal{M}(\psi)(1-2\sigma_0-2it)| \prod_{j=1}^{m}\prod_{k=1}^{m'} \frac{\left|\Gamma\left(\sigma_0+it-\frac{\overline{\mu_f(j)}+\mu_g(k)}{2}\right)\right|}{\left|\Gamma\left(1-\sigma_0-it-\frac{1+\mu_f(j)+\overline{\mu_g(k)}}{2}\right)\right|} dt < \infty.$$

Let $B \in (0,\infty)$ and split up the integral into three pieces: $\mathcal{I}_-, \mathcal{I}_0$, and $\mathcal{I}_+$ over $(-\infty,-B), (-B,B)$, and (B,∞), respectively. Since $\mathcal{I}_0 < \infty$ it remains to show $\mathcal{I}_\pm < \infty$. Here we choose B sufficiently large so that the claim above yields

$$\begin{aligned} \mathcal{I}_+ &= \int_{B}^{\infty} |\mathcal{M}(\psi)(1-2\sigma_0-2it)| \prod_{j=1}^{m}\prod_{k=1}^{m'} \frac{\left|\Gamma\left(\sigma_0+it-\frac{\overline{\mu_f(j)}+\mu_g(k)}{2}\right)\right|}{\left|\Gamma\left(1-\sigma_0-it-\frac{1+\mu_f(j)+\overline{\mu_g(k)}}{2}\right)\right|} dt \\ &\ll \int_{B}^{\infty} |\mathcal{M}(\psi)(1-2\sigma_0-2it)|\, |t|^{mm'(2\sigma_0-\frac{1}{2})} dt < \infty \end{aligned}$$

since $\mathcal{M}(1-2s)$ is of rapid decay. A similar argument shows that $\mathcal{I}_- < \infty$ and hence $\mathcal{I} < \infty$. Thus,

$$\sum_{n=1}^{\infty} \int_{(\sigma_0)} |D(n,s)| ds = \mathcal{I}|\pi^{2mm'}|^{-\sigma_0} \sum_{n=1}^{\infty} \frac{|\lambda_{\tilde{f}\times g}(n)|}{n^{2\sigma_0}} < \infty$$

wherever $L(2s, \tilde{f}\times g)$ converges absolutely. $\square$

We may therefore interchange the order of the summation and integration in equation (3.9) wherever $L(2s,\tilde{f}\times g)$ converges absolutely; i.e. for $\Re(s) > \frac{1}{2}$. Doing so and replacing $s \mapsto -s$ in (3.9) we have

$$\begin{aligned} &2\pi^{\frac{mm'}{2}} \cdot \frac{1}{2\pi i} \sum_{n=1}^{\infty} \int_{(-1)} \lambda_{\tilde{f}\times g}(n) \left(\pi^{2mm'} n^2\right)^s \\ &\qquad \times \mathcal{M}(\psi)(1+2s) \prod_{j=1}^{m}\prod_{k=1}^{m'} \frac{\Gamma\left(-s-\frac{\overline{\mu_f(j)}+\mu_g(k)}{2}\right)}{\Gamma\left(1+s-\frac{1+\mu_f(j)+\overline{\mu_g(k)}}{2}\right)} ds. \quad (3.10) \end{aligned}$$

Note that at this point a suitable change of variables would yield the Voronoi-type summation formula given in Theorem 5.1 of Goldfeld and Li [11] (see Remark 3.1 below).

The next step is to replace the Mellin transform with its integral and interchange the two integrals. To do so we require the absolute convergence of the resulting integral over s which will be a Meijer G-function. From Definition 2.6 we see that this integral will converge absolutely provided the contour L is such that all of the poles of $\Gamma\left(-s-\frac{\overline{\mu_f(j)}+\mu_g(k)}{2}\right)$ lie to the right of the contour L, and $\sigma+i\tau\in L$ satisfies $\sigma>\frac{1}{2\ell}-\frac{1}{4}$ as $\tau\to\pm\infty$. Observe that the poles of the Gamma function occur in the right half-plane $\Re(s)\geq-\frac{1}{2}$ by the trivial bound of the parameters at infinity (this can be improved to $\Re(s)\geq 0$ if one has the Selberg eigenvalue conjecture). Moreover, this contour deformation is justified because the integrand decays rapidly as $|t|\to\infty$.

Hence, putting in the integral expression for $\mathcal{M}(\psi)(1+2s)$, deforming the contour if necessary, and interchanging the integrals we obtain

$$2\pi^{\frac{mm'}{2}}\sum_{n=1}^{\infty}\lambda_{\tilde{f}\times g}(n)\int_0^{\infty}\psi(x)\cdot\frac{1}{2\pi i}\int_L\left(\pi^{2mm'}x^2n^2\right)^s\prod_{j=1}^{m}\prod_{k=1}^{m'}\frac{\Gamma\left(-s-\frac{\overline{\mu_f(j)}+\mu_g(k)}{2}\right)}{\Gamma\left(1+s-\frac{1+\mu_f(j)+\overline{\mu_g(k)}}{2}\right)}dsdx.$$

Rewriting this as a Meijer G-function and equating the right and left hand sides of equation (3.8) we obtain the summation formula

$$\sum_{n=1}^{\infty}\lambda_{f\times\tilde{g}}(n)\psi(n)=2\pi^{\frac{\ell}{2}}\sum_{n=1}^{\infty}\lambda_{\tilde{f}\times g}(n)\int_0^{\infty}\psi(x)G_{\tilde{f}\times g}(\pi^{2\ell}x^2n^2)dx$$

given in equation (3.6) where $\ell=mm'$ and

$$G_{\tilde{f}\times g}(x)=G^{\ell,0}_{0,2\ell}\left(\begin{matrix}-\\ -\frac{\overline{\mu_f(j)}+\mu_g(k)}{2},\ \frac{1+\mu_f(j)+\overline{\mu_g(k)}}{2}\end{matrix}\ \middle|\ x\right). \tag{3.11}$$

It is understood that the indicies j and k take all possible values.

Remark 3.1. *The explicit relationship between Theorem 3.1 and Theorem 5.1 of Goldfeld and Li [11] is as follows. First, note the difference in the definitions of the*

Rankin-Selberg L function: $L(s, g \times \tilde{f}) = L_{g\times f}(s)$. *The change of variables* $s \mapsto 1+2s$ *in equation (5.12) of [11] shows that* $\Phi(n)/n$ *is equal to* $\lambda_{\tilde{g}\times f}(n)$ *times the expression in equation (3.10). Using this equality and setting* $n = m_1^l m_2^{l-1}\cdots m_l$ *so that*

$$\lambda_{g\times\tilde{f}}(n) = \sum_{n=m_1^l m_2^{l-1}\cdots m_l} B_g(m_2,\ldots,m_l)\overline{A_f(1,\ldots,1,m_1,\ldots,m_l)}$$

shows that the summation formula in equation (5.14) of [11] is equivalent to that obtained by concluding the proof at equation (3.10) above. In particular, both Theorems 3.1 and 3.2 may be restated explicitly in terms of the Fourier-Whittaker coefficients.

3.3 Resonance Sums

Choosing the test function $\psi(x) = e(\alpha x^\beta)\phi\left(\frac{x}{X}\right)$ in equation (3.6), together with the asymptotics of the Meijer G-function, yields case (II) of Theorem 1.2.

Theorem 3.2. *Let* f *and* g *be Maass cusp forms for* $SL_m(\mathbb{Z})$ *and* $SL_{m'}(\mathbb{Z})$, *respectively, with* $2 \le m \le m'$. *Let* $\lambda_{f\times\tilde{g}}(n)$ *be the coefficients of the Rankin-Selberg* L*-function* $L(s, f\times\tilde{g})$. *If* $m = m'$ *then suppose that* f *and* $\tilde{g}$ *are not twist equivalent so that* $L(s, f\times\tilde{g})$ *is entire. Let* $\phi(x)\in C^\infty(0,\infty)$ *with compact support in* $[1,2]$ *and* $\phi^{(j)}(x) \ll 1$ *for* $j \ge 1$. *Moreover, let* $mm' = \ell, X > 1,$ *and* $\alpha, \beta \ge 0$.

(i) If $2\max\{1, 2^{\beta-\frac{1}{\ell}}\}(\alpha\beta)^\ell \le X^{1-\beta\ell}$, *then*

$$\sum_{n=1}^{\infty} \lambda_{f\times\tilde{g}}(n)\phi\left(\frac{n}{X}\right)e(\pm\alpha n^\beta) \ll_{\ell,\beta,M} X^{-M}$$

holds for any $M > 0$.

(ii) If $2\max\{1, 2^{\beta-\frac{1}{\ell}}\}(\alpha\beta)^\ell > X^{1-\beta\ell}$, *then*

$$\sum_{n=1}^{\infty} \lambda_{f\times\tilde{g}}(n)\phi\left(\frac{n}{X}\right)e(\pm\alpha n^\beta) \ll_{\ell,\beta} (\alpha X^\beta)^{\frac{\ell}{2}}$$

holds for $\beta \neq \frac{1}{\ell}$, *and*

$$\sum_{n=1}^{\infty} \lambda_{f\times\tilde{g}}(n)\phi\left(\frac{n}{X}\right) e(\pm\alpha n^{\beta}) \ll_{\ell,\beta} (\alpha X^{\beta})^{\frac{1+\ell}{2}}$$

holds for $\beta = \frac{1}{\ell}$.

(iii) *If* $X > \alpha^{\frac{\ell(\ell-1)}{1-\ell\varepsilon}}$ *with* $0 < \varepsilon < \frac{1}{\ell}$, *then*

$$\sum_{n=1}^{\infty} \lambda_{f\times\tilde{g}}(n)\phi\left(\frac{n}{X}\right) e(\pm\alpha n^{\frac{1}{\ell}}) = \frac{\lambda_{\tilde{f}\times g}(n_\alpha)}{n_\alpha} \times \sum_{k=0}^{r} c^{-}_{k,\ell} I_k(n_\alpha; -)(n_\alpha X)^{\frac{1}{2\ell}+\frac{1}{2}-\frac{k}{\ell}} + O_{\ell,r,\varepsilon}\left(X^{\frac{1}{2\ell}+\frac{1}{2}-\frac{r+1}{\ell}}\right)$$

for any $r > \frac{\ell-1}{2}$. *Here* n_α *is the unique positive integer satisfying* $\left(\frac{\alpha}{\ell}\right)^{\ell} - n_\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right]$,

$$I_k(n_\alpha; -) = \int_0^{\infty} \phi(t^{\ell}) e\left(\left(\alpha - n_\alpha^{\frac{1}{\ell}}\ell\right) X^{\frac{1}{\ell}} t\right) t^{\frac{\ell}{2}-\frac{1}{2}-k} dt,$$

and $c^{-}_{k,\ell}$ *are constants depending on* f, g, k, *and* ℓ.

(iv) *In particular, if* q *is a positive integer and* $0 < \varepsilon < \frac{1}{\ell}$, *then for* $X > \left(\ell^{\ell} q\right)^{\frac{\ell-1}{1-\ell\varepsilon}}$ *we have*

$$\sum_{n=1}^{\infty} \lambda_{f\times\tilde{g}}(n)\phi\left(\frac{n}{X}\right) e(\pm\ell(qn)^{\frac{1}{\ell}}) = \frac{\lambda_{\tilde{f}\times g}(q)}{q} \times \sum_{k=0}^{r} c^{-}_{k,\ell} I_k(q; -)(qX)^{\frac{1}{2\ell}+\frac{1}{2}-\frac{k}{\ell}} + O_{\ell,r,\varepsilon}\left(X^{\frac{1}{2\ell}+\frac{1}{2}-\frac{r+1}{\ell}}\right)$$

for any $r > \frac{\ell-1}{2}$ *where*

$$I_k(q; -) = \frac{1}{\ell} \int_0^{\infty} \phi(x) x^{\frac{1}{2\ell}-\frac{1}{2}-\frac{k}{\ell}} dx$$

and $c^{-}_{k,\ell}$ *are constants depending on* f, g, k, *and* ℓ.

We begin by setting

$$\psi(x) = e(\alpha x^{\beta})\phi\left(\frac{x}{X}\right)$$

and applying Lemma 2.1 to equation (3.11) to obtain the asymptotic expansion

$$G_{\tilde{f}\times g}(x) = A^{\ell,0}_{\ \ 2\ell} H_{0,2\ell}\left(xe^{i\pi\ell}; r\right) + \bar{A}^{\ell,0}_{\ \ 2\ell} H_{0,2\ell}\left(xe^{-i\pi\ell}; r\right).$$

This is valid for any integer $r \geq 0$ and real $x \to +\infty$ where

$$A^{\ell,0}_{\ \ 2\ell} = \left(-\frac{1}{2\pi i}\right)^{\ell} \exp\left(i\pi \sum_{j=1}^{m}\sum_{k=1}^{m'} \frac{1+\mu_f(j)+\overline{\mu_g(k)}}{2}\right) = \left(-\frac{1}{2\pi}\right)^{\ell},$$

$$H_{0,2\ell}(x; r) = \frac{(2\pi)^{\ell}}{2\sqrt{\pi\ell}} \exp\left(-2\ell x^{\frac{1}{2\ell}}\right) x^{\theta} \sum_{k=0}^{r} M_k x^{-\frac{k}{2\ell}} + O\left(x^{\theta-\frac{r+1}{2\ell}}\right),$$

$$\theta = \frac{1}{4\ell}\left(1-\ell+2i\sum_{j=1}^{m}\sum_{k=1}^{m'} \Im\left(\mu_f(j)\right) - \Im\left(\mu_g(k)\right)\right) = \frac{1-\ell}{4\ell},$$

and the M_k's are constants depending on $\ell, \mu_f(j)$ and $\mu_g(k)$. Here we have used the fact that the summations over $\mu_f(j)$ and $\mu_g(k)$ vanish; i.e.

$$\sum_{j=1}^{m} \mu_f(j) = 0 \qquad \text{and} \qquad \sum_{k=1}^{m'} \mu_g(k) = 0.$$

Thus, we have

$$\begin{aligned} G_{\tilde{f}\times g}\left(\pi^{2\ell}x^2n^2\right) &= \left(-\frac{1}{2\pi}\right)^{\ell} \sum_{\pm} H_{0,2\ell}\left(\pi^{2\ell}x^2n^2 e^{\mp i\pi\ell}; r\right) \\ &= \frac{(-1)^{\ell}}{2\sqrt{\pi\ell}} \sum_{\pm} e\left(\pm\ell\left(xn\right)^{\frac{1}{\ell}}\right) \sum_{k=0}^{r} M_k \left(\pi^{2\ell}x^2n^2e^{\mp i\pi\ell}\right)^{\theta-\frac{k}{2\ell}} + O\left((xn)^{2\theta-\frac{r+1}{\ell}}\right) \end{aligned}$$

and it follows that

$$\begin{aligned} \psi(x) G_{\tilde{f}\times g}(\pi^{2\ell}x^2n^2) = \frac{(-1)^{\ell}}{2\sqrt{\pi\ell}} &\sum_{\pm} \phi\left(\frac{x}{X}\right) e\left(\alpha x^{\beta} \pm \ell\left(xn\right)^{\frac{1}{\ell}}\right) \\ &\times \sum_{k=0}^{r} M_k \left(\pi^{2\ell}x^2n^2e^{\mp i\pi\ell}\right)^{\theta-\frac{k}{2\ell}} + O\left(\left|\phi\left(\frac{x}{X}\right)\right| (xn)^{2\theta-\frac{r+1}{\ell}}\right) \end{aligned}$$

where $\psi(x) = e(\alpha x^{\beta})\phi\left(\frac{x}{X}\right)$ as above. With this expansion the resonance sum in equation (3.6) becomes

$$\sum_{n=1}^{\infty} \lambda_{f\times\tilde{g}}(n)\psi(n) = \frac{(-1)^{\ell}}{\sqrt{\ell}} \pi^{\frac{\ell-1}{2}} \sum_{\pm} \sum_{n=1}^{\infty} \lambda_{\tilde{f}\times g}(n)$$
$$\times \int_0^{\infty} \phi\left(\frac{x}{X}\right) e\left(\alpha x^{\beta} \pm \ell\,(xn)^{\frac{1}{\ell}}\right) \sum_{k=0}^{r} M_k \left(\pi^{2\ell} x^2 n^2 e^{\mp i\pi\ell}\right)^{\theta-\frac{k}{2\ell}} dx$$
$$+ O\left(\sum_{n=1}^{\infty} |\lambda_{\tilde{f}\times g}(n)| \int_0^{\infty} \left|\phi\left(\frac{x}{X}\right)\right| (xn)^{2\theta-\frac{r+1}{\ell}} dx\right).$$

Making the change of variables $x \mapsto Xt^{\ell}$ and doing the integral in the error term yields

$$\sum_{n=1}^{\infty} \lambda_{f\times\tilde{g}}(n)\psi(n) = \sum_{k=0}^{r} X^{1+2\theta-\frac{k}{\ell}} \sum_{n=1}^{\infty} \frac{\lambda_{\tilde{f}\times g}(n)}{n^{\frac{k}{\ell}-2\theta}} \sum_{\pm} c_{k,\ell}^{\pm} I_k(n;\pm)$$
$$+ O\left(X^{1+2\theta-\frac{r+1}{\ell}} \sum_{n=1}^{\infty} |\lambda_{\tilde{f}\times g}(n)| n^{2\theta-\frac{r+1}{\ell}}\right) \quad (3.12)$$

where the constants have been condensed into $c_{k,\ell}^{\pm}$ and

$$I_k(n;\pm) := \int_0^{\infty} \phi(t^{\ell}) e\left(\alpha X^{\beta} t^{\ell\beta} \pm (Xn)^{\frac{1}{\ell}} \ell t\right) t^{2\ell\theta-k+\ell-1} dt.$$

Note that the error term here is $O_{\ell,r}\left(X^{1+2\theta-\frac{r+1}{\ell}}\right)$ for r such that $L_{\tilde{f}\times g}(\frac{r+1}{\ell} - 2\theta)$ converges absolutely; i.e. for $r > \frac{\ell-1}{2}$.

The similarities between equation (3.12) above and equation (5.2) in [28] enable us to follow the argument given there. In particular, the integrals $I_k(n;\pm)$ are identical and so repeated integration-by-parts gives

$$I_k(n;+) \ll_{\ell,j} (nX)^{-\frac{j}{\ell}}$$

for $j, n \geq 1$. We also need the bound

$$\sum_{n=1}^{X} |\lambda_{\tilde{f}\times g}(n)| \ll X \quad (3.13)$$

for $X > 0$ which follows from a Tauberian argument.

Lemma 3.3. *If the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges for $\Re(s) > 1$, is holomorphic on the line $\Re(s) = 1$ with possibly the exception of a simple pole at $s = 1$ of residue 1, and the coefficients satisfy $0 \leq a_n \in \mathbb{R}$, then $\sum_{n=1}^{X} a_n \ll X$.*

Proof. Section 3 in Chapter XV of Lang [21]. □

We now estimate the contribution of the terms in $\sum_{\pm}$ in equation (3.12). First, the contribution from $\sum_{+}$ is

$$\ll_{\ell,r} X^{1+2\theta-\frac{j}{\ell}} \sum_{n=1}^{\infty} \frac{|\lambda_{\tilde{f}\times g}(n)|}{n^{\frac{j}{\ell}-2\theta}} \ll_{\ell,r} X^{1+2\theta-\frac{r+1}{\ell}}$$

for $j > r+1$ since the sum converges. To estimate the terms in $\sum_{-}$ let

$$n_0 = \frac{1}{2}\min\{1, 2^{\beta-\frac{1}{\ell}}\}(\alpha\beta X^{\beta})^{\ell} X^{-1} \quad \text{and} \quad n_1 = 2\max\{1, 2^{\beta-\frac{1}{\ell}}\}(\alpha\beta X^{\beta})^{\ell} X^{-1}.$$

Then $I_k(n;-) \ll (nX)^{-\frac{j}{\ell}}$ when $n \notin (n_0, n_1)$ and the corresponding contribution is also $O_{\ell,r}\left(X^{1+2\theta-\frac{r+1}{\ell}}\right)$. Equation (3.12) is therefore reduced to

$$\sum_{n=1}^{\infty} \lambda_{f\times\tilde{g}}(n)\psi(n) = \sum_{k=0}^{r} X^{1+2\theta-\frac{k}{\ell}} \times \sum_{n_0<n<n_1} \frac{\lambda_{\tilde{f}\times g}(n)}{n^{\frac{k}{\ell}-2\theta}} c_{k,\ell}^{-} I_k(n;-) + O_{\ell,r}\left(X^{1+2\theta-\frac{r+1}{\ell}}\right). \quad (3.14)$$

Now either

$$2\max\{1, 2^{\beta-\frac{1}{\ell}}\}(\alpha\beta)^{\ell} \leq X^{1-\beta\ell} \quad \text{or} \quad 2\max\{1, 2^{\beta-\frac{1}{\ell}}\}(\alpha\beta)^{\ell} > X^{1-\beta\ell}.$$

In the former case $n_1 \leq 1$, the main term disappears, and we have

$$\sum_{n=1}^{\infty} \lambda_{f\times\tilde{g}}(n)\psi(n) \ll_{\ell,\beta,r} X^{1+2\theta-\frac{r+1}{\ell}} \ll_{\ell,\beta,M} X^{-M}$$

for any $M > 0$ by taking r sufficiently large in terms of M. This proves part (i) of Theorem 3.2. In the latter case $n_1 > 1$ and there are two subcases: $\beta \neq \frac{1}{\ell}$ or $\beta = \frac{1}{\ell}$. If $\beta \neq \frac{1}{\ell}$ then

$$(\alpha X^\beta t^{\ell\beta} - (nX)^{\frac{1}{\ell}} \ell t)'' = \alpha(\ell\beta)(\ell\beta - 1)X^\beta t^{\ell\beta-2} \gg_{\ell,\beta} \alpha X^\beta.$$

Applying Lemma 5.1.3 in Huxley [13] we have $I_k(n; -) \ll_{\beta,\ell} (\alpha X^\beta)^{-\frac{1}{2}}$, and using (3.13) the main term is

$$\ll_{\ell,\beta} X^{1+2\theta}(\alpha X^\beta)^{-\frac{1}{2}} \sum_{n_0<n<n_1} \frac{|\lambda_{\tilde{f}\times g}(n)|}{n^{-2\theta}} \ll_{\ell,\beta} (n_1 X)^{1+2\theta}(\alpha X^\beta)^{-\frac{1}{2}} \ll_{\ell,\beta} (\alpha X^\beta)^{\frac{\ell}{2}}.$$

For the subcase $\beta = \frac{1}{\ell}$ we take $I_k(n; -) \ll 1$ in equation (3.14) to obtain

$$\ll_{\ell,\beta} X^{1+2\theta} \sum_{n_0<n<n_1} \frac{|\lambda_{\tilde{f}\times g}(n)|}{n^{-2\theta}} \ll_{\ell,\beta} (n_1 X)^{1+2\theta} \ll_{\ell,\beta} (\alpha X^\beta)^{\frac{1+\ell}{2}}.$$

This proves part (ii) of Theorem 3.2.

Note that when $\beta = \frac{1}{\ell}$ we have $I := (n_0, n_1) = \left(\frac{1}{2}\left(\frac{\alpha}{\ell}\right)^\ell, 2\left(\frac{\alpha}{\ell}\right)^\ell\right)$ and

$$I_k(n; -) = \int_0^\infty \phi(t^\ell) e\left((\alpha - \ell n^{\frac{1}{\ell}})X^{\frac{1}{\ell}} t\right) t^{2\ell\theta-k+\ell-1} dt.$$

Now $n_1 > 1$ so $(\alpha/\ell)^\ell > 1/2$ and hence there is a unique integer $n_\alpha \geq 1$ such that

$$\left(\frac{\alpha}{\ell}\right)^\ell = n_\alpha + \lambda \quad \text{with} \quad -\frac{1}{2} < \lambda \leq \frac{1}{2}.$$

Moreover, $|n^{\frac{1}{\ell}} - \alpha/\ell| \gg_\ell |n - n_\alpha|\alpha^{1-\ell}$ for $n \in I, n \neq n_\alpha$, and repeated integration by parts gives

$$I_k(n; -) \ll_{\ell,j} \frac{1}{\left(|n - n_\alpha|\alpha^{1-\ell} X^{\frac{1}{\ell}}\right)^j}$$

for $j \geq 0$. Therefore, the contribution of the main terms in (3.14) without $n = n_\alpha$ is

$$\begin{aligned}
&\ll_\ell \quad X^{1+2\theta}\left(\alpha^{\ell-1}X^{-\frac{1}{\ell}}\right)^j \sum_{\substack{n_0<n<n_1\\ n\neq n_\alpha}} \frac{|\lambda_{\tilde{f}\times g}(n)|}{n^{-2\theta}} \cdot \frac{1}{|n-n_\alpha|^j} \\
&\ll_\ell \quad X^{1+2\theta}\left(\alpha^{\ell-1}X^{-\frac{1}{\ell}}\right)^j n_1^{\frac{1}{2\ell}} \ll_\ell X^{1+2\theta}\left(\alpha^{\ell-1}X^{-\frac{1}{\ell}}\right)^j \alpha^{\frac{1}{2}} \qquad (3.15)
\end{aligned}$$

for $j \geq 1$. Here we have used (3.13). Taking $0 < \varepsilon < \frac{1}{\ell}$ we have $\alpha^{\ell-1}X^{-\frac{1}{m}} < X^{-\varepsilon}$ whenever $X > \alpha^{\frac{m(m-1)}{1-m\varepsilon}}$. For j sufficiently large in terms of r this last expression is $\ll X^{1+2\theta-\frac{r+1}{\ell}}$ in (3.15). Hence,

$$\sum_{n=1}^{\infty} \lambda_{f\times\tilde{g}}(n)\psi(n) = \frac{\lambda_{\tilde{f}\times\tilde{g}}(n_\alpha)}{n_\alpha} \sum_{k=0}^{r} c_{k,\ell}^{-} I_k(n_\alpha;-)(n_\alpha X)^{1+2\theta-\frac{k}{\ell}} + O_{\ell,r,\varepsilon}\left(X^{1+2\theta-\frac{r+1}{\ell}}\right)$$

which proves part (iii) of Theorem 3.2. Finally, to prove part (iv), if $\left(\frac{\alpha}{\ell}\right)^\ell = q$ is an integer, then $\alpha = \ell q^{\frac{1}{\ell}}$ and $n_\alpha = q$. Therefore,

$$I_k(n_\alpha,-) = \int_0^\infty \phi(t^\ell)t^{2\ell\theta-k+\ell-1}dt = \frac{1}{\ell}\int_0^\infty \phi(x)x^{\frac{1}{2\ell}-\frac{1}{2}-\frac{k}{\ell}}dx.$$

This completes the proof of Theorem 3.2 with $e(\alpha n^\beta)$. The proof when $e(-\alpha n^\beta)$ is similar.

CHAPTER 4
THE $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ CASE

4.1 The Rankin Triple L-function

For $i = 1, 2, 3$, let

$$f_i(z) = \sum_{n=1}^{\infty} \lambda_{f_i}(n) n^{\frac{2k-1}{2}} e(nz)$$

be a holomorphic cusp form of weight $2k$ for $SL_2(\mathbb{Z})$ which is also a Hecke eigenform. Note that we have normalized the coefficients. Following Garrett [9], Kim and Shahidi [19], etc. we can form the Rankin triple product L-function

$$L(s, f_1 \times f_2 \times f_3) := \prod_p L_p(s, f_1 \times f_2 \times f_3) = \sum_{n=1}^{\infty} \frac{\lambda_{f_1 \times f_2 \times f_3}(n)}{n^s}. \tag{4.1}$$

Here the local L-factors are defined as

$$L_p(s, f_1 \times f_2 \times f_3) := \prod_{\substack{\eta_i = \pm 1 \\ 1 \leq i \leq 3}} (1 - \alpha_{1,p}^{\eta_1} \alpha_{2,p}^{\eta_2} \alpha_{3,p}^{\eta_2} p^{-s})^{-1}$$

where the product is taken over all $2^3 = 8$ possibilities of the exponents, and $\alpha_{i,p}$ are the parameters associated to f_i at place $p < \infty$. Initially the L-function converges absolutely for $\Re(s) \gg 1$, but we can improve this to $\Re(s) > 1$ since $|\alpha_{i,p}| = 1$ for $i = 1, 2, 3$ at each place p (i.e. the Ramanujan Conjecture is known in this case).

The completed L-function

$$\Lambda(s, f_1 \times f_2 \times f_3) := (2\pi)^{-4s-12k+6} \Gamma\left(s + k - \frac{1}{2}\right)^3 \Gamma\left(s + 3k - \frac{3}{2}\right) L(s, f_1 \times f_2 \times f_3)$$

satisfies the function equation

$$\Lambda(s, f_1 \times f_2 \times f_3) = -\Lambda(1 - s, \tilde{f}_1 \times \tilde{f}_2 \times \tilde{f}_3)$$

and is entire unless all of the π_i, the automorphic representations associated to f_i, are monomial (i.e. π_i is stable under twists by a nontrivial character of $\mathbb{Q}^\times / \mathbb{A}_{\mathbb{Q}}^\times$).

Using the duplication formula for the Gamma function

$$\Gamma(z) = (2\pi)^{-\frac{1}{2}} 2^{z-\frac{1}{2}} \Gamma\left(\frac{z}{2}\right) \Gamma\left(\frac{z+1}{2}\right)$$

one can show that

$$\Gamma\left(s+k-\frac{1}{2}\right)^3 \Gamma\left(s+3k-\frac{3}{2}\right) = (2\pi)^{-2} 2^{4s+6k-5}$$
$$\times \Gamma\left(\frac{s}{2}+\frac{k}{2}-\frac{1}{4}\right)^3 \Gamma\left(\frac{s}{2}+\frac{k}{2}+\frac{1}{4}\right)^3 \Gamma\left(\frac{s}{2}+\frac{3k}{2}-\frac{3}{4}\right) \Gamma\left(\frac{s}{2}+\frac{3k}{2}-\frac{1}{4}\right).$$

Thus, the completed L-function can then be rewritten as

$$\Lambda(s, f_1 \times f_2 \times f_3) = \pi^{-4s-12k+4} 2^{-6k-1} \Gamma(s, f_1 \times f_2 \times f_3) L(s, f_1 \times f_2 \times f_3)$$

where we have defined

$$\Gamma(s, f_1 \times f_2 \times f_3) :=$$
$$\Gamma\left(\frac{s}{2}+\frac{k}{2}-\frac{1}{4}\right)^3 \Gamma\left(\frac{s}{2}+\frac{k}{2}+\frac{1}{4}\right)^3 \Gamma\left(\frac{s}{2}+\frac{3k}{2}-\frac{3}{4}\right) \Gamma\left(\frac{s}{2}+\frac{3k}{2}-\frac{1}{4}\right)$$

to simplify notation.

4.2 The Summation Formula

In this section we prove Theorem 1.1 for case (III) where f is the product of three weight $2k$ holomorphic cusp forms for $SL_2(\mathbb{Z})$.

Theorem 4.1. *Let f_i be a holomorphic cusp form for $SL_2(\mathbb{Z})$ for $i = 1, 2, 3$. Let $\lambda_{f_1 \times f_2 \times f_3}(n)$ be the coefficients of the associated triple product L-function, and assume* *further that this L-function is entire. Let $\psi(x) \in C^\infty(0,\infty)$ with compact support, then*

$$\sum_{n=1}^{\infty} \lambda_{f_1\times f_2\times f_3}(n)\psi(n) = -2\pi^8 \sum_{n=1}^{\infty} \lambda_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(n) \int_0^\infty \psi(x) G_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(\pi^{16}x^2n^2)dx \quad (4.2)$$

where the function $G_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(x)$ is given by

$$G_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(x) = G^{8,0}_{0,16}\left(\begin{matrix} - \\ \boldsymbol{b}_1,\ldots,\boldsymbol{b}_{16} \end{matrix} \,\middle|\, x \right),$$

and the parameters $\boldsymbol{b}_j$, which depend on k, are given in equation (4.5).

From the functional equation we have

$$L(s, f_1\times f_2\times f_3) = -\pi^{4(2s-1)} \frac{\Gamma(1-s,\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3)}{\Gamma(s,f_1\times f_2\times f_3)} L(1-s,\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3).$$

Let $\psi(s)$ be a smooth function of compact support and let $\mathcal{M}(\psi)$ denote its Mellin transform. Then multiplying both sides by $\frac{1}{2\pi i}\mathcal{M}(\psi)(s)$ and integrating along the line $\Re(s) = 1+\varepsilon$ for some $\varepsilon > 0$ we obtain

$$\begin{aligned} \frac{1}{2\pi i}\int_{(1+\varepsilon)} L(s,f_1\times f_2\times f_3)\mathcal{M}(\psi)(s)ds = -\frac{1}{2\pi i}\int_{(1+\varepsilon)} \pi^{4(2s-1)}\mathcal{M}(\psi)(s) \\ \times \frac{\Gamma(1-s,\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3)}{\Gamma(s,f_1\times f_2\times f_3)} L(1-s,\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3)ds. \quad (4.3) \end{aligned}$$

Note that the integration is well defined since the L-function has at most polynomial growth in vertical strips of $\Re(s) > 0$ and $\mathcal{M}(\psi)(s)$ is of rapid decay as $\Im(s) \to \pm\infty$. We evaluate the left hand side of (4.3) by interchanging the summation and integration, which is valid by Lemma 3.1, to obtain the sum

$$\sum_{n=1}^{\infty} \lambda_{f_1\times f_2\times f_3}(n)\cdot\frac{1}{2\pi i}\int_{(1+\varepsilon)} \mathcal{M}(\psi)(s)n^{-s}ds = \sum_{n=1}^{\infty}\lambda_{f_1\times f_2\times f_3}(n)\psi(n).$$

To evaluate the right hand side of (4.3) we shift the contour left to $\Re(s) = -1$ and make the change of variables $s \mapsto 1-s$ to get

$$-\frac{1}{2\pi i}\int_{(2)} \pi^{4(1-2s)}\mathcal{M}(\psi)(1-s)\frac{\Gamma(s,\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3)}{\Gamma(1-s,f_1\times f_2\times f_3)}L(s,\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3)ds.$$

Observe that the integrand has no poles because, by the functional equation, it is equal to $L(1-s,f_1\times f_2\times f_3)\mathcal{M}(\psi)(1-s)$ which is entire. Also, the residual horizontal integrals vanish since $\mathcal{M}(\psi)$ has rapid decay. Putting in the series for $L(s,\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3)$ we have

$$-\frac{1}{2\pi i}\int_{(2)} \pi^{4(1-2s)}\mathcal{M}(\psi)(1-s)\frac{\Gamma(s,\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3)}{\Gamma(1-s,f_1\times f_2\times f_3)}\sum_{n=1}^{\infty}\frac{\lambda_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(n)}{n^s}ds$$

and with the change of variables $s \mapsto 2s$ this becomes

$$-2\pi^4\frac{1}{2\pi i}\int_{(1)}\sum_{n=1}^{\infty}\lambda_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(n)\left(\pi^8n^2\right)^{-s} \times \mathcal{M}(\psi)(1-2s)\frac{\Gamma(2s,\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3)}{\Gamma(1-2s,f_1\times f_2\times f_3)}ds. \quad (4.4)$$

Now interchanging the order of the summation and integration in (4.4), which is valid by Lemmas 3.1 and 3.2, and replacing $s \mapsto -s$ we have

$$-2\pi^4\frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{(-1)}\lambda_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(n)\left(\pi^8n^2\right)^{s}\mathcal{M}(\psi)(1+2s)\frac{\Gamma(-2s,\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3)}{\Gamma(1+2s,f_1\times f_2\times f_3)}ds.$$

To ensure absolute convergence of the integral we deform the line of integration to the contour L such that all of the poles in the numerator of $\Gamma(-2s,\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3)$ lie to the right of the contour L, and $s=\sigma+i\tau\in L$ satisfy $\sigma > -\frac{3}{16}$ as $\tau\to\pm\infty$. Finally, putting in the integral expression for $\mathcal{M}(\psi)(1+2s)$ we obtain

$$-2\pi^4\sum_{n=1}^{\infty}\lambda_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(n)\int_0^{\infty}\psi(x)\frac{1}{2\pi i}\int_L\left(\pi^8n^2x^2\right)^s\frac{\Gamma(-2s,\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3)}{\Gamma(1+2s,f_1\times f_2\times f_3)}dsdx.$$

The Gamma factors here are

$$\frac{\Gamma\left(-s+\frac{k}{2}-\frac{1}{4}\right)^3\Gamma\left(-s+\frac{k}{2}+\frac{1}{4}\right)^3\Gamma\left(-s+\frac{3k}{2}-\frac{3}{4}\right)\Gamma\left(-s+\frac{3k}{2}-\frac{1}{4}\right)}{\Gamma\left(s+1+\frac{k}{2}-\frac{3}{4}\right)^3\Gamma\left(s+1+\frac{k}{2}-\frac{1}{4}\right)^3\Gamma\left(s+1+\frac{3k}{2}-\frac{5}{4}\right)\Gamma\left(s+1+\frac{3k}{2}-\frac{3}{4}\right)},$$

so we may take L in the strip $-\frac{3}{16} < \Re(s) < \frac{1}{4}$ since all of the poles occur at or to the right of $\frac{1}{4}$. Thus, we can rewrite this using the Meijer G-function to obtain the summation formula

$$\sum_{n=1}^{\infty}\lambda_{f_1\times f_2\times f_3}(n)\psi(n) = -2\pi^4\sum_{n=1}^{\infty}\lambda_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(n)\int_0^{\infty}\psi(x)G_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(\pi^8x^2n^2)dx$$

where

$$G_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(x) = G^{8,0}_{0,16}\left(\left.\begin{matrix}\overline{\qquad\qquad}\\ \mathbf{b}_1,\ldots,\mathbf{b}_{16}\end{matrix}\,\right|\,x\right).$$

Here the indicies $\{\mathbf{b}_j\}_{j=1}^{16}$ are given by

$$\begin{aligned}\mathbf{b}_1=\mathbf{b}_2=\mathbf{b}_3=\frac{k}{2}-\frac{1}{4},\quad \mathbf{b}_4=\mathbf{b}_5=\mathbf{b}_6=\frac{k}{2}+\frac{1}{4},\\ \mathbf{b}_7=\mathbf{b}_1+k-\frac{1}{2},\quad \mathbf{b}_8=\mathbf{b}_4+k-\frac{1}{2}\\ \mathbf{b}_9=\mathbf{b}_{10}=\mathbf{b}_{11}=-\frac{k}{2}+\frac{3}{4},\quad \mathbf{b}_{12}=\mathbf{b}_{13}=\mathbf{b}_{14}=-\frac{k}{2}+\frac{1}{4},\\ \mathbf{b}_{15}=\mathbf{b}_9-k+\frac{1}{2},\quad \mathbf{b}_{16}=\mathbf{b}_{12}-k+\frac{1}{2}.\end{aligned} \tag{4.5}$$

It will be convenient in our application toward resonance sums to rewrite our summation formula as

$$\sum_{n=1}^{\infty}\lambda_{f_1\times f_2\times f_3}(n)\psi(n) = -2\pi^8\sum_{n=1}^{\infty}\lambda_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(n)\int_0^{\infty}\psi(x)G_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(\pi^{16}x^2n^2)dx.$$

4.3 Resonance Sums

The particular choice of $\psi(x) = e(\alpha x^{\beta})\phi\left(\frac{x}{X}\right)$ in equation (4.2), together with the known asymptotics of the Meijer G-function, yield case (III) of Theorem 1.2.

Theorem 4.2. *Let f_i be a holomorphic cusp form for $SL_2(\mathbb{Z})$ for $i = 1, 2, 3$. Let $\lambda_{f_1\times f_2\times f_3}(n)$ be the coefficients of the associated L-function, and assume further that this L-function is entire. Let $\phi(x) \in C^\infty(0,\infty)$ with compact support in $[1,2]$ and $\phi^{(j)}(x) \ll 1$ for $j \geq 1$. Moreover, let $X > 1$, and $\alpha, \beta \geq 0$.*

(i) If $2\max\{1, 2^{\beta-\frac{1}{8}}\}(\alpha\beta)^8 \leq X^{1-\frac{\beta}{8}}$, then

$$\sum_{n=1}^{\infty} \lambda_{f_1\times f_2\times f_3}(n)\phi\left(\frac{n}{X}\right) e(\pm\alpha n^\beta) \ll_{\beta,M} X^{-M}$$

holds for any $M > 0$.

(ii) If $2\max\{1, 2^{\beta-\frac{1}{8}}\}(\alpha\beta)^8 > X^{1-\frac{\beta}{8}}$, then

$$\sum_{n=1}^{\infty} \lambda_{f_1\times f_2\times f_3}(n)\phi\left(\frac{n}{X}\right) e(\pm\alpha n^\beta) \ll_{\beta} (\alpha X^\beta)^4$$

holds for $\beta \neq \frac{1}{8}$, and

$$\sum_{n=1}^{\infty} \lambda_{f_1\times f_2\times f_3}(n)\phi\left(\frac{n}{X}\right) e(\pm\alpha n^\beta) \ll_{\beta} (\alpha X^\beta)^{\frac{9}{2}}$$

holds for $\beta = \frac{1}{8}$.

(iii) If $X > \alpha^{\frac{56}{1-8\varepsilon}}$ with $0 < \varepsilon < \frac{1}{8}$, then

$$\sum_{n=1}^{\infty} \lambda_{f_1\times f_2\times f_3}(n)\phi\left(\frac{n}{X}\right) e(\pm\alpha n^{\frac{1}{8}}) = \frac{\lambda_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(n_\alpha)}{n_\alpha} \times \sum_{l=0}^{r} c_l^- I_l(n_\alpha; -)(n_\alpha X)^{\frac{9}{16}-\frac{l}{8}} + O_{r,\varepsilon}\left(X^{\frac{9}{16}-\frac{r+1}{8}}\right)$$

for any $r > \frac{7}{2}$. Here n_α is the unique positive integer satisfying $\left(\frac{\alpha}{8}\right)^8 - n_\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right]$,

$$I_l(n_\alpha; -) = \int_0^\infty \phi(t^8) e\left(\left(\alpha - 8n_\alpha^{\frac{1}{8}}\right) X^{\frac{1}{8}} t\right) t^{\frac{7}{2}-l} dt,$$

and c_l^- are constants depending on f and k.

(iv) In particular, if q is a positive integer and $0 < \varepsilon < \frac{1}{8}$, then for $X > (8^8 q)^{\frac{7}{1-8\varepsilon}}$ we have

$$\sum_{n=1}^{\infty} \lambda_{f_1\times f_2\times f_3}(n)\phi\left(\frac{n}{X}\right) e(\pm 8(qn)^{\frac{1}{8}}) = \frac{\lambda_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(q)}{q} \times \sum_{l=0}^{r} c_l^- I_l(q;-)(qX)^{\frac{9}{16}-\frac{l}{8}} + O_{r,\varepsilon}\left(X^{\frac{9}{16}-\frac{r+1}{8}}\right)$$

for any $r > \frac{7}{2}$ where

$$I_l(q;-) = \frac{1}{8}\int_0^{\infty} \phi(x) x^{\frac{7}{16}-\frac{l}{8}} dx$$

and c_l^- are constants depending on f and k.

The proof follows that of Section 3.3 with the major difference being the Meijer G-function and its parameters. We begin by applying Lemma 2.1 from Section 2.3 to obtain

$$G_{0,16}^{8,0}(x) = A_{16}^{8,0} H_{0,16}\left(xe^{8\pi i}; r\right) + \bar{A}_{16}^{8,0} H_{0,16}\left(xe^{-8\pi i}; r\right)$$

for any integer $r \geq 0$ and real $x \to +\infty$ where

$$A_{16}^{8,0} = \left(-\frac{1}{2\pi i}\right)^8 \exp\left(-i\pi \sum_{j=9}^{16} \mathbf{b}_j\right) = \left(-\frac{1}{2\pi i}\right)^8 e\left(\frac{6k-5}{2}\right),$$

$$H_{0,16}(x;r) = \frac{(2\pi)^8}{2\sqrt{8\pi}} \exp\left(-16x^{\frac{1}{16}}\right) x^{\theta} \sum_{l=0}^{r} M_l x^{-\frac{l}{16}} + O\left(x^{\theta-\frac{r+1}{16}}\right),$$

$$\theta = \frac{1}{32}\left(-15 + 2\sum_{j=1}^{16} \mathbf{b}_j\right) = -\frac{7}{32},$$

and the M_l's are constants depending on k. It follows that

$$\begin{aligned} H_{0,16}(\pi^{16}x^2n^2e^{\pm 8\pi i}; r) &= \frac{(2\pi)^8}{2\sqrt{8\pi}} \exp\left(-16\pi(xn)^{\frac{1}{8}}(\pm i)\right)\left(\pi^{16}x^2n^2e^{\pm 8\pi i}\right)^{-\frac{7}{32}} \\ &\times \sum_{l=0}^{r} M_l(\pm i\pi)^{-l}(xn)^{-\frac{l}{8}} + O\left((xn)^{-\frac{7}{16}-\frac{r+1}{8}}\right) \end{aligned}$$

and taking $\psi(x) = \phi(\frac{x}{X})e(\alpha x^\beta)$ we obtain

$$\psi(x)G^{8,0}_{0,16}(\pi^{16}x^2n^2) = \sum_{\pm}\phi\left(\frac{x}{X}\right)e\left(\alpha x^\beta \pm 8(xn)^{\frac{1}{8}}\right) \times \sum_{l=0}^{r} M_l^{\pm}(xn)^{-\frac{7+2l}{16}} + O\left(\left|\phi\left(\frac{x}{X}\right)\right|(xn)^{-\frac{9+2r}{16}}\right)$$

where we have condensed the constants into $M_l^{\pm}$. With this expansion the resonance sum in equation (4.2) becomes

$$\sum_{n=1}^{\infty}\lambda_{f_1\times f_2\times f_3}(n)\psi(n) = -2\pi^8\sum_{n=1}^{\infty}\sum_{\pm}\lambda_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(n) \times\int_0^\infty \phi\left(\frac{x}{X}\right)e\left(\alpha x^\beta \pm 8(xn)^{\frac{1}{8}}\right)\sum_{l=0}^{r}M_l^{\pm}(xn)^{-\frac{7+2l}{16}}dx + O\left(\sum_{n=1}^{\infty}|\lambda_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(n)|\int_0^\infty\left|\phi\left(\frac{x}{X}\right)\right|(xn)^{-\frac{9+2r}{16}}dx\right).$$

Making the variable change $x \mapsto Xt^8$ and simplifying error term yields

$$\sum_{n=1}^{\infty}\lambda_{f_1\times f_2\times f_3}(n)\psi(n) = \sum_{l=0}^{r}X^{\frac{9-2l}{16}}\sum_{n=1}^{\infty}\frac{\lambda_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(n)}{n^{\frac{7+2l}{16}}}\sum_{\pm}c_l^{\pm}I_l(n;\pm) + O(X^{\frac{7-2r}{16}})$$

where the constants have be condensed into $c_l^{\pm}$ and

$$I_l(n;\pm) := \int_0^\infty \phi(t^8)e\left(\alpha X^\beta t^{8\beta} \pm 8(Xn)^{\frac{1}{8}}t\right)t^{\frac{7-2l}{2}}dt.$$

Note that we have simplified the error term using the fact that $L_{\tilde{f}_1\times\tilde{f}_2\times\tilde{f}_3}(\frac{9+2r}{16})$ converges absolutely for $r > \frac{7}{2}$. The remainder of the calculations in the proof at this point are similar to those of Section 3.3 and will not be repeated here.

APPENDIX
THE $SL_m(\mathbb{Z})$ CASE

The Summation Formula

In this section we discuss the proof of Theorem 1.1 for case (I) where f is a single Maass cusp form for $SL_m(\mathbb{Z})$. We remark that similar results may be obtained for holomorphic cusp forms of $SL_2(\mathbb{Z})$ as well as for arbitrary congruence subgroups.

Theorem A.1. *Let f be a Maass cusp form for $SL_m(\mathbb{Z})$ with $m \geq 2$, and let $\lambda_f(n)$ be the coefficients of the associated L-function $L(s,f)$. Let $\psi(x) \in C^\infty(0,\infty)$ with compact support, then*

$$\sum_{n=1}^{\infty} \lambda_f(n)\psi(n) = 2\pi^{\frac{m}{2}} \sum_{n=1}^{\infty} \lambda_{\tilde{f}}(n) \int_0^\infty \psi(x) G_{\tilde{f}}(\pi^{2m}x^2n^2)dx \tag{A.1}$$

where $G_{\tilde{f}}(x)$ is given by

$$G_{\tilde{f}}(x) = G_{0,2m}^{m,0}\left(\left. \begin{matrix} \text{---} \\ -\frac{\overline{\mu_f(1)}}{2}, \ldots, -\frac{\overline{\mu_f(m)}}{2}, \frac{1+\mu_f(1)}{2}, \ldots, \frac{1+\mu_f(m)}{2} \end{matrix} \right| x \right).$$

Here $G_{0,2m}^{m,0}$ is the Meijer G-function and $\mu_f(j)$ are the parameters at ∞.

This Voronoi-type summation formula for a single Maass cusp form is effectively known via the full Voronoi summation formulas of Miller & Schmid [25, 26] and Goldfeld & Li [11, 12]. The full Voronoi summation formula for $SL_m(\mathbb{Z})$ is given by

$$\sum_{n\neq 0} A(c_{m-2}, c_{m-3}, \cdots, c_1, n) e\left(-\frac{nh}{q}\right)\psi(|n|)$$
$$= q \sum_{d_1|c_1} \sum_{d_2|\frac{c_1c_2q}{d_1}} \cdots \sum_{d_{m-2}|\frac{c_1\cdots c_{m-2}q}{d_1\cdots d_{m-3}}} \sum_{n\neq 0} \frac{A(n, d_{m-2}, \cdots, d_1)}{d_1\cdots d_{m-2}|n|}$$
$$\times S(n, h; q, c, d)\Psi\left(\frac{|n|}{q^m}\prod_{i=1}^{m-2}\frac{d_i^{m-i}}{c_i^{m-i-1}}\right) \quad \text{(A.2)}$$

where ψ is as above, $A(n_1, \ldots, n_{m-1})$ are the Fourier-Whittaker coefficients, $c = (c_1, \ldots, c_{m-2}), d = (d_1, \ldots, d_{m-2}), S(n, h; q, c, d)$ is the hyper-Kloosterman sum

$$S(n, h; q, c, d) =$$
$$\sum_{\substack{x_j \in (\mathbb{Z}\frac{c_1\cdots c_j q}{d_1\cdots d_j}\mathbb{Z})^* \\ j\leq m-2}} e\left(\frac{d_1x_1n}{q} + \frac{d_2x_2\overline{x_1}}{\frac{c_1q}{d_1}} + \cdots + \frac{d_{m-2}x_{m-2}\overline{x_{m-3}}}{\frac{c_1\cdots c_{m-3}q}{d_1\cdots d_{m-3}}} + \frac{h\overline{x_{m-2}}}{\frac{c_1\cdots c_{m-2}q}{d_1\cdots d_{m-2}}}\right),$$

and Ψ is given by

$$\Psi(x) = \frac{1}{2\pi i}\int_{(-\sigma)} \mathcal{M}(\psi)(s)\frac{\tilde{F}(1-s)}{F(s)}x^s ds$$

with

$$F(s) = \pi^{-\frac{ms}{2}}\prod_{i=1}^{m}\Gamma\left(\frac{s-\mu_f(j)}{2}\right)$$

and $\tilde{F}$ defined similarly with the parameters $\mu_{\tilde{f}}(j)$ associated to the dual $\tilde{f}$. When $q = 1$ the Voronoi summation formula has the particularly simple form

$$\sum_{n\neq 0} A(1, \ldots, 1, n)\psi(|n|) = \sum_{n\neq 0}\frac{A(n, 1, \ldots, 1)}{|n|}\Psi(|n|),$$

and this may be rewritten in the form of equation (A.1) following the ideas in the proof of Theorem 3.1.

We now proceed with the proof of Theorem A.1. Recall from Section 2.2 that L-function associated to f is given by

$$L(s,f)=\sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s}=\prod_p\prod_{j=1}^{m}(1-\alpha_{p,j}p^{-s})^{-1}$$

where $\lambda_f(n)=A_f(n,1,\ldots,1)\in\mathbb{C}$ are the normalized Fourier coefficients, and $\alpha_{p,j}\in\mathbb{C}$ are the parameters at $p<\infty$. Moreover, the completed L-function is given by

$$\Lambda(s,f)=\pi^{-\frac{ms}{2}}\prod_{j=1}^{m}\Gamma\left(\frac{s-\mu_f(j)}{2}\right)L(s,f)$$

where the $\mu_f(j)$ are the parameters at infinity. This completed L-function is entire (unless $m=1$ and $L(s,f)=\zeta(s)$) and it satisfies the functional equation

$$\Lambda(s,f)=\Lambda(1-s,\tilde{f})$$

where $\tilde{f}$ is the dual Maass form.

From this functional equation we have

$$L(s,f)=\pi^{ms-\frac{m}{2}}\prod_{j=1}^{m}\frac{\Gamma\left(\frac{1-s-\mu_{\tilde{f}}(j)}{2}\right)}{\Gamma\left(\frac{s-\mu_f(j)}{2}\right)}L(1-s,\tilde{f}).$$

Multiplying both sides by $\frac{1}{2\pi i}\mathcal{M}(\psi)(s)$ and integrating along the line $\Re(s)=1+\varepsilon$ for some $\varepsilon>0$ we have

$$\frac{1}{2\pi i}\int_{(1+\varepsilon)}L(s,f)\mathcal{M}(\psi)(s)ds$$
$$=\frac{1}{2\pi i}\int_{(1+\varepsilon)}\pi^{ms-\frac{m}{2}}\mathcal{M}(\psi)(s)\prod_{j=1}^{m}\frac{\Gamma\left(\frac{1-s-\mu_{\tilde{f}}(j)}{2}\right)}{\Gamma\left(\frac{s-\mu_f(j)}{2}\right)}L(1-s,\tilde{f})ds.$$

As in the previous sections we evaluate the left hand side by interchanging the summation and integration to get

$$\sum_{n=1}^{\infty}\lambda_f(n)\cdot\frac{1}{2\pi i}\int_{(1+\varepsilon)}\mathcal{M}(\psi)(s)n^{-s}ds=\sum_{n=1}^{\infty}\lambda_f(n)\psi(n).$$

To deal with the right hand side we shift the contour left to $\Re(s) = -1$ and make the change of variables $s \mapsto 1 - s$ to get

$$\frac{1}{2\pi i}\int_{(2)} \pi^{-ms+\frac{m}{2}}\mathcal{M}(\psi)(1-s)\prod_{j=1}^{m}\frac{\Gamma\left(\frac{s-\mu_{\tilde{f}}(j)}{2}\right)}{\Gamma\left(\frac{1-s-\mu_f(j)}{2}\right)}L(s,\tilde{f})ds.$$

Putting in the series for $L(s, \tilde{f})$ and making the change of variables $s \mapsto 2s$ this becomes

$$2\pi^{\frac{m}{2}}\cdot\frac{1}{2\pi i}\int_{(1)}\sum_{n=1}^{\infty}\lambda_{\tilde{f}}(n)\left(\pi^{2m}n^2\right)^{-s}\mathcal{M}(\psi)(1-2s)\prod_{j=1}^{m}\frac{\Gamma\left(s-\frac{\mu_{\tilde{f}}(j)}{2}\right)}{\Gamma\left(1-s-\frac{1+\mu_f(j)}{2}\right)}ds.$$

Now interchanging the order of the summation and integration, which is justified by Lemmas 3.1 & 3.2, shifting the contour back to $\Re(s) = 1$, and then writing $\mathcal{M}(\psi)(1 - 2s)$ as an integral we have

$$2\pi^{\frac{m}{2}}\sum_{n=1}^{\infty}\lambda_{\tilde{f}}(n)\int_0^{\infty}\psi(x)\cdot\frac{1}{2\pi i}\int_{(1)}\left(\pi^{2m}x^2n^2\right)^{-s}\prod_{j=1}^{m}\frac{\Gamma\left(s-\frac{\mu_{\tilde{f}}(j)}{2}\right)}{\Gamma\left(1-s-\frac{1+\mu_f(j)}{2}\right)}dsdx.$$

Finally, using that $\mu_{\tilde{f}}(j) = \overline{\mu_f(j)}$ and taking $s \mapsto -s$ so that we may identify the integral with the Meijer G-function we obtain

$$2\pi^{\frac{m}{2}}\sum_{n=1}^{\infty}\lambda_{\tilde{f}}(n)\int_0^{\infty}\psi(x)\cdot\frac{1}{2\pi i}\int_{L}\left(\pi^{2m}x^2n^2\right)^{s}\prod_{j=1}^{m}\frac{\Gamma\left(-s-\frac{\overline{\mu_f(j)}}{2}\right)}{\Gamma\left(1+s-\frac{1+\mu_f(j)}{2}\right)}dsdx.$$

Here the contour has been deformed so that all the poles of $\Gamma\left(-s-\frac{\overline{\mu_f(j)}}{2}\right)$ lie on the right to ensure convergence as in Definition 2.6. Note that the integral converges absolutely provided $\sigma + i\tau \in L$ satisfies $\sigma > \frac{1}{2m} - \frac{1}{4}$ as $\tau \to \pm\infty$. Thus, we obtain the summation formula

$$\sum_{n=1}^{\infty}\lambda_f(n)\psi(n) = 2\pi^{\frac{m}{2}}\sum_{n=1}^{\infty}\lambda_{\tilde{f}}(n)\int_0^{\infty}\psi(x)G_{\tilde{f}}(\pi^{2m}x^2n^2)dx \tag{A.3}$$

where the kernel

$$G_{\tilde{f}}(x) = G^{m,0}_{0,2m}\left(\begin{array}{c} - \\ -\frac{\overline{\mu_f(1)}}{2}, \ldots, -\frac{\overline{\mu_f(m)}}{2}, \frac{1+\mu_f(1)}{2}, \ldots, \frac{1+\mu_f(m)}{2} \end{array} \middle|\, x \right)$$

is a Meijer G-function.

Resonance Sums

The particular choice of $\psi(x) = e(\alpha x^{\beta})\phi\left(\frac{x}{X}\right)$ in equation (A.1), together with the known asymptotics of the Meijer G-function, yield case (I) of Theorem 1.2. This theorem was first proved by Ren & Ye [28] after deriving the asymptotics of $\Psi(x)$ in the Voronoi summation formula of equation (A.2).

Theorem A.2. *Let f be a Maass cusp form for $SL_m(\mathbb{Z})$ with $m \geq 2$, and let $\lambda_f(n)$ be the coefficients of the associated L-function $L(s, f)$. Let $\phi(x) \in C^{\infty}(0, \infty)$ with compact support in $[1, 2]$ and $\phi^{(j)}(x) \ll 1$ for $j \geq 1$. Moreover, let $X > 1$, and $\alpha, \beta \geq 0$.*

(i) If $2\max\{1, 2^{\beta-\frac{1}{m}}\}(\alpha\beta)^m \leq X^{1-\beta m}$, then

$$\sum_{n=1}^{\infty} \lambda_f(n)\phi\left(\frac{n}{X}\right) e(\pm\alpha n^{\beta}) \ll_{m,\beta,M} X^{-M}$$

holds for any $M > 0$.

(ii) If $2\max\{1, 2^{\beta-\frac{1}{m}}\}(\alpha\beta)^m > X^{1-\beta m}$, then

$$\sum_{n=1}^{\infty} \lambda_f(n)\phi\left(\frac{n}{X}\right) e(\pm\alpha n^{\beta}) \ll_{m,\beta} (\alpha X^{\beta})^{\frac{m}{2}}$$

holds for $\beta \neq \frac{1}{m}$, and

$$\sum_{n=1}^{\infty} \lambda_f(n)\phi\left(\frac{n}{X}\right) e(\pm\alpha n^{\beta}) \ll_{m,\beta} (\alpha X^{\beta})^{\frac{1+m}{2}}$$

holds for $\beta = \frac{1}{m}$.

(iii) If $X > \alpha^{\frac{m(m-1)}{1-m\varepsilon}}$ *with* $0 < \varepsilon < \frac{1}{m}$*, then*

$$\sum_{n=1}^{\infty} \lambda_f(n)\phi\left(\frac{n}{X}\right) e(\pm\alpha n^{\frac{1}{m}}) = \frac{\lambda_{\tilde{f}}(n_\alpha)}{n_\alpha}$$
$$\times \sum_{k=0}^{r} c_{k,m}^{-} I_k(n_\alpha; -)(n_\alpha X)^{\frac{1}{2m}+\frac{1}{2}-\frac{k}{m}} + O_{m,r,\varepsilon}\left(X^{\frac{1}{2m}+\frac{1}{2}-\frac{r+1}{m}}\right)$$

for any $r > \frac{m-1}{2}$*. Here* n_α *is the unique positive integer satisfying* $\left(\frac{\alpha}{m}\right)^m - n_\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right]$*,*

$$I_k(n_\alpha; -) = \int_0^\infty \phi(t^m) e\left(\left(\alpha - n_\alpha^{\frac{1}{m}} m\right) X^{\frac{1}{m}} t\right) t^{\frac{m}{2}-\frac{1}{2}-k} dt,$$

and $c_{k,m}^{-}$ *are constants depending on* $f, k,$ *and* m*.*

(iv) In particular, if q *is a positive integer and* $0 < \varepsilon < \frac{1}{m}$*, then for* $X > (m^m q)^{\frac{m-1}{1-m\varepsilon}}$ *we have*

$$\sum_{n=1}^{\infty} \lambda_f(n)\phi\left(\frac{n}{X}\right) e(\pm m(qn)^{\frac{1}{m}}) = \frac{\lambda_{\tilde{f}}(q)}{q}$$
$$\times \sum_{k=0}^{r} c_{k,m}^{-} I_k(q; -)(qX)^{\frac{1}{2m}+\frac{1}{2}-\frac{k}{m}} + O_{m,r,\varepsilon}\left(X^{\frac{1}{2m}+\frac{1}{2}-\frac{r+1}{m}}\right)$$

for any $r > \frac{m-1}{2}$ *where*

$$I_k(q; -) = \frac{1}{m}\int_0^\infty \phi(x) x^{\frac{1}{2m}-\frac{1}{2}-\frac{k}{m}} dx$$

and $c_{k,m}^{-}$ *are constants depending on* $f, k,$ *and* m*.*

The proof is similar to that of Section 3.3 with the major difference being the Meijer G-function and its parameters. We begin by applying Lemma 2.1 from Section 2.3 to obtain

$$G_{\tilde{f}}(x) = A^{m,0}{}_{2m} H_{0,2m}\left(xe^{i\pi m}; r\right) + \bar{A}^{m,0}{}_{2m} H_{0,2m}\left(xe^{-i\pi m}; r\right)$$

valid for any integer $r \geq 0$ and real $x \to +\infty$ where

$$A^{m,0}_{\ \ 2m} = \left(-\frac{1}{2\pi i}\right)^m \exp\left(i\pi \sum_{j=1}^{m} \frac{1+\mu_f(j)}{2}\right) = \left(-\frac{1}{2\pi}\right)^m,$$

$$H_{0,2m}(x;r) = \frac{(2\pi)^m}{2\sqrt{\pi m}} \exp\left(-2mx^{\frac{1}{2m}}\right) x^{\theta} \sum_{k=0}^{r} M_k x^{-\frac{k}{2m}} + O\left(x^{\theta-\frac{r+1}{2m}}\right),$$

$$\theta = \frac{1}{4m}\left(1-m+2i\sum_{j=1}^{m} \Im\left(\mu_f(j)\right)\right) = \frac{1-m}{4m},$$

and the M_k's are constants depending on m and $\mu_f(j)$. Here we have used that the summation over $\mu_f(j)$ vanishes. Thus,

$$\begin{aligned} G_{\tilde{f}}\left(\pi^{2m}x^2n^2\right) &= \left(-\frac{1}{2\pi}\right)^m \sum_{\pm} H_{0,2m}\left(\pi^{2m}x^2n^2e^{\mp i\pi m}; r\right) \\ &= \frac{(-1)^m}{2\sqrt{\pi m}} \sum_{\pm} e\left(\pm m\,(xn)^{\frac{1}{m}}\right) \sum_{k=0}^{r} M_k \left(\pi^{2m}x^2n^2e^{\mp i\pi m}\right)^{\theta-\frac{k}{2m}} + O\left((xn)^{2\theta-\frac{r+1}{m}}\right) \end{aligned}$$

and it follows that

$$\begin{aligned} \psi(x)G_{\tilde{f}}(\pi^{2m}x^2n^2) = \frac{(-1)^m}{2\sqrt{\pi m}} &\sum_{\pm} \phi\left(\frac{x}{X}\right) e\left(\alpha x^{\beta} \pm m\,(xn)^{\frac{1}{m}}\right) \\ &\times \sum_{k=0}^{r} M_k \left(\pi^{2m}x^2n^2e^{\mp i\pi m}\right)^{\theta-\frac{k}{2m}} + O\left(\left|\phi\left(\frac{x}{X}\right)\right| (xn)^{2\theta-\frac{r+1}{m}}\right) \end{aligned}$$

where $\psi(x) = e(\alpha x^{\beta})\phi\left(\frac{x}{X}\right)$ as above. With this expansion the resonance sum in equation (A.1) becomes

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_f(n)\psi(n) = \frac{(-1)^m}{\sqrt{m}} \pi^{\frac{m-1}{2}} &\sum_{\pm} \sum_{n=1}^{\infty} \lambda_{\tilde{f}}(n) \\ &\times \int_0^{\infty} \phi\left(\frac{x}{X}\right) e\left(\alpha x^{\beta} \pm m\,(xn)^{\frac{1}{m}}\right) \sum_{k=0}^{r} M_k \left(\pi^{2m}x^2n^2e^{\mp i\pi m}\right)^{\theta-\frac{k}{2m}} dx \\ &+ O\left(\sum_{n=1}^{\infty} |\lambda_{\tilde{f}}(n)| \int_0^{\infty} \left|\phi\left(\frac{x}{X}\right)\right| (xn)^{2\theta-\frac{r+1}{m}} dx\right). \end{aligned}$$

Making the change of variables $x \mapsto Xt^m$ and doing the integral in the error term yields

$$\sum_{n=1}^{\infty} \lambda_f(n)\psi(n) = \sum_{k=0}^{r} X^{1+2\theta-\frac{k}{m}} \sum_{n=1}^{\infty} \frac{\lambda_{\tilde{f}}(n)}{n^{\frac{k}{m}-2\theta}} \sum_{\pm} c_{k,m}^{\pm} I_k(n;\pm)$$
$$+ O\left(X^{1+2\theta-\frac{r+1}{m}} \sum_{n=1}^{\infty} |\lambda_{\tilde{f}}(n)| n^{2\theta-\frac{r+1}{m}} \right)$$

where the constants have been condensed into $c_{k,m}^{\pm}$ and

$$I_k(n;\pm) := \int_0^{\infty} \phi(t^m) e\left(\alpha X^{\beta} t^{m\beta} \pm (Xn)^{\frac{1}{m}} mt\right) t^{2m\theta-k+m-1} dt.$$

Note that the error term here is $O_{m,r}\left(X^{1+2\theta-\frac{r+1}{m}}\right)$ for r such that $L_{\tilde{f}}(\frac{r+1}{m} - 2\theta)$ converges absolutely; i.e. for $r > \frac{m-1}{2}$. The remainder of the calculations at this point are similar to those of Section 3.3 and will not be repeated here.

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