
Theses and Dissertations

Spring 2016

A combinatorial approach to the Cabling Conjecture

Colin Michael Grove
University of Iowa

Copyright 2016 Colin Grove

This dissertation is available at Iowa Research Online: <http://ir.uiowa.edu/etd/3091>

Recommended Citation

Grove, Colin Michael. "A combinatorial approach to the Cabling Conjecture." PhD (Doctor of Philosophy) thesis, University of Iowa, 2016.
<http://ir.uiowa.edu/etd/3091>.

Follow this and additional works at: <http://ir.uiowa.edu/etd>



Part of the [Mathematics Commons](#)

A COMBINATORIAL APPROACH TO THE
CABLING CONJECTURE

by

Colin Michael Grove

A thesis submitted in partial fulfillment
of the requirements for the Doctor of
Philosophy degree in Mathematics
in the Graduate College of
The University of Iowa

May 2016

Thesis Supervisor: Associate Professor Maggy Tomova

Copyright by
COLIN MICHAEL GROVE

2016

All Rights Reserved

Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Colin Michael Grove

has been approved by the Examining Committee
for the thesis requirement for the Doctor of Philosophy
degree in Mathematics at the May 2016 graduation.

Thesis Committee:

Maggy Tomova, Thesis Supervisor

Benjamin Cooper

Oguz Durumeric

Charles Frohman

Keiko Kawamuro

To my family, whose idea of a recreational activity is working on a Millenium problem: thank you for allowing space for and encouraging my curiosity, and teaching me that regardless of success, difficult things are worth attempting.

ACKNOWLEDGEMENTS

I wish to thank Maggy Tomova, both for her mathematical guidance and for a great deal of advice, widely varied in topic but consistently excellent.

ABSTRACT

Dehn surgery and the notion of reducible manifolds are both important tools in the study of 3-manifolds. The Cabling Conjecture of Francisco González-Acuña and Hamish Short describes the purported circumstances under which Dehn surgery can produce a reducible manifold. This thesis extends the work of James Allen Hoffman, who proved the Cabling Conjecture for knots of bridge number up to four.

Hoffman built upon the combinatorial machinery used by Cameron Gordon and John Luecke in their solution to the knot complement problem. The combinatorial approach starts with the graphs of intersection of a thin level sphere of the knot and the reducing sphere in the surgered manifold. Gordon and Luecke's proof then proceeds by induction on certain cycles. Hoffman provides more insight into the structure of the base case of the induction (i.e. in an innermost cycle or a graph containing no such cycles). Hoffman uses this structure in a case-by-case proof of the Cabling Conjecture for knots of bridge number up to four.

We find trees with specific properties in the graph of intersection, and use them to prove the existence of structure which provides lower bounds on the number of the aforementioned innermost cycles. Our results combined with a recent lower bound on the number of vertices inside the innermost cycles succinctly prove the conjecture for bridge number up to five and suggests an approach to the conjecture for knots of higher bridge number.

PUBLIC ABSTRACT

The Earth's curvature is invisible to us, with our feet planted firmly on the ground. Yet the Earth is round. By the very roundness of the Earth, by our successes at leaving its surface, and by our study of the vastness through which it travels, we are compelled to be curious about the nature of 3-dimensional space.

Before we recognized the Earth as round, we first had to recognize that round things existed, and consider their properties. Such steps are also crucial in considering 3-dimensional spaces. A useful method for determining properties of a 3-dimensional space is to ascertain how precisely that space differs from normal 3-dimensional space, and to ask what types of differences lead to which kinds of properties.

The Cabling Conjecture is a suggested answer to a question of this form. In this work, the conjecture is proven (and the answer is therefore verified) for certain cases, and a framework is outlined which may be useful in proving the conjecture in general.

TABLE OF CONTENTS

List of Figures	vii
1 Introduction	1
2 Setup	5
3 Intersection Graphs	7
3.1 Basic Definitions	7
3.2 Dual Orientation	14
3.3 Index	20
4 Gordon-Luecke Proof Deconstruction	27
4.1 Good, Bad, and Ugly Corners	27
4.2 Proof of Theorem 3.2	29
5 Rotation-Free Graph	33
6 The Cabling Conjecture	37
References	42

LIST OF FIGURES

Figure

1	The essential sphere obtained from longitudinal surgery on the unknot.	1
2	A section of the cabling annulus on the trefoil knot.	3
3	Fat vertices in G_S , of each sign.	8
4	An example of a great web in G_Q	9
5	Model Vertices.	15
6	Example of part of G_P for Lemma 3.17.	16
7	Switch labels.	16
8	The positive (clockwise) derivative of a star.	18
9	A negative edge and a switch edge.	21
10	Edge index calculations.	21
11	Directed graph index calculations.	22
12	Index calculations.	25
13	Good atoms.	27
14	Corner for Lemma 4.1.	28
15	The star T and its conjugate \widehat{T}	35

1 Introduction

Let $k \subset S^3$ be a knot, and let $N(k) \subset S^3$ be the open neighborhood of k . A **slope** is an isotopy class of nontrivial simple closed curves on $\partial\overline{N(k)}$. If π is a slope, π -**Dehn surgery** on k consists of drilling out $N(k)$, and gluing a solid torus T to $M = S^3 \setminus N(k)$ such that a meridian curve on ∂T is mapped to π .

The usefulness of Dehn surgery in the study of 3-manifolds is established by the following theorem.

Theorem 1.1 (Lickorish-Wallace, [16] and [12]). *Any closed, orientable, connected 3-manifold can be obtained from S^3 by a finite collection of Dehn surgeries.*

If M is a 3-manifold such that every embedded 2-sphere in M bounds an embedded 3-ball, M is **irreducible**. An embedded 2-sphere in M that bounds no 3-ball is called **essential** (or a **reducing sphere**), and the existence of an essential sphere makes M **reducible**. One naturally wonders about the relationship between these two important tools, i.e., when does Dehn surgery produce a reducible manifold?

Suppose $k \subset S^3$ is the unknot, m the meridian slope on $\partial\overline{N(k)}$, and l the longitudinal slope (i.e. the slope which bounds a disk in $S^3 \setminus N(k)$). Then m -Dehn surgery produces S^3 , l -Dehn surgery produces $S^2 \times S^1$ (see Figure 1), and Dehn surgery with any other slope produces a lens space.

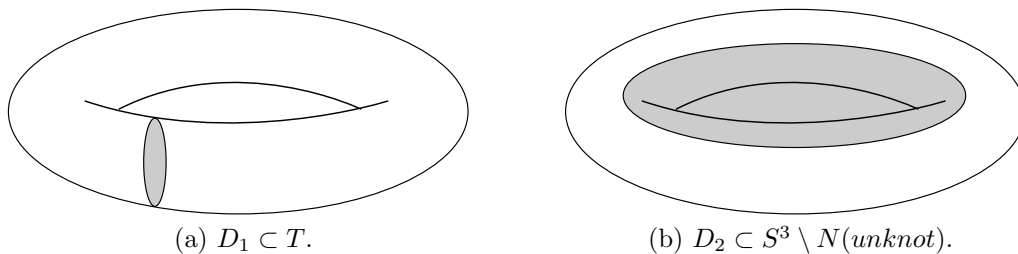


Figure 1: After longitudinal surgery, $D_1 \cup D_2$ is an essential sphere.

The Cabling Conjecture addresses when surgery on a nontrivial knot produces a reducible manifold. It should first be noted that David Gabai showed that only

separating spheres need be considered, by showing that surgery on a nontrivial knot produces neither $S^2 \times S^1$ (thereby proving the Property R Conjecture) nor any manifold with an $S^2 \times S^1$ summand (proving the Poenaru Conjecture) [3].

The Cabling Conjecture (presented in 1983) makes a claim about the specific circumstances under which Dehn surgery on a nontrivial $k \subset S^3$ produces a reducible manifold.

Conjecture 1 (González-Acuña and Short, [4]). *Let $k \subset S^3$ be a nontrivial knot. Then π -Dehn surgery on k produces a reducible manifold if and only if k is a (p, q) -cable knot and the surgery slope π equals pq .*

In order to understand the Cabling Conjecture we must define cable knots. A nontrivial knot $k \subset S^3$ is a (p, q) -**torus knot** if k can be isotoped to a $\frac{p}{q}$ -curve in the boundary of an unknotted solid torus $T \subset S^3$. Let $e : T \rightarrow S^3$ be an embedding of T into S^3 . If $k \subset \partial T$ is a (p, q) -torus knot with $p > 1$, then $e(k)$ is a (p, q) -**cable knot**.

One direction of the Cabling Conjecture is known: a pq -Dehn surgery on a (p, q) -cable knot produces a reducible manifold. To see this, let T be the solid (possibly knotted) torus on which the (p, q) -cable knot k lies. Cutting ∂T along k produces an annulus A , and both components of ∂A represent the same slope on $\overline{\partial N(k)}$. It follows that π -Dehn surgery, with slope π equal to the slope represented by ∂A , will produce a reducible manifold, since meridian disks D_1, D_2 of the filling torus V can be found such that ∂D_1 and ∂D_2 are precisely ∂A . Thus $A \cup D_1 \cup D_2$ is a sphere, and essential since none of D_1, D_2 , or A are boundary parallel to ∂V . A is called the **cabling annulus**.

To see that the appropriate slope π is pq , we first note that since the boundary of A on ∂T is k , each component of ∂A traverses k along a longitude once, so each component of ∂A has integral slope on $\overline{\partial N(k)}$. Each time k goes once around the longitude of T , a ∂A component goes q times around the meridian of $\overline{\partial N(k)}$. Since k goes p times around the longitude of T , the slope of ∂A on $\overline{\partial N(k)}$ is pq .

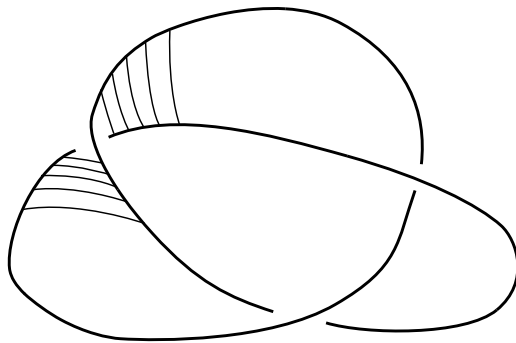


Figure 2: A section of the cabling annulus on the trefoil knot.

The Cabling Conjecture has been proven for many classes of knots, including:

- composite knots by Gordon in 1983 [6];
- satellite knots by Scharlemann in 1989 [15];
- strongly invertible knots by Eudave-Muñoz in 1992 [2];
- alternating knots by Menasco and Thistlethwaite in 1992 [13];
- arborescent knots by Wu in 1994 [17];
- knots of bridge number up to 4 in 1995 [11];
- symmetric knots by Hayashi and Shimokawa in 1998 [9];
- knots of bridge number at least 6 and distance at least 3, by Blair, Campisi, Johnson, Taylor and Tomova in 2012 [1].

Due to the results of Hoffman [11] and Blair et. al. [1], proving the Cabling Conjecture for 5-bridge knots restricts remaining cases to knots with low distance.

We assume the existence of a reducing surgery on a knot that does not satisfy the Cabling Conjecture, and find a planar surface in the knot exterior such that its intersection with the reducing sphere has certain desirable properties. The arcs of intersection are treated as edges in graphs on the planar surface and the reducing

sphere, and we use combinatorial methods to show the existence of various structure in each graph. The structure we find, combined with recent results of Zufelt [18], show that the bridge number must be at least 6.

It should be noted that in [10], Hoffman mentions having unpublished notes proving the Cabling Conjecture for 5-bridge knots.

2 Setup

Let $k \subset S^3$ be a knot, and let $M = S^3 - N(k)$ be the exterior of k . Let γ be the meridional slope in ∂M . Given a slope π in ∂M , let $M(\pi)$ be the 3-manifold obtained by performing π -Dehn surgery on k .

Let P be a 2-sphere in $M(\pi)$ which intersects k' transversely, and let $\tilde{P} = P \cap M$. Then each component of $\partial \tilde{P}$ has slope π . Let γ be the meridian slope of $\overline{N(k)}$. Then γ -Dehn surgery is trivial, so $M(\gamma) \cong S^3$. If Q is a 2-sphere in $M(\gamma)$ intersecting k transversely, then each component of $\partial \tilde{Q}$ has slope γ .

We use Gabai thin position from [3] (this setup follows [11]). Define a height function for $k \subset S^3$, $h : S^3 \setminus \{x, y\} \cong S^2 \times \mathbb{R} \rightarrow \mathbb{R}$. Let $Q_\alpha = h^{-1}(\alpha) = S^2 \times \alpha$ be the **level sphere** at height α . A **generic presentation** of k is an embedding $f : S^1 \rightarrow S^3$ such that $h \circ f$ is a Morse function with 2β critical points occurring at distinct levels. Assuming a generic presentation of k , let $c_1 < c_2 < \dots < c_{2\beta}$ be the critical levels and select levels h_i such that $c_i < h_i < c_{i+1}$ for $i = 1, \dots, 2\beta - 1$. Define Q_i to be the level sphere at level h_i . The **width** is

$$w(k) = \min \left\{ \frac{1}{2} \sum |Q_i \cap f(S^1)| \mid f \text{ is a generic presentation of } k \right\}.$$

A **thin presentation** is one which realizes $w(k)$ (i.e. a generic presentation which minimizes the sum).

Assume a thin presentation of k . The following is an combination of results of Gabai [3] with results of Gordon and Luecke [8], as stated in [11].

Proposition 2.1 (Proposition 1.2.1 in [11]). *Suppose $P \subset M(\pi)$ is an essential 2-sphere (with $\tilde{P} = P \cap M$) such that P meets k' transversely and minimally. Then there is a (level) 2-sphere $Q \subset M(\gamma)$ (with $\tilde{Q} = Q \cap M$) such that*

- (i) $\partial \tilde{P} \subset \partial M$ (resp. $\partial \tilde{Q} \subset \partial M$) consists of parallel copies of π (resp. γ);

- (ii) \check{P} and \check{Q} intersect transversely;
- (iii) no arc of $\check{P} \cap \check{Q}$ is boundary-parallel in either \check{P} or \check{Q} ; and
- (iv) each component $\partial\check{P}$ meets each component of $\partial\check{Q}$ exactly once.

This relates to the Cabling conjecture as shown below. Let $p = |P \cap k'|$ and $q = |Q \cap k|$.

Proposition 2.2 (Proposition 1.2.3 from [11]). *If P is a 2-sphere in $M(\pi)$ such that P meets k' transversely and $p = 2$, then the knot k is a cabled knot and \check{P} is the cabling annulus.*

Our goal is to prove the Cabling Conjecture for bridge number $b \leq 5$. We would like to claim that because k is in thin position, $q \leq 2b$. To see this, note that with a b -bridge knot k in (generic) bridge position \mathcal{B} ,

$$\sum |Q_i \cap f(S^1)| = 2 + 4 + \dots + (2b - 2) + 2b + (2b - 2) + \dots + 4 + 2.$$

If k is put in a different generic position \mathcal{P} such that some level sphere intersects k in $q' > 2b$ points, we can compare the widths of the two presentations

$$w_{\mathcal{P}}(k) = 2 + 4 + \dots + (q' - 2) + q' + (q' - 2) + \dots + 4 + 2 + \dots \quad (1)$$

$$w_{\mathcal{B}}(k) = 2 + 4 + \dots + (2b - 2) + 2b + (2b - 2) + \dots + 4 + 2. \quad (2)$$

Clearly $w_{\mathcal{B}}(k) < w_{\mathcal{P}}(k)$, so \mathcal{P} is not a thin presentation of k .

Thus we assume $p > 2$ and want to show that $q > 10$.

3 Intersection Graphs

We will henceforth assume that $p > 2$.

3.1 Basic Definitions

We now define the graphs $G_P \subset P$ and $G_Q \subset Q$. Many definitions and theorems will apply to both spheres P and Q . We will use $\{\mathcal{S}, \mathcal{T}\} = \{P, Q\}$ when we wish to make statements which may apply to either sphere. A **fat vertex** of $G_{\mathcal{S}}$ is a component of $\partial\check{\mathcal{S}}$, and each arc component of $\check{P} \cap \check{Q}$ is an edge in $G_{\mathcal{S}}$. Select a fat vertex in $G_{\mathcal{S}}$ to label 1, and follow along the appropriate knot (k' if $\mathcal{S} = P$, k if $\mathcal{S} = Q$), labeling the remaining fat vertices in the order in which they are encountered. We will denote by $V_{\mathcal{S}}$ the vertex set of $G_{\mathcal{S}}$.

Since edges of $G_{\mathcal{S}}$ are arc components of $\check{P} \cap \check{Q}$, they meet fat vertices of $G_{\mathcal{T}}$ at components of $\partial\check{\mathcal{T}}$. This can be used to **label** the points at which edges meet fat vertices in $G_{\mathcal{S}}$ with the labels of the appropriate components of $\partial\check{\mathcal{T}}$. We will frequently refer to subsets of V_P by variations of V , and subsets of the labels in G_P by variations of L . Since vertices in $G_{\mathcal{S}}$ are labels in $G_{\mathcal{T}}$, this means we will frequently refer to subsets of V_Q by variations of L (being labels in G_P), and subsets of the labels of G_Q by variations of V (being vertices in G_P).

Note that by Proposition 2.1(iv), each label from $\partial\check{\mathcal{T}}$ appears precisely once on each fat vertex in $G_{\mathcal{S}}$. Furthermore, since components of $\partial\check{\mathcal{T}}$ are curves of the same slope on ∂M , the labels from $\partial\check{\mathcal{T}}$ will appear in the same order around every fat vertex of $G_{\mathcal{S}}$, though in either orientation (clockwise or counter-clockwise). Fat vertices may be assigned a **sign** depending on the orientation of the labels from $\partial\check{\mathcal{T}}$ (see Figure 3). Vertices in $G_{\mathcal{S}}$ are **parallel** if they have the same sign, and **antiparallel** if their signs differ. Let V (V') be a set of vertices of $G_{\mathcal{S}}$ such that all vertices of V (V') are parallel. Then we call V parallel to V' if the sign which the vertices in V share

matches the sign of the vertices in V' , and we call V antiparallel to V' if the signs are opposite.

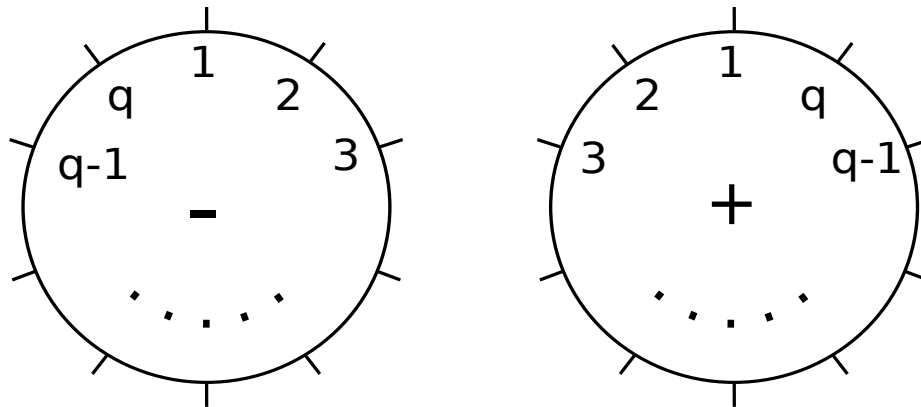


Figure 3: Fat vertices in G_S , of each sign.

To each label x on a fat vertex we associate a *parity*, which is the sign of x as a vertex in the other graph. Every label and vertex pair (x, v) may therefore be assigned a *character*, where $\text{char}(x, v) = (\text{parity } x)(\text{sign } v)$.

It follows from orientability of M , P , and Q that edges in either graph connect (x, v) pairs of opposite character. This is known as the *parity rule*.

We will frequently refer to various subgraphs of G_S . If E is a collection of edges, $G_S(E)$ is the subgraph of G_S containing all the fat vertices of G_S , and edges E . Let $V, W \subset V_S$ be sets of vertices of G_S . Then $[V, W]$ is the set of edges in G_S between a V vertex and a W vertex, or equivalently the set of edges in G_T between a V label and a W label. Let L be a set of labels of G_S . We will often consider the subgraph $G_S([L, V_T])$, which we will abbreviate as $G_S(L)$. Note that for an arbitrary label set L , $G_S(L)$ may (in fact, generally will) have edges meeting vertices at labels outside L . Such labels are *exceptional labels*.

An *x -cycle* in G_S is a directed cycle on parallel vertices such that the tail of each edge in the cycle is at the label x on each vertex. A *great x -cycle* is an x -cycle which bounds a disk such that all vertices inside the disk are parallel to the x -cycle. A *Scharlemann cycle* is a great x -cycle bounding a disk containing in its interior

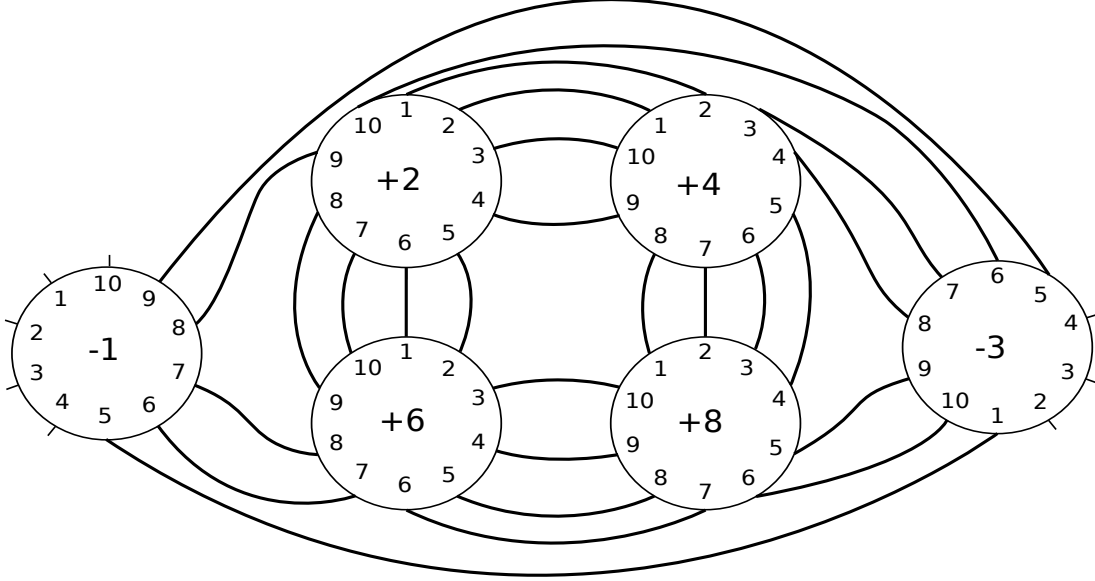


Figure 4: An example of part of G_Q . There is a Scharlemann cycle on the vertices $\{2, 4\}$ and a new 5-cycle σ on the vertices $\{1, 3\}$. The σ -disk depicted as bounded contains the σ -set $\{2, 4, 6, 8\}$, which also happens to be an innermost (+)-set and a great web.

no vertices or edges. A **corner** of a face F is the part of the boundary of a fat vertex which is incident to F . If x and y are labels on a vertex v of G_S , the two complementary pieces of the boundary of v extending from the label x to the label y are both $\langle x, y \rangle$ corners. Since a Scharlemann cycle bounds a disk containing no vertices or edges of G_S , each corner in a Scharlemann cycle must be an $\langle x, x + 1 \rangle$ corner. We may therefore refer to an $x, x + 1$ -Scharlemann cycle. The **order** of a Scharlemann cycle is the number of vertices the cycle traverses. A **new x -cycle** is an x -cycle which is not a Scharlemann cycle.

Lemma 3.1 (Lemma 2.1.2 of [11]). *If, in G_Q , Σ_1 is an x_1, x_2 -Scharlemann cycle of order m and Σ_2 is a y_1, y_2 -Scharlemann cycle of order n , then $\langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle$ and $m = n$.*

The key technical result in the resolution of the Knot Complement Problem was the existence of Scharlemann cycles. Gordon and Luecke's proof is described in a later section, as many lemmas in their proof are integral to our result, which explores

the graph structure indicated by their inductive proof.

Theorem 3.2 ([7]). *G_Q contains a Scharlemann cycle.*

By Theorem 3.2, G_Q must have a Scharlemann cycle. By convention the labels of G_Q and therefore the vertices of G_P are labeled such that Scharlemann cycles of G_Q are 1,2-Scharlemann cycles, since by Lemma 3.1 all Scharlemann cycles are on the same labels. The vertices 1 and 2 in G_P are the **special vertices**. The remaining vertices V_r of G_P are the **regular vertices**.

Hoffman's important result, meanwhile, was the nonexistence of new great x -cycles.

Theorem 3.3 (Lemma 3.0.3 of [11]). *If $p > 2$, then G_Q does not contain a new great x -cycle.*

A vertex x of G_Q is **isolated** if each edge incident to x leads to an antiparallel vertex.

Lemma 3.4 (2.4.1 from [11]). *There are no isolated vertices in G_Q .*

A **great m -web** Λ is a collection of parallel vertices in G_S such that

1. Λ lies in a disk D_λ of \mathcal{S} such that every vertex in D_λ is a vertex of Λ , and
2. precisely m edges leave Λ .

A great $(p - 2)$ -web in G_Q will be referred to simply as a **great web**. The proof of Theorem 3.2 in [7] shows the existence of a great web in G_Q .

Only the relevant parts of the following results have been kept.

Theorem 3.5 (Part of Proposition 3.1 from [18]). *Let Λ be a great web, $V(\Lambda)$ the vertices of the great web, and $v = |V(\Lambda)|$. Let n be the Scharlemann order. Then n divides v .*

Theorem 3.6 (Corollary 3.2 from [18]). *Under the hypotheses from Theorem 3.5, $n \neq v$. Hence the Cabling Conjecture holds for knots with bridge number ≤ 3 , and modulo the case $n = 2, v = 4$, for knots with $b \leq 5$.*

Corollary 3.7. *If Λ is a great web, $|V(\Lambda)| \geq 4$.*

Proof. The Scharlemann order n must be at least 2, and $|V(\Lambda)| > n$ must be a multiple of n . □

Our method is to establish minimums for q in different cases by finding several great webs, and to then apply Corollary 3.7.

Let V be a set of vertices of $G_{\mathcal{S}}$. Suppose σ is a cycle on parallel vertices in $G_{\mathcal{S}}$ bounding a disk D which contains the vertices V . Then D is a σ -**disk** and V is a σ -**set**. We call D **nontrivial** if V contains elements of the opposite sign of σ , and σ nontrivial if it bounds no trivial σ -disk. Let $s \in \{+, -\}$. If every vertex in V has sign s and every edge leaving V goes to a vertex of sign $-s$, V is an (s) -**set** of vertices. An (s) -**disk** is a disk $D \subset \mathcal{S}$ containing a nonempty connected (s) -set V and no other vertices, such that all edges $[V, V]$ are also contained in D . An (s) -set contained in an (s) -disk is called an **innermost s -set**. V^* will denote the set of labels at which edges leave V .

Let V be a subset of vertices in $G_{\mathcal{S}}$, and L a subset of labels. $G_{\mathcal{S}}$ has the parallel property $\mathbb{P}(V, L)$ if for each vertex x of V there exists a label $y(x)$ of L such that the edge of $G_{\mathcal{S}}$ incident to x at $y(x)$ goes to a vertex parallel to x . $G_{\mathcal{S}}$ has the antiparallel property $\mathbb{A}(V, L)$ if for each label y of L there exists a vertex $x(y)$ of V such that the edge of G incident to x at the label $y(x)$ connects x to an antiparallel vertex. Note that if V_1 is a vertex subset of $G_{\mathcal{S}}$ and V_2 is a vertex subset of $G_{\mathcal{T}}$, then $G_{\mathcal{S}}$ has $\mathbb{A}(V_1, V_2)$ if and only if $G_{\mathcal{T}}$ has $\mathbb{P}(V_2, V_1)$.

By basic graph theory,

Proposition 3.8. *If G contains no cycles, G contains more vertices than edges.*

Proposition 3.9. *Let G be a directed graph, let V be a subset of the vertices of G . If for each $v \in V$ there is a unique edge e_v between vertices of V , such that the tail of e_v is at v , then G has an directed cycle on the vertices V .*

Proof. Since $|\{e_v\}| = |V|$, by Proposition 3.8, G contains a cycle. The orientations of the edges in the cycle must agree by uniqueness of each e_v . \square

Corollary 3.10. *Let $V \subset V_Q$ be a set of uniform sign. If there is a label $x_0 \in V_P$ such that no edge leaves V at a label x_0 , then V contains an x_0 -cycle.*

Proof. Apply Proposition 3.9 to the components of $G_Q(\{x_0\})$ on the vertices in V . \square

Lemma 3.11. *Let V be an innermost (s) -set in G_Q . Then all of the following hold:*

1. $|V| \geq 2$,
2. $V^* \supset V_r$,
3. G_P has $\mathbb{P}(V_r, V)$.

Furthermore, if V does not have a Scharlemann cycle, then $V^ = V_P$.*

Proof. Suppose $x_0 \in V_r$ is not in V^* . Then there is a new x_0 -cycle Σ on V , by Corollary 3.10. By definition of (s) -disk, Σ is a great new x_0 -cycle, a contradiction with Theorem 3.3. Therefore no such x_0 exists. If $V = \{v\}$, then since v cannot be isolated (Lemma 3.4), there is a loop on v , which must also be a new great x -cycle. Thus $|V| \geq 2$. Since $V^* \supset V_r$, G_Q has $\mathbb{A}(V, V_r)$, so G_P has $\mathbb{P}(V_r, V)$.

Suppose V does not have a Scharlemann cycle. Then V cannot have any x -cycles, so $V^* = V_P$, by Corollary 3.10. \square

Lemma 3.12. *Let V be an innermost (s) -set. If $|[V, V_Q \setminus V]| = p - 2$, then for each $x \in V_r$, there are precisely $|V| - 1$ edges from $[V, V]$ adjacent to x .*

Proof. By Lemma 3.11, $V^* \supset V_r$. So if $|[V, V_Q \setminus V]| = p - 2$, precisely one edge leaves V from each label $x \in V_r$. Fix $x_0 \in V_r$. Each of the $|V|$ vertices in V have an x_0 label, so $|V| - 1$ of the edges adjacent to x_0 labels must be in $[V, V]$. \square

Note that a consequence of Corollary 3.10 and Theorem 3.3 is that if Λ is a great web, there is precisely one edge leaving Λ from each label in V_r , and Λ must contain a Scharlemann cycle. Lemma 3.11, meanwhile, shows that for an innermost (s) -set V , there is at least one edge leaving V from each label in V_r . The two kinds of sets are similar, but an innermost (s) -set V has the property that all edges leaving V go to antiparallel vertices but there are an unknown number of such edges, while a great web Λ has the property that those number and configuration of edges leaving Λ are well known, but those edges may go to parallel vertices.

Lemma 3.13. *Let V be any set of vertices in G_Q . Let $E_{antipar}$ be the set of edges joining vertices of V to antiparallel vertices (possibly also in V). If $|E_{antipar}| > p - 2$, then $G_P(V)$ has a cycle on parallel vertices.*

Furthermore, if for each $x \in V_r$, there is a $v \in V$ and $e \in E_{antipar}$ such that e meets v at label x , then the cycle can be oriented with the tail of each edge meeting the vertex in $G_P(V)$ at a label in V .

Proof. If $|E_{antipar}| > p - 2$, then $G_P(V)$ contains more than $p - 2$ edges between parallel vertices. Without loss of generality we may therefore assume that $G_P(V)$ contains at least $\frac{p}{2}$ edges between the $\frac{p}{2}$ positive vertices. Let G be the subgraph of $G_P(V)$ obtained by discarding edges between antiparallel vertices. Some positive component of G has at least as many edges as vertices, and by Proposition 3.8, the component will have a cycle. \square

Let $X, Y \subset V_Q \times V_P$, $r \in V_P$. A $(X \rightarrow Y \rightarrow r)$ **tree** is a subtree of G_P rooted at r such that each edge is directed from a (vertex, label) pair in X to a (vertex, label)

pair in Y . If $V, W \subset V_Q$, a $(V \rightarrow W \rightarrow r)$ tree means a $(V \times V_P \rightarrow W \times V_P \rightarrow r)$ tree. If $X \subset V_Q \times V_P$, a $(X \rightarrow r)$ tree means a $(X \rightarrow V_Q \times V_P \rightarrow r)$ tree.

Lemma 3.14. *Let V be an innermost (s) -set in G_Q . Let W be the vertices in $V_Q \setminus V$ which have edges to V . Then any component of $G_P([V, W])$ that is not in a $(V \rightarrow W \rightarrow r)$ tree on parallel vertices with r one of the special vertices, must have a cycle on parallel vertices.*

Proof. By Lemma 3.11, $V^* \supset V_r$, which means that $|[V, W]| \geq p - 2$. Since $V^* \supset V_r$, for each $x \in V_r$ there exists an edge e_x leaving x at a V label and going to a parallel vertex at a W label. Suppose that there does not exist a path from some negative $v_0 \in V_r$ to 1 along edges with tails at labels in V and heads at labels in W . Since every $v \in V_r$ has such an edge, the absence of such a path implies that there is an oriented cycle of such edges. \square

Lemma 3.15. *Nontrivial σ -disks contain (s) -disks.*

Proof. Let D_i be a nontrivial σ_i -disk containing the σ_i -set L_i , and let V_i be an (s_i) -set contained in L_i . Note that such an (s_i) -set must exist, or D_i is trivial. If V_i is not contained in an (s_i) -disk, then there is a cycle σ_{i+1} on V_i that bounds a nontrivial σ_{i+1} -disk $D_{i+1} \subset D_i$. The σ_{i+1} -set L_{i+1} contains an (s_{i+1}) -set V_{i+1} .

Since $\sigma_i \cap \sigma_{i+1} = \emptyset$, this can only be repeated finitely many times, until an (s_n) -set V_n is reached that is contained in an (s_n) -disk. \square

Corollary 3.16. *If Σ is a new x -cycle, any Σ -disk contains an (s) -disk.*

3.2 Dual Orientation

A pair of *model fat vertices* are abstract fat vertices V_{\pm} with marks on the boundary representing edges incident to each vertex, labels at each mark, and a sign \pm indicated by the subscript. V_- is the reflection of V_+ . If L is a subset of labels on an abstract

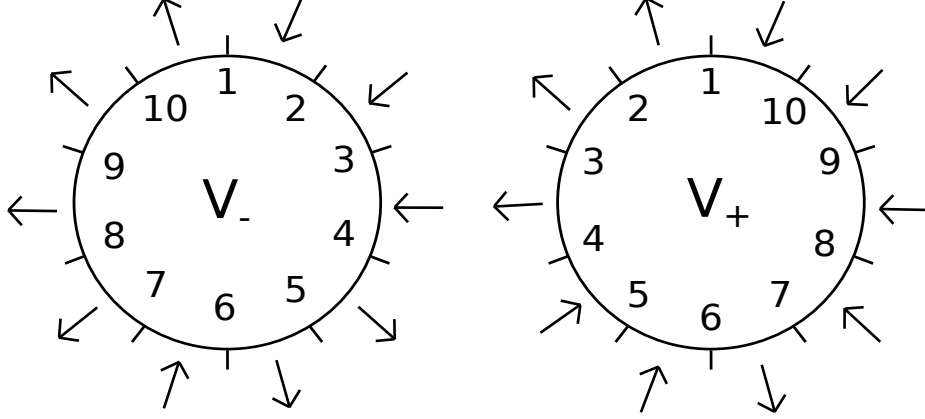


Figure 5: Model vertices. Note that $A = \{4, 7\}$, $C = \{1, 6\}$ on both vertices.

fat vertex V , an L -interval is an $\langle x, y \rangle$ corner where no labels of L are in the interior of $\langle x, y \rangle$. A **dual orientation** on an L -interval is indicated by an arrow pointing either into (a sink) or out from (a source) the corner. If all dual orientations on a vertex are the same, the vertex is said to have **uniform dual orientation**, or be a sink or source (depending on the dual orientation). A **star** T is an ordered triple $(V(T), L(T), \omega(T))$, where $V(T) = V_{\pm}$, $L(T)$ is a subset of the labels around $V(T)$, and $\omega(T)$ is an assignment of dual orientations to each $L(T)$ -interval around $V(T)$. Given a star T , \bar{T} is the same star with reversed dual orientations, and $-T$ is the mirror image of \bar{T} .

Let Δ be a nonempty face of G_P (meaning its interior contains vertices). The set of labels on the corner of vertex i in $\partial\Delta$ will be denoted $X_i(\Delta)$. Furthermore $X_-(\Delta) = \bigcup X_i(\Delta)$ where the union is taken over negative vertices in $\partial\Delta$. $X_+(\Delta)$ is defined similarly.

Lemma 3.17. *Let E be the edges of a Scharlemann cycle in G_Q . Let Δ be a disk face of $G_P(E)$ with no edges of $G_P(E)$ in its interior. Then considered relative to G_P , $X_-(\Delta) \cap X_+(\Delta) = \emptyset$.*

Proof. Note first that Δ is a 2-corner face with corners on the special vertices. Let the edges in $\partial\Delta$ be e_a and e_b . The edge e_a meets 1 and 2 at labels l_a^1 and l_a^2 respectively,

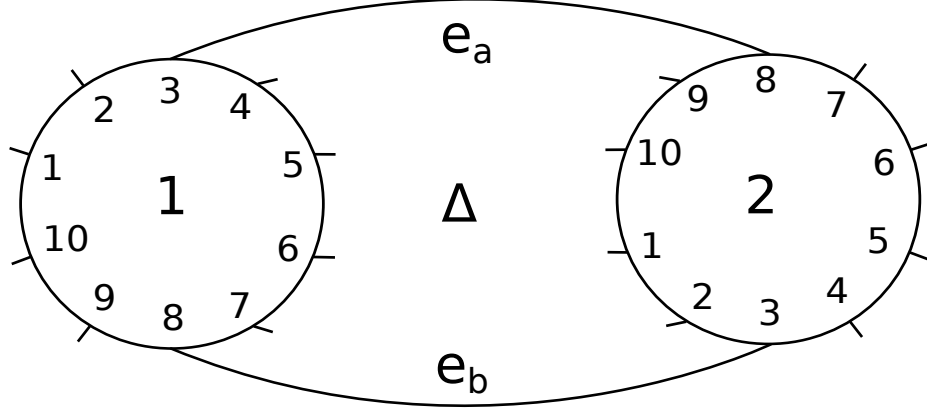


Figure 6: Example of part of G_P for Lemma 3.17. The edges e_a and e_b come from a Scharlemann cycle of order 2 in G_Q on the vertices 3 and 8.

and the edge e_b meets 1 and 2 at labels l_b^1 and l_b^2 . Suppose that $x \in X_-(\Delta) \cap X_+(\Delta)$. Then the labels corresponding to G_Q vertices in the Scharlemann cycle which are on either side of x on the star T , are $\{s_1, s_2\}$. But then $\{s_1, s_2\} = \{l_a^1, l_b^1\} = \{l_a^2, l_b^2\}$. Since 1 is a negative vertex and 2 is positive, the order of the labels is reversed, thus for x to be in $X_-(\Delta) \cap X_+(\Delta)$, we must have $l_a^1 = l_a^2$ and $l_b^1 = l_b^2$, implying that e_a and e_b are two loops in G_Q rather than a Scharlemann cycle. \square

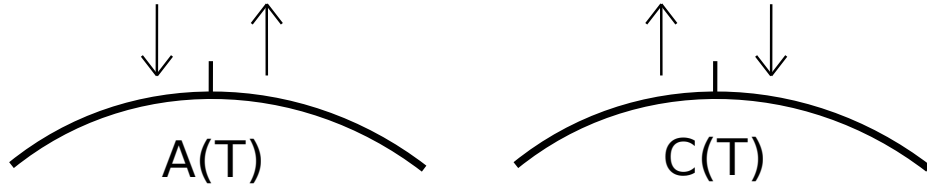


Figure 7: Switch labels.

We define a label of a fat vertex to be an **anticlockwise switch** if the dual orientation in the clockwise direction from the label is out while the dual orientation in the counter-clockwise direction from the label is in. We define a label of a fat vertex to be a **clockwise switch** if the dual orientations are opposites those of an anticlockwise switch. For a star T , $A(T) = \{l \in L(T) | l \text{ is an anticlockwise switch}\}$. Likewise, $C(T) = \{l \in L(T) | l \text{ is a clockwise switch}\}$. Let $S(T) = A(T) \cup C(T)$. For $l \in L(T) \setminus S(T)$, let $\phi(l) = +$ if both adjacent dual orientations are out, and $\phi(l) = -$

if both adjacent dual orientations are in. Define

$$B_s(T) = \{l \in L(T) \setminus S(T) \mid (\text{sign } T)(\text{parity } l)(\phi(l)) = s\}$$

for $s \in \{+, -\}$. The labels on T are partitioned into $A(T)$, $C(T)$, $B_+(T)$, and $B_-(T)$. A star T is **coherent** if all elements of $A(T)$ have the same parity and all elements of $C(T)$ have the same parity. Note that $A(T) = A(-T)$ and $C(T) = C(-T)$.

An m -type is a tuple of signs $(\{+, -\})$ of length m . A **trivial** m -type is one of uniform sign. Let L be a set of labels, and l the number of L -intervals on a star T . An L -type is an l -type where each sign corresponds to a unique L -interval. Let \mathbb{L}_0 be a subset of the L -intervals, and let τ be an L -type. Then $\tau|_{\mathbb{L}_0}$ is the $|\mathbb{L}_0|$ -type obtained by restricting τ to the L -intervals in \mathbb{L}_0 . Let T be a star and τ be an $L(T)$ -type. If it is possible to assign distinct signs to the dual orientations such that $\omega(T)$ and τ match (e.g. sink equals $-$ and source equals $+$, or switched), then T **represents** τ , and we write $[T] = \tau$. We will say that an L -type τ is coherent if there exists a coherent star such that $[T] = \tau$.

Let D be a disk face of $G_P(L)$ and let τ be an L -type. D **represents** τ if there is a star T with $[T] = \tau$ such that with the dual orientation of $G_P(L)$ inherited from T , D is a sink or source (i.e. all corners on ∂D have matching dual orientations). $G_P(L)$ represents τ if it has a disk face which represents τ .

The **positive** and **negative (clockwise) derivatives of T** are

$$d^\pm(T) = (V(T), C(T), \omega(d^\pm))$$

where $\omega(d^\pm)$ on a $C(T)$ -interval I is defined as follows. Let $a \in A(T)$ be the unique anticlockwise switch in I . Then $\omega(d^+T)$ is a source if $\text{char } a = +$, and a sink if $\text{char } a = -$. On the other hand, $\omega(d^-T)$ is a sink if $\text{char } a = +$, and a source if $\text{char } a = -$. By d we mean either d^+ or d^- .

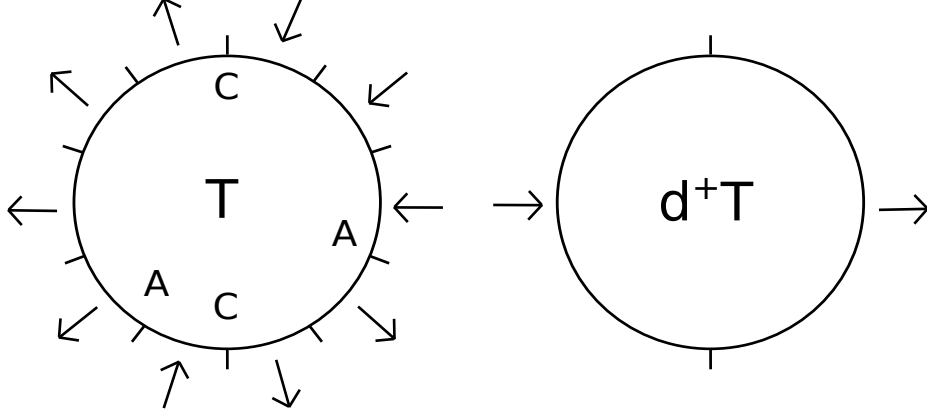


Figure 8: The positive (clockwise) derivative of a star.

Let L_0 be a subset of the labels on $V(T)$. Then the (\pm) -**derivative of T relative to L_0** is

$$d_{L_0}T = d_0T = (V(T), C(T) \cup L_0, \omega(d_0T))$$

where $\omega(d_0T)$ on a $C(T) \cup L_0$ -interval I is defined by any $a \in A(T)$ in I as above, or if none exist, $\omega(d_0T)$ is defined to match the orientation of T on I . Define $\tilde{A}(T) = A(T) - L_0$.

Proposition 3.18 (2.1.2 from [7]). *Let D be any composition of d^+ 's and d^- 's, and D_0 the corresponding composition of d_0^+ 's and d_0^- 's. Then*

$$C(DT) \subset C(D_0T), \text{ and } \tilde{A}(DT) \supset \tilde{A}(D_0T)$$

A **graph with dual orientation** is a pair $\Gamma = (G(\Gamma), \omega(\Gamma))$ where $G(\Gamma)$ is a subgraph of a fat vertex graph, and $\omega(\Gamma)$ is an assignment of dual orientation to each corner of $G(\Gamma)$. Note that we will sometimes consider G_i to simply have a dual orientation, rather than discussing Γ .

Given G_P and G_Q , a star T such that $L(T) \subset V_Q$ generates a dual orientation on $G_P(L(T))$ as follows. Vertices of the same sign as $V(T)$ are given dual orientations of $\omega(T)$, while vertices of the opposite sign are given dual orientations of $-\omega(T)$. If an $L(T)$ -interval on $V(T)$ corresponds to a corner in $G_P(L(T))$ which has edges in

its interior, each subcorner in $G_P(L(T))$ is given the dual orientation inherited from the corresponding $L(T)$ -interval on $V(T)$. Note that the definition of derivative(s) of a star extends naturally to derivative(s) of a graph with dual orientation. We define $\delta\Gamma$ to be the graph with dual orientation obtained by taking the derivative d at each fat vertex, and for a subgraph G_0 of Γ , we define $\delta_0\Gamma = \delta_{G_0}\Gamma$ to be the graph with dual orientation obtained by taking the derivative $d_{L(G_0)|v}$ relative to $L(G_0)|v$ at each fat vertex, where $L(G_0)|v$ is the set of labels at v which meet edges of G_0 .

Lemma 3.19 (2.2.2 from [7]). *If the exceptional labels of $G(L_0)$ are contained in $C(X)$, then $\delta\Gamma(T) = \Gamma(d_0T)$. (Here $d_0 = d_{L_0}$, $\delta_0 = \delta_{G(L_0)}$.)*

If Γ is a graph with dual orientation, we define the **dual graph** Γ^* (note that this definition is from [7] and is not the standard definition of the dual of a graph). For each disk face F of Γ , Γ^* has a dual vertex in $\text{int } F$. The vertices of Γ^* are the dual vertices *and* the vertices of Γ (i.e. the fat vertices, though they are treated as regular vertices in Γ^*). For each corner X of the disk face F of Γ , place an edge e from the (fat) vertex adjacent to X to the dual vertex corresponding to F , and orient e to match the dual orientation of X in Γ . When this is done to each corner of each disk face, the directed graph Γ^* is obtained.

Lemma 3.20 (2.4.1 from [7]). *If $(\delta_0\Gamma)^*$ has a sink or source at a dual vertex corresponding to a face E of $\delta_0\Gamma$, then Γ^* has a sink or source at a dual vertex corresponding to a face of Γ contained in E .*

Corollary 3.21 (2.4.2 from [7]). *Let τ be an L -type and T a star with $[T] = \tau$. If $G(C(T))$ represents $[dT]$, then $G(L)$ represents τ .*

Definition 3.22. *Let τ be a nontrivial L -type. Let (T_1, \dots, T_n) be a sequence of stars such that*

1. $[T_1] = \tau$, $[T_i]$ is nontrivial for $1 \leq i \leq n$;

2. $T_i = d_i T_{i-1}$, where $d_i = d^\pm$, for $2 \leq i \leq n$;

3. $[T_n]$ is coherent.

Then the sequence (T_1, \dots, T_n) is a **sequence of coherence** for τ .

Proposition 3.23. *Any nontrivial L-type τ has a sequence of coherence.*

Proof. Let T_1 be one of the two stars with sign $V(T_1) = +$ and $[T_1] = \tau$. If τ is coherent, let $n = 1$. If not, we may assume that not all elements of $A(T_1)$ have the same sign (replacing T_1 with $\overline{T_1}$ if necessary). Let m be the smallest positive integer such that $A((d^+)^m T_1)$ has uniform sign.

If $C((d^+)^m T_1)$ also has uniform sign, let $n = m + 1$, $T_i = d^+ T_{i-1}$, $2 \leq i \leq n$.

Otherwise, let $n = m + 2$, with

1. $T_i = d^+ T_{i-1}$, for $2 \leq i \leq m$,

2. $T_{m+1} = d^- T_m (= \overline{(d^+)^m T_1})$,

3. $T_{m+2} = d^+ T_{m+1}$.

Clearly all $A(T_n)$ elements have the same sign. If $n = 1$ or $n = m + 1$, it is immediate that $[T_n]$ is coherent. If $n = m + 2$, then $T_{m+1} = d^- T_m = \overline{(d^+)^m T_1}$. Thus $C(T_{m+1}) = A((d^+)^m T_1)$ is of uniform sign by definition of m . Since $n = m + 2$ only when $[(d^+)^m T_1]$ is not coherent, $[T_{m+1}]$ is not coherent. But $T_n = T_{m+2}$ is coherent. Note that since $[T_i]$ is coherent only for $i = n$, $[T_i]$ is nontrivial for all $i \leq n$.

The sequence (T_1, \dots, T_n) is thus a sequence of coherence for τ . □

3.3 Index

A **negative edge** (Figure 9a) is an edge such that all four dual orientations adjacent to the edge are identical (for example, all point out of the two vertices). A **switch edge** (Figure 9b, sometimes referred to as a positive edge, particularly in [7]) is an

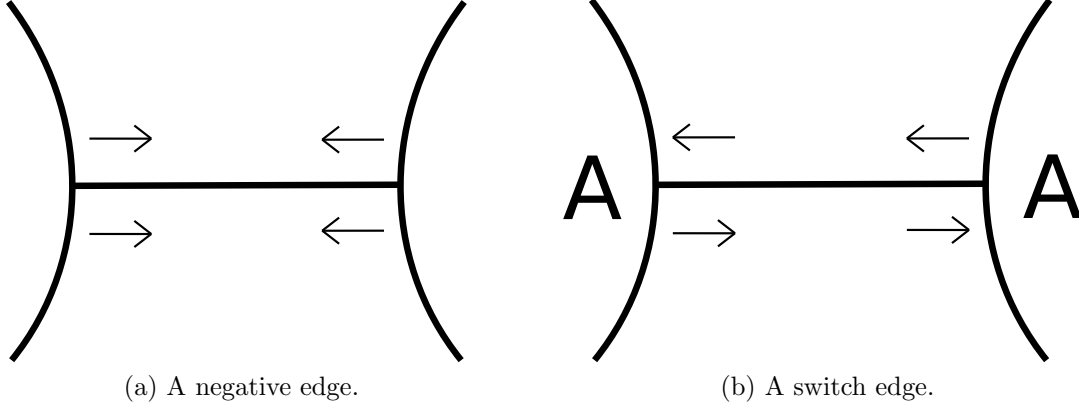


Figure 9

edge such that both ends meet vertices at switch labels of the same orientation (i.e., both anticlockwise switches or both clockwise switches).

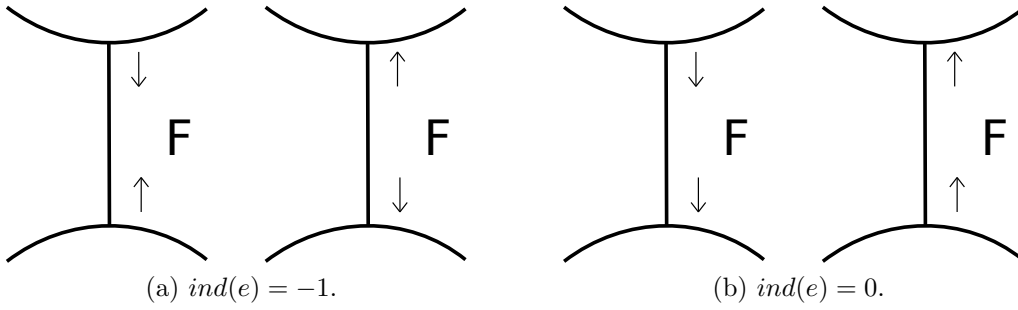


Figure 10

A **corner** X is $(V(X), I(X), L(X), \omega(X))$ where $V(X)$ is a model fat vertex, $I(X)$ is an interval on $V(X)$, $L(X)$ a subset of labels in $I(X)$, and $\omega(X)$ a dual orientation on the $L(X)$ -intervals in $I(X)$. We define the **index** of a corner $ind(X) = 1 - s(X)$ where $s(X)$ is the number of switches in the corner. For an edge $e \subset \partial D$, $ind(e) = -1$ if the dual orientations on the corners of D adjacent to e agree (i.e. are both out or both in, Figure 10a), while $ind(e) = 0$ if the dual orientations disagree (Figure 10b). Finally, for a disk face D of a subgraph of G_i , define

$$index \partial D = \sum_{X \text{ a corner of } F} ind(X) + \sum_{e \subset \partial F} ind(e).$$

Define

1. $i = \frac{|S(T)|}{2} - 1$,
2. u to be the number of negative edges of $G_P(L)$,
3. r to be the number of disk faces of $G_P(L)$ representing T ,
4. s to be the number of switch edges in $G_P(L)$.

Let \mathbb{G} be a directed graph. For a vertex v of \mathbb{G} , we define $s(v)$ to be the number of times the orientation switches on edges leaving v (see Figure 11a). The *index* of a vertex v is $I(v) = 1 - \frac{s(v)}{2}$. The *index* of a face is $I(F) = \chi(F) - \frac{s(F)}{2}$, where $s(F)$ is the number of switches around F (see Figure 11b). When \mathbb{G} is actually the dual graph Γ^* , define $t = r - \sum I(v)$, where the sum is taken over all dual vertices of Γ^* .

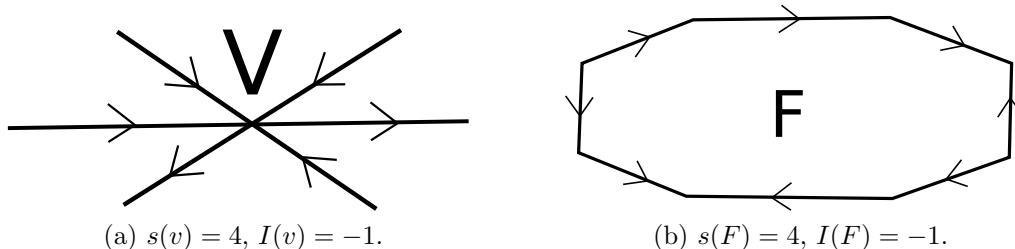


Figure 11: Directed graph index calculations.

Let L be a set of labels with $|L| \geq 2$ and τ a nontrivial L -type. Let T be a star with $L(T) = L$ and $[T] = \tau$.

Lemma 3.24 ([11], 2.3.1). *Suppose that*

- (i) *all elements of $C(T)$ have the same parity;*
- (ii) *all elements of $A(T)$ have the same parity;*
- (iii) *$G_P(L)$ does not represent τ .*

Then either

- (1) there exists a new x -cycle Σ in G_Q such that the vertices of Σ are a subset of either $C(T)$ or $A(T)$; or
- (2) there is no new x -cycle as described in (1), and the following conditions hold:
- (a) $G_P(L)$ is a connected graph;
 - (b) $t = u = 0$;
 - (c) each special vertex of $G_P(L)$ is adjacent to $|S(T)|$ switch edges;
 - (d) each regular vertex of $G_P(L)$ is adjacent to exactly $|C(T)| - 1$ clockwise switch edges and exactly $|A(T)| - 1$ anticlockwise switch edges;
 - (e) there exists 1, 2-Scharlemann cycles in G_Q on $C(T)$ and $A(T)$;
 - (f) $n \leq |C(T)| = |A(T)|$, where n is the order of the Scharlemann cycle;
 - (g) $|[C(T), L \setminus C(T)]| = |[A(T), L \setminus A(T)]| = p - 2$; and
 - (h) the vertices of $C(T)$ (also $A(T)$) are connected in G_Q .

For a directed graph \mathbb{G} ,

Lemma 3.25 ([7], 2.3.1). $\sum_{\text{vertices}} I(v) + \sum_{\text{faces}} I(F) = 2$.

Lemma 3.26 (Extension of [7], 2.3.3). *Let Γ be a graph with dual orientation, F a disk face of a subgraph of Γ such that $\text{ind}_\Gamma(\partial F) \leq 0$. Then Γ has one of the following in F :*

1. A switch edge;
2. A dual source or sink face;
3. A fat vertex with uniform dual orientation.

Proof. This proof is adopted from Gordon and Luecke's proof. Let $2F$ be the double of F , i.e. $2F = F \cup_{\partial F} -F$. Let $2\Gamma^*$ denote the double of $\Gamma^* \subset 2F$. By Lemma 3.25,

$$2 \sum_{v \in \Gamma^* \cap F} I(v) + 2 \sum_{f \text{ face of } \Gamma^* \cap F} I(f) + \sum_{X \text{ a corner of } F} \text{ind}(X) + \sum_{e \subset \partial F} \text{ind}(e) = 2$$

Therefore

$$2 \sum I(v) + 2 \sum I(f) + \text{ind}_\Gamma(\partial F) = 2.$$

Thus $\sum I(v) + \sum I(f) > 0$. Note that a positive index vertex of Γ^* implies the presence of either a fat vertex sink or source or a dual sink or source face, while a positive index face of Γ^* corresponds to a switch edge of Γ . \square

Lemma 3.27 and Lemma 3.28 show that the existence of certain types of cycles in $G_P(V)$ imply that $G_P(V)$ represents certain V -types. Note that Lemma 3.28 can cover the trivial type, but Lemma 3.27 is provided since in the case of the trivial type, the result can be obtained in a slightly more general setting.

Theorems 3.29 and 3.30 show the existence of trees in G_P when particular V -types are not represented by $G_P(V)$.

Lemma 3.28 and Theorem 3.30 are an extension of Theorem 2.4.2 from [11].

Lemma 3.27. *Let V be any set of vertices in G_Q . Suppose G_P has $\mathbb{P}(V_r, V)$. If $G_P(V)$ has a cycle on negative (or positive) vertices, then $G_P(V)$ represents the trivial V -type.*

Proof. Since G_P has $\mathbb{P}(V_r, V)$, every regular vertex in G_P is adjacent to a parallel vertex at a label in V . Let σ be an innermost cycle on vertices of uniform sign in $G_P(V)$. Without loss of generality, suppose σ is negative. Since G_Q contains a Scharlemann cycle, σ cannot separate the special vertices from one another. Thus σ bounds a unique disk $D \subset \Delta$ for some 1,2-bigon face Δ . Suppose D contains a positive vertex. Every positive vertex x in D has an edge e_x meeting x at a label in V , such that e_x connects x to another positive vertex. By the parity rule, there cannot be an edge from x to another positive vertex y meeting both vertices at labels in V . Thus by Proposition 3.8, there is a cycle among the positive vertices in D , which contradicts the choice of σ . Thus all vertices in D are negative. Since σ is a negative

cycle in $G_P(V)$ that bounds a disk with only negative vertices, $G_P(V)$ represents the trivial V -type. \square

Lemma 3.28. *Suppose $V \subset V_Q$ has uniform sign and uniform dual orientation type, that is, V is entirely contained in A , C , B_+ or B_- . Let $W \subset V_Q$ be a set of opposite sign from V . Assume that $G_Q[V, W]$ has $\mathbb{A}(V, V_r)$. If G_P has an oriented cycle where each edge has its tail at a label in V and its head at a label in W , then G_P represents τ .*

Proof. Suppose some component of $G_P[V, W]$ has such an oriented cycle σ . Since the special vertices are connected by edges from the Scharlemann cycles in G_Q , the special vertices must be in the same σ -disk. Assume σ is innermost in the σ -disk D not containing the special vertices, and assume σ is positive. Let V_D be the σ -set in D , and suppose that V_D contains a vertex antiparallel to σ . Since $G_Q[V, W]$ has $\mathbb{A}(V, V_r)$, $G_P[V, W]$ has $\mathbb{P}(V_r, V)$. Thus for each negative vertex $x \in V_D$, there is an edge e_x which meets x at a label in V and meets another negative vertex in V_D at a label in W . Thus there must be an oriented cycle of such edges, contradicting the innermost assumption of σ .

So V_D has no negative vertices. Since every vertex is adjacent to an antiparallel vertex via a switch edge, $V_D = \emptyset$. Let x_0 be a vertex on ∂D , and let e_0 be the edge leaving x_0 at a V label. Without loss of generality we assume that the dual orientation of the corner in D adjacent to e is into x_0 . Let X_0 be the corner of D at x_0 .

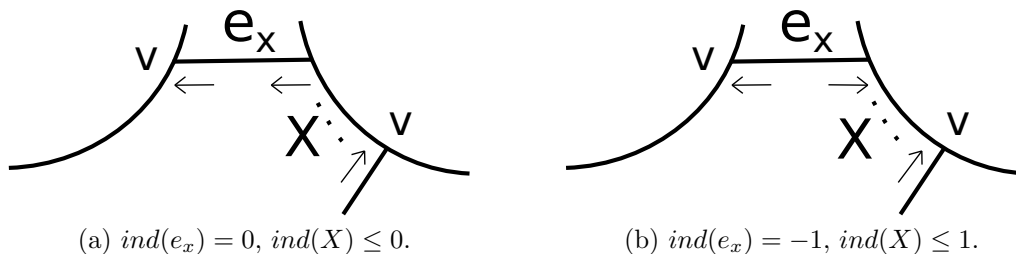


Figure 12: Index calculations.

Now note that if $\text{ind } e_X = 0$, then $\text{ind } X \leq 0$. If $\text{ind}(e_X) = -1$, then $\text{ind}(X) \leq 1$ (Figure 12). Either way, $\text{ind}(e_X) + \text{ind}(X) \leq 0$. Therefore $\text{index}(\partial D) \leq 0$. By Lemma 3.26, there is a face in $G_P(L)$ in D that represents τ . \square

Theorem 3.29. *Let V be an (s) -set contained in an (s) -disk. Let W be the set of vertices in G_Q of sign $-s$ to which V have edges. If $G_P(V)$ does not represent the trivial V -type, the following all hold:*

1. $V^* = V_r$;
2. $|[V, W]| = p - 2$;
3. V has a Scharlemann cycle;
4. $G_P([V, W])$ consists entirely of a $(V \rightarrow W \rightarrow 1)$ tree and a $(V \rightarrow W \rightarrow 2)$ tree.

Proof. By Lemma 3.11, G_P has $P(V_r, V)$ and $V^* \supset V_r$, implying that $[V, W] \geq p - 2$. By Lemma 3.27, $G_P(V)$ cannot have a cycle on parallel vertices, so by Lemma 3.13, $[V, W] = p - 2$. This means that $V^* = V_r$, so by Lemma 3.11 V has a Scharlemann cycle.

Finally, Lemma 3.14 gives the desired trees. \square

Theorem 3.30 (Extension of [11] 2.4.2). *Suppose $V \subset V_Q$ has uniform sign and uniform dual orientation type, that is, V is entirely contained in A , C , B_+ or B_- . Let $W \subset V_Q$ be a set of opposite sign from V . If $G_Q[V, W]$ has $\mathbb{A}(V, V_r)$, then G_P has a negative $(V \rightarrow W \rightarrow 1)$ tree and a positive $(V \rightarrow W \rightarrow 2)$ tree.*

Proof. Any component of $G([V, W])$ that is not in a $(V \rightarrow W \rightarrow r)$ tree with r a special vertex must have an oriented cycle such that each edge has its tail at a V label and its head at a W label. \square

4 Gordon-Luecke Proof Deconstruction

In this section, the proof from [7] is deconstructed. *This is not a new proof.* It is merely the proof of Gordon and Luecke rewritten so as to more easily extract lemmas which will be used later.

4.1 Good, Bad, and Ugly Corners

A corner X is **ugly** if there exist $A(X)$ elements with differing parities. Since $A(X) = A(-X)$, X is ugly if and only if $-X$ is ugly. If X is not ugly, define $\text{char } A(X) = \text{char}(a, V(X))$ for any $a \in A(X)$. Choose a clockwise character $\eta_c = \pm$ and an anticlockwise character $\eta_a = \pm$. Two-color the faces of G_P . Choose the B/W coloring so that a pair (l, v) of character η_c is WB (going counterclockwise). We refer to (l, v) pairs as WB or BW. For $C_1, C_2 \in \{B, W\}$, a corner X is C_1C_2 if the leftmost label of X ($\in \partial I(X)$) is C_2C_1 , and the rightmost label is C_1C_2 . For example, in Figure 13, the first corner is BB , the second WW , and the third and fourth BW .

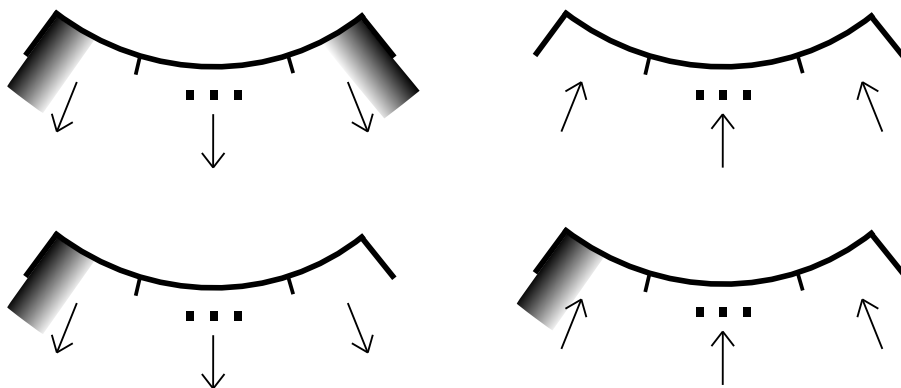


Figure 13: Good atoms.

An **atom** is a corner with no switch labels. The atoms in Figure 13 are **good**, and all others are **bad**. Note that

1. an atom X is good if and only if $-X$ is bad, and

2. atoms on either side of a clockwise switch are both good if $\text{char}(l, v) = \eta_c$; otherwise, both atoms are bad.

If $\text{char } A(X) = \eta_a$, then X is **good** if and only if all maximal atoms in X are good. If $\text{char } A(X) = -\eta_a$, X is **good** if and only if *some* maximal atom in X is good. X is **bad** if it is neither good nor ugly. A clockwise (anticlockwise) switch is **double-sided** if it has character η_c (η_a) and **single-sided** if it has character $-\eta_c$ ($-\eta_a$).

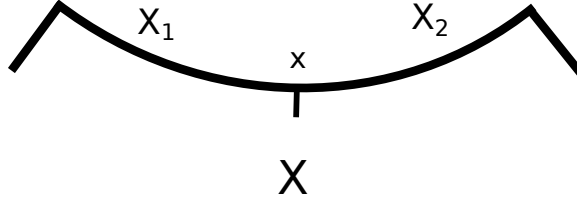


Figure 14: Corner for Lemma 4.1.

Lemma 4.1 (2.6.1 from [7]). *Let $x \in S(X)$, and X , X_1 , and X_2 be as Figure 14. Assume X is not ugly.*

- (i) *If x is double-sided then X is good if and only if X_1 and X_2 are good.*
- (ii) *If x is single-sided then X is good if and only if X_1 or X_2 is good.*

Lemma 4.2 (2.6.2 from [7]). *If $\text{char } A(X) = -\eta_a$ and there exists $c \in C(X)$ with $\text{char } c = \eta_c$ then X is good.*

Lemma 4.3 (2.7.1 from [7]). *Let F be a disk face of a subgraph of Γ such that each corner of F with respect to Γ is good. Then $\text{ind}_\Gamma \partial F \leq 0$.*

Definition 4.4. *Let τ be a nontrivial L -type, and let (T_1, \dots, T_n) be a sequence of coherence for τ . Let $L_0 \subset L$, and let (R_1, \dots, R_n) be the sequence obtained by letting $R_1 = T_1$, $R_i = d_0^\pm R_{i-1}$, where the sign of d_0^\pm is chosen to match that of the corresponding d^\pm . Then (R_1, \dots, R_n) is a L_0 -sequence for τ .*

Note that by Proposition 3.18, $C(T_n) \subset C(R_n)$ and $\tilde{A}(T_n) \supset \tilde{A}(R_n)$. Let I be an L_0 -interval at V_+ .

Proposition 4.5. *The corner $R_n|I$ is not ugly.*

Proof. $\tilde{A}(R_n) \subset \tilde{A}(T_n) \subset A(T_n)$. Since T_n is coherent, $\tilde{A}(R_n)$ is of uniform sign. \square

Let $\eta_c = \text{char}(C(T_n), V(T_n))$ and $\eta_a = -\text{char}(\tilde{A}(R_n), V(R_n))$. If $\tilde{A}(R_n)$, η_a can be chosen arbitrarily. Recall that $V(T_n) = V(R_n) = V_+$. For each L_0 -interval I , define $\epsilon(I) = \pm$ by requiring $\epsilon(I)(R_n|I)$ to be good. Define the L_0 -type τ_0 by $\tau_0 = (\epsilon(I) : I \text{ an } L_0\text{-interval})$. We call τ_0 an ***inherited L_0 -type*** of τ . Note that we have not shown, nor do we use, that τ_0 is independent of the coherence sequence chosen.

4.2 Proof of Theorem 3.2

For the duration of this section, let D be a disk in Q that is either the complement of a small open disk disjoint from G_Q , or a disk bounded by a new x -cycle Σ in G_Q . Let L be the set of vertices of G_Q in $\text{int } D$. Note that $|L| \geq 2$, as there can be no new great x -cycles, nor isolated vertices, in G_Q .

Lemma 4.6. *Let τ be a nontrivial L -type with sequence of coherence (T_1, \dots, T_n) . If $G_Q \cap D$ contains no x -cycle on vertices of $A(T_n)$ or $C(T_n)$, then $G_P(L)$ represents τ .*

Proof. If $G(L_n)$ does not represent $[T_n]$, then by Lemma 3.24, G_Q contains an x_0 -cycle Σ_0 with vertices either in $A(T_n)$ or $C(T_n)$, contradicting the hypothesis. Thus $G_P(L_n)$ represents $[T_n]$, so $G_P(L)$ represents τ by Corollary 3.21. \square

Lemma 4.7. *Suppose that $G_Q \cap D$ contains no x -cycle. Then $G_P(L)$ represents the trivial L -type.*

Proof. Let $J \subset L$ be all vertices of opposite sign from Σ (or, if $\Sigma = \emptyset$, choose J to be the positive elements of L). Σ cannot be a great new x -cycle by Theorem 3.3, so $J \neq \emptyset$.

Suppose first that for some vertex x_0 in G_P , every label $y \in J$ on x_0 is adjacent to an edge leading to an antiparallel vertex. These edges must all lead to parallel labels, implying that the labels on both ends of each edge are in J . We thus have an x_0 -cycle in D , contrary to our assumption.

Therefore for each vertex x in G_P there exists a label $y(x) \in J$ such that the edge $e(x)$ leaving vertex x at label y goes to a vertex parallel to x . Note that the label at the other end of $e(x)$ must have opposite sign from $y(x)$ by the parity rule. If $E = \{e(x)\}$, $G_P(E)$ will have circuits on every connected component. Every connected component will have uniform sign (since edges $e(x)$ connect parallel vertices). An innermost circuit on parallel vertices will therefore bound a disk E with only vertices parallel to the circuit. E thus contains a disk face representing the trivial L -type. \square

Lemma 4.8. *Suppose L is an arbitrary set of vertices of G_Q and $L' \subset L$. If $G_P(L')$ represents the trivial L' -type, then $G_P(L)$ represents the trivial L -type.*

Proof. $G_P(L')$ represents the trivial L' -type, and thus has a disk face E' representing the trivial L' -type. Any face E of $G(L)$ in the disk E' thus represents the trivial L -type. \square

Lemma 4.9. *Let τ be a nontrivial L -type, with sequence of coherence (T_1, \dots, T_n) . Suppose $G_Q \cap D$ contains a cycle Σ_0 on vertices of $A(T_n)$ (respectively $C(T_n)$), such that the Σ_0 -disk $D_0 \subset D$ contains vertices which are not in $A(T_n)$ (respectively $C(T_n)$). Let L_0 be the Σ_0 -set contained in D_0 . Suppose $G_P(L_0)$ represents τ_0 , an inherited L_0 -type of τ obtained from the above sequence of coherence. Then $G_P(L)$ represents τ .*

Proof. We assume that the vertices of Σ_0 are in $C(T_n)$ by possibly replacing T_n with $\overline{T_n}$. This can be achieved by replacing d_n by its negative in the sequence of coherence, and replacing τ_0 with $-\tau_0$. Let E be a face of $G_P(L_0)$ that represents τ_0 . Then there exists $\eta = \pm$ such that if a corner of E at a vertex v is in the L_0 -interval I , then

sign $v = \eta\epsilon(I)$ (recall the definition of η from the end of section 4.1). Let J be the subinterval of I corresponding to the corner, and let (R_1, \dots, R_n) be the L_0 -sequence for τ obtained from (T_1, \dots, T_n) .

The remainder of the proof is the same as the end of the proof of Lemma 2.8.2 in [7], but is included here for completeness.

Claim 4.10. $\eta(\text{sign } v)R_n|J$ is good.

Proof. Since $\eta(\text{sign } v) = \epsilon(I)$, we are done if J is an L_0 -interval, by the definition of τ_0 . So assume this is not the case. Then J is a subinterval of an L_0 -interval I with at least one endpoint of J an exceptional label in $G_P(L_0)$. Since the exceptional labels are contained in $C(T_n)$, $\epsilon(I) = +$ by Lemma 4.2. Now $R_n|J$ is good by Lemma 4.1(i). \square

Let $\Gamma_i = \Gamma(R_i)$, $1 \leq i \leq n$, and let $\delta_0 = \delta_{G(L_0)}$. Since

$$\{\text{exceptional labels of } G(L_0)\} \subset \{\text{vertices of } \Sigma_0\} \subset C(T_n) \subset C(T_i) \subset C(R_i),$$

for $1 \leq i \leq n$, we have from Lemma 3.19 that $\Gamma_i = \delta_0 \Gamma_{i-1}$, $2 \leq i \leq n$.

By Claim 4.10, ηX is good for any corner X of E in Γ_n . A face of $\Gamma_n^* \cap E$ of positive index corresponds to a switch-edge e of $\Gamma_n \cap E$. Since the endpoints of e have opposite characters, one endpoint of e will be a double-sided switch, the other single-sided. Split E along e , i.e., let $E = E_1 \cup_e E_2$. By Lemma 4.1, for some $i \in \{1, 2\}$, ηX is good for every corner X of E_i in Γ_n . We repeat until we get a disk $F \subset E$, bounded by edges of Γ_n , such that

- (a) for each Γ_n -corner X of F , ηX is good;
- (b) Γ_n^* has no faces of positive index in F .

Lemma 4.3 and a imply that $\text{ind}_{\Gamma_n} \partial F \leq 0$. Lemma 3.26 and b imply that Γ_n^* has a sink or source in F . Since $\Gamma_i = \delta_0 \Gamma_{i-1}$, $2 \leq i \leq n$, By Lemma 3.20, $\Gamma_1^* = \Gamma(T_1)^*$ has

a sink or source at a dual vertex in F . Thus $G_P(L)$ represents τ .

□

Theorem 4.11. $G_P(L)$ represents all L -types or there exists a Scharlemann cycle in $\text{int } D$.

Proof. Assume no Scharlemann cycle exists in $\text{int } D$. We proceed by induction on the number of new x -cycles. If no new x -cycles exist in $\text{int } D$, then by Lemmas 4.6 and 4.7 $G_P(L)$ represents all types.

We now make the inductive assumption that any disks containing fewer new x -cycles than D represent all types. Suppose that $G_P(L)$ does not represent the L -type τ . By Lemma 4.8, τ is nontrivial. Let (T_1, \dots, T_n) be a sequence of coherence for τ . By Lemma 4.6, $\text{int } D$ contains an x -cycle on $A(T_n)$ or $C(T_n)$. By Lemma 4.9 and the inductive assumption, $G_P(L)$ represents τ . □

Although the proof of the following theorem is in [14], the explanation of the homological implication is in Section 4 of [5].

Theorem 4.12 (Consequence of [14]). *If G_P represents all V_Q -types, then $H_1(M(\gamma))$ contains a nontrivial torsion element.*

Theorem 4.12 contradicts $M(\gamma) = S^3$, concluding the proof of Theorem 3.2.

5 Rotation-Free Graph

Assume τ is an L -type, T a coherent star with $[T] = \tau$. Let $G_P(L)$ have the dual orientation inherited from τ . Let \mathbb{F} be a collection of faces of $G_P(L)$. We define $Rev(G_P(L), \mathbb{F})$ to be the graph obtained by reversing the dual orientations on each $F \in \mathbb{F}$. We define a fat vertex graph with dual orientation to be **representative** if it has a face that is a source or a sink with respect to the dual orientation.

Lemma 5.1. *For any collection of faces \mathbb{F} , $G_P(L)$ is representative if and only if $Rev(G_P(L), \mathbb{F})$ is representative.*

Proof. Any source or sink in one graph will correspond to a source or sink in the other. □

Proposition 5.2. *Let G be a graph. If every vertex of G has even degree, the faces of G can be two-colored such that each edge has each color on exactly one side.*

Proof. The statement clearly holds if G has zero edges. Given a G satisfying the hypothesis that has more than zero edges, select a disk face D of G (possibly containing vertices in its interior) and remove all edges in ∂D to obtain G' . G' has strictly fewer edges, but still satisfies the hypothesis, proving the statement by induction. □

We assume that $G_P(L)$ is not representative, $[T]$ is nontrivial, and that neither $A(T)$ nor $C(T)$ have a new x -cycle. Let E_{switch} be the switch edges of $G_P(L)$. Then by Lemma 3.24 and Proposition 5.2, $G_P(E_{switch})$ can be two-colored black and white, with every edge between a white face and a black face. $G_P(L)$ inherits a coloring from $G_P(E_{switch})$. Let \mathbb{F}_{black} be the black faces of $G_P(L)$.

Lemma 5.3. *$Rev(G_P(L), \mathbb{F}_{black})$ has no switch edges and is not representative.*

Proof. $G = Rev(G_P(L), \mathbb{F}_{black})$ is not representative by Lemma 5.1. Since each switch edge of $G_P(L)$ has a black face on exactly one side, the dual orientations on exactly one side of every switch edge of $G_P(L)$ gets reversed in G .

All edges in $G_P(L)$ which are not switch edges are adjacent to two faces of the same color, and therefore are not switch edges in G . \square

We will henceforth define $RF_P = Rev(G_P(L), \mathbb{F}_{black})$. We will define

$$A(RF) = \{(l, v) \in V_Q \times V_P \mid l \text{ on } v \text{ is an anticlockwise switch label, in } RF_P\}, \quad (3)$$

$$C(RF) = \{(l, v) \in V_Q \times V_P \mid l \text{ on } v \text{ is a clockwise switch label, in } RF_P\}. \quad (4)$$

Lemma 5.4. *RF_P does not have a directed cycle with each tail at a label from $A(RF)$. Similarly for $C(RF)$.*

Proof. By Lemma 3.26, any such directed cycle σ would bound faces each containing a switch edge, a source/sink face, or a source/sink fat vertex. Since RF_P has no switch edges or source/sink faces, each face bounded by σ contains a source/sink face.

However by Lemma 3.24, only the special vertices will be source/sink vertices in RF_P . Since the special vertices have edges between them (from any Scharlemann cycle), they must be on the same side of σ , a contradiction. \square

The following can be seen as a generalization of Theorem 3.30. Note that if T is a trivial type, this section does not apply, but we get similar trees from Theorem 3.29.

Corollary 5.5. *RF_P has disjoint $(A(RF) \rightarrow 1)$ and $(A(RF) \rightarrow 2)$ trees. Similarly for $C(RF)$.*

Proof. This follows immediately from Lemma 3.24 and Lemma 5.4. \square

Suppose now that τ is a V_Q -type. We can 2-color the entire graph G_P , since q is always even. Let \mathbb{F} be the black faces. The parity rule allows us to partition the corners of T into black corners and white corners. Thus $Rev(G_P(L), \mathbb{F})$ defines a star \widehat{T} , the *conjugate* of T . The above results therefore imply that G_P represents τ if and only if G_P represents $\widehat{\tau} = [\widehat{T}]$. Although the conjugate is defined up to a

choice of coloring, G_P represents \hat{T} if and only if G_P represents $\overline{(\hat{T})}$, so generally the choice is not specified. A coherent V_Q -star T is **bicoherent** if \hat{T} is also coherent, or (equivalently) if $B_+(T)$ and $B_-(T)$ are each of uniform sign. Note that for some $s \in \{+, -\}$, $A(T) = B_s(\hat{T})$ and $C(T) = B_{-s}(\hat{T})$. We are therefore describing a symmetry between the $A(T), C(T)$ sets and the $B_+(T), B_-(T)$ sets.

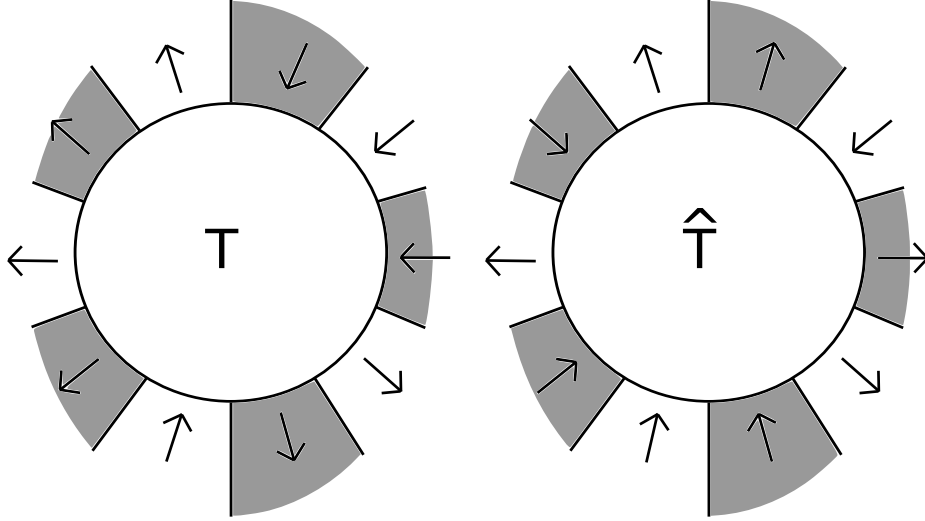


Figure 15: The star T and its conjugate \hat{T} .

Theorem 5.6. *Let T be bicoherent with $[T]$ a nontrivial V_Q -type, such that $[\hat{T}]$ is also nontrivial. If G_P has no new x -cycle on $A(T)$ or $C(T)$, then G_P has a new x -cycle on $B_+(T)$ or $B_-(T)$.*

Proof. Suppose G_P has no new x -cycle on B_+ or B_- . Suppose v is a leaf in the $(A(RF) \rightarrow 1)$ tree which exists by Corollary 5.5. This means that no B_+ or B_- label on v can be adjacent to an edge going to an $A(RF)$ label. But B_+ and B_- are each of uniform sign, and G_Q cannot have a new v -cycle, so at least one edge leaving v at a B_+ label must meet another vertex at a $C(RF)$ label, and similarly for B_- . Therefore two edges of the $(C(RF) \rightarrow 2)$ tree come into v . Hence v is a vertex of the $(C(RF) \rightarrow 2)$ tree with at least two children.

Since v was an arbitrary leaf of the $(A(RF) \rightarrow 1)$ tree, every leaf of the $(A(RF) \rightarrow 1)$ and $(A(RF) \rightarrow 2)$ trees must have two children in the $(C(RF) \rightarrow j)$ trees. Thus

if the $(A(RF) \rightarrow i)$ trees together have r leaves, the $(C(RF) \rightarrow j)$ trees will together have more than r leaves. But every leaf of the $(C(RF) \rightarrow j)$ tree can similarly be shown to have at least two children in the $(A(RF) \rightarrow i)$ trees, implying that the $(A(RF) \rightarrow i)$ trees have more than r leaves, a contradiction. \square

Corollary 5.7. *Let T be bicoherent with $[T]$ a nontrivial V_Q -type, such that \widehat{T} is also nontrivial, and such that G_P has no new x -cycle on $A(T)$ or $C(T)$. Then there is a new x -cycle on $A(\widehat{T})$ or $C([\widehat{T}])$.*

Proof. Taking the conjugate of a type swaps B_s with A and B_{-s} with C for some choice of sign s . \square

6 The Cabling Conjecture

Theorem 6.1. *The cabling conjecture is true for b -bridge knots with $b \leq 5$.*

Proof. Recall that by Theorem 4.12, if G_P represents all V_Q -types, then $H_1(M(\gamma))$ has torsion. But $M(\gamma)$ is S^3 , so G_P cannot represent all V_Q -types. We will show that if a V_Q -type τ exists which G_P does not represent, then $q > 2v + 2$, where v is the number of vertices in the smallest great web in G_Q . By Corollary 3.7, $v \geq 4$, so $q > 10$.

If G_P does not represent the V_Q -type $[T] = \tau$, then τ falls in one of the following cases:

1. τ is the trivial V_Q -type;
2. τ is coherent and nontrivial, and no cycle on $A(T)$ ($C(T)$) bounds two disks each containing vertices which are not in $A(T)$ ($C(T)$);
3. τ is incoherent with sequence of coherence (T_1, \dots, T_n) , and no cycle on $A(T_n)$ ($C(T_n)$) bounds two disks each containing vertices which are not in $A(T_n)$ ($C(T_n)$);
4. τ is nontrivial with sequence of coherence (T_1, \dots, T_n) , and there exists a cycle on $A(T_n)$ ($C(T_n)$) which bounds two disks both containing vertices not in $A(T_n)$ ($C(T_n)$).

Theorem 6.2. *G_P represents the trivial V_Q -type.*

Proof. Suppose G_P does not represent the trivial V_Q -type. By Lemma 3.13, if G_Q contains more than $p - 2$ edges between antiparallel vertices, then G_P has a cycle on parallel vertices. G_P has $\mathbb{P}(V_r, V_Q)$ by Lemma 3.4, so by Lemma 3.27, G_P cannot have a cycle on parallel vertices. Thus G_Q contains at most $p - 2$ edges between antiparallel vertices.

G_Q contains at least two innermost (s)-sets, say V and W . By Lemma 4.8, $G_P(V)$ does not represent the trivial V -type and $G_P(W)$ does not represent the trivial W -type. Theorem 3.29(2) shows that precisely $p - 2$ edges leave V and precisely $p - 2$ edges leave W . But since V (respectively W) is an (s)-set, any edge leaving V (respectively W) is an edge between antiparallel vertices. Thus V must contain all (we may assume) positive vertices and W must contain all negative vertices. By Theorem 3.29(1), $V^* = W^* = V_r$. This implies that there can be no edge between V and W which meets the label 1 at either end, a contradiction with Theorem 3.29(4). Thus G_P must represent the trivial V_Q -type. \square

Note that by Corollary 5.7, if τ is in case 2, then $\hat{\tau}$ is in case 3 or 4. Since G_P does not represent τ if and only if G_P does not represent $\hat{\tau}$, it is sufficient to consider cases 3 and 4.

Case 3: Suppose τ is incoherent with sequence of coherence (T_1, \dots, T_n) , with no cycle on $A(T_n)$ ($C(T_n)$) bounding two disks each containing vertices which are not in $A(T_n)$ ($C(T_n)$). Since T_n is the first coherent type in the sequence, $n > 1$ and T_n is nontrivial. By Lemma 3.24, $A(T_n)$ and $C(T_n)$ are great webs, so $|L(T_n)| \geq 2v$. But notice that by the definition of derivative type, either $A(T_n) \cup C(T_n) \subset A(T)$ or $A(T_n) \cup C(T_n) \subset C(T)$. Thus $|A(T)| = |C(T)| \geq 2v$. This immediately implies that $q \geq 4v$. In fact, if $q = 4v$, then $|A(T)| = |C(T)| = 2v$, so $\hat{\tau}$ is the trivial V_Q -type. Theorem 6.2 implies that G_P represents $\hat{\tau}$ and therefore τ , contrary to our assumptions. Thus $q \geq 4v + 2$.

Case 4: Suppose τ is nontrivial and has a sequence of coherence (T_1, \dots, T_n) and a cycle σ on $A(T_n)$ ($C(T_n)$) such that σ bounds two disks each containing vertices other than $A(T_n)$ ($C(T_n)$). By Corollary 3.21, $G_P(L(T_n))$ does not represent $[T_n]$. Let D_1 and D_2 be the two σ -disks, with σ -sets V_1 and V_2 . By Lemma 4.9, $G_P(V_1)$ does not represent some V_1 -type τ_1 and $G_P(V_2)$ does not represent some V_2 -type τ_2 .

Consider just $D^1 = D_1$ and $V^1 = V_1$, and let $\sigma^1 = \sigma$. For $[T^i] = \tau^i$ a V^i -type

with sequence of coherence (T_1^i, \dots, T_n^i) , we are interested in defining σ^{i+1} as follows, if possible (i.e. if such cycles exist):

1. a new x -cycle on $A(T_n^i)$ or $C(T_n^i)$ if τ^i is nontrivial; or
2. a cycle on $A(T_n^i)$ ($C(T_n^i)$) such that σ^{i+1} bounds two disks each containing vertices other than $A(T_n^i)$ ($C(T_n^i)$), if τ^i is nontrivial and no new x -cycle exists on $A(T_n^i)$ or $C(T_n^i)$; or
3. a new x -cycle, if τ^i is trivial.

When we find such a σ^{i+1} , let D^{i+1} be the σ^{i+1} -disk contained in D_1 , and let V^{i+1} be the corresponding σ^{i+1} -set. By Lemmas 4.8 and 4.9, there is some V^{i+1} -type τ^{i+1} such that $G_P(V^{i+1})$ does not represent τ^{i+1} . By finiteness we can find $T^\alpha, D^\alpha \subset D_1$, σ^α , and $V^\alpha \subset V_1$ such that no qualifying $\sigma^{\alpha+1}$ cycle exists. In the same way, we can find $T^\beta, D^\beta \subset D_2$, σ^β , and $V^\beta \subset V_2$ such that no qualifying $\sigma^{\beta+1}$ cycle exists.

Lemma 6.3. *D^α and D^β are nontrivial σ^α and σ^β disks, respectively.*

Proof. For $\gamma \in \{\alpha, \beta\}$, if σ^γ is a new x -cycle, the result follows immediately from Theorem 3.3. The only way σ^γ is not a new x -cycle is if $\tau^{\gamma-1}$ is nontrivial and there is no new x -cycle on $A(T_n^{\gamma-1})$ or $C(T_n^{\gamma-1})$. Thus the second part of Lemma 3.24 applies, and since σ^γ is on vertices of either $A(T_n^{\gamma-1})$ or $C(T_n^{\gamma-1})$ (without loss of generality, suppose $A(T_n^{\gamma-1})$), at most $p - 2$ edges leave V^γ .

If D^γ is trivial, then every V^γ vertex has the same sign. By Corollary 3.10 and Theorem 3.3, exactly $p - 2$ edges leave V^γ , one at each label in V_r . Thus every edge leaving $A(T_n^{\gamma-1})$ goes to a label in V_r , a contradiction of Theorem 3.30. \square

We break case 4 from above into the following three subcases:

- (i) $[T_n^\alpha], [T_n^\beta]$ are both nontrivial;
- (ii) Exactly one of $[T_n^\alpha], [T_n^\beta]$ is nontrivial;

(iii) $[T_n^\alpha], [T_n^\beta]$ are both trivial.

Case i:

Part 2 of Lemma 3.24 can be applied to both disks, implying the existence of at least 2 great webs in each. This implies that $|V^\alpha| \geq 2v$ and $|V^\beta| \geq 2v$. Since Σ is disjoint from both, we have $q > 4v$.

Case ii:

Suppose that τ_n^α is nontrivial and τ_n^β is trivial. Then as in Case i, $|V^\alpha| \geq 2v$. Meanwhile, Lemma 6.3 implies that V^β is antiparallel to σ^β . Thus by Theorem 3.29, V^β is a great web. Therefore by Corollary 3.7, $|V^\beta| \geq v$. Since T_n^α is coherent, either both $A(T_n^\alpha)$ and $C(T_n^\alpha)$ are the same sign, or one is the same sign as V^β . Therefore at least $2v$ vertices must exist of each sign, so $q \geq 4v$.

Case iii:

Suppose both τ_n^α and τ_n^β are trivial. As in Case ii, $|V^\alpha| \geq v$ and $|V^\beta| \geq v$. Since σ is disjoint from both, $q > 2v$.

Suppose $q = 2v + 2$. Then there is an antiparallel vertex x^α to V^α with a loop bounding a disk containing only V^α . By Theorem 3.29, $G([V^\alpha, \{x^\alpha\}])$ consists entirely of a negative ($V^\alpha \rightarrow \{x^\alpha\} \rightarrow 1$) tree and a positive ($V^\alpha \rightarrow \{x^\alpha\} \rightarrow 2$) tree. Note that the edges between the special vertices 1 and 2 split G_P into several disks Δ_i . By Lemma 3.17, the negative ($V^\alpha \rightarrow \{x^\alpha\} \rightarrow 1$) and the positive ($V^\alpha \rightarrow \{x^\alpha\} \rightarrow 2$) must be in distinct Δ_i . Hence there can be no edges between parallel regular vertices in G_Q , contradicting the existence of the great web V^α . Thus $q > 2v + 2$.

We have shown that if G_P does not represent a V_Q -type τ , $q > 2v + 2$. By Corollary 3.7, $v \geq 4$, so $q > 10$. Thus either G_P represents all types or the bridge number $b > 5$. By Theorem 4.12, G_P cannot represent all types, hence $b > 5$. \square

Remark 6.4. *Note that $q = 12$ is only possible if the type τ which is not represented by G_P falls in case 4(iii), and $b = 6$ is only possible if $q = 12$ and every thin presentation*

of k is also bridge position. Similarly, $q = 14$ (and hence $b = 7$) is only possible if τ falls in case 4(iii).

References

- [1] Ryan Blair et al. *Exceptional and Cosmetic Surgeries on Knots*. Version 2. Mar. 13, 2014. arXiv: 1209.0197v2 [math.GT].
- [2] Mario Eudave-Muñoz. “Band sums of links which yield composite links. The cabling conjecture for strongly invertible knots”. In: *Trans. Amer. Math. Soc.* 330.2 (1992), pp. 463–501. ISSN: 0002-9947. DOI: 10.2307/2153918. URL: <http://dx.doi.org/10.2307/2153918>.
- [3] David Gabai. “Foliations and the topology of 3-manifolds. III”. In: *J. Differential Geom.* 26.3 (1987), pp. 479–536. ISSN: 0022-040X. URL: <http://projecteuclid.org/euclid.jdg/1214441488>.
- [4] Francisco González-Acuña and Hamish Short. “Knot surgery and primeness”. In: *Math. Proc. Cambridge Philos. Soc.* 99.1 (1986), pp. 89–102. ISSN: 0305-0041. DOI: 10.1017/S0305004100063969. URL: <http://dx.doi.org/10.1017/S0305004100063969>.
- [5] C. McA. Gordon. “Combinatorial methods in Dehn surgery”. In: *Lectures at KNOTS '96 (Tokyo)*. Vol. 15. Ser. Knots Everything. World Sci. Publ., River Edge, NJ, 1997, pp. 263–290. DOI: 10.1142/9789812796097_0010. URL: http://dx.doi.org/10.1142/9789812796097_0010.
- [6] C. McA. Gordon. “Dehn surgery and satellite knots”. In: *Trans. Amer. Math. Soc.* 275.2 (1983), pp. 687–708. ISSN: 0002-9947. DOI: 10.2307/1999046. URL: <http://dx.doi.org/10.2307/1999046>.
- [7] C. McA. Gordon and J. Luecke. “Knots are determined by their complements”. In: *J. Amer. Math. Soc.* 2.2 (1989), pp. 371–415. ISSN: 0894-0347. DOI: 10.2307/1990979. URL: <http://dx.doi.org/10.2307/1990979>.
- [8] C. McA. Gordon and J. Luecke. “Only integral Dehn surgeries can yield reducible manifolds”. In: *Math. Proc. Cambridge Philos. Soc.* 102.1 (1987), pp. 97–101. ISSN: 0305-0041. DOI: 10.1017/S0305004100067086. URL: <http://dx.doi.org/10.1017/S0305004100067086>.
- [9] Chuichiro Hayashi and Koya Shimokawa. “Symmetric knots satisfy the cabling conjecture”. In: *Math. Proc. Cambridge Philos. Soc.* 123.3 (1998), pp. 501–529. ISSN: 0305-0041. DOI: 10.1017/S0305004197002399. URL: <http://dx.doi.org/10.1017/S0305004197002399>.
- [10] James A. Hoffman. “There are no strict great x -cycles after a reducing or P^2 surgery on a knot”. In: *J. Knot Theory Ramifications* 7.5 (1998), pp. 549–569. ISSN: 0218-2165. DOI: 10.1142/S0218216598000309. URL: <http://dx.doi.org/10.1142/S0218216598000309>.
- [11] James Allen Hoffman. *Reducing spheres and embedded projective planes after Dehn surgery on a knot*. Thesis (Ph.D.)—The University of Texas at Austin. ProQuest LLC, Ann Arbor, MI, 1995, p. 100. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:9603863.

- [12] W. B. R. Lickorish. “A representation of orientable combinatorial 3-manifolds”. In: *Ann. of Math. (2)* 76 (1962), pp. 531–540. ISSN: 0003-486X.
- [13] William W. Menasco and Morwen B. Thistlethwaite. “Surfaces with boundary in alternating knot exteriors”. In: *J. Reine Angew. Math.* 426 (1992), pp. 47–65. ISSN: 0075-4102.
- [14] Walter Parry. “All types implies torsion”. In: *Proc. Amer. Math. Soc.* 110.4 (1990), pp. 871–875. ISSN: 0002-9939. DOI: 10.2307/2047731. URL: <http://dx.doi.org/10.2307/2047731>.
- [15] Martin Scharlemann. “Producing reducible 3-manifolds by surgery on a knot”. In: *Topology* 29.4 (1990), pp. 481–500. ISSN: 0040-9383. DOI: 10.1016/0040-9383(90)90017-E. URL: [http://dx.doi.org/10.1016/0040-9383\(90\)90017-E](http://dx.doi.org/10.1016/0040-9383(90)90017-E).
- [16] Andrew H. Wallace. “Modifications and cobounding manifolds”. In: *Canad. J. Math.* 12 (1960), pp. 503–528. ISSN: 0008-414X.
- [17] Ying-Qing Wu. “Dehn surgery on arborescent knots”. In: *J. Differential Geom.* 43.1 (1996), pp. 171–197. ISSN: 0022-040X. URL: <http://projecteuclid.org/euclid.jdg/1214457901>.
- [18] Nicholas Zufelt. *Divisibility of Great Webs and Reducible Dehn Surgery*. Version 1. Oct. 13, 2014. arXiv: 1410.3442v1 [math.GT].