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# Non-commutative deformation rings

Benjamin Paul Margolin  
*University of Iowa*

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NON-COMMUTATIVE DEFORMATION RINGS

by

Benjamin Paul Margolin

A thesis submitted in partial fulfillment of the  
requirements for the Doctor of Philosophy  
degree in Mathematics  
in the Graduate College of  
The University of Iowa

May 2016

Thesis Supervisor: Professor Frauke Bleher

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Iowa City, Iowa

CERTIFICATE OF APPROVAL

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PH.D. THESIS

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This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the  
thesis requirement for the Doctor of Philosophy degree  
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Existence and uniqueness are more than abstract mathematical concepts. They are human concepts as well. While this thesis proves powerful mathematical results, it is perhaps more significant to note that this thesis is proof of the power of family, persistence, and the notion that unsolved problems are not necessarily ones that are impossible to solve.

While this thesis was physically written by me, I am a mere asterisk in the list of its contributors. For existence, uniqueness, and everything in between, I thank my parents, Mom and Dad, whose support is one of the confounding and wonderful manifestations of the infinite. Not of any lesser cardinality, I want to thank my sisters, Lisa and Becky, for always making sure I knew that I was never alone in solving the problems of life.

To my wife, Adri, you are the reason I went back to school. Far more importantly, however, you are proof that dreams are worth dreaming, and for that I am forever thankful. I love you.

I owe Professor Dan Anderson a very special kind of gratitude, as he provided me with a solution to the problem of second chances. During a time when my lifestyle might have been converging to unenviable values, you opened up a neighborhood of knowledge, support and trust which allowed me to redefine myself in ways which I had once forgotten existed. None of this would have happened without your willingness to take a chance on me. Thank you.

To Professor Frauke Bleher, what can I begin to say. That is a sentence which expresses a statement of fact, so please do not try and amend the punctuation, as you have made sufficiently many corrections to my thesis, already, all of which I am thankful for. Without your knowledge and expertise, I am not ashamed to say that my fingerprint on this subject would not have found the light of day. Your willingness to take me on as a student in trying times—for multiple reasons—was both motivating and humbling. You are an amazing advisor, an amazing supporter and defender of my work and my life interests, but above all you are an amazing teacher. I express profound gratitude to you for the time I have been able to learn from you, and I thank you for always having my best interests at heart.

At the end of the day, this thesis is dedicated to the subject of Mathematics. Whenever a mathematician is asked what usefulness his or her project or work provides, may he or she confidently respond that it likely addresses either a current intellectual need or is in advance of a future one, and perhaps even provides its inventor with a non-zero amount of happiness. What else could we hope for?

## ABSTRACT

The goal of this thesis is to study non-commutative deformation rings of representations of algebras. The main motivation is to provide a generalization of the deformation theory over commutative local rings studied by B. Mazur, M. Schlessinger and others. The latter deformation theory has played an important role in number theory, and in particular in the proof of Fermat's Last Theorem.

The thesis is divided into two parts.

In the first part,  $A$  is an arbitrary  $\lambda$ -algebra for a complete local commutative Noetherian ring  $\lambda$  with residue field  $k$ . A category  $\hat{\mathcal{C}}$  is defined whose objects are complete local  $\lambda$ -algebras  $R$  with residue field  $k$  such that  $R$  is a quotient ring of a power series algebra over  $\lambda$  in finitely many non-commuting variables. If  $V$  is a finite dimensional  $k$ -vector space that is also a left  $A$ -module and that satisfies a natural finiteness condition, it is proved that  $V$  has a so-called versal deformation ring  $R(A, V)$ . More precisely,  $R(A, V)$  is an object in  $\hat{\mathcal{C}}$  such that the isomorphism class of every lift of  $V$  over an object  $R$  in  $\hat{\mathcal{C}}$  arises from a morphism  $\alpha : R(A, V) \rightarrow R$  in  $\hat{\mathcal{C}}$  and  $\alpha$  is unique if  $R$  is the ring of dual numbers  $k[\epsilon]$ .

In the second part, two particular examples of  $\lambda$ ,  $A$  and  $V$  are studied and the versal deformation ring  $R(A, V)$  is determined in each of these cases. In the first example,  $\lambda = k$ ,  $A$  is a series of non-commutative  $k$ -algebras depending on a parameter  $r \geq 2$ , and  $V$  is a particular quotient module of  $A$ ; it is shown that  $R(A, V)$  is isomorphic to  $A$ . The second example builds on the first example when

$r = 2$  and uses that, if additionally the characteristic of  $k$  is 2, then  $A$  is isomorphic to the group ring  $k[D_8]$  of a dihedral group  $D_8$  of order 8. It is shown that if  $k$  is perfect and  $W$  is the ring of infinite Witt vectors over  $k$ , then  $R(W[D_8], V)$  is isomorphic to  $W[D_8]$ .

## PUBLIC ABSTRACT

The goal of this thesis is to study non-commutative deformation rings of representations of algebras. The main motivation is to provide a generalization of the deformation theory over commutative local rings studied by B. Mazur, M. Schlessinger and others. The latter deformation theory has played an important role in number theory, and in particular in the proof of Fermat's Last Theorem.

The thesis is divided into two parts.

In the first part, a theory of deformations of representations of an algebra  $A$  over a certain class of complete local rings with a fixed residue field  $k$  is developed. It is shown that every finite dimensional  $k$ -vector space  $V$  that is also a left  $A$ -module and that satisfies a natural finiteness condition has a so-called versal deformation ring  $R(A, V)$ , which is such a complete local ring with residue field  $k$ .

In the second part, two particular examples of non-commutative algebras  $A$  and modules  $V$  are studied and the versal deformation ring  $R(A, V)$  is shown to be isomorphic to  $A$  in each of these cases.



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# CHAPTER 1

## INTRODUCTION

### 1.1 Motivation

In this thesis, we study non-commutative deformation rings of representations of algebras. Our main motivation is to provide a generalization of the deformation theory over commutative local rings studied by Mazur [8], Schlessinger [10] and others. The latter deformation theory has played an important role in number theory, and in particular in the proof of Fermat's Last Theorem. For a related approach to non-commutative deformation theory, see [6].

Let  $k$  be a field, let  $\lambda$  be a complete local commutative Noetherian ring with residue field  $k$ , and let  $A$  be an arbitrary  $\lambda$ -algebra. Suppose  $V$  is a finite dimensional vector space over  $k$  such that  $V$  is also a left  $A$ -module.

The first goal of this thesis is to prove that if  $V$  satisfies a natural finiteness condition, then  $V$  has a versal deformation ring  $R(A, V)$  that is a complete local  $\lambda$ -algebra with residue field  $k$  such that the maximal ideal of  $R(A, V)$  modulo its square is finitely generated as an  $R(A, V)$ -module. Note that  $R(A, V)$  can be expressed as a quotient ring of a power series algebra over  $\lambda$  in finitely many non-commuting variables. The main idea behind the proof of this result is to modify Schlessinger's arguments in [10] so that they can be applied to the current situation. The second goal of this thesis is to apply this non-commutative deformation theory to particular examples of  $\lambda$ ,  $A$  and  $V$  for which the versal deformation ring  $R(A, V)$

is non-commutative.

## 1.2 Overview

In Chapter 2, we provide the necessary background on non-commutative complete local rings with residue field  $k$  as used in the remainder of the thesis. In particular, we introduce the category  $\hat{\mathcal{C}}$  of all rings over which we later define deformations of  $V$ . We prove basic properties about the category  $\hat{\mathcal{C}}$  in Theorem 2.8.

In Chapter 3, we define lifts and deformations of  $V$  over objects in  $\hat{\mathcal{C}}$  and also define versal and universal deformation rings. We prove our main result, Theorem 3.16, which shows that if  $V$  satisfies a natural finiteness condition then it has a versal deformation ring in  $\hat{\mathcal{C}}$ . More precisely, the finiteness condition, which is the same that was used by Schlessinger in [10], is that the set of all deformations of  $V$  over the dual numbers  $k[\epsilon]$  defines a finite dimensional vector space over  $k$ . In Section 3.2, we describe an example of a versal deformation ring of a one-dimensional representation, and we also discuss the connection of non-commutative versal deformation rings to commutative ones.

In Chapter 4, we consider two particular examples of  $\lambda$ ,  $A$  and  $V$ . Namely, in Section 4.1,  $\lambda = k$ ,  $r \geq 2$  is an integer,  $A = k\langle\langle x, y \rangle\rangle / (x^2, y^2, (xy)^r - (yx)^r)$ , and  $V$  is a particular quotient module of  $A$  of  $k$ -dimension  $2r + 1$ . We prove that  $R(A, V)$  is isomorphic to  $A$  in this case. In Section 4.2, we build on this example when  $r = 2$  and the characteristic of  $k$  is 2. In this case, the  $k$ -algebra  $A$  from Section 4.1 is isomorphic to the group ring  $k[D_8]$  of a dihedral group  $D_8$  of order 8. We show that

if  $k$  is perfect and  $W$  is the ring of infinite Witt vectors over  $k$ , then  $R(W[D_8], V)$  is isomorphic to  $W[D_8]$ .

## CHAPTER 2 BACKGROUND

In this chapter, we provide the necessary background on non-commutative complete local rings, as used in the remainder of this thesis. Throughout this thesis, all rings are assumed to be associative and to contain a multiplicative identity. We say a ring  $R$  is local if and only if  $R$  contains a unique maximal left ideal, which we denote by  $m_R$ . We say  $R$  is a local ring with residue field  $k$  if and only if  $R$  is a local ring and  $R/m_R$  is isomorphic to a given field  $k$ .

**Remark 2.1.** The above definition of a local ring is the same as used in [4, Section 19]. For a different definition of a local ring, see for example [1, Section 4.1], where our local rings are called scalar local. Note that if  $R$  is a local ring in our sense, then the full matrix ring  $Mat_n(R)$  is not local in our sense for  $n \geq 2$ .

The following lemma lists some obvious properties of local rings.

**Lemma 2.2.** *Let  $R$  be a non-zero ring with group of units  $U(R)$  and Jacobson radical  $rad R$ .*

(a) *The following statements are equivalent:*

- (i)  *$R$  is local, i.e.  $R$  has a unique maximal left ideal  $m_R$ .*
- (ii)  *$R$  has a unique maximal right ideal.*
- (iii)  *$R/rad R$  is a division ring.*
- (iv)  *$R - U(R)$  is a two-sided ideal of  $R$ .*

(v)  $R - U(R)$  is a group under addition.

(b) Let  $R$  be a local ring. Then  $R$  has a unique maximal two-sided ideal. Moreover,  $R$  is Dedekind-finite (i.e., if  $a \in R$  has a left inverse, then  $a \in U(R)$ ), and  $R$  has no nontrivial idempotents.

(c) Suppose  $R \neq 0$ , and every  $a \in R - U(R)$  is nilpotent, then  $R$  is a local ring.

*Proof.* See [4, Theorem (19.1) and Propositions (19.2) and (19.3)]. □

We next discuss the topology on a ring induced by a two-sided ideal and the corresponding completion of a ring.

**Definition 2.3.** Let  $R$  be a ring and let  $I$  be a two-sided ideal of  $R$ .

(a) The  $I$ -adic topology on  $R$  has a fundamental system of neighborhoods of any  $r \in R$  given by the cosets of  $r + I^\ell$ , for all  $\ell \geq 0$ . The  $I$ -adic completion of  $R$  is defined to be the inverse limit  $\varprojlim_{\ell} (R/m_R^\ell)$ . We say  $R$  is  $I$ -adically complete if and only if the natural map  $R \rightarrow \varprojlim_{\ell} (R/m_R^\ell)$  is an isomorphism of rings.

(b) If  $R$  is a local ring with unique maximal left ideal  $m_R$ , we say  $R$  is complete if and only if  $R$  is  $m_R$ -adically complete.

**Remark 2.4.** Let  $R$  be a local ring with unique maximal left ideal  $m_R$ . Then  $R$  is complete if and only if, with respect to the  $m_R$ -adic topology, every Cauchy sequence in  $R$  converges and  $R$  is Hausdorff. Note that the  $m_R$ -adic topology is Hausdorff if and only if  $\bigcap_{\ell \geq 0} (m_R)^\ell = 0$ .

**Lemma 2.5.** *Let  $R$  be a complete local ring with unique maximal left ideal  $m_R$ . Let  $n \geq 1$  be an integer. Then  $\text{Mat}_n(R)$  is complete with respect to the  $\text{Mat}_n(m_R)$ -adic topology.*

*Proof.* We need to show that the natural map  $\text{Mat}_n(R) \rightarrow \varprojlim_{\ell} \text{Mat}_n(R)/\text{Mat}_n(m_R)^\ell$  is bijective. Note that  $\text{Mat}_n(m_R)^\ell = \text{Mat}_n(m_R^\ell)$  and

$$\text{Mat}_n(R)/\text{Mat}_n(m_R^\ell) = \text{Mat}_n(R/m_R^\ell)$$

for all  $\ell \geq 1$ . Consider any sequence  $(B_\ell + \text{Mat}_n(m_R^\ell))_\ell$  in  $\varprojlim_{\ell} \text{Mat}_n(R)/\text{Mat}_n(m_R)^\ell$ . For all  $1 \leq i, j \leq n$ , the  $(i, j)$ -entries of the matrices  $B_\ell$ ,  $\ell \geq 1$ , result in a sequence  $((B_\ell)_{ij} + m_R^\ell)_\ell$  in the inverse limit  $\varprojlim_{\ell} (R/m_R^\ell)$ . Since  $R$  is  $m_R$ -adically complete, we can find unique elements  $a_{ij}$  in  $R$  such that  $a_{ij} - (B_\ell)_{ij}$  lies in  $m_R^\ell$  for all  $\ell \geq 1$ . Therefore,  $A = [a_{ij}]$  is the unique matrix in  $\text{Mat}_n(R)$  such that  $A - B_\ell$  lies in  $\text{Mat}_n(m_R^\ell)$  for all  $\ell \geq 1$ . This shows that  $\text{Mat}_n(R)$  is complete with respect to the  $\text{Mat}_n(m_R)$ -adic topology.  $\square$

For the remainder of the thesis, we fix the following definitions and notations.

Let  $k$  be a field, let  $\lambda$  be a complete, local, commutative ring with residue field isomorphic to  $k$ , and let  $\mu = \mu_\lambda$  denote the unique maximal ideal of  $\lambda$ . Moreover, assume that  $\mu/\mu^2$  is finitely generated as a left  $\lambda$ -module, and let  $\pi_\lambda : \lambda \rightarrow k$  be a fixed surjection from  $\lambda$  onto its residue field. For any ring  $R$ , we let  $Z(R)$  denote its center. If  $R$  is a local  $\lambda$ -algebra with unique maximal left ideal  $m_R$ , we let  $j_R$  denote the ring homomorphism  $j_R : \lambda \rightarrow R$  which satisfies  $j_R(1_\lambda) = 1_R$ ,  $j_R(\lambda) \subseteq Z(R)$  and which gives  $R$  its structure as a  $\lambda$ -algebra. We define  $\mu_R = j_R(\mu)$ . Finally, we view



$R$  as a topological ring with respect to the  $m_R$ -adic topology.

We are now ready to define the category  $\hat{\mathcal{C}}$ , which plays a key role in this thesis.

**Definition 2.6.** Let  $R$  be a complete, local  $\lambda$ -algebra with residue field isomorphic to  $k$ , such that  $m_R/m_R^2$  is finitely generated as a left  $\lambda$ -module, by which we more rigorously mean that it is finitely generated as a left  $j_R(\lambda)$ -module. Fix a surjective ring homomorphism  $\pi_R : R \rightarrow k$  from  $R$  onto its residue field.

Define the category  $\hat{\mathcal{C}}$  as follows: The objects of  $\hat{\mathcal{C}}$  are ordered pairs  $(R, \pi_R)$ , where  $R$  and  $\pi_R$  have the properties described above, and where  $\pi_R \circ j_R = \pi_\lambda$ , as shown in Figure 2.1.

Figure 2.1: The condition  $\pi_R \circ j_R = \pi_\lambda$ .

$$\begin{array}{ccc}
 \lambda & \xrightarrow{j_R} & R \\
 & \searrow \pi_\lambda & \swarrow \pi_R \\
 & & k
 \end{array}$$

If  $(R, \pi_R)$  and  $(S, \pi_S)$  are both objects of  $\hat{\mathcal{C}}$ , then a  $\hat{\mathcal{C}}$ -morphism from  $(R, \pi_R)$  to  $(S, \pi_S)$  is defined to be a  $\lambda$ -algebra homomorphism  $f : R \rightarrow S$  such that  $f(1_R) = 1_S$ , and  $\pi_S \circ f = \pi_R$ .

It should be noted that we will often suppress the residue maps when speaking of objects in  $\hat{\mathcal{C}}$ . Moreover, we denote the natural extension of a morphism  $f : R \rightarrow S$  in  $\hat{\mathcal{C}}$  to a  $\lambda$ -algebra homomorphism  $Mat_n(R) \rightarrow Mat_n(S)$  also by  $f$ .

**Remark 2.7.** Let  $R$  be an object of  $\hat{\mathcal{C}}$ , and let  $r$  be an element of  $R$ . Since  $\pi_R \circ j_R = \pi_\lambda$  is surjective, there exists some element  $l$  in  $\lambda$  such that

$$\pi_R(r) = \pi_\lambda(l) = (\pi_R \circ j_R)(l).$$

Therefore,  $r - j_R(l)$  is in  $ker(\pi_R) = m_R$ , and hence there exists some  $m$  in  $m_R$  such that  $r = j_R(l) + m$ . Note that if  $s + m_R^2$  is any element of the left  $R$ -module  $m_R/m_R^2$ , then  $s$  is in  $m_R$  and

$$r \cdot (s + m_R^2) = (j_R(l) + m) \cdot (s + m_R^2) = j_R(l)s + m_R^2 = j_R(l) \cdot (s + m_R^2).$$

Since  $j_R(\lambda) \subseteq Z(R)$ , we have

$$j_R(l)s + m_R^2 = sj_R(l) + m_R^2 = (s + m_R^2) \cdot j_R(l) = (s + m_R^2) \cdot r.$$

Hence,  $m_R/m_R^2$  can be spoken about unambiguously as a left/right  $\lambda$ -module or as a left/right  $R$ -module, with the left/right module structure arising from left/right multiplication. From the above equations, it is clear that  $m_R/m_R^2$  is finitely generated as a left  $R$ -module  $\iff$  it is finitely generated as a left  $\lambda$ -module  $\iff$  it is finitely generated as a right  $\lambda$ -module  $\iff$  it is finitely generated as a right  $R$ -module.

We now prove some properties about the category  $\hat{\mathcal{C}}$ . We use the usual convention that a ring  $R$  is called Artinian (resp. Noetherian) if  $R$  is both left and right Artinian (resp. Noetherian). Moreover, we define  $\lambda\langle\langle x_1, x_2, \dots, x_n \rangle\rangle$  to be the ring of

all formal power series in the finitely many non-commuting variables  $x_1, x_2, \dots, x_n$  with coefficients in  $\lambda$ , where the variables commute with  $\lambda$ .

**Theorem 2.8.**

- (a) *For any object  $R$  of  $\hat{\mathcal{C}}$ ,  $R/m_R^\ell$  is an Artinian ring for all positive integers  $\ell$ .*
- (b) *If  $R$  and  $S$  are any two local  $\lambda$ -algebras, then a  $\lambda$ -algebra homomorphism  $f : R \rightarrow S$  which satisfies  $f(1_R) = 1_S$  is continuous with respect to the  $m_R$ -adic and  $m_S$ -adic topologies if and only if  $f(m_R) \subseteq m_S$ .*
- (c) *Let  $R$  and  $S$  be two objects of  $\hat{\mathcal{C}}$ , and let  $f : R \rightarrow S$  be a morphism in  $\hat{\mathcal{C}}$ . Then  $f$  is continuous with respect to the  $m_R$ -adic and  $m_S$ -adic topologies. Moreover, for each  $R$  in  $\hat{\mathcal{C}}$ , the ring homomorphism  $j_R : \lambda \rightarrow R$ , which defines the  $\lambda$ -algebra structure on  $R$ , is a morphism in  $\hat{\mathcal{C}}$  and is continuous.*
- (d) *If  $R$  is an object of  $\hat{\mathcal{C}}$ , then  $\mu_R + m_R^2$  is an ideal of  $R$ .*
- (e) *If  $R$  is a complete, local, commutative  $\lambda$ -algebra, then  $R$  is Noetherian if and only if  $m_R/m_R^2$  is finitely generated as a  $\lambda$ -module.*
- (f) *Let  $S$  be any object of  $\hat{\mathcal{C}}$  and let  $I$  be any ideal of  $S$ . Then  $S/I$  is an object of  $\hat{\mathcal{C}}$  (with the obvious induced morphisms  $j_{S/I}$  and  $\pi_{S/I}$ ) if and only if  $I$  is a proper, closed ideal.*
- (g) *Let  $R$  and  $S$  be two objects of  $\hat{\mathcal{C}}$ , and let  $f : R \rightarrow S$  be a morphism in  $\hat{\mathcal{C}}$ . Then  $f(R)$  is an object of  $\hat{\mathcal{C}}$ .*

- (h) Let  $S = \lambda\langle\langle x_1, x_2, \dots, x_n \rangle\rangle$  be the power series ring over  $\lambda$  in the finitely many non-commuting variables  $x_1, x_2, \dots, x_n$ . Then  $S$  is an object of  $\hat{\mathcal{C}}$ , where  $j_S : \lambda \rightarrow S$  sends  $l$  in  $\lambda$  to the constant power series  $l$ ,  $m_S$  is the ideal generated by  $\mu$  and  $x_1, x_2, \dots, x_n$ , and  $\pi_S : S \rightarrow k$  is the natural surjection which sends a formal power series with constant coefficient  $l$  in  $\lambda$  to  $\pi_\lambda(l)$ .
- (i) Let  $R$  be an object of  $\hat{\mathcal{C}}$ . Then there exists an integer  $n \geq 0$  and a surjective morphism  $\phi : \lambda\langle\langle x_1, \dots, x_n \rangle\rangle \rightarrow R$  in  $\hat{\mathcal{C}}$ .
- (j) Let  $R$  be an object of  $\hat{\mathcal{C}}$ . Then  $Z(R)$  is a local  $\lambda$ -algebra with residue field  $k$  and unique maximal ideal  $m_R \cap Z(R)$ . Moreover, we have an isomorphism  $Z(R) \cong \varprojlim_{\ell} Z(R/m_R^\ell)$  via the restriction of the natural isomorphism between  $R$  and  $\varprojlim_{\ell} (R/m_R^\ell)$ .
- (k) Let  $R$  be an object of  $\hat{\mathcal{C}}$  such that the maximal ideal  $m_R$  is nilpotent. Then  $Z(R)$  is an Artinian complete, local  $\lambda$ -algebra with residue field  $k$ . In particular,  $Z(R)$  is an object of  $\hat{\mathcal{C}}$ .
- (l) Let  $f : R \rightarrow S$  be a morphism in  $\hat{\mathcal{C}}$ . Then  $f$  is an isomorphism in  $\hat{\mathcal{C}}$  if and only if  $f$  is bijective. Moreover,  $f$  is surjective if and only if the induced map  $\bar{f} : m_R/(\mu_R + m_R^2) \rightarrow m_S/(\mu_S + m_S^2)$  is surjective. If  $R = S$ , then  $f$  is bijective if and only if  $f$  is surjective.

*Proof.*

- (a) By Remark 2.7,  $m_R/m_R^2$  is an Artinian left/right  $R$ -module. Since  $R$  is in  $\hat{\mathcal{C}}$ ,  $R/m_R \cong k$  and is therefore also an Artinian left/right  $R$ -module. We now use

induction on short exact sequences of left/right  $R$ -modules of the form

$$0 \rightarrow m_R^\ell/m_R^{\ell+1} \rightarrow R/m_R^{\ell+1} \rightarrow R/m_R^\ell \rightarrow 0.$$

Using the fact that the middle term is an Artinian left/right  $R$ -module if and only if both the left and right terms are Artinian left/right  $R$ -modules, proves that  $R/m_R^\ell$  is an Artinian left/right  $R$ -module for all positive integers  $\ell$ . This proves part (a).

- (b) Let  $R$  and  $S$  be local  $\lambda$ -algebras and let  $f : R \rightarrow S$  be a  $\lambda$ -algebra homomorphism with  $f(1_R) = 1_S$ . Then  $f$  is continuous if and only if the inverse image of every open set in  $S$  is open in  $R$ .

Suppose  $f$  is continuous. Then  $f^{-1}(m_S)$  is open in  $R$ . Since  $0_R$  is in  $f^{-1}(m_S)$ , it follows that there must exist some open set  $U$  in  $R$  which contains  $0_R$ , such that  $U \subseteq f^{-1}(m_S)$ . This means there exists an integer,  $\ell$ , such that  $m_R^\ell \subseteq f^{-1}(m_S)$ . From this it follows that  $f(m_R^\ell) \subseteq m_S$ . But since  $f$  is a ring homomorphism,  $f(m_R^\ell) = (f(m_R))^\ell$ . Suppose  $f(m_R)$  is not contained in  $m_S$ . Then there is some element  $x \in m_R$  and some unit  $u \in S$  such that  $f(x) = u$ . But then  $f(x^\ell) = (f(x))^\ell = u^\ell$ . However,  $x^\ell$  is in  $m_R^\ell$ , which means that  $f(x^\ell) = (f(x))^\ell = u^\ell$  is an element of  $m_S$ . But  $u^\ell$  is a unit in  $S$ , giving a contradiction. Therefore, if  $f$  is continuous,  $f(m_R) \subseteq m_S$ .

Now suppose  $f(m_R) \subseteq m_S$ . Then  $m_R \subseteq f^{-1}(m_S)$ . Suppose that there exists some element  $v$  of  $R$  which is not in  $m_R$ , such that  $f(v)$  is in  $m_S$ . Since  $R$  is local, this means that  $v$  is a unit. Because we assume that  $f$  sends  $1_R$  to  $1_S$ , we then have

that  $1_S = f(1_R) = f(vv^{-1}) = f(v)f(v^{-1})$ , which means that  $f(v)$  is a unit in  $S$ , contradicting the assumption that  $f(v)$  is in  $m_S$ . Therefore,  $m_R = f^{-1}(m_S)$ , showing that  $f^{-1}(m_S)$  is open in  $R$ . It follows then that  $m_R^\ell \subseteq f^{-1}(m_S^\ell)$  for all  $\ell \geq 1$ . Let  $U$  be an arbitrary open set in  $S$ , and let  $x \in U$  and  $\ell_x \geq 1$  such that  $x + m_S^{\ell_x} \subseteq U$ . If  $r$  is any element in  $f^{-1}(x + m_S^{\ell_x})$ , then  $r + m_R^{\ell_x}$  is an open set in  $R$  which contains  $r$  and which is contained in  $f^{-1}(x + m_S^{\ell_x})$ . Therefore,  $f^{-1}(U)$  is open, and  $f$  is continuous.

- (c) Let  $f : R \rightarrow S$  be a morphism in  $\hat{\mathcal{C}}$ . Then  $\pi_S \circ f = \pi_R$ , and  $\ker(\pi_R) = m_R$  and  $\ker(\pi_S) = m_S$ . Let  $a$  be an element of  $m_R = \ker(\pi_R)$ . Then

$$0 = \pi_R(a) = (\pi_S \circ f)(a).$$

This implies that  $f(a)$  is in  $\ker(\pi_S) = m_S$ , proving that  $f(m_R) \subseteq m_S$ , which implies by part (b) that  $f$  is continuous with respect to the  $m_R$ -adic and  $m_S$ -adic topologies.

Notice that for each  $R$  in  $\hat{\mathcal{C}}$ , the ring homomorphism  $j_R : \lambda \rightarrow R$  satisfies  $j_R(1_\lambda) = 1_R$ , and by Definition 2.6,  $\pi_R \circ j_R = \pi_\lambda$ . Hence  $j_R$  is a morphism in  $\hat{\mathcal{C}}$ . Therefore, by the above arguments,  $j_R$  is continuous.

- (d) Let  $r$  be an element of  $R$ . By Remark 2.7, there exists some element  $l$  in  $\lambda$  and some element  $m$  in  $m_R$  such that  $r = j_R(l) + m$ . If  $x$  is in  $\mu_R + m_R^2$ , then there exists some element  $a$  in  $\mu$  and some element  $w$  in  $m_R^2$  such that  $x = j_R(a) + w$ . It follows that

$$rx = (j_R(l) + m)(j_R(a) + w) = j_R(la) + j_R(l)w + mj_R(a) + mw.$$

Since  $j_R$  is continuous,  $j_R(a)$  is in  $m_R$ , and hence  $j_R(l)w + mj_R(a) + mw$  is in  $m_R^2$ . But since  $\mu$  is an ideal in  $\lambda$ ,  $la$  is in  $\mu$ , and hence  $j_R(la)$  is in  $j_R(\mu) = \mu_R$ , proving that  $\mu_R + m_R^2$  is closed under left multiplication in  $R$ . Since both  $\mu$  and  $m_R^2$  are two sided ideals, similar calculations show that  $\mu_R + m_R^2$  is also closed under right multiplication in  $R$ . Since  $\mu_R + m_R^2$  is closed under subtraction and contains  $0_R$ , it follows that  $\mu_R + m_R^2$  is an ideal of  $R$ .

(e) See [7, Theorem 29.4].

(f) Let  $S$  be a ring in  $\hat{\mathcal{C}}$ . First notice that  $S/S$  is the zero-ring which does not have residue field isomorphic to  $k$ , and so  $S/S$  is not in  $\hat{\mathcal{C}}$ . Now let  $I$  be any proper ideal of  $S$ . Letting  $j_{S/I} : \lambda \rightarrow S/I$  be given by  $j_S : \lambda \rightarrow S$  followed by the natural projection  $S \rightarrow S/I$ , we see that  $S/I$  is a  $\lambda$ -algebra. Since  $S$  is local with unique maximal left ideal  $m_S$ , the ideal  $I$  is contained in  $m_S$ . Hence,  $S/I$  is local with unique maximal left ideal  $m_{S/I} = m_S/I$ . Moreover, the residue field of  $S/I$  is isomorphic to  $k$ , since

$$k \cong S/m_S \cong (S/I)/(m_S/I).$$

Letting  $\pi_{S/I} : S/I \rightarrow k$  be the surjection induced by  $\pi_S$ , we see that  $\pi_{S/I} \circ j_{S/I} = \pi_S \circ j_S = \pi_\lambda$ . Notice that, for all  $\ell \geq 1$ ,

$$m_{S/I}^\ell = (m_S/I)^\ell = \frac{m_S^\ell + I}{I}.$$

If

$$b_1 + m_S^2, b_2 + m_S^2, \dots, b_m + m_S^2$$

is a finite generating set for the  $S$ -module  $m_S/m_S^2$ , then

$$(b_1 + I) + m_{S/I}^2, (b_2 + I) + m_{S/I}^2, \dots, (b_m + I) + m_{S/I}^2$$

is a finite generating set for the  $S/I$ -module  $m_{S/I}/m_{S/I}^2$ . Therefore, to prove part (f), it suffices to show that  $S/I$  is complete with respect to the  $m_{S/I}$ -adic topology if and only if  $I$  is a closed ideal. But  $S/I$  will be complete if and only if the natural homomorphism from  $S/I$  to the inverse limit of the quotients  $(S/I)/m_{S/I}^\ell = (S/I)/((m_S^\ell + I)/I) \cong S/(m_S^\ell + I)$  is a bijection.

For any proper ideal  $I$  of  $S$ , this natural map will be surjective. To see this, consider any sequence  $(s_\ell + (m_S^\ell + I))_\ell$  in the inverse limit of  $S/(m_S^\ell + I)$ . Then, for all  $\ell \geq 1$ ,

$$s_{\ell+1} - s_\ell \in m_S^\ell + I.$$

Hence, there are elements  $j_\ell$  in  $I$  such that  $s_{\ell+1} + j_\ell - s_\ell \in m_S^\ell$ , for all  $\ell \geq 1$ .

Rewriting this as

$$(s_{\ell+1} + j_\ell + j_{\ell-1} + \dots + j_2 + j_1) - (s_\ell + j_{\ell-1} + j_{\ell-2} + \dots + j_2 + j_1) \in m_S^\ell,$$

it follows that  $((s_\ell + j_\ell + j_{\ell-1} + \dots + j_2 + j_1) + m_S^\ell)_\ell$  is an element in the inverse limit  $\varprojlim_\ell (S/m_S^\ell)$ . However, since  $S$  is complete with respect to the  $m_S$ -adic topology, this means that there exists a unique element  $s$  in  $S$  such that

$$s + m_S^\ell = (s_\ell + j_\ell + j_{\ell-1} + \dots + j_2 + j_1) + m_S^\ell$$

for all  $\ell$ . But this means that  $s - s_\ell$  is in  $m_S^\ell + I$  for all  $\ell$ , implying  $(s_\ell + (m_S^\ell + I))_\ell =$



$(s + (m_S^\ell + I))_\ell$  and proving that the natural map from  $S/I$  to the inverse limit of the quotients  $S/(m_S^\ell + I)$  is surjective.

Notice that this means that  $S/I$  will be complete if and only if this natural map is injective, which is if and only if  $\bigcap_\ell (m_S^\ell + I) = I$ . Clearly  $I \subseteq \bigcap_\ell (m_S^\ell + I)$ . Now suppose  $s$  is an element in  $\bigcap_\ell (m_S^\ell + I)$ . Then, for each  $\ell$ , there exist elements  $j_\ell$  in  $m_S^\ell$  and  $a_\ell$  in  $I$  such that  $s = j_\ell + a_\ell$ , which means  $s - j_\ell = a_\ell$  which is in  $I$ . However, the sequence  $(s - j_\ell)_\ell$  converges to  $s$  in the  $m_S$ -adic topology. But this implies that the sequence  $(a_\ell)_\ell$  is a sequence entirely in  $I$  which converges to  $s$ . Therefore, if  $I$  is closed,  $s$  must be in  $I$ , which means that  $\bigcap_\ell (m_S^\ell + I) = I$ , and so  $S/I$  is complete. Conversely, if  $I$  is not a closed ideal, then there exists a sequence  $(a_\ell)_\ell$  of elements entirely in  $I$  which converges to an element,  $x$ , which is not in  $I$ . Then there exist elements  $j_\ell$  in  $m_S^\ell$  such that  $x - a_\ell = j_\ell$ . Therefore,  $x = j_\ell + a_\ell$  is in  $m_S^\ell + I$  for all  $\ell$ , which means  $I$  is strictly smaller than  $\bigcap_\ell (m_S^\ell + I)$ , proving that the natural map is not injective. This completes the proof of part (f).

- (g) Since  $f(R) \cong R/\ker(f)$ , our proof will follow from part (f) if we can show that  $\ker(f)$  is a proper, closed ideal in  $R$ . Since  $f$  is a morphism in  $\hat{\mathcal{C}}$ ,  $\pi_R = \pi_S \circ f$ , and so  $f$  cannot be the zero morphism. Therefore,  $\ker(f)$  is a proper ideal of  $R$ . Moreover, by part (c),  $f$  is a continuous morphism, so the inverse image under  $f$  of a closed set in  $S$  must be a closed set in  $R$ . Since  $S$  is complete with respect to the  $m_S$ -adic topology,  $S$  is Hausdorff with respect to this topology, which

implies that one-point sets in  $S$  are closed. Since  $\ker(f)$  is exactly  $f^{-1}(\{0_S\})$ , the continuity of  $f$  proves that  $\ker(f)$  is closed in  $R$ , and hence  $\ker(f)$  is a proper and closed ideal in  $R$ .

(h) Define  $(x_1, x_2, \dots, x_n)$  to be the ideal of  $S = \lambda\langle\langle x_1, x_2, \dots, x_n \rangle\rangle$  generated by  $x_1, \dots, x_n$ , and define  $J = \mu + (x_1, x_2, \dots, x_n)$ . Then  $J$  is an ideal of  $S$ , namely the ideal generated by  $\mu$  and  $x_1, \dots, x_n$ . Let  $s$  be any element of  $S$ . We can then write  $s$  as

$$s = l_0 + \sum_i l_1^i x_i + \sum_{i'', j''} l_2^{i'' j''} x_{i''} x_{j''} + \sum_{i''', j''', k'''} l_3^{i''' j''' k'''} x_{i'''} x_{j'''} x_{k'''} + \dots$$

where all the indices range from 1 to  $n$ , and where the coefficients come from  $\lambda$ . It follows that  $s$  is not in  $J$  if and only if  $l_0$  is not in  $\mu$ . Since  $\lambda$  is assumed to be local,  $l_0$  is not in  $\mu$  if and only if  $l_0$  is a unit in  $\lambda$ . If

$$h = h_0 + \sum_i h_1^i x_i + \sum_{i'', j''} h_2^{i'' j''} x_{i''} x_{j''} + \sum_{i''', j''', k'''} h_3^{i''' j''' k'''} x_{i'''} x_{j'''} x_{k'''} + \dots$$

is another element of  $S$ , then  $s * h$  is a power series whose coefficient on the term  $x_a x_b \dots x_c x_d$  is

$$l_0 h_t^{ab \dots cd} + l_1^a h_{t-1}^{b \dots cd} + \dots + l_{t-1}^{ab \dots c} h_1^d + l_t^{ab \dots cd} h_0$$

where  $t$  is the number of variables occurring in the monomial  $x_a x_b \dots x_c x_d$ . Therefore, if  $s$  is a unit in  $S$ , then  $l_0$  must be a unit in  $\lambda$ , and hence  $s$  is not in  $J$ . Conversely, assume now  $s$  is not in  $J$ . Since then  $l_0$  is a unit, we can define  $h_0 = l_0^{-1}$ , and use a simple induction argument to successively solve the equations

$$l_0 h_t^{ab \dots cd} + l_1^a h_{t-1}^{b \dots cd} + \dots + l_{t-1}^{ab \dots c} h_1^d + l_t^{ab \dots cd} h_0 = 0$$

by setting

$$h_t^{ab\dots cd} = -l_0^{-1}(l_1^a h_{t-1}^{b\dots cd} + \dots + l_{t-1}^{ab\dots c} h_1^d + l_t^{ab\dots cd} h_0).$$

This proves that  $S - J = U(S)$ , or  $J = S - U(S)$ , and therefore proves that  $S$  is a local ring with unique maximal left ideal  $m_S = J$ .

We next prove that  $m_S/m_S^2$  is finitely generated as a left  $S$ -module. Let  $a_1 + \mu^2, a_2 + \mu^2, \dots, a_m + \mu^2$  be a finite generating set for the left  $\lambda$ -module  $\mu/\mu^2$ . Since any element  $s$  of  $m_S$  can be written as

$$s = l_0 + \sum l_1^i x_i + \sum l_2^{i''j''} x_{i''} x_{j''} + \sum l_3^{i'''j'''k'''} x_{i'''} x_{j'''} x_{k'''} + \dots$$

where  $l_0$  is in  $\mu$ , it follows that  $s + m_S^2$  is just  $(l_0 + \sum_{i=1}^n l_1^i x_i) + m_S^2$ . But since  $l_0$  is in  $\mu$ ,

$$l_0 + \mu^2 = \sum_{j=1}^m h_j a_j + \mu^2$$

for certain  $h_1, \dots, h_m$  in  $\lambda$ . Since

$$m_S^2 = (\mu + (x_1, x_2, \dots, x_n))^2 = \mu^2 + \mu * (x_1, x_2, \dots, x_n) + (x_1, x_2, \dots, x_n)^2,$$

it follows that  $s + m_S^2 = (\sum_{j=1}^m h_j a_j + \sum_{i=1}^n l_i x_i) + m_S^2$ . This proves that

$$a_1 + m_S^2, a_2 + m_S^2, \dots, a_m + m_S^2, \quad x_1 + m_S^2, x_2 + m_S^2, \dots, x_n + m_S^2$$

is a finite generating set for the left  $S$ -module  $m_S/m_S^2$ .

To prove that  $S$  is complete, we show that the natural homomorphism from  $S$  to the inverse limit  $\varprojlim_{\ell} (S/m_S^\ell)$  is a bijection. To see injectivity, let

$$s = l_0 + \sum l_1^i x_i + \sum l_2^{i''j''} x_{i''} x_{j''} + \sum l_3^{i'''j'''k'''} x_{i'''} x_{j'''} x_{k'''} + \dots$$

be any element of  $S$  which goes to 0 in the inverse limit. Then  $s$  lies in  $\bigcap_{\ell} m_S^{\ell}$ . Since  $m_S = \mu + (x_1, x_2, \dots, x_n)$ , and because  $\lambda$  commutes with each of the  $x_i$ , it follows that, for all  $\ell \geq 1$ ,

$$\begin{aligned} m_S^{\ell} &= \mu^{\ell} + \mu^{\ell-1}(x_1, x_2, \dots, x_n) + \mu^{\ell-2}(x_1, x_2, \dots, x_n)^2 \\ &+ \dots + \mu(x_1, x_2, \dots, x_n)^{\ell-1} + (x_1, x_2, \dots, x_n)^{\ell}. \end{aligned}$$

Therefore, if  $s$  as above lies in  $\bigcap_{\ell} m_S^{\ell}$ , then it follows that  $l_0$  and each of the coefficients  $l_1^i, l_2^{i''j''}, l_3^{i'''j'''k'''}, \dots$  lie in  $\bigcap_{\ell} \mu^{\ell}$ . By completeness of  $\lambda$ , this means that all the coefficients of  $s$  are 0, proving that  $s = 0$ .

To see that the natural map  $S \rightarrow \varprojlim_{\ell} (S/m_S^{\ell})$  is surjective, consider any element

$$y = (a_0 + m_S, (b_0 + \sum b_i x_i) + m_S^2, (c_0 + \sum c_1^i x_i + \sum c_2^{i''j''} x_{i''} x_{j''}) + m_S^3, \dots)$$

in the inverse limit  $\varprojlim_{\ell} (S/m_S^{\ell})$ . Because of our description of  $m_S^{\ell}$  as a sum of products of the form  $\mu^{\ell-j}(x_1, \dots, x_n)^j$  for  $0 \leq j \leq \ell$ , it follows that  $y$  gives rise to elements

$$(a_0 + \mu, b_0 + \mu^2, c_0 + \mu^3, \dots)$$

$$(0 + \mu, b_1^i + \mu^2, c_1^i + \mu^3, \dots), 1 \leq i \leq n,$$

$$(0 + \mu, 0 + \mu^2, c_2^{i''j''} + \mu^3, \dots), 1 \leq i'', j'' \leq n,$$

⋮

in the inverse limit  $\varprojlim_{\ell} (\lambda/\mu^{\ell})$ . But since  $\lambda$  is complete, all these sequences converge to elements in  $\lambda$ . This means that our original element  $y$  is the image of

the formal power series in  $S$  whose coefficients are given by these elements, completing the proof that  $S$  is complete. Using  $j_S$  and  $\pi_S$  as in the statement, this implies part (h).

- (i) Since  $R$  is in  $\hat{\mathcal{C}}$ , there exist finitely many elements  $b_1, b_2, \dots, b_n$  in  $m_R$  such that  $\{b_i + m_R^2 \mid 1 \leq i \leq n\}$  is a finite generating set for the left  $R$ -module  $m_R/m_R^2$ . Let  $S = \lambda\langle\langle x_1, x_2, \dots, x_n \rangle\rangle$ , and let  $\phi$  be the unique  $\lambda$ -algebra homomorphism

$$\phi : S = \lambda\langle\langle x_1, x_2, \dots, x_n \rangle\rangle \rightarrow R$$

given by  $\phi(x_i) = b_i$  for  $i = 1, \dots, n$ . Notice that this means that for all  $l$  in  $\lambda$ ,

$$\phi(l) = \phi(j_S(l)) = j_R(l).$$

It follows that  $\phi(1_S) = 1_R$ , and  $\phi(m_S) \subseteq m_R$ , since  $\phi(x_i) = b_i \in m_R$  and  $\phi(\mu) = j_R(\mu) \subseteq m_R$ . Therefore, writing  $s \in S$  as  $s = l_0 + m$  for some element  $l_0$  in  $\lambda$  and some element  $m$  in  $m_S$ , we obtain

$$\begin{aligned} (\pi_R \circ \phi)(s) &= \pi_R(\phi(l_0) + \phi(m)) = \pi_R(\phi(l_0)) = \pi_R(j_R(l_0)) \\ &= \pi_\lambda(l_0) = \pi_S(j_S(l_0)) = \pi_S(l_0) = \pi_S(l_0 + m) \\ &= \pi_S(s). \end{aligned}$$

Hence,  $\phi$  is a morphism in  $\hat{\mathcal{C}}$ , which means that to prove part (i), it suffices to prove that  $\phi$  is surjective. Since  $\phi \circ j_S = j_R$ , it follows from Remark 2.7 that we only need to prove that every element in  $m_R$  lies in the image of  $\phi$ .

Let now  $q$  be any element of  $m_R$ . Then there exist elements  $r_1, r_2, \dots, r_n$  in  $j_R(\lambda)$  such that

$$q \equiv \sum_{i=1}^n (r_i b_i) \pmod{m_R^2}.$$

Suppose inductively that for  $\ell \geq 1$ ,

$$q \equiv \sum_{i_1} (r_{i_1}^1 b_{i_1}) + \sum_{i_1, i_2} (r_{i_1 i_2}^2 b_{i_1} b_{i_2}) + \dots + \sum_{i_1, i_2, \dots, i_\ell} (r_{i_1 i_2 \dots i_\ell}^\ell b_{i_1} b_{i_2} \dots b_{i_\ell}) \pmod{m_R^{\ell+1}}$$

where all the coefficients  $r_{i_1}^1, r_{i_1 i_2}^2, \dots, r_{i_1 i_2 \dots i_\ell}^\ell$  lie in  $j_R(\lambda)$ . This means that there exists an element  $x$  in  $m_R^{\ell+1}$  so that

$$q = x + \sum (r_{i_1}^1 b_{i_1}) + \sum (r_{i_1 i_2}^2 b_{i_1} b_{i_2}) + \dots + \sum (r_{i_1 i_2 \dots i_\ell}^\ell b_{i_1} b_{i_2} \dots b_{i_\ell}).$$

But  $x$  is just a finite sum of products of  $(\ell + 1)$  elements in  $m_R$ , and any such product can be written as  $s_1 s_2 \dots s_{\ell+1}$  where all the  $s_i$  are in  $m_R$ . Therefore, for any  $1 \leq i \leq \ell + 1$ , there exist elements  $t_1^i, t_2^i, \dots, t_n^i$  in  $j_R(\lambda)$  and an element  $y_i$  in  $m_R^2$  so that

$$s_i = \sum_{j=1}^n (t_j^i b_j) + y_i.$$

But then it follows that each product

$$s_1 s_2 \dots s_{\ell+1} \equiv \sum_{i_1, i_2, \dots, i_{\ell+1}} (w_{i_1 i_2 \dots i_{\ell+1}}^{\ell+1} b_{i_1} b_{i_2} \dots b_{i_{\ell+1}}) \pmod{m_R^{\ell+2}},$$

where  $w_{i_1 i_2 \dots i_{\ell+1}}^{\ell+1}$  lies in  $j_R(\lambda)$ . But this implies

$$q \equiv \sum (r_{i_1}^1 b_{i_1}) + \sum (r_{i_1 i_2}^2 b_{i_1} b_{i_2}) + \dots + \sum (r_{i_1 i_2 \dots i_\ell}^{\ell+1} b_{i_1} b_{i_2} \dots b_{i_{\ell+1}}) \pmod{m_R^{\ell+2}},$$

where all the coefficients lie in  $j_R(\lambda)$ , completing the induction. In this way, we form a power series in the elements  $b_1, b_2, \dots, b_n$  over  $j_R(\lambda)$ . Since  $R$  is assumed

to be complete with respect to the  $m_R$ -adic topology, this power series converges to  $q$  in  $R$ . Since  $\phi \circ j_S = j_R$ , it follows that there exists a power series  $p$  in  $S = \lambda\langle x_1, \dots, x_n \rangle$  such that  $\phi(p) = q$ . In other words,  $\phi$  is surjective, which completes the proof of part (i).

- (j) Let  $R$  be an object in  $\hat{\mathcal{C}}$ , and let  $Z = Z(R)$ . If  $z$  is any non-unit in  $Z$ , then  $z$  is a non-unit in  $R$ , and hence lies in  $m_R$ . It follows that  $Z$  is a commutative local ring with unique maximal ideal  $m_Z = m_R \cap Z$ . Moreover, since  $j_R(\lambda) \subseteq Z$  and since  $\pi_R(R) = \pi_\lambda(\lambda) = (\pi_R \circ j_R)(\lambda)$ , it follows that  $Z/(m_R \cap Z) \cong (Z + m_R)/m_R \cong k$ . In other words,  $Z$  is a local  $\lambda$ -algebra with residue field  $k$ .

To show that  $Z \cong \varprojlim_{\ell} Z(R/m_R^\ell)$ , let  $\pi_\ell : R \rightarrow R/m_R^\ell$  be the natural surjection, for all  $\ell \geq 1$ . Since  $R$  is an object in  $\hat{\mathcal{C}}$ , the natural map  $\nu : R \rightarrow \varprojlim_{\ell} (R/m_R^\ell)$  defined by  $\nu(r) = (\pi_\ell(r))_\ell$  is an isomorphism. Note that since each  $\pi_\ell$  is surjective, it follows for any element  $z$  in  $Z$  that  $\pi_\ell(z)$  is an element in  $Z(R/m_R^\ell)$ . Therefore, the image of  $\nu$  restricted to  $Z$  lies in  $\varprojlim_{\ell} Z(R/m_R^\ell)$ . Since  $\nu$  is injective, so is this restriction. Now suppose  $(z_\ell + m_R^\ell)_\ell$  is an element of  $\varprojlim_{\ell} Z(R/m_R^\ell)$ . Then  $(z_\ell + m_R^\ell)_\ell$  is also an element of  $\varprojlim_{\ell} (R/m_R^\ell)$ , and so the surjectivity of  $\nu$  implies the existence of some  $z$  in  $R$  such that  $z - z_\ell$  lies in  $m_R^\ell$  for all  $\ell \geq 1$ . Then, if  $r$  is any element of  $R$ ,

$$rz + m_R^\ell = rz_\ell + m_R^\ell = z_\ell r + m_R^\ell = zr + m_R^\ell,$$

for all  $\ell \geq 1$ , and hence  $rz - zr \in \bigcap_{\ell} m_R^\ell = 0$ , which shows that  $z$  is in  $Z$ . Therefore, letting  $\nu'$  denote the restriction of  $\nu$  to  $Z$ , we see that  $\nu'$  is an isomorphism between

$Z$  and  $\varprojlim_{\ell} Z(R/m_R^{\ell})$ .

- (k) Let  $R$  be an object of  $\hat{\mathcal{C}}$  such that  $m_R$  is nilpotent. Then there exists some positive integer,  $\ell$ , such that  $m_R^{\ell} = 0$ . Let  $Z = Z(R)$ . We have seen in part (j) that  $Z$  is a local  $\lambda$ -algebra with residue field  $k$  and unique maximal ideal  $m_Z = m_R \cap Z$ . For each  $i \geq 1$ ,

$$(Z \cap m_R^i)/(Z \cap m_R^{i+1}) = (Z \cap m_R^i)/((Z \cap m_R^i) \cap m_R^{i+1}) \cong ((Z \cap m_R^i) + m_R^{i+1})/m_R^{i+1}.$$

Since  $m_R^i/m_R^{i+1}$  is a finitely generated  $R$ -module and since

$$m_R * (m_R^i/m_R^{i+1}) = 0,$$

$m_R^i/m_R^{i+1}$  is a finite dimensional  $k$ -vector space. Notice that  $((Z \cap m_R^i) + m_R^{i+1})/m_R^{i+1}$  is a  $Z$ -module which is annihilated by  $m_R$ , which implies that  $((Z \cap m_R^i) + m_R^{i+1})/m_R^{i+1}$  is also a  $k$ -vector space. Since  $((Z \cap m_R^i) + m_R^{i+1})/m_R^{i+1} \subseteq m_R^i/m_R^{i+1}$ , and since the action of  $k$  on both of these spaces is induced by the action of  $j_R(\lambda) \subseteq Z(R)$ , it follows that  $((Z \cap m_R^i) + m_R^{i+1})/m_R^{i+1}$  must also be a finite dimensional  $k$ -vector space. Note that each  $Z \cap m_R^i$  is an ideal of  $Z$ , and  $Z \cap m_R^{\ell} = 0$ .

Therefore, the filtration of ideals

$$Z \supseteq Z \cap m_R \supseteq Z \cap m_R^2 \supseteq \dots \supseteq Z \cap m_R^{\ell} = 0$$

gives rise to a finite composition series of  $Z$  as a  $Z$ -module, and hence  $Z$  is a commutative Artinian ring. This forces  $m_Z/m_Z^2$  to be a finitely generated left  $\lambda$ -module. Since  $m_R^{\ell} = 0$ , it follows that  $m_Z^{\ell} \subseteq Z \cap m_R^{\ell} = 0$ . Therefore, the natural map  $Z \rightarrow \varprojlim_i (Z/m_Z^i)$  is a bijection. In particular,  $Z$  satisfies all the conditions necessary to be an object of  $\hat{\mathcal{C}}$ .



- (1) The proof of part (l) follows closely the proof of [10, Lemma 1.1]. Let  $f : R \rightarrow S$  be a morphism in  $\hat{\mathcal{C}}$ . If  $f$  is an isomorphism in  $\hat{\mathcal{C}}$ , then  $f$  is certainly bijective. Conversely, if  $f$  is bijective, then  $f(m_R) = m_S$ , which implies that  $f^{-1}$  is also a morphism in  $\hat{\mathcal{C}}$ .

Let  $\bar{f} : m_R/(\mu_R + m_R^2) \rightarrow m_S/(\mu_S + m_S^2)$  be the induced map by  $f$ . Notice that  $f(\mu_R + m_R^2) \subseteq \mu_S + m_S^2$  and  $f(m_R) \subseteq m_S$ , so that  $\bar{f}$  is well-defined. If  $f$  is surjective, then  $f(m_R) = m_S$ , and hence  $\bar{f}$  is surjective.

Now suppose conversely that  $\bar{f}$  is surjective. Notice that for each object  $T$  of  $\hat{\mathcal{C}}$ , the map

$$\beta_T : \mu/\mu^2 \rightarrow \mu_T/(m_T^2 \cap \mu_T),$$

given by  $\beta_T(l + \mu^2) = j_T(l) + (m_T^2 \cap \mu_T)$ , for each  $l$  in  $\mu$ , is surjective. We have a commutative diagram of  $k$ -linear maps with exact rows as in Figure 2.2, where  $\tilde{f}_1$  and  $\tilde{\tilde{f}}_1$  are induced from  $f$ .

Figure 2.2: The commutative diagram with  $\bar{f}$ ,  $\tilde{f}_1$  and  $\tilde{\tilde{f}}_1$ .

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mu_R/(m_R^2 \cap \mu_R) & \longrightarrow & m_R/m_R^2 & \longrightarrow & m_R/(\mu_R + m_R^2) & \longrightarrow & 0 \\
& & \downarrow \tilde{\tilde{f}}_1 & & \downarrow \tilde{f}_1 & & \downarrow \bar{f} & & \\
0 & \longrightarrow & \mu_S/(m_S^2 \cap \mu_S) & \longrightarrow & m_S/m_S^2 & \longrightarrow & m_S/(\mu_S + m_S^2) & \longrightarrow & 0
\end{array}$$

Since  $\beta_S$  is surjective and  $f$  is a  $\lambda$ -algebra homomorphism, it follows that  $\tilde{\tilde{f}}_1$

is surjective. Since  $\bar{f}$  is surjective, it follows that  $\tilde{f}_1 : m_R/m_R^2 \rightarrow m_S/m_S^2$  is surjective. By Remark 2.7 and since  $\pi_S \circ f = \pi_R$ , it follows that  $\bar{f}_2 : R/m_R^2 \rightarrow S/m_S^2$ , given by  $\bar{f}_2(r + m_R^2) = f(r) + m_S^2$ , for all  $r \in R$ , is surjective. Moreover, if  $i \geq 2$  and  $y_1, \dots, y_i$  are elements in  $m_S$ , then there exist elements  $x_1, \dots, x_i \in m_R$  so that  $y_j - f(x_j) \in m_S^2$ , for all  $1 \leq j \leq i$ . But this implies that  $y_1 \dots y_i - f(x_1 \dots x_i) \in m_S^{i+1}$ , proving that the map  $\tilde{f}_i : m_R^i/m_R^{i+1} \rightarrow m_S^i/m_S^{i+1}$  induced from  $f$  is surjective for all  $i \geq 1$ . We obtain, for all  $i \geq 1$ , a commutative diagram with exact rows as in Figure 2.3, where  $\bar{f}_i$  and  $\bar{f}_{i+1}$  are induced from  $f$ .

Figure 2.3: The commutative diagram with  $\tilde{f}_i$ ,  $\bar{f}_{i+1}$  and  $\bar{f}_i$ .

$$\begin{array}{ccccccccc}
0 & \longrightarrow & m_R^i/m_R^{i+1} & \longrightarrow & R/m_R^{i+1} & \longrightarrow & R/m_R^i & \longrightarrow & 0 \\
& & \downarrow \tilde{f}_i & & \downarrow \bar{f}_{i+1} & & \downarrow \bar{f}_i & & \\
0 & \longrightarrow & m_S^i/m_S^{i+1} & \longrightarrow & S/m_S^{i+1} & \longrightarrow & S/m_S^i & \longrightarrow & 0
\end{array}$$

Hence we obtain inductively that for all  $\ell \geq 1$ ,  $\bar{f}_\ell : R/m_R^\ell \rightarrow S/m_S^\ell$ , given by  $\bar{f}_\ell(r + m_R^\ell) = f(r) + m_S^\ell$ , for all  $r \in R$ , is surjective. Since  $R$  is isomorphic to  $\varprojlim_{\ell} (R/m_R^\ell)$  and  $S$  is isomorphic to  $\varprojlim_{\ell} (S/m_S^\ell)$ , this implies that  $f$  is surjective.

Finally, suppose  $R = S$  and  $f$  is surjective. Then for all  $i \geq 1$ , the induced map  $\tilde{f}_i : m_R^i/m_R^{i+1} \rightarrow m_R^i/m_R^{i+1}$  is surjective. Since  $m_R^i/m_R^{i+1}$  is a finite dimensional  $k$ -vector space and  $\tilde{f}_i$  is  $k$ -linear, it follows that  $\tilde{f}_i$  is bijective for all  $i \geq 1$ .

Using induction and the commutative diagram in Figure 2.3, it follows that  $\bar{f}_\ell : R/m_R^\ell \rightarrow R/m_R^\ell$  is bijective for all  $\ell \geq 1$ , since  $\bar{f}_1$  is the identity morphism of  $R/m_R$ . But this means that if  $r$  lies in  $\ker(f)$  then  $r + m_R^\ell$  lies in  $\ker(\bar{f}_\ell)$  for all  $\ell$ , which implies that  $r \in \bigcap_{\ell} m_R^\ell$ . Since  $R$  is  $m_R$ -adically complete, it follows that  $r = 0$ , which means that  $f$  is bijective. □

**Remark 2.9.** Let  $R$  be an object of  $\hat{\mathcal{C}}$  and let  $Z = Z(R)$  be its center. We have proved in Theorem 2.8, part (j), that  $Z$  is a local ring with residue field  $k$  and unique maximal ideal  $m_Z = m_R \cap Z$ . Moreover,  $Z \cong \varprojlim_{\ell} Z(R/m_R^\ell)$ . Since by Theorem 2.8, part (k),  $Z(R/m_R^\ell)$  is an Artinian ring in  $\hat{\mathcal{C}}$  for all  $\ell \geq 1$ , it follows that  $Z$  is isomorphic to the inverse limit of complete local Artinian rings in  $\hat{\mathcal{C}}$ . However, we do not know whether  $Z$  is an object of  $\hat{\mathcal{C}}$  in general. The main issue is that  $m_Z^2 = (m_R \cap Z)^2$  is usually much smaller than  $m_R^2 \cap Z$ , which makes it hard to decide whether  $m_Z/m_Z^2$  is a finitely generated  $\lambda$ -module in general.

**Definition 2.10.** Let  $\mathcal{C}$  denote the full subcategory of  $\hat{\mathcal{C}}$  consisting of all  $\lambda$ -algebras in  $\hat{\mathcal{C}}$  whose maximal ideal is nilpotent. Let  $\hat{\mathcal{C}}_{comm}$  and  $\mathcal{C}_{comm}$  denote the full subcategories of  $\hat{\mathcal{C}}$  and  $\mathcal{C}$ , respectively, which consist exclusively of commutative rings.

The following result follows from Theorem 2.8.

**Lemma 2.11.** *The category  $\mathcal{C}$  is the same as the full subcategory of  $\hat{\mathcal{C}}$  consisting exclusively of Artinian rings. The category  $\hat{\mathcal{C}}_{comm}$  (resp.  $\mathcal{C}_{comm}$ ) is the same as the category of all complete local commutative Noetherian (resp. Artinian)  $\lambda$ -algebras*

with residue field  $k$  whose morphisms are local  $\lambda$ -algebra homomorphisms inducing the identity on  $k$ .

*Proof.* We first consider the category  $\mathcal{C}$ . Let  $R$  be an object of  $\hat{\mathcal{C}}$ . By Theorem 2.8, part (a), we know that  $R/m_R^\ell$  is Artinian for all  $\ell \geq 1$ . Hence it follows that if  $R$  is in  $\mathcal{C}$ , then  $R$  is Artinian. On the other hand, if  $R$  is Artinian then the decreasing chain of ideals

$$m_R \supseteq m_R^2 \supseteq \dots \supseteq m_R^\ell \supseteq \dots$$

must eventually stabilize. Since  $R$  is Artinian,  $R$  is Noetherian, and hence  $m_R^\ell$  is finitely generated as both a left and a right  $R$ -module, for all  $\ell \geq 1$ . Since these powers eventually stabilize, there exists a positive integer  $t$  such that  $m_R * (m_R^t) = m_R^t$ . We can then use Nakayama's Lemma to conclude that  $m_R^t = 0$ , and hence  $m_R$  is nilpotent. Therefore, an object  $R$  of  $\hat{\mathcal{C}}$  belongs to  $\mathcal{C}$  if and only if  $R$  is Artinian.

The statements about  $\hat{\mathcal{C}}_{comm}$  and  $\mathcal{C}_{comm}$ , respectively, follow now from Theorem 2.8, part (e), using that if  $(R, \pi_R)$  and  $(S, \pi_S)$  are commutative Noetherian rings in  $\hat{\mathcal{C}}$ , then a local  $\lambda$ -algebra homomorphism from  $R$  to  $S$  inducing the identity on  $k$  is, by definition, a  $\lambda$ -algebra homomorphism  $f : R \rightarrow S$  such that  $f(1_R) = 1_S$  and  $\pi_S \circ f = \pi_R$ . □

We conclude this chapter by discussing the ring of dual numbers, denoted by  $k[\epsilon]$ , which plays a vital part in our future discussion.

**Definition 2.12.** The ring of dual numbers,  $k[\epsilon]$ , is defined to be the ring  $k[[x]]/(x^2)$ ,

where we identify  $\epsilon$  with the image of  $x$  in  $k[[x]]/(x^2)$ . Specifically,

$$k[\epsilon] = \{a + b\epsilon \mid a, b \in k\},$$

where for all  $a_1, a_2, b_1, b_2 \in k$ ,

$$(a_1 + b_1\epsilon) + (a_2 + b_2\epsilon) = (a_1 + a_2) + (b_1 + b_2)\epsilon$$

and

$$(a_1 + b_1\epsilon)(a_2 + b_2\epsilon) = (a_1a_2) + (a_1b_2 + a_2b_1)\epsilon.$$

**Remark 2.13.** Notice that  $k[\epsilon]$  can be viewed as an object of  $\hat{\mathcal{C}}$ , where  $m_{k[\epsilon]} = \epsilon k[\epsilon] = \{b\epsilon \mid b \in k\}$  and  $\pi_{k[\epsilon]} : k[\epsilon] \rightarrow k$  is the natural surjection given by  $\pi_{k[\epsilon]}(a + b\epsilon) = a$  for all  $a, b \in k$ . Additionally, the  $\lambda$ -algebra structure on  $k[\epsilon]$  is given by  $j_{k[\epsilon]} : \lambda \rightarrow k[\epsilon]$  defined by  $j_{k[\epsilon]}(l) = \pi_\lambda(l) + 0\epsilon$  for all  $l \in \lambda$ .

### CHAPTER 3 NON-COMMUTATIVE DEFORMATION RINGS

We keep the same definitions and notations that were introduced in Chapter 2. In particular,  $k$  is a fixed field, and  $\lambda$  is a fixed complete, local, commutative, Noetherian ring with residue field  $k$  and unique maximal ideal  $\mu$ . Moreover, we fix the following additional definitions and notations.

Let  $A$  be a  $\lambda$ -algebra, and let  $V$  be a finite dimensional vector space over  $k$  such that  $V$  is also a left  $A$ -module. Let  $n = \dim_k V$ . After fixing a  $k$ -basis of  $V$  and recognizing that  $k$  is a  $\lambda$ -algebra, there exists a  $\lambda$ -algebra homomorphism,  $\rho : A \rightarrow \text{Mat}_n(k)$ , such that the module action of  $A$  on  $V$  is described as  $a \cdot v = \rho(a)(v)$  for all  $a \in A$  and  $v \in V$ . Conversely, if we are given a  $\lambda$ -algebra homomorphism  $\rho : A \rightarrow \text{Mat}_n(k)$ , we may associate with it an  $n$ -dimensional  $k$ -vector space,  $V$ , which is simultaneously a left  $A$ -module

In this chapter, we define lifts and deformations of  $V$  over objects in  $\hat{\mathcal{C}}$ . Note that we will interchangeably speak of lifts and deformations of  $V$  and lifts and deformations of  $\rho$  over objects in  $\hat{\mathcal{C}}$ . We also define versal and universal couples and versal and universal deformation rings. We prove our main result, Theorem 3.16, which shows that if  $V$  satisfies a natural finiteness condition, then it has a versal deformation ring in  $\hat{\mathcal{C}}$ . Many of our definitions, results and proof ideas are based on work by Schlessinger [10] and Mazur [8] and on Vélez-Marulanda's thesis [12]. The main difference is that [10], [8] and [12] worked with  $\hat{\mathcal{C}}_{comm}$  instead of  $\hat{\mathcal{C}}$ . This requires us to modify their definitions, results and proof ideas to fit our situation. We develop this

non-commutative deformation theory in Section 3.1, and we give some applications of this theory in Section 3.2. For a related approach to non-commutative deformations, see [6].

### 3.1 Non-commutative Deformation Theory

We first introduce lifts and deformations of  $V$  (or  $\rho$ ) over objects in  $\hat{\mathcal{C}}$ .

**Definition 3.1.** Let  $(R, \pi_R)$  and  $(S, \pi_S)$  be objects of  $\hat{\mathcal{C}}$ .

- (a) We define a lift of  $V$  (or  $\rho$ ) over  $(R, \pi_R)$  to be a  $\lambda$ -algebra homomorphism,  $\phi : A \rightarrow \text{Mat}_n(R)$ , such that  $\pi_R \circ \phi = \rho$ . We will often suppress the residue map and speak of lifts of  $V$  (or  $\rho$ ) over  $R$ , instead of lifts of  $V$  (or  $\rho$ ) over  $(R, \pi_R)$ .
- (b) Two lifts,  $\phi$  and  $\phi'$ , of  $V$  over  $R$  are said to be strictly equivalent if there exists an element  $P$  in  $GL_n(R)$  such that  $\pi_R(P)$  is the identity matrix in  $GL_n(k)$ , and such that  $P\phi(a)P^{-1} = \phi'(a)$  for all  $a \in A$ .
- (c) Strict equivalence forms a partition of the set of all lifts of  $V$  over  $R$ . We define a deformation of  $V$  over  $R$  to be an equivalence class of strictly equivalent lifts of  $V$  over  $R$ . We denote the deformation of  $V$  over  $R$  corresponding to a lift  $\phi$  by  $[\phi]$ .
- (d) If  $\phi$  is a lift of  $V$  over  $R$  and if  $\psi$  is a lift of  $V$  over  $S$ , then we say that  $f : \text{Mat}_n(R) \rightarrow \text{Mat}_n(S)$  is a morphism from  $\phi$  to  $\psi$  if  $f$  is the natural extension of a morphism  $f : R \rightarrow S$  in  $\hat{\mathcal{C}}$  such that  $f \circ \phi = \psi$ .

**Lemma 3.2.** *Let  $(R, \pi_R)$  and  $(S, \pi_S)$  be objects of  $\hat{\mathcal{C}}$ .*

(a) *If  $P$  is a matrix in  $Mat_n(R)$  such that  $P$  is congruent to the identity matrix modulo  $Mat_n(m_R)$ , then  $P$  is invertible, i.e.  $P$  lies in  $GL_n(R)$ .*

(b) *If  $\phi$  and  $\phi'$  are strictly equivalent lifts of  $V$  over  $R$ , and if  $f : R \rightarrow S$  is a morphism in  $\hat{\mathcal{C}}$ , then  $f \circ \phi$  and  $f \circ \phi'$  are strictly equivalent lifts of  $V$  over  $S$ .*

*Proof.* (a) By Lemma 2.5,  $Mat_n(R)$  is  $Mat_n(m_R)$ -adically complete. Let  $I$  denote the  $n \times n$  identity matrix in  $Mat_n(R)$ . It follows that for any matrix  $X$  in  $Mat_n(m_R)$ , the infinite sum  $I + X + X^2 + X^3 + \dots$  converges in  $Mat_n(R)$ , and it is clear that this element is the inverse matrix of  $I - X$ . Therefore, if  $P$  is a matrix in  $Mat_n(R)$  which is congruent to the identity matrix modulo  $Mat_n(m_R)$ , then  $P = I - X$  for some  $X$  in  $Mat_n(m_R)$ , and hence  $P$  is invertible.

(b) Strict equivalence of  $\phi$  and  $\phi'$  means there exists an invertible matrix  $P \in Mat_n(R)$  such  $P\phi P^{-1} = \phi'$ , where  $\pi_R(P)$  is the identity matrix in  $GL_n(k)$ . Since  $f$  is a ring homomorphism, we have

$$f \circ \phi' = f \circ (P\phi P^{-1}) = f(P)(f \circ \phi)f(P^{-1}).$$

Further, since  $f : R \rightarrow S$  is a morphism in  $\hat{\mathcal{C}}$ ,  $\pi_S \circ f = \pi_R$ , and so  $f$  sends  $P$  to a matrix in  $Mat_n(S)$  which reduces to the identity matrix in  $Mat_n(k)$ . By (a), this means that  $f(P)$  is a unit in  $Mat_n(S)$ . Therefore,  $f(P)$  lies in  $GL_n(S)$ . Since  $I = f(PP^{-1}) = f(P)f(P^{-1})$ , we see that  $f(P^{-1}) = f(P)^{-1}$ , and hence  $f \circ \phi$  and



$f \circ \phi'$  are conjugates by the matrix  $f(P)$ . This proves that  $f \circ \phi$  and  $f \circ \phi'$  are strictly equivalent lifts of  $V$  over  $S$ .

□

We next define the deformation functor and couples for  $V$ . This leads to the definition of a universal couple and of a universal deformation ring and deformation for  $V$ . Note that our definition of couples goes back to [10, Section 2].

**Definition 3.3.**

- (a) The deformation functor  $F_V : \hat{\mathcal{C}} \rightarrow Sets$  is defined to be the following covariant functor. For any object  $R$  of  $\hat{\mathcal{C}}$ , let  $F_V(R)$  be the set consisting of all deformations of  $V$  over  $R$ . If

$$f : R \rightarrow S$$

is a morphism in  $\hat{\mathcal{C}}$ , then

$$F_V(f) : F_V(R) \rightarrow F_V(S)$$

is defined by

$$F_V(f)([\phi]) = [f \circ \phi]$$

for any  $[\phi] \in F_V(R)$ . We note that we use the notation  $F_V$  regardless of whether we speak of deformations of  $V$  or of deformations of  $\rho$ .

- (b) For any object  $R$  of  $\hat{\mathcal{C}}$ , we define the covariant functor  $h_R : \hat{\mathcal{C}} \rightarrow Sets$  to be  $h_R = Hom_{\hat{\mathcal{C}}}(R, -)$ . In other words, for any object  $T$  of  $\hat{\mathcal{C}}$ , if

$$f : T \rightarrow S$$

is a morphism in  $\hat{\mathcal{C}}$ , then

$$h_R(f) : h_R(T) \rightarrow h_R(S)$$

is given by

$$h_R(f)(\alpha) = f \circ \alpha$$

for any morphism  $\alpha : R \rightarrow T$  in  $\hat{\mathcal{C}}$ .

- (c) Fix an object  $R$  of  $\hat{\mathcal{C}}$  and a deformation  $[\phi]$  in  $F_V(R)$ . Then the pair  $(R, [\phi])$  is called a couple for  $V$ . Define the natural transformation  $\tau = \tau(R, [\phi]) : h_R \rightarrow F_V$  as follows: For each object  $T$  of  $\hat{\mathcal{C}}$ , let

$$\tau_T : h_R(T) \rightarrow F_V(T)$$

be defined by

$$\tau_T(\alpha) = F_V(\alpha)([\phi]).$$

In the case when  $\tau$  is a natural isomorphism, the functor  $F_V$  is said to be representable, and  $(R, [\phi])$  is called a universal couple for  $V$ . Moreover,  $R$  is said to be the universal deformation ring of  $V$ , and  $[\phi]$  is said to be the universal deformation of  $V$ .

**Remark 3.4.** The definition of  $F_V$  in Definition 3.3, part (a), does indeed provide a functor. Namely, if  $f : R \rightarrow S$  and  $g : S \rightarrow T$  are morphisms in  $\hat{\mathcal{C}}$ , then

$$F_V(g \circ f)([\phi]) = [(g \circ f) \circ \phi] = [g \circ (f \circ \phi)] = F_V(g)([f \circ \phi]) = (F_V(g) \circ F_V(f))([\phi])$$

and hence  $F_V$  preserves composition. Moreover, for any object  $R$  of  $\hat{\mathcal{C}}$ , we have that

$$F_V(id_R)([\phi]) = [id_R \circ \phi] = [\phi]$$

and so  $F_V$  also preserves identity morphisms.

**Theorem 3.5.** *If  $(R, [\phi])$  and  $(R', [\phi'])$  are both universal couples for  $V$ , then they are isomorphic up to a unique isomorphism. In other words, there exists a morphism  $f : R \rightarrow R'$  in  $\hat{\mathcal{C}}$  such that  $f$  is unique with respect to the property that  $f$  is an isomorphism in  $\hat{\mathcal{C}}$  between  $R$  and  $R'$  satisfying  $[f \circ \phi] = [\phi']$ .*

*Proof.* Let  $\tau = \tau(R, [\phi])$  be the natural transformation from Definition 3.3, part (c). Since  $(R, [\phi])$  is a universal couple for  $V$ ,  $\tau_{R'} : h_R(R') \rightarrow F_V(R')$  is a bijection. Therefore, there exists a unique  $f \in h_R(R')$  such that  $[f \circ \phi] = [\phi']$ . However, since  $(R', [\phi'])$  is also a universal couple for  $V$ , a completely similar argument provides the existence of a unique  $f' \in h_{R'}(R)$  such that  $[f' \circ \phi'] = [\phi]$ . But then  $f' \circ f$  is an element in  $h_R(R)$  which gets sent to  $[\phi]$  by  $\tau_R$ . Since  $id_R$  also gets sent to  $[\phi]$  by  $\tau_R$ , and since  $\tau_R$  is a bijection, we must have that  $id_R = f' \circ f$ . By a completely similar argument, we have that  $id_{R'} = f \circ f'$ , proving that  $f : R \rightarrow R'$  is an isomorphism in  $\hat{\mathcal{C}}$  which satisfies  $[f \circ \phi] = [\phi']$ . But since  $[f \circ \phi] = \tau_R(f)$ , and since  $\tau_R$  is bijective,  $f$  is the unique morphism from  $R$  to  $R'$  in  $\hat{\mathcal{C}}$  which is both an isomorphism and satisfies  $[f \circ \phi] = [\phi']$ . □

In some cases, however, the functor  $F_V$  will fail to be representable, meaning that no universal couple for  $V$  will exist. Even so, while this means that there does not exist a couple for  $V$  which gives rise to a natural transformation  $\tau$  that is a natural isomorphism, there may still be a suitable couple for  $V$  which gives rise to a natural transformation  $\tau$  that makes  $F_V$  “close to representable.” The following definition

makes this more precise by defining smoothness, which leads to the definition of a versal couple and of a versal deformation ring and deformation of  $V$ . Note that these definitions again go back to [10, Section 2].

**Definition 3.6.** Fix a couple  $(R, [\phi])$  for  $V$ . Then the natural transformation  $\tau = \tau(R, [\phi]) : h_R \rightarrow F_V$  from Definition 3.3, part (c), is said to be smooth if the morphism,  $q$ , resulting from the pullback diagram in Figure 3.1 is surjective, whenever  $\alpha : B \rightarrow B'$  is a surjective morphism in  $\mathcal{C}$ . If  $\tau$  is smooth and if  $\tau_{k[\epsilon]}$  is bijective, then  $(R, [\phi])$  is said to be a versal couple for  $V$ . Moreover,  $R$  is said to be a versal deformation ring of  $V$ , and  $[\phi]$  is said to be a versal deformation of  $V$ .

Figure 3.1: The pullback diagram needed to define smoothness.

$$\begin{array}{ccc}
 h_R(B) & \xrightarrow{h_R(\alpha)} & h_R(B') \\
 \tau_B \downarrow & \swarrow q & \nearrow \\
 & h_R(B') \times_{F_V(B')} F_V(B) & \\
 & \swarrow & \searrow \\
 F_V(B) & \xrightarrow{F_V(\alpha)} & F_V(B') \\
 & \tau_{B'} \downarrow &
 \end{array}$$

Note that representability implies smoothness, and so in the event that  $(R, [\phi])$  is a universal couple for  $V$  and  $\tau = \tau(R, [\phi])$  is a natural isomorphism, then  $(R, [\phi])$

is a versal couple for  $V$  and  $\tau$  is smooth. While we will show in Theorem 3.11 that any two versal couples for  $V$  are isomorphic, there may not be a unique isomorphism which relates them, as there is in the universal case. To work our way up to the proof, we begin with a few preliminary results.

**Lemma 3.7.** *If  $(R, [\phi])$  is a versal couple for  $V$ , then the smoothness of  $\tau = \tau(R, [\phi])$  implies that  $\tau_T : h_R(T) \rightarrow F_V(T)$  is surjective for all  $T$  in  $\hat{\mathcal{C}}$ .*

*Proof.* Note that some of the proof ideas are similar to ideas in the proof of [3, Proposition A.1].

Fix an object  $(T, \pi_T)$  of  $\hat{\mathcal{C}}$ . First notice that  $F_V(k)$  consists of only a single element, namely  $[\rho]$ . Since  $k$  is isomorphic to  $T/m_T$ ,  $F_V(T/m_T)$  also contains only a single element. Since  $R$  and  $T$  share residue fields, there exists some element,  $f_1$ , in  $h_R(T/m_T)$ . Moreover, if  $[\gamma]$  is any element of  $F_V(T)$ , then, considering the natural projections

$$\pi_\ell : T \rightarrow T/m_T^\ell,$$

we can define deformations

$$[\gamma_\ell] \in F_V(T/m_T^\ell)$$

by

$$\gamma_\ell = \pi_\ell \circ \gamma.$$

Suppose, inductively, that we have found maps  $f_i \in h_R(T/m_T^i)$  for  $i \leq \ell$ , compatible in the sense that  $f_i$  projects onto  $f_{i-1}$  under the natural projection  $T/m_T^i \rightarrow T/m_T^{i-1}$ , and such that  $\tau_{T/m_T^i}(f_i) = [\gamma_i]$  (see Figure 3.2). Then the smoothness of  $\tau$  immedi-

ately implies the existence of some  $f_{\ell+1} \in h_R(T/m_T^{\ell+1})$  which projects onto  $f_\ell$ , and which also satisfies  $\tau_{T/m_T^{\ell+1}}(f_{\ell+1}) = [\gamma_{\ell+1}]$  (see Figure 3.3), completing the induction. Therefore, we can construct the inverse limit of the  $f_\ell$ , and by completeness of  $T$  we can associate this with an element  $f \in h_R(T)$ , such that  $f(r) = (f_\ell(r))_\ell$  for all  $r$  in  $R$ , where we identify  $T$  with the inverse limit  $\varprojlim_\ell (T/m_T^\ell)$ . Since  $\tau_{T/m_T^\ell}(f_\ell) = [\gamma_\ell]$  for all  $\ell \geq 1$ , there exists some  $P_\ell$  in  $Mat_n(T/m_T^\ell)$  which reduces to the identity in  $Mat_n(k)$ , such that  $P_\ell \gamma_\ell P_\ell^{-1} = f_\ell \circ \phi$ . Therefore, we have that

$$f \circ \phi = (f_\ell \circ \phi)_\ell = (P_\ell \gamma_\ell P_\ell^{-1})_\ell.$$

To show that  $[f \circ \phi] = [\gamma]$ , we need to find some  $P \in Mat_n(T)$  which reduces to the identity in  $Mat_n(k)$ , such that

$$(f_\ell(\phi))_\ell = P(\gamma_\ell)_\ell P^{-1}.$$

Figure 3.2: The induction hypothesis for the construction of  $f_{\ell+1}$ .

$$\begin{array}{ccccccc}
 & & T/m_T^\ell & \rightarrow & T/m_T^{\ell-1} & \rightarrow & \dots & \rightarrow & T/m_T \\
 & & & & & & & & \\
 h_R(T/m_T^\ell) & \longrightarrow & h_R(T/m_T^{\ell-1}) & \longrightarrow & \dots & \longrightarrow & h_R(T/m_T) \\
 \tau_{T/m_T^\ell} \downarrow & & \tau_{T/m_T^{\ell-1}} \downarrow & & & & \tau_{T/m_T} \downarrow \\
 F_V(T/m_T^\ell) & \longrightarrow & F_V(T/m_T^{\ell-1}) & \longrightarrow & \dots & \longrightarrow & F_V(T/m_T)
 \end{array}$$

Figure 3.3: The compatibility conditions for  $f_{\ell+1} \in h_R(T/m_T^{\ell+1})$ .

$$\begin{array}{ccc}
& T/m_T^{\ell+1} & \rightarrow & T/m_T^{\ell} \\
& \exists f_{\ell+1} & & f_{\ell} \\
h_R(T/m_T^{\ell+1}) & \longrightarrow & & h_R(T/m_T^{\ell}) \\
\downarrow \tau_{T/m_T^{\ell+1}} & & & \downarrow \tau_{T/m_T^{\ell}} \\
F_V(T/m_T^{\ell+1}) & \longrightarrow & & F_V(T/m_T^{\ell}) \\
[\gamma_{\ell+1}] & & & [\gamma_{\ell}]
\end{array}$$

To this end, we define, for each  $i \geq 1$ , the set

$$Z_i = \{X_i \in \text{Mat}_n(T/m_T^i) \mid X_i \gamma_i = \gamma_i X_i \text{ and all entries of } X_i \text{ are in } m_T/m_T^i\}.$$

For all  $j \geq i$ , let  $\sigma_i^j : \text{Mat}_n(T/m_T^j) \rightarrow \text{Mat}_n(T/m_T^i)$  be the natural projection. Note that  $T/m_T^i$  has a filtration of left  $(T/m_T^i)$ -submodules

$$T/m_T^i \supseteq m_T/m_T^i \supseteq m_T^2/m_T^i \supseteq \dots \supseteq m_T^i/m_T^i = 0$$

where each of the successive quotients is a finite dimensional  $k$ -vector space. Using that this is also a filtration of (left)  $j_{T/m_T^i}(\lambda)$ -submodules, we see that the successive quotients are Artinian  $j_{T/m_T^i}(\lambda)$ -modules, showing that  $T/m_T^i$  is an Artinian  $j_{T/m_T^i}(\lambda)$ -module. Therefore,  $\text{Mat}_n(T/m_T^i)$  is also an Artinian module over  $j_{T/m_T^i}(\lambda)$ . Since  $Z_i$  is a  $j_{T/m_T^i}(\lambda)$ -submodule of  $\text{Mat}_n(T/m_T^i)$ ,  $Z_i$  is again an Artinian module over  $j_{T/m_T^i}(\lambda)$ . Therefore, for each  $i \geq 1$ , the decreasing sequence of  $j_{T/m_T^i}(\lambda)$ -modules

$$Z_i \supseteq \sigma_i^{i+1}(Z_{i+1}) \supseteq \sigma_i^{i+2}(Z_{i+2}) \supseteq \dots$$

stabilizes. For each  $i \geq 1$ , define the integer  $N_i \geq i$  to be the smallest integer such that  $\sigma_i^{N_i}(Z_{N_i}) = \sigma_i^j(Z_j)$  for all  $j \geq N_i$ . Notice that by construction,  $N_i \leq N_{i+1}$  for all  $i$ .

For each  $i \geq 1$ , define:

$$\begin{aligned} S_i &= \bigcap_{j \geq i} \sigma_i^j(Z_j) = \sigma_i^{N_i}(Z_{N_i}), \\ Q_i &= \sigma_i^{N_i}(P_{N_i}^{-1})\sigma_i^{N_{i+1}}(P_{N_{i+1}}) - \text{Id}_{T/m_T^i} \in \text{Mat}_n(T/m_T^i), \end{aligned}$$

where  $\text{Id}_{T/m_T^i}$  denotes the identity matrix in  $\text{Mat}_n(T/m_T^i)$ .

Notice that since each  $P_\ell$  reduces to the identity in  $\text{Mat}_n(k)$ , all the entries of  $Q_i$  lie in  $m_T/m_T^i$ . Recall that, for all  $\ell \geq 1$ ,

$$P_\ell \gamma_\ell P_\ell^{-1} = f_\ell \circ \phi.$$

From the definition of the  $\gamma_\ell$ , we have  $\sigma_i^j(\gamma_j) = \gamma_i$  for all  $j \geq i$ , and the compatibility of the  $f_\ell$  means that  $\sigma_i^j(f_j) = f_i$  for all  $j \geq i$ . Therefore, for all  $j \geq i$ ,

$$\begin{aligned} \sigma_i^j(P_j)\gamma_i\sigma_i^j(P_j^{-1}) &= \sigma_i^j(P_j)\sigma_i^j(\gamma_j)\sigma_i^j(P_j^{-1}) \\ &= \sigma_i^j(P_j\gamma_jP_j^{-1}) \\ &= \sigma_i^j(f_j \circ \phi) \\ &= f_i \circ \phi = P_i\gamma_iP_i^{-1}. \end{aligned}$$

Hence, for all  $j \geq i$ ,  $P_i^{-1}\sigma_i^j(P_j)$  commutes with  $\gamma_i$ . From this, and from the fact that all the entries of  $Q_i$  lie in  $m_T/m_T^i$ , it follows that  $Q_i \in S_i$  for all  $i \geq 1$ .



Let  $j \geq 3$ . For  $2 \leq i \leq j$ , define

$$\rho_i^j : S_j \rightarrow S_i$$

to be the restriction of  $\sigma_i^j$  to  $S_j$ . Notice that

$$\rho_i^j(S_j) = \sigma_i^j(\sigma_j^{N_j}(Z_{N_j})) = \sigma_i^{N_j}(Z_{N_j}).$$

By the definition of  $N_i$  and the fact that  $N_j \geq N_i$ , we obtain, for all  $j \geq i$ ,

$$\sigma_i^{N_j}(Z_{N_j}) = \sigma_i^{N_i}(Z_{N_i}).$$

Since  $S_i = \sigma_i^{N_i}(Z_{N_i})$ , the above equations combine to give, for  $j \geq i$ ,

$$\rho_i^j(S_j) = S_i,$$

showing that each  $\rho_i^j$  is surjective.

We now proceed to inductively define a collection of matrices  $X_i^{(j)}$  in  $S_j$ , for  $j \geq 2$  and  $2 \leq i \leq j$ . For all  $j \geq 2$ , let  $X_j^{(j)} = Q_j$ . Fix now  $\ell \geq 2$  and suppose inductively that we have defined  $X_i^{(j)}$  for all  $i, j$  satisfying  $3 \leq j \leq \ell$  and  $2 \leq i \leq j - 1$ . We proceed to define  $X_i^{(\ell+1)}$ , for all  $2 \leq i \leq \ell$ , as follows. Since  $\rho_\ell^{\ell+1}$  is surjective, let  $X_i^{(\ell+1)}$  be a matrix in  $S_{\ell+1}$  such that  $\rho_\ell^{\ell+1}(X_i^{(\ell+1)}) = X_i^{(\ell)}$ . Hence, for all  $j \geq 3$  and all  $2 \leq i \leq j - 1$ , we have  $\rho_{j-1}^j(X_i^{(j)}) = X_i^{(j-1)}$ . Since, for all  $t \leq j - 1$ , we have  $\rho_t^j = \rho_t^{t+1} \circ \rho_{t+1}^{t+2} \circ \dots \circ \rho_{j-2}^{j-1} \circ \rho_{j-1}^j$ , we obtain inductively, for all  $j \geq 3$  and all  $2 \leq i \leq t \leq j - 1$ , that  $\rho_t^j(X_i^{(j)}) = X_i^{(t)}$ .

Next, define the following matrices  $L_\ell$ , for all  $\ell \geq 1$ :

$$L_1 = \sigma_1^{N_1}(P_{N_1}),$$

$$L_2 = \sigma_2^{N_2}(P_{N_2}),$$

$$L_\ell = \sigma_\ell^{N_\ell}(P_{N_\ell})(\text{Id}_{T/m_T^\ell} + X_{\ell-1}^{(\ell)})^{-1}(\text{Id}_{T/m_T^\ell} + X_{\ell-2}^{(\ell)})^{-1} \cdots (\text{Id}_{T/m_T^\ell} + X_2^{(\ell)})^{-1},$$

for  $\ell \geq 3$ . Notice that the  $L_\ell$  are compatible, in the sense that for  $j \geq i$ ,  $\sigma_i^j(L_j) = L_i$ .

To see this, first notice that

$$\begin{aligned} \sigma_{j-1}^j(L_j) &= \sigma_{j-1}^j(\sigma_j^{N_j}(P_{N_j})(\text{Id}_{T/m_T^j} + X_{j-1}^{(j)})^{-1}(\text{Id}_{T/m_T^j} + X_{j-2}^{(j)})^{-1} \\ &\quad \cdots (\text{Id}_{T/m_T^j} + X_2^{(j)})^{-1}) \\ &= \sigma_{j-1}^{N_j}(P_{N_j})\sigma_{j-1}^j((\text{Id}_{T/m_T^j} + X_{j-1}^{(j)})^{-1}(\text{Id}_{T/m_T^j} + X_{j-2}^{(j)})^{-1} \\ &\quad \cdots (\text{Id}_{T/m_T^j} + X_2^{(j)})^{-1}) \\ &= \sigma_{j-1}^{N_j}(P_{N_j})(\text{Id}_{T/m_T^{j-1}} + X_{j-1}^{(j-1)})^{-1}(\text{Id}_{T/m_T^{j-1}} + X_{j-2}^{(j-1)})^{-1} \\ &\quad \cdots (\text{Id}_{T/m_T^{j-1}} + X_2^{(j-1)})^{-1}) \\ &= \sigma_{j-1}^{N_j}(P_{N_j})(\text{Id}_{T/m_T^{j-1}} + Q_{j-1})^{-1}(\text{Id}_{T/m_T^{j-1}} + X_{j-2}^{(j-1)})^{-1} \\ &\quad \cdots (\text{Id}_{T/m_T^{j-1}} + X_2^{(j-1)})^{-1}) \\ &= \sigma_{j-1}^{N_j}(P_{N_j})[\sigma_{j-1}^{N_{j-1}}(P_{N_{j-1}})^{-1}\sigma_{j-1}^{N_j}(P_{N_j})]^{-1}(\text{Id}_{T/m_T^{j-1}} + X_{j-2}^{(j-1)})^{-1} \\ &\quad \cdots (\text{Id}_{T/m_T^{j-1}} + X_2^{(j-1)})^{-1}) \\ &= \sigma_{j-1}^{N_{j-1}}(P_{N_{j-1}})(\text{Id}_{T/m_T^{j-1}} + X_{j-2}^{(j-1)})^{-1} \cdots (\text{Id}_{T/m_T^{j-1}} + X_2^{(j-1)})^{-1}) \\ &= L_{j-1} \end{aligned}$$

where we use that  $\sigma_i^j$  coincides with  $\rho_i^j$  on  $S_j$ . Therefore, we have for all  $j \geq i$ ,

$$L_i = \sigma_i^{i+1}(L_{i+1}) = (\sigma_i^{i+1} \circ \sigma_{i+1}^{i+2})(L_{i+2}) = \dots = (\sigma_i^{i+1} \circ \sigma_{i+1}^{i+2} \circ \dots \circ \sigma_{j-1}^j)(L_j).$$

Since  $\sigma_i^j = \sigma_i^{i+1} \circ \sigma_{i+1}^{i+2} \circ \dots \circ \sigma_{j-1}^j$ , it follows that  $L_i = \sigma_i^j(L_j)$  for all  $i \leq j$ .

By construction,  $X_i^{(\ell)}$  lies in  $S_\ell$  for each  $i \leq \ell$ . Therefore,  $(\text{Id}_{T/m_T^\ell} + X_i^{(\ell)})$  commutes with  $\gamma_\ell$ , for all  $i \leq \ell$ . From this it follows, for all  $\ell \geq 1$ , that

$$\begin{aligned} L_\ell \gamma_\ell L_\ell^{-1} &= \sigma_\ell^{N_\ell}(P_{N_\ell}) \gamma_\ell \sigma_\ell^{N_\ell}(P_{N_\ell})^{-1} \\ &= \sigma_\ell^{N_\ell}(P_{N_\ell}) \sigma_\ell^{N_\ell}(\gamma_{N_\ell}) \sigma_\ell^{N_\ell}(P_{N_\ell})^{-1} \\ &= \sigma_\ell^{N_\ell}(P_{N_\ell} \gamma_{N_\ell} P_{N_\ell}^{-1}) \\ &= \sigma_\ell^{N_\ell}(f_{N_\ell} \circ \phi) = f_\ell \circ \phi. \end{aligned}$$

Since the matrices  $L_\ell$  form an element  $(L_\ell)_\ell$  in  $\varprojlim_{\ell} \text{Mat}_n(T/m_T^\ell)$ , we know from Lemma 2.5 that there exists a matrix  $L$  in  $\text{Mat}_n(T)$  such that, for all  $\ell \geq 1$ ,  $L$  projects onto  $L_\ell$  via the natural projection  $\text{Mat}_n(T) \rightarrow \text{Mat}_n(T/m_T^\ell)$ . Therefore,  $L\gamma L^{-1} = f \circ \phi$ . Moreover, because  $L$  projects onto  $L_\ell$  and because  $L_\ell$  reduces to the identity in  $GL_n(k)$ ,  $L$  also reduces to the identity in  $GL_n(k)$ . Hence

$$\tau_T(f) = [f \circ \phi] = [\gamma],$$

completing the proof that  $\tau_T$  is surjective. □

As a consequence of the above proof, we get:

**Corollary 3.8.** *Suppose  $\tau$  is as in Lemma 3.7. Then  $\tau$  is a natural isomorphism if and only if  $\tau_B$  is a bijection for all  $B$  in  $\mathcal{C}$ .*

*Proof.* Suppose  $\tau_B$  is a bijection for all  $B$  in  $\mathcal{C}$ . By the definition of completeness, any  $T$  in  $\hat{\mathcal{C}}$  can be identified with the inverse limit  $\varprojlim_{\ell} (T/m_T^{\ell})$ . Therefore, if  $\tau_T(\alpha) = \tau_T(\alpha')$ , then  $[\alpha \circ \phi] = [\alpha' \circ \phi]$ , which implies  $[\pi_{\ell} \circ \alpha \circ \phi] = [\pi_{\ell} \circ \alpha' \circ \phi]$ , where  $\pi_{\ell}$  is again taken to be the natural projection from  $T$  onto  $T/m_T^{\ell}$ , for all  $\ell \geq 1$ . However,

$$[\pi_{\ell} \circ \alpha \circ \phi] = [(\pi_{\ell} \circ \alpha) \circ \phi] = \tau_{T/m_T^{\ell}}(\pi_{\ell} \circ \alpha)$$

proving that we must have  $\tau_{T/m_T^{\ell}}(\pi_{\ell} \circ \alpha) = \tau_{T/m_T^{\ell}}(\pi_{\ell} \circ \alpha')$  for all  $\ell$ . But since each  $T/m_T^{\ell}$  is an object of  $\mathcal{C}$ , each  $\tau_{T/m_T^{\ell}}$  is a bijection by assumption, and hence  $\pi_{\ell} \circ \alpha = \pi_{\ell} \circ \alpha'$  for all  $\ell$ . Since  $T$  is complete, this proves that  $\alpha = \alpha'$ , since  $\alpha$  and  $\alpha'$  are exactly the inverse limits of the maps  $\pi_{\ell} \circ \alpha$  and  $\pi_{\ell} \circ \alpha'$ , respectively, in the identification of  $T$  with  $\varprojlim_{\ell} T/m_T^{\ell}$ . This shows that  $\tau_T$  is injective.

To see surjectivity of  $\tau_T$ , let  $[\eta]$  be any element in  $F_V(T)$ . Then  $[\pi_{\ell} \circ \eta]$  is an element in  $F_V(T/m_T^{\ell})$  for all  $\ell \geq 1$ . Since  $\tau_{T/m_T^{\ell}}$  is assumed to be surjective, there exists some  $\alpha_{\ell}$  in  $h_R(T/m_T^{\ell})$  so that  $\tau_{T/m_T^{\ell}}(\alpha_{\ell}) = [\pi_{\ell} \circ \eta]$ , which is to say that  $[\alpha_{\ell} \circ \phi] = [\pi_{\ell} \circ \eta]$ , for all  $\ell \geq 1$ . Letting  $\nu : T/m_T^{\ell+1} \rightarrow T/m_T^{\ell}$  be the natural projection, we have that

$$\begin{aligned} \tau_{T/m_T^{\ell}}(\alpha_{\ell}) &= [\alpha_{\ell} \circ \phi] = [\pi_{\ell} \circ \eta] = [\nu \circ \pi_{\ell+1} \circ \eta] = [\nu \circ (\pi_{\ell+1} \circ \eta)] \\ &= [\nu \circ (\alpha_{\ell+1} \circ \phi)] = [(\nu \circ \alpha_{\ell+1}) \circ \phi] = \tau_{T/m_T^{\ell}}(\nu \circ \alpha_{\ell+1}). \end{aligned}$$

However, since  $\tau_{T/m_T^{\ell}}$  is assumed to be a bijection, we must have that  $\alpha_{\ell} = \nu \circ \alpha_{\ell+1}$  for

all  $\ell \geq 1$ , which is to say that the  $\alpha_\ell$  are compatible in the sense that they all project onto each other under the natural projections. We then use completeness and mimic the proof of Lemma 3.7, replacing  $f_\ell$  with  $\alpha_\ell$  and  $\gamma$  with  $\eta$ , to find the existence of some  $\alpha : R \rightarrow T$  in  $\hat{\mathcal{C}}$  such that  $\pi_\ell \circ \alpha = \alpha_\ell$  for all  $\ell$ , and such that  $[\alpha \circ \phi] = [\eta]$ , proving that  $\tau_T$  is surjective, and completing the proof that  $\tau$  is a natural isomorphism.  $\square$

As a final preliminary to the proof that any two versal couples for  $V$  are isomorphic, it will be necessary to find certain elements in  $h_R(k[\epsilon])$ . We first prove the following result:

**Lemma 3.9.** *For any  $R$  in  $\hat{\mathcal{C}}$  with  $m_R \neq 0$ , there exists an integer  $m \geq 1$  and an isomorphism in  $\hat{\mathcal{C}}$ ,  $f : R/(\mu_R + m_R^2) \rightarrow k[\epsilon] \times_k k[\epsilon] \times_k \dots \times_k k[\epsilon]$ , the  $m$ -fold pullback of  $k[\epsilon]$  over  $k$ . More specifically,  $m_R/(\mu_R + m_R^2)$  is a vector space over  $k$  of  $k$ -dimension  $m$ . If  $b_1, \dots, b_m$  in  $m_R$  generate  $m_R$  modulo  $(\mu_R + m_R^2)$ , then  $f(l + \sum_{i=1}^m r_i b_i + (\mu_R + m_R^2)) = (\pi_R(l) + \pi_R(r_i)\epsilon)_i$  for all  $l$  in  $j_R(\lambda)$  and all  $r_1, \dots, r_m$  in  $R$ .*

*Proof.* We know from Theorem 2.8, part (d), that  $\mu_R + m_R^2$  is an ideal of  $R$ . Since any ring  $R$  in  $\hat{\mathcal{C}}$  has a maximal ideal  $m_R$  such that  $m_R/m_R^2$  is finitely generated as a left  $R$ -module,  $m_R/(\mu_R + m_R^2)$  is also finitely generated as a left  $R$ -module. Moreover,  $m_R$  annihilates  $m_R/(\mu_R + m_R^2)$ , and hence  $m_R/(\mu_R + m_R^2)$  is an  $m$ -dimensional  $k$ -vector space for some finite integer,  $m$ . Choose elements  $b_1, b_2, \dots, b_m$  in  $m_R$  so that their images form a  $k$ -basis of  $m_R/(\mu_R + m_R^2)$ . Then, letting  $\bar{r}$  denote the image of  $r$  in  $R/(\mu_R + m_R^2)$ , we can write any element  $\bar{r}$  of  $R/(\mu_R + m_R^2)$  as  $l + \overline{\sum_{i=1}^m r_i b_i}$ , where  $l$  is some element in  $j_R(\lambda)$ , and the  $r_1, \dots, r_m$  are certain elements of  $R$ . Therefore, we can

define a map

$$f : R/(\mu_R + m_R^2) \rightarrow k[\epsilon] \times_k k[\epsilon] \times_k \dots \times_k k[\epsilon]$$

by

$$f\left(\overline{l + \sum_{i=1}^m r_i b_i}\right) = (\pi_R(l) + \pi_R(r_i)\epsilon)_i.$$

To see this is well-defined, suppose  $l + \sum_{i=1}^m r_i b_i = l' + \sum_{i=1}^m r'_i b_i$ . Then

$$\overline{l - l'} = \overline{\sum_{i=1}^m r'_i b_i} - \overline{\sum_{i=1}^m r_i b_i} = \overline{\sum_{i=1}^m (r'_i - r_i) b_i}.$$

But since all the  $b_i$  lie in  $m_R$  and since both  $l$  and  $l'$  are in  $j_R(\lambda)$ ,

$$l - l' \in j_R(\lambda) \cap m_R = \mu_R,$$

and hence

$$\overline{l - l'} = \overline{\sum_{i=1}^m (r'_i - r_i) b_i} = \bar{0}.$$

But this shows that  $r'_i - r_i \in m_R$  for all  $i$ , since the  $\bar{b}_i$  form a  $k$ -basis of  $m_R/(\mu_R + m_R^2)$ .

Therefore, we have that

$$\pi_R(l) = \pi_R(l')$$

and

$$\pi_R(r_i) = \pi_R(r'_i)$$

for all  $i$ , and hence  $f$  is well-defined. It follows from the definition of  $f$  that  $f$  is

surjective. To see injectivity, suppose  $f\left(\overline{l + \sum_{i=1}^m r_i b_i}\right) = (0)_i$ . Then  $\pi_R(l) = 0$ , and  $\pi_R(r_i) = 0$  for all  $i$ , and hence  $l \in \mu_R$  and  $r_i \in m_R$  for all  $i$ . But then since  $b_i \in m_R$ , it is clear that  $l + \sum_{i=1}^m r_i b_i \in \mu_R + m_R^2$ , and hence  $l + \sum_{i=1}^m r_i b_i = \bar{0}$ , proving that  $f$  is

injective. Lastly, we must show that  $f$  is a morphism in  $\hat{\mathcal{C}}$ . To see that  $f$  is a ring homomorphism, it is clear that  $f$  preserves addition since  $\pi_R$  preserves addition. To see that it preserves the multiplicative structure, notice that

$$\begin{aligned}
\overline{f\left(l + \sum_{i=1}^m r_i b_i\right) * \left(l' + \sum_{i=1}^m r'_i b_i\right)} &= \overline{f\left(ll' + \sum_{i=1}^m (lr'_i + l'r_i) b_i\right)} \\
&= (\pi_R(ll') + \pi_R(lr'_i + l'r_i)\epsilon)_i \\
&= (\pi_R(l)\pi_R(l') + (\pi_R(l)\pi_R(r'_i) + \pi_R(l')\pi_R(r_i))\epsilon)_i \\
&= (\pi_R(l) + \pi_R(r_i)\epsilon)_i * (\pi_R(l') + \pi_R(r'_i)\epsilon)_i \\
&= \overline{f\left(l + \sum_{i=1}^m r_i b_i\right)} * \overline{f\left(l' + \sum_{i=1}^m r'_i b_i\right)}.
\end{aligned}$$

Therefore,  $f$  is indeed a ring homomorphism. To see that  $f$  is a  $\lambda$ -algebra homomorphism, notice that if  $x$  is in  $\lambda$ , then

$$\begin{aligned}
\overline{f\left(x\left(l + \sum_{i=1}^m r_i b_i\right)\right)} &= \overline{f\left(j_R(x)l + \sum_{i=1}^m j_R(x)r_i b_i\right)} \\
&= (\pi_R(j_R(x)l) + \pi_R(j_R(x)r_i)\epsilon)_i \\
&= (\pi_\lambda(x)\pi_R(l) + \pi_\lambda(x)\pi_R(r_i)\epsilon)_i \\
&= x(\pi_R(l) + \pi_R(r_i)\epsilon)_i \\
&= \overline{xf\left(l + \sum_{i=1}^m r_i b_i\right)}.
\end{aligned}$$

Since  $\pi_{k[\epsilon] \times_k \dots \times_k k[\epsilon]}((\pi_R(l) + \pi_R(r_i)\epsilon)_i) = \pi_R(l)$  and since  $\pi_{R/(\mu_R + m_R^2)}\left(\overline{l + \sum_{i=1}^m r_i b_i}\right) = \pi_R\left(l + \sum_{i=1}^m r_i b_i\right) = \pi_R(l)$ , we obtain  $\pi_{k[\epsilon] \times_k \dots \times_k k[\epsilon]} \circ f = \pi_{R/(\mu_R + m_R^2)}$ . Since  $f$  sends  $\bar{1}_R$  to

$1_{k[\epsilon] \times_k \dots \times_k k[\epsilon]}$ ,  $f$  is indeed an isomorphism in  $\hat{\mathcal{C}}$ .  $\square$

**Corollary 3.10.** *Let  $R$  be in  $\hat{\mathcal{C}}$  with  $m_R \neq 0$ . Suppose  $b_1, b_2, \dots, b_m$  are elements in  $m_R$  so that their images form a  $k$ -basis of  $m_R/(\mu_R + m_R^2)$ . Then, for  $1 \leq i \leq m$ , there exists an element  $\alpha_{b_i} \in h_R(k[\epsilon])$  such that  $\alpha_{b_i}(\mu_R + m_R^2) = 0$ ,  $\alpha_{b_i}(b_i) = \epsilon$  and  $\alpha_{b_i}(b_j) = 0$  for all  $j \neq i$ .*

*Proof.* For  $1 \leq i \leq m$ , let  $p_i : k[\epsilon] \times_k k[\epsilon] \times_k \dots \times_k k[\epsilon] \rightarrow k[\epsilon]$  be the  $i^{\text{th}}$  projection map, where  $k[\epsilon] \times_k \dots \times_k k[\epsilon]$  is the  $m$ -fold pullback of  $k[\epsilon]$  over  $k$ , and let  $\pi : R \rightarrow R/(\mu_R + m_R^2)$  be the natural projection. Then we can use the isomorphism  $f$  in Lemma 3.9 to see that  $\alpha_{b_i} = p_i \circ f \circ \pi : R \rightarrow k[\epsilon]$  is a morphism in  $\hat{\mathcal{C}}$ , for  $1 \leq i \leq m$ , which satisfies the properties in the statement of the corollary.  $\square$

**Theorem 3.11.** *Any two versal couples for  $V$  are isomorphic.*

*Proof.* The outline of the proof is similar to the proof of [10, Proposition 2.9]. However, we provide many more details.

Suppose  $(R, [\phi])$  and  $(R', [\phi'])$  are both versal couples for  $V$ . Then smoothness of the two associated natural transformations

$$\tau = \tau(R, [\phi]) : h_R \rightarrow F_V,$$

$$\tau' = \tau(R', [\phi']) : h_{R'} \rightarrow F_V$$

combined with Lemma 3.7 imply the existence of morphisms  $f : R \rightarrow R'$  and  $f' : R' \rightarrow R$  in  $\hat{\mathcal{C}}$  such that  $[f \circ \phi] = [\phi']$  and  $[f' \circ \phi'] = [\phi]$ . Putting this together we see that

$$[f' \circ f \circ \phi] = [\phi]$$



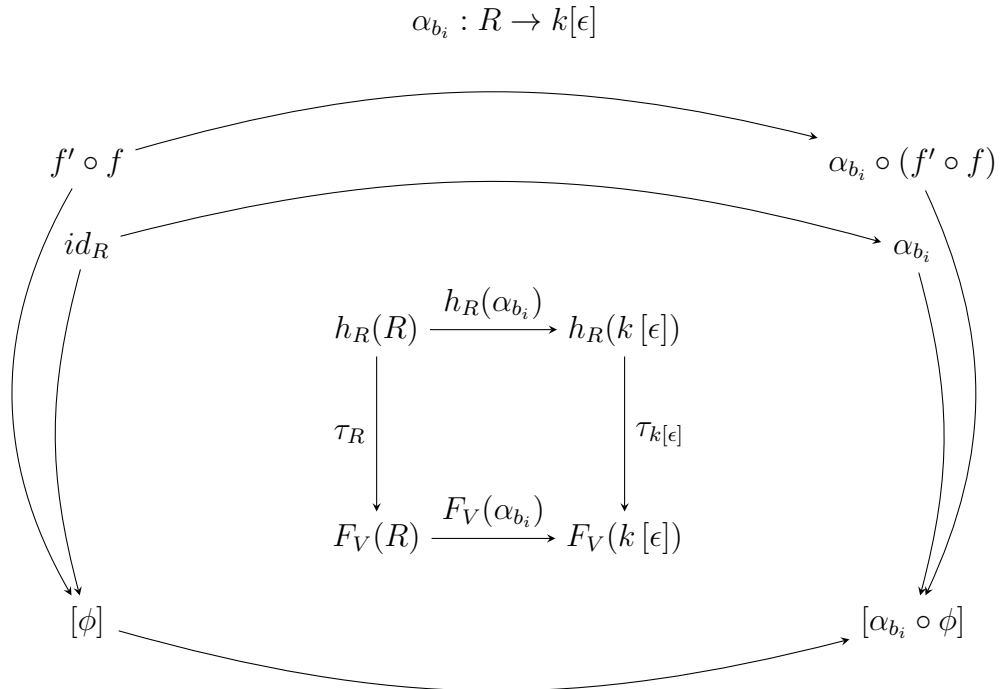
and

$$[f \circ f' \circ \phi'] = [\phi'].$$

Let  $b_1, b_2, \dots, b_m$  be elements of  $m_R$  such that their images form a  $k$ -basis of  $m_R/(\mu_R + m_R^2)$ . By Corollary 3.10, for each  $1 \leq i \leq m$ , there exists a morphism  $\alpha_{b_i} : R \rightarrow k[\epsilon]$  in  $\hat{\mathcal{C}}$  such that  $\alpha_{b_i}(\mu_R + m_R^2) = 0$ ,  $\alpha_{b_i}(b_i) = \epsilon$  and  $\alpha_{b_i}(b_j) = 0$  for all  $j \neq i$ .

Applying the functors  $h_R$  and  $F_V$  to these morphisms, we have the commutative diagrams in Figure 3.4. Since

Figure 3.4: The commutative diagrams obtained by applying  $h_R$  and  $F_V$  to  $\alpha_{b_i}$ .



$$[\phi] = \tau_R(id_R) = \tau_R(f' \circ f)$$

we must have that

$$F_V(\alpha_{b_i})(\tau_R(id_R)) = F_V(\alpha_{b_i})(\tau_R(f' \circ f)).$$

But by the commuting diagram in Figure 3.4,

$$F_V(\alpha_{b_i})(\tau_R(id_R)) = \tau_{k[\epsilon]}(\alpha_{b_i} \circ id_R) = \tau_{k[\epsilon]}(\alpha_{b_i})$$

and

$$F_V(\alpha_{b_i})(\tau_R(f' \circ f)) = \tau_{k[\epsilon]}(\alpha_{b_i} \circ (f' \circ f))$$

which shows that

$$\tau_{k[\epsilon]}(\alpha_{b_i}) = \tau_{k[\epsilon]}(\alpha_{b_i} \circ (f' \circ f)).$$

Since  $\tau_{k[\epsilon]}$  is bijective by the assumption that  $(R, [\phi])$  is a versal couple for  $V$ , it follows that  $\alpha_{b_i} = \alpha_{b_i} \circ f' \circ f$ , for  $1 \leq i \leq m$ . Therefore, for  $1 \leq i \leq m$ ,

$$\epsilon = \alpha_{b_i}(b_i) = (\alpha_{b_i} \circ f' \circ f)(b_i).$$

We next show that the images of  $(f' \circ f)(b_1), \dots, (f' \circ f)(b_m)$  also form a  $k$ -basis of  $m_R/(\mu_R + m_R^2)$ . As before, we let  $\bar{r}$  denote the image of  $r$  in  $R/(\mu_R + m_R^2)$ . Suppose there exist elements  $l_1, \dots, l_m$  in  $\lambda$  such that

$$\overline{\sum_{i=1}^m j_R(l_i)(f' \circ f)(b_i)} = \bar{0}.$$

Then since  $f' \circ f$  is a  $\lambda$ -algebra homomorphism, we see that

$$\sum_{i=1}^m j_R(l_i)(f' \circ f)(b_i) = (f' \circ f)\left(\sum_{i=1}^m (j_R(l_i)b_i)\right) \in \mu_R + m_R^2.$$

But then for each  $1 \leq i \leq m$ ,

$$\begin{aligned}
0 &= \alpha_{b_i}(0) = (\alpha_{b_i} \circ f' \circ f)\left(\sum_{j=1}^m (j_R(l_j)b_j)\right) \\
&= \alpha_{b_i}\left(\sum_{j=1}^m (j_R(l_j)b_j)\right) \\
&= \sum_{j=1}^m j_R(l_j)\alpha_{b_i}(b_j) \\
&= j_R(l_i) \cdot \epsilon
\end{aligned}$$

so that  $j_R(l_i)$  must be in  $\mu_R$ , and hence  $l_i$  must be in  $\mu$ , for all  $1 \leq i \leq m$ . Therefore, the images of

$$(f' \circ f)(b_1), (f' \circ f)(b_2), \dots, (f' \circ f)(b_m)$$

in  $R/(\mu_R + m_R^2)$  also form a basis for the  $k$ -vector space  $m_R/(\mu_R + m_R^2)$ . Hence,  $f' \circ f$  induces a bijection from  $m_R/(\mu_R + m_R^2)$  to itself. By Theorem 2.8, part (l), this implies that  $f' \circ f$  is an automorphism of  $R$  in  $\hat{\mathcal{C}}$ . By an entirely analagous argument, in which we replace  $R$  by  $R'$ , we see that  $f \circ f'$  is an automorphism of  $R'$  in  $\hat{\mathcal{C}}$ . This implies that both  $f$  and  $f'$  are isomorphisms, and hence  $R \cong R'$ , completing the proof that any two versal couples for  $V$  are isomorphic.  $\square$

**Definition 3.12.** If there exists a versal couple  $(R, [\phi])$  for  $V$ , we denote the versal deformation ring  $R$  by  $R(A, V)$  and the versal deformation  $[\phi]$  by  $[\phi_u]$ . Note that we also use this notation when  $(R, [\phi])$  is a universal couple for  $V$ .

Our goal is to prove that if  $F_V(k[\epsilon])$  is a finite dimensional  $k$ -vector space, then a versal couple for  $V$  exists. To see that this is a natural finiteness condition, we first

prove the following result:

**Lemma 3.13.** *The set  $F_V(k[\epsilon])$  is a  $k$ -vector space.*

*Proof.* The proof uses ideas from [9, §15] and [12, Section 3.4].

To define addition in  $F_V(k[\epsilon])$ , first consider the map  $\mathbb{P} : k[\epsilon] \times_k k[\epsilon] \rightarrow k[\epsilon]$  defined by

$$\mathbb{P}((x + \epsilon y, x + \epsilon z)) = x + \epsilon(y + z)$$

for all  $x, y, z$  in  $k$ . This is a  $k$ -algebra homomorphism since for all  $c, x_1, x_2, y_1, y_2, z_1, z_2$  in  $k$ , we have

$$\begin{aligned} & \mathbb{P}(c(x_1 + \epsilon y_1, x_1 + \epsilon z_1) + (x_2 + \epsilon y_2, x_2 + \epsilon z_2)) \\ = & \mathbb{P}((cx_1 + x_2 + \epsilon(cy_1 + y_2), cx_1 + x_2 + \epsilon(cz_1 + z_2))) \\ = & cx_1 + x_2 + \epsilon(cy_1 + y_2 + cz_1 + z_2) \\ = & c(x_1 + \epsilon(y_1 + z_1)) + (x_2 + \epsilon(y_2 + z_2)) \\ = & c\mathbb{P}((x_1 + \epsilon y_1, x_1 + \epsilon z_1)) + \mathbb{P}((x_2 + \epsilon y_2, x_2 + \epsilon z_2)) \end{aligned}$$

and

$$\begin{aligned}
& \mathbb{P}((x_1 + \epsilon y_1, x_1 + \epsilon z_1) * (x_2 + \epsilon y_2, x_2 + \epsilon z_2)) \\
= & \mathbb{P}((x_1 x_2 + \epsilon(x_1 y_2 + x_2 y_1), x_1 x_2 + \epsilon(x_1 z_2 + x_2 z_1))) \\
= & x_1 x_2 + \epsilon(x_1 y_2 + x_2 y_1 + x_1 z_2 + x_2 z_1) = x_1 x_2 + \epsilon(x_1(y_2 + z_2) + x_2(y_1 + z_1)) \\
= & (x_1 + \epsilon(y_1 + z_1)) * (x_2 + \epsilon(y_2 + z_2)) \\
= & \mathbb{P}((x_1 + \epsilon y_1, x_1 + \epsilon z_1)) * \mathbb{P}((x_2 + \epsilon y_2, x_2 + \epsilon z_2)).
\end{aligned}$$

Since the  $\lambda$ -algebra structures of  $k[\epsilon]$  and  $k[\epsilon] \times_k k[\epsilon]$  coincide with their  $k$ -algebra structures, it follows that  $\mathbb{P}$  is a  $\lambda$ -algebra homomorphism which sends  $1_{k[\epsilon] \times_k k[\epsilon]}$  to  $1_{k[\epsilon]}$ . Moreover,  $\pi_{k[\epsilon]} \circ \mathbb{P} = \pi_{k[\epsilon] \times_k k[\epsilon]}$ , which implies that  $\mathbb{P}$  is a morphism in  $\hat{\mathcal{C}}$ .

The morphism  $\mathbb{P}$  then extends to a ring homomorphism, which we also call  $\mathbb{P}$ , between the respective  $n \times n$  matrix rings. If  $\phi$  and  $\psi$  are any two lifts of  $V$  over  $k[\epsilon]$ , then

$$\pi_{k[\epsilon]} \circ \phi = \rho = \pi_{k[\epsilon]} \circ \psi.$$

Hence for all  $a$  in  $A$  and all  $1 \leq i, j \leq n$ , the  $(i, j)$ -entries  $\phi(a)_{ij}$  and  $\psi(a)_{ij}$  of  $\phi(a)$  and  $\psi(a)$ , respectively, define an element

$$(\phi(a)_{ij}, \psi(a)_{ij}) \in k[\epsilon] \times_k k[\epsilon].$$

By abuse of notation, let  $(\phi(a), \psi(a))$  denote the matrix in  $Mat_n(k[\epsilon] \times_k k[\epsilon])$  with  $(i, j)$ -entry given by  $(\phi(a)_{ij}, \psi(a)_{ij})$  for all  $1 \leq i, j \leq n$ . Therefore, we can define a lift  $\mathbb{P}(\phi, \psi)$  of  $V$  over  $k[\epsilon] \times_k k[\epsilon]$  by  $\mathbb{P}(\phi, \psi)(a) = \mathbb{P}((\phi(a), \psi(a)))$ , for all  $a$  in  $A$ .

Notice that this also extends to a well-defined map on deformations. Namely, using the same name,  $\mathbb{P}$ , we can construct a map

$$\mathbb{P} : F_V(k[\epsilon]) \times_{F_V(k)} F_V(k[\epsilon]) \rightarrow F_V(k[\epsilon])$$

as follows: If  $[\phi]$  and  $[\psi]$  are two deformations in  $F_V(k[\epsilon])$ , then  $([\phi], [\psi])$  is certainly in  $F_V(k[\epsilon]) \times_{F_V(k)} F_V(k[\epsilon])$  due to the fact that  $F_V(k)$  contains only a single element, namely  $[\rho]$ . We now define  $\mathbb{P}([\phi], [\psi])$  as  $[\mathbb{P}(\phi, \psi)]$ . To see that this is well-defined, suppose that  $[\phi] = [\phi']$  and  $[\psi] = [\psi']$ . Then, by definition of strictly equivalent lifts, there exist  $X$  and  $Y$  in  $GL_n(k[\epsilon])$  such that

$$\phi' = X\phi X^{-1},$$

$$\psi' = Y\psi Y^{-1},$$

and both  $X$  and  $Y$  reduce to the identity matrix in  $GL_n(k)$ . Notice that if  $X = I + \epsilon X'$  and  $Y = I + \epsilon Y'$  where  $I$  is the  $n \times n$  identity matrix in  $GL_n(k)$  and  $X'$  and  $Y'$  are suitable matrices in  $Mat_n(k)$ , then  $X^{-1} = I - \epsilon X'$  and  $Y^{-1} = I - \epsilon Y'$ . Moreover,  $\mathbb{P}(X, Y) = I + \epsilon(X' + Y')$  and  $\mathbb{P}(X^{-1}, Y^{-1}) = I - \epsilon(X' + Y') = (\mathbb{P}(X, Y))^{-1}$ . Therefore, we have

$$\begin{aligned} \mathbb{P}(\phi', \psi') &= \mathbb{P}(X\phi X^{-1}, Y\psi Y^{-1}) \\ &= \mathbb{P}(X, Y)\mathbb{P}(\phi, \psi)\mathbb{P}(X^{-1}, Y^{-1}) \\ &= \mathbb{P}(X, Y)\mathbb{P}(\phi, \psi)(\mathbb{P}(X, Y))^{-1}, \end{aligned}$$

which proves  $[\mathbb{P}(\phi, \psi)] = [\mathbb{P}(\phi', \psi')]$ . Hence  $\mathbb{P}$  extends to a well-defined map on deformations. We can therefore define addition in  $F_V(k[\epsilon])$  as  $[\phi] + [\psi] = \mathbb{P}([\phi], [\psi])$ .

We can also define scalar multiplication in  $F_V(k[\epsilon])$  by considering the map  $\mathbb{M} : k \times k[\epsilon] \rightarrow k[\epsilon]$ , given by  $\mathbb{M}(c, a + b\epsilon) = a + cb\epsilon$  for all  $a, b, c$  in  $k$ . Notice that for any fixed  $c \in k$ ,  $\mathbb{M}(c, *) : k[\epsilon] \rightarrow k[\epsilon]$  is a morphism in  $\hat{\mathcal{C}}$ . To see this, note that  $\mathbb{M}$  is a  $\lambda$ -algebra homomorphism since for all  $a_1, a_2, b_1, b_2, c_1$  in  $k$  we have

$$\begin{aligned}
& \mathbb{M}(c, c_1(a_1 + b_1\epsilon) + (a_2 + b_2\epsilon)) \\
&= \mathbb{M}(c, c_1a_1 + a_2 + \epsilon(c_1b_1 + b_2)) \\
&= c_1a_1 + a_2 + c\epsilon(c_1b_1 + b_2) \\
&= c_1(a_1 + cb_1\epsilon) + (a_2 + cb_2\epsilon) \\
&= c_1\mathbb{M}(c, a_1 + b_1\epsilon) + \mathbb{M}(c, a_2 + b_2\epsilon),
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{M}(c, (a_1 + b_1\epsilon) * (a_2 + b_2\epsilon)) \\
&= \mathbb{M}(c, a_1a_2 + \epsilon(a_1b_2 + a_2b_1)) \\
&= a_1a_2 + c\epsilon(a_1b_2 + a_2b_1) \\
&= (a_1 + cb_1\epsilon) * (a_2 + cb_2\epsilon) \\
&= \mathbb{M}(c, a_1 + b_1\epsilon) * \mathbb{M}(c, a_2 + b_2\epsilon).
\end{aligned}$$

Since  $\mathbb{M}(c, 1_{k[\epsilon]}) = 1_{k[\epsilon]}$  and  $\pi_{k[\epsilon]} \circ \mathbb{M} = \pi_{k[\epsilon]}$ , this implies that  $\mathbb{M}$  is a morphism in  $\hat{\mathcal{C}}$ .

Note that we can extend  $\mathbb{M}$  in the natural way to a map  $k \times \text{Mat}_n(k[\epsilon]) \rightarrow \text{Mat}_n(k[\epsilon])$ , which we also denote by  $\mathbb{M}$ . If  $\phi$  is a lift of  $V$  over  $k[\epsilon]$ , we define  $\mathbb{M}(c, \phi)$  to be the

lift of  $V$  over  $k[\epsilon]$  given by  $\mathbb{M}(c, \phi)(a) = \mathbb{M}(c, \phi(a))$ , for all  $a$  in  $A$ . Suppose that  $\phi$  and  $\phi'$  are two strictly equivalent lifts of  $V$  over  $k[\epsilon]$ . Then  $\phi' = X\phi X^{-1}$  for some  $X \in GL_n(k[\epsilon])$  which reduces to the identity matrix in  $GL_n(k)$ . Notice that if  $X = I + \epsilon X'$ , then  $\mathbb{M}(c, X) = I + \epsilon c X'$  and  $\mathbb{M}(c, X^{-1}) = I - \epsilon c X' = (\mathbb{M}(c, X))^{-1}$ .

Hence we have

$$\mathbb{M}(c, \phi') = \mathbb{M}(c, X\phi X^{-1}) = \mathbb{M}(c, X)\mathbb{M}(c, \phi)\mathbb{M}(c, X^{-1}) = \mathbb{M}(c, X)\mathbb{M}(c, \phi)\mathbb{M}(c, X)^{-1},$$

which proves  $[\mathbb{M}(c, \phi)] = [\mathbb{M}(c, \phi')]$ . Therefore, we can define scalar multiplication in  $F_V(k[\epsilon])$  as  $c \cdot [\phi] = [\mathbb{M}(c, \phi)]$ . The vector space axioms for  $F_V(k[\epsilon])$  using this addition and scalar multiplication are readily checked, since  $\mathbb{P}$  and  $\mathbb{M}$  are morphisms in  $\hat{\mathcal{C}}$ . Hence  $F_V(k[\epsilon])$  is a  $k$ -vector space.  $\square$

We also need to consider ring extensions in  $\mathcal{C}$  with “small” kernels, which we now define.

**Definition 3.14.** We call a morphism  $\alpha : B \rightarrow B'$  in  $\mathcal{C}$  a small extension if  $\alpha$  is surjective and  $\ker(\alpha)$  has length 1, as a left ideal. Note that there exists then an element  $t \in B$  so that  $\ker(\alpha) = Bt = tB$  and  $m_{Bt} = 0 = tm_B$ . In particular, each element in  $Bt$  can be written as  $lt$  for some  $l$  in  $j_B(\lambda)$ .

Define  $k[Bt] = k \oplus Bt = \{a + lt | a \in k, l \in j_B(\lambda)\}$ . Then  $k[Bt]$  is a ring in  $\mathcal{C}$  which is isomorphic to  $k[\epsilon]$  by the isomorphism  $\omega : k[\epsilon] \rightarrow k[Bt]$ , defined by

$$\omega(a + b\epsilon) = a + \hat{b}t$$

for all  $a, b$  in  $k$ , where  $\hat{b}$  is any element in  $j_B(\lambda)$  satisfying  $\pi_B(\hat{b}) = b$ .



The next result will be important when showing smoothness of the natural transformation  $\tau = \tau(R, [\phi]) : h_R \rightarrow F_V$ , defined in Definition 3.3, part (c) (see also Definition 3.6).

**Lemma 3.15.** *Let  $\alpha : B \rightarrow B'$  be a small extension in  $\mathcal{C}$ , and let  $\ker(\alpha) = Bt$ , where  $t \in B$  satisfies  $m_B t = tm_B = 0$ .*

(a) *If  $\beta$  and  $\gamma$  are two lifts of  $V$  over  $B$  such that  $\alpha \circ \beta = \alpha \circ \gamma$ , then  $\gamma - \beta$  defines a lift  $\psi_{\gamma\beta}$  of  $V$  over  $k[Bt]$ , by defining*

$$\psi_{\gamma\beta}(a) = \rho(a) + (\gamma - \beta)(a)$$

*for all  $a$  in  $A$ .*

(b) *For every lift  $\phi : A \rightarrow \text{Mat}_n(k[Bt])$  of  $V$  over  $k[Bt]$ , define  $\tilde{\phi}(a) = \phi(a) - \rho(a) \in \text{Mat}_n(Bt)$ , for all  $a \in A$ . If  $\beta$  is a lift of  $V$  over  $B$ , then for every lift  $\gamma$  of  $V$  over  $B$  satisfying  $[\alpha \circ \beta] = [\alpha \circ \gamma]$ , there exists a lift  $\psi$  of  $V$  over  $k[Bt]$  such that  $\gamma$  is strictly equivalent to the lift  $\beta_\psi : A \rightarrow \text{Mat}_n(B)$ , defined by*

$$\beta_\psi(a) = \beta(a) + \tilde{\psi}(a)$$

*for all  $a \in A$ .*

*Proof.* (a) Since  $\pi_{k[Bt]} \circ \psi_{\gamma\beta} = \rho$ , it suffices to show that  $\psi_{\gamma\beta}$  is a  $\lambda$ -algebra homo-

morphism. Let  $a_1, a_2 \in A$  and  $l \in \lambda$ . Then

$$\begin{aligned}
\psi_{\gamma\beta}(la_1 + a_2) &= \rho(la_1 + a_2) + (\gamma - \beta)(la_1 + a_2) \\
&= \pi_\lambda(l)\rho(a_1) + \rho(a_2) + j_B(l)(\gamma - \beta)(a_1) + (\gamma - \beta)(a_2) \\
&= l(\rho(a_1) + (\gamma - \beta)(a_1)) + (\rho(a_2) + (\gamma - \beta)(a_2)) \\
&= l\psi_{\gamma\beta}(a_1) + \psi_{\gamma\beta}(a_2)
\end{aligned}$$

and, using the isomorphism  $\omega$  from Definition 3.14 and using Definition 2.12,

$$\begin{aligned}
\psi_{\gamma\beta}(a_1a_2) &= \rho(a_1a_2) + (\gamma - \beta)(a_1a_2) \\
&= \rho(a_1)\rho(a_2) + \gamma(a_1)\gamma(a_2) - \beta(a_1)\beta(a_2) \\
&= \rho(a_1)\rho(a_2) + \gamma(a_1)(\gamma - \beta)(a_2) + (\gamma - \beta)(a_1)\beta(a_2) \\
&= (\rho(a_1) + (\gamma - \beta)(a_1)) * (\rho(a_2) + (\gamma - \beta)(a_2)) \\
&= \psi_{\gamma\beta}(a_1) * \psi_{\gamma\beta}(a_2).
\end{aligned}$$

Hence  $\psi_{\gamma\beta}$  is a lift of  $V$  over  $k[Bt]$ .

- (b) Let  $\beta$  and  $\gamma$  be lifts of  $V$  over  $B$  satisfying  $[\alpha \circ \beta] = [\alpha \circ \gamma]$ . This means there exists a matrix  $P'$  in  $GL_n(B')$  which reduces to the identity matrix in  $GL_n(k)$  such that

$$\alpha \circ \beta = P'(\alpha \circ \gamma)(P')^{-1}.$$

Let  $P$  be any matrix in  $Mat_n(B)$  so that  $\alpha(P) = P'$ . Then  $P$  reduces to the

identity in  $Mat_n(k)$ , and hence is a unit in  $Mat_n(B)$ , and

$$\alpha \circ \beta = \alpha \circ (P\gamma P^{-1}).$$

Define  $\gamma' = P\gamma P^{-1}$ , so that  $\gamma$  and  $\gamma'$  are strictly equivalent. Define  $\psi : A \rightarrow Mat_n(k[Bt])$  to be equal to the lift  $\psi_{\gamma'\beta}$  defined in part (a). Then it follows that  $\tilde{\psi}(a) = (\gamma' - \beta)(a)$ , for all  $a$  in  $A$ . Therefore,

$$\gamma'(a) = \beta(a) + (\gamma' - \beta)(a) = \beta(a) + \tilde{\psi}(a)$$

for all  $a$  in  $A$ , which means that  $\gamma' = \beta_\psi$ . Hence,  $[\gamma] = [\gamma'] = [\beta_\psi]$ .

□

We now prove our main result by constructing a versal couple for  $V$ .

**Theorem 3.16.** *If  $F_V(k[\epsilon])$  is finite dimensional as a  $k$ -vector space, then there exists a versal couple for  $V$ . In particular,  $V$  then has a versal deformation ring  $R(A, V)$ .*

The proof of this theorem consists of several steps and uses similar arguments as in the proof of [10, Theorem 2.11]. We will first construct a certain ring  $R$  in  $\hat{\mathcal{C}}$ , and a certain lift  $\phi$  of  $V$  over  $R$ . We will then prove the theorem by proving that  $(R, [\phi])$  is a versal couple for  $V$ .

*Construction of  $(R, [\phi])$ :*

If  $F_V(k[\epsilon])$  is an  $m$ -dimensional  $k$ -vector space for some  $m \geq 1$ , we can select a finite basis  $\{\psi_i\}_{i=1}^m$  of  $F_V(k[\epsilon])$ . Due to the fact that each lift  $\psi_i$  must satisfy

$\pi_{k[\epsilon]} \circ \psi_i = \rho$ , we can then define a lift,  $\phi'_2$ , of  $V$  over the  $m$ -fold pullback of  $k[\epsilon]$  over  $k$ ,  $k[\epsilon] \times_k k[\epsilon] \times_k \dots \times_k k[\epsilon]$ , by

$$\phi'_2(a) = (\psi_1(a), \dots, \psi_m(a)),$$

for all  $a$  in  $A$ . Let  $R'_2$  be the  $m$ -fold pullback of  $k[\epsilon]$  over  $k$ , let  $S$  be the power series ring over  $\lambda$  in  $m$  non-commuting variables  $x_1, x_2, \dots, x_m$ , and let  $I_2$  be the ideal  $\mu_S + m_S^2$  in  $S$ . Notice that since  $j_S : \lambda \rightarrow S$  is simply the inclusion map, we commit a slight abuse of notation and write  $\mu$  to refer to  $\mu_S$ . Explicitly,

$$R'_2 = k[\epsilon] \times_k k[\epsilon] \times_k \dots \times_k k[\epsilon],$$

$$S = \lambda \langle \langle x_1, \dots, x_m \rangle \rangle,$$

and

$$I_2 = \mu + (x_1, \dots, x_m)^2.$$

By Lemma 3.9, we know that  $R'_2 \cong S/I_2$  via the isomorphism

$$f : S/I_2 \rightarrow R'_2,$$

$$f\left(l + \sum_{i=1}^m (l_i x_i) + I_2\right) = (\pi_\lambda(l) + \epsilon \pi_\lambda(l_i))_i,$$

where  $l, l_1, \dots, l_m$  lie in  $\lambda$ , and we use that  $\pi_S \circ j_S = \pi_\lambda$ . Note that the inverse of  $f$  is the map

$$g : R'_2 \rightarrow S/I_2$$

given by

$$g((c + \epsilon c_1, \dots, c + \epsilon c_m)) = l + \sum_{i=1}^m (l_i x_i) + I_2$$

where  $c, c_1, \dots, c_m$  lie in  $k$  and  $l$  and  $l_1, \dots, l_m$  are any elements of  $\lambda$  so that  $\pi_\lambda(l) = c$  and  $\pi_\lambda(l_i) = c_i$  for  $1 \leq i \leq m$ . Now define

$$R_2 = S/I_2$$

and

$$\phi_2 = g \circ \phi'_2.$$

The couple  $(R_2, [\phi_2])$  for  $V$  begins the construction of the versal couple for  $V$ . To continue the construction, we proceed by induction: Let  $\ell \geq 2$ . Define  $R_i = S/I_i$ , for  $2 \leq i \leq \ell$ , where we assume that we have found ideals  $I_2 \supseteq I_3 \supseteq \dots \supseteq I_\ell$  of  $S$ , along with lifts  $\phi_i$  of  $V$  over  $R_i$  for  $2 \leq i \leq \ell$ , such that  $\pi_i \circ \phi_i = \phi_{i-1}$  for  $3 \leq i \leq \ell$  when  $\pi_i : R_i \rightarrow R_{i-1}$  is the natural projection. Assume further that  $m_S^i \subseteq I_i$  for  $2 \leq i \leq \ell$ . We then define  $I_{\ell+1}$  to be the minimal ideal among all two-sided ideals,  $J$ , of  $S$ , such that the following two conditions hold:

- (a)  $m_S * I_\ell + I_\ell * m_S \subseteq J \subseteq I_\ell$ .
- (b) There exists a lift  $\phi_J$  of  $V$  over  $S/J$  so that  $\pi_J \circ \phi_J = \phi_\ell$ , where  $\pi_J : S/J \rightarrow R_\ell$  is defined to be the natural projection.

Notice that the set of ideals satisfying conditions (a) and (b) is non-empty, since  $I_\ell$  satisfies both requirements. To see that a unique minimal ideal exists, we first show that the set of all such ideals is closed under pairwise intersection. Supposing both  $J$  and  $J'$  satisfy the two conditions (a) and (b), then it follows that  $m_S * I_\ell + I_\ell * m_S \subseteq J \cap J' \subseteq I_\ell$ , and hence  $J \cap J'$  satisfies the first condition (a). Since  $I_\ell / (m_S * I_\ell + I_\ell * m_S)$

Figure 3.5: The pullback diagram used to show that  $J \cap J'$  satisfies condition (b).

$$\begin{array}{ccc}
\phi_{J \cap J'} & & \\
S/(J \cap J') & \xrightarrow{\cong} & \\
& \searrow^{\phi_J \times_{\phi_\ell} \phi_{J'}} & \phi_{J'} \\
& & S/J \times_{S/I_\ell} S/J' \longrightarrow S/J' \\
& & \downarrow \qquad \qquad \downarrow \\
& & S/J \longrightarrow S/I_\ell \\
& \phi_J & \phi_\ell
\end{array}$$

is a left/right  $S$ -module on which  $m_S$  acts trivially, it is a  $k$ -vector space. Moreover, since  $m_S^\ell \subseteq I_\ell$  by assumption, we know that

$$m_S^{\ell+1} \subseteq m_S * I_\ell + I_\ell * m_S.$$

Because  $S/m_S^{\ell+1}$  is Artinian by Theorem 2.8, part (a), it follows that  $S/(m_S * I_\ell + I_\ell * m_S)$  is Artinian, and therefore  $I_\ell/(m_S * I_\ell + I_\ell * m_S)$  is a finite dimensional  $k$ -vector space. Take a  $k$ -basis of  $(J + J')/(m_S * I_\ell + I_\ell * m_S)$ , say  $\{d_j + (m_S * I_\ell + I_\ell * m_S)\}_{j=1}^u$ , and extend it to a  $k$ -basis of  $I_\ell/(m_S * I_\ell + I_\ell * m_S)$ , say  $\{d_j + (m_S * I_\ell + I_\ell * m_S)\}_{j=1}^v$ , where  $v \geq u$  and  $d_1, \dots, d_u$  lie in  $J + J'$  and  $d_{u+1}, \dots, d_v$  lie in  $I_\ell - (J + J')$ . Consider the two-sided ideal  $J'' = J' + (d_{u+1}, \dots, d_v)$  in  $S$ . Then  $J \cap J'' = J \cap J'$  since this equality holds modulo  $(m_S * I_\ell + I_\ell * m_S)$ . Therefore, we can replace  $J'$  by  $J''$  and  $\phi_{J'}$  by its projection onto  $S/J''$ , to be able to assume that  $J + J' = I_\ell$ . Hence, we have

the pullback diagram in Figure 3.5, where all maps are natural. Note that

$$S/J \times_{S/I_\ell} S/J' = S/J \times_{S/(J+J')} S/J' \cong S/(J \cap J')$$

and that

$$\phi_J \times_{\phi_\ell} \phi_{J'}$$

is a lift of  $V$  over  $S/J \times_{S/I_\ell} S/J'$  which projects onto  $\phi_\ell$ . We then compose this lift with the natural isomorphism between  $S/J \times_{S/I_\ell} S/J'$  and  $S/(J \cap J')$  to arrive at a lift,  $\phi_{J \cap J'}$ , of  $V$  over  $S/(J \cap J')$  which projects onto  $\phi_\ell$ , showing that  $J \cap J'$  also satisfies condition (b). This proves that the set of all ideals satisfying conditions (a) and (b) is closed under pairwise intersections. Moreover, since we have already seen that  $I_\ell/(m_S * I_\ell + I_\ell * m_S)$  is a finite dimensional  $k$ -vector space, it follows that an intersection of any arbitrary collection of two-sided ideals satisfying condition (a) must in fact be able to be written as a finite intersection of such ideals. Since our set of ideals in question is closed under pairwise intersection, we have proved the existence of our minimal ideal,  $I_{\ell+1}$ . In addition,  $m_S^{\ell+1} \subseteq I_{\ell+1}$  since  $m_S^{\ell+1} \subseteq m_S * I_\ell + I_\ell * m_S$ . Let  $\phi_{\ell+1}$  be any lift of  $V$  over  $R_{\ell+1} = S/I_{\ell+1}$  such that  $\pi_{\ell+1} \circ \phi_{\ell+1} = \phi_\ell$ , where  $\pi_{\ell+1} : R_{\ell+1} \rightarrow R_\ell$  is the natural projection. The construction of  $I_{\ell+1}$  and  $\phi_{\ell+1}$  completes the inductive step. Define  $I$  to be the intersection of all the  $I_\ell$ , and define  $R$  to be  $S/I$ . We claim that  $R$  is a versal deformation ring for  $V$ . However, before proving that, we first show that  $R$  is an object of  $\hat{\mathcal{C}}$ , and also construct a lift  $\phi$  of  $V$  over  $R$ .

*Proof that  $R$  is in  $\hat{\mathcal{C}}$ :*

Since  $m_S^\ell \subseteq I_\ell$  for all  $\ell \geq 2$ ,  $I_\ell$  can be written as a union of cosets of the form  $b + m_S^\ell$ , where the union is taken over all  $b$  in  $I_\ell$ . Since the cosets of  $m_S^\ell$  partition  $S$ , the complement of  $I_\ell$  can be written as the union of cosets of the form  $b' + m_S^\ell$ , where the union is taken over all  $b'$  in  $S - I_\ell$ . Hence, the complement of  $I_\ell$  is open, which means that  $I_\ell$  is closed for each  $\ell \geq 2$ . Therefore,  $I$  is the intersection of closed subsets of  $S$ , and hence  $I$  is closed. Moreover, since each  $I_\ell \subseteq m_S$ ,  $I$  is a proper ideal of  $S$ . The fact that  $R = S/I$  is an object of  $\hat{\mathcal{C}}$  now follows from Theorem 2.8, parts (f) and (h).

*Construction of a particular  $\phi$  of  $V$  over  $R$ :*

In order to complete the construction of the desired couple, it will be necessary to find a deformation  $[\phi]$  of  $V$  over  $R$  and show it is versal. We now construct a lift  $\phi$  of  $V$  over  $R$ , as follows. Recall that the  $\phi_\ell$  were chosen so that  $\pi_{\ell+1} \circ \phi_{\ell+1} = \phi_\ell$  for all  $\ell \geq 2$ , where  $\pi_{\ell+1} : R_{\ell+1} \rightarrow R_\ell$  is the natural projection. Therefore, we know that for all  $a$  in  $A$ , the sequence  $(\phi_\ell(a))_\ell$  is in the inverse limit  $\varprojlim_\ell Mat_n(R_\ell) = \varprojlim_\ell Mat_n(S/I_\ell)$ . Consider the following inverse systems  $(S/I_\ell, \pi_\ell) = (S/I_\ell)$  and  $(S/m_S^\ell, \xi_\ell)$ , where  $\xi_\ell : S/m_S^\ell \rightarrow S/m_S^{\ell-1}$  is the natural projection. Then  $(I/m_S^\ell, \tilde{\xi}_\ell) = (I_\ell/m_S^\ell)$  is also an inverse system, where  $\tilde{\xi}_\ell : I_\ell/m_S^\ell \rightarrow I_{\ell-1}/m_S^{\ell-1}$  is the restriction of  $\xi_\ell$  using that  $I_\ell \subseteq I_{\ell-1}$ . By extending  $\pi_\ell$ ,  $\xi_\ell$ , and  $\tilde{\xi}_\ell$  to matrix rings, we obtain inverse systems  $(Mat_n(S/I_\ell), \pi_\ell) = (Mat_n(S/I_\ell))$ ,  $(Mat_n(S/m_S^\ell), \xi_\ell) = (Mat_n(S/m_S^\ell))$  and  $(Mat_n(I_\ell/m_S^\ell), \tilde{\xi}_\ell) = (Mat_n(I_\ell/m_S^\ell))$ . Since  $S/m_S^\ell$  is an Artinian ring for each  $\ell$ ,



$Mat_n(S/m_S^\ell)$  is also an Artinian ring for each  $\ell$ . Therefore, the collection of ideals  $\{Mat_n((I_{\ell+i} + m_S^\ell)/m_S^\ell)\}_{i \geq 1}$  stabilizes for each  $\ell$ . In other words, the inverse system  $(Mat_n(I_\ell/m_S^\ell))$  satisfies the Mittag-Leffler condition. It follows from [5, Proposition III.10.3] that the following sequence of inverse limits is exact, where all the maps are natural:

$$0 \rightarrow \varprojlim_{\ell} Mat_n(I_\ell/m_S^\ell) \rightarrow \varprojlim_{\ell} Mat_n(S/m_S^\ell) \rightarrow \varprojlim_{\ell} Mat_n(S/I_\ell) \rightarrow 0.$$

Therefore, there exists an element  $(\bar{B}_\ell)_\ell$  in the inverse limit  $\varprojlim_{\ell} Mat_n(S/m_S^\ell)$  such that  $\bar{B}_\ell$  is in  $Mat_n(S/m_S^\ell)$  and  $\beta_\ell(\bar{B}_\ell) = \phi_\ell(a)$  for all  $\ell \geq 2$ , where

$$\beta_\ell : Mat_n(S/m_S^\ell) \rightarrow Mat_n(S/I_\ell)$$

is the natural projection. By Lemma 2.5,  $Mat_n(S)$  is  $Mat_n(m_S)$ -adically complete. Therefore, we can find a matrix  $B$  in  $Mat_n(S)$  such that  $q_\ell^S(B) = \bar{B}_\ell$  for all  $\ell$ , where

$$q_\ell^S : Mat_n(S) \rightarrow Mat_n(S/m_S^\ell)$$

is the natural projection. If  $(\bar{B}'_\ell)_\ell$  is any other element in  $\varprojlim_{\ell} Mat_n(S/m_S^\ell)$  such that  $\bar{B}'_\ell$  is in  $Mat_n(S/m_S^\ell)$  and  $\beta_\ell(\bar{B}'_\ell) = \phi_\ell(a)$  for all  $\ell \geq 2$ , then by a similar argument we can find some matrix  $B'$  in  $Mat_n(S)$  such that  $q_\ell^S(B') = \bar{B}'_\ell$  for all  $\ell$ . But then  $\beta_\ell(q_\ell^S(B - B')) = 0$  for all  $\ell \geq 2$ , and so  $B - B'$  lies in  $Mat_n(I_\ell)$  for all  $\ell \geq 2$ , from which it follows that  $B - B'$  is in  $Mat_n(I)$ . Thus, there exists a unique element  $B + Mat_n(I)$  in  $Mat_n(S/I)$  such that  $q_\ell^I(B + Mat_n(I)) = \phi_\ell(a)$  for all  $\ell \geq 2$ , where

$$q_\ell^I : Mat_n(S/I) \rightarrow Mat_n(S/I_\ell)$$

is the natural projection.

Therefore, we obtain a lift  $\phi$  of  $V$  over  $S/I$  by defining, for all  $a \in A$ ,  $\phi(a)$  to be equal to this unique element  $B + \text{Mat}_n(I)$  in  $\text{Mat}_n(S/I)$ , which for all  $\ell$  projects onto  $\phi_\ell(a)$  via the natural projections.

Finally, before proving that the constructed couple  $(R, [\phi])$  is a versal couple, we prove two useful relationships between the constructed ideals,  $I_\ell$ , and their intersections,  $I$ . Namely, we show

$$I_{\ell+1} = I + m_S * I_\ell + I_\ell * m_S \quad (3.1)$$

for all  $\ell \geq 2$ , and

$$I_{\ell+i} \subseteq I + m_S^{i+1} \quad (3.2)$$

for all  $\ell \geq 2$  and  $i \geq 0$ .

To prove (3.1), first note that

$$m_S * I_\ell + I_\ell * m_S \subseteq I + m_S * I_\ell + I_\ell * m_S \subseteq I_\ell$$

for all  $\ell \geq 2$ . Moreover,  $\phi$  is a lift of  $V$  over  $R = S/I$  which projects onto the lift  $\phi_\ell$  of  $V$  over  $R_\ell = S/I_\ell$ , for all  $\ell$ . Projecting  $\phi$  onto  $S/(I + m_S * I_\ell + I_\ell * m_S)$ , we can find a lift of  $V$  over  $S/(I + m_S * I_\ell + I_\ell * m_S)$  which projects onto  $\phi_\ell$ . Since  $I_{\ell+1}$  is the minimal ideal satisfying both these conditions, it follows that

$$I_{\ell+1} \subseteq I + m_S * I_\ell + I_\ell * m_S.$$

But since both  $I$  and  $m_S * I_\ell + I_\ell * m_S$  are contained in  $I_{\ell+1}$ , we obtain (3.1).

To prove (3.2), we fix  $\ell \geq 2$  and use induction on  $i \geq 0$ . Since  $I_\ell \subseteq m_S$ , it follows that  $I_\ell \subseteq I + m_S$ , which means that (3.2) holds for  $i = 0$ . Assume (3.2) is

true for  $0 \leq i \leq r$ . Then, using (3.1), we have

$$\begin{aligned} I_{\ell+r+1} &= I + m_S * I_{\ell+r} + I_{\ell+r} * m_S \\ &\subseteq I + m_S * (I + m_S^{r+1}) + (I + m_S^{r+1}) * m_S = I + m_S^{r+2} \end{aligned}$$

which proves (3.2).

*Proof of Theorem 3.16:*

It remains to show that  $(R, [\phi])$  is a versal couple for  $V$ . To see this, we prove the required properties of the natural transformation  $\tau = \tau(R, [\phi]) : h_R \rightarrow F_V$ , recalling that for all  $B$  in  $\hat{\mathcal{C}}$

$$\tau_B : h_R(B) \rightarrow F_V(B)$$

is defined as

$$\tau_B(\alpha) = [\alpha \circ \phi]$$

for all  $\alpha : R \rightarrow B$  in  $\hat{\mathcal{C}}$ .

First, we will show that  $\tau_{k[\epsilon]}$  is a bijection. Recall that  $R = S/I$  and  $R_2 = S/I_2$ . Then  $h_R(k[\epsilon]) = h_{S/I}(k[\epsilon])$  is naturally identified with  $h_{R_2}(k[\epsilon]) = h_{S/I_2}(k[\epsilon])$  due to the fact that

$$\mu_R + m_R^2 = (\mu + m_S^2)/I = I_2/I$$

is contained in the kernel of any morphism  $R = S/I \rightarrow k[\epsilon]$  in  $\hat{\mathcal{C}}$ . For  $1 \leq i \leq m$ , let  $b_i = x_i + I_2$ . Then  $b_1, \dots, b_m$  are elements in  $m_{R_2}$  so that their images form a  $k$ -basis

of  $m_{R_2}/(\mu_{R_2} + m_{R_2}^2)$ . Let  $\alpha_{b_1}, \dots, \alpha_{b_m} \in h_{R_2}(k[\epsilon])$  be the morphisms constructed in Corollary 3.10. Recall that, for  $1 \leq i \leq m$ ,  $\alpha_{b_i}(\mu_{R_2} + m_{R_2}^2) = 0$ ,  $\alpha_{b_i}(b_i) = \epsilon$  and  $\alpha_{b_i}(b_j) = 0$  for all  $j \neq i$ . From this it follows that  $\{\alpha_{b_i}\}_{i=1}^m$  is a  $k$ -basis of  $h_{R_2}(k[\epsilon])$ . Letting  $p_2 : R \rightarrow R_2$  be the natural projection, we obtain that  $\{\alpha_{b_i} \circ p_2\}_{i=1}^m$  is a  $k$ -basis of  $h_R(k[\epsilon])$ . Recall that our lift  $\phi_2 = p_2 \circ \phi$  of  $V$  over  $R_2 = S/I_2$  has the property that

$$\phi_2 = g \circ \phi'_2 = g \circ (\psi_1, \dots, \psi_m),$$

where  $g$  is the inverse of  $f$  and  $\{[\psi_i]\}_{i=1}^m$  is a  $k$ -basis of  $F_V(k[\epsilon])$ . From this and the properties of  $\alpha_{b_i}$ ,  $1 \leq i \leq m$ , it follows that

$$\tau_{k[\epsilon]}(\alpha_{b_i} \circ p_2) = [\alpha_{b_i} \circ p_2 \circ \phi] = [\alpha_{b_i} \circ \phi_2] = [\psi_i]$$

for  $1 \leq i \leq m$ . As  $\tau_{k[\epsilon]}$  maps basis elements to basis elements, it follows that  $\tau_{k[\epsilon]}$  is a bijection.

In order to show that  $\tau$  is smooth, consider any surjective morphism  $\alpha : B \rightarrow B'$  in the category  $\mathcal{C}$ . We must show that the natural map  $q$  in Figure 3.1 is surjective. This is equivalent to the following statement, outlined in Figure 3.6: If  $[\beta]$  is in  $F_V(B)$  and  $u$  is in  $h_R(B')$  such that  $[u \circ \phi] = [\alpha \circ \beta]$ , we need to find an element  $\delta$  in  $h_R(B)$  such that  $\alpha \circ \delta = u$  and  $[\delta \circ \phi] = [\beta]$ . Since  $[u \circ \phi] = [\alpha \circ \beta]$  there exists a matrix  $P'$  in  $GL_n(B')$  which reduces to the identity matrix in  $GL_n(k)$  such that

$$u \circ \phi = P'(\alpha \circ \beta)(P')^{-1}.$$

Let  $P$  in  $Mat_n(B)$  be any matrix such that  $\alpha(P) = P'$ . Then  $P$  reduces to the

Figure 3.6: The smoothness conditions on  $\delta$ .

$$\begin{array}{ccc}
 & & [\phi] \\
 & & R \\
 & \delta & \downarrow u \\
 B & \xrightarrow{\alpha} & B' \\
 [\beta] & & [\beta']
 \end{array}$$

$$[\alpha \circ \beta] = [u \circ \phi]$$

identity in  $Mat_n(k)$  and

$$u \circ \phi = \alpha \circ (P\beta P^{-1}).$$

Replacing  $\beta$  by the strict equivalent  $P\beta P^{-1}$ , we can assume that

$$u \circ \phi = \alpha \circ \beta.$$

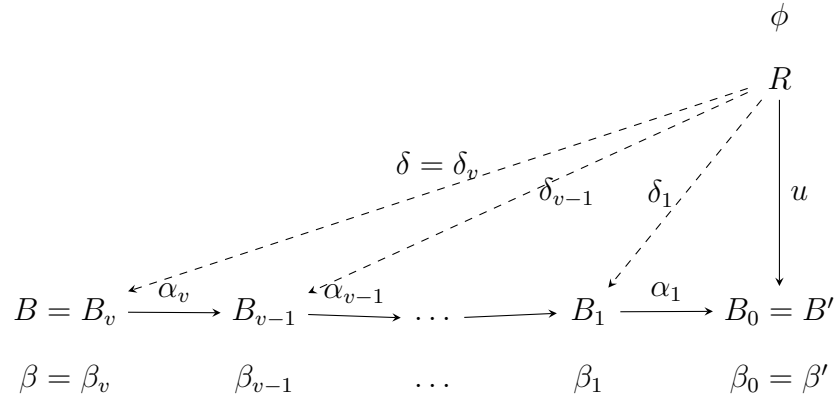
Define  $\beta' = u \circ \phi = \alpha \circ \beta$ .

Since  $B$  is Artinian and  $\alpha$  is surjective, the kernel of  $\alpha$  has finite length as a left ideal. Hence we can factor  $\alpha$  into finitely many compositions of surjections, each of which has a kernel of length 1, as left ideals. By a simple induction argument, suggested by Figure 3.7, we can assume that the kernel of  $\alpha$  is of length 1 as a left ideal. Note that in Figure 3.7, each morphism  $B_i \xrightarrow{\alpha_i} B_{i-1}$  has a kernel of length 1 as a left ideal, and  $\beta_{i-1} = \alpha_i \circ \beta_i$ , for  $1 \leq i \leq v$ .

Hence, we assume now that  $\alpha$  is a small extension, as defined in Definition

3.14. In particular,  $m_B * \ker(\alpha) = 0 = \ker(\alpha) * m_B$ . By (3.2),  $I_{2+i} \subseteq I + m_S^{i+1}$  for any  $i \geq 0$ , which we can rewrite as  $I_i \subseteq I + m_S^{i-1}$  for any  $i \geq 2$ .

Figure 3.7: The induction argument used to show that the kernel of  $\alpha$  can be assumed to have length 1.



Since  $B'$  is Artinian, there exists an integer  $\ell \geq 2$  such that  $m_{B'}^{\ell-1} = 0$ . From  $I_\ell \subseteq I + m_S^{\ell-1}$  it follows that  $u : S/I = R \rightarrow B'$  factors through a morphism  $\bar{u} : S/I_\ell \rightarrow B'$  in  $\hat{\mathcal{C}}$ . Let  $w : S \rightarrow B'$  be the composition of  $u$  with the natural projection  $S \rightarrow S/I$ , as indicated in the vertical right paths of Figure 3.8. For  $1 \leq i \leq m$ , let  $y_i$  be an element of  $B$  belonging to  $\alpha^{-1}(w(x_i))$ . Since  $\alpha$  and  $w$  are morphisms in  $\hat{\mathcal{C}}$ ,  $y_i$  is in  $m_B$  for all  $i$ . We can then define  $\eta(x_i) = y_i$  for all  $1 \leq i \leq m$ . Since  $S = \lambda\langle\langle x_1, \dots, x_m \rangle\rangle$ , we can extend  $\eta$  naturally to a morphism  $\eta : S \rightarrow B$  in  $\hat{\mathcal{C}}$  such that  $\alpha \circ \eta = w$ . Since  $w(I_\ell) = 0$ , we see that  $\eta(I_\ell) \subseteq \ker(\alpha)$ . Recalling that

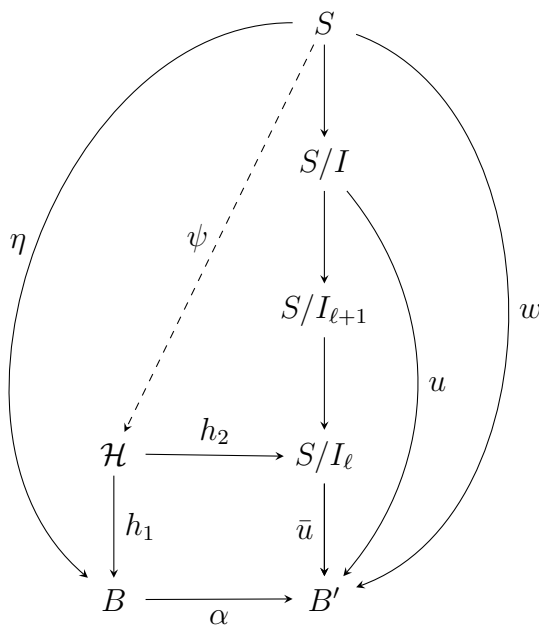
$$m_B * \ker(\alpha) = 0 = \ker(\alpha) * m_B$$

we then have that

$$\eta(m_S * I_\ell + I_\ell * m_S) \subseteq m_B * \eta(I_\ell) + \eta(I_\ell) * m_B \subseteq m_B * \ker(\alpha) + \ker(\alpha) * m_B = 0$$

and hence  $m_S * I_\ell + I_\ell * m_S \subseteq \ker(\eta)$ .

Figure 3.8: The diagram used to find a morphism  $\delta' : R = S/I \rightarrow B$  with  $\alpha \circ \delta' = u$ .



Let  $\mathcal{H}$  be the pullback of  $\alpha$  and  $\bar{u}$ , as indicated in Figure 3.8. Then there exists a morphism  $\psi : S \rightarrow \mathcal{H}$  in  $\hat{\mathcal{C}}$  such that  $\psi(s) = (\eta(s), s + I_\ell)$  for all  $s \in S$ . It follows from above that  $m_S * I_\ell + I_\ell * m_S \subseteq \ker(\psi)$ , and it is clear from the definition of  $\psi$  that  $\ker(\psi) \subseteq I_\ell$ . Notice that  $h_2 : \mathcal{H} \rightarrow S/I_\ell$  is surjective, since  $h_2 \circ \psi : S \rightarrow S/I_\ell$  is surjective. Furthermore, the kernel of  $h_2$  is the set of all elements of  $\mathcal{H}$  of the form  $(x, 0 + I_\ell)$ , where  $x$  is an element of the kernel of  $\alpha$ . Since we assumed that the kernel

of  $\alpha$  has length 1, we see that the kernel of  $h_2$  must also be of length 1. Therefore,

$$\text{length}(S/I_\ell) \leq \text{length}(\text{Im}(\psi)) \leq \text{length}(\mathcal{H}) = 1 + \text{length}(S/I_\ell).$$

If  $\psi$  is not surjective, then  $\text{Im}(\psi)$ , which is in  $\hat{\mathcal{C}}$  by Theorem 2.8, part (g), is a proper subalgebra of  $\mathcal{H}$ . Hence, by a length argument,  $\text{Im}(\psi)$  must be isomorphic to  $S/I_\ell$  via the restriction of  $h_2$ . Letting  $\sigma : S/I_\ell \rightarrow \text{Im}(\psi)$  be the morphism in  $\hat{\mathcal{C}}$  with  $h_2 \circ \sigma = \text{id}_{S/I_\ell}$  and letting  $\pi_\ell^I : R = S/I \rightarrow S/I_\ell$  be the natural projection, we set  $\delta'$  equal to  $h_1 \circ \sigma \circ \pi_\ell^I$ . However, if  $\psi$  is surjective, then  $\mathcal{H}$  is isomorphic to  $S/\ker(\psi)$ .

Recall that we assume  $u \circ \phi = \alpha \circ \beta = \beta'$ . Since  $u \circ \phi = \bar{u} \circ \phi_\ell$ , we obtain that the pullback of lifts  $\beta \times_{\beta'} \phi_\ell$  is a lift over  $\mathcal{H}$  which projects onto  $\phi_\ell$  via  $h_2$ . Hence there is also a lift over  $S/\ker(\psi)$  that projects onto  $\phi_\ell$ . Moreover, we have already established that  $m_S * I_\ell + I_\ell * m_S \subseteq \ker(\psi) \subseteq I_\ell$ . But by the minimality of  $I_{\ell+1}$ , we therefore have that  $I_{\ell+1} \subseteq \ker(\psi)$ , and hence  $\psi$  factors through a morphism  $\bar{\psi} : S/I_{\ell+1} \rightarrow \mathcal{H}$ . Letting  $\pi_{\ell+1}^I : R = S/I \rightarrow S/I_{\ell+1}$  be the natural projection, we set  $\delta'$  equal to  $h_1 \circ \bar{\psi} \circ \pi_{\ell+1}^I$ , in this case.

In both cases, it follows that  $\alpha \circ \delta' = u$ . Define  $\gamma = \delta' \circ \phi$ , so that  $\gamma$  is a lift of  $V$  over  $B$ . Then we have

$$\alpha \circ \gamma = \alpha \circ \delta' \circ \phi = u \circ \phi = \alpha \circ \beta.$$

By Lemma 3.15, part (a),  $\beta - \gamma$  defines a lift  $\psi_{\beta\gamma}$  of  $V$  over  $k[Bt]$ , where

$$\psi_{\beta\gamma}(a) = \rho(a) + (\beta - \gamma)(a)$$

for all  $a$  in  $A$ . Since we have shown that  $\tau_{k[\epsilon]} : h_R(k[\epsilon]) \rightarrow F_V(k[\epsilon])$  is a bijection, we can use the isomorphism  $\omega : k[\epsilon] \rightarrow k[Bt]$  in  $\hat{\mathcal{C}}$ , from Definition 3.14, to see that there



exists a unique morphism  $\delta_{\beta\gamma} : R \rightarrow k[Bt]$  in  $\hat{\mathcal{C}}$  such that  $[\delta_{\beta\gamma} \circ \phi] = [\psi_{\beta\gamma}]$ . Hence there exists  $X \in \text{Mat}_n(B)$  so that

$$\delta_{\beta\gamma} \circ \phi = (1 + Xt)\psi_{\beta\gamma}(1 - Xt).$$

Define the map  $\tilde{\delta}_{\beta\gamma} : R \rightarrow Bt$  by

$$\tilde{\delta}_{\beta\gamma}(r) = \delta_{\beta\gamma}(r) - \pi_R(r)$$

for all  $r$  in  $R$ , and define  $\delta : R \rightarrow B$  by

$$\delta(r) = \delta'(r) + \tilde{\delta}_{\beta\gamma}(r)$$

for all  $r$  in  $R$ . Then  $\delta$  is a  $\lambda$ -algebra homomorphism, since  $\delta'$ ,  $\delta_{\beta\gamma}$  and  $\pi_R$  are  $\lambda$ -algebra homomorphisms, and  $\tilde{\delta}_{\beta\gamma}(1_R) = 0t$ . Moreover, we have for all  $r_1, r_2 \in R$ ,  $\delta_{\beta\gamma}(r_1 r_2) = \delta_{\beta\gamma}(r_1)\delta_{\beta\gamma}(r_2)$ . Using the isomorphism  $\omega$  and Definition 2.12, this implies

$$\tilde{\delta}_{\beta\gamma}(r_1 r_2) = \widehat{\pi_R(r_1)}\tilde{\delta}_{\beta\gamma}(r_2) + \tilde{\delta}_{\beta\gamma}(r_1)\widehat{\pi_R(r_2)},$$

where, for  $i \in \{1, 2\}$ ,  $\widehat{\pi_R(r_i)}$  denotes any element  $q_i$  in  $B$  with  $\pi_B(q_i) = \pi_R(r_i)$ . Note that since  $m_B t = 0 = t m_B$ , it does not matter which elements  $q_1, q_2$  we choose.

Therefore, we obtain for all  $r_1, r_2$  in  $R$ ,

$$\begin{aligned} \delta(r_1 r_2) &= \delta'(r_1 r_2) + \tilde{\delta}_{\beta\gamma}(r_1 r_2) \\ &= \delta'(r_1)\delta'(r_2) + \delta'(r_1)\tilde{\delta}_{\beta\gamma}(r_2) + \tilde{\delta}_{\beta\gamma}(r_1)\delta'(r_2) \\ &= (\delta'(r_1) + \tilde{\delta}_{\beta\gamma}(r_1)) * (\delta'(r_2) + \tilde{\delta}_{\beta\gamma}(r_2)) \end{aligned}$$

where the last equality follows since  $\tilde{\delta}_{\beta\gamma}(r_1)\tilde{\delta}_{\beta\gamma}(r_2) = 0$  because  $(Bt)^2 = 0$ . Additionally, we have  $\pi_B \circ \delta = \pi_B \circ \delta' = \pi_R$ , which implies that  $\delta$  is a morphism in  $\hat{\mathcal{C}}$ . Moreover, using that  $(Bt)^2 = 0$  and that  $m_B t = 0 = t m_B$ , we have for all  $a \in A$ ,

$$\begin{aligned}
(\delta \circ \phi)(a) &= \delta'(\phi(a)) + \tilde{\delta}_{\beta\gamma}(\phi(a)) \\
&= \gamma(a) + \delta_{\beta\gamma}(\phi(a)) - \pi_R(\phi(a)) \\
&= \gamma(a) + (1 + Xt)\psi_{\beta\gamma}(a)(1 - Xt) - \rho(a) \\
&= \gamma(a) + (\psi_{\beta\gamma}(a) - \rho(a)) + Xt\psi_{\beta\gamma}(a) - \psi_{\beta\gamma}(a)Xt \\
&= \gamma(a) + (\beta - \gamma)(a) + Xt\beta(a) - \beta(a)Xt \\
&= \beta(a) + Xt\beta(a) - \beta(a)Xt \\
&= (1 + Xt)\beta(a)(1 - Xt).
\end{aligned}$$

Since  $t \in m_B$ ,  $1 + Xt$  is a matrix in  $GL_n(B)$  which reduces to the identity in  $GL_n(k)$ .

Therefore,  $[\delta \circ \phi] = [\beta]$ . Since  $\tilde{\delta}_{\beta\gamma}(r) \in Bt = \ker(\alpha)$  for all  $r \in R$ , we also obtain that

$$\alpha \circ \delta = \alpha \circ \delta' = u.$$

This completes the proof that  $(R, [\phi])$  is a versal couple for  $V$ , and hence the proof of Theorem 3.16.

### 3.2 Applications

We will now compute an example of a versal deformation ring of a one-dimensional representation. Let  $A = S/I$ , where  $S = \lambda\langle\langle x_1, \dots, x_m \rangle\rangle$  and where  $I$  is any proper, closed ideal of  $S$ . In particular,  $A$  is in  $\hat{\mathcal{C}}$  by Theorem 2.8, parts (f)

and (h). Let  $\pi_I : S \rightarrow S/I = A$  be the natural surjection. Then  $j_A : \lambda \rightarrow A$  is given as  $j_A = \pi_I \circ j_S$ . Moreover,  $\pi_A : A \rightarrow k$  satisfies  $\pi_A \circ \pi_I = \pi_S$ .

Let  $V$  be a one-dimensional  $k$ -vector space, and give  $V$  a left  $A$ -module structure by defining  $a \cdot v = \pi_A(a)(v)$  for all  $v \in V$ . In other words,  $\pi_A = \rho$ .

Notice that  $\pi_A \circ id_A = \pi_A$ , and since  $\pi_A = \rho$ , we also have  $\pi_A \circ id_A = \rho$ . This means that  $id_A : A \rightarrow A$  is both a morphism in  $\hat{\mathcal{C}}$  as well as a lift of  $V$  over  $A$ .

The claim is that  $(A, [id_A])$  is a versal couple for  $V$ . To see this, let  $T$  be any object of  $\hat{\mathcal{C}}$  and let  $[\psi]$  be any element of  $F_V(T)$ . Then by virtue of  $\psi$  being a lift of  $V$  over  $A$ ,  $\psi(1_A) = 1_T$  and  $\pi_T \circ \psi = \rho$ . But since  $\rho = \pi_A$ , this gives  $\pi_T \circ \psi = \pi_A$ , and hence  $\psi : A \rightarrow T$  is also a morphism in  $\hat{\mathcal{C}}$ .

Since  $[\psi] = [\psi \circ id_A]$ ,  $\tau_T : h_A(T) \rightarrow F_V(T)$  is surjective, where  $\tau_T(\psi) = [\psi \circ id_A]$  for all  $\psi$  in  $h_A(T)$ . Suppose  $\alpha$  and  $\alpha'$  are elements in  $h_A(T)$  such that  $\tau_T(\alpha) = \tau_T(\alpha')$ . Then  $[\alpha \circ id_A] = [\alpha' \circ id_A]$ , which means  $[\alpha] = [\alpha']$ . Therefore, there exists some  $t$  in  $Mat_1(T) = T$  which reduces to the identity in  $Mat_1(k) = k$ , such that  $t \cdot \alpha \cdot t^{-1} = \alpha'$ . Hence, if  $T$  is commutative, in particular if  $T = k[\epsilon]$ ,  $\tau_T$  is bijective.

It remains to show  $\tau$  is smooth. Let  $\alpha : B \rightarrow B'$  be a surjective morphism in  $\mathcal{C}$ , let  $[\beta] \in F_V(B)$  and let  $u : A \rightarrow B'$  be a morphism in  $\hat{\mathcal{C}}$  such that  $[u \circ id_A] = [\alpha \circ \beta]$ . Hence there exists some  $b'$  in  $B'$  with  $\pi_{B'}(b') = 1$  such that

$$u = b'(\alpha \circ \beta)(b')^{-1}.$$

Letting  $b \in B$  be any element with  $\beta(b) = b'$ , we obtain that  $\pi_B(b) = 1$ . Hence

$\tilde{\beta} = b\beta b^{-1}$  is strictly equivalent to  $\beta$ , and

$$u = \alpha \circ \tilde{\beta}.$$

Since  $\tilde{\beta} : A \rightarrow B$  is a morphism in  $\hat{\mathcal{C}}$ , we can take  $\delta = \tilde{\beta}$  to see that  $\alpha \circ \delta = u$  and  $[\delta \circ id_A] = [\tilde{\beta}] = [\beta]$ . This proves that  $(A, [id_A])$  is a versal couple for  $V$ .

**Remark 3.17.** This example shows that it is easier to obtain universal couples, and hence universal deformation rings, when restricting the functor  $F_V$  to  $\hat{\mathcal{C}}_{comm}$ , i.e., to the full subcategory of  $\hat{\mathcal{C}}$  consisting of the commutative objects of  $\hat{\mathcal{C}}$ . Moreover, we see that every object of  $\hat{\mathcal{C}}$  is a versal deformation ring of a one-dimensional representation, and every object of  $\hat{\mathcal{C}}_{comm}$  is a universal deformation ring of a one-dimensional representation.

Let now  $F_{V,c} : \hat{\mathcal{C}}_{comm} \rightarrow Sets$  be the restriction of  $F_V$  to  $\hat{\mathcal{C}}_{comm}$ . For any object  $R$  of  $\hat{\mathcal{C}}$ , let  $[R, R] = \{ab - ba | a, b \in R\}$ , and define  $R_c = R/[R, R]$ . Let  $\pi_c : R \rightarrow R_c$  be the natural projection.

**Theorem 3.18.** *Let  $(R, [\phi])$  be a versal couple for  $V$  with respect to the category  $\hat{\mathcal{C}}$ .*

*Then  $(R_c, [\pi_c \circ \phi])$  is a versal couple of  $V$  with respect to the category  $\hat{\mathcal{C}}_{comm}$ .*

*Proof.* Since  $k[\epsilon]$  is a commutative object in  $\hat{\mathcal{C}}$ , the sets  $h_{R_c}(k[\epsilon])$  and  $h_R(k[\epsilon])$  are in bijection via the map that sends  $\alpha : R_c \rightarrow k[\epsilon]$  to  $\alpha \circ \pi_c : R \rightarrow k[\epsilon]$ . Since  $(R, [\phi])$  is a versal couple for  $V$ , the sets  $h_R(k[\epsilon])$  and  $F_V(k[\epsilon])$  are in bijection via the map that sends  $f : R \rightarrow k[\epsilon]$  to  $[f \circ \phi]$ . Therefore, the sets  $h_{R_c}(k[\epsilon])$  and  $F_V(k[\epsilon])$  are also in bijection via the map that sends  $\alpha : R_c \rightarrow k[\epsilon]$  to  $[\alpha \circ (\pi_c \circ \phi)]$ . Therefore, all that

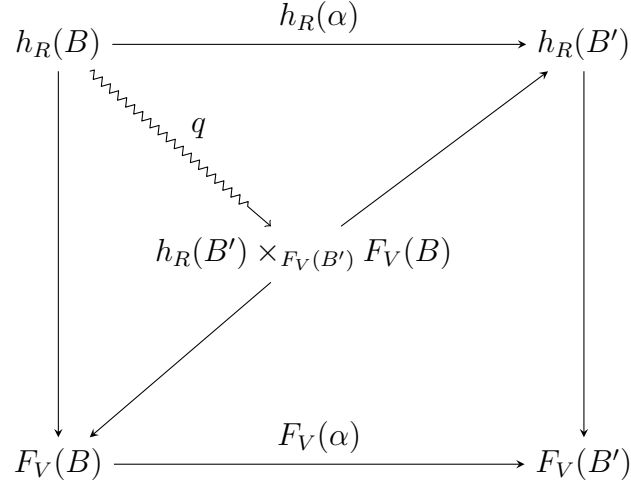
remains is to prove a smoothness condition. Let  $B$  and  $B'$  be objects of  $\mathcal{C}_{comm}$ , and let  $\alpha : B \rightarrow B'$  be a morphism in  $\mathcal{C}_{comm}$ . We must show that the map  $q'$  resulting from the pullback diagram in Figure 3.9 is surjective.

Figure 3.9: The pullback diagram for  $q'$ .

$$\begin{array}{ccc}
 h_{R_c}(B) & \xrightarrow{h_{R_c}(\alpha)} & h_{R_c}(B') \\
 \downarrow & \searrow^{q'} & \nearrow \\
 & h_{R_c}(B') \times_{F_{V,c}(B')} F_{V,c}(B) & \\
 \downarrow & \swarrow & \downarrow \\
 F_{V,c}(B) & \xrightarrow{F_{V,c}(\alpha)} & F_{V,c}(B')
 \end{array}$$

However, since  $(R, [\phi])$  is a versal couple for  $V$  with respect to the category  $\hat{\mathcal{C}}$ , we know that the map  $q$  resulting from the pullback diagram in Figure 3.10 is surjective.

Any morphism in  $\hat{\mathcal{C}}$  from  $R$  to a commutative ring in  $\hat{\mathcal{C}}_{comm}$  must factor through  $\pi_c : R \rightarrow R_c$ . Specifically, since  $B$  and  $B'$  are both commutative,  $h_R(B) \cong h_{R_c}(B)$  and  $h_R(B') \cong h_{R_c}(B')$ . From this, it follows that the surjectivity of  $q$  implies the surjectivity of  $q'$ , completing the proof that  $(R_c, [\pi_c \circ \phi])$  is a versal couple for  $V$  with respect to the category  $\hat{\mathcal{C}}_{comm}$ .  $\square$

Figure 3.10: The pullback diagram for  $q$ .

**Theorem 3.19.** *Let  $(R, [\phi])$  be a universal couple for  $V$  with respect to the category  $\hat{\mathcal{C}}$ . Then  $(R_c, [\pi_c \circ \phi])$  is a universal couple for  $V$  with respect to the category  $\hat{\mathcal{C}}_{comm}$ .*

*Proof.* Let  $B$  be an object of  $\mathcal{C}_{comm}$ . As a consequence of Theorem 3.18 and Corollary 3.8, it suffices to prove that the map from  $h_{R_c}(B)$  to  $F_{V,c}(B)$  which sends  $\alpha : R_c \rightarrow B$  in  $\hat{\mathcal{C}}_{comm}$  to  $[\alpha \circ (\pi_c \circ \phi)]$  is injective. Suppose that  $f : R_c \rightarrow B$  and  $g : R_c \rightarrow B$  are two elements of  $h_{R_c}(B)$  such that  $[f \circ (\pi_c \circ \phi)] = [g \circ (\pi_c \circ \phi)]$ , which we can rewrite as  $[(f \circ \pi_c) \circ \phi] = [(g \circ \pi_c) \circ \phi]$ . Since  $(R, [\phi])$  is a universal couple for  $V$  with respect to the category  $\hat{\mathcal{C}}$ , the map from  $h_R(B)$  to  $F_V(B)$  which sends  $h : R \rightarrow B$  in  $\hat{\mathcal{C}}$  to  $[h \circ \phi]$  must be injective, implying that  $f \circ \pi_c = g \circ \pi_c$ . In other words,

$$0 = f \circ \pi_c - g \circ \pi_c = (f - g) \circ \pi_c.$$

Since  $h_{R_c}(B)$  and  $h_R(B)$  are in bijection via the map which sends  $\alpha : R_c \rightarrow B$  to  $\alpha \circ \pi_c : R \rightarrow B$ , it follows that  $f - g = 0$ , and hence  $(R_c, [\pi_c \circ \phi])$  is a universal couple

for  $V$  with respect to the category  $\hat{\mathcal{C}}_{comm}$ .

□

## CHAPTER 4 TWO PARTICULAR EXAMPLES

In this chapter, we consider two particular examples of  $\lambda$ ,  $A$ , and  $V$  and determine the versal deformation ring  $R(A, V)$  and a corresponding versal couple  $(R(A, V), [\phi_u])$ .

Let  $k$  be any field, let  $r \geq 2$  be a fixed integer, and define

$$\Omega_r = k\langle\langle x, y \rangle\rangle / (x^2, y^2, (xy)^r - (yx)^r).$$

We consider a particular representation

$$\rho_r : \Omega_r \rightarrow \text{Mat}_{2r+1}(k).$$

Since  $r$  can be arbitrarily large, it becomes necessary to find an efficient way to write down and multiply arbitrarily large matrices. The following two remarks introduce our notation for such large matrices.

**Remark 4.1.** Let  $S$  be an object of  $\hat{\mathcal{C}}$ . Let  $c_i$  denote the  $n \times 1$  column vector with 1 in the  $i^{\text{th}}$  row, and zeros everywhere else. Using this notation, we can define matrices in  $\text{Mat}_n(S)$  by writing each column as an  $S$ -linear combination of the  $c_i$ . For example, if  $n = 3$  and

$$B = \begin{bmatrix} 0 & 0 & a \\ d & f & 0 \\ g & 0 & s \end{bmatrix} \in \text{Mat}_3(S),$$



then, using the above notation, we would write  $B$  as

$$B = \left[ \begin{array}{c|c|c} dc_2 & fc_2 & ac_1 \\ \hline +gc_3 & & +sc_3 \end{array} \right].$$

Additionally, if  $B$  represents a matrix with elements in the ring  $S/J$  for some ideal  $J$  of  $S$ , the coefficients of the  $c_i$  in the above notation are understood to be taken modulo  $J$ .

**Remark 4.2.** Let  $S$  be again an object of  $\hat{\mathcal{C}}$ , and let  $B$  and  $D$  be two matrices in  $Mat_n(S)$ . Let  $B_{i,*}$  denote the  $i^{th}$  row of  $B$ , and let  $D_{*,j}$  denote the  $j^{th}$  column of  $D$ . Then the  $j^{th}$  column of  $BD$  is  $(BD)_{*,j} = B \cdot D_{*,j}$ . If  $D_{*,j} = s_1c_1 + s_2c_2 + \dots + s_nc_n$  with each  $s_i$  in  $S$ , then  $(BD)_{*,j} = B_{*,1} \cdot s_1 + B_{*,2} \cdot s_2 + \dots + B_{*,n} \cdot s_n$ .

Similarly, the  $i^{th}$  row of  $BD$  is  $(BD)_{i,*} = B_{i,*} \cdot D$ . Notice that we can write row vectors as the transpose,  $^T$ , of column vectors, and vice versa. If  $B_{i,*} = (r_1c_1 + r_2c_2 + \dots + r_nc_n)^T$  with each  $r_i$  in  $S$ , then

$$(BD)_{i,*} = r_1 \cdot D_{1,*} + r_2 \cdot D_{2,*} + \dots + r_n \cdot D_{n,*} = (r_1 \cdot D_{1,*}^T + r_2 \cdot D_{2,*}^T + \dots + r_n \cdot D_{n,*}^T)^T.$$

To define our representation  $\rho_r : \Omega_r \rightarrow Mat_{2r+1}(k)$ , we need the following two matrices in  $Mat_{2r+1}(k)$ :

$$\begin{aligned} X &= \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & c_1 & c_4 & 0 & c_6 & 0 & \dots & c_{2r} & 0 & 0 \end{array} \right] \\ &= \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c} c_2 & 0 & 0 & c_3 & 0 & c_5 & \dots & 0 & c_{2r-1} & 0 \end{array} \right]^T \end{aligned}$$

and

$$\begin{aligned} Y &= \left[ 0 \mid c_3 \mid 0 \mid c_5 \mid 0 \mid c_7 \mid \dots \mid 0 \mid c_{2r+1} \mid 0 \right] \\ &= \left[ 0 \mid 0 \mid c_2 \mid 0 \mid c_4 \mid 0 \mid \dots \mid c_{2r-2} \mid 0 \mid c_{2r} \right]^T. \end{aligned}$$

Let

$$\bar{x} = x + (x^2, y^2, (xy)^r - (yx)^r) \in \Omega_r,$$

and

$$\bar{y} = y + (x^2, y^2, (xy)^r - (yx)^r) \in \Omega_r.$$

Since  $X^2 = Y^2 = (XY)^r - (YX)^r = 0$ , we may define a  $k$ -algebra homomorphism

$$\rho_r : \Omega_r \rightarrow \text{Mat}_{2r+1}(k)$$

by letting

$$\rho_r(\bar{x}) = X,$$

$$\rho_r(\bar{y}) = Y,$$

and extending this definition  $k$ -linearly. Let  $V_r$  be the associated left  $\Omega_r$ -module which is a  $(2r + 1)$ -dimensional  $k$ -vector space and on which  $\Omega_r$  acts by the representation  $\rho_r$ .

In the following two sections, we consider the following two situations. In Section 4.1, we let  $\lambda = k$ ,  $A = \Omega_r$ , and  $V = V_r$  and compute  $R(\Omega_r, V_r)$  for any  $r \geq 2$ . In Section 4.2, we take  $k$  to be a perfect field of characteristic 2, and we let  $W$  be

the ring of infinite Witt vectors over  $k$ . We take  $\lambda = W$  and we take  $A$  to be the group ring  $W[D_8]$  of a dihedral group  $D_8$  of order 8. In this case, it turns out that  $A/2A \cong \Omega_2$ , which means that we can view  $V_2$  as a 5-dimensional representation of  $D_8$  over  $k$ . We determine  $R(W[D_8], V_2)$ .

#### 4.1 The case $\lambda = k$ , $A = \Omega_r$ , and $V = V_r$

In this section, let  $r \geq 2$  be arbitrary, let  $\lambda = k$ ,  $A = \Omega_r$ ,  $V = V_r$ , and  $\rho = \rho_r$ .

Our goal is to prove the following result.

**Theorem 4.3.** *The versal deformation ring  $R(\Omega_r, V_r)$  is isomorphic to  $\Omega_r$  for all  $r \geq 2$ .*

We first prove the following lemma.

**Lemma 4.4.** *The  $k$ -vector space  $F_V(k[\epsilon])$  is 2-dimensional.*

*Proof.* Any  $k$ -algebra homomorphism  $\psi : \Omega_r \rightarrow \text{Mat}_{2r+1}(k[\epsilon])$  is completely determined by its values on  $\bar{x}$  and  $\bar{y}$ . Therefore, may view a lift  $\psi$  of  $V$  over  $k[\epsilon]$  as a pair of matrices  $(X_\psi, Y_\psi) \in \text{Mat}_{2r+1}(k[\epsilon]) \times \text{Mat}_{2r+1}(k[\epsilon])$  such that

$$X_\psi^2 = Y_\psi^2 = (X_\psi Y_\psi)^r - (Y_\psi X_\psi)^r = 0 \in \text{Mat}_{2r+1}(k[\epsilon]).$$

Conversely, if we have any two matrices,  $B$  and  $D$ , in  $\text{Mat}_{2r+1}(k[\epsilon])$  which satisfy the above equations, then we may define a lift of  $V$  over  $k[\epsilon]$  by sending  $\bar{x}$  to  $B$ ,  $\bar{y}$  to  $D$ , and extending this  $k$ -linearly. Since  $\psi$  must reduce to  $\rho$  via the natural projection  $\text{Mat}_{2r+1}(k[\epsilon]) \rightarrow \text{Mat}_{2r+1}(k)$ , it follows that there exist matrices  $L = [l_{ij}]$

and  $L' = [l'_{ij}]$  in  $Mat_{2r+1}(m_{k[\epsilon]})$  such that

$$X_\psi = X + L,$$

and

$$Y_\psi = Y + L'.$$

Because  $X_\psi^2$  and  $Y_\psi^2$  must equal 0, we see that  $(X + L)^2 = (Y + L')^2 = 0$ . But since  $m_{k[\epsilon]}^2 = 0$ , it follows that  $L^2$  and  $L'^2$  are both 0. Since  $X^2 = Y^2 = 0$ , it then follows that

$$X_\psi^2 = XL + LX = 0$$

and

$$Y_\psi^2 = YL' + L'Y = 0.$$

Since

$$\begin{aligned} X &= \left[ 0 \mid c_1 \mid c_4 \mid 0 \mid c_6 \mid 0 \mid \dots \mid c_{2r} \mid 0 \mid 0 \right] \\ &= \left[ c_2 \mid 0 \mid 0 \mid c_3 \mid 0 \mid c_5 \mid \dots \mid 0 \mid c_{2r-1} \mid 0 \right]^T \end{aligned}$$

and

$$\begin{aligned} Y &= \left[ 0 \mid c_3 \mid 0 \mid c_5 \mid 0 \mid c_7 \mid \dots \mid 0 \mid c_{2r+1} \mid 0 \right] \\ &= \left[ 0 \mid 0 \mid c_2 \mid 0 \mid c_4 \mid 0 \mid \dots \mid c_{2r-2} \mid 0 \mid c_{2r} \right]^T, \end{aligned}$$

it follows that

$$\begin{aligned}
0 = X_\psi^2 = XL + LX &= \left[ \begin{array}{c|c|c|c|c|c|c|c} L_{2,*}^T & 0 & 0 & L_{3,*}^T & 0 & L_{5,*}^T & \dots & 0 & L_{2r-1,*}^T & 0 \end{array} \right]^T \\
&+ \left[ \begin{array}{c|c|c|c|c|c|c|c} 0 & L_{*,1} & L_{*,4} & 0 & L_{*,6} & 0 & \dots & L_{*,2r} & 0 & 0 \end{array} \right], \quad (4.1)
\end{aligned}$$

and

$$\begin{aligned}
0 = Y_\psi^2 = YL' + L'Y &= \left[ \begin{array}{c|c|c|c|c|c|c|c} 0 & 0 & L'_{2,*} & 0 & L'_{4,*} & 0 & \dots & L'_{2r-2,*} & 0 & L'_{2r,*} \end{array} \right]^T \\
&+ \left[ \begin{array}{c|c|c|c|c|c|c|c} 0 & L'_{*,3} & 0 & L'_{*,5} & 0 & L'_{*,7} & \dots & 0 & L'_{*,2r+1} & 0 \end{array} \right]. \quad (4.2)
\end{aligned}$$

These equations, in addition to the equation

$$(X_\psi Y_\psi)^r - (Y_\psi X_\psi)^r = 0,$$

serve as restrictions as to what our lifts may actually look like. Before proving that  $F_V(k[\epsilon])$  is 2-dimensional for all  $r$ , we first look explicitly at the case  $r = 2$ , to help guide the reader in the general case.

Let  $r = 2$ . Then

$$\begin{aligned}
0 = X_\psi^2 &= XL + LX \\
&= \left[ \begin{array}{c|c|c|c} L_{2,*}^T & 0 & 0 & L_{3,*}^T & 0 \end{array} \right]^T + \left[ \begin{array}{c|c|c|c} 0 & L_{*,1} & L_{*,4} & 0 & 0 \end{array} \right]
\end{aligned}$$

and

$$\begin{aligned}
0 = Y_\psi^2 &= YL' + L'Y \\
&= \left[ 0 \mid 0 \mid L'_{2,*}{}^T \mid 0 \mid L'_{4,*}{}^T \right]^T + \left[ 0 \mid L'_{*,3} \mid 0 \mid L'_{*,5} \mid 0 \right],
\end{aligned}$$

which we may rewrite as

$$0 = XL + LX = \begin{bmatrix} l_{21} & l_{22} + l_{11} & l_{23} + l_{14} & l_{24} & l_{25} \\ 0 & l_{21} & l_{24} & 0 & 0 \\ 0 & l_{31} & l_{34} & 0 & 0 \\ l_{31} & l_{32} + l_{41} & l_{33} + l_{44} & l_{34} & l_{35} \\ 0 & l_{51} & l_{54} & 0 & 0 \end{bmatrix} \quad (4.3)$$

and

$$0 = YL' + L'Y = \begin{bmatrix} 0 & l'_{13} & 0 & l'_{15} & 0 \\ 0 & l'_{23} & 0 & l'_{25} & 0 \\ l'_{21} & l'_{22} + l'_{33} & l'_{23} & l'_{24} + l'_{35} & l'_{25} \\ 0 & l'_{43} & 0 & l'_{45} & 0 \\ l'_{41} & l'_{42} + l'_{53} & l'_{43} & l'_{44} + l'_{55} & l'_{45} \end{bmatrix}. \quad (4.4)$$

Let  $E$  be any matrix in  $Mat_5(m_{k[\epsilon]})$ . Then we obtain

$$XE - EX = \begin{bmatrix} e_{21} & e_{22} - e_{11} & e_{23} - e_{14} & e_{24} & e_{25} \\ 0 & -e_{21} & -e_{24} & 0 & 0 \\ 0 & -e_{31} & -e_{34} & 0 & 0 \\ e_{31} & e_{32} - e_{41} & e_{33} - e_{44} & e_{34} & e_{35} \\ 0 & -e_{51} & -e_{54} & 0 & 0 \end{bmatrix} \quad (4.5)$$

and

$$YE - EY = \begin{bmatrix} 0 & -e_{13} & 0 & -e_{15} & 0 \\ 0 & -e_{23} & 0 & -e_{25} & 0 \\ e_{21} & e_{22} - e_{33} & e_{23} & e_{24} - e_{35} & e_{25} \\ 0 & -e_{43} & 0 & -e_{45} & 0 \\ e_{41} & e_{42} - e_{53} & e_{43} & e_{44} - e_{55} & e_{45} \end{bmatrix}. \quad (4.6)$$

We note that any lift  $\zeta = (X_\zeta, Y_\zeta)$  of  $V$  over  $k[\epsilon]$  which is strictly equivalent to the original lift  $\rho = (X, Y)$  must be of the form

$$X_\zeta = (1 - E)X(1 + E) = X + XE - EX,$$

$$Y_\zeta = (1 - E)Y(1 + E) = Y + YE - EY,$$

for some matrix  $E = [e_{ij}]$  in  $Mat_5(m_{k[\epsilon]})$ . The conditions imposed on  $L$  by the equation  $XL + LX = 0$  are not sufficient to force  $L$  to be of the form  $L = XE - EX$ , for some matrix  $E = [e_{ij}]$  in  $Mat_5(m_{k[\epsilon]})$ . Similarly, the conditions imposed on  $L'$  by the equation  $YL' + L'Y = 0$  are not sufficient to force  $L'$  to be of the form  $L' = YE - EY$ , for some matrix  $E = [e_{ij}]$  in  $Mat_5(m_{k[\epsilon]})$ . However, we will now investigate what is forced by those equations.

We begin by examining the implications of equations (4.3) and (4.5). Notice that the requirement that  $XL + LX = 0$  forces 8 elements of  $L$  to be 0, namely  $l_{21}, l_{31}, l_{51}, l_{24}, l_{34}, l_{54}, l_{25}, l_{35}$ . Moreover, if we look at these respective entries in the matrix  $XE - EX$  we find that they are all 0. The only entry in  $XE - EX$  which is 0 but is not “referred to” by the requirement that  $XL + LX = 0$  is the entry  $(XE - EX)_{55}$ . In other words,  $(XE - EX)_{55} = 0$ , but the element  $l_{55}$  does not appear in the matrix  $XL + LX$ . Additionally, there are 4 duplicate entries in the matrix  $XL + LX$ , namely  $l_{21}, l_{24}, l_{31}, l_{34}$  appear twice. Notice that the sums appearing as entries in the matrix  $XL + LX$  exactly prescribe the locations of the duplicate entries in  $XL + LX$ . For example,  $l_{22} + l_{11}$  appears as an entry in  $XL + LX$ , and we note that

$$(XL + LX)_{11} = (XL + LX)_{22}.$$

Moreover, the conditions imposed on  $L$  by the sums found in  $XL + LX = 0$  are satisfied in the matrix  $XE - EX$ . As an example, we take the condition  $l_{22} + l_{11} = 0$  and note that  $(XE - EX)_{22} + (XE - EX)_{11} = 0$ . Finally, with the single exception that  $(XE - EX)_{55} = 0$ , there are no other dependencies found in  $XE - EX$  which are not prescribed by the conditions found in  $XL + LX = 0$ . Therefore, any  $L$  satisfying  $XL + LX = 0$  will be of the form  $XE - EX$ , with the possible exception of differing by an element in  $m_{k[e]}$  in entry  $L_{55}$ .

Next, we continue by examining the implications of equations (4.4) and (4.6). By looking at the conditions imposed on  $L'$  in equation (4.4), in addition to observing the form of equation (4.6), it follows that any  $L'$  satisfying  $YL' + L'Y = 0$  will be of



the form  $YE - EY$ , with the possible exception of differing by an element in  $m_{k[\epsilon]}$  in entry  $L'_{11}$ .

Let  $Q_{ij}$  be the matrix in  $Mat_5(m_{k[\epsilon]})$  with  $\epsilon$  in the  $(i, j)$ -entry and zeros everywhere else. What we know now is that there exist matrices  $F$  and  $F'$  in  $Mat_5(m_{k[\epsilon]})$  such that any lift  $(X + L, Y + L')$  of  $V$  over  $k[\epsilon]$  is of the form

$$X + L = (1 - F)X(1 + F) + cQ_{55}, \quad (4.7)$$

$$Y + L' = (1 - F')Y(1 + F') + c'Q_{11}, \quad (4.8)$$

for some  $c$  and  $c'$  in  $k$ .

We now show that  $F_V(k[\epsilon])$  is 2-dimensional. This means that we need to find two lifts  $\beta_1$  and  $\beta_2$  of  $\rho$  over  $k[\epsilon]$  such that any other such lift,  $\psi$ , satisfies  $[\psi] = b_1[\beta_1] + b_2[\beta_2]$  for some  $b_1$  and  $b_2$  in  $k$ . Note that we use here the addition and multiplication in  $F_V(k[\epsilon])$  introduced in Lemma 3.13. As lifts, this means that there exists a matrix  $E$  in  $Mat_5(m_{k[\epsilon]})$ , depending on  $\psi$ , such that  $\psi = (1 - E)\mathbb{P}(\mathbb{M}(b_1, \beta_1), \mathbb{M}(b_2, \beta_2))(1 + E)$ , where  $\mathbb{P}$  and  $\mathbb{M}$  are the maps introduced in the proof of Lemma 3.13. Translating this into matrices, we must find matrices  $M_1, M_2, M_3, M_4$  in  $Mat_5(m_{k[\epsilon]})$  such that

$$\beta_1 = (X + M_1, Y + M_3),$$

$$\beta_2 = (X + M_2, Y + M_4),$$

and so that for any other lift  $\psi = (X + L, Y + L')$ , we can find some  $E$  in  $Mat_5(m_{k[\epsilon]})$ , depending on  $\psi$ , such that

$$X + L = (1 - E)(X + b_1M_1 + b_2M_2)(1 + E) = (1 - E)X(1 + E) + b_1M_1 + b_2M_2,$$

$$Y + L = (1 - E)(Y + b_1M_3 + b_2M_4)(1 + E) = (1 - E)Y(1 + E) + b_1M_3 + b_2M_4.$$

Moreover, we must show that the deformations  $[\beta_1]$  and  $[\beta_2]$  are  $k$ -linearly independent. We begin by defining our lifts,  $\beta_1$  and  $\beta_2$ , and showing that the resulting deformations are  $k$ -linearly independent.

Define

$$\beta_1 = (\beta_1(\bar{x}), \beta_1(\bar{y})) = \left( \begin{array}{c} \left[ \begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon \end{array} \right], \left[ \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \epsilon & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right] \\ \left[ \begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right], \left[ \begin{array}{ccccc} \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right] \end{array} \right).$$

In other words,

$$\beta_1 = (X + Q_{55}, Y + Q_{31})$$

and

$$\beta_2 = (X, Y + Q_{11}).$$

It is a straightforward calculation to see that, for  $i \in \{1, 2\}$ ,

$$\beta_i(\bar{x})^2 = 0,$$

$$\beta_i(\bar{y})^2 = 0,$$

$$(\beta_i(\bar{x})\beta_i(\bar{y}))^2 - (\beta_i(\bar{y})\beta_i(\bar{x}))^2 = 0,$$

and so both  $\beta_1$  and  $\beta_2$  give lifts of  $\rho$  over  $k[\epsilon]$ . We now show that  $[\beta_1]$  and  $[\beta_2]$  are  $k$ -linearly independent. If they are not, then there exists some non-zero element  $c$  in  $k$  such that  $c[\beta_1] = [\beta_2]$ . In other words, there exists some matrix  $E$  in  $Mat_5(m_{k[\epsilon]})$  such that  $\mathbb{M}(c, \beta_1) = (1 - E)\beta_2(1 + E)$ . But notice that

$$(1 - E)\beta_2(\bar{x})(1 + E) = \beta_2(\bar{x}) + \beta_2(\bar{x})E - E\beta_2(\bar{x}).$$

Since the  $5^{th}$  row and  $5^{th}$  column of  $\beta_2(\bar{x})$  is 0, it follows that the  $(5, 5)$ -entry

$$(\beta_2(\bar{x}) + \beta_2(\bar{x})E - E\beta_2(\bar{x}))_{55} = (\beta_2(\bar{x}))_{55} = 0.$$

Since  $(\beta_1(\bar{x}))_{55} = \epsilon$ , it is impossible to find a non-zero element  $c$  in  $k$  such that

$$\mathbb{M}(c, \beta_1)(\bar{x}) = (1 - E)\beta_2(\bar{x})(1 + E).$$

Therefore,  $[\beta_1]$  and  $[\beta_2]$  are  $k$ -linearly independent deformations.

Now that we have established that the dimension of  $F_V(k[\epsilon])$  is greater than or equal to 2, we proceed to show its dimension is exactly 2. Let

$$\psi = (\psi(\bar{x}), \psi(\bar{y})) = (X + L, Y + L')$$

be any lift of  $\rho$  over  $k[\epsilon]$ . We recall the equations (4.3) and (4.5), and begin to define  $E$  so that  $XE - EX$  is as close to  $L$  as possible. Looking at the 6 negative single entries in  $XE - EX$ , as well as the 2 single entries in the 5<sup>th</sup> column, we begin to define the matrix  $E$ :

$$E = \begin{bmatrix} * & * & * & * & * \\ -l_{22} & * & * & -l_{23} & l_{15} \\ -l_{32} & * & * & -l_{33} & l_{45} \\ * & * & * & * & * \\ -l_{52} & * & * & -l_{53} & * \end{bmatrix}.$$

Next, we recall the equations (4.4) and (4.6). We continue to define the matrix  $E$  by looking at the 5 negative single entries in  $YE - EY$ , excluding the (2, 4)-entry, as well as all of the single entries in the 1<sup>st</sup> column, except for the (3, 1)-entry. We exclude these two particular entries, because we note that both  $e_{25}$  and  $e_{21}$  have already been defined. Setting  $e_{13} = -l'_{12}$ ,  $e_{15} = -l'_{14}$ ,  $e_{23} = -l'_{22}$ ,  $e_{43} = -l'_{42}$ ,  $e_{45} = -l'_{44}$ ,  $e_{41} = l'_{51}$ , we continue to fill in the matrix  $E$  as:

$$E = \begin{bmatrix} * & * & -l'_{12} & * & -l'_{14} \\ -l_{22} & * & -l'_{22} & -l_{23} & l_{15} \\ -l_{32} & * & * & -l_{33} & l_{45} \\ l'_{51} & * & -l'_{42} & * & -l'_{44} \\ -l_{52} & * & * & -l_{53} & * \end{bmatrix}.$$

Next, we define the remaining 11 undetermined entries of  $E$  by examining the entries which contain a difference expression in  $XE - EX$  and  $YE - EY$ . Noting

that  $e_{22}$  has yet to be defined, we choose to set  $e_{22} = 0$ . Then we define  $e_{11}$  as  $e_{11} = e_{22} - l_{12} = -l_{12}$ , which we obtain by noticing that  $e_{22} - e_{11} = (XE - EX)_{12}$ . Similarly, we set  $e_{33} = e_{22} - l'_{32} = -l'_{32}$ , which we obtain by noticing that the expression  $e_{22} - e_{33} = (YE - EY)_{32}$ . We may then continue in this manner, defining  $e_{14} = e_{23} - l_{14} = -l'_{22} - l_{13}$ ,  $e_{32} = e_{41} + l_{42} = l'_{51} + l_{42}$ ,  $e_{44} = e_{33} - l_{43} = -l'_{32} - l_{43}$ ,  $e_{55} = e_{44} - l'_{54} = -l'_{32} - l_{43} - l'_{54}$ . Also, since neither  $e_{53}$  nor  $e_{42}$  have yet been defined, we choose to set  $e_{53} = 0$ , which results in  $e_{42} = e_{53} + l'_{52} = l'_{52}$ . The only difference expression we have yet to address is the expression  $e_{24} - e_{35} = (YE - EY)_{34}$ , due to the fact that both  $e_{24}$  and  $e_{35}$  have previously been defined.

To summarize, all but 3 of the expressions found in  $XE - EX$  and  $YE - EY$  have been utilized to define elements of  $E$ . In particular, we were unable to use expressions found in  $(YE - EY)_{24}$ ,  $(YE - EY)_{31}$ ,  $(YE - EY)_{34}$ . For all the elements of  $E$  that we have yet to define, we choose to set them all equal to 0, which results in the matrix:

$$E = \begin{bmatrix} -l_{12} & 0 & -l'_{12} & -l'_{22} - l_{13} & -l'_{14} \\ -l_{22} & 0 & -l'_{22} & -l_{23} & l_{15} \\ -l_{32} & l'_{51} + l_{42} & -l'_{32} & -l_{33} & l_{45} \\ l'_{51} & l'_{52} & -l'_{42} & -l'_{32} - l_{43} & -l'_{44} \\ -l_{52} & 0 & 0 & -l_{53} & -l'_{32} - l_{43} - l'_{54} \end{bmatrix}.$$

Recall that  $Q_{ij}$  represents the matrix in  $Mat_5(m_{k[\epsilon]})$  with  $\epsilon$  in the  $(i, j)$ -entry and zeros everywhere else. Using equations (4.7) and (4.8), we now know the following: If  $L = [l_{ij}]$  and  $L' = [l'_{ij}]$  are any matrices in  $Mat_5(m_{k[\epsilon]})$  such that  $(X+L, Y+L')$

is a lift of  $V$  over  $k[\epsilon]$ , then there exist elements  $a_1, a_2, a_3, a_4, a_5$  in  $k$  such that

$$X + L = (1 - E)X(1 + E) + a_1Q_{55},$$

$$Y + L' = (1 - E)Y(1 + E) + a_2Q_{11} + a_3Q_{24} + a_4Q_{31} + a_5Q_{34}.$$

However, we have yet to consider the conditions imposed on  $L$  and  $L'$  by the equation

$$((X + L)(Y + L'))^2 - ((Y + L')(X + L))^2 = 0.$$

Notice that because  $L$  and  $L'$  have entries lying in  $m_{k[\epsilon]}$ ,

$$\begin{aligned} & ((X + L)(Y + L'))^2 - ((Y + L')(X + L))^2 = \\ & XYXY + LYXY + XL'XY + XYLY + XYXL' \\ & - (YXYX + L'XYX + YLYX + YXL'X + YXYL). \end{aligned}$$

We then compute the matrices  $X, Y, XY, YX, XYX, YXY, XYXY, YXYX$  to examine the second column of  $((X + L)(Y + L'))^2 - ((Y + L')(X + L))^2$ . Since  $YX = \begin{bmatrix} 0 & 0 & c_5 & 0 & 0 \end{bmatrix}$ , the matrices  $YXYX, L'XYX$  and  $YLYX$  have a zero second column. Moreover,  $XYX$  is the zero matrix so that  $XYXY$  and  $XYXL'$  also

have a zero second column. On the other hand,

$$\begin{aligned}
(LYXY)_{*,2} &= L_{*,5}, \\
(XL'XY)_{*,2} &= (l'_{24}, 0, 0, l'_{34}, 0)^T, \\
(XYLY)_{*,2} &= (0, 0, 0, l_{23}, 0)^T, \\
-(YXL'X)_{*,2} &= (0, 0, 0, 0, -l'_{31})^T, \\
-(YXYL)_{*,2} &= (0, 0, 0, 0, -l_{22}).
\end{aligned}$$

Therefore, we obtain

$$(((X+L)(Y+L'))^2 - ((Y+L')(X+L))^2)_{*,2} = (l_{15} + l'_{24}, l_{25}, l_{35}, l_{45} + l_{23} + l'_{34}, l_{55} - l_{22} - l'_{31})^T.$$

Since  $((X+L)(Y+L'))^2 - ((Y+L')(X+L))^2$  must equal 0, it follows that all the elements in its second column are zero. Notice that this gives the 3 following dependencies between elements of  $L$  and  $L'$ :

$$\begin{aligned}
l'_{24} &= -l_{15} \\
l'_{34} &= -l_{45} - l_{23} \\
l'_{31} &= l_{55} - l_{22}.
\end{aligned}$$

Therefore, since these three elements of  $L'$  are determined by  $L$ , we must set  $a_4 = a_1$  and  $a_3 = a_5 = 0$  in the expression for  $L'$  to arrive at the following expressions

for  $L$  and  $L'$ :

$$\begin{aligned} X + L &= (1 - E)X(1 + E) + a_1 Q_{55}, \\ Y + L' &= (1 - E)Y(1 + E) + a_2 Q_{11} + a_1 Q_{31}. \end{aligned}$$

Recalling our expressions for  $\beta_1$  and  $\beta_2$ , we may now write

$$[\psi] = [(\psi(\bar{x}), \psi(\bar{y}))] = [(X + L, Y + L')] = a_1[\beta_1] + a_2[\beta_2].$$

Therefore, due to the fact that  $[\beta_1]$  and  $[\beta_2]$  have already been shown to be  $k$ -linearly independent, we conclude that  $F_V(k[\epsilon])$  is a 2-dimensional  $k$ -vector space when  $r = 2$ .

In the general case, for arbitrary  $r \geq 2$ , we proceed in a similar manner. Let  $Q_{ij}$  be the matrix in  $Mat_{2r+1}(m_{k[\epsilon]})$  with  $\epsilon$  in the  $(i, j)$ -entry and zeros everywhere else. We define the following two lifts of  $V$  over  $k[\epsilon]$ :

$$\begin{aligned} \beta_1 = (\beta_1(\bar{x}), \beta_1(\bar{y})) &= (X + Q_{(2r+1)(2r+1)}, Y + Q_{31}) = & (4.9) \\ & \left( \begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & c_1 & c_4 & 0 & c_6 & 0 & \dots & c_{2r} & 0 & \epsilon c_{2r+1} \\ \hline \epsilon c_3 & c_3 & 0 & c_5 & 0 & c_7 & \dots & 0 & c_{2r+1} & 0 \end{array} \right) \end{aligned}$$



and

$$\beta_2 = (\beta_2(\bar{x}), \beta_2(\bar{y})) = (X, Y + Q_{11}) = \quad (4.10)$$

$$\left( \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & c_1 & c_4 & 0 & c_6 & 0 & \dots & c_{2r} & 0 & 0 \\ \hline \epsilon c_1 & c_3 & 0 & c_5 & 0 & c_7 & \dots & 0 & c_{2r+1} & 0 \end{array} \right] \right).$$

As in the case when  $r = 2$ , we see that the deformations  $[\beta_1]$  and  $[\beta_2]$  are  $k$ -linearly independent. Let

$$\psi = (\psi(\bar{x}), \psi(\bar{y})) = (X + L, Y + L')$$

be any lift of  $V$  over  $k[\epsilon]$ . We seek elements  $b_1$  and  $b_2$  in  $k$  such that we can write

$$[\psi] = b_1[\beta_1] + b_2[\beta_2].$$

This will be accomplished if we can find a matrix  $E$  in  $Mat_{2r+1}(m_{k[\epsilon]})$  such that

$$X + L = (1 - E)X(1 + E) + b_1 Q_{(2r+1)(2r+1)}, \quad (4.11)$$

$$Y + L' = (1 - E)Y(1 + E) + b_2 Q_{11} + b_1 Q_{31}. \quad (4.12)$$

Note that, as in the equations (4.7) and (4.8) when  $r = 2$ , we see that there exist matrices  $F$  and  $F'$  in  $Mat_{2r+1}(m_{k[\epsilon]})$  such that

$$X + L = (1 - F)X(1 + F) + cQ_{(2r+1)(2r+1)}, \quad (4.13)$$

$$Y + L' = (1 - F')Y(1 + F') + c'Q_{11} \quad (4.14)$$

for certain  $c$  and  $c'$  in  $k$ . To find  $E$  in (4.11) and (4.12), we use the structure of the equations (4.1) and (4.2) to obtain

$$\begin{aligned} XE - EX &= \left[ \begin{array}{c|c|c|c|c|c|c|c|c} E_{2,*}^T & 0 & 0 & E_{3,*}^T & 0 & E_{5,*}^T & \dots & 0 & E_{2r-1,*}^T & 0 \end{array} \right]^T \\ &- \left[ \begin{array}{c|c|c|c|c|c|c|c|c} 0 & E_{*,1} & E_{*,4} & 0 & E_{*,6} & 0 & \dots & E_{*,2r} & 0 & 0 \end{array} \right] \end{aligned} \quad (4.15)$$

and

$$\begin{aligned} YE - EY &= \left[ \begin{array}{c|c|c|c|c|c|c|c|c} 0 & 0 & E_{2,*}^T & 0 & E_{4,*}^T & 0 & \dots & E_{2r-2,*}^T & 0 & E_{2r,*}^T \end{array} \right]^T \\ &- \left[ \begin{array}{c|c|c|c|c|c|c|c|c} 0 & E_{*,3} & 0 & E_{*,5} & 0 & E_{*,7} & \dots & 0 & E_{*,2r+1} & 0 \end{array} \right]. \end{aligned} \quad (4.16)$$

We then begin to define entries in  $E$  in terms of entries from  $L$  by first examining all of the negative, single-element entries in the matrix  $XE - EX$ , along with all of the single-element entries in the  $(2r + 1)^{th}$  column of  $XE - EX$ . We then continue to define entries of  $E$  in terms of entries from  $L'$  by looking at all of the negative, single-element entries in the matrix  $YE - EY$ , in addition to all of the single-element entries in the  $1^{st}$  column of  $YE - EY$ , but excluding any of those entries which contain entries in  $E$  that have already been defined. Finally, we continue to define the remaining entries of  $E$  in terms of elements from  $L$  and  $L'$  by examining the entries consisting of differences in the matrices  $XE - EX$  and  $YE - EY$ . This can be done in a manner so that, excluding element  $L_{(2r+1)(2r+1)}$  by (4.13), every element of  $L$  is either 0, as determined by equation (4.1), or is defined in terms of an entry found in  $E$ . While this method does not result in knowing every non-zero

element of  $L'$  in terms of  $E$ , it does result in the following: With the exception of  $L'_{11}$  by equation (4.14), those entries of  $L'$  that are currently not determined by  $E$  are in fact determined by the equation

$$((X + L)(Y + L'))^r - ((Y + L')(X + L))^r = 0.$$

Notice that since  $L$  and  $L'$  are matrices in  $Mat_{2r+1}(m_{k[\epsilon]})$ , and because  $m_{k[\epsilon]}^2 = 0$ , it follows that

$$\begin{aligned} & ((X + L)(Y + L'))^r - ((Y + L')(X + L))^r = \\ & ((XY)^r + LY(XY)^{r-1} + XL'(XY)^{r-1} + \dots (XY)^{r-1}XL') \\ & - ((YX)^r + L'X(YX)^{r-1} + YL(YX)^{r-1} + \dots (YX)^{r-1}YL). \end{aligned}$$

Setting the above equation equal to zero allows us to solve for the currently undetermined entries of  $L'$  in terms of entries of  $L$ , and hence in terms of entries of  $E$ . From this, we are able to find elements  $b_1$  and  $b_2$  in  $k$  such that

$$\begin{aligned} X + L &= (1 - E)X(1 + E) + b_1Q_{(2r+1)(2r+1)}, \\ Y + L' &= (1 - E)Y(1 + E) + b_2Q_{11} + b_1Q_{31}, \end{aligned}$$

and so we can write

$$[\psi] = b_1[\beta_1] + b_2[\beta_2].$$

Since  $[\beta_1]$  and  $[\beta_2]$  are  $k$ -linearly independent, this concludes the proof of Lemma 4.4. □

*Proof of Theorem 4.3:*

We prove this by constructing a versal couple for  $V$ , using the construction in the proof of Theorem 3.16. Define

$$S = k\langle\langle s, t \rangle\rangle.$$

Since  $F_V(k[\epsilon])$  is a 2-dimensional vector space, we know from Theorem 3.16 that a versal couple for  $V$  exists, and we further know that the versal deformation ring,  $R = R(A, V)$ , will be a quotient of  $S$  by some proper closed ideal,  $I$ .

In the process of constructing the versal couple, we know we can find ideals  $I_\ell$  of  $S$ , for all integers  $\ell \geq 2$ , which satisfy

$$m_S * I_\ell + I_\ell * m_S \subseteq I_{\ell+1}.$$

Moreover, we can find compatible lifts  $\phi_\ell$  of  $V$  over  $S/I_\ell$  for all  $\ell$ . Notice that any lift of  $V$  over  $S/I_\ell$  is completely determined by its values on  $\bar{x}$  and  $\bar{y}$ . Namely, these lifts correspond to pairs of matrices,

$$(\phi_\ell(\bar{x}), \phi_\ell(\bar{y})) = (X_\ell, Y_\ell) \in \text{Mat}_{2r+1}(S/I_\ell) \times \text{Mat}_{2r+1}(S/I_\ell)$$

such that

$$(X_\ell)^2 = (Y_\ell)^2 = (X_\ell Y_\ell)^r - (Y_\ell X_\ell)^r = 0 \in \text{Mat}_{2r+1}(S/I_\ell). \quad (4.17)$$

Let

$$\pi_\ell : S \rightarrow S/I_\ell$$

be the natural surjections, and let  $\mathcal{X}_\ell$  and  $\mathcal{Y}_\ell$  be any elements in  $\pi_\ell^{-1}(X_\ell)$  and  $\pi_\ell^{-1}(Y_\ell)$ , respectively, where we have extended  $\pi_\ell$  naturally to a surjection of matrix rings.

The relations in equation (4.17) involving  $X_\ell$  and  $Y_\ell$  then translate into the following relations regarding  $\mathcal{X}_\ell$  and  $\mathcal{Y}_\ell$ :

$$(\mathcal{X}_\ell)^2, (\mathcal{Y}_\ell)^2, (\mathcal{X}_\ell \mathcal{Y}_\ell)^r - (\mathcal{Y}_\ell \mathcal{X}_\ell)^r \in I_\ell. \quad (4.18)$$

Since we know the lifts  $\phi_\ell$  will be compatible, each  $X_{\ell+1}$  will project onto  $X_\ell$ , and  $Y_{\ell+1}$  will project onto  $Y_\ell$  via the natural projection of  $Mat_{2r+1}(S/I_{\ell+1})$  onto  $Mat_{2r+1}(S/I_\ell)$ . Notice that this precisely means that  $\pi_{\ell+1}^{-1}(X_{\ell+1}) \subseteq \pi_\ell^{-1}(X_\ell)$  and  $\pi_{\ell+1}^{-1}(Y_{\ell+1}) \subseteq \pi_\ell^{-1}(Y_\ell)$ , for all  $\ell$ . These set containments then immediately yield:

$$\mathcal{X}_{\ell+1} - \mathcal{X}_\ell \in I_\ell,$$

$$\mathcal{Y}_{\ell+1} - \mathcal{Y}_\ell \in I_\ell.$$

We proceed in the construction of the versal couple of  $V$  by finding the ideals  $I_\ell$  and appropriate matrices  $\mathcal{X}_\ell$  and  $\mathcal{Y}_\ell$ , for each  $\ell$ .

Following the constructive proof of the existence of a versal couple in the proof of Theorem 3.16, we take

$$I_2 = (s^2, t^2, st, ts).$$

To find our matrices  $\mathcal{X}_2$  and  $\mathcal{Y}_2$ , we recall equations (4.9) and (4.10) from Lemma 4.4. Since  $[\beta_1]$  and  $[\beta_2]$  are  $k$ -linearly independent elements of the 2-dimensional  $k$ -vector space  $F_V(k[\epsilon])$ , they form a basis of  $F_V(k[\epsilon])$ . Therefore, as in the proof of Theorem 3.16, we take the pullback of  $\beta_1$  and  $\beta_2$  over  $Mat_{2r+1}(k)$  to arrive at our desired lift  $\phi_2 = (X_2, Y_2)$  of  $V$  over  $S/I_2$ :

$$X_2 = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & c_1 & c_4 & 0 & c_6 & 0 & \dots & c_{2r} & 0 & sc_{2r+1} \end{array} \right]$$

and

$$Y_2 = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c} tc_1 & c_3 & 0 & c_5 & 0 & c_7 & \dots & 0 & c_{2r+1} & 0 \\ +sc_3 & & & & & & & & & \end{array} \right].$$

Define an element  $b$  in  $S$  by

$$b = t(-st)^{r-1}, \quad (4.19)$$

and define two matrices  $\mathcal{X}$  and  $\mathcal{Y}$  in  $Mat_{2r+1}(S)$  by

$$\mathcal{X} = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & c_1 & c_4 & 0 & c_6 & 0 & \dots & c_{2r} & 0 & bc_1 \\ & & & & & & & & & -bsc_2 \\ & & & & & & & & & +sc_{2r+1} \end{array} \right]$$

and by

$$\mathcal{Y} = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c} tc_1 & c_3 & 0 & c_5 & 0 & c_7 & \dots & 0 & c_{2r+1} & 0 \\ -stc_2 & & & & & & & & & \\ +sc_3 & & & & & & & & & \end{array} \right].$$

Notice that  $\mathcal{X} \bmod I_2 = X_2$ , and  $\mathcal{Y} \bmod I_2 = Y_2$ . Now let

$$(\mathcal{X}_2, \mathcal{Y}_2) \in Mat_{2r+1}(S) \times Mat_{2r+1}(S)$$

be a pair of matrices such that

$$\pi_2(\mathcal{X}_2, \mathcal{Y}_2) = (X_2, Y_2)$$

where we have extended  $\pi_2$  to the natural map between Cartesian products.

Suppose inductively that we have found ideals  $I_2, I_3, \dots, I_\ell$  of  $S$  and pairs of matrices

$$(\mathcal{X}_2, \mathcal{Y}_2), (\mathcal{X}_3, \mathcal{Y}_3), \dots, (\mathcal{X}_\ell, \mathcal{Y}_\ell) \in \text{Mat}_{2r+1}(S) \times \text{Mat}_{2r+1}(S)$$

such that

$$\mathcal{X}_i^2, \mathcal{Y}_i^2, (\mathcal{X}_i \mathcal{Y}_i)^r - (\mathcal{Y}_i \mathcal{X}_i)^r \in \text{Mat}_{2r+1}(I_i)$$

for all  $2 \leq i \leq \ell$ . Further assume that it is permissible to take  $\mathcal{X}_\ell = \mathcal{X}$  and  $\mathcal{Y}_\ell = \mathcal{Y}$ .

We seek an ideal  $I_{\ell+1}$  minimal among all ideals  $J$  of  $S$  that satisfy the following two conditions:

- (a)  $m_S * I_\ell + I_\ell * m_S \subseteq J \subseteq I_\ell$ , and
- (b) there exists a pair of matrices  $(\mathcal{X}_J, \mathcal{Y}_J)$  in  $\text{Mat}_{2r+1}(S) \times \text{Mat}_{2r+1}(S)$  such that  $\pi_J((\mathcal{X}_J, \mathcal{Y}_J))$  defines a lift of  $V$  over  $S/J$ , where  $\pi_J : S \rightarrow S/J$  is the natural projection extended to the Cartesian product of matrix rings, and so that  $\mathcal{X}_J - \mathcal{X}_\ell$  and  $\mathcal{Y}_J - \mathcal{Y}_\ell$  both lie in  $\text{Mat}_{2r+1}(I_\ell)$ .

Since  $\mathcal{X}_J - \mathcal{X}_\ell$  and  $\mathcal{Y}_J - \mathcal{Y}_\ell$  must both lie in  $\text{Mat}_{2r+1}(I_\ell)$ , there are matrices  $H_X$  and  $H_Y$  in  $\text{Mat}_{2r+1}(I_\ell)$  such that  $\mathcal{X}_J = \mathcal{X}_\ell + H_X$  and  $\mathcal{Y}_J = \mathcal{Y}_\ell + H_Y$ . Moreover, since we are assuming  $\mathcal{X}_\ell = \mathcal{X}$  and  $\mathcal{Y}_\ell = \mathcal{Y}$ , we must have  $\mathcal{X}_J = \mathcal{X} + H_X$  and  $\mathcal{Y}_J = \mathcal{Y} + H_Y$ . Letting  $(X)_{i,*}$  and  $(X)_{*,i}$  denote the  $i^{\text{th}}$  row and column, respectively, of  $\mathcal{X}$ , we notice the following:

$$(\mathcal{X})_{2r+1,*} = (0, 0, \dots, 0, s),$$

$$(\mathcal{X})_{*,2r+1} = (b, -bs, 0, 0, \dots, 0, s)^T.$$

Furthermore,

$$\begin{aligned} (X_J^2)_{2r+1,2r+1} &= ((\mathcal{X})_{2r+1,*} + (H_X)_{2r+1,*}) \cdot ((\mathcal{X})_{*,2r+1} + (H_X)_{*,2r+1}) \\ &= (\mathcal{X})_{2r+1,*} \cdot (\mathcal{X})_{*,2r+1} + (\mathcal{X})_{2r+1,*} \cdot (H_X)_{*,2r+1} \\ &\quad + (H_X)_{2r+1,*} \cdot (\mathcal{X})_{*,2r+1} + (H_X)_{2r+1,*} \cdot (H_X)_{*,2r+1}. \end{aligned}$$

However, recall that

$$m_S * I_\ell + I_\ell * m_S \subseteq J \subseteq I_\ell.$$

Additionally, notice that every entry of the matrix  $H_X$  lies in  $I_\ell$  and every entry of  $(\mathcal{X})_{2r+1,*}$  and  $(\mathcal{X})_{*,2r+1}$  lies in  $m_S$ . Therefore, letting

$$a = (\mathcal{X})_{2r+1,*} \cdot (H_X)_{*,2r+1} + (H_X)_{2r+1,*} \cdot (\mathcal{X})_{*,2r+1} + (H_X)_{2r+1,*} \cdot (H_X)_{*,2r+1}$$

we see that

$$(\mathcal{X}_J^2)_{2r+1,2r+1} = (\mathcal{X})_{2r+1,*} \cdot (\mathcal{X})_{*,2r+1} + a = (\mathcal{X}^2)_{2r+1,2r+1} + a$$

where  $a$  lies in  $J$ . Notice that since  $(\mathcal{X}_J, \mathcal{Y}_J)$  constitutes a lift of  $V$  over  $S/J$ ,  $\mathcal{X}_J^2$  must be a matrix in  $Mat_{2r+1}(J)$ . The  $(2r+1, 2r+1)$ -entry of this matrix is exactly  $(\mathcal{X}_J^2)_{2r+1,2r+1}$ , which means that

$$(\mathcal{X}_J^2)_{2r+1,2r+1} = (\mathcal{X}^2)_{2r+1,2r+1} + a$$

must lie in  $J$ . Therefore,  $(\mathcal{X}^2)_{2r+1,2r+1}$  must lie in  $J$ . Since  $(\mathcal{X}^2)_{2r+1,2r+1} = s^2$ , we see that  $s^2$  must lie in  $J$ . Since this is true for all  $J$  satisfying the two conditions





$$\mathcal{X}\mathcal{Y} = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c} -stc_1 & c_4 & 0 & c_6 & 0 & \dots & c_{2r} & 0 & bc_1 & 0 \\ \hline +sc_4 & & & & & & & & -bsc_2 & \\ \hline & & & & & & & & +sc_{2r+1} & \end{array} \right],$$

and

$$\mathcal{Y}\mathcal{X} = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & tc_1 & c_5 & 0 & c_7 & 0 & \dots & c_{2r+1} & 0 & tbc_1 \\ \hline -stc_2 & & & & & & & & & -stbc_2 \\ \hline +sc_3 & & & & & & & & & +(sb - bs)c_3 \end{array} \right].$$

Let  $X_J = \mathcal{X} \bmod J$  and  $Y_J = \mathcal{Y} \bmod J$ . Furthermore, let us recall that when

we write

$$X_J = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & c_1 & c_4 & 0 & c_6 & 0 & \dots & c_{2r} & 0 & bc_1 \\ \hline & & & & & & & & & -bsc_2 \\ \hline & & & & & & & & & +sc_{2r+1} \end{array} \right],$$

we implicitly mean that the coefficients of the  $c_i$  are to be taken modulo  $J$ . Since  $s^2$

and  $t^2$  are in  $J$ , we see that

$$X_J^2 = Y_J^2 = 0$$

in  $Mat_{2r+1}(S/J)$ . Moreover, we have

$$X_J Y_J = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c} -stc_1 & c_4 & 0 & c_6 & 0 & \dots & c_{2r} & 0 & bc_1 & 0 \\ \hline +sc_4 & & & & & & & & -bsc_2 & \\ \hline & & & & & & & & +sc_{2r+1} & \end{array} \right],$$

$$Y_J X_J = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c|c} 0 & tc_1 & c_5 & 0 & c_7 & 0 & \dots & c_{2r+1} & 0 & (sb - bs)c_3 \\ \hline & -stc_2 & & & & & & & & & \\ \hline & +sc_3 & & & & & & & & & \end{array} \right],$$

since  $b = t(-st)^{r-1}$  by equation (4.19), and hence  $tb$  and  $-stb$  are zero in  $S/J$ .

Let

$$w = sb - bs. \quad (4.20)$$

Looking at small powers of  $X_J Y_J$ , we get

$$(X_J Y_J)^2 = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} (-st)^2 c_1 & c_6 & 0 & c_8 & 0 & \dots & c_{2r} & 0 & bc_1 & 0 & wc_4 & 0 \\ \hline & +sc_6 & & & & & & & -bsc_2 & & & \\ \hline & & & & & & & & +sc_{2r+1} & & & \end{array} \right],$$

$$(X_J Y_J)^3 = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c} (-st)^3 c_1 & c_8 & 0 & c_{10} & 0 & \dots & c_{2r} & 0 & bc_1 & 0 & wc_4 & 0 & wc_6 & 0 \\ \hline & +sc_8 & & & & & & & -bsc_2 & & & & & \\ \hline & & & & & & & & +sc_{2r+1} & & & & & \end{array} \right].$$

We see inductively that

$$(X_J Y_J)^{r-1} = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c} (-st)^{r-1} c_1 & c_{2r} & 0 & bc_1 & 0 & wc_4 & 0 & wc_6 & \dots & 0 & wc_{2(r-1)} & 0 \\ \hline & +sc_{2r} & & -bsc_2 & & & & & & & & \\ \hline & & & +sc_{2r+1} & & & & & & & & \end{array} \right],$$

and

$$(X_J Y_J)^r = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c} ((-st)^r + bs)c_1 & bc_1 & 0 & wc_4 & 0 & wc_6 & \dots & 0 & wc_{2r} & 0 \\ \hline & -bsc_2 & & & & & & & & \\ \hline & & +sc_{2r+1} & & & & & & & \end{array} \right].$$

Rewriting  $(-st)^r = (-s)(t)(-st)^{r-1} = -sb$ , we have  $(-st)^r + bs = -sb + bs = -w$ .

Therefore we can write

$$(X_J Y_J)^r = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c} -wc_1 & bc_1 & 0 & wc_4 & 0 & wc_6 & \dots & 0 & wc_{2r} & 0 \\ \hline & -bsc_2 & & & & & & & & \\ \hline & & +sc_{2r+1} & & & & & & & \end{array} \right].$$

We similarly look at powers of  $Y_J X_J$ . We have

$$(Y_J X_J)^2 = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & t(-st)c_1 & c_7 & 0 & c_9 & \dots & 0 & c_{2r+1} & 0 & wc_3 & 0 & wc_5 \\ \hline & +(-st)^2 c_2 & & & & & & & & & & \\ \hline & & +sc_5 & & & & & & & & & \end{array} \right].$$

Inductively, we see that

$$(Y_J X_J)^{r-1} = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & t(-st)^{r-2}c_1 & c_{2r+1} & 0 & wc_3 & 0 & wc_5 & \dots & 0 & wc_{2(r-1)+1} \\ \hline & +(-st)^{r-1}c_2 & & & & & & & & \\ \hline & & +sc_{2(r-1)+1} & & & & & & & \end{array} \right],$$

and

$$(Y_J X_J)^r = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & t(-st)^{r-1}c_1 & wc_3 & 0 & wc_5 & \dots & 0 & wc_{2(r-1)+1} & 0 & wc_{2r+1} \\ \hline & +(-st)^r c_2 & & & & & & & & \\ \hline & & +sc_{2r+1} & & & & & & & \end{array} \right].$$

Recalling from equation (4.19) that  $b = t(-st)^{r-1}$  and hence  $(-st)^r = -sb$ , we see that

$$(Y_J X_J)^r = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & bc_1 & wc_3 & 0 & wc_5 & \dots & 0 & wc_{2(r-1)+1} & 0 & wc_{2r+1} \\ \hline & -sbc_2 & & & & & & & & \\ \hline & +sc_{2r+1} & & & & & & & & \end{array} \right].$$

Therefore, using that  $w = sb - bs$  by equation (4.20), we obtain

$$(X_J Y_J)^r - (Y_J X_J)^r = \left[ \begin{array}{c|c|c|c|c|c} -wc_1 & wc_2 & -wc_3 & \dots & wc_{2r} & -wc_{2r+1} \end{array} \right]. \quad (4.21)$$

Recall that  $X_J = \mathcal{X} \bmod J$  and  $Y_J = \mathcal{Y} \bmod J$ . Equation (4.21), combined with our calculation of  $\mathcal{X}^2$  and  $\mathcal{Y}^2$  results in the following conclusion: The pair of matrices  $(\mathcal{X}, \mathcal{Y})$  defines a lift of  $V$  over  $S/(s^2, t^2, w)$ .

Since the first row of  $(X_J Y_J)^{r-1}$  is  $((-st)^{r-1}, 0, 0, b, 0, \dots, 0)$ , it follows that the first row of  $(\mathcal{X} \mathcal{Y})^{r-1}$  is

$$((-st)^{r-1}, 0, 0, b, 0, \dots, 0) + (j_1, j_2, \dots, j_{2r+1}),$$

where each  $j_i$  is in  $J \subseteq I_\ell$ . However, recall that  $\mathcal{X}_{\ell+1}$  must be of the form  $\mathcal{X} + H_X$ , where  $H_X$  is a matrix in  $Mat_{2r+1}(I_\ell)$ , and that  $\mathcal{Y}_{\ell+1}$  must be of the form  $\mathcal{Y} + H_Y$ , where  $H_Y$  is a matrix in  $Mat_{2r+1}(I_\ell)$ . Since  $J \subseteq I_\ell$ , it follows that the first row of  $(\mathcal{X}_{\ell+1} \mathcal{Y}_{\ell+1})^{r-1}$  is

$$((-st)^{r-1}, 0, 0, b, 0, \dots, 0) + (a_1, a_2, \dots, a_{2r+1}),$$

where each  $a_i$  is in  $I_\ell$ . Similarly, since the first column of  $(X_J Y_J)$  is

$(-st, 0, 0, s, 0, 0, \dots, 0)^T$ , it follows that the first column of  $(\mathcal{X}_{\ell+1} \mathcal{Y}_{\ell+1})$  is

$$(-st, 0, 0, s, 0, 0, \dots, 0)^T + (a'_1, a'_2, \dots, a'_{2r+1})^T,$$

where each of the  $a'_i$  lies in  $I_\ell$ . Since  $m_S * I_\ell + I_\ell * m_S \subseteq I_{\ell+1}$ , it follows that there exists an element  $d$  in  $I_{\ell+1}$  so that

$$(\mathcal{X}_{\ell+1}\mathcal{Y}_{\ell+1})_{11}^r = (-st)^r + bs + d = -w + d.$$

If we can show that there exists an element  $d'$  in  $I_{\ell+1}$  so that  $(\mathcal{Y}_{\ell+1}\mathcal{X}_{\ell+1})_{11}^r = d'$ , then  $(\mathcal{X}_{\ell+1}\mathcal{Y}_{\ell+1})_{11}^r - (\mathcal{Y}_{\ell+1}\mathcal{X}_{\ell+1})_{11}^r = w + d - d'$ . Since this entry is required to be in  $I_{\ell+1}$ , it then follows that  $w$  must be in  $I_{\ell+1}$ .

To this end, notice that the first row of  $(Y_J X_J)$  is  $(0, t, 0, \dots, 0)$ , and the first column of  $(Y_J X_J)^{r-1}$  is  $(0, 0, \dots, 0)^T$ . This implies that there exist elements  $a''_i$  and  $a'''_i$  in  $I_\ell$  for  $1 \leq i \leq 2r+1$  so that the first row of  $\mathcal{Y}_{\ell+1}\mathcal{X}_{\ell+1}$  is  $(0, t, 0, \dots, 0) + (a''_1, \dots, a''_{2r+1})$ , and the first column of  $(\mathcal{Y}_{\ell+1}\mathcal{X}_{\ell+1})^{r-1}$  is  $(0, 0, \dots, 0)^T + (a'''_1, \dots, a'''_{2r+1})^T$ . Once again recalling that  $m_S * I_\ell + I_\ell * m_S \subseteq I_{\ell+1}$ , it follows that there exists an element  $d'$  in  $I_{\ell+1}$  so that  $(\mathcal{Y}_{\ell+1}\mathcal{X}_{\ell+1})_{11}^r = d'$ , completing the proof that  $w$  must lie in  $I_{\ell+1}$ .

Therefore, we obtain that  $(\mathcal{X}, \mathcal{Y})$  defines a lift of  $V$  over  $S/(s^2, t^2, w)$  and that  $(s^2, t^2, w) + m_S * I_\ell + I_\ell * m_S$  is the minimal ideal satisfying conditions (a) and (b). Therefore,

$$I_{\ell+1} = (s^2, t^2, w) + m_S * I_\ell + I_\ell * m_S \subseteq I_\ell, \quad (4.22)$$

and we may take  $\mathcal{X}_{\ell+1} = \mathcal{X}$  and  $\mathcal{Y}_{\ell+1} = \mathcal{Y}$ , completing the induction step.

Notice that  $I_2 = m_S^2 = (s^2, t^2, w) + m_S^2$ . If we suppose inductively that  $I_i = (s^2, t^2, w) + m_S^i$  for  $2 \leq i \leq \ell$ , then applying equation (4.22) yields

$$\begin{aligned}
I_{\ell+1} &= (s^2, t^2, w) + m_S * I_\ell + I_\ell * m_S \\
&= (s^2, t^2, w) + m_S * ((s^2, t^2, w) + m_S^\ell) + ((s^2, t^2, w) + m_S^\ell) * m_S \\
&= (s^2, t^2, w) + m_S^{\ell+1},
\end{aligned}$$

completing the induction. Therefore,

$$I_\ell = (s^2, t^2, w) + m_S^\ell \tag{4.23}$$

for all  $\ell \geq 2$ . We know from the proof of Theorem 3.16 that the versal deformation ring  $R(A, V)$  of  $V$  is isomorphic to  $R = S/I$ , where  $I = \bigcap I_\ell$ . Since  $m_S^\ell \subseteq (s^2, t^2, w)$  for all  $\ell \geq 2r + 1$ , applying equation (4.23) yields

$$I = \bigcap_{\ell} I_\ell = \bigcap_{\ell} ((s^2, t^2, w) + m_S^\ell) = (s^2, t^2, w).$$

In particular,  $R(A, V) = R(\Omega_r, V_r) \cong \Omega_r$ , proving Theorem 4.3.

Additionally, we have

$$\begin{aligned}
(X_\ell, Y_\ell) &= (\mathcal{X}_\ell \pmod{I_\ell}, \mathcal{Y}_\ell \pmod{I_\ell}) \\
&= (\mathcal{X} \pmod{I_\ell}, \mathcal{Y} \pmod{I_\ell})
\end{aligned}$$

for all  $\ell$ . Therefore, it follows from the construction of a versal deformation in the proof of Theorem 3.16 that the versal deformation  $[\phi_u]$  of  $V$  over  $S/I$  corresponds to the lift  $\phi_u$  with

$$(\phi_u(\bar{x}), \phi_u(\bar{y})) = (\mathcal{X} \pmod{I}, \mathcal{Y} \pmod{I}).$$

## 4.2 The case $\lambda = W$ , $A = W[D_8]$ and $V = V_2$

In this section, suppose that  $k$  is a perfect field of characteristic 2, and let  $\lambda = W$  be the ring of infinite Witt vectors over  $k$ . Note that  $W$  is a complete, local, commutative, Noetherian ring with residue field isomorphic to  $k$ , whose unique maximal ideal is the principal ideal  $\mu = (2)$ . We let  $A = W[D_8]$  be the group ring over  $W$  of a dihedral group of order 8. We first prove the following connection between group rings over  $W$  of dihedral groups of order  $2^{m+2}$  and  $\Omega_{2^m}$ , for  $m \geq 1$ . Recall that

$$\Omega_{2^m} = k\langle\langle x, y \rangle\rangle / (x^2, y^2, (xy)^{2^m} - (yx)^{2^m})$$

and that  $\bar{x}$  and  $\bar{y}$  denote the images in  $\Omega_{2^m}$  of  $x$  and  $y$ , respectively

**Lemma 4.5.** *For  $m \geq 1$ , let  $D_{2^{m+2}}$  be a dihedral group of order  $2^{m+2}$  with presentation  $D_{2^{m+2}} = \langle u, v \mid u^2 = v^2 = 1, (uv)^{2^m} = (vu)^{2^m} \rangle$ . Define*

$$\widehat{\Omega}_{2^m} = W\langle\langle x, y \rangle\rangle / (x^2 - 2x, y^2 - 2y, ((x-1)(y-1))^{2^m} - ((y-1)(x-1))^{2^m})$$

and let  $\hat{x}$  and  $\hat{y}$  denote the images in  $\widehat{\Omega}_{2^m}$  of  $x$  and  $y$ , respectively. Then the map  $f : W[D_{2^{m+2}}] \rightarrow \widehat{\Omega}_{2^m}$ , defined by  $f(u) = \hat{x} - 1$ ,  $f(v) = \hat{y} - 1$ , and extending  $W$ -linearly, is a  $W$ -algebra isomorphism. Furthermore,  $f$  induces a  $k$ -algebra isomorphism

$$\bar{f} : k[D_{2^{m+2}}] \cong W[D_{2^{m+2}}]/2W[D_{2^{m+2}}] \rightarrow \Omega_{2^m}$$

given by  $\bar{f}(u) = \bar{x} - 1$ ,  $\bar{f}(v) = \bar{y} - 1$ , and extending  $k$ -linearly.

*Proof.* Since  $u^2 = v^2 = 1$  and  $(uv)^{2^m} = (vu)^{2^m}$ , it follows that  $f$  is a bijective  $W$ -algebra homomorphism. Since  $W/2W \cong k$ , we have  $k[D_{2^{m+2}}] \cong W[D_{2^{m+2}}]/2W[D_{2^{m+2}}]$ .



To complete the proof of the lemma, it suffices to show that the ideals

$$(x^2 - 2x, y^2 - 2y, ((x-1)(y-1))^{2^m} - ((y-1)(x-1))^{2^m})$$

and

$$(x^2, y^2, (xy)^{2^m} - (yx)^{2^m})$$

are equal in  $k\langle\langle x, y \rangle\rangle$ .

Note that since the characteristic of  $k$  is 2,  $2x = 2y = 0$  in  $k\langle\langle x, y \rangle\rangle$ , and so we can already write

$$\begin{aligned} &(x^2 - 2x, y^2 - 2y, ((x-1)(y-1))^{2^m} - ((y-1)(x-1))^{2^m}) \\ &= (x^2, y^2, ((x-1)(y-1))^{2^m} - ((y-1)(x-1))^{2^m}). \end{aligned}$$

Let  $\ell$  be a positive integer, and define the ideal  $L$  of  $k\langle\langle x, y \rangle\rangle$  as

$$L = (x^2, y^2).$$

Notice that since the characteristic of  $k$  is 2,

$$(x-1)^2 + L = (y-1)^2 + L = 1 + L = -1 + L.$$

We also note that

$$(x-1)(y-1) - (y-1)(x-1) = (xy - x - y + 1) - (yx - x - y + 1) = xy - yx.$$

Suppose inductively that for  $i \leq \ell - 1$ ,

$$((x-1)(y-1))^i - ((y-1)(x-1))^i + L = (xy)^i - (yx)^i + L.$$

Letting  $a = ((x-1)(y-1))^{\ell-1} - ((y-1)(x-1))^{\ell-1}$ , we have the following equations:

$$\begin{aligned}
& (xy)^\ell - (yx)^\ell + L \\
= & (xy - x - y + 1)((xy)^{\ell-1} - (yx)^{\ell-1}) - ((xy)^{\ell-1} - (yx)^{\ell-1})(yx - x - y + 1) + L \\
= & (x-1)(y-1)((xy)^{\ell-1} - (yx)^{\ell-1}) - ((xy)^{\ell-1} - (yx)^{\ell-1})(y-1)(x-1) + L \\
& = (x-1)(y-1)a - a(y-1)(x-1) + L \\
= & ((x-1)(y-1))^\ell - ((y-1)(x-1))^\ell + L.
\end{aligned}$$

Therefore, the ideals

$$(x^2, y^2, ((x-1)(y-1))^{2m} - ((y-1)(x-1))^{2m})$$

and

$$(x^2, y^2, (xy)^{2m} - (yx)^{2m})$$

of  $k\langle\langle x, y \rangle\rangle$  are equal, which proves the lemma.  $\square$

By Lemma 4.5, we can identify  $A = W[D_8]$  with the  $W$ -algebra

$$\widehat{\Omega}_2 = W\langle\langle x, y \rangle\rangle / (x^2 - 2x, y^2 - 2y, ((x-1)(y-1))^2 - ((y-1)(x-1))^2),$$

and  $A/2A \cong k[D_8]$  with

$$\Omega_2 = k\langle\langle x, y \rangle\rangle / (x^2, y^2, (xy)^2 - (yx)^2).$$

In this way, we can view  $V_2$  as a 5-dimensional  $k$ -vector space which is also a left  $A$ -module. More precisely, using the representation  $\rho_2$  of  $\Omega_2$  from the beginning of Chapter 4, we define the following matrices in  $Mat_5(k)$ :

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \left[ 0 \mid c_1 \mid c_4 \mid 0 \mid 0 \right]$$

and

$$Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \left[ 0 \mid c_3 \mid 0 \mid c_5 \mid 0 \right].$$

We then obtain a  $W$ -algebra homomorphism

$$\widehat{\rho}_2 : \widehat{\Omega}_2 \rightarrow \text{Mat}_5(k)$$

by defining  $\widehat{\rho}_2(\hat{x}) = X$ ,  $\widehat{\rho}_2(\hat{y}) = Y$ , and extending  $W$ -linearly. In other words, if

$\pi : \widehat{\Omega}_2 \rightarrow \Omega_2$  is the natural projection sending  $\hat{x}$  to  $\bar{x}$  and  $\hat{y}$  to  $\bar{y}$ , then  $\widehat{\rho}_2 = \rho_2 \circ \pi$ .

Let  $\widehat{V}_2$  denote the corresponding  $\widehat{\Omega}_2$ -module.

Our goal is to prove the following result.

**Theorem 4.6.** *The versal deformation ring  $R(\widehat{\Omega}_2, \widehat{V}_2)$  is isomorphic to  $\widehat{\Omega}_2$ .*

*Proof.* Let  $V = \widehat{V}_2$  and  $\rho = \widehat{\rho}_2$ . Since  $\widehat{\rho}_2 = \rho_2 \circ \pi$ , it follows from Lemma 4.4 that

$F_V(k[\epsilon])$  is a two-dimensional  $k$ -vector space. Therefore, a versal couple for  $V$  exists,

and the versal deformation ring  $R = R(\widehat{\Omega}_2, V)$  will be a quotient of  $W\langle\langle s, t \rangle\rangle$ . In finding this versal couple we closely follow the methods and notations of Section 4.1.

Define  $\widehat{S} = W\langle\langle s, t \rangle\rangle$ , and define the following matrices in  $\widehat{S}/(2, s^2, t) \cong k[\epsilon]$ :

$$X_s = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & s \end{bmatrix} = \left[ \begin{array}{c|c|c|c|c} 0 & c_1 & c_4 & 0 & sc_5 \end{array} \right]$$

and

$$Y_s = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ s & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \left[ \begin{array}{c|c|c|c|c} sc_3 & c_3 & 0 & c_5 & 0 \end{array} \right].$$

Also, define the following matrices in  $\widehat{S}/(2, s, t^2) \cong k[\epsilon]$ :

$$X_t = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \left[ \begin{array}{c|c|c|c|c} 0 & c_1 & c_4 & 0 & 0 \end{array} \right]$$

and

$$Y_t = \begin{bmatrix} t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \left[ tc_1 \mid c_3 \mid 0 \mid c_5 \mid 0 \right].$$

It follows from the arguments in Section 4.1 that the pairs of matrices  $(X_s, Y_s)$  and  $(X_t, Y_t)$  define non-isomorphic lifts of  $V$  over  $k[\epsilon]$ . Let

$$\widehat{I}_2 = \mu + m_{\widehat{S}}^2 = (2, s^2, t^2, st, ts).$$

Taking the pullback over  $k$  of these pairs of matrices, we arrive at the following matrices in  $Mat_5(\widehat{S}/\widehat{I}_2)$ :

$$X_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & s \end{bmatrix} = \left[ 0 \mid c_1 \mid c_4 \mid 0 \mid sc_5 \right]$$

and

$$Y_2 = \begin{bmatrix} t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ s & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \left[ tc_1 + sc_3 \mid c_3 \mid 0 \mid c_5 \mid 0 \right].$$

Then  $(X_2, Y_2)$  defines a lift of  $V$  over  $\widehat{S}/\widehat{I}_2$ . We define the following elements of  $\widehat{S}$ :

$$b = -tst,$$

$$\alpha = b - 2t - 4s + 2st + 2ts,$$

$$q = \alpha s - s\alpha,$$

$$w = (st)^2 - (ts)^2.$$

We also define the following matrices in  $Mat_5(\widehat{S})$ :

$$\mathcal{X} = \begin{bmatrix} 2 & 1 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & -\alpha s \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & s \end{bmatrix},$$

$$\mathcal{Y} = \begin{bmatrix} t & 0 & 0 & 0 & 0 \\ 2s - st & 2 & 0 & 4 & -8 \\ s & 1 & 0 & 2 & -4 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Let  $\pi_2 : \widehat{S} \rightarrow \widehat{S}/\widehat{I}_2$  be the natural projection. Then  $\pi_2((\mathcal{X}, \mathcal{Y})) = (X_2, Y_2)$ .

To proceed with the construction of the versal couple for  $V$ , suppose inductively that we have found ideals of  $\widehat{I}_2, \widehat{I}_3, \dots, \widehat{I}_\ell$  of  $\widehat{S}$  and pairs of matrices

$$(\mathcal{X}_2, \mathcal{Y}_2), (\mathcal{X}_3, \mathcal{Y}_3), \dots, (\mathcal{X}_\ell, \mathcal{Y}_\ell) \in \text{Mat}_5(\widehat{S}) \times \text{Mat}_5(\widehat{S})$$

such that

$$\mathcal{X}_i^2 - 2\mathcal{X}_i, \mathcal{Y}_i^2 - 2\mathcal{Y}_i, ((\mathcal{X}_i - 1)(\mathcal{Y}_i - 1))^2 - ((\mathcal{Y}_i - 1)(\mathcal{X}_i - 1))^2 \in \text{Mat}_{2r+1}(\widehat{I}_i)$$

for all  $2 \leq i \leq \ell$ . Further assume that it is permissible to take  $\mathcal{X}_\ell = \mathcal{X}$ , and  $\mathcal{Y}_\ell = \mathcal{Y}$ .

We seek an ideal  $\widehat{I}_{\ell+1}$ , minimal among all ideals  $J$  of  $\widehat{S}$  that satisfy the following two conditions:

(a)  $m_{\widehat{S}} * \widehat{I}_\ell + \widehat{I}_\ell * m_{\widehat{S}} \subseteq J \subseteq \widehat{I}_\ell$ , and

(b) there exists a pair of matrices  $(\mathcal{X}_J, \mathcal{Y}_J)$  in  $Mat_5(\widehat{S}) \times Mat_5(\widehat{S})$  such that

$\pi_J((\mathcal{X}_J, \mathcal{Y}_J))$  defines a lift of  $V$  over  $\widehat{S}/J$ , where  $\pi_J : \widehat{S} \rightarrow \widehat{S}/J$  is the natural projection extended to the Cartesian product of matrix rings, and so that  $\mathcal{X}_J - \mathcal{X}_\ell$  and  $\mathcal{Y}_J - \mathcal{Y}_\ell$  both lie in  $Mat_5(\widehat{I}_\ell)$ .

Since  $\mathcal{X}_J - \mathcal{X}_\ell$  and  $\mathcal{Y}_J - \mathcal{Y}_\ell$  must both lie in  $Mat_5(\widehat{I}_\ell)$ , there are matrices  $G_X$  and  $G_Y$  in  $Mat_5(\widehat{I}_\ell)$  such that  $\mathcal{X}_J = \mathcal{X}_\ell + G_X$  and  $\mathcal{Y}_J = \mathcal{Y}_\ell + G_Y$ . Moreover, since we are assuming  $\mathcal{X}_\ell = \mathcal{X}$  and  $\mathcal{Y}_\ell = \mathcal{Y}$ , we must have  $\mathcal{X}_J = \mathcal{X} + G_X$  and  $\mathcal{Y}_J = \mathcal{Y} + G_Y$ . Letting  $\mathcal{X}_{i,*}$  and  $\mathcal{X}_{*,i}$  denote the  $i^{\text{th}}$  row and column, respectively, of  $\mathcal{X}$ , we notice the following:

$$(\mathcal{X})_{5,*} = (0, 0, 0, 0, s),$$

$$(\mathcal{X})_{*,5} = (\alpha, -\alpha s, 0, 0, s)^T.$$

Furthermore,

$$\begin{aligned} (\mathcal{X}_J^2)_{5,5} &= ((\mathcal{X})_{5,*} + (G_X)_{5,*}) \cdot ((\mathcal{X})_{*,5} + (G_X)_{*,5}) \\ &= (\mathcal{X})_{5,*} \cdot (\mathcal{X})_{*,5} + (\mathcal{X})_{5,*} \cdot (G_X)_{*,5} + (G_X)_{5,*} \cdot (\mathcal{X})_{*,5} + (G_X)_{5,*} \cdot (G_X)_{*,5}. \end{aligned}$$

However, recall that

$$m_{\widehat{S}} * \widehat{I}_\ell + \widehat{I}_\ell * m_{\widehat{S}} \subseteq J \subseteq \widehat{I}_\ell.$$



Additionally, notice that every entry of the matrix  $G_X$  lies in  $\widehat{I}_\ell$  and every entry of  $(\mathcal{X})_{5,*}$  and  $(\mathcal{X})_{*,5}$  lies in  $m_{\widehat{S}}$ . Therefore, letting

$$a = (\mathcal{X})_{5,*} \cdot (G_X)_{*,5} + (G_X)_{5,*} \cdot (\mathcal{X})_{*,5} + (G_X)_{5,*} \cdot (G_X)_{*,5}$$

we see that

$$(\mathcal{X}_J^2)_{5,5} = (\mathcal{X})_{5,*} \cdot (\mathcal{X})_{*,5} + a = (\mathcal{X}^2)_{5,5} + a$$

where  $a$  lies in  $J$ . Additionally,  $2\mathcal{X}_J = 2\mathcal{X} + 2G_X$ . Since  $2 \in m_{\widehat{S}}$  and

$$m_{\widehat{S}} * \widehat{I}_\ell + \widehat{I}_\ell * m_{\widehat{S}} \subseteq J \subseteq \widehat{I}_\ell,$$

it follows that there exists some  $h$  in  $J$  such that

$$(2\mathcal{X}_J)_{5,5} = (2\mathcal{X})_{5,5} + h.$$

Therefore,

$$(\mathcal{X}_J^2 - 2\mathcal{X}_J)_{5,5} - (\mathcal{X}^2 - 2\mathcal{X})_{5,5} \in \text{Mat}_5(J).$$

Notice that since  $(\mathcal{X}_J, \mathcal{Y}_J)$  defines a lift of  $V$  over  $\widehat{S}/J$ , it follows that  $\mathcal{X}_J^2 - 2\mathcal{X}_J$  is a matrix in  $\text{Mat}_5(J)$ . Therefore, it follows that

$$(\mathcal{X}^2 - 2\mathcal{X})_{5,5} = s^2 - 2s \in J.$$

Since this is true for all  $J$  satisfying the conditions (a) and (b) above, it follows that it is in particular true for the minimal  $J$  satisfying these conditions. Therefore,  $s^2 - 2s$  must lie in  $\widehat{I}_{\ell+1}$ .

Next, notice that

$$(\mathcal{Y})_{1,*} = (t, 0, 0, 0, 0),$$

$$(\mathcal{Y})_{*,1} = (t, 2s - st, s, 0, 0)^T,$$

where every entry lies in  $m_{\widehat{S}}$ . Therefore, when looking at  $(\mathcal{Y}_J^2)_{1,1}$ , an analogous argument can be used to show that  $t^2 - 2t$  must also lie in  $\widehat{I}_{\ell+1}$ . Specifically, because all the entries of  $(\mathcal{Y})_{1,*}$  and  $(\mathcal{Y})_{*,1}$  lie in  $m_{\widehat{S}}$ , there exists an  $a'$  in  $J$  such that

$$(\mathcal{Y}_J^2)_{1,1} = (\mathcal{Y})_{1,1}^2 + a' = t^2 + a'.$$

Additionally, since  $2\widehat{I}_\ell \subseteq m_{\widehat{S}} * \widehat{I}_\ell \subseteq J$ , there exists an  $h'$  in  $J$  such that

$$(2\mathcal{Y}_J)_{1,1} = (2\mathcal{Y})_{1,1} + h' = 2t + h'.$$

Therefore, since  $\mathcal{Y}_J^2 - 2Y_J$  is a matrix in  $Mat_5(J)$ , it must be the case that  $t^2 - 2t$  is in  $J$  for all  $J$  satisfying the conditions (a) and (b). Hence,  $t^2 - 2t$  must lie in  $\widehat{I}_{\ell+1}$ .

Let  $N$  be the ideal of  $\widehat{S}$  generated by  $s^2 - 2s$  and  $t^2 - 2t$ . Since  $N \subseteq J$ , it will be useful to compute

$$((\mathcal{X} - 1)(\mathcal{Y} - 1))^2 - ((\mathcal{Y} - 1)(\mathcal{X} - 1))^2$$

modulo  $N$ . To do so, we first note that

$$\begin{aligned}
 \mathcal{X}^2 &= \begin{bmatrix} 4 & 2 & 0 & 0 & 2\alpha \\ 0 & 0 & 0 & 0 & -\alpha s^2 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & s^2 \end{bmatrix}, \\
 2\mathcal{X} &= \begin{bmatrix} 4 & 2 & 0 & 0 & 2\alpha \\ 0 & 0 & 0 & 0 & -2\alpha s \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2s \end{bmatrix}, \\
 \mathcal{Y}^2 &= \begin{bmatrix} t^2 & 0 & 0 & 0 & 0 \\ 2st - st^2 + 2(2s - st) & 4 & 0 & 8 & -16 \\ 2s & 2 & 0 & 4 & -8 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}, \\
 2\mathcal{Y} &= \begin{bmatrix} 2t & 0 & 0 & 0 & 0 \\ 2(2s - st) & 4 & 0 & 8 & -16 \\ 2s & 2 & 0 & 4 & -8 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix},
 \end{aligned}$$

and so  $\mathcal{X}^2 - 2\mathcal{X}$  and  $\mathcal{Y}^2 - 2\mathcal{Y}$  are each 0 modulo  $N$ . Therefore,

$$((\mathcal{X} - 1)(\mathcal{Y} - 1))^2 - ((\mathcal{Y} - 1)(\mathcal{X} - 1))^2 \equiv (\mathcal{X}\mathcal{Y})^2 - (\mathcal{Y}\mathcal{X})^2 - 2(\mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X}) \pmod{N}.$$

To compute

$$(\mathcal{X}\mathcal{Y})^2 - (\mathcal{Y}\mathcal{X})^2 - 2(\mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X}) \pmod{N},$$

we first perform the following initial calculations in  $\widehat{S}/N$ , using the fact that  $s^2 - 2s$  and  $t^2 - 2t$  are in  $N$ :

$$t\alpha + N = -4t + N,$$

$$\alpha s + N = bs + 2ts - 8s + 2sts + N,$$

$$q + N = w - (2st - 2ts) + N = ((s - 1)(t - 1))^2 - ((t - 1)(s - 1))^2 + N.$$

Additionally, we compute

$$\mathcal{X}\mathcal{Y} = \begin{bmatrix} 2t + 2s - st & 2 & 0 & \alpha + 4 & -8 \\ 0 & 0 & 0 & -\alpha s & 0 \\ 2s & 2 & 0 & 4 & -8 \\ s & 1 & 0 & 2 & -4 \\ 0 & 0 & 0 & s & 0 \end{bmatrix},$$

$$\mathcal{Y}\mathcal{X} = \begin{bmatrix} 2t & t & 0 & 0 & t\alpha \\ 4s - 2st & 2s - st & 4 & 0 & -2q + 4st - 8s \\ 2s & s & 2 & 0 & -q - 4s \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Using these relations, we see that

$$((\mathcal{X} - 1)(\mathcal{Y} - 1))^2 - ((\mathcal{Y} - 1)(\mathcal{X} - 1))^2 \equiv \begin{bmatrix} q & 0 & 0 & -2q & 2tq \\ -2q & -q & 2q & 0 & (4s - 2st + 4)q \\ 0 & 0 & q & -2q & 2sq \\ 0 & 0 & 0 & -q & 2q \\ 0 & 0 & 0 & 0 & q \end{bmatrix}$$

modulo  $N$ . Therefore, the pair  $(\mathcal{X}, \mathcal{Y})$  defines a lift of  $V$  over

$$\widehat{S}/(N + (q)) = \widehat{S}/(s^2 - 2s, t^2 - 2t, ((s - 1)(t - 1))^2 - ((t - 1)(s - 1))^2).$$

We now examine the first rows and columns of the matrices  $\mathcal{X}\mathcal{Y}$  and  $\mathcal{Y}\mathcal{X}$ , to

see that  $q$  must also be in  $\widehat{I}_{\ell+1}$ . Recall that

$$((\mathcal{X}_J - 1)(\mathcal{Y}_J - 1))^2 - ((\mathcal{Y}_J - 1)(\mathcal{X}_J - 1))^2$$

must be a matrix in  $Mat_5(J)$ . However, since  $\mathcal{X}_J^2 - 2\mathcal{X}_J$  and  $\mathcal{Y}_J^2 - 2\mathcal{Y}_J$  are also matrices in  $Mat_5(J)$ , we get

$$((\mathcal{X}_J - 1)(\mathcal{Y}_J - 1))^2 - ((\mathcal{Y}_J - 1)(\mathcal{X}_J - 1))^2 \equiv (\mathcal{X}_J\mathcal{Y}_J)^2 - (\mathcal{Y}_J\mathcal{X}_J)^2 - 2(\mathcal{X}_J\mathcal{Y}_J - \mathcal{Y}_J\mathcal{X}_J)$$

modulo  $J$ . Recall that  $\mathcal{X} - \mathcal{X}_J$  and  $\mathcal{Y} - \mathcal{Y}_J$  must both be matrices in  $Mat_5(\widehat{I}_\ell)$ , and that  $m_{\widehat{S}} * \widehat{I}_\ell + \widehat{I}_\ell * m_{\widehat{S}} \subseteq J$ . Since 2 is in  $m_{\widehat{S}}$ ,

$$2(\mathcal{X}_J\mathcal{Y}_J - \mathcal{Y}_J\mathcal{X}_J) \equiv 2(\mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X}) \pmod{J}.$$

Looking at the  $(1, 1)$ -entries of these matrices yields

$$2(\mathcal{X}_J\mathcal{Y}_J - \mathcal{Y}_J\mathcal{X}_J)_{1,1} \equiv 2(\mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X})_{1,1} \pmod{J}.$$

We next look at the  $(1, 1)$ -entry of  $(\mathcal{X}_J\mathcal{Y}_J)^2 - (\mathcal{Y}_J\mathcal{X}_J)^2$ . Notice that our earlier computations show that all entries in the first row and first column of both  $\mathcal{X}\mathcal{Y}$  and  $\mathcal{Y}\mathcal{X}$  lie in  $m_{\widehat{S}}$ . Since  $m_{\widehat{S}} * \widehat{I}_\ell + \widehat{I}_\ell * m_{\widehat{S}} \subseteq J$ , and because each entry of  $\mathcal{X} - \mathcal{X}_J$  and  $\mathcal{Y} - \mathcal{Y}_J$  must lie in  $\widehat{I}_\ell$ , it follows that the  $(1, 1)$ -entries of the matrices

$$(\mathcal{X}_J\mathcal{Y}_J)^2 - (\mathcal{Y}_J\mathcal{X}_J)^2$$

and

$$(\mathcal{X}\mathcal{Y})^2 - (\mathcal{Y}\mathcal{X})^2$$

only differ by an element of  $J$ .

Therefore, the  $(1, 1)$ -entries of the matrices

$$(\mathcal{X}_J \mathcal{Y}_J)^2 - (\mathcal{Y}_J \mathcal{X}_J)^2 - 2(\mathcal{X}_J \mathcal{Y}_J - \mathcal{Y}_J \mathcal{X}_J)$$

and

$$(\mathcal{X} \mathcal{Y})^2 - (\mathcal{Y} \mathcal{X})^2 - 2(\mathcal{X} \mathcal{Y} - \mathcal{Y} \mathcal{X})$$

are congruent to each other modulo  $J$ . Since

$$(\mathcal{X}_J \mathcal{Y}_J)^2 - (\mathcal{Y}_J \mathcal{X}_J)^2 - 2(\mathcal{X}_J \mathcal{Y}_J - \mathcal{Y}_J \mathcal{X}_J)$$

must be a matrix in  $Mat_5(J)$ , and because the  $(1, 1)$ -entry of

$$(\mathcal{X} \mathcal{Y})^2 - (\mathcal{Y} \mathcal{X})^2 - 2(\mathcal{X} \mathcal{Y} - \mathcal{Y} \mathcal{X})$$

is congruent to  $q$  modulo  $N$  and  $N \subseteq J$ , it follows that  $q$  must lie in  $J$  for all  $J$  satisfying the conditions (a) and (b). Hence  $q$  must lie in  $\widehat{I}_{\ell+1}$ .

We obtain that  $(\mathcal{X}, \mathcal{Y})$  defines a lift of  $V$  over  $\widehat{S}/(s^2 - 2s, t^2 - 2t, q)$ . Moreover, we know that  $(s^2 - 2s, t^2 - 2t, q) + m_{\widehat{S}} * \widehat{I}_\ell + \widehat{I}_\ell * m_{\widehat{S}}$  is the minimal ideal satisfying the conditions (a) and (b). Therefore,

$$\widehat{I}_{\ell+1} = (s^2 - 2s, t^2 - 2t, q) + m_{\widehat{S}} * \widehat{I}_\ell + \widehat{I}_\ell * m_{\widehat{S}}, \quad (4.24)$$

and we may take  $\mathcal{X}_{\ell+1} = \mathcal{X}$  and  $\mathcal{Y}_{\ell+1} = \mathcal{Y}$ , completing the induction step.

Notice that  $\widehat{I}_2 = \mu + m_{\widehat{S}}^2 = (s^2 - 2s, t^2 - 2t, q) + \mu + m_{\widehat{S}}^2$ . If we suppose inductively that  $\widehat{I}_i = (s^2 - 2s, t^2 - 2t, q) + \mu m_{\widehat{S}}^{i-2} + m_{\widehat{S}}^i$  for  $2 \leq i \leq \ell$ , then applying

equation (4.24) yields

$$\begin{aligned}
\widehat{I}_{\ell+1} &= (s^2 - 2s, t^2 - 2t, q) + m_{\widehat{S}} * \widehat{I}_\ell + \widehat{I}_\ell * m_{\widehat{S}} \\
&= (s^2 - 2s, t^2 - 2t, q) + m_{\widehat{S}} * ((s^2 - 2s, t^2 - 2t, q) + \mu m_{\widehat{S}}^{\ell-2} + m_{\widehat{S}}^\ell) \\
&\quad + ((s^2 - 2s, t^2 - 2t, q) + \mu m_{\widehat{S}}^{\ell-2} + m_{\widehat{S}}^\ell) * m_{\widehat{S}} \\
&= (s^2 - 2s, t^2 - 2t, q) + \mu m_{\widehat{S}}^{\ell-1} + m_{\widehat{S}}^{\ell+1},
\end{aligned}$$

completing the induction. Therefore,

$$\widehat{I}_\ell = (s^2 - 2s, t^2 - 2t, q) + \mu m_{\widehat{S}}^{\ell-2} + m_{\widehat{S}}^\ell \quad (4.25)$$

for all  $\ell \geq 2$ . We know from Theorem 3.16 that the versal deformation ring  $R(\widehat{\Omega}_2, V)$  of  $V$  is isomorphic to  $\widehat{R} = \widehat{S}/\widehat{I}$ , where  $\widehat{I} = \bigcap_{\ell \geq 2} \widehat{I}_\ell$ . Let  $\widehat{L} = (s^2 - 2s, t^2 - 2t, q)$ ,  $\widehat{T} = \widehat{S}/\widehat{L}$ , and consider the natural projection  $\widehat{p}: \widehat{S} \rightarrow \widehat{S}/\widehat{L} = \widehat{T}$ . Then, for all  $\ell \geq 2$ ,  $\widehat{p}$  maps  $\widehat{L} + \mu m_{\widehat{S}}^{\ell-2} + m_{\widehat{S}}^\ell$  into  $\mu m_{\widehat{T}}^{\ell-2} + m_{\widehat{T}}^\ell$ . Therefore,  $\bigcap_{\ell \geq 2} (\widehat{L} + \mu m_{\widehat{S}}^{\ell-2} + m_{\widehat{S}}^\ell)$  is sent by  $\widehat{p}$  into the set

$$\bigcap_{\ell \geq 2} (\mu m_{\widehat{T}}^{\ell-2} + m_{\widehat{T}}^\ell) \subseteq \bigcap_{\ell \geq 2} m_{\widehat{T}}^{\ell-1} = \{0_{\widehat{T}}\}.$$

Applying equation (4.25), this implies that

$$\widehat{I} = \bigcap_{\ell \geq 2} \widehat{I}_\ell = \bigcap_{\ell \geq 2} (\widehat{L} + \mu m_{\widehat{S}}^{\ell-2} + m_{\widehat{S}}^\ell) = \widehat{L} = (s^2 - 2s, t^2 - 2t, q).$$

In particular,  $R(\widehat{\Omega}_2, V) = R(\widehat{\Omega}_2, \widehat{V}_2) \cong \widehat{\Omega}_2$ , proving Theorem 4.6.  $\square$



Additionally, we have

$$\begin{aligned} (X_\ell, Y_\ell) &= (\mathcal{X}_\ell \bmod \widehat{I}_\ell, \mathcal{Y}_\ell \bmod \widehat{I}_\ell) \\ &= (\mathcal{X} \bmod \widehat{I}_\ell, \mathcal{Y} \bmod \widehat{I}_\ell) \end{aligned}$$

for all  $\ell$ . Therefore, it follows from the construction of a versal deformation in the proof of Theorem 3.16 that the versal deformation  $[\phi_u]$  of  $V = \widehat{V}_2$  over  $\widehat{S}/\widehat{I}$  corresponds to the lift  $\phi_u$  with

$$(\phi_u(\widehat{x}), \phi_u(\widehat{y})) = (\mathcal{X} \bmod \widehat{I}, \mathcal{Y} \bmod \widehat{I}).$$

## REFERENCES

- [1] K. Ardakov and K. A. Brown. Ring-Theoretic Properties of Iwasawa Algebras: A Survey. *Doc. Math.*, (Extra Vol.):7–33, 2006.
- [2] F.M. Bleher. Universal Deformation Rings and Dihedral 2-Groups. *J. Lond. Math. Soc. (2)*, 79(1):225–237, 2009.
- [3] F.M. Bleher and J.A. Vélez-Marulanda. Deformations of Complexes for Finite Dimensional Algebras. Preprint, ArXiv:1511.08081, 2015.
- [4] T. Y. Lam. *A First Course in Noncommutative Rings*, volume 131 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.
- [5] S. Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.
- [6] O.A. Laudal. Noncommutative Deformations of Modules. *Homology Homotopy Appl.*, 4(2, part 2):357–396, 2002. The Roos Festschrift volume, 2.
- [7] H. Matsumura. *Commutative Ring Theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Translated from the Japanese by M. Reid.
- [8] B. Mazur. Deforming Galois Representations. In *Galois Groups over  $\mathbf{Q}$  (Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 385–437. Springer, New York, 1989.
- [9] B. Mazur. An Introduction to the Deformation Theory of Galois Representations. In *Modular Forms and Fermat’s Last Theorem (Boston, MA, 1995)*, pages 243–311. Springer, New York, 1997.
- [10] M. Schlessinger. Functors of Artin Rings. *Trans. Amer. Math. Soc.*, 130:208–222, 1968.
- [11] E. Sernesi. *Deformations of Algebraic Schemes*, volume 334 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.
- [12] J.A. Vélez-Marulanda. *Universal Deformation Rings of Modules over Self-Injective Algebras*. ProQuest LLC, Ann Arbor, MI, 2010. Thesis (Ph.D.)—The University of Iowa.