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Essays in economic theory

Wei He
University of Iowa

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Wei He

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Thesis Supervisor: Professor Nicholas Yannelis

Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Wei He

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Economics at the May 2016 graduation.

Thesis Committee: ______________________________
Nicholas Yannelis, Thesis Supervisor

____
Rabah Amir

____
Anne Villamil

____
Luciano de Castro

____
Kyungmin Kim

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ABSTRACT

This thesis is composed of three chapters. Chapter 1 considers the existence of equilibria in games with complete information, where players may have non-ordered and discontinuous preferences. Chapter 2 studies the issues on the existence of pure and behavioral strategy equilibria in games with incomplete information and discontinuous payoffs. We consider the standard setting with Bayesian preferences as well as the case in which players may face ambiguity. Chapter 3 extends the classical results on the Walras-core existence and equivalence to an ambiguous asymmetric information economy, where agents maximize maximin expected utilities (MEU). These results are based on the papers He and Yannelis (2014, 2015a,b,c, 2016a,b).

In the first chapter, we propose the condition of "continuous inclusion property" to handle the difficulty of discontinuous payoffs in various general equilibrium and game theory models. Such discontinuities arise naturally in economic situations, including auction, price competition of firms and also patent races. Based on the continuous inclusion property, we establish the equilibrium existence result in a very general framework with discontinuous payoffs. On one hand, this condition is sufficiently general from the methodological point of view, as it unifies almost all special conditions proposed in the literature. On the other hand, our condition is also potentially useful from the realistic point of view, as it could be applied to deal with many economic models which cannot be studied before because of the presence of the discontinuity.

In the second chapter, I study the existence problem of pure and behavioral strategy equilibria in discontinuous games with incomplete information. The framework of games with incomplete information is standard as in the literature, except for that we allow players' payoffs to be discontinuous. We illustrate by examples that the Bayesian equilibria may not exist in such games and the previous results are not applicable to handle this problem. We propose some general conditions to retain the existence of both pure strategy and behavioral strategy Bayesian equilibrium, and show that our condition is tight. In addition, we study the equilibrium existence problem in discontinuous games under incomplete information and ambiguity, and show that the maximin framework solves the equilibrium existence issue without introducing any additional condition.

In the last chapter, I study a general equilibrium model with incomplete information by adopting the maximin expected utilities. The model is powerful enough to describe the behaviors of ambiguity averse agents that cannot be explained by the standard assumption of subjective expected utilities. I use this new formulation to extend many classical results in general equilibrium theory by incorporating ambiguity into the model. In addition, the desirable incentive compatibility property is shown in our model with maximin expected utilities, while this property will typically fail in the traditional setup. Specifically, the existence results are shown for various equilibrium notions in a general equilibrium model, and the incentives can be guaranteed when all agents use the maximin expected utilities.

PUBLIC ABSTRACT

This thesis contributes to economic theory on economies and games with discontinuous payoffs and ambiguity. In Chapter 1, we consider economic environments with complete information, where agents' payoffs may exhibit discontinuities. To handle this difficulty, we propose the condition of "continuous inclusion property", and prove the existence of equilibria in a very general framework. Chapter 2 addresses the issues on the existence of pure and behavioral strategy equilibria in games with incomplete information and discontinuous payoffs. We provide several examples to show that a Bayesian equilibrium may not exist and previous results are not directly applicable. We introduce some general conditions to retain the existence of both pure strategy and behavioral strategy Bayesian equilibria, and apply our results to analyze all-pay auctions with general tie-breaking rules. Chapter 3 studies a general equilibrium model with incomplete information and ambiguity aversion. We assume that agents adopt the maximin preferences, and show the existence of maximin expectations equilibrium and maximin core. Importantly, we prove that the desirable incentive compatibility property can be guaranteed for efficient allocations, which typically fails in the conventional approach.

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CHAPTER 1
DISCONTINUOUS GAMES AND ECONOMIES WITH COMPLETE INFORMATION

1.1 Introduction

The classical equilibrium existence theorems of Nash (1950), Debreu (1952), Arrow and Debreu (1954) and McKenzie (1954) have been generalized to games/abstract economies where agents' preferences need not be transitive or complete, and therefore need not be representable by utility functions.[1] The need to drop the transitivity assumption from equilibrium theory was motivated by behavioral/experimental works which demonstrate that consumers do not necessarily behave in a transitive way. A different line of literature pioneered by Dasgupta and Maskin (1986) and Reny (1999) necessitated the need to drop the continuity assumption on the payoff function of each agent. Their works were motivated by many realistic applications (for example, Bertrand competition and auctions), and generalizations of the Nash-Debreu equilibrium existence theorems were obtained where payoff functions need not be continuous. In other words, a new literature emerged on equilibrium existence theorems with discontinuous payoffs.[2]

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[1]See for example, Mas-Colell (1974), Shafer and Sonnenschein (1975), Gale and Mas-Colell (1975), Borglin and Keiding (1976), Shafer (1976), Yannelis and Prabhakar (1983), and Wu and Shen (1996) among others.

[2]A number of authors have extended their results in different directions; see, for example, Lebrun (1996), Bagh and Jofre (2006), Monteiro and Page (2007), Bich (2009), Carbonell-Nicolau (2011), de Castro (2011), Carmona (2011b, 2014), He and Yannelis (2014, 2015a), Carmona and Podczeck (2015), Prokopovych (2011, 2015), Prokopovych and Yannelis (2014), Reny (1999, 2011, 2015a), Nessah and Tian (2015), and Scalzo (2011, 2015a).

The aim of this chapter is to provide new existence results for discontinuous games and economies. Towards this end, we introduce the notion of "continuous inclusion property". The correspondences satisfying the continuous inclusion property could be neither lower nor upper hemicontinuous, actually they may be discontinuous. The continuous inclusion property is a very weak condition in the sense that any correspondence, which has either an open graph, or open lower sections, or the local intersection property, or it is upper hemicontinuous, will automatically satisfy this property.

Firstly, we obtain an extension of the fixed point theorems of Fan (1952) and Glicksberg (1952), which also generalizes the Browder (1968)'s fixed point theorem in locally convex spaces. In addition, we substantially generalize the fixed point theorem of Gale and Mas-Colell (1975). We also (1) show the nonemptiness of demand correspondences for non-ordered and discontinuous preferences, which generalizes the theorem of Sonnenschein (1971), and (2) prove the existence of Nash equilibrium for discontinuous games with non-ordered preferences, which extends the results in Reny (1999) to the setting with non-ordered preferences.

Secondly, we generalize the equilibrium existence theorems of Shafer and Sonnenschein (1975) and Yannelis and Prabhakar (1983) by dispensing with the continuity assumption on the preference correspondences. Although the proof of our equilibrium existence theorem in an abstract economy follows the approach of Yannelis and Prabhakar (1983), we cannot rely on the continuous selections results, as it was the case in their work (and even earlier in Gale and Mas-Colell (1975)). Indeed, the preference correspondence may not admit any the continuous selection in our setting.[3]

Thirdly, we obtain the existence of Walrasian equilibria in an exchange economy where the preference correspondences could be discontinuous, non-transitive, incomplete, interdependent and price-dependent. An additional point we would like to emphasize is that contrary to the standard existence results in the literature, we do not impose the assumption that the initial endowment is an interior point of the consumption set.

Lastly, we extend the classical Gale-Debreu-Nikaido lemma (see Debreu (1956)) by allowing for discontinuous demand correspondences. Our extension generalizes the Gale-Debreu-Nikaido lemma to infinite dimensional spaces, and also extends the results of Aliprantis and Brown (1983) and Yannelis (1985). To show that our generalization is non-vacuous, an example on Walrasian equilibrium with discontinuous preferences is provided, which cannot be covered by any existence result in the literature. However, our version of the Gale-Debreu-Nikaido lemma can be applied to this example.

This chapter is based on He and Yannelis (2014, 2015a), and proceeds as follows. Section 1.2 collects some basic notations and definitions, and discusses the preservation and the failure of the continuous inclusion property under some usual operations. Section 1.3 proves a fixed-point theorem and a generalization of the fixed-point theorem of Gale and Mas-Colell (1975), and obtains the existence of Nash

[3] Independently of our work, Reny (2015b) has also obtained related results.

equilibrium in games with discontinuous preferences. Section 1.4 provides a proof of the existence of equilibrium for an abstract economy, which extends the results of Shafer and Sonnenschein (1975) and Yannelis and Prabhakar (1983). The existence of Walrasian equilibrium with finite and infinite dimensional commodity spaces is proved and discussed in Sections 1.5. Section 1.6 presents a generalization of the Gale-Debreu-Nikaido lemma to the setting with discontinuous preferences in infinite dimensional spaces.

1.2 Basics

1.2.1 Definitions

Let X and Y be linear topological spaces, and ψ a correspondence from X to Y. Then ψ is said to be **lower hemicontinuous** if the lower inverse $\psi^l(V) = \{x \in X \colon \psi(x) \cap V \neq \emptyset\}$ is open in X for every open subset V of Y, **upper hemicontinuous** if the upper inverse $\psi^u(V) = \{x \in X \colon \psi(x) \subseteq V\}$ is open in X for every open subset V of Y, and **upper demicontinuous** if the upper inverse of every open half space in Y is open in X. In addition, if the set

$$G = \{(x, y) \in X \times Y \colon y \in \psi(x)\}$$

is open (resp. closed) in $X \times Y$, then we say that ψ has an **open (resp. closed) graph**. If $\psi^l(y)$ is open for each $y \in Y$, then ψ is said to have **open lower sections**. At some $x \in X$, if there exists an open set O_x such that $x \in O_x$ and $\cap_{x' \in O_x} \psi(x') \neq \emptyset$, then we say ψ has the local intersection property. Furthermore, ψ is said to have the **local intersection property** if this property holds for every $x \in X$. Given a linear topological space X, its dual is the space X^* of all continuous linear functionals on X.

Clearly, every nonempty correspondence with open lower sections has the local intersection property. Yannelis and Prabhakar (1983) proved a continuous selection theorem and several fixed-point theorems by assuming that ψ has open lower sections. Based on the local intersection property, Wu and Shen (1996) generalized the results of Yannelis and Prabhakar (1983).[4] Recently, Scalzo (2015a) proposed the "local continuous selection property", and proved that this condition is necessary and sufficient for the existence of continuous selections.

We now introduce the "continuous inclusion property".

Definition 1. *A correspondence ψ from X to Y is said to have the* ***continuous inclusion property*** *at x if there exists an open neighborhood O_x of x and a nonempty correspondence $F_x\colon O_x \to 2^Y$ such that $F_x(z) \subseteq \psi(z)$ for any $z \in O_x$ and coF_x[5] has a closed graph.*[6]

The continuous inclusion property is motivated by the majorization idea in general equilibrium (see the KF-majorization in Borglin and Keiding (1976), and L-majorization in Yannelis and Prabhakar (1983)), and also the "multiply security"

[4]Mappings with the local intersection property have found applications in mathematical economics and game theory, see Wu and Shen (1996) and Prokopovych (2011) among others.

[5]For a correspondence F, co$F$ denotes the convex hull of F.

[6]If the sub-correspondence F_x has a closed graph and X is finite dimensional, then coF_x still has a closed graph since the convex hull of a closed set is closed in finite dimensional spaces. However, this may not be true if one works with infinite dimensional spaces. One can easily see that assuming the sub-correspondence F_x is convex valued and has a closed graph would suffice for our aim.

condition of McLennan, Monteiro and Tourky (2011), the "continuous security" condition of Barelli and Meneghel (2013), and the "correspondence security" condition of Reny (2015a) in the context of discontinuous games.

Remark 1. *If the correspondence ψ from X to Y has the local intersection property at x, then F_x can be chosen as a constant correspondence which only contains a singe point of $\cap_{x' \in O_x} \psi(x')$, and hence ψ also has the continuous inclusion property at x. As a result, any nonempty correspondence with open lower sections has the continuous inclusion property. Furthermore, any upper hemicontinuous, convex and compact valued correspondence satisfies the continuous inclusion property.*

1.2.2 Operations on Correspondences

In this subsection, we consider the preservation and the failure of the continuous inclusion property under some usual operations, including union, inclusion, addition and product.

Let X, Y, Z, $\{X_j\}_{j \in J}$ and $\{Y_j\}_{j \in J}$ be linear topological spaces, where J is an index set. Given a family of correspondences $\{\psi_j\}_{j \in J}$ from X to Y, we define the union and intersection of this family pointwise. That is, $\cup_{j \in J} \psi_j$ maps x to $\cup_{j \in J} \psi_j(x)$, and $\cap_{j \in J} \psi_j$ maps x to $\cap_{j \in J} \psi_j(x)$.

Let ψ_1 and ψ_2 be two correspondences from X to Y, and α and β be two nonzero numbers. The linear combination $\alpha\psi_1 + \beta\psi_2$ of ψ_1 and ψ_2 is defined as

$$(\alpha\psi_1 + \beta\psi_2)(x) = \{\alpha y_1 + \beta y_2 \colon y_1 \in \psi_1(x), y_2 \in \psi_2(x)\}.$$

The product of a family of correspondences $\{\psi_j \colon X_j \to 2^{Y_j}\}_{j \in J}$ is the correspondence

$\prod_{j \in J} \psi_j$ from $\prod_{j \in J} X_j$ to $\prod_{j \in J} Y_j$, defined naturally by $(\prod_{j \in J} \psi_j)(x) = \prod_{j \in J} \psi_j(x_j)$ for each $x = \{x_j\}_{j \in J}$.

In the next proposition, we consider the preservation and the failure of the continuous inclusion property under some regularity conditions.

Proposition 1. *1. Let $\psi_1 \colon X \to 2^Y$ be a correspondence having the continuous inclusion property, and $\{\phi_j \colon X \to 2^Y\}_{j \in J}$ be a family of arbitrary correspondences. Then their union $(\cup_{j \in J} \phi_j) \cup \psi_1$ also has the continuous inclusion property.*

2. *Let $\psi_1 \colon [0,1] \to 2^{[0,1]}$ and $\psi_2 \colon [0,1] \to 2^{[0,1]}$ be two correspondences both having the continuous inclusion property, their intersection may not have the continuous inclusion property.*

3. *If Y is a compact Hausdorff space, and $\psi, \phi \colon X \to 2^Y$ are convex valued correspondences with the continuous inclusion property, then $\alpha\psi + \beta\phi$ has the continuous inclusion property for any nonzero α and β.*

4. *Let $\{\psi_i \colon X_i \to 2^{Y_i}\}_{1 \le i \le n}$ be a finite family of correspondences having the continuous inclusion property. Then their product $\prod_{1 \le i \le n} \psi_i$ also has the continuous inclusion property.*

Proof. (1) Fix $x \in X$. Since ψ_1 has the continuous inclusion property, there exists an open neighborhood O_x of x and a nonempty correspondence $F_x \colon O_x \to 2^Y$ such that $F_x(x') \subseteq \psi_1(x')$ for any $x' \in O_x$ and $\mathrm{co}F_x$ has a closed graph. Since ψ_1 is a sub-correspondence of $(\cup_{j \in J} \phi_j) \cup \psi_1$, the rest is clear.

(2) Let $\psi_1\colon [0,1] \to 2^{[0,1]}$ and $\psi_2\colon [0,1] \to 2^{[0,1]}$ be defined as follows:

$$\psi_1(x) = \begin{cases} \{x, 0\}, & 0 \le x \le \frac{1}{2}, \\ \{x, 1\}, & \frac{1}{2} < x \le 1; \end{cases} \qquad \psi_2(x) = \begin{cases} \{1-x, 0\}, & 0 \le x \le \frac{1}{2}; \\ \{1-x, 1\}, & \frac{1}{2} < x \le 1. \end{cases}$$

It is obvious that ψ_1 and ψ_2 satisfy the continuous inclusion property since both of them have continuous selections. However, their intersection is

$$\psi_1 \cap \psi_2(x) = \begin{cases} \{0\}, & 0 \le x < \frac{1}{2}; \\ \{0, \frac{1}{2}\}, & x = \frac{1}{2}; \\ \{1\}, & \frac{1}{2} < x \le 1. \end{cases}$$

It is clear that the correspondence $\psi_1 \cap \psi_2$ does not satisfy the continuous inclusion property at the point $\frac{1}{2}$.

(3) Fix $x \in X$. Since ψ and ϕ are convex valued and have the continuous inclusion property at x, there exist open neighborhoods O_x^1 and O_x^2 of x, and nonempty convex valued correspondences $F_x^1\colon O_x^1 \to 2^Y$ and $F_x^2\colon O_x^2 \to 2^Y$ such that $F_x^1(x') \subseteq \psi(x')$ for any $x' \in O_x^1$ and $F_x^2(x') \subseteq \phi(x')$ for any $x' \in O_x^2$, and both F_x^1 and F_x^2 have closed graphs. Let $O_x = O_x^1 \cap O_x^2$ and $G_x = \alpha F_x^1 + \beta F_x^2$. Then O_x is an open neighborhood of x, G_x is convex valued, and $G_x(x') \subseteq (\alpha\psi + \beta\phi)(x')$ for any $x' \in O_x$. Since Y is a compact Hausdorff space and F_x^1 (resp. F_x^2) has a closed graph, F_x^1 (resp. F_x^2) is upper hemicontinuous and compact valued. As a result, G_x is upper hemicontinuous and compact valued, and hence has a closed graph. This proves our claim.

(4) This property is obvious. $\square$

1.3 Some Fixed-Point Theorems

1.3.1 A Fixed-Point Theorem

Below, we prove a fixed-point theorem based on the continuous inclusion property. This theorem replaces the upper hemicontinuity condition on the fixed point theorems of Fan (1952) and Glicksberg (1952) by the continuous inclusion property. Since an upper hemicontinuous, convex and compact valued correspondence has the continuous inclusion property, our fixed point theorem improves the fixed point theorems of Fan (1952) and Glicksberg (1952).

Theorem 1. *Let X be a nonempty, compact, convex subset of a Hausdorff locally convex linear topological space Y, and $\psi\colon X \to 2^X$ be a correspondence which is nonempty and convex valued, and has the continuous inclusion property. Then there exists a point $x^* \in X$ such that $x^* \in \psi(x^*)$.*

Proof. Since ψ has the continuous inclusion property, for each $x \in X$, there exists an open neighborhood O_x and a nonempty correspondence $F_x\colon O_x \to 2^X$ such that $F_x(z) \subseteq \psi(z)$ for any $z \in O_x$ and coF_x has a closed graph.

The collection $\mathscr{C} = \{O_x\colon x \in X\}$ is an open cover of X. Since X is compact, there is a finite set $\{x_1, \ldots, x_n\}$ such that $X \subseteq \cup_{1 \le i \le n} O_{x_i}$. Let $\{E_{x_i}\}_{1 \le i \le n}$ be a closed refinement; that is, $E_{x_i} \subseteq O_{x_i}$, E_{x_i} is closed and $X = \cup_{1 \le i \le n} E_{x_i}$ (see Michael (1953, Lemma 1)).

For each $x \in X$, let $I(x) = \{1 \le i \le n\colon x \in E_{x_i}\}$, and $F(x) = \mathrm{co}\left(\cup_{i \in I(x)} \mathrm{co} F_{x_i}(x)\right)$. Then it is obvious that F is nonempty and convex valued. Moreover, F is also compact valued; see Hildenbrand (1974, p.37). For each x and $i \in I(x)$, $F_{x_i}(x) \subseteq \psi(x)$.

Since ψ is convex valued, $\mathrm{co}F_{x_i}(x) \subseteq \psi(x)$, which implies that $\cup_{i \in I(x)} \mathrm{co}F_{x_i}(x) \subseteq \psi(x)$. Again by the convexity of $\psi(x)$, we have $F(x) = \mathrm{co}\left(\cup_{i \in I(x)} \mathrm{co}F_{x_i}(x)\right) \subseteq \psi(x)$.

Since $\mathrm{co}F_{x_i}$ has a closed graph in E_{x_i} and X is a compact Hausdorff space, it is upper hemicontinuous in E_{x_i}. We can slightly abuse the notation by assuming that $\mathrm{co}F_{x_i}$ is empty when $x_i \notin E_{x_i}$. As E_{x_i} is a closed set, the correspondence $\mathrm{co}F_{x_i}$ is upper hemicontinuous on the whole space. For each x, $I(x)$ is finite, which implies that $\cup_{i \in I(x)} \mathrm{co}F_{x_i}(x)$ is the union of a finite family of upper hemicontinuous correspondences, and hence is upper hemicontinuous (see Hildenbrand (1974, p.22)). Since $F(x)$ is the convex hull of $\cup_{i \in I(x)} \mathrm{co}F_{x_i}(x)$ and it is compact valued, it is also upper hemicontinuous (see Proposition 6 in Hildenbrand (1974, p.26)). By Fan-Glicksberg's fixed-point theorem (see Fan (1952) and Glicksberg (1952)), there is a point $x^* \in X$ such that $x^* \in F(x^*) \subseteq \psi(x^*)$. $\square$

Remark 2. *Browder (1968, Theorem 1) and Yannelis and Prabhakar (1983, Theorem 3.3) proved a fixed point theorem by assuming that Y a Hausdorff linear topological space (not necessarily locally convex) and the correspondence ψ has open lower sections. In Wu and Shen (1996, Theorem 2), Y is required to be locally convex and ψ has the local intersection property. Since the local intersection property implies the continuous inclusion property, our result covers the theorem of Wu and Shen (1996) as a corollary.*

1.3.2 A Generalization of the Gale and Mas-Colell's Fixed-Point Theorem

Below, we will generalize the fixed-point theorem of Gale and Mas-Colell (1975) based on our continuous inclusion property.

Theorem 2. *Let I be a countable set and for each $i \in I$, X_i be a nonempty, compact, convex and metrizable subset of a Hausdorff locally convex linear topological space, and $X = \prod_{i \in I} X_i$. For each $i \in I$, let $\psi_i \colon X \to 2^{X_i}$ be a convex valued correspondence, and $I(x) = \{i \in I \colon \psi_i(x) \neq \emptyset, x_i \notin \psi_i(x)\}$. Suppose that for every $x \in X$ with $I(x) \neq \emptyset$, there is some $i \in I(x)$ such that ψ_i has the continuous inclusion property at x. Then there exists a point $x^* \in X$ such that for each i, either $x_i^* \in \psi_i(x^*)$ or $\psi_i(x^*) = \emptyset$.*

We first present two preparatory lemmas.

Lemma 1. *Suppose that the conditions in Theorem 2 hold. For each i, let*

$$U_i = \{x \in X \colon \psi_i \text{ has the continuous inclusion property at } x\}.$$

If $U_i = \emptyset$ for all i, then the result of Theorem 2 is true.

Proof. Since $U_i = \emptyset$ for each i, $I(x) = \emptyset$ for all x by the conditions in Theorem 2, which implies that for each i, either $x_i \in \psi_i(x)$ or $\psi_i(x) = \emptyset$. □

Lemma 2. *Under conditions of Theorem 2, for each i such that $U_i \neq \emptyset$, there exists a nonempty, convex and compact valued, upper hemicontinuous correspondence $\phi_i \colon U_i \to 2^{X_i}$ such that $\phi_i(x) \subseteq \psi_i(x)$ for each $x \in U_i$.*

Proof. Suppose that $U_i \neq \emptyset$. Since ψ_i has the continuous inclusion property at each $x \in U_i$, there exists an open subset $O_x^i \subseteq X$ such that $x \in O_x^i$ and a correspondence $F_x^i \colon O_x^i \to 2^{X_i}$ with nonempty values such that $F_x^i(z) \subseteq \psi_i(z)$ for any $z \in O_x^i$ and $\mathrm{co}F_x^i$ is closed. Then $O_x^i \subseteq U_i$, which implies that U_i is open. Since X is metrizable, U_i is paracompact (see for example, Michael (1956, p. 831)). Moreover, the collection $\mathscr{C}_i = \{O_x^i \colon x \in X\}$ is an open cover of U_i. There is a closed locally finite refinement $\mathcal{F}_i = \{E_k^i \colon k \in K\}$, where K is an index set and E_k^i is a closed set in X (see Michael (1953, Lemma 1)).

For each $k \in K$, choose $x_k \in X$ such that $E_k^i \subseteq O_{x_k}^i$. For each $x \in U_i$, let $I_i(x) = \{k \in K \colon x \in E_k^i\}$. Then $I_i(x)$ is finite for each $x \in U_i$. Let $\phi_i(x) = \mathrm{co}\left(\cup_{k \in I_i(x)} \mathrm{co}F_{x_k}^i(x)\right)$ for $x \in U_i$. For each x and $k \in I_i(x)$, $F_{x_k}^i(x) \subseteq \psi_i(x)$. Thus, $\mathrm{co}F_{x_k}^i(x) \subseteq \psi_i(x)$, which implies that $\cup_{k \in I_i(x)} \mathrm{co}F_{x_k}^i(x) \subseteq \psi_i(x)$. As a result, we have $\phi_i(x) = \mathrm{co}\left(\cup_{k \in I_i(x)} \mathrm{co}F_{x_k}^i(x)\right) \subseteq \psi_i(x)$.

Since $\mathrm{co}F_{x_k}^i$ has a closed graph in E_k^i and X_i is a compact Hausdorff space, $\mathrm{co}F_{x_k}^i$ is compact valued and upper hemicontinuous. For each x, $I_i(x)$ is finite, which implies that $\cup_{k \in I_i(x)} \mathrm{co}F_{x_k}^i(x)$ is the union of values for a finite family of compact valued and upper hemicontinuous correspondences, and hence is also compact valued and upper hemicontinuous at the point x. Since each $\mathrm{co}F_{x_k}^i(x)$ is convex and compact, and $\phi_i(x)$ is the convex hull of the finite union $\cup_{k \in I_i(x)} \mathrm{co}F_{x_k}^i(x)$, $\phi_i(x)$ is also compact, which implies that $\phi_i(x)$ is upper hemicontinuous at the point $x \in U_i$. This completes the proof. $\square$

Now we are ready to prove Theorem 2.

Proof of Theorem 2. By Lemma 1, we only need to consider the case that there exists some i such that $U_i \neq \emptyset$.

Define a correspondence

$$H_i(x) = \begin{cases} \phi_i(x), & x \in U_i; \\ X_i, & \text{otherwise.} \end{cases}$$

Then it is obvious that H_i is nonempty, convex and compact valued. Since U_i is open and ϕ_i is upper hemicontinuous by Lemma 2, H_i is upper hemicontinuous on the whole space. Let $H = \prod_{i \in I} H_i$. Since H is nonempty, convex and compact valued, and upper hemicontinuous, by the Fan-Glicksberg fixed point theorem (see Fan (1952) and Glicksberg (1952)), there exists a point $x^* \in X$ such that $x^* \in H(x^*)$.

Let $J = \{i \in I : x_i^* \notin \psi_i(x^*)\}$. Then $I(x^*) \subseteq J$. If $x^* \in U_j$ for some $j \in J$, then $x_j^* \in \phi_j(x^*) \subseteq \psi_j(x^*)$, which is a contradiction. Thus, we have $x^* \notin U_j$ for every $j \in J$, which implies that $I(x^*) = \emptyset$. Therefore, for every $j \in J$, $\psi_j(x^*) = \emptyset$. For every $i \in I \setminus J$, $x_i^* \in \psi_i(x^*)$. The proof is complete. □

Remark 3. *In Gale and Mas-Colell (1975), X_i is finite dimensional and ψ_i is lower hemicontinuous for each i. Then the continuous selection theorem of Michael (1956, Theorem 3.1"')) implies that ψ_i has a continuous selection ϕ_i on U_i, which can be regarded as a continuous sub-correspondence of ψ_i. Thus, the continuous inclusion property holds and the result follows.*

In addition, our result implies that one can further weaken the lower hemicontinuity condition of Gale and Mas-Colell (1975). Specifically, at each $x \in U_i$, suppose that there exists an open neighborhood O_x^i of x and a nonempty convex valued, lower *hemicontinuous correspondence* $F_x^i\colon O_x^i \to X_i$ *with* $F_x^i(z) \subseteq \psi_i(z)$ *for* $z \in O_x^i$. *Then the continuous inclusion property still holds. However, in this case* X_i *is still required to be finite dimensional since the continuous selection theorem of Michael (1956) is needed.*

1.3.3 Existence of Maximal Elements

Suppose that X is a nonempty subset of a linear topological space. Let $P(x) = \{y \in X\colon (y,x) \in \mathcal{P}\}$ for all $x \in X$, where $\mathcal{P}$ is some binary relation on X. Then P is a preference correspondence induced by $\mathcal{P}$ on X. If $P(x^*) = \emptyset$ for some $x^* \in X$, then x^* is said to be a **maximal element** in X.

Corollary 1. *Let* X *be a compact, convex subset of a Hausdorff locally convex linear topological space and* $P\colon X \to 2^X$ *be a correspondence such that for all* $x \in X$, $x \notin coP(x)$. *If* P *has the continuous inclusion property at each* $x \in X$ *such that* $P(x) \neq \emptyset$, *then there exists a point* $x^* \in X$ *such that* $P(x^*) = \emptyset$.

Proof. By way of contradiction, suppose that $P(x) \neq \emptyset$ for all $x \in X$. Then the correspondence $\psi(x) = \text{co}P(x)$ is convex and nonempty valued. It is clear that ψ has the continuous inclusion property. By Theorem 1, there exists a fixed point $x^* \in X$ such that $x^* \in \psi(x^*) = \text{co}P(x^*)$, a contradiction. □

Remark 4. *Theorem 5.1 of Yannelis and Prabhakar (1983) proved the existence of maximal element when* X *is a compact, convex subset of a Hausdorff linear topological space and the correspondence* P *has open lower sections. This result generalizes Lemma 4 of Fan (1962). In our Corollary 1, the condition on the correspondence is* *more general while X is required to be locally convex.*

Below, we shall illustrate the usefulness of the above corollary.

Let $\triangle$ and X be two Hausdorff locally convex linear topological spaces, where $\triangle$ is the set of all price vectors and X is the set of goods. Let the correspondence $B\colon \triangle \to 2^X$ denote the *budget set* which is assumed to be nonempty, convex and compact valued. The *preference correspondence* is denoted by $P\colon X \to 2^X$ and satisfies the condition that $x \notin \mathrm{co}P(x)$ for any $x \in X$. Let $\psi(p,x) = B(p) \cap P(x)$, and define the *demand correspondence* $D\colon \triangle \to 2^X$ by $D(p) = \{x \in B(p)\colon \psi(p,x) = \emptyset\}$.

Corollary 2. *If $\psi(p,\cdot)$ has the continuous inclusion property for each $p \in \triangle$,[7] then the demand correspondence D is nonempty valued.*

Proof. Fix $p_0 \in \triangle$. Since $x \notin \mathrm{co}P(x)$ for any $x \in X$, it follows that $x \notin \mathrm{co}\psi(p_0, x)$ for any $x \in B(p_0)$. Since $\psi(p_0,\cdot)\colon B(p_0) \to 2^{B(p_0)}$ has the continuous inclusion property, $B(p_0)$ is nonempty, convex and compact, by Corollary 1, there exists a point $x_0 \in B(p_0)$ such that $\psi(p_0, x_0) = \emptyset$. That is, $x_0 \in D(p_0)$, which implies that D is nonempty valued. $\square$

The above corollary generalizes the corresponding theorem in Sonnenschein (1971) by relaxing the continuity assumption.

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[7] The continuity inclusion property captures the case that the preference could be discontinuous. For example, people's preference on food could dramatically change if the amount goes to the zero: people will be sick or even die.

1.3.4 Existence of Nash Equilibrium

Below, we obtain the existence of a Nash equilibrium in games with (possibly) nontransitive, incomplete, discontinuous preferences as a direct corollary of Theorem 2. Notice that the preference need not be representable by a utility function.

Let I be a set of countable players, and the game is $\Gamma = \{(X_i, P_i) \colon i \in I\}$, where X_i is the **action space** of player i, $X = \prod_{i \in I} X_i$, and the **preference correspondence** of player i is $P_i \colon X \to 2^{X_i}$. If the preference P_i can be represented by a utility function $u_i \colon X \to \mathbb{R}$, then

$$P_i(x) = \{y_i \in X_i \colon u_i(y_i, x_{-i}) > u_i(x)\}.$$

Corollary 3. *Let $\Gamma = \{(X_i, P_i) \colon i \in I\}$ be a game such that for each $i \in I$:*

i *X_i is a nonempty, compact, convex, metrizable subset of a Hausdorff locally convex linear topological space;*

ii *Let $I(x) = \{i \in I \colon P_i(x) \neq \emptyset\}$. Suppose that for every $x \in X$ with $I(x) \neq \emptyset$, there exists an agent $i \in I(x)$ such that P_i has the continuous inclusion property at x and $x_i \notin coP_i(x)$.*

Then Γ has a Nash equilibrium; that is, there exists some $x^* \in X$ such that for any $i \in I$, $P_i(x^*) = \emptyset$.

Proof. Denote $\psi_i = \text{co}P_i$ for each $i \in I$. Let $I'(x) = \{i \in I \colon \psi_i(x) \neq \emptyset, x_i \notin \psi_i(x)\}$. Then for every $x \in X$ with $I'(x) \neq \emptyset$, $I(x) \neq \emptyset$. By condition (ii), there exists an agent $i \in I'(x)$ such that ψ_i has the continuous inclusion property at x.

By Theorem 2, there exists a point $x^* \in X$ such that for each i, either $x_i^* \in$

$\psi_i(x^*)$ or $\psi_i(x^*) = \emptyset$. Then $I(x^*) = \{i \in I \colon x_i^* \in \psi_i(x^*)\}$. If $I(x^*) \neq \emptyset$, then by condition (ii), there is an agent $i \in I(x^*)$ such that P_i has the continuous inclusion property at x^* and $x_i^* \notin \psi_i(x^*)$, which is a contradiction. As a result, $I(x^*) = \emptyset$. That is, $\psi_i(x^*) = \emptyset$ for each $i \in I$, which implies that x^* is a Nash equilibrium in the game Γ. □

The following result is an immediate corollary of Corollary 3. The continuous inclusion property is directly assumed for each player.

Corollary 4. *Let $\Gamma = \{(X_i, P_i) \colon i \in I\}$ be a game such that for each $i \in I$:*

i *X_i is a nonempty, compact, convex, metrizable subset of a Hausdorff locally convex linear topological space;*

ii *P_i has the continuous inclusion property at each $x \in X = \times_{i \in I} X_i$ with $P_i(x) \neq \emptyset$;*

iii *$x_i \notin coP_i(x)$ for all $x \in X$.*

Then Γ has a Nash equilibrium; that is, $\exists x^* \in X$ such that for any $i \in I$, $P_i(x^*) = \emptyset$.

Remark 5. *Suppose that for each $i \in I$, the utility function u_i satisfy the generalized payoff security condition of Carmona (2011b), and define the value function $g_i \colon X_{-i} \to \mathbb{R}$ by $g_i(x_{-i}) = \sup_{x_i \in X_i} u_i(x_i, x_{-i})$. Fix $\epsilon > 0$. For each $i \in I$, consider the correspondence*

$$M_i^\epsilon(x_{-i}) = \{x_i \in X_i \colon u_i(x_i, x_{-i}) > g_i(x_{-i}) - \epsilon\}.$$

Then it is easy to see that M_i^ϵ has the continuous inclusion property. Following the argument in Prokopovych (2011), one can impose standard conditions (e.g., qua-

siconcavity and transfer reciprocal upper semicontinuity) to prove the existence of approximate and exact Nash equilibrium.

Remark 6. *Reny (1999) proved the existence of a pure strategy Nash equilibrium in games with discontinuous payoffs based on some payoff security type condition. Our corollaries 3 and 4 extend his results to non-ordered preferences, but do not imply his and vice versa. However, to verify the conditions of theorems in the above paper, one has to work with the non-equilibrium point, and check for all players at every point in a neighborhood of this non-equilibrium point. To the contrary, we can check the preference correspondence for each agent separately, as shown in Corollary 4.*[8]

1.4 Equilibria in Abstract Economies

1.4.1 Results

In this section we prove the existence of equilibrium for an abstract economy with an infinite number of commodities and a countable number of agents.

An **abstract economy** is a set of ordered triples $\Gamma = \{(X_i, A_i, P_i) \colon i \in I\}$, where

- I is a countable set of **agents**.
- X_i is a nonempty set of **actions** for agent i. Set $X = \prod_{i \in I} X_i$.
- $A_i \colon X \to 2^{X_i}$ is the **constraint correspondence** of agent i.
- $P_i \colon X \to 2^{X_i}$ is the **preference correspondence** of agent i.

An **equilibrium** of Γ is a point $x^* \in X$ such that for each $i \in I$:

[8] For further results, see Reny (2015a) and Carmona and Podczeck (2015).

1. $x_i^* \in \overline{A_i}(x^*)$, where $\overline{A_i}$ denotes the closure of A_i, and

2. $P_i(x^*) \cap A_i(x^*) = \emptyset$.

If $A_i \equiv X_i$ for all $i \in I$, then the point x^* is a Nash equilibrium.

For each $i \in I$, let $\psi_i(x) = A_i(x) \cap P_i(x)$ for all $x \in X$.

Theorem 3. *Let $\Gamma = \{(X_i, A_i, P_i) \colon i \in I\}$ be an abstract economy such that for each $i \in I$:*

i *X_i is a nonempty, compact, convex, metrizable subset of a Hausdorff locally convex linear topological space;*

ii *A_i is nonempty and convex valued;*

iii *the correspondence $\overline{A_i}$ is upper hemicontinuous;*

iv *ψ_i has the continuous inclusion property at each $x \in X$ with $\psi_i(x) \neq \emptyset$;*

v *$x_i \notin co\psi_i(x)$ for all $x \in X$.*

Then Γ has an equilibrium.

Proof. Fix $i \in I$. Let $U_i = \{x \in X \colon \psi_i(x) \neq \emptyset\}$.[9] Since ψ_i has the continuous inclusion property at each $x \in U_i$, there exist an open set $O_x^i \subseteq X$ such that $x \in O_x^i$ and a correspondence $F_x^i \colon O_x^i \to 2^{X_i}$ with nonempty values such that $F_x^i(z) \subseteq \psi_i(z)$ for any $z \in O_x^i$ and $\mathrm{co}F_x^i$ is closed. Then $O_x^i \subseteq U_i$, which implies that U_i is open. Since X is metrizable, U_i is paracompact (see Michael (1956, p. 831)). Moreover, the collection $\mathscr{C}_i = \{O_x^i \colon x \in X\}$ is an open cover of U_i. There is a closed locally finite

[9] If $U_i = \emptyset$ for all i, then the correspondence $\overline{A} = \prod_{i \in I} \overline{A_i}$ is nonempty, convex valued and upper hemicontinuous. As a result, there exists a fixed-point x^* of $\overline{A}$ which is an equilibrium.

refinement $\mathcal{F}_i = \{E^i_k \colon k \in K\}$, where K is an index set and E^i_k is a closed set in X (see Michael (1953, Lemma 1)).

For each $k \in K$, choose $x_k \in X$ such that $E^i_k \subseteq O^i_{x_k}$. For each $x \in U_i$, let $I_i(x) = \{k \in K \colon x \in E^i_k\}$. Then $I_i(x)$ is finite for each $x \in U_i$. Let $\phi_i(x) = \mathrm{co}\left(\cup_{k \in I_i(x)} \mathrm{co} F^i_{x_k}(x)\right)$ for $x \in U_i$. For each x and $k \in I_i(x)$, $F^i_{x_k}(x) \subseteq \psi_i(x)$. Thus, $\mathrm{co}F^i_{x_k}(x) \subseteq \mathrm{co}\psi_i(x)$, which implies that $\cup_{k \in I_i(x)} \mathrm{co}F^i_{x_k}(x) \subseteq \mathrm{co}\psi_i(x)$. As a result, we have $\phi_i(x) = \mathrm{co}\left(\cup_{k \in I_i(x)} \mathrm{co}F^i_{x_k}(x)\right) \subseteq \mathrm{co}\psi_i(x)$.

Define the correspondence

$$H_i(x) = \begin{cases} \phi_i(x) & x \in U_i; \\ \\ \overline{A_i}(x) & \text{otherwise.} \end{cases}$$

Then it is obvious that H_i is nonempty and convex valued. Moreover, H_i is also compact valued (see Lemma 5.29 in Aliprantis and Border (2006)).

Since $\mathrm{co}F^i_{x_k}$ has a closed graph in E^i_k and E^i_k is a compact Hausdorff space, it is upper hemicontinuous. For each x, $I_i(x)$ is finite, which implies that $\cup_{k \in I_i(x)} \mathrm{co}F^i_{x_k}(x)$ is the union of values for a finite family of upper hemicontinuous correspondences, and hence is upper hemicontinuous at the point x (see Aliprantis and Border (2006, Theorem 17.27)). Then $\phi_i(x)$ is the convex hull of $\cup_{k \in I_i(x)} \mathrm{co}F^i_{x_k}(x)$ and it is compact for all $x \in U_i$, hence it is upper hemicontinuous on U_i (see Aliprantis and Border (2006, Theorem 17.35)). Note that $H_i(x)$ is $\phi_i(x)$ when $x \in U_i$, and $\overline{A_i}(x)$ when $x \notin U_i$. Since U_i is open, analogous to the argument in Yannelis and Prabhakar (1983, Theorem 6.1), H_i is upper hemicontinuous on the whole space. Let $H = \prod_{i \in I} H_i$. Since H is nonempty, convex and closed valued, by the Fan-Glicksberg fixed point theorem, there exists a point $x^* \in X$ such that $x^* \in H(x^*)$.

Since $\phi_i(x) \subseteq \overline{A_i}(x)$ for $x \in U_i$, $H_i(x) \subseteq \overline{A_i}(x)$ for any x, which implies that $x_i^* \in \overline{A_i}(x^*)$. Note that if $x^* \in U_i$ for some $i \in I$, then $x_i^* \in \text{co}\left(\cup_{k \in I_i(x^*)} \text{co} F^i_{x_k}(x^*)\right) \subseteq \text{co}\psi_i(x^*)$, a contradiction to assumption (v). Thus, we have $x^* \notin U_i$ for all $i \in I$. Therefore, $\psi_i(x^*) = \emptyset$, which implies that $A_i(x^*) \cap P_i(x^*) = \emptyset$. That is, x^* is an equilibrium for Γ. □

Below, we show that the theorem of Shafer and Sonnenschein (1975) and Theorem 6.1 of Yannelis and Prabhakar (1983) on the existence of equilibrium in an abstract economy can be obtained as corollaries. Note that in Shafer and Sonnenschein (1975) the correspondence A_i is compact valued for each $i \in I$, and therefore there is no need to work with the closure of A_i. That is, an equilibrium x^* should satisfy $x_i^* \in A_i(x^*)$ and $P_i(x^*) \cap A_i(x^*) = \emptyset$. In Yannelis and Prabhakar (1983), the equilibrium notion is the same as defined above.

Corollary 5. *[Shafer and Sonnenschein (1975)]*

Let $\Gamma = \{(X_i, A_i, P_i)\colon i \in I\}$ be an abstract economy such that for each $i \in I$:

i *X_i is a nonempty, compact, convex subset of $\mathbb{R}^l_+$;*

ii *A_i is nonempty, convex and compact valued;*

iii *A_i is a continuous correspondence;*

v *P_i has an open graph;*

vi *$x_i \notin co\psi_i(x)$ for all $x \in X$.*[10]

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[10]Shafer and Sonnenschein (1975) assume that $x_i \notin \text{co}P_i(x)$ for all $x \in X$, but their

Then Γ *has an equilibrium* x^**; that is, for any* $i \in I$, $x_i^* \in A_i(x^*)$ *and* $P_i(x^*) \cap A_i(x^*) = \emptyset$.

Proof. For each $i \in I$, define a mapping $U_i \colon \mathrm{Gr}(A_i) \to \mathbb{R}$ by $U_i(y, x_i) = \mathrm{dist}((y, x_i), \mathrm{Gr}^C(P_i))$, where $\mathrm{Gr}(A_i)$ is the graph of A_i, $\mathrm{Gr}^C(P_i)$ denotes the complement of the graph of P_i and $\mathrm{dist}(\cdot, \cdot)$ denotes the usual distance on $\mathbb{R}^l_+$. Since P_i has an open graph, U_i is continuous. Let $m_i(x) = \max_{z \in A_i(x)} U_i(x, z)$ and $\phi_i(x) = \{z \in A_i(x) \colon U_i(x, z) = m_i(x)\}$ for each $x \in X$. Since A_i is continuous, by the Berge Maximum Theorem (see Aliprantis and Border (2006, Theorem 17.31)), ϕ_i is nonempty, compact valued and upper hemicontinuous. At any point x such that $\psi_i(x) = P_i(x) \cap A_i(x) \neq \emptyset$, we have $m_i(x) > 0$, and hence $\phi_i(x) \subseteq \psi_i(x)$. Thus, the continuous inclusion property holds and by Theorem 3, there is an equilibrium. □

Corollary 6. *[Yannelis and Prabhakar (1983, Theorem 6.1)]*

Let $\Gamma = \{(X_i, A_i, P_i) \colon i \in I\}$ *be an abstract economy such that for each* $i \in I$*:*

i X_i *is a nonempty, compact, convex, metrizable subset of a Hausdorff locally convex linear topological space;*

ii A_i *is nonempty and convex valued;*

iii *the correspondence* $\overline{A_i}$ *is upper hemicontinuous;*

iv A_i *has open lower section;*

v P_i *has open lower section;*

---

proof still holds under this more general condition. The same comment is also valid for the existence theorem of Yannelis and Prabhakar (1983), see condition (vi) of Corollary 6 below.

vi *$x_i \notin co\psi_i(x)$ for all $x \in X$.*

Then Γ has an equilibrium x^*; that is, for each $i \in I$, $x_i^* \in \overline{A}_i(x^*)$ and $P_i(x^*) \cap A_i(x^*) = \emptyset$.

Proof. By Fact 6.1 in Yannelis and Prabhakar (1983), ψ_i has open lower sections. As a result, ψ_i has the continuous inclusion property at each $x \in X$ when $\psi_i(x) \neq \emptyset$. Then the result follows from Theorem 3. □

Remark 7. *Note that our Theorem 3 also covers Theorem 10 of Wu and Shen (1996). Wu and Shen (1996) did not impose the metrizability condition on X_i, but directly assumed that U_i is paracompact. Our proof still holds under this condition.*

Remark 8. *In condition (iv) of Theorem 3, we assume that ψ_i has the continuous inclusion property at each $x \in X$ with $\psi_i(x) \neq \emptyset$. It is natural to ask whether we can impose conditions on the correspondences P_i and A_i separately, and then verify that their intersection ψ_i has the continuous inclusion property (for example, see conditions (iv) and (v) in Yannelis and Prabhakar (1983, Theorem 6.1)). However, a simple example can be constructed to show that a combination of the following two conditions cannot guarantee our condition (iv):*

1. *P_i has the continuous inclusion property at x when $P_i(x) \neq \emptyset$;*
2. *A_i has an open graph.*

Suppose that there is only one agent and $X = [0,1]$, $A(x) = (0,1]$ and

$$P(x) = \begin{cases} [0,1], & x = 1; \\ \{0\}, & x \in [0,1). \end{cases}$$

Then it is obvious that P has the continuous inclusion property and A has an open graph. However,

$$\psi(x) = \begin{cases} (0,1], & x = 1, \\ \emptyset, & x \in [0,1); \end{cases}$$

does not have the continuous inclusion property.

1.4.2 Some Remarks

Subsequent to our results, Carmona and Podczeck (2015) dropped the metrizability condition on X_i and generalized our conditions (4) and (5) as follows.

Let $I(x) = \{i \in I \colon \psi_i(x) \neq \emptyset\}$. For every $x \in X$ such that $I(x) \neq \emptyset$ and $x_i \in \overline{A}_i(x)$ for all $i \in I$, there is an agent $i \in I(x)$,

1. ψ_i has the continuous inclusion property at x;
2. $x_i \notin \mathrm{co}\psi_i(x)$.

Notice that our proof above still goes through under this condition by slightly modifying the definition of the set U_i as

$$\{x \in X \colon \psi_i \text{ has the continuous inclusion property at } x\}.$$

The metrizability condition in our Theorem 3 is not needed. Following a similar argument as in Borglin and Keiding (1976) and Toussaint (1984), we provide an alternative proof for Theorem 3 in which the set of agents can be any arbitrary (finite or infinite set) and X_i need not to be metrizable for each i.[11]

[11] It should be noted that using the existence of maximal element theorem for L-majorized correspondences (see Yannelis and Prabhakar (1983)), it is known that the metrizability

Alternative proof of Theorem 3. For each $i \in I$, define a correspondence H_i from X to X_i as follows:

$$H_i(x) = \begin{cases} \psi_i(x), & x_i \in \overline{A}_i(x); \\ \\ \overline{A}_i(x), & x_i \notin \overline{A}_i(x). \end{cases}$$

We will show that H_i has the continuous inclusion property at each x such that $H_i(x) \neq \emptyset$.

1. If $x_i \in \overline{A}_i(x)$, then $\psi_i(x) = H_i(x) \neq \emptyset$, which implies that there exists an open neighborhood O_x of x and a nonempty correspondence $F_x \colon O_x \to 2^{X_i}$ such that $F_x(z) \subseteq \psi_i(z)$ for any $z \in O_x$ and coF_x has a closed graph. For any $z \in O_x$, $F_x(z) \subseteq \psi_i(z) = H_i(z)$ if $z_i \in \overline{A}_i(z)$, and $F_x(z) \subseteq \psi_i(z) \subseteq \overline{A}_i(z) = H_i(z)$ if $z_i \notin \overline{A}_i(z)$.

2. Consider the case that $x_i \notin \overline{A}_i(x)$. Since the correspondence $\overline{A}_i$ is upper hemicontinuous and closed valued, it has a closed graph. As a result, one can find an open neighborhood O_x of x such that $z_i \notin \overline{A}_i(z)$ and hence $H_i(z) = \overline{A}_i(z)$ for any $z \in O_x$. As $\overline{A}_i$ is upper hemicontinuous, closed and convex valued, H_i has the continuous inclusion property.

Let $I(x) = \{i \in I \colon H_i(x) \neq \emptyset\}$. Define a correspondence $H \colon X \to 2^X$ as

$$H(x) = \begin{cases} (\prod_{i \in I(x)} H_i(x)) \times (\prod_{j \in I \setminus I(x)} X_j), & I(x) \neq \emptyset; \\ \\ \emptyset, & I(x) = \emptyset. \end{cases}$$

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assumption is not needed. Indeed, the proof of Borglin and Keiding (1976) remains valid if one replaces the KF-majorization by L-majorization. The existence of maximal element theorem for correspondences having the continuous inclusion property can be used to show that the metrizability in our Theorem 3 is not needed.

It can be easily checked that $H(x)$ has the continuous inclusion property at each x such that $H(x) \neq \emptyset$.

In addition, one can easily show that $x \notin \mathrm{co}H(x)$ for any $x \in X$. Indeed, fix any $x \in X$. If $I(x) = \emptyset$, then $H(x) = \emptyset$, which implies that $x \notin \mathrm{co}H(x)$. If $I(x) \neq \emptyset$, then there exists an agent i such that $H_i(x) \neq \emptyset$. If $x_i \in \overline{A}_i(x)$, then $x_i \notin \mathrm{co}\psi(x) = \mathrm{co}H_i(x)$. If $x_i \notin \overline{A}_i(x)$, then $x_i \notin \mathrm{co}H_i(x)$ as $H_i(x) = \overline{A}_i(x)$ (since $\overline{A}_i(x)$ is convex). Hence, $x \notin \mathrm{co}H(x)$.

It is easy to see that there exists a point $x^* \in X$ such that $H(x^*) = \emptyset$, which implies that $I(x^*) = \emptyset$. That is, for any i, $H_i(x^*) = \emptyset$, which implies that $x_i^* \in \overline{A}_i(x^*)$ and $\psi_i(x^*) = H_i(x^*) = \emptyset$. □

Remark 9. *The previous proof adapted in Theorem 3 seems to be suitable to cover the case where the set of agents is a measure space as in Yannelis (1987). It is not clear whether the above proof can be easily extended to a measure space of agents.*

1.5 Existence of Walrasian Equilibria

1.5.1 Existence of Free/Non-free Disposal Walrasian Equilibrium

An **exchange economy** $\mathcal{E}$ is a set of triples $\{(X_i, P_i, e_i) \colon i \in I\}$, where

- I is a finite set of agents;
- $X_i \subseteq \mathbb{R}^l_+$ is the **consumption set** of agent i, and $X = \prod_{i \in I} X_i$;
- $P_i \colon X \times \triangle \to 2^{X_i}$ is the **preference correspondence** of agent i, where $\triangle$ is the set of all possible prices;[12]

[12] We allow for very general preferences, which can be interdependent and price-dependent.

- $e_i \in X_i$ is the **initial endowment** of agent i, where $e = \sum_{i \in I} e_i \neq 0$.

Let $\triangle = \{p \in \mathbb{R}^l_+ \colon \sum_{k=1}^l p_k = 1\}$. Given a price $p \in \triangle$, the **budget set** of agent i is $B_i(p) = \{x_i \in X_i \colon p \cdot x_i \leq p \cdot e_i\}$. Let $\psi_i(p, x) = B_i(p) \cap P_i(x, p)$ for each $i \in I$, $x \in X$ and $p \in \triangle$. Then $\psi_i(p, x)$ is the set of all allocations in the budget set of agent i at price p that he prefers to x.

A **free disposal Walrasian equilibrium** for the exchange economy $\mathcal{E}$ is $(p^*, x^*) \in \triangle \times X$ such that

1. for each $i \in I$, $x_i^* \in B_i(p^*)$ and $\psi_i(p^*, x^*) = \emptyset$;

2. $\sum_{i \in I} x_i^* \leq \sum_{i \in I} e_i$.

Theorem 4. *Let $\mathcal{E}$ be an exchange economy satisfying the following assumptions: for each $i \in I$,*

1. *X_i is a nonempty compact convex subset of $\mathbb{R}^l_+$;*[13]

2. *ψ_i has the continuous inclusion property at each $(p, x) \in \triangle \times X$ with $\psi_i(p, x) \neq \emptyset$, and $x_i \notin co\psi_i(p, x)$.*

Then $\mathcal{E}$ has a free disposal Walrasian equilibrium.

Proof. The proof follows the idea of Arrow and Debreu (1954), which introduces a fictitious player; see also Shafer (1976).

For each $i \in I$, $p \in \triangle$ and $x \in X$, let $A_i(p, x) = B_i(p)$. Define the correspon-

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See McKenzie (1955) and Shafer and Sonnenschein (1975) for more discussions. For agent i, $y_i \in P_i(x, p)$ means that y_i is strictly preferred to x_i provided that all other components are unchanged at the price $p \in \triangle$.

[13] The commodity space X_i can be sufficiently large. For example, we can let $X_i = \{x_i \in \mathbb{R}^l_+ \colon x_i \leq K \cdot \sum_{i \in I} e_i\}$, where K is an arbitrarily large positive number.

dences $A_0(p,x) = \triangle$ and $P_0(p,x) = \{q \in \triangle \colon q \cdot \left(\sum_{i\in I}(x_i - e_i)\right) > p \cdot \left(\sum_{i\in I}(x_i - e_i)\right)\}$. Let $I_0 = I \cup \{0\}$. Then for any $i \in I_0$, A_i is nonempty, convex valued, and upper hemicontinuous on $\triangle \times X$.

Note that $\psi_i(p,x) = A_i(p,x) \cap P_i(p,x)$ has the continuous inclusion property for each $i \in I$. Moreover, let $\psi_0(p,x) = A_0(p,x) \cap P_0(p,x) = P_0(p,x)$. Fix any $(p,x) \in \triangle \times X$ such that $\psi_0(p,x) \neq \emptyset$, pick $q \in \psi_0(p,x)$, then $(q-p) \cdot \left(\sum_{i\in I}(x_i - e_i)\right) > 0$. Since the left side of the inequality is continuous, there is an open neighborhood O of (p,x) such that for any $(p',x') \in O$, $(q-p') \cdot \left(\sum_{i\in I}(x'_i - e_i)\right) > 0$, which implies that the correspondence ψ_0 has the continuous inclusion property. In addition, it is obvious that ψ_0 is convex valued and $p \notin \psi_0(p,x)$ for any $(p,x) \in \triangle \times X$.

Thus, we can view the exchange economy $\mathcal{E}$ as an abstract economy $\Gamma = \{(X_i, A_i, P_i) \colon i \in I_0\}$ which satisfies all the conditions of Theorem 3. Therefore, there exists a point $(p^*, x^*) \in \triangle \times X$ such that

1. $x_i^* \in A_i(p^*, x^*) = B_i(p^*)$ and $\psi_i(p^*, x^*) = \emptyset$ for each $i \in I$, and
2. $P_0(p^*, x^*) = \psi_0(p^*, x^*) = \emptyset$.

Let $z = \sum_{i\in I}(x_i^* - e_i)$. Then (1) implies that $p^* \cdot z \leq 0$ and (2) implies that $q \cdot z \leq p^* \cdot z$ for any $q \in \triangle$, and hence $q \cdot z \leq p^* \cdot z \leq 0$. Suppose that $z \notin \mathbb{R}^l_-$. Thus, there exists some $k \in \{1, \ldots, l\}$ such that $z_k > 0$. Let $q' = \{q_j\}_{1 \leq j \leq l}$ such that $q_j = 0$ for any $j \neq k$ and $q_k = 1$. Then $q' \in \triangle$ and $q' \cdot z = z_k > 0$, a contradiction. Therefore, $z \in R^l_-$, which implies that $\sum_{i\in I} x_i^* \leq \sum_{i\in I} e_i$.

Therefore, (p^*, x^*) is a free disposal Walrasian equilibrium. $\square$

Remark 10. *We have imposed the compactness condition on the consumption set. It* *is not clear to us at this stage if this condition can be dispensed with. When agents' preferences are continuous, one can work with a sequence of economies with compact consumption sets, which are the truncations of the original consumption set. Then the existence of Walrasian equilibrium allocations and prices can be proved in each truncated economy. Since the set of feasible allocations and the price set are both compact, there exists a convergent point. By virtue of the continuity of preferences, one can show that this is indeed a Walrasian equilibrium of the original economy. The convergence argument fails in our setting as we do not require the continuity assumption on preferences. Consequently, relaxing the compactness assumption seems to be an open problem.*[14]

We must add that the compactness assumption is not unreasonable at all. The world is finite, and the initial endowment for each good is also finite. Thus, by assuming that for each good, $\|x_i\| \leq K \cdot \sum_{i \in I} \|e_i\|$, where K is a sufficiently large number and I is the set of all agents in the world, no real restriction on the attainability of the consumption of each good is imposed.

Note that in Theorem 4 we allowed for free disposal. Below we prove the existence of a non-free disposal Walrasian equilibrium following the proof of Shafer (1976).

Hereafter we allow for negative prices: $\triangle' = \{p \in \mathbb{R}^l \colon \|p\| = \sum_{k=1}^{l} |p_k| \leq 1\}$

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[14]One could allow $X_i = \mathbb{R}^l$ by assuming that if $x_i \in X_i$ and $x_i' \in P_i(x)$, then also $(1-\lambda)x_i + \lambda x_i' \in P_i(x)$ for all $0 < \lambda < 1$. With this assumption one needs to consider only one truncation of the consumption sets (any truncation which contains the feasible consumption points as interior points).

is the set of all possible prices. Let $B_i(p) = \{x_i \in X_i \colon p \cdot x_i \leq p \cdot e_i + 1 - \|p\|\}$ and $\psi_i(p, x) = P_i(x, p) \cap B_i(p)$ for each $i \in I$, $x \in X$ and $p \in \triangle'$. Let $K = \{x \colon \sum_{i \in I} x_i = \sum_{i \in I} e_i\}$, and $\mathrm{pr}_i \colon X \to X_i$ be the projection mapping for each $i \in I$.

A **(non-free disposal) Walrasian equilibrium** for the exchange economy $\mathcal{E}$ is $(p^*, x^*) \in \triangle' \times X$ such that

1. $\|p^*\| = 1$;
2. for each $i \in I$, $x_i^* \in B_i(p^*)$ and $\psi_i(p^*, x^*) = \emptyset$;
3. $\sum_{i \in I} x_i^* = \sum_{i \in I} e_i$.

If p^* is a Walrasian equilibrium price, then $\|p^*\| = 1$ and $B_i(p^*) = \{x_i \in X_i \colon p^* \cdot x_i \leq p^* \cdot e_i\}$, which is the standard budget set of agent i.

Theorem 5. *Let $\mathcal{E}$ be an exchange economy satisfying the following assumptions: for each $i \in I$,*

1. *X_i is a nonempty compact convex subset of $\mathbb{R}^l_+$;*
2. *ψ_i has the continuous inclusion property at each $(p, x) \in \triangle' \times X$ with $\psi_i(p, x) \neq \emptyset$, and $x_i \notin co\psi_i(p, x)$.*
3. *for each $x_i \in pr_i(K)$ and $p \in \triangle'$, $x_i \in bdP_i(x, p)$, where bd denotes boundary.*

Then $\mathcal{E}$ has a Walrasian equilibrium.

Proof. Repeating the arguments in the first two paragraphs of the proof of Theorem 4, one could show that there exists a point $(p^*, x^*) \in \triangle' \times X$ such that

1. $x_i^* \in A_i(p^*, x^*) = B_i(p^*)$ for each $i \in I$, which implies that $p^* \cdot x_i^* \leq p^* \cdot e_i + 1 - \|p^*\|$;

2. $\psi_i(p^*, x^*) = \emptyset$ for each $i \in I$;

3. $P_0(p^*, x^*) = \psi_0(p^*, x^*) = \emptyset$.

Let $z = \sum_{i \in I}(x_i^* - e_i)$. We must show that $z = 0$. Suppose that $z \neq 0$. From (3), it follows that $q \cdot z \leq p^* \cdot z$ for any $q \in \triangle'$. Let $q = \frac{z}{\|z\|}$. Then $q \in \triangle'$ and $p^* \cdot z \geq q \cdot z > 0$. Let $q^* = \frac{p^*}{\|p^*\|}$. Since $\frac{p^*}{\|p^*\|} \cdot z \geq p^* \cdot z \geq q^* \cdot z$, it follows that $\|p^*\| = 1$. As a result, $p^* \cdot x_i^* \leq p^* \cdot e_i$ (since $x_i^* \in A_i(p^*, x^*)$), which implies that $p^* \cdot z = p^* \cdot \sum_{i \in I}(x_i^* - e_i) \leq 0$, a contradiction. Thus, $z = 0$; that is, $\sum_{i \in I} x_i^* = \sum_{i \in I} e_i$, $x^* \in K$.

Note that $x_i^* \in \text{pr}_i(K)$ implies that $x_i^* \in \text{bd}P_i(x^*, p^*)$. Since $x_i^* \in B_i(p^*)$ and $x_i^* \notin \text{co}\psi_i(p^*, x^*)$, $x_i^* \notin P_i(x^*, p^*)$. If there exists some i such that $p^* \cdot x_i^* < p^* \cdot e_i + 1 - \|p^*\|$, then due to assumption (3), $x_i^* \in \text{bd}P_i(x^*, p^*)$ implies that one can find a point $y_i \in P_i(x^*, p^*)$ such that x_i^* and y_i are sufficiently close, and $p^* \cdot y_i < p^* \cdot e_i + 1 - \|p^*\|$. Thus, $y_i \in \psi_i(p^*, x^*)$, which contradicts (2). Therefore, $p^* \cdot x_i^* = p^* \cdot e_i + 1 - \|p^*\|$ for each $i \in I$, and summing up over all i yields $\|p^*\| = 1$.

Therefore, (p^*, x^*) is a Walrasian equilibrium. □

Remark 11. *Shafer (1976) proved the existence of non-free disposal Walrasian equilibrium based on the equilibrium existence result of Shafer and Sonnenschein (1975) (see Corollary 5 above). Thus, the main theorem of Shafer (1976) follows from our Corollary 5 and Theorem 5.*

Below, we provide an alternative proof of the theorem of Shafer (1976) without invoking the norm of the price $\|p\|$ into the budget set. It requires the nonsatiation condition for one agent only. Furthermore, the proof below remains unchanged if the consumption set is a nonempty, norm compact and convex subset of a Hausdorff locally convex topological vector space. This is not the case in Shafer (1976)'s proof, since the norm of prices is part of the budget set. Recall that the price space $\triangle'$ is weak* compact by Alaoglu's theorem, and $\triangle'$ may not be metrizable unless the space of allocations is separable.

Theorem 6. *Let $\mathcal{E}$ be an exchange economy satisfying the following assumptions:*

1. *for each $i \in I$, let X_i be a nonempty compact convex set of $\mathbb{R}^l_+$;*
2. *for each $i \in I$, ψ_i has the continuous inclusion property at each $(p, x) \in \triangle' \times X$ with $\psi_i(p, x) \neq \emptyset$, and for any $x_i \in X_i$, $x_i \notin co\psi_i(p, x)$;*
3. *for any $p \in \triangle'$ and x in the set of feasible allocations*

$$\mathcal{A} = \{x \in X \colon \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} e_i\},$$

there exists an agent $i \in I$ such that $P_i(x, p) \neq \emptyset$.

Then $\mathcal{E}$ has a Walrasian equilibrium (p^*, x^*); that is,

1. *$p^* \neq 0$;*
2. *for each $i \in I$, $x_i^* \in B_i(p^*)$ and $\psi_i(p^*, x^*) = \emptyset$;*
3. *$\sum_{i \in I} x_i^* = \sum_{i \in I} e_i$.*

Most of the proof proceeds as in Theorem 4. We repeat the argument here for the sake of completeness.

Proof. For each $i \in I$, $p \in \triangle'$ and $x \in X$, let $A_i(p, x) = B_i(p)$. Denote $X_0 = \triangle'$, and define the correspondences $A_0(p, x) \equiv \triangle'$ and $P_0(p, x) = \{q \in \triangle' \colon q\left(\sum_{i \in I}(x_i - e_i)\right) >$

$p\left(\sum_{i\in I}(x_i - e_i)\right)\}$.[15] Let $I_0 = I \cup \{0\}$. Let $\psi_0(p,x) = A_0(p,x) \cap P_0(p,x) = P_0(p,x)$. As shown in the proof of Theorem 4, for each $i \in I_0$, the correspondence ψ_i is convex valued, $(p,x) \notin \psi_i(p,x)$ for any $(p,x) \in \triangle' \times X$, and has the continuous inclusion property.

We have constructed an abstract economy $\Gamma = \{(X_i, P_i, A_i)\colon i \in \{0\} \cup I\}$. By Theorem 3, there exists a point $(p^*, x^*) \in \triangle' \times X$ such that

1. $x_i^* \in A_i(p^*, x^*) = B_i(p^*)$ and $\psi_i(p^*, x^*) = \emptyset$ for each $i \in I$;
2. $P_0(p^*, x^*) = \psi_0(p^*, x^*) = \emptyset$.

Let $z = \sum_{i\in I}(x_i^* - e_i)$. Then (1) implies that $p^*(z) \leq 0$, and (2) implies that $q(z) \leq p^*(z)$ for any $q \in \triangle'$, and hence $q(z) \leq p^*(z) \leq 0$. As a result, $z = 0$;[16] that is, $x^* \in \mathcal{A}$. To complete the proof we must show that $p^* \neq 0$. Suppose otherwise; that is, $p^* = 0$. Then $B_i(p^*) = X_i$ and $\psi_i(p^*, x^*) = P_i(x^*, p^*) = \emptyset$ for each $i \in I$, a contradiction to condition (3). Therefore, (p^*, x^*) is a Walrasian equilibrium. $\square$

Remark 12. *In Theorems 4, 5 and 6, the condition that ψ_i has the continuous inclusion property at each $(p,x) \in \triangle \times X$ with $\psi_i(p,x) \neq \emptyset$, and $x_i \notin co\psi_i(p,x)$ for each i can be weakened following the argument in Subsection 1.4.2. In particular, one can let $I(x) = \{i \in I\colon \psi_i(p,x) \neq \emptyset\}$ and assume that for every $x \in X$ such that $I(x) \neq \emptyset$ and $x_i \in A_i(p,x)$ for all $i \in I$, there is an agent $i \in I(x)$,*

1. *ψ_i has the continuous inclusion property at (p,x);*

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[15] The function $q(x)$ is viewed as the inner product $q \cdot x$ when q is a price vector and x is an allocation.

[16] If $z \neq 0$, then there exists a point $q \in \triangle'$ such that $q(z) < 0$, which implies that $-q(z) > 0$. However, $-q \in \triangle'$, a contradiction.

2. $x_i \notin co\psi_i(p, x)$.

In other words, the continuous inclusion property is not required for all agents, but only for some agents. The proofs of Theorems 4, 5 and 6 can still go through under this new condition.[17] *For pedagogical reasons, we work with condition (2) in Theorem 4.*

1.5.2 Further Remarks

Remark 13. *Theorem 6 can be extended to a more general setting with an infinite dimensional commodity space. In particular, the commodity space can be any normed linear space whose positive cone may not have an interior point, and the set of prices is a subset of its dual space. If the consumption sets are nonempty, norm compact and convex, and the price space is weak* compact, then the proof of Theorem 6 remains unchanged.*

Remark 14. *To prove the existence of a Walrasian equilibrium in economies with infinite dimensional commodity spaces, Mas-Colell (1986) proposed the "uniform properness" condition when the preferences are transitive, complete and convex. Yannelis and Zame (1986) and Podczeck and Yannelis (2008) proved the existence result with non-ordered preferences using the "extreme desirability" condition. All the above results impose on the commodity space a lattice structure. Our Theorem 6 does not require the extreme desirability or uniform properness condition, and no ordering or lattice structure is needed on the commodity space. It should be noticed that the proof of our Theorem 6 requires that the evaluation map* $(p, x_i) \to p(x_i)$ *from* $\triangle' \times X_i$ *to* $\mathbb{R}$

[17]Such a remark has been also made by Carmona and Podczeck (2015).

is continuous for $\triangle'$ with the weak topology, while this joint continuity property of the evaluation map is not required in the papers above.*

Mas-Colell (1986) provided an example of a single agent economy in which the preference is reflexive, transitive, complete, continuous, convex and monotone, but there is no quasi-equilibrium.[18] *We show that his example does not satisfies our condition (2) of Theorem 6 when the commodity space is compact.*

In the example of Mas-Colell (1986), the commodity space is the space of signed bounded countably additive measures $L = ca(K)$ with the bounded variation norm $\|\cdot\|_{BV}$, where $K = Z_+ \cup \{\infty\}$ is the compactification of the positive integers. Let $x_i = x(\{i\})$ for $x \in L$ and $i \in K$. For every $i \in K$, define a function $u_i\colon [0,\infty) \to [0,\infty)$ by

$$u_i(t) = \begin{cases} 2^i t & t \le \frac{1}{2^{2i}}; \\ \\ \frac{1}{2^i} - \frac{1}{2^{2i}} + t & t > \frac{1}{2^{2i}}. \end{cases}$$

The preference relation P is given by $U(x) = \sum_{i=1}^{i=\infty} u_i(x_i)$, which is concave, strictly monotone and weak continuous.*

Suppose that $X = \{x \in L_+\colon \|x\|_{BV} \le M\}$ for some sufficiently large positive integer M. Fix the initial endowment $e = (0, M, 0, \ldots, 0) \in X$ and the price $p_0 = 0$. Then $\psi(p_0, e) = B(p_0) \cap P(e) \neq \emptyset$, as $y = (M, 0, \ldots, 0) \in \psi(p_0, e)$. For each $i \in K$, let $w_i(\{j\}) = 1$ if $j = i$ and 0 otherwise. Fix a linear functional $p \in L'$ such that $p(w_2) = 0$ and $p(w_i) > 0$ for $i \neq 2$. Set $p_n = \frac{p}{n}$. Then $B(p_n) = \{0, m, 0, \ldots, 0\}$, where

[18]The pair (p^*, x^*) is called a free (non-free) disposal quasi equilibrium if: (1) for each $i \in I$, $x_i^* \in B_i(p^*)$; (2) $x_i \in P_i(x^*, p^*)$ implies that $p^* \cdot x_i \ge p^* \cdot e_i$; (3) $\sum_{i\in I} x_i^* \le \sum_{i\in I} e_i$ ($\sum_{i\in I} x_i^* = \sum_{i\in I} e_i$).

$0 \leq m \leq M$. However, for any $z \in B(p_n)$, $z \notin P(e)$. Consequently, $\psi(p_n, e) = \emptyset$. This implies that the correspondence ψ does not have the continuous inclusion property when the commodity space is compact, as $p_n \to 0$ when $n \to \infty$. Therefore, the example of Mas-Colell (1986) violates condition (2) of our Theorem 6.

Remark 15. *If we interpret the infinite dimensional commodity space as goods over an infinite time horizon, the weak, Mackey and weak* topologies on preferences imply that agents are impatient, because those topologies are generated by finitely many continuous linear functionals and they impose a form of "myopia" (that is, tails do not matter, see for example Bewley (1972) and Araujo, Novinski and Páscoa (2011) among others). As our theorems drop the continuity assumption, it will be interesting to see if one can prove the existence theorem with patient agents relying on such discontinuous preferences.*

Remark 16. *Contrary to the standard existence results of Walrasian equilibrium, in the above theorems we do not impose the assumptions that the initial endowment is an interior point of the consumption set, or the preference has an open graph/open lower sections. Below we give an example in which the preferences are discontinuous, and a Walrasian equilibrium exists. Notice that none of the classical existence theorems cover the example below.*

Example 1. *Consider the following 2-agent 2-good economy:*

1. *The set of available allocations for both agents is $X_1 = X_2 = [0, 1] \times [0, 1]$.*

2. Agent 1's preference correspondence depends on $x_1 = (x_1^1, x_1^2)$ *and* $x_2 = (x_2^1, x_2^2)$*:*

$$P_1(x_1, x_2) =$$

$$\{(y_1^1, y_1^2) \in X_1 \colon y_1^1 \cdot y_1^2 > x_1^1 \cdot x_1^2\} \setminus \{(y_1^1, y_1^2) \in X_1 \colon y_1^1 - x_1^1 = y_1^2 - x_1^2, y_1^1 < \frac{3}{2}x_1^1\}.$$[19]

The preference of agent 2 is defined similarly.

3. The initial endowments are given by $e_1 = (\frac{1}{3}, \frac{2}{3})$ *and* $e_2 = (\frac{2}{3}, \frac{1}{3})$*.*

Note that P_i *does not have open lower sections for any* $i = 1, 2$*. For example,*

$$P_i^l(\frac{1}{2}, \frac{1}{2}) =$$

$$\{(y_i^1, y_i^2) \in [0,1] \times [0,1] \colon y_i^1 \cdot y_i^2 < \frac{1}{4}, y_i^1 \neq y_i^2\} \cup \{(z, z) \colon 0 \leq z \leq \frac{1}{3}\}$$

which is neither open nor closed. As a result, P_i *does not have an open graph.*

We show that the conditions of Theorem 4 hold. Pick any point $(p, x) \in \triangle \times X$ *such that* $\psi_i(p, x) \neq \emptyset$*, then there exists a point* $y_i \in \psi_i(p, x) = B_i(p) \cap P_i(x)$*. Since* $y_i \in P_i(x)$*, it follows that* $y_i^1 \cdot y_i^2 > x_i^1 \cdot x_i^2$*. Thus, one can pick a point* $z_i = (z_i^1, z_i^2)$ *such that* $z_i^j < y_i^j$ *for* $j = 1, 2$ *and* z_i *is an interior point of* $P_i(x)$*.*[20] *Consequently, there exists an open neighborhood* O_i *of* x_i *such that* $(z_i^1, z_i^2) \in P(x_i', x_{-i})$ *for any* $x_i' \in O_i$ *and* $x_{-i} \in X_{-i}$*. Furthermore, due to the fact that* $z_i^j < y_i^j$ *for* $j = 1, 2$*, we have* $0 < p \cdot z_i < p \cdot y_i \leq p \cdot e_i$*, which implies that there exists a neighborhood*

[19]Given an allocation $x = (x_1, x_2) = ((x_1^1, x_1^2), (x_2^1, x_2^2))$ in the edgeworth box, the set of allocations which is preferred to x for agent 1 is the set of all points above the curve $y_1^1 \cdot y_1^2 = x_1^1 \cdot x_1^2$ such that the segment $\{(y_1^1, y_1^2) \colon y_1^1 - x_1^1 = y_1^2 - x_1^2, x_1^1 \leq y_1^1 < \frac{3}{2}x_1^1\}$ is removed.

[20]For example, one can choose the point $z_i = (y_i^1 - \epsilon, y_i^2 - 2\epsilon)$, where ϵ is a positive number. It is easy to see that if ϵ is sufficiently small, then z_i is an interior point of $P_i(x)$.

O_p of p, $z_i \in B_i(p')$ for any $p' \in O_p$. Define the correspondence $F_{(p,x)}$ as follows: $F_{(p,x)}(p', x') \equiv \{z_i\}$ for any $(p', x') \in O_p \times (O_i \times X_{-i})$.

Then we have:

1. *$O_p \times (O_i \times X_{-i})$ is an open neighborhood of (p, x);*

2. *$F_{(p,x)}(p', x') \equiv \{z_i\} \subseteq \psi_i(p', x')$ for any $(p', x') \in O_p \times (O_i \times X_{-i})$;*

3. *$F_{(p,x)}$ is a single-valued constant correspondence, and hence is closed.*

Therefore, ψ has the continuous inclusion property at (p, x). In addition, it is easy to see that $x_i \notin co\psi_i(p, x)$. By Theorem 4 above, there exists a Walrasian equilibrium. Indeed, it can be easily checked that (p^*, x^*) is a unique Walrasian equilibrium, where $p^* = (p_1^*, p_2^*) = (\frac{1}{2}, \frac{1}{2})$, and $x_1^* = x_2^* = (\frac{1}{2}, \frac{1}{2})$. Notice that even if the endowment is on the boundary $e_1 = (0, 1)$ and $e_2 = (1, 0)$, the equilibrium still remains the same.

Remark 17. *A natural question that arises is whether or not the continuous inclusion property is easily verifiable for an economy. In the example above we have demonstrated that it is easily verifiable, and it can be used to obtain the existence of a Walrasian equilibrium. Below we present another example in which one can easily check that the continuous inclusion property does not hold, and there is no Walrasian equilibrium. In this example, the preferences are continuous, and the initial endowment is not an interior point of the consumption set.*

Example 2. *There are two agents $I = \{1, 2\}$, and two goods x and y. The payoff functions are given by $u_1(x, y) = x + y$ and $u_2(x, y) = y$, which are continuous. The initial endowments are $e_1 = (\frac{1}{2}, 0)$ and $e_2 = (\frac{1}{2}, 1)$. The consumption sets for both agents are $[0, 2] \times [0, 2]$. In this example, one can easily see that there is no Walrasian* *equilibrium, but a quasi equilibrium* $((x^*, y^*), p^*)$ *exists, where* $(x^*, y^*) = (x_i^*, y_i^*)_{i \in I}$, *and* $(x_1^*, y_1^*) = (1, 0)$, $(x_2^*, y_2^*) = (0, 1)$, $p^* = (0, 1)$.

In this example, the continuity inclusion property does not hold. Consider agent 1 *in the above quasi equilibrium. Since* $p^* \times e_1 = 0$, *the budget set of agent* 1 *is* $B_1(p^*) = \{(x_1, 0)\colon x_1 \in [0, 2]\}$. *In addition, the set of allocations for agent* 1 *which are preferred to* (x_1^*, y_1^*) *is* $P_1(x^*, y^*) = \{(x_1, y_1) \in [0, 2] \times [0, 2]\colon x_1 + y_1 > x_1^* + y_1^* = 1 + 0 = 1\}$. *Thus,* $\psi_1(p^*, (x^*, y^*)) = B_1(p^*) \cap P_1(x^*, y^*) = \{(x_1, 0)\colon x_1 \in (1, 2]\}$, *which is nonempty.*

However, if we slightly perturb the price p^* *by assuming that it is* $q = (\epsilon, 1 - \epsilon)$ *for sufficiently small* $0 < \epsilon < \frac{1}{4}$, *then the budget set of agent* 1 *is* $B_1(q) = \{(x_1, y_1) \in [0, 2] \times [0, 2]\colon x_1 \cdot \epsilon + y_1 \cdot (1 - \epsilon) \leq \frac{1}{2}\epsilon\}$, *which implies that* $x_1 \leq \frac{1}{2}$ *and* $y_1 \leq \frac{1}{2}\frac{\epsilon}{1-\epsilon} < \frac{1}{6}$. *Thus,* $x_1 + y_1 < \frac{1}{2} + \frac{1}{6} = \frac{2}{3} < 1$ *for all* $(x_1, y_1) \in B_1(q)$, *which implies that* $\psi_1(q, (x^*, y^*)) = B_1(q) \cap P_1(x^*, y^*) = \emptyset$.

Therefore, in any neighborhood O *of* $((x^*, y^*), p^*)$, *there is a point* $((x^*, y^*), q) \in O$ *such that* $\psi_1(q, (x^*, y^*)) = \emptyset$, *which implies that the continuity inclusion property does not hold. It can be easily checked that the weaker condition discussed in Remark 12 still fails in this example.*

1.6 A Generalization of the GDN Lemma

In this section, using the fixed point theorem (Theorem 1), we provide a generalization of the Gale-Debreu-Nikaido (GDN) lemma to an infinite-dimensional commodity space with discontinuous excess demand correspondences.

1.6.1 Results

Let X be a Hausdorff locally convex linear topological space, and E a closed, convex cone of X having an interior point e. Denote $E^* = \{p \in X^* \colon p \cdot x \leq 0 \text{ for all } x \in E\} \neq \{0\}$; that is, E^* is the dual cone of E. Let $\triangle = \{p \in E^* \colon p{\cdot}e = -1\}$ and $Z \colon \triangle \to 2^X$ be an excess demand correspondence. Given $p \in \triangle$, let $Y(p) = \{x \in X \colon p \cdot x \leq 0\}$ and $\Gamma(p) = Y(p) \cap Z(p)$.

Theorem 7. *If Γ is nonempty and convex valued, and satisfies the continuous inclusion property, where X^* is endowed with the weak* topology, then $\exists p^* \in \triangle$ such that $Z(p^*) \cap E \neq \emptyset$.*

Proof. Define a correspondence Π from E to $\triangle$ as follows: for each $x \in E$,

$$\Pi(x) = \operatorname{argmax}_{p \in \triangle}(p \cdot x).$$

By Alaoglu's Theorem, $\triangle$ is weak* compact. By Berge's maximum theorem (see Berge (1963)), Π is nonempty, convex and weak* compact valued, and upper hemicontinuous.

Define the correspondence Ψ from $E \times \triangle$ to $E \times \triangle$ as $\Psi(x, p) = \Gamma(p) \times \Pi(x)$ for each $(x, p) \in E \times \triangle$. It is obvious that Ψ is nonempty and convex valued. For each $p_0 \in \triangle$, since Γ is convex valued and has the continuous inclusion property, there exists a weak* open neighborhood O_{p_0} of p_0, and a nonempty, convex valued and weak* upper hemicontinuous correspondence $F_{p_0} \colon O_{p_0} \to 2^E$ such that $F_{p_0}(q) \subseteq \Gamma(q)$ for any $q \in O_{p_0}$. Let $\Phi(x, p) = F_{p_0}(p) \times \Pi(x)$ for $(x, p) \in E \times O_{p_0}$. Then Φ is a sub-correspondence of Ψ on $E \times O_{p_0}$, which is nonempty, convex-valued and upper hemicontinuous. Therefore, Ψ has the continuous inclusion property.

By Theorem 1, there exists $(x^*, p^*) \in E \times \triangle$ such that $(x^*, p^*) \in \Psi(x^*, p^*)$. That is,

1. $p^* \cdot x^* \geq p \cdot x^*$ for any $p \in \triangle$;

2. $x^* \in Z(p^*)$ and $p^* \cdot x^* \leq 0$.

Combining (1) and (2), we have $p \cdot x^* \leq p^* \cdot x^* \leq 0$ for any $p \in \triangle$, which implies that $x^* \in E$. Therefore, $Z(p^*) \cap E \neq \emptyset$ for some $p^* \in \triangle$. $\square$

Below, we provide an alternative proof using Corollary 1.

Alternative Proof. Since Γ has the continuous inclusion property, for each $p \in \triangle$, there exists an open neighborhood O_p and a nonempty correspondence $G_p \colon O_p \to 2^X$ such that $G_p(q) \subseteq \Gamma(q)$ for any $q \in O_p$ and coG_p has a closed graph. As in the proof of Theorem 1, one can find a nonempty, convex and compact valued, weak* upper hemicontinuous correspondence $G \colon \triangle \to 2^X$ which is a sub-correspondence of Γ. Define the correspondence $F \colon \triangle \to 2^{\triangle}$ by $F(p) = \{q \in \triangle \colon q \cdot x > 0 \text{ for all } x \in G(p)\}$. Fix $q \in \triangle$. As in the proof of Yannelis (1985, Theorem 3.1), one can easily show that $W = F^l(q)$, where $W = \{p \in \triangle \colon G(p) \subseteq V_q\}$ and $V_q = \{x \in X \colon q \cdot x > 0\}$. The set W is weak* open since G is weak* upper hemicontinuous. Consequently, F has weak* open lower sections, and hence has the continuous inclusion property.[21] In addition, by the definition of F, $p \notin F(p)$ for every $p \in \triangle$. Since $\triangle$ is nonempty, convex and weak* compact, by Corollary 1, there exists a point $p^* \in \triangle$ such that $F(p^*) = \emptyset$; that

[21]The continuous inclusion property of F holds on the subset $\triangle$ of X^*, which is endowed with the weak* topology.

is,

$$\text{for any } q \in \triangle, \exists x \in G(p^*), q \cdot x \leq 0. \tag{1.1}$$

We will show that (1.1) implies $Z(p^*) \cap E \neq \emptyset$ for some $p^* \in \triangle$. Suppose otherwise, then there exists a continuous linear functional which strictly separates the convex compact set $G(p^*) \subseteq Z(p^*)$ from the closed convex set E; that is,

$$\begin{gathered}\text{there exists } r \in X^*, r \neq 0 \text{ such that} \\ \inf_{x \in G(p^*)} r \cdot x > \sup_{x \in E} r \cdot x \geq 0.\end{gathered} \tag{1.2}$$

Without loss of generality, we can take r to be in $\triangle$.[22] It follows from (1.2) that $r \cdot x > 0$ for any $x \in G(p^*)$, a contradiction to (1.1).

Therefore, $Z(p^*) \cap E \neq \emptyset$ for some $p^* \in \triangle$. □

1.6.2 An Example

Below, we provide an example which indicates how Theorem 7 can be used to prove the existence of an equilibrium. Notice that the preferences of both agents below are neither upper hemicontinuous nor lower hemicontinuous, and hence none of the previous equilibrium existence theorems in the literature are applicable. However, an equilibrium exists by virtue of our Theorem 7

Example 3. *Consider the following* 2*-agent* 2*-good economy:*

1. *The set of available allocations for both agents are* $X_1 = X_2 = [0, 1] \times [0, 1]$, $X = X_1 \times X_2$.

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[22]If $r \notin \triangle$, then the fact e is an interior point of E implies that $r \cdot e < 0$, and we can replace r by $\frac{r}{-r \cdot e}$.

2. The initial endowments are given by $e_1 = (\frac{2}{3}, \frac{1}{3})$ *and* $e_2 = (\frac{1}{3}, \frac{2}{3})$*.*

3. For agent 1 *and an allocation* $x_1 = (x_1^1, x_1^2)$ *and* $x_2 = (x_2^1, x_2^2)$*, agent* 1*'s preference only depends on his own allocation:*

a *if* $x_1^1 > x_1^2$*, then* $P_1(x_1, x_2) = \{(y, z) \colon z > y \geq 0, y + z \geq 1\}$*;*

b *if* $x_1^1 < x_1^2$*, then* $P_1(x_1, x_2) = \{(y, z) \colon y > z \geq 0, y + z \geq 1\}$*;*

c *if* $x_1^1 = x_1^2$*, then* $P_1(x_1, x_2) = \{(y, y) \colon y > x_1^1\}$*.*

The preference of agent 2 *is defined similarly.*

Note that P_i *is neither upper hemicontinuous nor lower hemicontinuous,* $i = 1, 2$*.*

For any price $p = (p_1, 1 - p_1)$*, the budget set of agent* 1 *is*

$$B_1(p) = \{(x_1^1, x_1^2) \in X_1 \colon p_1 \cdot x_1^1 + (1 - p_1) \cdot x_1^2 \leq \frac{1}{3}(1 + p_1)\},$$

and the budget set of agent 2 *is*

$$B_2(p) = \{(x_2^1, x_2^2) \in X_2 \colon p_1 \cdot x_2^1 + (1 - p_1) \cdot x_2^2 \leq \frac{1}{3}(2 - p_1)\}.$$

The demand correspondence for agent i *is defined as*

$$D_i(p) = \{x \in B_i(p) \colon P_i(x) \cap B_i(p) = \emptyset\}.$$

It is easy to see that D_i *is nonempty and convex valued. In addition, given any price* p*,* $x_1 = (\frac{1+p_1}{3}, \frac{1+p_1}{3}) \in D_1(p)$ *and* $x_2 = (\frac{2-p_1}{3}, \frac{2-p_1}{3}) \in D_2(p)$*.*

One can define functions ψ_1 *and* ψ_2 *such that* $\psi_1(p) = (\frac{1+p_1}{3}, \frac{1+p_1}{3})$ *and* $\psi_2(p) = (\frac{2-p_1}{3}, \frac{2-p_1}{3})$*. Since* $\psi_i(p) \in D_i(p)$ *for every* p*,* D_i *has the continuous inclusion property for any* i*. Then* $D_1 + D_2$ *also satisfies the continuous inclusion property, Theorem 7 can be used to show the existence of an equilibrium. Indeed,* $(x_1^1, x_1^2) = (x_2^1, x_2^2) = (\frac{1}{2}, \frac{1}{2})$ *and* $(p_1, p_2) = (\frac{1}{2}, \frac{1}{2})$ *is an equilibrium.*

1.6.3 Remarks

We show that Theorem 7 implies the standard Gale-Debreu-Nikaido lemma, see Debreu (1956).

Corollary 7. *Let $X = \mathbb{R}^l$ and $Z\colon \triangle \to 2^X$ be an excess demand correspondence satisfying the following conditions:*

1. *Z is nonempty, convex and compact valued, and upper hemicontinuous;*
2. *for every $p \in \triangle$, $\exists z \in Z(p)$ such that $p \cdot z \leq 0$.*

Then, $\exists p^* \in \triangle$ such that $Z(p^*) \cap \mathbb{R}^l_- \neq \emptyset$.

Proof. Given $p \in \triangle$, let $Y(p) = \{z \in \mathbb{R}^l \colon p \cdot z \leq 0\}$ and $X(p) = Y(p) \cap Z(p)$. Due to (2), X is nonempty. Since both Y and Z are convex valued and upper hemicontinuous, X is also convex valued and upper hemicontinuous. Thus, X has the continuous inclusion property. Then the result follows from Theorem 7. □

Remark 18. *Yannelis (1985) proved the market equilibrium theorem of Gale-Debreu-Nikaido for an infinite dimensional commodity space by assuming that the excess demand correspondence is upper demicontinuous. In our theorem, the excess demand correspondence may not be continuous, hence not upper demicontinuous.*

Suppose that X is an AM-space with the unit e, X_+ be the positive cone and $\triangle = \{p \in X_+^* \colon p \cdot e = 1\}$. Aliprantis and Brown (1983) worked with an excess demand function $Z\colon \triangle \to X$ instead of an excess demand correspondence, and proved the following result.

Corollary 8 (Aliprantis and Brown (1983))**.** *Suppose that*

1. *there exists a consistent locally convex topology on X such that Z is weak* continuous;*

2. *Z satisfies the Walras law, that is, $p \cdot Z(p) = 0$ for all $p \in \triangle$.*

Then there exists a point $p \in \triangle$ such that $Z(p) \leq 0$.

It is obvious that this result is covered by our Theorem 7, since $\Gamma(p) = Z(p)$ in their setting. As a consequence, Γ is a weak* continuous function and the continuous inclusion property automatically holds.

CHAPTER 2
DISCONTINUOUS GAMES WITH INCOMPLETE INFORMATION

Games with discontinuous payoffs have been used to model a variety of important economic problems; for example, Hotelling location games, Bertrand competition, and various auction models. The seminal work by Reny (1999) proposes the "better reply security" condition and proves the equilibrium existence in quasiconcave compact games with discontinuous payoffs. Since the hypotheses are sufficiently simple and easily verified, the increasing applications of his results has widened significantly in recent years, as evidenced by Jackson and Swinkels (2005) and Monteiro and Page (2008) among others.[1]

In the previous chapter, we consider discontinuous games with complete information. In this chapter, we consider discontinuous games with incomplete information; that is, games with a finite set of players and each of whom is characterized by his own private information, a strategy set, a state dependent (random) utility function and a prior. This problem arises naturally in situations where privately informed agents behave strategically. Because of its importance, the research trend in this field has been quite active since Harsanyi's seminal work. The purpose of this chapter is to provide new results on pure/behavioral strategy equilibria for games with incomplete information and discontinuous payoffs.

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[1] A number of papers appeared in the topic of discontinuous games and further extensions have been obtained in several directions. See the introduction of Chapter 1 for more references.

Firstly, we consider the existence of pure strategy equilibria in games where all players adopt the Bayesian reasoning. We introduce the notions of finite/finite* payoff security and adopt the aggregate upper semicontinuity condition in the ex post games. We show that the (ex ante) Bayesian game is payoff secure and reciprocal upper semicontinuous, and hence Reny's theorem is applicable and a pure strategy Bayesian equilibrium exists. A key issue here is that the quasiconcavity of the Bayesian game cannot be guaranteed even if all ex post games are quasiconcave. We show by means of counterexamples that the concavity and finite payoff security conditions of the ex post games are both necessary for the existence of a pure strategy Bayesian equilibrium.[2]

Secondly, we study the equilibrium existence problem in discontinuous games under incomplete information and ambiguity. The Bayesian paradigm has been constantly criticized since Ellsberg (1961), and the non-expected utility theory has received much attention. In the framework of Bayesian preferences, the results on the existence of pure/behavioral strategy equilibria in discontinuous games with incomplete information (see He and Yannelis (2015b, 2016a), and Carbonell-Nicolau and McLean (2015)) typically build on the equilibrium existence result of Reny (1999). However, the equilibrium existence result in the incomplete information framework is not a straightforward adaptation of the result of Reny (1999). In order to generalize the result of Reny (1999) to asymmetric information, one has to introduce some exogenous assumptions. In this chapter, we adopt the maximin expected utility of

[2]Based on a different approach using the communication device, Jackson et al. (2002) studied discontinuous games with asymmetric information.

Gilboa and Schmeidler (1989) (see also de Castro and Yannelis (2009)).[3] Our result shows that by working with the maximin preferences, the existence of equilibrium in games with incomplete information follows directly from the existence of equilibrium for every ex post game. As a result, the maximin framework solves the equilibrium existence issue without introducing any additional conditions. To demonstrate the usefulness of this result, we present a timing game with asymmetric information as an illustrative example, which has an equilibrium when players have maximin preferences, but has no equilibrium when the Bayesian reasoning is adopted.

Finally, we provide a new existence result on behavioral strategy equilibria for Bayesian games with discontinuous payoffs. Our result is based on a Bayesian generalization of the clever condition called "disjoint payoff matching", which was introduced by Allison and Lepore (2014) for a normal form game. The advantage of this condition is that one only needs to check the payoff at each strategy profile itself. The standard payoff security-type condition forces one to check the payoffs in the neighborhood of each strategy profile, which is more demanding. Thus, our condition is relatively straightforward, and the equilibrium existence result can be easily verified for a large class of Bayesian games. Our result widens the applications in economics as we can cover situations that previous results in the literature are not readily applicable. As an illustrative example, we provide an application to an all-pay auction with general tie-breaking rules.

[3]For some recent applications of maximin preference in general equilibrium theory and game theory, see, for example, de Castro, Pesce and Yannelis (2011), de Castro, Liu and Yannelis (2015), and He and Yannelis (2015c).

This chapter is based on He and Yannelis (2015b, 2016a,b), and is organized as follows. Section 2.1 proves the existence of a pure strategy Bayesian equilibrium, and collects some discussions and possible extensions. Section 2.2 presents the result on the existence of equilibrium when players adopt maximin preferences. Section 2.3 provides the result on the existence of behavioral strategy equilibria.

2.1 Existence of Pure Strategy Equilibria: An Extension of Reny's Existence Theorem

2.1.1 Model

2.1.1.1 Discontinuous games with asymmetric information

We consider an **asymmetric information game**

$$G = \{\Omega, (u_i, X_i, \mathcal{F}_i)_{i\in I}\}.$$

- There is a finite set of players, $I = \{1, 2, \cdots, N\}$.
- Ω is a countable state space representing the **uncertainty** of the world, $\mathcal{F}$ is the power set of Ω.
- $\mathcal{F}_i$ is a partition of Ω, denoting the **private information** of player i. $\mathcal{F}_i(\omega)$ denotes the element of $\mathcal{F}_i$ including the state ω.
- Player i's action space X_i is a nonempty, compact, convex subset of a topological vector space, $X = \prod_{i\in I} X_i$.
- For every $i \in I$, $u_i : X \times \Omega \to \mathbb{R}$ is a **random utility function** representing the (ex post) preference of player i.

A game G is called a **compact** game if u_i is bounded for every $i \in I$; that is,

$\exists M > 0$, $|u_i(x, \omega)| \leq M$ for all $x \in X$ and $\omega \in \Omega$, $1 \leq i \leq N$. A game G is said to be **quasiconcave (resp. concave)** if $u_i(\cdot, x_{-i}, \omega)$ is quasiconcave (resp. concave) for every $i \in I$, $x_{-i} \in X_{-i}$ and $\omega \in \Omega$. For every $\omega \in \Omega$, the **ex post game** is $G_\omega = (u_i(\cdot, \omega), X_i)_{i \in I}$. Suppose that each player has a **private prior** π_i on $\mathcal{F}$ such that $\pi_i(E) > 0$ for any $E \in \mathcal{F}_i$. The **weighted ex post game** is $G'_\omega = (w_i(\cdot, \omega), X_i)_{i \in I}$, where $w_i(\cdot, \omega)$ is a mapping from X to $\mathbb{R}$ and $w_i(\cdot, \omega) = u_i(\cdot, \omega)\pi_i(\omega)$ for each $\omega \in \Omega$.

For every player i, a **strategy** is an $\mathcal{F}_i$-measurable mapping from Ω to X_i. Let

$$L_i = \{f_i : \Omega \to X_i : f_i \text{ is } \mathcal{F}_i\text{-measurable}\},$$

then L_i is a convex and compact set endowed with the product topology. $L = \prod_{i \in I} L_i$. Given a strategy profile $f \in L$, the **expected utility** of player i is

$$U_i(f) = \sum_{\omega \in \Omega} u_i(f_i(\omega), f_{-i}(\omega), \omega)\pi_i(\omega),$$

then $U_i(\cdot)$ is also bounded by M. Therefore, the **(ex ante) Bayesian game** of G is $G_0 = (U_i, L_i)_{1 \leq i \leq N}$, which is compact and concave if the game G is compact and concave. A strategy profile $f \in L$ is said to be a **Bayesian equilibrium** if for each player i and any $g_i \in L_i$,

$$U_i(f) \geq U_i(g_i, f_{-i}).$$

Remark 19. *It is well known that quasiconcavity may not be preserved under summation or integration. Thus, the Bayesian game G_0 may not be quasiconcave even if G is quasiconcave.*

2.1.1.2 Deterministic case

Hereafter, $G_d = (X_i, u_i)_{i=1}^N$ will denote a **deterministic discontinuous game**; that is, Ω is a singleton. The following definitions strengthen the notion of payoff security in Reny (1999).

Definition 2. *In the game G_d, player i can secure an n-dimensional payoff $(\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$ at $(x_i, x_{-i}^1, \cdots, x_{-i}^n) \in X_i \times X_{-i}^n$ if there is $\overline{x_i} \in X_i$, such that $u_i(\overline{x_i}, y_{-i}^k) \geq \alpha_k$ for all y_{-i}^k in some open neighborhood of x_{-i}^k, $1 \leq k \leq n$.*

Definition 3. *The game G_d is n-**payoff secure** if for every $i \in I$ and $(x_i, x_{-i}^1, \cdots, x_{-i}^n) \in X_i \times X_{-i}^n$, $\forall \epsilon > 0$, player i can secure an n-dimensional payoff*

$$\big(u_i(x_i, x_{-i}^1) - \epsilon, \cdots, u_i(x_i, x_{-i}^n) - \epsilon\big)$$

*at $(x_i, x_{-i}^1, \cdots, x_{-i}^n) \in X_i \times X_{-i}^n$. The game G_d is said to be **finitely payoff secure** if it is n-payoff secure for any $n \in \mathbb{N}$.*[4]

*If $n = 1$, it is called **payoff secure**.*

Given $x \in X$, let $u(x) = (u_1(x), \cdots, u_N(x))$ be the payoff vector of the game G_d. Define $\Gamma_d = \{(x, u(x)) \in X \times \mathbb{R} : x \in X\}$; that is, the graph of the payoff vector $u(\cdot)$, then $\overline{\Gamma_d}$ denotes the closure of Γ_d.

The following definition is due to Reny (1999).

[4]It is clear that the uniform payoff security condition of Monteiro and Page (2007) implies our finite payoff security condition. See Subsection 2.1.3.2 for further discussion of this point.

Definition 4. *The game G_d is **better-reply secure** if whenever $(x^*, \alpha^*) \in \overline{\Gamma_d}$ and x^* is not a Nash equilibrium, some player j can secure a payoff strictly above α_i^* at x^*.*

In their pioneer paper, Dasgupta and Maskin (1986) proposed the following condition which is weaker than the upper semicontinuity condition of the utility functions.

Definition 5. *A game G_d is said to be **aggregate upper semicontinuous** if the summation of the utility functions of all players is upper semicontinuous.*[5]

The following generalization is due to Simon (1987), which is called complimentary discontinuity or reciprocal upper semicontinuity.

Definition 6. *A game G_d is **reciprocal upper semicontinuous** if for any $(x, \alpha) \in \overline{\Gamma_d} \setminus \Gamma_d$, there is a player i such that $u_i(x) > \alpha_i$.*

Reny (1999) showed that the game G_d is better reply secure if it is payoff secure and reciprocal upper semicontinuous.

Theorem 8 (Reny (1999))**.** *Every compact, quasiconcave and better-reply secure deterministic game has a Nash equilibrium.*

We will use this theorem to establish our existence results. One may easily develop analogous definitions of "n-payoff security" in the framework of many recent papers.

[5] Carmona (2009) proved the existence of Nash equilibria in compact, quasiconcave games via weak versions of both upper semicontinuity and payoff security.

2.1.2 Existence of Pure Strategy Bayesian Equilibrium

In this section, we will show the existence of pure strategy Bayesian equilibrium in discontinuous games with asymmetric information.

First, we shall prove Propositions 2 and 3, which provide sufficient conditions to guarantee the payoff security of a Bayesian game.

Proposition 2. *If the weighted ex post game G'_ω is finitely payoff secure at every state $\omega \in \Omega$ and $u_i(x, \cdot)$ is $\mathcal{F}_i$-measurable for every $x \in X$ and $i \in I$, then the Bayesian game G_0 is payoff secure.*

Proof. For any $i \in I$, suppose that $\mathcal{F}_i = \{E_1, \cdots, E_k, \cdots\}$ is the information partition of player i, M is the bound for u_i. Given any $\epsilon > 0$, there exists a positive integer $K > 0$ such that $\pi_i(\cup_{k=1}^K E_k) > 1 - \frac{\epsilon}{6M}$. For $1 \leq k \leq K$, there exists a finite subset $E'_k \subseteq E_k$ such that $\pi_i(E_k \setminus E'_k) < \frac{\epsilon}{6KM}$ and $\pi_i(E'_k) > 0$.

Fix $\omega_k \in E'_k$ such that $\pi_i(\omega_k) > 0$. Given any $f \in L$, because $u_i(x, \cdot)$ and $f_i(\cdot)$ are both $\mathcal{F}_i$-measurable,

$$u_i(f_i(\omega), f_{-i}(\omega), \omega) = u_i(f_i(\omega_k), f_{-i}(\omega), \omega_k)$$

for any $\omega \in E_k$, $1 \leq k \leq K$.

Since G'_{ω_k} is finitely payoff secure, there exists a point $y_i^k \in X_i$, such that

$$w_i(y_i^k, y_{-i}^\omega, \omega_k) \geq w_i(f_i(\omega_k), f_{-i}(\omega), \omega_k) - \frac{\epsilon}{3}\pi_i(\omega_k)$$

for all y_{-i}^ω in some open neighborhood O_ω of $f_{-i}(\omega)$, $\forall \omega \in E'_k$.

Let

$$g_i(\omega) = \begin{cases} y_i^k, & \text{if } \omega \in E_k \text{ for } 1 \le k \le K, \\ \\ f_i(\omega), & \text{otherwise.} \end{cases}$$

Then by construction g_i is $\mathcal{F}_i$-measurable.

Choose the open set O in L_{-i} such that $O = \left(\prod_{1\le k\le K}(\prod_{\omega\in E'_k} O_\omega \times X_{-i}^{E_k\setminus E'_k})\right)\times X_{-i}^{\Omega\setminus\cup_{1\le k\le K}E_k}$,

$$\begin{aligned}
U_i(g_i, g'_{-i}) &= \sum_{E\in\mathcal{F}_i}\sum_{\omega\in E} w_i(g_i(\omega), g'_{-i}(\omega), \omega) \\
&\ge \sum_{k=1}^{K}\sum_{\omega\in E'_k} w_i(g_i(\omega), g'_{-i}(\omega), \omega) - M\left(\pi_i(\Omega\setminus(\cup_{k=1}^K E_k)) + \sum_{k=1}^{K}\pi_i(E_k\setminus E'_k)\right) \\
&\ge \sum_{k=1}^{K}\sum_{\omega\in E'_k} w_i(y_i^k, g'_{-i}(\omega), \omega_k)\frac{\pi_i(\omega)}{\pi_i(\omega_k)} - \frac{\epsilon}{3} \\
&\ge \sum_{k=1}^{K}\sum_{\omega\in E'_k} [w_i(f_i(\omega_k), f_{-i}(\omega), \omega_k) - \frac{\epsilon}{3}\pi_i(\omega_k)]\frac{\pi_i(\omega)}{\pi_i(\omega_k)} - \frac{\epsilon}{3} \\
&\ge \sum_{k=1}^{K}\sum_{\omega\in E'_k} w_i(f_i(\omega_k), f_{-i}(\omega), \omega) - \frac{2\epsilon}{3} \\
&\ge \sum_{E\in\mathcal{F}_i}\sum_{\omega\in E} w_i(f_i(\omega_k), f_{-i}(\omega), \omega) - \frac{2\epsilon}{3} - M\left(\pi_i(\Omega\setminus(\cup_{k=1}^K E_k)) + \sum_{k=1}^{K}\pi_i(E_k\setminus E'_k)\right) \\
&> U_i(f) - \epsilon
\end{aligned}$$

for every $g'_{-i} \in O$. Thus, the game G_0 is payoff secure. □

Remark 20. *Note that the finitely payoff security of the weighted ex post game $G'_\omega = (w_i(\cdot,\omega), X_i)_{i\in I}$ is slightly weaker than the finitely payoff security of the ex post game $G_\omega = (u_i(\cdot,\omega), X_i)_{i\in I}$, where u_i is the ex post payoff function and $w_i(\cdot,\omega) = u_i(\cdot,\omega)\cdot\pi_i(\omega)$ for every $i \in I$. These two conditions will be equivalent if $\pi_i(\omega) > 0$ for any $i \in I$ and $\omega \in \Omega$.*

In Proposition 2, the ex post utility functions are required to be private information measurable. This assumption can be dropped if the finitely payoff security condition is strengthened accordingly.

Definition 7. *An asymmetric information game G is n^****-payoff secure** *if for every $i \in I$, every $(x_i, x_{-i}^1, \cdots, x_{-i}^n) \in X_i \times X_{-i}^n$ and every $(\omega_1, \cdots, \omega_n) \subseteq D$ for some $D \in \mathcal{F}_i$, $\forall \epsilon > 0$, there is $\overline{x_i} \in X_i$, such that $u_i(\overline{x_i}, y_{-i}^k, \omega_k) \geq u_i(x_i, x_{-i}^k, \omega_k) - \epsilon$ for all y_{-i}^k in some open neighborhood of x_{-i}^k, $1 \leq k \leq n$.*

The game G is said to be ***finitely*** * ***payoff secure*** *if it is n^*-payoff secure for any $n \in \mathbb{N}$.*

Proposition 3. *The Bayesian game G_0 is payoff secure if G is finitely*-payoff secure.*

Proof. As in the proof of Proposition 2, we could find some positive integer K and finite set E_k' for each $1 \leq k \leq K$ satisfying the same conditions therein.

Given any $f \in L$. Since G is finitely* payoff secure, for each $1 \leq k \leq K$, there exists a point $y_i^k \in X_i$, such that

$$u_i(y_i^k, y_{-i}^{\omega}, \omega) \geq u_i(f_i(\omega), f_{-i}(\omega), \omega) - \frac{\epsilon}{3}$$

for all y_{-i}^{ω} in some open neighborhood O_ω of $f_{-i}(\omega)$, $\forall \omega \in E_k'$.

Let

$$g_i(\omega) = \begin{cases} y_i^k, & \text{if } \omega \in E_k \text{ for } 1 \leq k \leq K, \\ f_i(\omega), & \text{otherwise.} \end{cases}$$

Then the rest of the proof proceeds similarly as in the proof of Proposition 2. $\square$

Proposition 4. *In the game G, if the weighted ex post game G'_ω is aggregate upper semicontinuous at every state $\omega \in \Omega$, then the Bayesian game G_0 is reciprocal upper semicontinuous.*

Proof. By way of contradiction, suppose that the Bayesian game G_0 is not reciprocal upper semicontinuous. Then there exists a sequence $\{f^n\} \subseteq L$, $f^n \to f$ and $U(f^n) \to \alpha$ as $n \to \infty$, where $U(f) = (U_1(f), \cdots, U_N(f))$ and $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{R}^N$. $U_i(f) \leq \alpha_i$ for $1 \leq i \leq N$ and $U(f) \neq \alpha$.

Denote $\epsilon = \max_{1 \leq i \leq N}(\alpha_i - U_i(f))$, $\epsilon > 0$. Thus,

$$\sum_{i \in I} U_i(f) \leq \sum_{i \in I} \alpha_i - \epsilon.$$

There exists a finite subset $E \subseteq \Omega$ such that $\pi_i(\Omega \setminus E) < \frac{\epsilon}{2NM}$ for every $i \in I$, where M is the bound of u_i for all i.

Then for any $i \in I$, $U_i(f^n)$ can be divided into two parts: $\mu_i^n = \sum_{\omega \in E} w_i(f^n(\omega), \omega)$ and $\nu_i^n = \sum_{\omega \notin E} w_i(f^n(\omega), \omega)$, $U_i(f^n) = \mu_i^n + \nu_i^n$. Let $\mu^n = \{\mu_i^n\}_{i \in I}$, since $\{\mu^n\}_{n \in \mathbb{N}}$ is bounded, there is a subsequence, say itself, which converges to some $\mu \in \mathbb{R}^N$. Since $\nu_i^n \leq M\pi_i(\Omega \setminus E) < \frac{\epsilon}{2N}$ for any $i \in I$ and $n \in \mathbb{N}$, $\mu_i \geq \alpha_i - \frac{\epsilon}{2N}$ for every $i \in I$.

At each state $\omega \in E$ and $i \in I$, since $w_i(f^n(\omega), \omega)$ is bounded, there is a subsequence which converges to some β_i^ω. Since there are only finitely many players and states, we can assume without loss of generality that $w_i(f^n(\omega), \omega) \to \beta_i^\omega$ as $n \to \infty$, then $\sum_{\omega \in E} \beta_i^\omega = \mu_i$.

Since $f^n(\omega) \to f(\omega)$ for every $\omega \in E$ and G'_ω is aggregate upper semicontinuous,

$$\sum_{i \in I} w_i(f(\omega), \omega) \geq \sum_{i \in I} \beta_i^{\omega}.$$

Thus,

$$\sum_{i \in I} U_i(f) \geq \sum_{i \in I} \sum_{\omega \in E} w_i(f(\omega), \omega) \geq \sum_{i \in I} \sum_{\omega \in E} \beta_i^{\omega} = \sum_{i \in I} \mu_i \geq \sum_{i \in I} \alpha_i - \frac{\epsilon}{2},$$

which is a contradiction. □

By combining Theorem 8, Proposition 2/3 and Proposition 4, we obtain the following result which is an extension of Reny (1999) to Bayesian games with discontinuous payoffs.

Theorem 9. *Suppose that an asymmetric information game G is compact, the corresponding Bayesian game G_0 is quasiconcave, and the weighted ex post game G'_ω is aggregate upper semicontinuous at each state ω. Then a Bayesian equilibrium exists if either of the following conditions holds.*

1. *The weighted ex post game G'_ω is finitely payoff secure at every state $\omega \in \Omega$ and $u_i(x, \cdot)$ is $\mathcal{F}_i$-measurable for every $x \in X$ and $i \in I$.*
2. *The game G is finitely* payoff secure.*

Remark 21. *Note that the (ex ante) Bayesian game G_0 is assumed to be quasiconcave. However, Example 4 below indicates that the theorem may fail if we only require that G is quasiconcave. To impose conditions in the primitive stage, one possible alternative is to require that G be concave. However, the concavity of the utility function implies that it is continuous on the interior of its domain, and hence the discontinuity* *only arises on the boundary. This is a rather strong assumption and will deter many possible applications.*

2.1.3 Discussion

In this section, we shall first provide two examples to show the necessity of the quasiconcavity and the finite payoff security conditions. In addition, we shall also compare our notion of finite payoff security and the uniform payoff security condition of Monteiro and Page (2007), and discuss the possible extension of Theorem 9 to the setting of a continuum of states based on the uniform payoff security condition.

2.1.3.1 Two counterexamples

To guarantee the existence of a Bayesian equilibrium, the expected utility of each player is required to be quasiconcave in Theorem 9. Example 4 below shows that this condition cannot be dropped, even if all other conditions are satisfied and the ex post utility function is quasiconcave itself.

Example 4 (Importance of concavity)**.**

Consider the following game G. There are two players $I = \{1, 2\}$ competing for a object. The strategy spaces for players 1 and 2 are respectively X and Y, $X = Y = [0, 1]$. Player 1 has only one possible private value 1, and player 2 has two possible private values 0 and 1.

Denote $a = (1, 1)$ and $b = (1, 0)$ (the first component is the private value of player 1 and the second component is the private value of player 2). The state space *is* $\Omega = \{a, b\}$. *The information partitions and priors are as follows:*

$$\mathcal{F}_1 = \{\{a, b\}\}, \pi_1(a) = \pi_1(b) = \frac{1}{2};$$

$$\mathcal{F}_2 = \{\{a\}, \{b\}\}, \pi_2(a) = \pi_2(b) = \frac{1}{2}.$$

For $\omega = a, b$, *the utility function of player* 1 *is*

$$u_1(x, y, \omega) = \begin{cases} 1 - x, & \text{if } x \geq y \\ \\ 0, & \text{otherwise} \end{cases}.$$

Then $u_1(x, y, \cdot)$ *is measurable with respect to* $\mathcal{F}_1$ *for any* $(x, y) \in X \times Y$.

The utility function of player 2 *is*

$$u_2(x, y, a) = \begin{cases} 1 - y, & \text{if } y > x \\ \\ 0, & \text{if } y \leq x \end{cases}$$

and

$$u_2(x, y, b) = \begin{cases} -y, & \text{if } y > x \\ \\ \frac{-y}{2}, & \text{if } y \leq x \end{cases}.$$

1. *At both states, when there is a tie, player* 1 *will take the good and player* 2 *gets nothing.*

2. *At state* b, *the private value of player* 2 *is* 0, *bidding for positive price will harm both, thus player* 2 *will be punished when he bids more than* 0 *even if he loses the game.*

The ex post games G_a *and* G_b *are* 2*-payoff secure. Consider the ex post game* G_a *and player* 1. *Given* $\epsilon > 0$, $x \in X$ *and* $(y_1, y_2) \in Y \times Y$. *Assume* $y_1 \geq y_2$ *without loss of generality. There are three possible cases.*

1. *If* $y_1 \leq x$*, then let player* 1 *bid* $\overline{x} = \min\{x+\frac{\epsilon}{2}, 1\}$*. For* $i = 1, 2$*,* $y_i' \leq \min\{x+\frac{\epsilon}{2}, 1\}$ *for any* y_i' *in a small neighborhood of* y_i*, hence the payoff of player* 1 *is at least* $1 - x - \frac{\epsilon}{2}$*.*

2. *If* $y_2 > x$*, then let player* 1 *bid* $\overline{x} = x$ *and his payoff cannot be worse off.*

3. *If* $y_2 \leq x < y_1$*, then let player* 1 *bid* $\overline{x} = x + \delta$ *such that* $x + \delta < y_1$ *and* $0 < \delta < \epsilon$*.*

Similarly, one can show the 2*-payoff security of player* 2 *at state* a *and* b*. Therefore, the ex post game is* 2*-payoff secure at each state. It is easy to see that the summations of ex post utility functions are upper semicontinuous at both states, and the assumptions of quasiconcavity and compactness are satisfied. Thus, there are Nash equilibria for both ex post games. At state* a*, the unique equilibrium is* $(1, 1)$*; at state* b*, the unique equilibrium is* $(0, 0)$*.*

However, there is no Bayesian equilibrium in this game.[6] *Suppose* (x, y) *is an equilibrium, where* $y = (y(a), y(b))$*. In state* b*, player* 2 *will always choose* $y(b) = 0$*, thus player* 1 *can guarantee himself a positive payoff by choosing* $x = 0$*. But if* $x < 1$*, player* 2 *has no optimal strategy at state* a*. Thus, player* 1 *has to choose* $x = 1$ *and gets* 0 *payoff, a contradiction.*

Remark 22. *In Example 4, although the ex post utility function is quasiconcave at both states, the expected utility function is not quasiconcave, and hence there is no*

[6] Note that there is mixed strategy equilibria for this game: for example, Bidder 1's strategy is $\frac{1}{2}\delta_0 + \frac{1}{2}U([0, \frac{1}{2}])$, Bidder 2's strategy is 0 when his value is 0, and $U([0, \frac{1}{2}])$ when his value is 1, where δ_0 is the delta measure concentrated at 0 and $U([0, \frac{1}{2}])$ is the uniform distribution on $[0, \frac{1}{2}]$. However, we only focus on pure strategies.

Bayesian equilibrium.

In Theorem 9, we strengthen the payoff security of Reny (1999) to finite payoff security . The second example shows that the payoff security of every ex post game cannot guarantee the payoff security of the Bayesian game.

Example 5 (Ex post payoff security does not imply ex ante payoff security).

Consider the following game: the player space is $I = \{1, 2, 3\}$, *the state space is* $\Omega = \{a, b\}$, *and the information partitions of all players are* $\mathcal{F}_1 = \mathcal{F}_2 = \{\{a, b\}\}$ *and* $\mathcal{F}_3 = \{\{a\}, \{b\}\}$. *Players have common prior* $\pi(a) = \pi(b) = \frac{1}{2}$. *The action space of player* i *is* $X_i = [0, 1]$, $i = 1, 2, 3$. *The games* L *and* R *are listed below.*

In both states, players 1 *and* 2 *will play the game* L *if* $x_3 = 0$ *and the game* R *otherwise. Player* 1*'s action is in the left and player* 2*'s action is in the top. The*

	0	$(0,1)$	1
0	$(1,6)$	$(0,7)$	$(3,7)$
$(0,1)$	$(5,4)$	$(4,5)$	$(3,7)$
1	$(6,3)$	$(6,3)$	$(4,5)$

	0	$(0,1)$	1
0	$(14,14)$	$(16,10)$	$(16,10)$
$(0,1)$	$(13,15)$	$(14,14)$	$(14,14)$
1	$(13,15)$	$(12,15)$	$(12,15)$

Figure 2.1: Games L & R

utility function of player 3 *is defined as follow:*

$$u_3(x_1, x_2, x_3, \omega) = \begin{cases} 1, & \textit{if } x_3 = 0 \textit{ at } \omega = a \textit{ or } x_3 \in (0,1] \textit{ at } \omega = b; \\ 0, & \textit{otherwise.} \end{cases}$$

Below we study the ex post game G_a *and show that it is payoff secure but not* 2*-payoff* *secure. The same result holds for the ex post game G_b. However, the Bayesian game is not payoff secure.*

In the game L, player 1 can choose the dominant strategy $x_1 = 1$ and player 2 can choose the dominant strategy $x_2 = 1$, thus the game L is payoff secure. In the game R, player 1 can choose the dominant strategy $x_1 = 0$ and player 2 can choose the dominant strategy $x_2 = 0$, thus the game R is payoff secure.

Suppose state a realizes. The payoff of player 3 is secured since he can always choose $x_3 = 0$, which could guarantee his highest payoff. For players 1 and 2, if player 3's action $x_3 = 0$, then players 1 and 2 will play the game L and it is payoff secure since if x_3 deviates in a small neighborhood, then players 1 and 2 will play the game R and their payoffs are strictly higher; if x_3 stays unchanged and they are still in game L, then the payoff security of the game L supports our claim. If player 3's action $x_3 \in (0, 1]$, they will play game R and it is payoff secure since a sufficiently small neighborhood of x_3 is still included in $(0, 1]$ and the game R itself is payoff secure. Therefore, the ex post game G_a is payoff secure.

However, this game is not 2-payoff secure. For example, let $x_1 = 0$, $(x_2^1, x_3^1) = (1, 0)$ and $(x_2^2, x_3^2) = (1, 1)$, there is no action which could guarantee that player 1 can secure the 2 dimensional payoff vector $(3, 16)$. Similarly, one could show that the ex post game G_b is also payoff secure but not 2-payoff secure.

Finally, we verify our claim that the Bayesian game is not payoff secure. Let the strategy of player 3 be $x_3 = (x_3(a), x_3(b)) = (0, 1)$, the expected utilities for players 1 and 2 are listed as the following game E. Then player 1 cannot secure his payoff if

	0	$(0,1)$	1
0	$(\frac{15}{2},10)$	$(8,\frac{17}{2})$	$(\frac{19}{2},\frac{17}{2})$
$(0,1)$	$(9,\frac{19}{2})$	$(9,\frac{19}{2})$	$(\frac{17}{2},\frac{21}{2})$
1	$(\frac{19}{2},9)$	$(9,9)$	$(8,10)$

Figure 2.2: Game E

$x_1 = 1$ *and* $x_2 = 0$*, and player* 2 *cannot secure his payoff if* $x_1 = 0$ *and* $x_2 = 0$*.*

Moreover, this game does not have a Bayesian equilibrium. It is easy to see that player 3 *will choose* $x_3(a) = 0$ *and* $x_3(b) \in (0,1]$*. Consequently, the expected payoff matrix of players* 1 *and* 2 *is* E*. However, the game* E *has no equilibrium.*

Remark 23. *The game in Example 5 is obviously compact and satisfies the private information measurability requirement. We need to show that the Bayesian game is quasiconcave. It is clear that the expected utility of player* 3 *is quasiconcave. Now we consider players* 1 *and* 2*. Given* $x_3 = (x_3(a), x_3(b))$*. If* $x_3 = (0,0)$*, then players* 1 *and* 2 *will play the game* L *in both states. Their expected payoff matrix is* L*, which is quasiconcave. If* $x_3 \in (0,1] \times (0,1]$*, players* 1 *and* 2 *will play the game* R *in both states, and hence their expected payoff matrix is the quasiconcave game* R*. Otherwise, players* 1 *and* 2 *will play the game* L *at one state and the game* R *at the other state. That is, their expected payoff matrix is* E*, which is also quasiconcave.*

2.1.3.2 Comparison with Monteiro and Page (2007)

Below, we compare our notion of finite payoff security with the uniform payoff security of Monteiro and Page (2007).

The following condition is due to Monteiro and Page (2007).

Definition 8. *The game G_d is **uniform payoff secure** if for every $i \in I$ and $x_i \in X_i$, $\forall \epsilon > 0$, there is $\overline{x_i} \in X_i$ such that for every $x_{-i} \in X_{-i}$, $u_i(\overline{x_i}, y_{-i}) \geq u_i(x_i, x_{-i}) - \epsilon$ for all y_{-i} in some open neighborhood of x_{-i}.*

A game G is uniformly payoff secure if each player starting at any strategy $x_i \in X_i$ has a strategy $\overline{x_i} \in X_i$ he can move to in order to secure a payoff of $u_i(x_i, x_{-i})$ against all possible small deviations of all strategy profiles of others. It is obvious that the uniform payoff security condition is stronger than our finite payoff security condition. Below, we provide an example which shows that the uniform payoff security is strictly stronger than the finite payoff security condition.

Example 6. *Given a deterministic game G such that $I = \{1, 2\}$, $X_1 = X_2 = [0, 1]$,*

$$u_1(x_1, x_2) = \begin{cases} -1, & \text{if } x_1 < x_2 < \frac{1}{2}(x_1 + 1); \\ 0, & \text{if } x_1 = x_2 \text{ or } x_1 = 2x_2 - 1; \\ 1, & \text{otherwise.} \end{cases}$$

and $u_2 \equiv 0$.

We shall show that this game is finitely payoff secure, but not uniformly payoff secure. We only need to verify this for player 1. Fix arbitrary $n \in \mathbb{N}$. Pick $(x_1, x_2^1, \ldots, x_2^n) \in X_1 \times X_2^n$. Without loss of generality, assume that $x_2^1 < x_2^2 < \cdots < x_2^n$. If $x_2^n < 1$, then choose $\overline{x_1} = 1$; if $x_2^n = 1$, then choose $\overline{x_1}$ sufficiently close to 1 such that $x_2^{n-1} < \overline{x_1} < 1$. In all these cases we can find a neighborhood $O_{x_2^k}$ of x_2^k such that $u_1(\overline{x_1}, y_2^k) \geq u_1(x_1, x_2^k)$ for all $y_2^k \in O_{x_2^k}$, $1 \leq k \leq n$.

However, the uniform payoff security condition is not satisfied in this game. Thus, the uniform payoff security condition is strictly stronger than the finite payoff security condition.

2.1.3.3 Extension of Theorem 9 to a continuum of states

By modifying the uniform payoff security condition of Monteiro and Page (2007) and adopting the standard absolute continuity condition of Milgrom and Weber (1985), Carbonell-Nicolau and McLean (2015) proved the existence of behavioral/distributional strategy Bayesian equilibrium in the setting of a continuum of states. They do not need to impose the quasiconcavity condition on the payoffs since the concavity property is automatic by working with behavioral/distributional strategies. We will show that our Theorem 9 can be extended to the setting of a continuum of states by strengthening the finite payoff security to uniform payoff security.

The model of **Bayesian games with a continuum of states** is as follows.

- The set of players: $I = \{1, 2, \ldots, N\}$.
- The set of **actions** available to each player: $\{X_i\}_{i\in I}$. Each X_i is a nonempty compact metric space endowed with the Borel σ-algebra $\mathcal{B}(X_i)$. Let $X = \times_{i=1}^{n} X_i$ and $\mathcal{B}(X) = \otimes_{i\in I}\mathcal{B}(X_i)$.
- The **(private) information space** for each player: T_i. Each T_i is a measurable space endowed with a σ-algebra $\mathcal{T}_i$. Let $T = \times_{i=1}^{n} T_i$ and $\mathcal{T} = \otimes_{i=1}^{n}\mathcal{T}_i$.
- The **payoff functions**: $\{u_i\}_{i\in I}$. Each $u_i \colon X \times T_i \to \mathbb{R}$ is a bounded measurable mapping.

- The **information structure**: λ, a probability measure on the measurable space $(T, \mathcal{T})$ with marginal λ_i on T_i for each $i \in I$.

The following condition is an extension of Definition 8 to the case of incomplete information games, and it is due to Carbonell-Nicolau and McLean (2015). Based on this condition, Carbonell-Nicolau and McLean (2015) proved the existence of a behavioral strategy equilibrium (see Theorem 1 therein).

Definition 9. *The Bayesian game is uniformly payoff secure if for each $i \in I$, $\epsilon > 0$, and a behavioral strategy f_i, there exists another behavioral strategy g_i such that for all (t, x_{-i}), there exists a neighborhood $V_{x_{-i}}$ of x_{-i} such that*

$$u_i(t_i, g_i(t_i), y_{-i}) > u_i(t_i, f_i(t_i), x_{-i}) - \epsilon$$

for all $y_{-i} \in V_{x_{-i}}$.

Below, we consider the purification of behavioral strategies. He and Sun (2014) proposed the "relative diffuseness" condition as a characterization of the relation between two kinds of diffuseness of information, and proved a purification theorem for Bayesian games based on this condition.

For each $i \in I$, let $(T_i, \mathcal{T}_i, \lambda_i)$ be the private information space, and $\mathcal{F}_i \subseteq \mathcal{T}_i$ be the smallest σ-algebra relative to which u_i is measurable. The σ-algebras $\mathcal{T}_i$ and $\mathcal{F}_i$ will represent the diffuseness of information from the aspect of strategies and from the aspect of payoffs, respectively. The probability spaces $(T_i, \mathcal{T}_i, \lambda_i)$ and $(T_i, \mathcal{F}_i, \lambda_i)$ will be used to model the information space and the payoff-relevant information space.

For any nonnegligible subset $D \in \mathcal{T}_i$, the restricted probability space $(D, \mathcal{F}_i^D, \lambda_i^D)$

is defined as follows: $\mathcal{F}_i^D$ is the σ-algebra $\{D \cap D' : D' \in \mathcal{F}_i\}$ and λ_i^D the probability measure re-scaled from the restriction of λ_i to $\mathcal{F}_i^D$. Furthermore, $(D, \mathcal{T}_i^D, \lambda_i^D)$ can be defined similarly.

Definition 10. *Following the notations above, $\mathcal{F}_i$ is said to be **setwise coarser** than $\mathcal{T}_i$ if for every $D \in \mathcal{T}_i$ with $\lambda_i(D) > 0$, there exists a $\mathcal{T}_i$-measurable subset D_0 of D such that $\lambda_i(D_0 \triangle D_1) > 0$ for any $D_1 \in \mathcal{F}_i^D$.*

The following assumption due to He and Sun (2014) states that on any non-negligible set $D \subseteq T_i$, $\mathcal{T}_i^D$ is always larger than $\mathcal{F}_i^D$. That is, the strategy-relevant diffuseness of information is essentially richer than the payoff-relevant diffuseness of information.

Assumption 1 (RD)**.** *For each $i \in I$, $(T_i, \mathcal{T}_i, \lambda_i)$ is atomless and $\mathcal{F}_i$ is setwise coarser than $\mathcal{T}_i$.*

Now we are ready to prove the existence of a pure strategy Bayesian equilibrium with a continuum of states.

Theorem 10. *Suppose that*

1. *Assumption (RD) holds,*[7] *u_i is measurable with respect to $\mathcal{F}_i$ for each $i \in I$, and $\lambda = \otimes_{i \in I} \lambda_i$;*
2. *the Bayesian game is uniformly payoff secure and that each $t \in T$, the map $\sum_{i \in I} u_i(t_i, \cdot) \colon X \to \mathbb{R}$ is upper semicontinuous.*

[7] Instead, one can assume that $(T_i, \mathcal{T}_i, \lambda_i)$ is an atomless Loeb space/saturated space for each $i \in I$. The purification result for Bayesian games still holds, see Loeb and Sun (2006) and Wang and Zhang (2012).

Then there exists a pure strategy Bayesian equilibrium.

Proof. By Theorem 1 of Carbonell-Nicolau and McLean (2015), there exists a behavioral strategy Bayesian equilibrium f, and by Theorem 2 of He and Sun (2014), f has a purification g, which is a pure strategy Bayesian equilibrium. □

Remark 24. *In an incomplete information game with finitely many states, one can work with the k-payoff security condition, where k could be the number of all states in the Bayesian game. However, as we consider a Bayesian game with countably many states, we need to extend the k-payoff security to finite payoff security as we may need to use a Bayesian game with arbitrarily finitely many states to approximate the original Bayesian game. Monteiro and Page (2007) proved the existence of a mixed strategy equilibrium $m = (m_1, \ldots, m_n)$ with the stronger condition of uniform payoff security in a simple deterministic setting. Indeed, their result can be understood as an existence result of a pure strategy Bayesian equilibrium in a Bayesian game with uncountable states and state-irrelevant payoffs. Thus, they need to further strengthen the condition due to the larger size of the state space.*

In particular, suppose that each player can only observe his own private signal from the unit interval $[0,1]$, which is endowed with the uniform distribution η. Let $\Omega = [0,1]^n$ be the state space. The payoff of each player only depends on the action profile, but not on the state. Then the deterministic game is reformulated as a Bayesian game with uncountable states and state-irrelevant payoffs.

The mixed strategy m_i of player i in the deterministic game can be realized by his private signal (like a randomization device) to be a pure strategy f_i in the sense *that* $m_i = \eta \circ f_i^{-1}$. *It is easy to check that* $f = (f_1, \ldots, f_n)$ *is a pure strategy Bayesian equilibrium in this Bayesian game.*

If we view a deterministic discontinuous game as such a Bayesian game, then $\mathcal{F}_i = \{\emptyset, [0,1]\}$ *for each* $i \in I$ *since players' payoffs do not depend on the states. Thus, our Assumption 1 holds trivially, and our result goes beyond Monteiro and Page (2007) by allowing for the payoffs to be state-dependent.*

Remark 25. *If one views a deterministic game as a Bayesian game with uncountable states and let* f_i *and* g_i *be pure strategies and state irrelevant in Definition 9, then this condition reduces to the uniform payoff security in the sense of Monteiro and Page (2007). In Theorem 10, we adopt the uniform payoff security condition in the sense of Carbonell-Nicolau and McLean (2015) since our payoffs are state dependent, and thus the best response of each player is typically state dependent. Therefore, one needs to compare the state dependent strategies for each player; for more discussions, see Carbonell-Nicolau and McLean (2015).*

2.2 Discontinuous Games with Incomplete Information and Ambiguity

2.2.1 Existence of Equilibrium under Ambiguity

The framework is the same as specified in the previous section.

2.2.1.1 General case under ambiguity

In the following, we shall consider discontinuous games with incomplete information and ambiguity. Suppose that players could have multiple priors and are ambiguous. We follow the non-expected utility approach by adopting the notion of maximin preferences of Gilboa and Schmeidler (1989).

For each player i, let M_i be the set of his possible priors such that for any priors $\pi_i, \pi_i' \in M_i$, $\pi_i(E) = \pi_i'(E)$ for any $E \in \mathcal{F}_i$. That is, priors must be consistent with each other on player i's private information partition. Without loss of generality, we assume that $\pi_i(E) > 0$ for any $E \in \mathcal{F}_i$ and $\pi_i \in M_i$.

Given a strategy profile $f \in \mathcal{L}$, the **maximin expected utility (MEU)** of player i is

$$V_i(f) = \inf_{\pi_i \in M_i} \sum_{\omega \in \Omega} u_i(f(\omega), \omega)\pi_i(\omega).$$

The **ex ante game** is denoted by $G_0 = (V_i, \mathcal{L}_i)_{i \in I}$.

Definition 11. 1. *When players have maximin preferences, a strategy profile $f \in \mathcal{L}$ is said to be an equilibrium if it is a Nash equilibrium in the game G_0.*

2. *Suppose that M_i is a singleton set, and player i is restricted to choose f_i which is measurable with respect to $\mathcal{F}_i$ for each i. Then f is said to be a Bayesian equilibrium if it is a Nash equilibrium in the ex ante game.*

Remark 26. *If M_i is a singleton set for each agent i, then the maximin expected utility above reduces to the standard Bayesian expected utility. If M_i is the set of all probability measures on $\mathcal{F}$ which agree with each other on $\mathcal{F}_i$, then it is the maximin expected utility considered in de Castro and Yannelis (2009).*

In games with maximin preferences, priors must be consistent on the information partition of each player. The information asymmetry is captured by the MEU, and hence it is natural to relax the restriction of private information measurability.

On the contrary, the information asymmetry in a Bayesian model is captured by the assumption of private information measurability of the strategy set of each player; that is, each f_i is assumed to be private information measurable. If the private information measurability condition is relaxed in the Bayesian setup, then the game is reduced to be symmetric information.

It is demonstrated via counterexamples in the previous section that a Bayesian equilibrium may not exist in a discontinuous game with Bayesian preferences. They resolved this issue by proposing the "finite payoff security" condition. The following result shows that if we adopt the maximin preferences, then the existence of equilibria in the ex ante game follows immediately from the conditions that could guarantee the existence of Nash equilibria in ex post games.

Proposition 5. *If an asymmetric information game G is compact and quasiconcave, every ex post game G_ω is better payoff secure, and players are maximin preference maximizes, then there exists an equilibrium in the ex ante game G_0.*

Proof. Since the ex post game G_ω is compact, quasiconcave and better-reply secure, there exists a Nash equilibrium $f(\omega)$ in G_ω. We claim that f is an equilibrium in the ex ante game G_0.

Suppose otherwise. Then there exists some player i and strategy g_i such that $V_i(f) < V_i(g_i, f_{-i})$. There exists a prior $\pi_i \in M_i$ such that

$$\begin{aligned}\sum_{\omega\in\Omega} u_i(f(\omega),\omega)\pi_i(\omega) &< \inf_{\pi_i'\in M_i}\sum_{\omega\in\Omega} u_i(g_i(\omega), f_{-i}(\omega),\omega)\pi_i'(\omega)\\ &\leq \sum_{\omega\in\Omega} u_i(g_i(\omega), f_{-i}(\omega),\omega)\pi_i(\omega),\end{aligned}$$

which implies that there exists a state $\omega_1 \in E$ such that $\pi_i(\omega_1) > 0$ and

$$u_i(f(\omega_1), \omega_1) < u_i(g_i(\omega_1), f_{-i}(\omega_1), \omega_1).$$

This is a contradiction. Therefore, f is an equilibrium of G_0. □

Remark 27. *We would like to emphasize that in the setting where players adopt the Bayesian preferences and each ex post game is compact, quasiconcave and better payoff secure, even if we do not require the private information measurability for any player's strategy, Reny (1999)'s theorem is still not applicable to conclude the existence of an equilibrium in the ex ante game. Indeed, the condition that every ex post game is quasiconcave is not sufficient to guarantee the quasiconcavity of the ex ante game.*

2.2.2 Timing Games with Asymmetric Information

We study a class of two-person, non-zero-sum, noisy timing games with asymmetric information. Such games can be used to model behavior in duels as well as in R&D and patent races.

Let G be an asymmetric information timing game. The state space is Ω. For player i, the information partition is denoted as $\mathcal{F}_i$ and the private prior π_i is defined on $\mathcal{F}_i$. The action space for both players is $[0, 1]$. At state ω, the payoff of player i is given by

$$u_i(a_i, a_{-i}, \omega) = \begin{cases} p_i(x_i, \omega), & \text{if } x_i < x_{-i} \\ q_i(x_i, \omega), & \text{if } x_i = x_{-i} \\ h_i(x_{-i}, \omega), & \text{otherwise} \end{cases}.$$

Suppose that the following conditions hold for $i = 1, 2$, $\omega \in \Omega$ and $x \in [0, 1]$,

1. $p_i(\cdot, \omega)$ and $h_i(\cdot, \omega)$ are both continuous and $p_i(\cdot, \omega)$ is nondecreasing,
2. $q_i(x, \omega) \in \mathrm{co}\{p_i(x, \omega), h_i(x, \omega)\}$,[8]
3. if $q_i(x, \omega) + q_{-i}(x, \omega) < p_i(x, \omega) + h_{-i}(x, \omega)$, then $\mathrm{sgn}(p_i(x, \omega) - q_i(x, \omega)) = \mathrm{sgn}(q_{-i}(x, \omega) - h_{-i}(x, \omega))$.[9]

As shown in Reny (1999), each ex post game is compact, quasiconcave and payoff secure. We claim that each ex post game is reciprocal upper semicontinuous. If $q_i(x, \omega) + q_{-i}(x, \omega) \leq p_i(x, \omega) + h_{-i}(x, \omega)$, then we have that $\mathrm{sgn}(p_i(x, \omega) - q_i(x, \omega)) = \mathrm{sgn}(q_{-i}(x, \omega) - h_{-i}(x, \omega))$. This case has already been shown in Reny (1999), we only need to consider the case that $q_i(x, \omega) + q_{-i}(x, \omega) > p_i(x, \omega) + h_{-i}(x, \omega)$. The reciprocal upper semicontinuity in the latter case is obvious since there must be some $i \in \{1, 2\}$ such that $q_i(x, \omega) > p_i(x, \omega)$ or $q_i(x, \omega) > h_i(x, \omega)$. Therefore, if the conditions above hold and players are maximin preference maximizers, then this asymmetric information timing game has an ex ante equilibrium due to Proposition 5.

The following example shows that an asymmetric information timing game

[8]The notation $\mathrm{co}(A)$ denotes the convex hull of the set A.

[9]Notice that this condition is slightly weaker than the corresponding condition in Example 3.1 of Reny (1999). Example 7 satisfies our condition, but does not satisfy the condition of Reny (1999).

may not possess an equilibrium if players have Bayesian preferences. However, this example has an equilibrium when all players have maximin preferences.

Example 7 (Nonexistence of Bayesian equilibria)**.**

The state space is $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, *where*

$$\omega_1 = (\frac{1}{2}, \frac{1}{2}),\ \omega_2 = (\frac{1}{2}, 1),\ \omega_3 = (1, 1),\ \omega_4 = (1, \frac{1}{2}).$$

The information partitions are

$$\mathcal{F}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\},\ \mathcal{F}_2 = \{\{\omega_1, \omega_4\}, \{\omega_2, \omega_3\}\}.$$

The ex post utility functions of players at state $\omega = (t_1, t_2)$ *are given as in the general model, where* $p_i(x, \omega) = x - t_i$, $h_i(x, \omega) \equiv 0$ *and*

$$q_i(x, \omega) = \begin{cases} x - t_i, & \text{if } t_i < t_{-i}; \\ \frac{x - t_i}{2}, & \text{if } t_i = t_{-i}; \\ 0, & \text{if } t_i > t_{-i}. \end{cases}$$

Players 1 *and* 2 *hold the common prior*

$$\pi(\omega_1) = \pi(\omega_2) = \pi(\omega_3) = \frac{1}{3}, \quad \pi(\omega_4) = 0.$$

It is easy to see that this game satisfies all the specified conditions, and hence by Proposition 5, it possesses an equilibrium when both players are maximin preference maximizers. We claim that there is no Bayesian equilibrium in this game. By way of contradiction, suppose that (x_1, x_2) *is a Bayesian equilibrium.*

We shall first show that $x_i(\omega) \geq t_i$ *at state* $\omega = (t_1, t_2)$ *for* $i = 1, 2$. *It is clear that* $x_1(\omega), x_2(\omega) \geq \frac{1}{2}$ *for any* $\omega \in \Omega$, *hence we only need to show* $x_1(\omega_3) = x_1(\omega_4) = 1$

and $x_2(\omega_2) = x_2(\omega_3) = 1$. *Suppose that* $x_1(\omega_3) = x_1(\omega_4) < 1$. *If* $x_2(\omega_3) < x_1(\omega_3)$, *then player* 2 *gets a negative payoff at the event* $\{\omega_2, \omega_3\}$, *and he can choose* $x_2(\omega_2) = x_2(\omega_3) = 1$ *to be strictly better off. If* $x_2(\omega_3) \geq x_1(\omega_3)$, *then player* 1 *gets a negative payoff at the event* $\{\omega_3, \omega_4\}$, *and he can choose* $x_1(\omega_3) = x_1(\omega_4) = 1$ *to be strictly better off. Thus,* $x_1(\omega_3) = x_1(\omega_4) = 1$. *Similarly, we can check that* $x_2(\omega_2) = x_2(\omega_3) = 1$, *as player* 2 *will otherwise get a negative payoff at the event* $\{\omega_2, \omega_3\}$.

Now we consider the choice of player 2 *at state* ω_1.

1. *If* $x_2(\omega_1) = \frac{1}{2}$, *then the best response of player* 1 *at the event* $\{\omega_1, \omega_2\}$ *is to choose the strategy* $x_1(\omega_1) = x_1(\omega_2) = 1$. *However, in this case, there is no best response for player* 2 *at the state* ω_1.

2. *If* $x_2(\omega_1) = 1$, *then there is no best response for player* 1 *at the event* $\{\omega_1, \omega_2\}$.

3. *Suppose that* $x_2(\omega_1) = a \in (\frac{1}{2}, 1)$. *If* $x_1(\omega_1) = x_1(\omega_2) \in [\frac{1}{2}, a)$, *then player* 1 *can always slightly increase his strategy to be strictly better off. If* $x_1(\omega_1) = x_1(\omega_2) = a$, *then player* 1 *can always slightly decrease his strategy to be strictly better off. If* $x_1(\omega_1) = x_1(\omega_2) \in (a, 1]$, *then the best response of player* 1 *must be* $x_1(\omega_1) = x_1(\omega_2) = 1$, *which implies that there is no best response for player* 2 *as shown in point (1).*

Therefore, there is no Bayesian equilibrium.

2.3 Behavioral Strategy Bayesian Equilibria in Discontinuous Games with Incomplete Information

2.3.1 Model

2.3.1.1 Bayesian game and behavioral-strategy equilibrium

We consider a **Bayesian game** as follows:

$$G = \{u_i, X_i, (T_i, \mathcal{T}_i), \lambda\}_{i \in I}.$$

- There is **a finite set of players**, $I = \{1, 2, \ldots, n\}$.
- Player i's **action space** X_i is a nonempty compact metric space, which is endowed with the Borel σ-algebra $\mathcal{B}(X_i)$. Denote $X = \prod_{i \in I} X_i$ and $\mathcal{B}(X) = \otimes_{i \in I} \mathcal{B}(X_i)$; that is, $\mathcal{B}(X)$ is the product Borel σ-algebra.
- The measurable space $(T_i, \mathcal{T}_i)$ represents the **private information space** of player i. Let $T = \prod_{i \in I} T_i$ and $\mathcal{T} = \otimes_{i \in I} \mathcal{T}_i$.
- The **common prior** λ is a probability measure on the measurable space $(T, \mathcal{T})$.
- For every player $i \in I$, $u_i : X \times T \to \mathbb{R}_+$ is a $\mathcal{B}(X) \otimes \mathcal{T}$-measurable function representing the **payoff** of player i, which is bounded by some $\gamma > 0$.[10]

As usual, we write t_{-i} for an information profile of all players other than i, and T_{-i} as the space of all such information profiles. We adopt similar notation for action profiles, strategy profiles and payoff profiles.

For every player $i \in I$, a **pure strategy** is a $\mathcal{T}_i$-measurable function from T_i to X_i. Let $\mathcal{L}_i$ be **the set of all possible pure strategies** of player i, and $\mathcal{L} = \prod_{i \in I} \mathcal{L}_i$.

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[10]Since u_i is bounded, we can assume that u_i takes values in $\mathbb{R}_+$ without loss of generality.

A **behavioral strategy** of player i is a $\mathcal{T}_i$-measurable function from T_i to $\triangle(X_i)$, where $\triangle(X_i)$ denotes the space of all Borel probability measures on X_i under the topology of weak convergence.[11] A pure strategy can be viewed as a special case of a behavioral strategy by considering it as a Dirac measure for every t_i. **The set of all behavioral strategies** of player i is denoted by $\mathcal{M}_i$, and $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i$.

Given a behavioral strategy profile $f = (f_1, f_2, \ldots, f_n) \in \mathcal{M}$, the **expected payoff** of player i is

$$U_i(f) = \int_T \int_{X_1} \cdots \int_{X_n} u_i(x_1, \ldots, x_n, t_1, \ldots, t_n) f_n(\mathrm{d}x_n|t_n) \ldots f_1(\mathrm{d}x_1|t_1) \lambda(\mathrm{d}t).$$

Definition 12. *A* ***behavioral-strategy equilibrium*** *is a behavioral strategy profile $f^* = (f_1^*, f_2^*, \ldots, f_n^*) \in \mathcal{M}$ such that f_i^* maximizes $U_i(f_i, f_{-i}^*)$ for any $f_i \in \mathcal{M}_i$ and each player $i \in I$.*[12]

We impose the following assumption on the information structure. Let λ_i be the marginal probability of λ on $(T_i, \mathcal{T}_i)$ for each $i \in I$. Suppose that $(T, \mathcal{T}, \lambda)$ and $(T_i, \mathcal{T}_i, \lambda_i)$ are complete probability measure spaces.

Assumption 2 (Absolue Continuity (AC))**.** *The probability measure λ is absolutely*

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[11] That is, a behavioral strategy f_i is a transition probability from $(T_i, \mathcal{T}_i)$ to $(X_i, \mathcal{B}(X_i))$ such that $f_i(\cdot|t_i)$ is a probability measure on $(X_i, \mathcal{B}(X_i))$ for all $t_i \in T_i$, and $f_i(B|\cdot)$ is a $\mathcal{T}_i$-measurable function for every $B \in \mathcal{B}(X_i)$. If λ_i is a probability measure on $(T_i, \mathcal{T}_i)$, then $\lambda_i \diamond f_i$ denotes a probability measure on $T_i \times X_i$ such that $\lambda_i \diamond f_i(A \times B) = \int_A f_i(B|t_i) \lambda_i(\mathrm{d}t_i)$ for any measurable subsets $A \subseteq T_i$ and $B \subseteq X_i$.

[12] Milgrom and Weber (1985) considered distributional strategies and Balder (1988) extended their results to behavioral strategies. As remarked in Milgrom and Weber (1985), every behavioral strategy gives rise to a natural distributional strategy, and every distributional strategy corresponds to an equivalent class of behavioral strategies defined as the induced regular conditional probabilities. We consider behavioral strategies for simplicity, but all the results can be easily extended to distributional strategies.

continuous with respect to $\otimes_{i\in I}\lambda_i$ *with the corresponding Radon-Nikodym derivative* $\psi\colon T \to \mathbb{R}_+$.

This assumption is widely adopted in the setting of Bayesian games; see, for example, Milgrom and Weber (1985), Balder (1988), Jackson et al. (2002), Loeb and Sun (2006) and Carbonell-Nicolau and McLean (2015). Notice that the (AC) assumption is imposed even when the payoff function is continuous in the action variables. If players have independent priors in the sense that $\lambda = \otimes_{i\in I}\lambda_i$, then the (AC) assumption holds trivially.

2.3.1.2 Normal form game

Below, we convert a Bayesian game G to an (ex ante) normal form game G_0. If one can show the existence of a Nash equilibrium in the game G_0, then this equilibrium corresponds to a behavioral-strategy equilibrium in the original Bayesian game G.

A normal form game G_d is a collection $(X_i, u_i)_{i\in I}$, where X_i and u_i are the action space and payoff function of player i, respectively. We view a Bayesian game G as a normal norm game and denote it by $G_0 = (\mathcal{M}_i, U_i)_{i\in I}$, where $\mathcal{M}_i$ is the set of all possible behavioral strategies, and U_i is the expected payoff function of player i.

A **Nash equilibrium** in the game G_0 is a strategy profile $f^* = (f_1^*, f_2^*, \ldots, f_n^*) \in \mathcal{M}$ such that f_i^* maximizes $U_i(f_i, f_{-i}^*)$ for any $f_i \in \mathcal{M}_i$ and each player $i \in I$. Thus, if f^* is a Nash equilibrium in the game G_0, then it is also a behavioral-strategy equilibrium in the original Bayesian game G.

2.3.1.3 Main result

To prove that the mixed extension of a normal form game is payoff secure, Allison and Lepore (2014) introduced the interesting notion of "disjoint payoff matching" in games with complete information. Below, we extend this notion to the setting of Bayesian games, and show that the ex ante game G_0 is payoff secure.

Consider the points at which a player's payoff function is discontinuous in other players' strategies. In particular, let $D_i \colon T_i \times X_i \to T_{-i} \times X_{-i}$ be defined by

$$D_i(t_i, x_i) = \{(t_{-i}, x_{-i}) \in T_{-i} \times X_{-i} \colon u_i(x_i, \cdot, t_i, t_{-i}) \text{ is discontinuous in } x_{-i}\}.$$

Suppose that D_i has a $\mathcal{B}(X) \otimes \mathcal{T}$-measurable graph for each $i \in I$. Given a pure strategy f_i of player i, denote $D_i^{f_i}(t_i) = D_i(t_i, f_i(t_i))$.

Remark 28. *In many applications such as auctions and price competition, the discontinuity arises due to the action variables, and independently of the state variables. That is, the correspondence D_i does not depend on T in the sense that if $(t, x) \in Gr(D_i)$, then $(t', x) \in Gr(D_i)$ for any $t' \in T$. It is usually easy to check that D_i has a measurable graph in such cases.*[13]

Definition 13. *A Bayesian game G is said to satisfy the condition of "random disjoint payoff matching" if for any $f_i \in \mathcal{L}_i$, there exists a sequence of deviations $\{g_i^k\}_{k=1}^{\infty} \subseteq \mathcal{L}_i$ such that the following conditions hold:*

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[13] If A is a correspondence from a space Y to Z, then $\mathrm{Gr}(A) \subseteq Y \times Z$ denotes the graph of A.

1. *for λ-almost all $t = (t_i, t_{-i}) \in T$ and all $x_{-i} \in X_{-i}$,*

$$\liminf_{k\to\infty} u_i(g_i^k(t_i), x_{-i}, t_i, t_{-i}) \geq u_i(f_i(t_i), x_{-i}, t_i, t_{-i});$$

2. $\limsup_{k\to\infty} D_i(t_i, g_i^k(t_i)) = \emptyset$ *for any $i \in I$ and λ_i-almost all $t_i \in T_i$.*[14]

When T_i is a singletons set for each player $i \in I$, the above definition reduces to be the notion of disjoint payoff matching introduced by Allison and Lepore (2014) in a complete information environment.

The following theorem shows that the random disjoint payoff matching condition of a Bayesian game G could guarantee the payoff security of the game G_0. Its proof is provided in Section 2.3.2.

Theorem 11. *Under Assumption (AC), if a Bayesian game G satisfies the random disjoint payoff matching condition, then the game G_0 is payoff secure.*

2.3.1.4 Existence of behavioral-strategy equilibria

Theorem 11 above shows that the random disjoint payoff matching condition of a Bayesian game G guarantees the payoff security of the ex ante game G_0. Reny (1999) showed that a payoff secure game has a pure-strategy Nash equilibrium provided that the game has compact action spaces, and each player's payoff function is quasiconcave in his own actions and satisfies some upper semicontinuity condition.

In the following theorem, we prove the existence of behavioral-strategy equilibria in Bayesian games based on Theorem 11.

[14]For a sequence of sets $\{A_k\}$, $\limsup_{k\to\infty} A_k = \cap_{k=1}^{\infty} \cup_{j=k}^{\infty} A_j$ and $\liminf_{k\to\infty} A_k = \cup_{k=1}^{\infty} \cap_{j=k}^{\infty} A_j$.

Theorem 12. *Suppose that a Bayesian game G satisfies the random disjoint payoff matching condition and Assumption (AC). Furthermore, suppose that the mapping $\sum_{i\in I} u_i(\cdot, t)\colon X \to \mathbb{R}$ is upper semicontinuous for each $t \in T$. Then the game G_0 has a Nash equilibrium, which is a behavioral-strategy equilibrium for the Bayesian game G.*

Proof. By Theorem 11, the game G_0 is payoff secure. Then applying Lemma 3 in Carbonell-Nicolau and McLean (2015), the mapping

$$f \in \mathcal{M} \to \sum_{i\in I} U_i(f)$$

is upper semicontinuous. By Proposition 3.2 and Theorem 3.1 in Reny (1999), the game G_0 has a Nash equilibrium, which implies that G has a behavioral-strategy equilibrium. □

Remark 29. *By extending the uniform payoff security condition of Monteiro and Page (2007) and adopting the (AC) assumption, Carbonell-Nicolau and McLean (2015) proved the existence of behavioral/distributional-strategy equilibria in Bayesian games with discontinuous payoffs. In particular, they showed that the ex ante game G_0 is payoff secure when the Bayesian game G satisfies their uniform payoff security condition. Our result does not cover the result of Carbonell-Nicolau and McLean (2015) and vice versa. Notice that our condition only needs to check the payoffs at each strategy profile itself, but not for those payoffs in the neighborhood of the strategy profile.*

Remark 30. *By adopting the "relative diffuseness" condition of He and Sun (2014)* *and the "uniform payoff security" condition of Carbonell-Nicolau and McLean (2015), He and Yannelis (2015b) presented a purification result for behavioral-strategy equilibrium in Bayesian games with discontinuous payoffs. It is straightforward to check that one can also obtain the existence of pure-strategy equilibria here via a similar purification result based on the "relative diffuseness" condition and Theorem 12.*

2.3.2 Proof of Theorem 11

2.3.2.1 Preparatory results

The proof of Theorem 11 is based on a clever argument of Allison and Lepore (2014). However, our incomplete information framework introduces several subtle difficulties and necessitates new arguments that are far from trivial. Below, we present some technical results needed for the proof of Theorem 11.

We first consider the topology on the space $\mathcal{M}_i$ for each $i \in I$. Let $\mathcal{H}_i$ be the space of uniformly finite transition measures from $(T_i, \mathcal{T}_i, \lambda_i)$ to $(X_i, \mathcal{B}(X_i))$.

Definition 14. *The weak topology on $\mathcal{H}_i$ is the weakest topology with respect to which the functional*

$$\nu \to \int_{T_i} \int_{X_i} c(t_i, x_i)\nu(\mathrm{d}x_i|t_i)\lambda_i(\mathrm{d}t_i)$$

is continuous on $\mathcal{H}_i$ for every integrably bounded Carathéodory function c.[15]

The set $\mathcal{M}_i$ can be viewed as a subspace of $\mathcal{H}_i$ endowed with its relative

[15]The function c is said to be a Carathéodory function if $c(\cdot, x_i)$ is $\mathcal{T}_i$-measurable for each $x_i \in X_i$ and $c(t_i, \cdot)$ is continuous on X_i for each $t_i \in T_i$. In addition, c is called integrably bounded if there exists a λ_i-integrable function $\chi\colon T_i \to \mathbb{R}_+$ such that $|c(t_i, x_i)| \le \chi(t_i)$ for all $(t_i, x_i) \in T_i \times X_i$.

topology. The Cartesian product $\mathcal{M} = \Pi_{i\in I}\mathcal{M}_i$ is endowed with the usual product topology.

The following lemma shows that each player i in the game G_0 has a nonempty, convex and weakly compact strategy space $\mathcal{M}_i$.

Lemma 3. *$\mathcal{M}_i$ is a nonempty, convex and weakly compact subset of the topological vector space $\mathcal{H}_i$.*

Proof. It is obvious that $\mathcal{M}_i$ is nonempty and convex. The weak compactness of $\mathcal{M}_i$ is from Theorem 2.3 of Balder (1988). $\square$

Lemma 4. *If $\mathcal{M}_i$ is viewed as a subspace of $\mathcal{H}_i$ endowed with its relative topology, then the functional*

$$\nu \to \int_{T_i}\int_{X_i} c(t_i, x_i)\nu(\mathrm{d}x_i|t_i)\lambda_i(\mathrm{d}t_i)$$

is lower semicontinuous for every function $c\colon T_i \times X_i \to (-\infty, +\infty]$ such that

1. *$c(t_i, \cdot)$ is lower semicontinuous on X_i for every $t_i \in T_i$;*
2. *c is $\mathcal{T}_i \otimes \mathcal{B}(X_i)$-measurable;*
3. *c is integrably bounded from below in the sense that there exists some integrable function $h\colon T_i \to \mathbb{R}$ such that $c(t_i, x_i) \geq h(t_i)$ for all $t_i \in T_i$ and $x_i \in X_i$.*

Proof. Lemma 4 is from Theorem 2.2 (a) in Balder (1988). $\square$

For any nonempty subset $J \subseteq I$, let $\tilde{\mathcal{M}}_J$ be the space of transition probabilities from $(\prod_{j\in J} T_j, \otimes_{j\in J}\mathcal{T}_j, \otimes_{j\in J}\lambda_j)$ to $\prod_{j\in J} X_j$, and $\tilde{\mathcal{H}}_J$ the space of uniformly finite transition measures from $(\prod_{j\in J} T_j, \otimes_{j\in J}\mathcal{T}_j, \otimes_{j\in J}\lambda_j)$ to $\prod_{j\in J} X_j$. Suppose that $\tilde{\mathcal{H}}_J$ is endowed with the weak topology as defined in Definition 14, and $\tilde{\mathcal{M}}_J$ is viewed as a subset of $\tilde{\mathcal{H}}_J$ endowed with the relative topology.

Lemma 5. *The mapping $(f_j)_{j\in J} \to \otimes_{j\in J} f_j$ from $\prod_{j\in J} \mathcal{M}_j$ to $\tilde{\mathcal{M}}_J$ is continuous.*

Proof. Theorem 2.5 in Balder (1988) considers the case that J has two elements. It is obvious that his argument still holds for any finite set J. $\square$

In the proof of our Theorem 11, we need to deal with some subtle measurability issues based on the projection theorem and Aumann's measurable selection theorem. These theorems are stated below for the convenience of the reader.

Projection Theorem: Let X be a Polish space and $(S, \mathcal{S}, \mu)$ a complete finite measure space. If a set E belongs to $\mathcal{S} \otimes \mathcal{B}(X)$, then the projection of E on S belongs to $\mathcal{S}$.

Aumann's measurable selection theorem: Let X be a Polish space and $(S, \mathcal{S}, \mu)$ a complete finite measure space. Suppose that F is a nonempty valued correspondence from S to X having an $\mathcal{S} \otimes \mathcal{B}(X)$-measurable graph. Then F admits a measurable selection; that is, there is a measurable function f from S to X such that $f(s) \in F(s)$ for μ-almost all $s \in S$.

2.3.2.2 Proof

We now proceed with the proof of Theorem 11.

Fix a behavioral strategy profile $(f_1, \ldots, f_n) \in \mathcal{M}$, a player $i \in I$ and $\epsilon > 0$.

Let $S_i\colon T_i \to X_i$ be a correspondence defined by

$$S_i(t_i) = \{x_i \in X_i\colon \int_{T_{-i}} \int_{X_{-i}} u_i(x_i, x_{-i}, t_i, t_{-i}) \psi(t_i, t_{-i}) f_{-i}(\mathrm{d}x_{-i}|t_{-i}) \otimes_{j\neq i} \lambda_i(\mathrm{d}t_{-i})$$
$$\geq \int_{T_{-i}} \int_X u_i(x_i, x_{-i}, t_i, t_{-i}) \psi(t_i, t_{-i}) f(\mathrm{d}x|t_i, t_{-i}) \otimes_{j\neq i} \lambda_i(\mathrm{d}t_{-i})\}.$$

It is obvious that for each fixed t_i, $S_i(t_i)$ is nonempty. Since u_i is jointly measurable, and f and ψ are measurable, the correspondence S_i has a $\mathcal{B}(X_i)\otimes\mathcal{T}_i$-measurable graph. By the Aumann's measurable selection theorem, S_i has a $\mathcal{T}_i$-measurable selection f_i'. Therefore, we have that

$$\int_T \int_{X_{-i}} u_i(f_i'(t_i), x_{-i}, t_i, t_{-i}) f_{-i}(\mathrm{d}x_{-i}|t_{-i}) \lambda(\mathrm{d}t) \geq \int_T \int_X u_i(x_i, x_{-i}, t_i, t_{-i}) f(\mathrm{d}x|t) \lambda(\mathrm{d}t).$$

By the random disjoint payoff matching condition, there exists a sequence of pure strategies $\{g_i^k\} \subseteq \mathcal{L}_i$ such that for λ-almost all $t = (t_i, t_{-i}) \in T$ and all $x_{-i} \in X_{-i}$,

$$\liminf_{k\to\infty} u_i(g_i^k(t_i), x_{-i}, t_i, t_{-i}) \geq u_i(f_i'(t_i), x_{-i}, t_i, t_{-i}),$$

and $\limsup_{k\to\infty} D_i(t_i, g_i^k(t_i)) = \emptyset$ for any $i \in I$ and λ_i-almost all $t_i \in T_i$.

Let

$$E_i^k(t_i) = \{(t_{-i}, x_{-i})\colon u_i(g_i^k(t_i), x_{-i}, t_i, t_{-i}) > u_i(f_i'(t_i), x_{-i}, t_i, t_{-i}) - \epsilon\}.$$

Since the functions u_i, g_i^k and f_i' are all measurable, the correspondence E_i^k has a $\mathcal{B}(X_{-i}) \otimes \mathcal{T}$-measurable graph. For λ-almost all $t \in T$ and all $x_{-i} \in X_{-i}$, since

$$\liminf_{k\to\infty} u_i(g_i^k(t_i), x_{-i}, t_i, t_{-i}) \geq u_i(f_i'(t_i), x_{-i}, t_i, t_{-i}),$$

we have $(t, x_{-i}) \in \liminf_{k\to\infty} \mathrm{Gr}(E_i^k)$. As a result,

$$\lambda \diamond f_{-i} \left(\liminf_{k\to\infty} \mathrm{Gr}(E_i^k)\right) = 1,$$

which implies that

$$\liminf_{k\to\infty} \lambda \diamond f_{-i}\left(\mathrm{Gr}(E_i^k)\right) \geq \lambda \diamond f_{-i}\left(\liminf_{k\to\infty} \mathrm{Gr}(E_i^k)\right) = 1.$$

Thus, $\lim_{k\to\infty} \lambda \diamond f_{-i}\left(\mathrm{Gr}(E_i^k)\right) = 1$.

Notice that the t_i-section of the set $\limsup_{k\to\infty} \mathrm{Gr}(D_i^{g_i^k})$ is $\limsup_{k\to\infty} D_i(t_i, g_i^k(t_i))$, which is the empty set for λ_i-almost all $t_i \in T_i$. Thus, $\lambda \diamond f_{-i}\left(\limsup_{k\to\infty} \mathrm{Gr}(D_i^{g_i^k})\right) = 0$, and

$$\limsup_{k\to\infty} \lambda \diamond f_{-i}\left(\mathrm{Gr}(D_i^{g_i^k})\right) \leq \lambda \diamond f_{-i}\left(\limsup_{k\to\infty} \mathrm{Gr}(D_i^{g_i^k})\right) = 0.$$

As a result, $\lim_{k\to\infty} \lambda \diamond f_{-i}\left(\mathrm{Gr}(D_i^{g_i^k})\right) = 0$.

Therefore, $\lim_{k\to\infty} \lambda \diamond f_{-i}\left(\mathrm{Gr}(E_i^k) \setminus \mathrm{Gr}(D_i^{g_i^k})\right) = 1$. There exists some positive integer $K > 0$ such that for any $k \geq K$,

$$\lambda \diamond f_{-i}\left(\mathrm{Gr}(E_i^k) \setminus \mathrm{Gr}(D_i^{g_i^k})\right) > 1 - \epsilon.$$

Let $g_i = g_i^K$ and $F = \mathrm{Gr}(E_i^K) \setminus \mathrm{Gr}(D_i^{g_i^K})$. Then we have

$$\begin{aligned} &\int_F u_i(g_i(t_i), x_{-i}, t_i, t_{-i}) \lambda \diamond f_{-i}(\mathrm{d}(t_i, t_{-i}, x_{-i})) \\ \geq &\int_F u_i(f_i'(t_i), x_{-i}, t_i, t_{-i}) \lambda \diamond f_{-i}(\mathrm{d}(t_i, t_{-i}, x_{-i})) - \epsilon, \end{aligned}$$

which implies that[16]

$$
\begin{aligned}
&\int_{T\times X_{-i}} u_i(g_i(t_i), x_{-i}, t_i, t_{-i})\lambda \diamond f_{-i}(\mathrm{d}(t_i, t_{-i}, x_{-i})) \\
&= \int_F u_i(g_i(t_i), x_{-i}, t_i, t_{-i})\lambda \diamond f_{-i}(\mathrm{d}(t_i, t_{-i}, x_{-i})) \\
&+ \int_{F^c} u_i(g_i(t_i), x_{-i}, t_i, t_{-i})\lambda \diamond f_{-i}(\mathrm{d}(t_i, t_{-i}, x_{-i})) \\
&\geq \int_F u_i(f_i'(t_i), x_{-i}, t_i, t_{-i})\lambda \diamond f_{-i}(\mathrm{d}(t_i, t_{-i}, x_{-i})) - \epsilon \\
&+ \int_{F^c} u_i(f_i'(t_i), x_{-i}, t_i, t_{-i})\lambda \diamond f_{-i}(\mathrm{d}(t_i, t_{-i}, x_{-i})) - \gamma \cdot \epsilon \\
&= \int_{T\times X_{-i}} u_i(f_i'(t_i), x_{-i}, t_i, t_{-i})\lambda \diamond f_{-i}(\mathrm{d}(t_i, t_{-i}, x_{-i})) - (\gamma + 1)\epsilon.
\end{aligned}
$$

Since X_{-i} is a compact metric space, it is second countable (see Royden and Fitzpatrick (2010, Proposition 25, p.204)). Thus, we can find a countable base $\{V_m\}_{m\geq 1}$ for X_{-i}. Let

$$
h_i^m(x_{-i}, t) = \begin{cases} \inf_{x'_{-i}\in V_m} u_i(g_i(t_i), x'_{-i}, t_i, t_{-i}), & \text{if } x_{-i} \in V_m, \\ -2\gamma, & \text{otherwise.} \end{cases}
$$

It is easy to see that $h_i^m(\cdot, t)$ is lower semicontinuous on X_{-i} for each fixed $t \in T$ and $m \geq 1$. It can be easily checked that h_i^m is a jointly measurable function. Indeed, it suffices to show that for any $c \geq 0$, the set $\{(x_{-i}, t) \in X_{-i} \times T \colon h_i^m(x_{-i}, t) < c\}$ is $\mathcal{B}(X_{-i}) \otimes \mathcal{T}$-measurable. Since u_i is jointly measurable and g_i is $\mathcal{T}_i$-measurable, the set

$$
\{(x_{-i}, t) \in V_m \times T \colon u_i(g_i(t_i), x_{-i}, t_i, t_{-i}) < c\}
$$

[16]For any subset E, E^c denotes the complement of the set E.

is $\mathcal{B}(X_{-i}) \otimes \mathcal{T}$-measurable. By the Projection Theorem, the projection of the above set on T, denoted as T_m, is a $\mathcal{T}$-measurable subset. Notice that

$$\{(x_{-i}, t) \in X_{-i} \times T \colon h_i^m(x_{-i}, t) < c\} = (V_m \times T_m) \cup (V_m^c \times T),$$

which is $\mathcal{B}(X_{-i}) \otimes \mathcal{T}$-measurable. Thus, h_i^m is a jointly measurable function.

Let $\underline{u}_i(x_{-i}, t) = \sup_{m \geq 1} h_i^m(x_{-i}, t)$. For each fixed $t \in T$, as in the proof of Reny (1999, Theorem 3.1), $\underline{u}_i(\cdot, t)$ is the pointwise supremum of a sequence of lower semicontinuous function, which is also lower semicontinuous on X_{-i}. In addition, $\underline{u}_i$ is the supremum of a sequence of $\mathcal{B}(X_{-i}) \otimes \mathcal{T}$-measurable functions, which is also $\mathcal{B}(X_{-i}) \otimes \mathcal{T}$-measurable. Define a function $H_i^l \colon \prod_{j \neq i} \mathcal{M}_j \to \mathbb{R}$ as follows: for $g_{-i} = (g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n) \in \prod_{j \neq i} \mathcal{M}_j$,

$$H_i^l(g_{-i}) = \int_T \int_{X_{-i}} \underline{u}_i(x_{-i}, t)\psi(t) \otimes_{j \neq i} g_j(\mathrm{d}x_j | t_j) \otimes_{i \in I} \lambda_i(\mathrm{d}t).$$

Let $\phi(x_{-i}, t_{-i}) = \int_{T_i} \underline{u}_i(x_{-i}, t)\psi(t)\lambda_i(\mathrm{d}t_i)$. Since $\underline{u}_i(x_{-i}, t)\psi(t)$ is lower semicontinuous in x_{-i}, jointly measurable and integrably bounded, ϕ is also lower semicontinuous in x_{-i}, jointly measurable and integrably bounded. By Lemma 5, the functional $g_{-i} = (g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n) \to \otimes_{j \neq i} g_j$ from $\prod_j \mathcal{M}_{j \neq i}$ to $\tilde{\mathcal{M}}_{-i}$ is continuous. Then by Lemma 4, the functional

$$\nu \to \int_{T_{-i}} \int_{X_{-i}} \phi(x_{-i}, t_{-i}) \nu(\mathrm{d}x_{-i} | t_{-i}) \lambda_{-i}(\mathrm{d}t_{-i})$$

is lower semicontinuous on $\tilde{\mathcal{M}}_{-i}$. Since H_i^l is the composition of these two functionals, it is lower semicontinuous. As a result, there is an open neighborhood $\mathcal{N}_{-i}(f_{-i}) \subseteq$

$\prod_{j\neq i}\mathcal{M}_j$ of f_{-i} such that for any $g_{-i}\in\mathcal{N}_{-i}(f_{-i})$,

$$\begin{aligned}&\int_T\int_{X_{-i}}\underline{u}_i(x_{-i},t)\psi(t)g_{-i}(\mathrm{d}x_{-i}|t_{-i})\otimes_{i\in I}\lambda_i(\mathrm{d}t)\\ \geq&\int_T\int_{X_{-i}}\underline{u}_i(x_{-i},t)\psi(t)f_{-i}(\mathrm{d}x_{-i}|t_{-i})\otimes_{i\in I}\lambda_i(\mathrm{d}t)-\epsilon.\end{aligned}$$

That is,

$$\begin{aligned}&\int_T\int_{X_{-i}}\underline{u}_i(x_{-i},t)g_{-i}(\mathrm{d}x_{-i}|t_{-i})\lambda(\mathrm{d}t)\\ \geq&\int_T\int_{X_{-i}}\underline{u}_i(x_{-i},t)f_{-i}(\mathrm{d}x_{-i}|t_{-i})\lambda(\mathrm{d}t)-\epsilon.\end{aligned}$$

Recall that $F=\mathrm{Gr}(E_i^K)\setminus\mathrm{Gr}(D_i^{g_i^K})$. Since $u_i(t,g_i(t_i),\cdot)$ is continuous on the t-section $\{x_{-i}\in X_{-i}\colon(x_{-i},t)\in F\}$ of F, we have $\underline{u}_i(x_{-i},t)=u_i(g_i(t_i),x_{-i},t)$ for any $(x_{-i},t)\in F$. As a result,

$$\begin{aligned}&\int_T\int_{X_{-i}}\underline{u}_i(x_{-i},t)f_{-i}(\mathrm{d}x_{-i}|t_{-i})\lambda(\mathrm{d}t)\\ =&\int_F\underline{u}_i(x_{-i},t)\lambda\diamond f_{-i}(\mathrm{d}(t,x_{-i}))+\int_{F^c}\underline{u}_i(x_{-i},t)\lambda\diamond f_{-i}(\mathrm{d}(t,x_{-i}))\\ \geq&\int_F u_i(g_i(t_i),x_{-i},t)\lambda\diamond f_{-i}(\mathrm{d}(t,x_{-i}))\\ >&\int_F u_i(g_i(t_i),x_{-i},t)\lambda\diamond f_{-i}(\mathrm{d}(t,x_{-i}))+\int_{F^c}u_i(g_i(t_i),x_{-i},t)\lambda\diamond f_{-i}(\mathrm{d}(t,x_{-i}))-\gamma\cdot\epsilon\\ =&\int_T\int_{X_{-i}}u_i(g_i(t_i),x_{-i},t)f_{-i}(\mathrm{d}x_{-i}|t_{-i})\lambda(\mathrm{d}t)-\gamma\cdot\epsilon.\end{aligned}$$

Therefore, for any $g_{-i} \in \mathcal{N}_{-i}(f_{-i})$, we have

$$
\begin{aligned}
&\int_T \int_{X_{-i}} u_i(g_i(t_i), x_{-i}, t) g_{-i}(\mathrm{d}x_{-i}|t_{-i}) \lambda(\mathrm{d}t) \\
\geq & \int_T \int_{X_{-i}} \underline{u}_i(x_{-i}, t) g_{-i}(\mathrm{d}x_{-i}|t_{-i}) \lambda(\mathrm{d}t) \\
\geq & \int_T \int_{X_{-i}} \underline{u}_i(x_{-i}, t) f_{-i}(\mathrm{d}x_{-i}|t_{-i}) \lambda(\mathrm{d}t) - \epsilon \\
\geq & \int_T \int_{X_{-i}} u_i(g_i(t_i), x_{-i}, t) f_{-i}(\mathrm{d}x_{-i}|t_{-i}) \lambda(\mathrm{d}t) - (\gamma + 1) \cdot \epsilon \\
\geq & \int_T \int_{X_{-i}} u_i(f_i'(t_i), x_{-i}, t) f_{-i}(\mathrm{d}x_{-i}|t_{-i}) \lambda(\mathrm{d}t) - 2(\gamma + 1) \cdot \epsilon \\
\geq & \int_T \int_{X} u_i(x_i, x_{-i}, t) f(\mathrm{d}x|t) \lambda(\mathrm{d}t) - 2(\gamma + 1) \cdot \epsilon,
\end{aligned}
$$

and consequently, the game G_0 is payoff secure.

2.3.3 An Application

Below, we provide an example of an all-pay auction with general tie-breaking rules to demonstrate the usefulness of our result.[17]

Suppose that N bidders compete for an object. Each bidder's valuation of the object is given by a measurable function $v\colon \Pi_{i\in I} T_i \to [0,1]$, where T_i is the state space, $i = 1, \ldots, N$. The common prior is λ, and λ is absolutely continuous with respect to $\otimes_{i\in I}\lambda_i$. Bidder i observes his own state t_i and submits a bid $x_i \in X_i = [0,1]$. The

[17] Jackson et al. (2002) showed the existence of a distributional-strategy equilibrium for discontinuous games with incomplete information by proposing a solution concept where the payoff is "endogenously defined" at the discontinuities; see also Araujo and De Castro (2009). Araujo, De Castro and Moreira (2008) first considered non-monotonic functions in auctions and showed that an all-pay auction tie-breaking rule is sufficient for the existence of pure-strategy equilibrium for a class of auctions. Carbonell-Nicolau and McLean (2015) considered an all-pay auction with the standard tie-breaking rule that the winning players share the object with equal probability. The results of this section are not covered by any of the above papers.

bidder who submits the highest bid wins the object and all bidders need to pay their bids. If multiple bidders submit the highest bid simultaneously, then the tie is broken as follows:

$$u_i(x_1, \ldots, x_N, t_1, \ldots, t_N) =$$

$$\begin{cases} -x_i, & x_i < \max_{j \in I} x_j, \\ \frac{\xi_i(x_1,\ldots,x_N)}{\sum_{k \in I: x_k = \max_{j \in I} x_j} \xi_k(x_1,\ldots,x_N)} \cdot v(t_1, \ldots, t_N) - x_i, & x_i = \max_{j \in I} x_j; \end{cases}$$

where $\xi = (\xi_1, \ldots, \xi_N) \colon [0,1]^N \to (0,1]^N$ is a continuous function which assesses the relative importance of each bidder's position when breaking the tie. In particular, if $\xi_i \equiv 1$ for any i, then the tie is broken via the equal probability rule. However, this is not necessary.

Proposition 6. *An all-pay auction with general tie-breaking rules satisfies the random disjoint payoff matching condition.*

Proof. Given any bidder i and $f_i \in \mathcal{L}_i$, let

$$g_i^k(t_i) = \begin{cases} \min\{f_i(t_i) + \frac{1}{k}, 1\}, & f_i(t_i) < 1, \\ \frac{1}{k}, & f_i(t_i) = 1. \end{cases}$$

It is obvious that $g_i^k \in \mathcal{L}_i$ for any $k \geq 1$.

Fix any $t \in T$ and $x_{-i} \in X_{-i}$. If $f_i(t_i) = 1$, then $u_i(f_i(t_i), x_{-i}, t_i, t_{-i}) \leq 0$ and $\liminf_{k \to \infty} u_i(g_i^k(t_i), x_{-i}, t_i, t_{-i}) \geq 0$. If $f_i(t_i) < 1$, we need to consider three possible cases.

1. If bidder i is the unique winner, then he is still the unique winner by adopting the strategy $g_i^k(t_i)$ since $g_i^k(t_i) > f_i(t_i)$. Since $g_i^k(t_i) \to f_i(t_i)$ and ξ is a continuous

function, we have $\lim_{k\to\infty} u_i(g_i^k(t_i), x_{-i}, t_i, t_{-i}) = u_i(f_i(t_i), x_{-i}, t_i, t_{-i})$.

2. If bidder i is one of the multiple winners, then he becomes the unique winner by adopting the strategy $g_i^k(t_i)$. Then

$$\lim_{k\to\infty} u_i(g_i^k(t_i), x_{-i}, t_i, t_{-i}) = v_i(t_i, t_{-i}) - f_i(t_i) \geq u_i(f_i(t_i), x_{-i}, t_i, t_{-i}).$$

3. If bidder i does not get the object, then he still loses the game by adopting $g_i^k(t_i)$ for sufficiently large k. As a result, $\lim_{k\to\infty} u_i(g_i^k(t_i), x_{-i}, t_i, t_{-i}) = u_i(f_i(t_i), x_{-i}, t_i, t_{-i})$.

Thus, we have

$$\liminf_{k\to\infty} u_i(g_i^k(t_i), x_{-i}, t_i, t_{-i}) \geq u_i(f_i(t_i), x_{-i}, t_i, t_{-i}),$$

which implies that condition (1) of Definition 13 is satisfied. In addition, for all $t_i \in T_i$, $D_i(t_i, g_i^k(t_i)) = \left\{[0, g_i^k(t_i)]^{N-1} \setminus [0, g_i^k(t_i))^{N-1}\right\} \times T_{-i}$. Since $g_i^k(t_i) \neq g_i^{k'}(t_i)$ for sufficiently large k and k', we have

$$\limsup_{k\to\infty} D_i(t_i, g_i^k(t_i)) = \emptyset$$

for any $t_i \in T_i$. Thus, condition (2) of Definition 13 also holds.

Therefore, an all-pay auction with general tie-breaking rules satisfies the random disjoint payoff matching condition. $\square$

Since $\sum_{i\in I} u_i(t, x) = v(t) - \sum_{i\in I} x_i$, $\sum_{i\in I} u_i(t, \cdot)$ is upper semicontinuous for every t. Thus, the existence of a behavioral-strategy equilibrium follows immediately by combining Theorem 12 and Proposition 6.

Corollary 9. *A behavioral-strategy equilibrium exists in an all-pay auction with general tie-breaking rules.*

Remark 31. *Allison and Lepore (2014) presented a Bertrand-Edgeworth oligopoly model which has general specifications of costs, residual demand rationing, and tie-breaking rules. They showed that this price competition problem satisfies the disjoint payoff matching condition and a mixed-strategy equilibrium exists. One can easily extend their model to an incomplete information environment and formulate the problem as a Bayesian game. Then by referring to our Theorems 11 and 12, one can prove the existence of a behavioral-strategy equilibrium. For further applications on Bayesian games with discontinuous payoffs including the war of attrition, Cournot competition and rent seeking, see Carbonell-Nicolau and McLean (2015).*

CHAPTER 3
EQUILIBRIUM THEORY UNDER AMBIGUITY

3.1 Introduction

Modeling the market with uncertainty is of important academic significance and realistic value in economics as most decision making is made under uncertainty. Towards this direction, the Arrow-Debreu "state contingent model" allows the state of nature of the world to be involved in the initial endowments and payoff functions, which is an enhancement of the deterministic general equilibrium model of Arrow-Debreu-McKenzie. According to Arrow-Debreu, agents make contacts ex ante (in period one) before the state of nature is realized and once the state is realized (in period two) the contract is executed and consumption takes place. The issue of incentive compatibility doesn't arise in this model, as all the information is symmetric. However, for the state contingent model to make sense one must assume that there is an exogenous court or government that enforces the contract ex post, otherwise agents may find it beneficial to renege. Radner (1968, 1982) extended the analysis of Arrow and Debreu by introducing asymmetric (differential) information. In particular, each agent is now characterized by his own private information, a random initial endowment, a random utility function and a prior. The private information is modeled as a partition of a finite state space and the allocation of each agent is assumed to be measurable with respect to his own private information. This means that each agent only knows the atom of his partition including the true state, but cannot distinguish those states within the same atom when making decisions. The Walrasian equilibrium notion in this model is called 'Walrasian expectations equilibrium', or WEE in short. Along this line, Yannelis (1991) proposed a core concept, which is called private core.

The Walrasian expectations equilibrium and private core share some interesting properties (in fact, the Walrasian expectations equilibrium is a strict subset of the private core): without the assumption of free disposal, whenever agents are Bayesian expected utility maximizers and allocations are private information measurable, the two above notions are both Bayesian incentive compatible and private information measurable efficient (see Koutsougeras and Yannelis (1993) and Krasa and Yannelis (1994)). However, these solution concepts are only efficient in the second best sense; that is, they are only private information measurable efficient allocations and may result in a possible welfare loss (recall that from Holmstrom and Myerson (1983), we know that with the Bayesian expected utility it is not possible to have allocations which are both first best efficient and also incentive compatible). The existence of WEE in a free disposal economy can be found in Radner (1968, 1982). However, the free disposal WEE allocations may be not incentive compatible (see Glycopantis and Yannelis (2005)). Furthermore, if we require non-free disposal, then a WEE may not exist (see Einy and Shitovitz (2001)). Therefore, a natural question arises:

> Can one find an appropriate framework in the asymmetric information economy such that the existence of equilibrium and core notions continues to hold and furthermore, these notions are both incentive compatible and first best efficient?

A crucial assumption in the frameworks of Radner (1968, 1982) and Yannelis

(1991) is that agents maximize Bayesian expected utilities. Nevertheless, from Ellsberg (1961) (see also de Castro and Yannelis (2014)), there is a huge literature which criticizes the Bayesian paradigm and explores the non-expected utility theory. The maximin expected utility of Gilboa and Schmeidler (1989) is one of the successful alternatives. Indeed, recently de Castro, Pesce and Yannelis (2011, 2014) and de Castro and Yannelis (2009) applied the maximin expected utility to an asymmetric information economy with a finite number of states of nature,[1] and introduced various core and Walrasian equilibrium notions. With the maximin expected utilities, agents take into account the worst possible state that can occur and choose the best possible allocations. de Castro, Pesce and Yannelis (2011) proved that the ex ante equilibrium and core notions based on the maximin expected utility, which are called maximin expectations equilibrium (MEE) and maximin core (MC) therein, are incentive compatible in the economy without free disposal. Moreover, it is noteworthy that since the allocations are not required to be measurable with respect to agents' private information, MEE and MC allocations are also first best efficient. Therefore, the conflict between efficiency and incentive compatibility is solved in this new approach. More importantly, de Castro and Yannelis (2009) showed that the conflict of incentive compatibility and first best efficiency is inherent in the standard expected utility decision making (Bayesian) and it is resolved only when agents maximize the maximin expected utility (MEU). In particular, they proved that the MEU is a nec-

[1]MEU is first applied to a general equilibrium model of an asymmetric information economy by Correia-da-Silva and Hervés-Beloso (2009). They proved the existence of the ex ante Walrasian equilibrium in an asymmetric information economy with maximin preferences and a finite state space. However, their setup is different from ours and they do not consider the issue of incentive compatibility; see also Correia-da-Silva and Hervés-Beloso (2012, 2014).

essary condition for efficient allocations to be incentive compatible. The above work implies the fact that one has to work with MEU if the first best efficiency is desirable. As a result, a natural question arises:

> Can one obtain the classical core-Walras existence and equivalence results for asymmetric information economies where agents are ambiguous (that is, MEU maximizers) and also the state space is not necessarily finite?

An affirmative answer to this question is of great importance because not only this way one develops a new equilibrium theory where there is no conflict between efficiency and incentive compatibility, but also such positive results could become the main tool for applications in other fields of economics.

The first aim of this chapter is to prove the existence of the maximin expectations equilibrium and maximin core in a non-free disposal economy with countably many states of nature.[2] Since there is a countable number of states in the economy, the allocations are infinite dimensional. An advantage of the ambiguous economy modeling is that it allows us to view an asymmetric information economy as a deterministic economy with infinite dimensional commodity spaces. Thus, we can directly apply known results in the literature to obtain the existence of maximin expectations equilibrium.[3] As a corollary, we obtain that the consistency between incentive

[2]For a general equilibrium model with countably many states, see, for example, Hervés-Beloso, Martins-da-Rocha and Monteiro (2009).

[3]On the contrary, one can not readily convert an asymmetric information economy with Bayesian expected utility maximizers to an economy with infinite dimensional commodity spaces due to the restriction of the private information measurability requirement. For some papers with infinite dimensional commodity spaces, see, for example, Bewley (1972) and Podczeck and Yannelis (2008).

compatibility and efficiency also holds with a countable number of states.

The second aim of this chapter is to prove a core equivalence theorem for an economy with asymmetric information where agents are ambiguous (that is, maximize MEU). In a finite agent framework and complete information, Debreu and Scarf (1963) considered a sequence of replicated economy and showed that the set of non-blocked allocations in every replicated economy converges to the set of Walrasian equilibria. In Section 3.4, we follow the Debreu-Scarf approach and establish a similar equivalence result for an equal treatment economy with asymmetric information, a countable number of states and MEU preferences. In an atomless economy with complete information, Schmeidler (1972), Grodal (1972) and Vind (1972) improved the core-Walras equivalence theorem of Aumann (1964), by showing that if an allocation is not in the core, then it can be blocked by a non-negligible coalition with any given measure less than 1. Hervés-Beloso, Moreno-García and Yannelis (2005a,b) first extended this result to an asymmetric information economy with the equal treatment property and with an infinite dimensional commodity space by appealing to the finite dimensional Lyapunov's theorem. Bhowmik and Cao (2012, 2013a) obtained further extensions based on an infinite dimensional version of Lyapunov's theorem. All the above results rely on the Bayesian expected utility formulation and therefore the conflict of efficiency and incentive compatibility still holds despite the non atomic measure space of agents.[4] Our Theorem 18 is an extension of Vind's theorem to the

[4]As the work of Sun and Yannelis (2008) indicates, even with an atomless measure space of agents we cannot guarantee that WEE allocations are incentive compatible.

asymmetric information economy with the equal treatment property and a countable number of states of nature, where agents behave as maximin expected utility maximizers. Thus, our new core equivalence theorem for the MEU framework, resolves the inconsistency of efficiency and incentive compatibility.

Finally, we provide two characterizations for maximin expectations equilibrium. In the complete information economy with finite agents, Aubin (1979) introduced a new approach that at a first glance seems to be different from the Debreu-Scarf; however one can show that they are essentially equivalent. Aubin considered a veto mechanism in the economy when a coalition is formed; in particular, agents are allowed to participate with any proportion of their endowments. The core notions defined by the veto mechanism, is called Aubin core and it coincides with the Walrasian equilibrium allocations. The approach of Aubin has been extended to an asymmetric information economy to characterize the Walrasian expectations equilibrium (see for example Graziano and Meo (2005), Hervés-Beloso, Moreno-García and Yannelis (2005b) and Bhowmik and Cao (2013a)). Another approach to characterize the Walrasian expectations equilibrium is due to Hervés-Beloso, Moreno-García and Yannelis (2005a,b). They showed that the Walrasian expectations equilibrium allocation cannot be privately blocked by the grand coalition in any economy with the initial endowment redistributed along the direction of the allocation itself. This approach has been extended to a pure exchange economy with an atomless measure space of agents and finitely many commodities, and an asymmetric information economy with an infinite dimensional commodity space (e.g., see Hervés-Beloso and Moreno-García

(2008), Bhowmik and Cao (2013a,b)). Our Theorem 14 and 15 extended these two characterizations to the asymmetric information economy with ambiguous agents and with countably many states of nature.

This chapter is based on He and Yannelis (2015c), and is organized as follows. Section 3.2 states the model of ambiguous asymmetric information economies with a countable number of states and discusses main assumptions. Section 3.3 introduces the maximin expectations equilibrium and maximin core and proves their existence, and contains two different characterizations of maximin expectations equilibrium by using the maximin blocking power of the grand coalition. Section 3.4 extends the maximin expectations equilibrium and maximin core to an economy with a continuum of agents, and interprets the asymmetric information economy with finite agents as a continuum economy with finite types. In addition, two core-Walras equivalence theorems and an extension of Vind's result are given for an asymmetric information economy with a countable number of states. Section 3.5 shows that maximin efficient allocations are incentive compatible in economies with finite agents and atomless economies with the equal treatment property. Section 3.6 collects some concluding remarks and open questions. Section 3.7 contains some proofs.

3.2 Ambiguous Asymmetric Information Economy

We define an exchange economy with uncertainty and asymmetric information. The **uncertainty** is represented by a measurable space $(\Omega, \mathcal{F})$, where $\Omega = \{\omega_n\}_{n \in \mathbb{N}}$ is a countable set and $\mathcal{F}$ is the power set of Ω. Let $\mathbb{R}^l_+$ be the commodity space, and

$I = \{1, 2, \cdots, s\}$ the set of agents.

For each $i \in I$, $\mathcal{F}_i$ is the σ-algebra on Ω generated by the partition Π_i of agent i, which represents the private information. Let $\Pi_i(\omega)$ be the element in the partition Π_i which contains ω. Therefore, if any state $\omega \in \Omega$ is realized, then agent i can only observe the event $\Pi_i(\omega)$. The **prior** π_i of agent i is defined on $\mathcal{F}_i$ such that $\sum_{E \in \Pi_i} \pi_i(E) = 1$ and $\pi_i(E) > 0$ for every $E \in \Pi_i$. Notice that π_i is incomplete; that is, the probability of each element in the information partition Π_i is well defined, but not the probability of the event $\{\omega\}$ for every $\omega \in \Omega$. Let $u_i(\omega, \cdot) \colon \mathbb{R}^l_+ \to \mathbb{R}_+$ be the positive **ex post utility function** of agent i at state ω from the consumption space to the positive real line, and $e_i : \Omega \to \mathbb{R}^l_+$ be i's **random initial endowment**.

Let $\mathcal{E}$ be an **ambiguous asymmetric information economy**, where

$$\mathcal{E} = \{(\Omega, \mathcal{F}); (\mathcal{F}_i, u_i, e_i, \pi_i) : i \in I = \{1, \ldots, s\}\}.$$

A **price vector** p is a nonzero function from Ω to $\mathbb{R}^l$.[5] We assume that $\triangle$ denotes the set of all price vectors, where

$$\triangle = \{p \in (\mathbb{R}^l)^\infty : |\sum_{\omega \in \Omega} \sum_{j=1}^{l} p(\omega, j)| = 1\},$$

and $p(\omega, j)$ is the price of the commodity j at the state ω.

There are three stages in this economy: at the ex ante stage (t=0), the information partition and the economy structure are common knowledge; at the interim stage (t=1), each individual i learns his private information $\Pi_i(\omega)$ which includes the

[5]The vector p is said to be nonzero if p is not a constant function of value 0, but it is possible that $p(\omega) = 0$ for some ω.

true state ω, and makes his consumption plan; at the ex post stage (t=2), agent i receives the endowment and consumes according to his plan.[6]

An **allocation** is a mapping x from $I \times \Omega$ to $\mathbb{R}^l_+$. For each $i \in I$, let

$$L_i = \{x_i : x_i(\omega) \in \mathbb{R}^l_+ \text{ and uniformly bounded for all } \omega \in \Omega\}$$

be the **set of all random allocations** of agent i.[7] If $x_i \in L_i$ and $p \in \triangle$, we denote $\sum_{\omega \in \Omega} p(\omega) \cdot x_i(\omega)$ as $p \cdot x_i$.

Suppose that x is an allocation. Then $x_i(\omega)$ is a vector in $\mathbb{R}^l_+$ for each $i \in I$, which represents the allocation at the state ω. In addition, $x_i(\omega, j)$ denotes the allocation of commodity j at the state ω. An allocation x is said to be **feasible** if $\sum_{i \in I} x_i = \sum_{i \in I} e_i$. That is, for each $\omega \in \Omega$,

$$\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega).$$

The feasibility here indicates that the economy has no free disposal.

Assumption 3 (E). *1. For each $i \in I$, e_i is $\mathcal{F}_i$-measurable.*[8]

2. There exists some $\beta > 0$ such that for any $\omega \in \Omega$ and $1 \leq j \leq l$, $e_i(\omega, j) \geq \beta$.

3. There exists some $\gamma > 0$ such that for any $\omega \in \Omega$ and $1 \leq j \leq l$, $\sum_{i \in I} e_i(\omega, j) \leq \gamma$.

Assumption (E) is about the endowment. Condition (1) says that each agent's endowment should be measurable with respect to his private information, otherwise

[6] The production sector can be included in the analysis and the results should still hold. For simplicity of the exposition, we have not included production.

[7] That is, $L_i = l^\infty_+$ for each $i \in I$.

[8] Clearly, if e_i is independent of ω, then it is $\mathcal{F}_i$-measurable.

the agent may disclose the true state from his endowment. Condition (2) implies that for every agent i, e_i is an interior point of $(\mathbb{R}^l_+)^\infty$ under the sup-norm topology. Condition (3) implies that $e_i \in L_i$; that is, the resource of the economy is limited no matter what the state is. This condition will be automatically satisfied if there are only finitely many states.[9]

Assumption 4 (U). *1. For each $\omega \in \Omega$ and $i \in I$, $u_i(\omega, \cdot)$ is continuous, strictly increasing and concave.*

2. For each $i \in I$ and $x \in \mathbb{R}^l_+$, $u_i(\cdot, x)$ is $\mathcal{F}_i$-measurable.[10]

3. For any $a \in \mathbb{R}^l_+$ and $K_0 > 0$ such that $a(j) \leq K_0$ for $1 \leq j \leq l$, there exists some $K > 0$ such that $0 \leq u_i(\omega, a) \leq K$ for any $i \in I$ and $\omega \in \Omega$. Let $u_i(\omega, 0) = 0$ for all $i \in I$ and $\omega \in \Omega$.

Assumption (U) is about the utility. Conditions (1) and (2) are standard in the literature. Condition (3) basically says that agents' utility cannot be arbitrarily large with limited goods. This condition can be removed if Ω is finite: for each $i \in I$ and $\omega \in \Omega$, $u_i(\omega, a)$ is continuous at a, if a is bounded, then $u_i(\omega, \cdot)$ is bounded; since there are only finitely many states, $u_i(\omega, \cdot)$ is uniformly bounded among all ω. Moreover, the condition $u_i(\omega, 0) = 0$ means that agents have no payoff if they have no consumption.

For every agent i, his private prior may be incomplete and the allocation in

[9] Since the initial endowment is bounded, the value $p \cdot e_i$ of the initial endowment e_i is finite for any agent i and price p.

[10] If u_i is state independent, then it is automatically $\mathcal{F}_i$-measurable.

L_i is not required to be $\mathcal{F}_i$-measurable. Thus, agents cannot evaluate the allocation based on the Bayesian expected utility. In this chapter, we will consider the maximin preference axiomatized by Gilboa and Schmeidler (1989).[11]

Let $\mathcal{M}_i$ be the set of all probability measures on $\mathcal{F}$ which agree with π_i on $\mathcal{F}_i$. That is,

$$\mathcal{M}_i = \{\mu : \mathcal{F} \to [0,1] : \mu(E) = \pi_i(E), \forall E \in \mathcal{F}_i\}.$$

Let P_i be a nonempty and convex subset of $\mathcal{M}_i$, which is the set of priors of agent i.

We assume that agent i is ambiguous on the set P_i and will take the worst possible scenario when evaluating his payoff. In particular, for any two allocations $x_i, y_i \in L_i$, agent i prefers the allocation x_i to the allocation y_i if

$$\inf_{\mu \in P_i} \sum_{\omega \in \Omega} u_i(\omega, x_i(\omega))\mu(\omega) \geq \inf_{\mu \in P_i} \sum_{\omega \in \Omega} u_i(\omega, y_i(\omega))\mu(\omega).$$

For any allocation $\{x_i\}_{i \in I}$, the **maximin ex ante utility** of agent i is:

$$V_i(x_i) = \inf_{\mu \in P_i} \sum_{\omega \in \Omega} u_i(\omega, x_i(\omega))\mu(\omega).$$

The **maximin interim utility** of agent i with allocation x_i at the state ω is

$$v_i(\omega, x_i) = \frac{1}{\pi_i(\Pi_i(\omega))} \inf_{\mu \in P_i} \sum_{\omega_1 \in \Pi_i(\omega)} u_i(\omega_1, x_i(\omega_1))\mu(\omega_1).$$

We will slightly abuse the notations by writing $v_i(\omega, x_i) = v_i(E, x_i)$ for $\omega \in E \in \mathcal{F}_i$.

Remark 32. *If P_i is a singleton set for each agent i, then the maximin expected utility above reduces to the standard Bayesian expected utility. If $P_i = \mathcal{M}_i$, the set*

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[11] We can adopt the more general variational preferences axiomatized by Maccheroni, Marinacci and Rustichini (2006), and all the results in Sections 3 and 4 will still go through.

of all probability measures on $\mathcal{F}$ which agree with π_i on $\mathcal{F}_i$, then it is the maximin expected utility considered in de Castro and Yannelis (2009). In the latter case, de Castro and Yannelis (2009) showed that for any two allocations $x_i, y_i \in L_i$, agent i prefers the allocation x_i to the allocation y_i if:

$$\sum_{E_i \in \Pi_i} [\inf_{\omega \in E_i} u_i(\omega, x_i(\omega))]\pi_i(E_i) \geq \sum_{E_i \in \Pi_i} [\inf_{\omega \in E_i} u_i(\omega, y_i(\omega))]\pi_i(E_i).^{12} \tag{3.1}$$

Remark 33. *It should be noted that the asymmetric information in a Bayesian model comes from the private information measurability of allocations. For example, if allocations are not required to be private information measurable, then the framework of Radner (1968) reduces to the standard Arrow-Debreu state-contingent model. In other words, the private information measurability of allocations captures the information asymmetry in a Bayesian model. Furthermore, despite the fact that the Walrasian expectations equilibrium is incentive compatible (see Koutsougeras and Yannelis (1993)), it may be only second best efficient due to the private information measurability requirement of the allocations, which is pointed out in this chapter (see Example 9 below) as well as de Castro and Yannelis (2009).*

In an ambiguity model, the information asymmetry is captured by the maximin expected utility itself. In particular, priors are defined on the information partition of each agent (while they are defined on the whole state space Ω in a Bayesian model). Thus, it is natural to relax the restriction of private information measurability of

[12] First, we use 'inf' in these two inequalities instead of 'min' used in de Castro and Yannelis (2009), since there are infinite states here. The existence of infimum is guaranteed since the ex post utility function is nonnegative. Thus the ex ante utility V_i is well defined. Second, although de Castro and Yannelis (2009) only argued that these two inequalities are equivalent when there are finitely many states, this observation is still true in our context.

allocations in an ambiguity model. In addition, we show that the maximin expectations equilibrium is both first best efficient and incentive compatible.

The proposition below indicates that the maximin ex ante utility function satisfies several desirable properties.

Proposition 7. *If Assumption (U) holds, then V_i is increasing and concave, continuous in the sup-norm topology, and lower semicontinuous in the weak* topology.*

Proof. See Section 3.7. $\square$

3.3 Maximin Expectations Equilibrium and Maximin Core

3.3.1 Existence of MEE and MC

In this section, we define the notions of maximin core (MC) and maximin expectations equilibrium (MEE).

Given a price vector p, the budget set of agent i is defined as follows:

$$B_i(p) = \{x_i \in L_i : \sum_{\omega \in \Omega} p(\omega) \cdot x_i(\omega) \leq \sum_{\omega \in \Omega} p(\omega) \cdot e_i(\omega)\}.$$

Definition 15. *An allocation x is said to be a* ***maximin expectations equilibrium*** *allocation for the economy $\mathcal{E}$, if there exists a price vector p such that for any agent $i \in I$,*

1. *x_i maximizes $V_i(\cdot)$ subject to the budget set $B_i(p)$;*
2. *x is feasible.*

The following definition of a core concept in the current context implies that coalitions of agents cannot cooperate to become better off in terms of MEU.

Definition 16. *A feasible allocation x is said to be a* ***maximin core*** *allocation for the economy $\mathcal{E}$, if there do not exist a coalition $C \subseteq I$, $C \neq \emptyset$, and an allocation $\{y_i \in L_i\}_{i \in C}$ such that*

(i) $V_i(y_i) > V_i(x_i)$ *for all* $i \in C$;

(ii) $\sum_{i \in C} y_i(\omega) = \sum_{i \in C} e_i(\omega)$ *for all* $\omega \in \Omega$.

The allocation is said to be **maximin efficient** if $C = I$.

Remark 34. *The notions of maximin expectations equilibrium, maximin core and maximin efficiency in the above definitions correspond to the concepts of Walrasian equilibrium, core and efficiency in the standard model. If Bayesian expected utilities, instead of maximin expected utilities, are used in Definition 15, and the private information measurability assumption is imposed on allocations, then the solution concept is Walrasian expectations equilibrium defined in Radner (1968, 1982). In particular, the Walrasian expectations equilibrium is defined as follows: an allocation $x = (x_1, \ldots, x_s)$ is said to be a* ***Walrasian expectations equilibrium*** *allocation for the economy $\mathcal{E}$, if x_i is an $\mathcal{F}_i$-measurable mapping for each agent i and there exists a price vector p such that for any agent $i \in I$,*

1. *x_i maximizes agent i's expected utility subject to the budget set $B_i(p)$;*
2. $\sum_{i \in I} x_i \leq \sum_{i \in I} e_i$.

The following example shows that MEE provides strictly higher efficiency than the (free disposal) WEE allocations. Furthermore, we show that the MEE is incentive compatible.

Example 8. [13] *Consider the following economy with one commodity, the agent space is* $I = \{1,2\}$ *and the state space is* $\Omega = \{a,b,c\}$. *The initial endowments and information partitions of agents are given by*

$$e_1 = (5,5,0), \Pi_1 = \{\{a,b\},\{c\}\};$$

$$e_2 = (5,0,5), \Pi_2 = \{\{a,c\},\{b\}\}.$$

It is also assumed that for $i \in I$, $u_i(\omega, x_i) = \sqrt{x_i}$, *which is strictly concave and monotone in* x_i, *and the priors for both agents are the same:* $\mu(\{\omega\}) = \frac{1}{3}$ *for every* $\omega \in \Omega$.

Suppose that agents are both Bayesian expected utility maximizers. It can be easily checked that there is no (non-free disposal) WEE. If we allow for free disposal, $x_1 = (4,4,1)$ *and* $x_2 = (4,1,4)$ *is a (free disposal) WEE allocation with the equilibrium price* $p(a) = 0$ *and* $p(b) = p(c) = \frac{1}{2}$. *However, this allocation is not incentive compatible (see Example 9 in Section 3.5 for details).*

If $P_i = \mathcal{M}_i$ *for each* i, *and agents are maximin expected utility maximizers, then there exists an MEE* (y,p), *where* $y_1 = (5,4,1)$, $y_2 = (5,1,4)$ *and* $p(a) = 0$, $p(b) = p(c) = \frac{1}{2}$.

If state b *or* c *realizes, the ex post utility of agent* 1 *will be the same in both Bayesian preference setting and maximin preference setting, since* $x_1(b) = y_1(b)$ *and*

[13] This example has been analyzed in Glycopantis and Yannelis (2005) in Bayesian preference setting for the existence and incentive compatibility of Walrasian expectations equilibrium and private core, and in de Castro, Liu and Yannelis (2015) in maximin preference setting for the existence and incentive compatibility of maximin core. See also Bhowmik, Cao and Yannelis (2014).

$x_1(c) = y_1(c)$. *But if state* a *occurs, the ex post utility of agent* 1 *with maximin preference will be strictly higher than that in the Bayesian preference setting, since*

$$x_1(a) = 4 < 5 = y_1(a).$$

Therefore, the maximin preference allows agents to reach higher efficiency.

The following lemma is standard, which shows that the set of maximin expectations equilibrium allocations is included in the set of maximin core allocations.

Lemma 6. *The set of MEE allocations is a subset of the MC allocations, and hence any maximin expectations equilibrium allocation is maximin efficient.*

This inclusion can be strict. It is clear that both the Arrow-Debreu 'state contingent model' and the deterministic general equilibrium model are special cases of our model: if $\mathcal{F}_i = \mathcal{F} = 2^{\Omega}$ for every $i \in I$, then the maximin expected utility coincides with the Bayesian expected utility and $\mathcal{E}$ is indeed the state contingent model; if Ω is a singleton, then $\mathcal{E}$ is the deterministic model. Moreover, it is well known that in those two models, the set of core allocations could strictly contain the set of Walrasian equilibrium allocations.

We now turn to the issue of the existence of MEE.

Theorem 13. *For an ambiguous asymmetric information economy* $\mathcal{E}$*, if Assumptions (E) and (U) hold, then there exists an MEE.*

Proof. See Section 3.7. □

Based on Theorem 13 and Lemma 6, it is straightforward to show that the set of maximin core allocations is also nonempty.

Corollary 10. *Under the conditions of Theorem 13, a maximin core allocation exists.*

3.3.2 Equivalence Theorems

For the economy $\mathcal{E}$, Hervés-Beloso, Moreno-García and Yannelis (2005b) provided two equivalence results for the Walrasian expectations equilibrium in terms of the private blocking power of the grand coalition, and Bhowmik and Cao (2013a) extended this result to an asymmetric information economy whose commodity space is a Banach lattice. We will follow this approach and characterize the maximin expectations equilibrium. The two theorems below correspond to Theorem 4.1 and 4.2 of Hervés-Beloso, Moreno-García and Yannelis (2005b). The proofs are omitted since the same argument can be followed here.

For an allocation $x = \{x_i\}_{i \in I}$ and a vector $a = (a_1, \cdots, a_s) \in [0,1]^s$, consider the ambiguous asymmetric information economy $\mathcal{E}(a,x)$ which is identical with $\mathcal{E}$ except for the random initial endowment of each agent i given by the convex combination $e_i(a_i, x_i) = a_i e_i + (1 - a_i) x_i$.

Definition 17. *An allocation z is **maximin dominated** (or **maximin blocked** by the grand coalition) in the economy $\mathcal{E}(a,x)$ if there exists a feasible allocation y in $\mathcal{E}(a,x)$ such that $V_i(y_i) > V_i(z_i)$ for every $i \in I$.*

Theorem 14. *The allocation x is an MEE in $\mathcal{E}$ if and only if x is not a maximin dominated allocation in every economy $\mathcal{E}(a,x)$.*

Definition 18. *A coalition $S \subseteq I$ maximin blocks an allocation x in the sense of Aubin via $y = \{y_i\}_{i \in S}$ if for all $i \in S$, there is some $\alpha_i \in (0,1]$ such that $V_i(y_i) > V_i(x_i)$*

and $\sum_{i\in S}\alpha_i y_i \le \sum_{i\in S}\alpha_i e_i$. *The* ***Aubin maximin core*** *is the set of all feasible allocations that cannot be maximin blocked by any coalition in the sense of Aubin. An allocation* x *is called Aubin non-dominated if* x *is not maximin blocked by the grand coalition in the sense of Aubin.*

Theorem 15. *The allocation* x *is an MEE in* $\mathcal{E}$ *if and only if* x *is not a maximin dominated allocation in the sense of Aubin in the economy* $\mathcal{E}$.

3.4 A Continuum Approach

3.4.1 Basics

In this section, we introduce the maximin expectations equilibrium and maximin core for an atomless economy. Let the atomless probability space $(T, \mathcal{T}, \lambda)$ denote the agent space. We define an **atomless ambiguous asymmetric information economy** as follows:

$$\mathcal{E}_0 = \{(\Omega, \mathcal{F}); (\mathcal{F}_t, u_t, e_t, \pi_t) : t \in T\}.$$

An **allocation** in the continuum economy $\mathcal{E}_0$ is a mapping f from $T \times \Omega$ to $\mathbb{R}^l_+$ such that $f(\cdot, \omega)$ is integrable for every $\omega \in \Omega$ and $f(t, \cdot) \in l^\infty_+$ for λ-almost all $t \in T$. The allocation is said to be **feasible** if $\int_T f(t,\omega)\,\mathrm{d}\lambda(t) = \int_T e(t,\omega)\,\mathrm{d}\lambda(t)$ for every $\omega \in \Omega$.

A coalition in T is a measurable set $S \in \mathcal{T}$ such that $\lambda(S) > 0$. An allocation f is **maximin blocked** by a coalition S in the economy $\mathcal{E}_0$ if there exists some $g : S \times \Omega \to \mathbb{R}^l_+$ such that $\int_S g(t,\omega)\,\mathrm{d}\lambda(t) = \int_S e(t,\omega)\,\mathrm{d}\lambda(t)$ for every $\omega \in \Omega$, and $V_t(g(t)) > V_t(f(t))$ for λ-almost every $t \in S$.

Definition 19. *An allocation f is said to be the* ***maximin core*** *for the economy $\mathcal{E}_0$ if it is not maximin blocked by any coalition.*

Definition 20. *An allocation f is said to be a* ***maximin expectations equilibrium*** *allocation for the economy $\mathcal{E}_0$, if there exists a price vector p such that*

1. *f_t maximizes $V_t(\cdot)$ subject to the budget set $B_t(p)$ for λ-almost all $t \in T$;*
2. *f is feasible.*

3.4.2 A Continuum Interpretation of the Finite Economy

We associate an atomless economy $\mathcal{E}_c$ with the discrete economy $\mathcal{E}$ as in García-Cutrín and Hervés-Beloso (1993), Hervés-Beloso, Moreno-García and Yannelis (2005a,b) and Bhowmik and Cao (2013a). The space of agents in $\mathcal{E}_c$ is the Lebesgue unit interval $(T, \mathcal{T}, \mu)$ such that $T = \cup_{i=1}^{s} T_i$, where $T_i = [\frac{i-1}{s}, \frac{i}{s})$ for $i = 1, \cdots, s-1$ and $T_s = [\frac{s-1}{s}, 1]$. For each agent $t \in T_i$, set $\mathcal{F}_t = \mathcal{F}_i$, $\pi_t = \pi_i$, $u_t = u_i$ and $e_t = e_i$. Thus, the maximin ex ante utility V_t of agent t is V_i. We refer to T_i as the set of agents of type i, and

$$\mathcal{E}_c = \{(\Omega, \mathcal{F}); (T, \mathcal{F}_i, V_i, e_i, \pi_i) : i \in I = \{1, \cdots, s\}\}$$

is the **economy with the equal treatment property**. The allocations in $\mathcal{E}$ and $\mathcal{E}_c$ are closely related: for any allocation f in $\mathcal{E}_c$, there is an corresponding allocation x in $\mathcal{E}$, where $x_i(\omega) = \frac{1}{\mu(T_i)} \int_{T_i} f(t, \omega)\, \mathrm{d}\mu(t)$ for all $i \in I$ and $\omega \in \Omega$; conversely, an allocation x in $\mathcal{E}$ can be interpreted as an allocation f in $\mathcal{E}_c$, where $f(t, \omega) = x_i(\omega)$ for all $t \in T_i$, $\omega \in \Omega$ and $i \in I$. f is said to be a step allocation if $f(\cdot, \omega)$ is a constant function on T_i for any $\omega \in \Omega$ and $i \in I$.

Analogously to the theorems in Hervés-Beloso, Moreno-García and Yannelis (2005a,b), the next proposition shows that the maximin expectations equilibrium can be considered equivalent in discrete and continuum approaches.

Proposition 8. *Suppose that Assumption (U) holds. Then we have the following properties:*

- *If (x, p) is an MEE for the economy $\mathcal{E}$, then (f, p) is the MEE for the associated continuum economy $\mathcal{E}_c$, where $f(t, \omega) = x_i(\omega)$ if $t \in T_i$.*
- *If (f, p) is an MEE for the economy $\mathcal{E}_c$, then (x, p) is the MEE for the economy $\mathcal{E}$, where $x_i(\omega) = \frac{1}{\mu(T_i)} \int_{T_i} f(t, \omega)\, \mathrm{d}\mu$ for any $\omega \in \Omega$.*

The proof is straightforward, interested readers may refer to Theorem 3.1 of Hervés-Beloso, Moreno-García and Yannelis (2005b).

3.4.3 Core Equivalence with A Countable Number of States

The core-Walras equivalence theorem has been recently extended to a Bayesian asymmetric information economy. Specifically, Einy, Moreno and Shitovitz (2001) showed that the Walrasian expectations equilibrium is equivalent to the private core for atomless economies with a finite number of commodities in a free disposal setting, Angeloni and Martins-da-Rocha (2009) completed the discussion by proposing appropriate conditions which guarantees the core equivalence result in non-free disposal context. Hervés-Beloso, Moreno-García and Yannelis (2005a,b) and Bhowmik and Cao (2013a) followed the Debreu- Scarf approach and showed that the set of Walrasian expectations equilibrium allocations coincides with the private core in the asymmetric information economy with the equal treatment property, finitely many states and infinitely many commodities.

However, all these discussions focus on the asymmetric information economy with Bayesian expected utilities and a finite state space. Our aim here is to examine whether this result is still true when agents are ambiguous (have maximin expected utilities) and the state space is countable. The theorems below show that the core equivalence theorem holds with either of the following conditions:

1. Maximin expected utility and finitely many states;

2. Maximin expected utility, countably many states and the equal treatment property holds.

Theorem 16. *Let Ω be finite in the atomless economy $\mathcal{E}_0$. Assume that (E) and (U) hold. Then the set of MC allocations coincides with the set of MEE allocations.*

We omit the proof since it is standard, interested readers may check that the proof of the core equivalence theorem in Hildenbrand (1974) with minor modifications still holds.

Theorem 17. *Suppose Assumptions (E) and (U) hold. Let the step allocation f be feasible in the associated continuum economy $\mathcal{E}_c$. Then f is an MEE allocation if and only if f is an MC allocation.*

Proof. See Section 3.7. □

3.4.4 An Extension of Vind's Theorem

Hervés-Beloso, Moreno-García and Yannelis (2005a,b) and Bhowmik and Cao

(2013a) extended Vind's theorem to an asymmetric information economy with the equal treatment property. Sun and Yannelis (2007) established this theorem in an economy with a continuum of agents and negligible asymmetric information. Below, we extend this result to the atomless ambiguous asymmetric information economy with a countable number of states of nature.

Theorem 18. *Suppose that Assumptions (E) and (U) hold. If the feasible step allocation f is not in the MC of the associated continuum economy $\mathcal{E}_c$, then for any α, $0 < \alpha < 1$, there exists a coalition S such that $\mu(S) = \alpha$, which maximin blocks f.*

Proof. See Section 3.7. □

3.5 Efficiency and Incentive Compatibility under Ambiguity

In this section, we will define a notion of maximin incentive compatibility, and then prove that any maximin efficient allocation is maximin incentive compatible.

First, we illustrate the incentive compatibility issue when agents adopt Bayesian preferences.

Example 9. *[Example 8 with Bayesian preference]*

Recall Example 8 in Section 3.3.1: the agent space is $I = \{1, 2\}$ and the state space is $\Omega = \{a, b, c\}$. The initial endowments and information partitions of agents are given by

$$e_1 = (5, 5, 0), \Pi_1 = \{\{a, b\}, \{c\}\};$$

$$e_2 = (5, 0, 5), \Pi_2 = \{\{a, c\}, \{b\}\}.$$

It is also assumed that for $i \in I$, $u_i(\omega, x_i) = \sqrt{x_i}$, *which is strictly concave and monotone in* x_i, *and the priors for both agents are the same:* $\mu(\{\omega\}) = \frac{1}{3}$ *for every* $\omega \in \Omega$.

Suppose that agents are Bayesian expected utility maximizers, and all allocations are required to be private information measurable. The no-trade allocation $x_1 = (5, 5, 0)$ *and* $x_2 = (5, 0, 5)$ *is in the private core and it is incentive compatible. Indeed, it has been shown in Koutsougeras and Yannelis (1993) that private core allocations are always CBIC provided that the utility functions are monotone and continuous.*

This conclusion is not true in free disposal economies. Glycopantis and Yannelis (2005) pointed out that private core and Walrasian expectations equilibrium allocations need not be incentive compatible in an economy with free disposal. In this example, $x_1 = (4, 4, 1)$ *and* $x_2 = (4, 1, 4)$ *is a (free disposal) WEE allocation with the equilibrium price* $p(a) = 0$ *and* $p(b) = p(c) = \frac{1}{2}$, *and hence in the (free disposal) private core. However, this allocation is not incentive compatible. Indeed, if agent 1 observes* $\{a, b\}$, *he has an incentive to report state* c *to become better off. Note that agent 2 cannot distinguish the state* a *from the state* c. *In particular, if state* a *occurs, agent 1 has an incentive to report state* c *because his utility is* $u_1(e_1(a) + x_1(c) - e_1(c))$, *which is greater than the utility* $u_1(x_1(a))$ *when he truthfully reports state* a. *That is,*

$$u_1(e_1(a) + x_1(c) - e_1(c)) = u_1(5 + 1 - 0) = \sqrt{6} > \sqrt{4} = u_1(x_1(a)).$$

Hence, the free disposal WEE allocation is not incentive compatible.

Note that in the above example, when agent 1 reports $\{c\}$ and agent 2 reports

$\{b\}$, there will be incompatible reports. To rule out such situations, we make the following assumption.

Assumption 5 (R)**.** *For any $i \in I$ and $E_i \in \Pi_i$, $\cap_{i \in I} E_i = \{\omega\}$ for some $\omega \in \Omega$.*

Remark 35. *This assumption is only needed in this section. Assumption (R) above guarantees that there are no incompatible reports. The assumption that the intersection is a singleton set is without loss of generality. If $\{a, b\} \subseteq \cap_{i \in I} E_i$ for two states a and b, then no one can distinguish these two states and hence they can be combined as one state.*

de Castro and Yannelis (2009) showed that their choice of maximin expected utility is both sufficient and necessary for the incentive compatibility of maximin Pareto efficient allocations. In this section, we shall adopt the maximin expected utility considered in de Castro and Yannelis (2009). That is, as in Remark 1, for any two allocation $x_i, y_i \in L_i$, agent i prefers the allocation x_i to the allocation y_i if

$$\sum_{E_i \in \Pi_i} [\inf_{\omega \in E_i} u_i(\omega, x_i(\omega))] \pi_i(E_i) \geq \sum_{E_i \in \Pi_i} [\inf_{\omega \in E_i} u_i(\omega, y_i(\omega))] \pi_i(E_i).$$

Below, we propose a notion of maximin incentive compatibility.

Definition 21. *An allocation x is said to be* ***maximin incentive compatible*** *(MIC) if the following does not hold:*

1. *there exists an agent $i \in I$, and two events $E_i^1, E_i^2 \in \Pi_i$;*
2. *$e_i(\omega) + x_i(b(\omega)) - e_i(b(\omega)) \in \mathbb{R}^l_+$ for each $\omega \in E_i^1$ and $\{b(\omega)\} = (\cap_{j \neq i} \Pi_j(\omega)) \cap E_i^2$;*
3.

$$\inf_{\omega_1 \in E_i^1} u_i(\omega_1, y_i(\omega_1)) > \inf_{\omega_1 \in E_i^1} u_i(\omega_1, x_i(\omega_1)),$$

where

$$y_i(\omega) = \begin{cases} e_i(\omega) + x_i(b(\omega)) - e_i(b(\omega)), & \text{if } \omega \in E_i^1; \\ \\ x_i(\omega), & \text{otherwise.} \end{cases}$$

In other words, an allocation is maximin incentive compatible if it is impossible for any agent to misreport the realized event and become better off. That is, if the true event is E_i^1 and agent i reports E_i^2, then the allocation y_i under the misreported event E_i^2 will not make him better off.

In this chapter, we consider a partition model for the information structure. Alternatively, one can also consider a type model.

Let $\Omega = \Theta = \prod_{i \in I} \Theta_i$, where Θ_i is the private information set of agent i. For any state $\omega \in \Omega$, $\omega = (\theta_1, \theta_2, \ldots, \theta_s)$, let $\Pi_i(\omega) = \{\theta_i\} \times \Theta_{-i}$, where Θ_{-i} is the set of states for all agents other than i. Then the maximin incentive compatibility can be described as follows, and Definitions 21 and 22 are equivalent.

Definition 22. *An allocation x is MIC if for every agent i and two distinct points $\tilde{\theta}_i, \hat{\theta}_i$ in Θ_i such that for every $\theta_{-i} \in \Theta_{-i}$,*

$$y_i^{\tilde{\theta}_i}(\hat{\theta}_i, \theta_{-i}) = e_i(\tilde{\theta}_i) + x_i(\hat{\theta}_i, \theta_{-i}) - e_i(\hat{\theta}_i) \in \mathbb{R}_+^l$$

and

$$\inf_{\theta_{-i} \in \Theta_{-i}} u_i(\tilde{\theta}_i, x_i(\tilde{\theta}_i, \theta_{-i})) \geq \inf_{\theta_{-i} \in \Theta_{-i}} u_i(\tilde{\theta}_i, y_i^{\tilde{\theta}_i}(\hat{\theta}_i, \theta_{-i})).$$

Thus, an agent i cannot become better off in terms of maximin expected utility by reporting $\hat{\theta}_i$ when his true state is $\tilde{\theta}_i$.

The following theorem shows that any maximin efficient allocation is maximin incentive compatible.

Theorem 19. *If Assumptions (E), (U) and (R) hold, then any maximin efficient allocation in $\mathcal{E}$ is MIC.*

Proof. See Section 3.7. □

Corollary 11. *Under the conditions of Theorem 19, any MC or MEE allocation is maximin incentive compatible.*

Remark 36. *There is a substantial literature on the mechanism design under ambiguity; see, for example, Bodoh-Creed (2012) and de Castro and Yannelis (2009). Bodoh-Creed (2012) considers a standard mechanism design environment except that agents are ambiguity averse with preferences of the maxmin expected utility. In particular, Bodoh-Creed (2012) assumes that each agent knows his valuation but has ambiguous beliefs about the distribution of valuations of the other agents which can be modeled by a convex set of priors, while we consider the particular case that this set contains all possible priors. There are significant differences between Bodoh-Creed's paper and ours. In particular, Bodoh-Creed (2012) focuses on the payoff equivalence theorem and characterizes the revenue maximizing mechanism, which could be constrained efficient (that is, second best efficient). On the contrary, we study the issue between the first best efficiency and incentive compatibility.*

Remark 37. *One could extend the result of Angelopoulos and Koutsougeras (2015) on maximin value allocations to an ambiguous asymmetric information economy with countably many states. By standard arguments, one could show that the maximin value allocation is maximin efficient, and therefore, it is maximin incentive compatible* *by the above corollary.*

3.6 Concluding Remarks

We presented a new asymmetric information economy framework, where agents face ambiguity (that is, they are MEU maximizers) and also the state space is not necessarily finite. This new set up allowed us to derive new core -Walras existence and equivalence results. It should be noted that contrary to the Bayesian asymmetric information economy framework, our core and Walrasian equilibrium concepts formulated in an ambiguous asymmetric information economy framework are now incentive compatible and obviously efficient. For this reason, we believe that our new results will be useful to other fields in economics.

We would like to conclude by saying that the continuum of states and modeling perfect competition as in Sun and Yannelis (2007, 2008), Sun, Wu and Yannelis (2012, 2013) and Qiao, Sun and Zhang (2014), or modeling the idea of informational smallness (that is, approximate perfect competition) in countable replica economies as in McLean and Postlewaite (2003), or characterizing cores in economies where agents' information can be altered by coalitions as in Hervés-Beloso, Meo and Moreno-García (2014) in the presence of ambiguity remain open questions and further research in this direction seems to be needed.

3.7 Proofs

3.7.1 Proof of Proposition 7

It is clear that V_i is increasing and concave, we first show that it is weak* lower semicontinuous. Suppose that the sequence $\{z^k\}_{k\geq 0} \subseteq L_i$, and $z^k \to z^0$ in the weak* topology as $k \to \infty$. Fix $\epsilon > 0$. Since $z^0 \in L_i = l^\infty_+$, there exists some positive number $K_0 > 0$ such that $z^0(\omega, j) < K_0$ for each $1 \leq j \leq l$ and $\omega \in \Omega$. By Assumption (U.3), there exists some $K > 0$ such that $u_i(\omega, z^0(\omega)) \leq K$ for any $\omega \in \Omega$.

Suppose that $\Pi_i = \{E_m\}_{m\in\mathbb{N}}$. Then there exists some m_0 sufficiently large such that $\pi_i(\cup_{1\leq m\leq m_0} E_m) > 1 - \frac{\epsilon}{2K}$. Let $\Omega^{m_0} = \cup_{1\leq m\leq m_0} E_m$. Then we have

$$
\begin{aligned}
V_i(z^k) - V_i(z^0) &= \inf_{\mu\in P_i} \sum_{\omega\in\Omega} u_i(\omega, z^k(\omega))\mu(\omega) - \inf_{\mu\in P_i} \sum_{\omega\in\Omega} u_i(\omega, z^0(\omega))\mu(\omega) \\
&\geq \inf_{\mu\in P_i} \sum_{\omega\in\Omega^{m_0}} u_i(\omega, z^k(\omega))\mu(\omega) - \inf_{\mu\in P_i} \sum_{\omega\in\Omega^{m_0}} u_i(\omega, z^0(\omega))\mu(\omega) \\
&\quad - \inf_{\mu\in P_i} \sum_{\omega\notin\Omega^{m_0}} u_i(\omega, z^0(\omega))\mu(\omega).
\end{aligned}
$$

For the third term, we have

$$
\inf_{\mu\in P_i} \sum_{\omega\notin\Omega^{m_0}} u_i(\omega, z^0(\omega))\mu(\omega) \leq K\pi_i(\Omega \setminus \Omega^{m_0}) < \frac{\epsilon}{2}.
$$

Since z^k weak* converges to z^0 and Ω^{m_0} is finite, $z^k(\omega)$ converges to $z^0(\omega)$ for each $\omega \in \Omega^{m_0}$. Thus, we have

$$
\left| \inf_{\mu\in P_i} \sum_{\omega\in\Omega^{m_0}} u_i(\omega, z^k(\omega))\mu(\omega) - \inf_{\mu\in P_i} \sum_{\omega\in\Omega^{m_0}} u_i(\omega, z^0(\omega))\mu(\omega) \right| < \frac{\epsilon}{2}
$$

for k sufficiently large. As a result, $V_i(z^k) - V_i(z^0) > -\epsilon$ for k sufficiently large, which implies that $V_i(\cdot)$ is weak* lower semicontinuous.

Next we show that V_i is continuous in the sup-norm topology. The proof is similar as the argument above.

Suppose that the sequence $\{z^k\}_{k\geq 0} \subseteq L_i$, and $z^k \to z^0$ in the sup-norm topology. Then $\{z^k\}_{k\geq 0}$ is uniformly bounded by some K_0. By Assumption (U.3), there exists some $K > 0$ such that $u_i(\omega, z^k(\omega)) \leq K$ for any $k \geq 0$ and $\omega \in \Omega$. Following an analogous argument as in the proof of the weak* lower semicontinuity, one can obtain a finite subset Ω^{m_0} such that $\pi_i(\Omega^{m_0}) > 1 - \frac{\epsilon}{2K}$. Then we have

$$\begin{aligned}
\left|V_i(z^k) - V_i(z^0)\right| &= \left|\inf_{\mu\in P_i}\sum_{\omega\in\Omega} u_i(\omega, z^k(\omega))\mu(\omega) - \inf_{\mu\in P_i}\sum_{\omega\in\Omega} u_i(\omega, z^0(\omega))\mu(\omega)\right| \\
&\leq \left|\inf_{\mu\in P_i}\sum_{\omega\in\Omega^{m_0}} u_i(\omega, z^k(\omega))\mu(\omega) - \inf_{\mu\in P_i}\sum_{\omega\in\Omega^{m_0}} u_i(\omega, z^0(\omega))\mu(\omega)\right| \\
&\quad + \sup_{\mu\in P_i}\sum_{\omega\notin\Omega^{m_0}} u_i(\omega, z^k(\omega))\mu(\omega) + \sup_{\mu\in P_i}\sum_{\omega\notin\Omega^{m_0}} u_i(\omega, z^0(\omega))\mu(\omega).
\end{aligned}$$

As in the above argument,

$$\sup_{\mu\in P_i}\sum_{\omega\notin\Omega^{m_0}} u_i(\omega, z^k(\omega))\mu(\omega),\ \sup_{\mu\in P_i}\sum_{\omega\notin\Omega^{m_0}} u_i(\omega, z^0(\omega))\mu(\omega) < \frac{\epsilon}{2};$$

and

$$\left|\inf_{\mu\in P_i}\sum_{\omega\in\Omega^{m_0}} u_i(\omega, z^k(\omega))\mu(\omega) - \inf_{\mu\in P_i}\sum_{\omega\in\Omega^{m_0}} u_i(\omega, z^0(\omega))\mu(\omega)\right| < \frac{\epsilon}{2}$$

for k sufficiently large. As a result, $\left|V_i(z^k) - V_i(z^0)\right| \leq \epsilon$ for k sufficiently large, which implies that $V_i(\cdot)$ is continuous in the sup-norm topology.

3.7.2 Proofs in Sections 3.3 and 3.4

One can view an ambiguous asymmetric information economy $\mathcal{E}$ as a complete information economy $\mathcal{E}_d = \{(l^\infty_+, V_i, e_i) \colon i \in I\}$ with the agent space I.[14] That is, each agent i has the utility function V_i and the infinite dimensional commodity space l^∞_+. Given the initial endowment $e_i \colon \Omega \to \mathbb{R}^l_+$ in the economy $\mathcal{E}$, since Ω is countable, e_i can be viewed as a point in the infinite dimensional commodity space l^∞_+ of the deterministic economy $\mathcal{E}_d$. By Proposition 7, the utility function V_i is increasing, concave and norm continuous, and lower semicontinuous in the weak* topology.

Given an allocation $x = (x_1, \ldots, x_s) \in l^\infty_+$ and a price $p \in (l^\infty)^\circ$, for any agent $i \in I$,

$$p \cdot x_i = \int_\Omega x_i(\omega) p(\mathrm{d}\omega).$$

An equilibrium in $\mathcal{E}_d$ is a pair $(x = (x_1, \ldots, x_s), p)$ with $x_i \in l^\infty_+$ for each $i \in I$ and $p \in (l^\infty)^\circ$ such that

1. $x_i \in B_i(p) = \{y \in l^\infty_+ \colon p \cdot y \leq p \cdot e_i\}$;
2. x_i maximizes $V_i(\cdot)$ on the budget set $B_i(p)$;
3. $\sum_{i \in I} x_i = \sum_{i \in I} e_i$.

It can be easily checked that if $p \in l^1$, then the equilibrium (x, p) in the economy $\mathcal{E}_d$ is also an MEE in the ambiguous asymmetric information economy $\mathcal{E}$.

Since V_i is norm continuous, it is Mackey continuous with respect to the Mackey topology $\tau(l^\infty, (l^\infty)^\circ)$ by Corollary 6.23 in Aliprantis and Border (2006).

[14] Let l^∞ and l^1 represent the spaces of all bounded sequences and all absolutely summable sequences, respectively. Denote by $(l^\infty)^\circ$ the topological dual space of l^∞.

Then the economy $\mathcal{E}_d$ has a competitive equilibrium (x^*, p^*) by Propositions 5.2.3 and 5.3.1 in Florenzano (2003), where $p^* \in (l^\infty)^\circ$. Since V_i is lower semicontinuous in the weak* topology, it is also lower semicontinuous in the Mackey topology $\tau(l^\infty, l^1)$. By Theorem 2 in Bewley (1972), we know that p^* is indeed in l^1. One can then normalize p^* such that $\|p^*\|_1 = 1$. Then it is clear that (x^*, p^*) is also a maximin expectations equilibrium in the ambiguous asymmetric information economy $\mathcal{E}$, which proves Theorem 13.

If $\mathcal{E}_c$ is an atomless ambiguous asymmetric information economy, one can also view $\mathcal{E}_c$ as an atomless complete information economy $\mathcal{E}_{cd}$ as above. Then Theorems 17 and 18 follow from Theorems 3.2 and 3.3 in Hervés-Beloso, Moreno-García and Yannelis (2005b).

3.7.3 Proof of Theorem 19

Recall that for any agent i, allocation $z \in L_i$ and event $E \in \Pi_i$, $v_i(E, z) = \inf_{\omega \in E} u_i(\omega, z(\omega))$. Let $\{x_i\}_{i \in I}$ be a maximin efficient allocation, and assume that it is not maximin incentive compatible. Then there exist an agent $i \in I$, and two events $E_i^1, E_i^2 \in \Pi_i$ such that

$$v_i(E_i^1, y_i) > v_i(E_i^1, x_i),$$

where

$$y_i(\omega) = \begin{cases} e_i(\omega) + x_i(b) - e_i(b), & \text{if } \omega \in E_i^1, \{b\} = (\cap_{j \neq i} \Pi_j(\omega)) \cap E_i^2; \\ x_i(\omega), & \text{otherwise.} \end{cases}$$

For each $j \neq i$, define $y_j : \Omega \to \mathbb{R}^l_+$ as follows:

$$y_j(\omega) = \begin{cases} e_j(\omega) + x_j(b) - e_j(b), & \text{if } \omega \in E_i^1, \{b\} = (\cap_{j\neq i}\Pi_j(\omega)) \cap E_i^2; \\ \\ x_j(\omega), & \text{otherwise.} \end{cases}$$

It can be easily checked that $\{y_i\}_{i\in I}$ is feasible:

1. If $\omega \in E_i^1$ and $\{b\} = (\cap_{j\neq i}\Pi_j(\omega)) \cap E_i^2$, then $\sum_{j\in I} y_j(\omega) = \sum_{j\in I} e_j(\omega) + \sum_{j\in I} x_j(b) - \sum_{j\in I} e_j(b) = \sum_{j\in I} e_j(\omega)$, since $\sum_{j\in I} e_j(b) = \sum_{j\in I} x_j(b)$.
2. If $\omega \notin E_i^1$, then $\sum_{j\in I} y_j(\omega) = \sum_{j\in I} x_j(\omega) = \sum_{j\in I} e_j(\omega)$.

We now show that agent i is better off and all other agents are not worse off if considering the allocation y instead of x.

For agent i, if $\omega \notin E_i^1$, then $v_i(\omega, y_i) = v_i(\omega, x_i)$. In addition, $v_i(E_i^1, y_i) > v_i(E_i^1, x_i)$. Therefore, $V_i(y_i) = \sum_{E_i\in\Pi_i} v_i(E_i, y_i)\pi_i(E_i) > \sum_{E_i\in\Pi_i} v_i(E_i, x_i)\pi_i(E_i) = V_i(x_i)$.

For $j \neq i$ and event E_j, if $\omega \in E_i^1$, then there exists a point $b(\omega) \in E_j \cap E_i^2$ such that $e_j(b(\omega)) = e_j(\omega)$ and $y_j(\omega) = e_j(\omega) + x_j(b(\omega)) - e_j(b(\omega)) = x_j(b(\omega))$. Notice that $u_j(\omega, y_j(\omega)) = u_j(\omega, x_j(b(\omega))) = u_j(b(\omega), x_j(b(\omega)))$. If $\omega \notin E_i^1$, then $y_j(\omega) = x_j(\omega)$.

Thus, we have

$$
\begin{aligned}
v_j(E_j, y_j) &= \min\left(\inf_{\omega\in E_j, \omega\in E_i^1} u_j(\omega, y_j(\omega)), \inf_{\omega\in E_j, \omega\notin E_i^1} u_j(\omega, y_j(\omega))\right) \\
&= \min\left(\inf_{\omega\in E_j, \omega\in E_i^1} u_j(b(\omega), x_j(b(\omega))), \inf_{\omega\in E_j, \omega\notin E_i^1} u_j(\omega, x_j(\omega))\right) \\
&= \inf_{\omega\in E_j, \omega\notin E_i^1} u_j(\omega, x_j(\omega)) \\
&\geq \inf_{\omega\in E_j} u_j(\omega, x_j(\omega)) \\
&= v_j(E_j, x_j).
\end{aligned}
$$

Then $V_j(y_j) = \sum_{E_j\in\Pi_j} v_j(E_j, y_j)\pi_j(E_j) \geq \sum_{E_j\in\Pi_j} v_j(E_j, x_j)\pi_j(E_j) = V_j(x_j)$ for all $j \neq i$.

Since $\epsilon y_i \to y_i$ as $\epsilon \to 1$ in $(\mathbb{R}^l_+)^\infty$ and V_i is continuous, there exists $\epsilon \in (0, 1)$ such that

$$V_i(\epsilon y_i) > V_i(x_i) \text{ for all } i \in C.$$

For all $\omega \in \Omega$, define

$$
z_j(\omega) = \begin{cases} \epsilon y_j(\omega) & \text{if } j = i; \\ y_j(\omega) + \frac{1-\epsilon}{\|I-1\|} y_i(\omega) & \text{if } j \neq i. \end{cases}
$$

Then $V_i(z_i) = V_i(\epsilon y_i) > V_i(x_i)$. Moreover, since $u_i(\omega, \cdot)$ is strongly monotone, for all $j \neq i$

$$V_j(z_j) = V_j(y_j + \frac{1-\epsilon}{\|I-1\|} y_i) > V_j(y_j) \geq V_j(x_j). \tag{3.2}$$

Notice that for every $\omega \in \Omega$,

$$\sum_{i \in I} z_i(\omega) = \epsilon y_i(\omega) + \sum_{j \neq i} y_j(\omega) + (1 - \epsilon) y_i(\omega)$$

$$= \sum_{i \in I} y_i(\omega) = \sum_{i \in I} e_i(\omega).$$

That is, z is feasible and by (3.2), $V_i(z_i) > V_i(x_i)$ for any i. Thus, $\{x_i\}_{i \in I}$ is not maximin efficient, a contradiction.

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