Spring 2016

Non-commutative deformation rings

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NON-COMMUTATIVE DEFORMATION RINGS

by

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A thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

May 2016

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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ACKNOWLEDGEMENTS

Existence and uniqueness are more than abstract mathematical concepts. They are human concepts as well. While this thesis proves powerful mathematical results, it is perhaps more significant to note that this thesis is proof of the power of family, persistence, and the notion that unsolved problems are not necessarily ones that are impossible to solve.

While this thesis was physically written by me, I am a mere asterisk in the list of its contributors. For existence, uniqueness, and everything in between, I thank my parents, Mom and Dad, whose support is one of the confounding and wonderful manifestations of the infinite. Not of any lesser cardinality, I want to thank my sisters, Lisa and Becky, for always making sure I knew that I was never alone in solving the problems of life.

To my wife, Adri, you are the reason I went back to school. Far more importantly, however, you are proof that dreams are worth dreaming, and for that I am forever thankful. I love you.

I owe Professor Dan Anderson a very special kind of gratitude, as he provided me with a solution to the problem of second chances. During a time when my lifestyle might have been converging to unenviable values, you opened up a neighborhood of knowledge, support and trust which allowed me to redefine myself in ways which I had once forgotten existed. None of this would have happened without your willingness to take a chance on me. Thank you.

To Professor Frauke Bleher, what can I begin to say. That is a sentence which expresses a statement of fact, so please do not try and amend the punctuation, as you have made sufficiently many corrections to my thesis, already, all of which I am thankful for. Without your knowledge and expertise, I am not ashamed to say that my fingerprint on this subject would not have found the light of day. Your willingness to take me on as a student in trying times–for multiple reasons–was both motivating and humbling. You are an amazing advisor, an amazing supporter and defender of my work and my life interests, but above all you are an amazing teacher. I express profound gratitude to you for the time I have been able to learn from you, and I thank you for always having my best interests at heart.

At the end of the day, this thesis is dedicated to the subject of Mathematics. Whenever a mathematician is asked what usefulness his or her project or work provides, may he or she confidently respond that it likely addresses either a current intellectual need or is in advance of a future one, and perhaps even provides its inventor with a non-zero amount of happiness. What else could we hope for?

ABSTRACT

The goal of this thesis is to study non-commutative deformation rings of representations of algebras. The main motivation is to provide a generalization of the deformation theory over commutative local rings studied by B. Mazur, M. Schlessinger and others. The latter deformation theory has played an important role in number theory, and in particular in the proof of Fermat's Last Theorem.

The thesis is divided into two parts.

In the first part, A is an arbitrary λ-algebra for a complete local commutative Noetherian ring λ with residue field k. A category $\hat{\mathcal{C}}$ is defined whose objects are complete local λ-algebras R with residue field k such that R is a quotient ring of a power series algebra over λ in finitely many non-commuting variables. If V is a finite dimensional k-vector space that is also a left A-module and that satisfies a natural finiteness condition, it is proved that V has a so-called versal deformation ring $R(A,V)$. More precisely, $R(A,V)$ is an object in $\hat{\mathcal{C}}$ such that the isomorphism class of every lift of V over an object R in $\hat{\mathcal{C}}$ arises from a morphism $\alpha : R(A,V) \to R$ in $\hat{\mathcal{C}}$ and α is unique if R is the ring of dual numbers $k[\epsilon]$.

In the second part, two particular examples of λ, A and V are studied and the versal deformation ring $R(A,V)$ is determined in each of these cases. In the first example, $\lambda = k$, A is a series of non-commutative k-algebras depending on a parameter $r \geq 2$, and V is a particular quotient module of A; it is shown that $R(A,V)$ is isomorphic to A. The second example builds on the first example when

$r = 2$ and uses that, if additionally the characteristic of k is 2, then A is isomorphic to the group ring $k[D_8]$ of a dihedral group D_8 of order 8. It is shown that if k is perfect and W is the ring of infinite Witt vectors over k, then $R(W[D_8], V)$ is isomorphic to $W[D_8]$.

PUBLIC ABSTRACT

The goal of this thesis is to study non-commutative deformation rings of representations of algebras. The main motivation is to provide a generalization of the deformation theory over commutative local rings studied by B. Mazur, M. Schlessinger and others. The latter deformation theory has played an important role in number theory, and in particular in the proof of Fermat's Last Theorem.

The thesis is divided into two parts.

In the first part, a theory of deformations of representations of an algebra A over a certain class of complete local rings with a fixed residue field k is developed. It is shown that every finite dimensional k-vector space V that is also a left A-module and that satisfies a natural finiteness condition has a so-called versal deformation ring $R(A, V)$, which is such a complete local ring with residue field k.

In the second part, two particular examples of non-commutative algebras A and modules V are studied and the versal deformation ring $R(A, V)$ is shown to be isomorphic to A in each of these cases.

TABLE OF CONTENTS

LIST OF FIGURES

Figure

CHAPTER 1
INTRODUCTION

1.1 Motivation

In this thesis, we study non-commutative deformation rings of representations of algebras. Our main motivation is to provide a generalization of the deformation theory over commutative local rings studied by Mazur [8], Schlessinger [10] and others. The latter deformation theory has played an important role in number theory, and in particular in the proof of Fermat's Last Theorem. For a related approach to non-commutative deformation theory, see [6].

Let k be a field, let λ be a complete local commutative Noetherian ring with residue field k, and let A be an arbitrary λ-algebra. Suppose V is a finite dimensional vector space over k such that V is also a left A-module.

The first goal of this thesis is to prove that if V satisfies a natural finiteness condition, then V has a versal deformation ring $R(A,V)$ that is a complete local λ-algebra with residue field k such that the maximal ideal of $R(A,V)$ modulo its square is finitely generated as an $R(A,V)$-module. Note that $R(A,V)$ can be expressed as a quotient ring of a power series algebra over λ in finitely many non-commuting variables. The main idea behind the proof of this result is to modify Schlessinger's arguments in [10] so that they can be applied to the current situation. The second goal of this thesis is to apply this non-commutative deformation theory to particular examples of λ, A and V for which the versal deformation ring $R(A,V)$

is non-commutative.

1.2 Overview

In Chapter 2, we provide the necessary background on non-commutative complete local rings with residue field k as used in the remainder of the thesis. In particular, we introduce the category $\hat{\mathcal{C}}$ of all rings over which we later define deformations of V. We prove basic properties about the category $\hat{\mathcal{C}}$ in Theorem 2.8.

In Chapter 3, we define lifts and deformations of V over objects in $\hat{\mathcal{C}}$ and also define versal and universal deformation rings. We prove our main result, Theorem 3.16, which shows that if V satisfies a natural finiteness condition then it has a versal deformation ring in $\hat{\mathcal{C}}$. More precisely, the finiteness condition, which is the same that was used by Schlessinger in [10], is that the set of all deformations of V over the dual numbers $k[\epsilon]$ defines a finite dimensional vector space over k. In Section 3.2, we describe an example of a versal deformation ring of a one-dimensional representation, and we also discuss the connection of non-commutative versal deformation rings to commutative ones.

In Chapter 4, we consider two particular examples of λ, A and V. Namely, in Section 4.1, $\lambda = k$, $r \geq 2$ is an integer, $A = k\langle\langle x, y\rangle\rangle/(x^2, y^2, (xy)^r - (yx)^r)$, and V is a particular quotient module of A of k-dimension $2r+1$. We prove that $R(A, V)$ is isomorphic to A in this case. In Section 4.2, we build on this example when $r = 2$ and the characteristic of k is 2. In this case, the k-algebra A from Section 4.1 is isomorphic to the group ring $k[D_8]$ of a dihedral group D_8 of order 8. We show that if k is perfect and W is the ring of infinite Witt vectors over k, then $R(W[D_8], V)$ is isomorphic to $W[D_8]$.

CHAPTER 2
BACKGROUND

In this chapter, we provide the necessary background on non-commutative complete local rings, as used in the remainder of this thesis. Throughout this thesis, all rings are assumed to be associative and to contain a multiplicative identity. We say a ring R is local if and only if R contains a unique maximal left ideal, which we denote by m_R. We say R is a local ring with residue field k if and only if R is a local ring and R/m_R is isomorphic to a given field k.

Remark 2.1. The above definition of a local ring is the same as used in [4, Section 19]. For a different definition of a local ring, see for example [1, Section 4.1], where our local rings are called scalar local. Note that if R is a local ring in our sense, then the full matrix ring $Mat_n(R)$ is not local in our sense for $n \geq 2$.

The following lemma lists some obvious properties of local rings.

Lemma 2.2. *Let R be a non-zero ring with group of units $U(R)$ and Jacobson radical rad R.*

(a) The following statements are equivalent:

(i) R is local, i.e. R has a unique maximal left ideal m_R.

(ii) R has a unique maximal right ideal.

(iii) $R/radR$ is a division ring.

(iv) $R - U(R)$ is a two-sided ideal of R.

(v) $R - U(R)$ *is a group under addition.*

(b) Let R *be a local ring. Then* R *has a unique maximal two-sided ideal. Moreover,* R *is Dedekind-finite (i.e., if* $a \in R$ *has a left inverse, then* $a \in U(R)$*), and* R *has no nontrivial idempotents.*

(c) Suppose $R \neq 0$*, and every* $a \in R - U(R)$ *is nilpotent, then* R *is a local ring.*

Proof. See [4, Theorem (19.1) and Propositions (19.2) and (19.3)]. □

We next discuss the topology on a ring induced by a two-sided ideal and the corresponding completion of a ring.

Definition 2.3. Let R be a ring and let I be a two-sided ideal of R.

(a) The I-adic topology on R has a fundamental system of neighborhoods of any $r \in R$ given by the cosets of $r + I^\ell$, for all $\ell \geq 0$. The I-adic completion of R is defined to be the inverse limit $\varprojlim_{\ell}(R/m_R^\ell)$. We say R is I-adically complete if and only if the natural map $R \to \varprojlim_{\ell}(R/m_R^\ell)$ is an isomorphism of rings.

(b) If R is a local ring with unique maximal left ideal m_R, we say R is complete if and only if R is m_R-adically complete.

Remark 2.4. Let R be a local ring with unique maximal left ideal m_R. Then R is complete if and only if, with respect to the m_R-adic topology, every Cauchy sequence in R converges and R is Hausdorff. Note that the m_R-adic topology is Hausdorff if and only if $\bigcap_{\ell \geq 0} (m_R)^\ell = 0$.

Lemma 2.5. *Let R be a complete local ring with unique maximal left ideal m_R. Let $n \geq 1$ be an integer. Then $Mat_n(R)$ is complete with respect to the $Mat_n(m_R)$-adic topology.*

Proof. We need to show that the natural map $Mat_n(R) \to \varprojlim_{\ell} Mat_n(R)/Mat_n(m_R)^\ell$ is bijective. Note that $Mat_n(m_R)^\ell = Mat_n(m_R^\ell)$ and

$$Mat_n(R)/Mat_n(m_R^\ell) = Mat_n(R/m_R^\ell)$$

for all $\ell \geq 1$. Consider any sequence $(B_\ell + Mat_n(m_R^\ell))_\ell$ in $\varprojlim_{\ell} Mat_n(R)/Mat_n(m_R^\ell)$. For all $1 \leq i, j, \leq n$, the (i, j)-entries of the matrices B_ℓ, $\ell \geq 1$, result in a sequence $((B_\ell)_{ij} + m_R^\ell)_\ell$ in the inverse limit $\varprojlim_{\ell}(R/m_R^\ell)$. Since R is m_R-adically complete, we can find unique elements a_{ij} in R such that $a_{ij} - (B_\ell)_{ij}$ lies in m_R^ℓ for all $\ell \geq 1$. Therefore, $A = [a_{ij}]$ is the unique matrix in $Mat_n(R)$ such that $A - B_\ell$ lies in $Mat_n(m_R^\ell)$ for all $\ell \geq 1$. This shows that $Mat_n(R)$ is complete with respect to the $Mat_n(m_R)$-adic topology. □

For the remainder of the thesis, we fix the following definitions and notations.

Let k be a field, let λ be a complete, local, commutative ring with residue field isomorphic to k, and let $\mu = \mu_\lambda$ denote the unique maximal ideal of λ. Moreover, assume that μ/μ^2 is finitely generated as a left λ-module, and let $\pi_\lambda : \lambda \to k$ be a fixed surjection from λ onto its residue field. For any ring R, we let $Z(R)$ denote its center. If R is a local λ-algebra with unique maximal left ideal m_R, we let j_R denote the ring homomorphism $j_R : \lambda \to R$ which satisfies $j_R(1_\lambda) = 1_R$, $j_R(\lambda) \subseteq Z(R)$ and which gives R its structure as a λ-algebra. We define $\mu_R = j_R(\mu)$. Finally, we view

R as a topological ring with respect to the m_R-adic topology.

We are now ready to define the category $\hat{\mathcal{C}}$, which plays a key role in this thesis.

Definition 2.6. Let R be a complete, local λ-algebra with residue field isomorphic to k, such that m_R/m_R^2 is finitely generated as a left λ-module, by which we more rigorously mean that it is finitely generated as a left $j_R(\lambda)$-module. Fix a surjective ring homomorphism $\pi_R : R \to k$ from R onto its residue field.

Define the category $\hat{\mathcal{C}}$ as follows: The objects of $\hat{\mathcal{C}}$ are ordered pairs (R, π_R), where R and π_R have the properties described above, and where $\pi_R \circ j_R = \pi_\lambda$, as shown in Figure 2.1.

Figure 2.1: The condition $\pi_R \circ j_R = \pi_\lambda$.

$$\begin{tikzcd}
\lambda \arrow[rr, "j_R"] \arrow[dr, "\pi_\lambda"'] & & R \arrow[dl, "\pi_R"] \\
& k &
\end{tikzcd}$$

If (R, π_R) and (S, π_S) are both objects of $\hat{\mathcal{C}}$, then a $\hat{\mathcal{C}}$-morphism from (R, π_R) to (S, π_S) is defined to be a λ-algebra homomorphism $f : R \to S$ such that $f(1_R) = 1_S$, and $\pi_S \circ f = \pi_R$.

It should be noted that we will often suppress the residue maps when speaking of objects in $\hat{\mathcal{C}}$. Moreover, we denote the natural extension of a morphism $f : R \to S$ in $\hat{\mathcal{C}}$ to a λ-algebra homomorphism $Mat_n(R) \to Mat_n(S)$ also by f.

Remark 2.7. Let R be an object of $\hat{\mathcal{C}}$, and let r be an element of R. Since $\pi_R \circ j_R = \pi_\lambda$ is surjective, there exists some element l in λ such that

$$\pi_R(r) = \pi_\lambda(l) = (\pi_R \circ j_R)(l).$$

Therefore, $r - j_R(l)$ is in $ker(\pi_R) = m_R$, and hence there exists some m in m_R such that $r = j_R(l) + m$. Note that if $s + m_R^2$ is any element of the left R-module m_R/m_R^2, then s is in m_R and

$$r \cdot (s + m_R^2) = (j_R(l) + m) \cdot (s + m_R^2) = j_R(l)s + m_R^2 = j_R(l) \cdot (s + m_R^2).$$

Since $j_R(\lambda) \subseteq Z(R)$, we have

$$j_R(l)s + m_R^2 = sj_R(l) + m_R^2 = (s + m_R^2) \cdot j_R(l) = (s + m_R^2) \cdot r.$$

Hence, m_R/m_R^2 can be spoken about unambiguously as a left/right λ-module or as a left/right R-module, with the left/right module structure arising from left/right multiplication. From the above equations, it is clear that m_R/m_R^2 is finitely generated as a left R-module $\iff$ it is finitely generated as a left λ-module $\iff$ it is finitely generated as a right λ-module $\iff$ it is finitely generated as a right R-module.

We now prove some properties about the category $\hat{\mathcal{C}}$. We use the usual convention that a ring R is called Artinian (resp. Noetherian) if R is both left and right Artinian (resp. Noetherian). Moreover, we define $\lambda\langle\langle x_1, x_2, ..., x_n\rangle\rangle$ to be the ring of all formal power series in the finitely many non-commmuting variables $x_1, x_2, ..., x_n$ with coefficients in λ, where the variables commute with λ.

Theorem 2.8.

(a) For any object R of $\hat{\mathcal{C}}$, R/m_R^ℓ is an Artinian ring for all positive integers ℓ.

(b) If R and S are any two local λ-algebras, then a λ-algebra homomorphism $f : R \to S$ which satisfies $f(1_R) = 1_S$ is continuous with respect to the m_R-adic and m_S-adic topologies if and only if $f(m_R) \subseteq m_S$.

(c) Let R and S be two objects of $\hat{\mathcal{C}}$, and let $f : R \to S$ be a morphism in $\hat{\mathcal{C}}$. Then f is continuous with respect to the m_R-adic and m_S-adic topologies. Moreover, for each R in $\hat{\mathcal{C}}$, the ring homomorphism $j_R : \lambda \to R$, which defines the λ-algebra structure on R, is a morphism in $\hat{\mathcal{C}}$ and is continuous.

(d) If R is an object of $\hat{\mathcal{C}}$, then $\mu_R + m_R^2$ is an ideal of R.

(e) If R is a complete, local, commutative λ-algebra, then R is Noetherian if and only if m_R/m_R^2 is finitely generated as a λ-module.

(f) Let S be any object of $\hat{\mathcal{C}}$ and let I be any ideal of S. Then S/I is an object of $\hat{\mathcal{C}}$ (with the obvious induced morphisms $j_{S/I}$ and $\pi_{S/I}$) if and only if I is a proper, closed ideal.

(g) Let R and S be two objects of $\hat{\mathcal{C}}$, and let $f : R \to S$ be a morphism in $\hat{\mathcal{C}}$. Then $f(R)$ is an object of $\hat{\mathcal{C}}$.

(h) *Let* $S = \lambda\langle\langle x_1, x_2, ..., x_n\rangle\rangle$ *be the power series ring over* λ *in the finitely many non-commuting variables* $x_1, x_2, ..., x_n$*. Then* S *is an object of* $\hat{\mathcal{C}}$*, where* $j_S : \lambda \to S$ *sends* l *in* λ *to the constant power series* l*,* m_S *is the ideal generated by* μ *and* $x_1, x_2, ..., x_n$*, and* $\pi_S : S \to k$ *is the natural surjection which sends a formal power series with constant coefficient* l *in* λ *to* $\pi_\lambda(l)$*.*

(i) *Let* R *be an object of* $\hat{\mathcal{C}}$*. Then there exists an integer* $n \geq 0$ *and a surjective morphism* $\phi : \lambda\langle\langle x_1, ..., x_n\rangle\rangle \to R$ *in* $\hat{\mathcal{C}}$*.*

(j) *Let* R *be an object of* $\hat{\mathcal{C}}$*. Then* $Z(R)$ *is a local* λ*-algebra with residue field* k *and unique maximal ideal* $m_R \cap Z(R)$*. Moreover, we have an isomorphism* $Z(R) \cong \varprojlim_\ell Z(R/m_R^\ell)$ *via the restriction of the natural isomorphism between* R *and* $\varprojlim_\ell (R/m_R^\ell)$*.*

(k) *Let* R *be an object of* $\hat{\mathcal{C}}$ *such that the maximal ideal* m_R *is nilpotent. Then* $Z(R)$ *is an Artinian complete, local* λ*-algebra with residue field* k*. In particular,* $Z(R)$ *is an object of* $\hat{\mathcal{C}}$*.*

(l) *Let* $f : R \to S$ *be a morphism in* $\hat{\mathcal{C}}$*. Then* f *is an isomorphism in* $\hat{\mathcal{C}}$ *if and only if* f *is bijective. Moreover,* f *is surjective if and only if the induced map* $\bar{f} : m_R/(\mu_R + m_R^2) \to m_S/(\mu_S + m_S^2)$ *is surjective. If* $R = S$*, then* f *is bijective if and only if* f *is surjective.*

Proof.

(a) By Remark 2.7, m_R/m_R^2 is an Artinian left/right R-module. Since R is in $\hat{\mathcal{C}}$, $R/m_R \cong k$ and is therefore also an Artinian left/right R-module. We now use induction on short exact sequences of left/right R-modules of the form

$$0 \to m_R^\ell/m_R^{\ell+1} \to R/m_R^{\ell+1} \to R/m_R^\ell \to 0.$$

Using the fact that the middle term is an Artinian left/right R-module if and only if both the left and right terms are Artinian left/right R-modules, proves that R/m_R^ℓ is an Artinian left/right R-module for all positive integers ℓ. This proves part (a).

(b) Let R and S be local λ-algebras and let $f : R \to S$ be a λ-algebra homomorphism with $f(1_R) = 1_S$. Then f is continuous if and only if the inverse image of every open set in S is open in R.

Suppose f is continuous. Then $f^{-1}(m_S)$ is open in R. Since 0_R is in $f^{-1}(m_S)$, it follows that there must exist some open set U in R which contains 0_R, such that $U \subseteq f^{-1}(m_S)$. This means there exists an integer, ℓ, such that $m_R^\ell \subseteq f^{-1}(m_S)$. From this it follows that $f(m_R^\ell) \subseteq m_S$. But since f is a ring homomorphism, $f(m_R^\ell) = (f(m_R))^\ell$. Suppose $f(m_R)$ is not contained in m_S. Then there is some element $x \in m_R$ and some unit $u \in S$ such that $f(x) = u$. But then $f(x^\ell) = (f(x))^\ell = u^\ell$. However, x^ℓ is in m_R^ℓ, which means that $f(x^\ell) = (f(x))^\ell = u^\ell$ is an element of m_S. But u^ℓ is a unit in S, giving a contradiction. Therefore, if f is continuous, $f(m_R) \subseteq m_S$.

Now suppose $f(m_R) \subseteq m_S$. Then $m_R \subseteq f^{-1}(m_S)$. Suppose that there exists some element v of R which is not in m_R, such that $f(v)$ is in m_S. Since R is local, this means that v is a unit. Because we assume that f sends 1_R to 1_S, we then have that $1_S = f(1_R) = f(vv^{-1}) = f(v)f(v^{-1})$, which means that $f(v)$ is a unit in S, contradicting the assumption that $f(v)$ is in m_S. Therefore, $m_R = f^{-1}(m_S)$, showing that $f^{-1}(m_S)$ is open in R. It follows then that $m_R^\ell \subseteq f^{-1}(m_S^\ell)$ for all $\ell \geq 1$. Let U be an arbitrary open set in S, and let $x \in U$ and $\ell_x \geq 1$ such that $x + m_S^{\ell_x} \subseteq U$. If r is any element in $f^{-1}(x + m_S^{\ell_x})$, then $r + m_R^{\ell_x}$ is an open set in R which contains r and which is contained in $f^{-1}(x + m_S^{\ell_x})$. Therefore, $f^{-1}(U)$ is open, and f is continuous.

(c) Let $f : R \to S$ be a morphism in $\hat{\mathcal{C}}$. Then $\pi_S \circ f = \pi_R$, and $\ker(\pi_R) = m_R$ and $\ker(\pi_S) = m_S$. Let a be an element of $m_R = \ker(\pi_R)$. Then

$$0 = \pi_R(a) = (\pi_S \circ f)(a).$$

This implies that $f(a)$ is in $\ker(\pi_S) = m_S$, proving that $f(m_R) \subseteq m_S$, which implies by part (b) that f is continuous with respect to the m_R-adic and m_S-adic topologies.

Notice that for each R in $\hat{\mathcal{C}}$, the ring homomorphism $j_R : \lambda \to R$ satisfies $j_R(1_\lambda) = 1_R$, and by Definition 2.6, $\pi_R \circ j_R = \pi_\lambda$. Hence j_R is a morphism in $\hat{\mathcal{C}}$. Therefore, by the above arguments, j_R is continuous.

(d) Let r be an element of R. By Remark 2.7, there exists some element l in λ and some element m in m_R such that $r = j_R(l) + m$. If x is in $\mu_R + m_R^2$, then there exists some element a in μ and some element w in m_R^2 such that $x = j_R(a) + w$. It follows that

$$rx = (j_R(l) + m)(j_R(a) + w) = j_R(la) + j_R(l)w + mj_R(a) + mw.$$

Since j_R is continuous, $j_R(a)$ is in m_R, and hence $j_R(l)w + mj_R(a) + mw$ is in m_R^2. But since μ is an ideal in λ, la is in μ, and hence $j_R(la)$ is in $j_R(\mu) = \mu_R$, proving that $\mu_R + m_R^2$ is closed under left multiplication in R. Since both μ and m_R^2 are two sided ideals, similar calculations show that $\mu_R + m_R^2$ is also closed under right multiplication in R. Since $\mu_R + m_R^2$ is closed under subtraction and contains 0_R, it follows that $\mu_R + m_R^2$ is an ideal of R.

(e) See [7, Theorem 29.4].

(f) Let S be a ring in $\hat{\mathcal{C}}$. First notice that S/S is the zero-ring which does not have residue field isomorphic to k, and so S/S is not in $\hat{\mathcal{C}}$. Now let I be any proper ideal of S. Letting $j_{S/I} : \lambda \to S/I$ be given by $j_S : \lambda \to S$ followed by the natural projection $S \to S/I$, we see that S/I is a λ-algebra. Since S is local with unique maximal left ideal m_S, the ideal I is contained in m_S. Hence, S/I is local with unique maximal left ideal $m_{S/I} = m_S/I$. Moreover, the residue field of S/I is isomorphic to k, since

$$k \cong S/m_S \cong (S/I)/(m_S/I).$$

Letting $\pi_{S/I} : S/I \to k$ be the surjection induced by π_S, we see that $\pi_{S/I} \circ j_{S/I} = \pi_S \circ j_S = \pi_\lambda$. Notice that, for all $\ell \geq 1$,

$$m_{S/I}^\ell = (m_S/I)^\ell = \frac{m_S^\ell + I}{I}.$$

If

$$b_1 + m_S^2, b_2 + m_S^2, ..., b_m + m_S^2$$

is a finite generating set for the S-module m_S/m_S^2, then

$$(b_1 + I) + m_{S/I}^2, (b_2 + I) + m_{S/I}^2, ..., (b_m + I) + m_{S/I}^2$$

is a finite generating set for the S/I-module $m_{S/I}/m_{S/I}^2$. Therefore, to prove part (f), it suffices to show that S/I is complete with respect to the $m_{S/I}$-adic topology if and only if I is a closed ideal. But S/I will be complete if and only if the natural homomorphism from S/I to the inverse limit of the quotients $(S/I)/m_{S/I}^\ell = (S/I)/((m_S^\ell + I)/I) \cong S/(m_S^\ell + I)$ is a bijection.

For any proper ideal I of S, this natural map will be surjective. To see this, consider any sequence $(s_\ell + (m_S^\ell + I))_\ell$ in the inverse limit of $S/(m_S^\ell + I)$. Then, for all $\ell \geq 1$,

$$s_{\ell+1} - s_\ell \in m_S^\ell + I.$$

Hence, there are elements j_ℓ in I such that $s_{\ell+1} + j_\ell - s_\ell \in m_S^\ell$, for all $\ell \geq 1$. Rewriting this as

$$(s_{\ell+1} + j_\ell + j_{\ell-1} + ... + j_2 + j_1) - (s_\ell + j_{\ell-1} + j_{\ell-2} + ... + j_2 + j_1) \in m_S^\ell,$$

it follows that $((s_\ell + j_\ell + j_{\ell-1} + ... + j_2 + j_1) + m_S^\ell)_\ell$ is an element in the inverse limit $\varprojlim_\ell (S/m_S^\ell)$. However, since S is complete with respect to the m_S-adic topology, this means that there exists a unique element s in S such that

$$s + m_S^\ell = (s_\ell + j_\ell + j_{\ell-1} + ... + j_2 + j_1) + m_S^\ell$$

for all ℓ. But this means that $s - s_\ell$ is in $m_S^\ell + I$ for all ℓ, implying $(s_\ell + (m_S^\ell + I))_\ell =$

$(s + (m_S^\ell + I))_\ell$ and proving that the natural map from S/I to the inverse limit of the quotients $S/(m_S^\ell + I)$ is surjective.

Notice that this means that S/I will be complete if and only if this natural map is injective, which is if and only if $\bigcap_\ell (m_S^\ell + I) = I$. Clearly $I \subseteq \bigcap_\ell (m_S^\ell + I)$. Now suppose s is an element in $\bigcap_\ell (m_S^\ell + I)$. Then, for each ℓ, there exist elements j_ℓ in m_S^ℓ and a_ℓ in I such that $s = j_\ell + a_\ell$, which means $s - j_\ell = a_\ell$ which is in I. However, the sequence $(s - j_\ell)_\ell$ converges to s in the m_S-adic topology. But this implies that the sequence $(a_\ell)_\ell$ is a sequence entirely in I which converges to s. Therefore, if I is closed, s must be in I, which means that $\bigcap_\ell (m_S^\ell + I) = I$, and so S/I is complete. Conversely, if I is not a closed ideal, then there exists a sequence $(a_\ell)_\ell$ of elements entirely in I which converges to an element, x, which is not in I. Then there exist elements j_ℓ in m_S^ℓ such that $x - a_\ell = j_\ell$. Therefore, $x = j_\ell + a_\ell$ is in $m_S^\ell + I$ for all ℓ, which means I is strictly smaller than $\bigcap_\ell (m_S^\ell + I)$, proving that the natural map is not injective. This completes the proof of part (f).

(g) Since $f(R) \cong R/\ker(f)$, our proof will follow from part (f) if we can show that $\ker(f)$ is a proper, closed ideal in R. Since f is a morphism in $\hat{\mathcal{C}}$, $\pi_R = \pi_S \circ f$, and so f cannot be the zero morphism. Therefore, $\ker(f)$ is a proper ideal of R. Moreover, by part (c), f is a continuous morphism, so the inverse image under f of a closed set in S must be a closed set in R. Since S is complete with respect to the m_S-adic topology, S is Hausdorff with respect to this topology, which implies that one-point sets in S are closed. Since $\ker(f)$ is exactly $f^{-1}(\{0_S\})$, the continuity of f proves that $\ker(f)$ is closed in R, and hence $\ker(f)$ is a proper and closed ideal in R.

(h) Define $(x_1, x_2, ..., x_n)$ to be the ideal of $S = \lambda\langle\langle x_1, x_2, ..., x_n\rangle\rangle$ generated by $x_1, ..., x_n$, and define $J = \mu + (x_1, x_2, ..., x_n)$. Then J is an ideal of S, namely the ideal generated by μ and $x_1, ..., x_n$. Let s be any element of S. We can then write s as

$$s = l_0 + \sum_i l_1^i x_i + \sum_{i'',j''} l_2^{i''j''} x_{i''} x_{j''} + \sum_{i''',j''',k'''} l_3^{i'''j'''k'''} x_{i'''} x_{j'''} x_{k'''} + ...$$

where all the indices range from 1 to n, and where the coefficients come from λ. It follows that s is not in J if and only if l_0 is not in μ. Since λ is assumed to be local, l_0 is not in μ if and only if l_0 is a unit in λ. If

$$h = h_0 + \sum h_1^i x_i + \sum h_2^{i''j''} x_{i''} x_{j''} + \sum h_3^{i'''j'''k'''} x_{i'''} x_{j'''} x_{k'''} + ...$$

is another element of S, then $s * h$ is a power series whose coefficient on the term $x_a x_b ... x_c x_d$ is

$$l_0 h_t^{ab...cd} + l_1^a h_{t-1}^{b...cd} + ... + l_{t-1}^{ab...c} h_1^d + l_t^{ab...cd} h_0$$

where t is the number of variables occurring in the monomial $x_a x_b ... x_c x_d$. Therefore, if s is a unit in S, then l_0 must be a unit in λ, and hence s is not in J. Conversely, assume now s is not in J. Since then l_0 is a unit, we can define $h_0 = l_0^{-1}$, and use a simple induction argument to successively solve the equations

$$l_0 h_t^{ab...cd} + l_1^a h_{t-1}^{b...cd} + ... + l_{t-1}^{ab...c} h_1^d + l_t^{ab...cd} h_0 = 0$$

by setting

$$h_t^{ab...cd} = -l_0^{-1}(l_1^a h_{t-1}^{b...cd} + ... + l_{t-1}^{ab...c} h_1^d + l_t^{ab...cd} h_0).$$

This proves that $S - J = U(S)$, or $J = S - U(S)$, and therefore proves that S is a local ring with unique maximal left ideal $m_S = J$.

We next prove that m_S/m_S^2 is finitely generated as a left S-module. Let $a_1 + \mu^2, a_2 + \mu^2, ..., a_m + \mu^2$ be a finite generating set for the left λ-module μ/μ^2. Since any element s of m_S can be written as

$$s = l_0 + \sum l_1^i x_i + \sum l_2^{i''j''} x_{i''} x_{j''} + \sum l_3^{i'''j'''k'''} x_{i'''} x_{j'''} x_{k'''} + ...$$

where l_0 is in μ, it follows that $s + m_S^2$ is just $(l_0 + \sum_{i=1}^{n} l_1^i x_i) + m_S^2$. But since l_0 is in μ,

$$l_0 + \mu^2 = \sum_{j=1}^{m} h_j a_j + \mu^2$$

for certain $h_1, ..., h_m$ in λ. Since

$$m_S^2 = (\mu + (x_1, x_2, ..., x_n))^2 = \mu^2 + \mu * (x_1, x_2, ..., x_n) + (x_1, x_2, ..., x_n)^2,$$

it follows that $s + m_S^2 = (\sum_{j=1}^{m} h_j a_j + \sum_{i=1}^{n} l_i x_i) + m_S^2$. This proves that

$$a_1 + m_S^2, a_2 + m_S^2, ..., a_m + m_S^2, \quad x_1 + m_S^2, x_2 + m_S^2, ..., x_n + m_S^2$$

is a finite generating set for the left S-module m_S/m_S^2.

To prove that S is complete, we show that the natural homomorphism from S to the inverse limit $\varprojlim_{\ell}(S/m_S^{\ell})$ is a bijection. To see injectivity, let

$$s = l_0 + \sum l_1^i x_i + \sum l_2^{i''j''} x_{i''} x_{j''} + \sum l_3^{i'''j'''k'''} x_{i'''} x_{j'''} x_{k'''} + ...$$

be any element of S which goes to 0 in the inverse limit. Then s lies in $\bigcap_\ell m_S^\ell$. Since $m_S = \mu + (x_1, x_2, ..., x_n)$, and because λ commutes with each of the x_i, it follows that, for all $\ell \geq 1$,

$$\begin{aligned} m_S^\ell &= \mu^\ell + \mu^{\ell-1}(x_1, x_2, ..., x_n) + \mu^{\ell-2}(x_1, x_2, ..., x_n)^2 \\ &+ \ldots + \mu(x_1, x_2, ..., x_n)^{\ell-1} + (x_1, x_2, ..., x_n)^\ell. \end{aligned}$$

Therefore, if s as above lies in $\bigcap_\ell m_S^\ell$, then it follows that l_0 and each of the coefficients $l_1^i, l_2^{i''j''}, l_3^{i'''j'''k'''}, ...$ lie in $\bigcap_\ell \mu^\ell$. By completeness of λ, this means that all the coefficients of s are 0, proving that $s = 0$.

To see that the natural map $S \to \varprojlim_\ell (S/m_S^\ell)$ is surjective, consider any element

$$y = (a_0 + m_S, \quad (b_0 + \sum b_i x_i) + m_S^2, \quad (c_0 + \sum c_1^i x_i + \sum c_2^{i''j''} x_{i''} x_{j''}) + m_S^3, ...)$$

in the inverse limit $\varprojlim_\ell (S/m_S^\ell)$. Because of our description of m_S^ℓ as a sum of products of the form $\mu^{\ell-j}(x_1, ..., x_n)^j$ for $0 \leq j \leq \ell$, it follows that y gives rise to elements

$$(a_0 + \mu, b_0 + \mu^2, c_0 + \mu^3, ...)$$

$$(0 + \mu, b_1^i + \mu^2, c_1^i + \mu^3, ...), 1 \leq i \leq n,$$

$$(0 + \mu, 0 + \mu^2, c_2^{i''j''} + \mu^3, ...), 1 \leq i'', j'' \leq n,$$

$$\vdots$$

in the inverse limit $\varprojlim_\ell (\lambda/\mu^\ell)$. But since λ is complete, all these sequences converge to elements in λ. This means that our original element y is the image of the formal power series in S whose coefficients are given by these elements, completing the proof that S is complete. Using j_S and π_S as in the statement, this implies part (h).

(i) Since R is in $\hat{\mathcal{C}}$, there exist finitely many elements $b_1, b_2, ..., b_n$ in m_R such that $\{b_i + m_R^2 | 1 \leq i \leq n\}$ is a finite generating set for the left R-module m_R/m_R^2. Let $S = \lambda\langle\langle x_1, x_2, ..., x_n\rangle\rangle$, and let ϕ be the unique λ-algebra homomorphism

$$\phi : S = \lambda\langle\langle x_1, x_2, ..., x_n\rangle\rangle \to R$$

given by $\phi(x_i) = b_i$ for $i = 1, ..., n$. Notice that this means that for all l in λ,

$$\phi(l) = \phi(j_S(l)) = j_R(l).$$

It follows that $\phi(1_S) = 1_R$, and $\phi(m_S) \subseteq m_R$, since $\phi(x_i) = b_i \in m_R$ and $\phi(\mu) = j_R(\mu) \subseteq m_R$. Therefore, writing $s \in S$ as $s = l_0 + m$ for some element l_0 in λ and some element m in m_S, we obtain

$$\begin{aligned} (\pi_R \circ \phi)(s) &= \pi_R(\phi(l_0) + \phi(m)) = \pi_R(\phi(l_0)) = \pi_R(j_R(l_0)) \\ &= \pi_\lambda(l_0) = \pi_S(j_S(l_0)) = \pi_S(l_0) = \pi_S(l_0 + m) \\ &= \pi_S(s). \end{aligned}$$

Hence, ϕ is a morphism in $\hat{\mathcal{C}}$, which means that to prove part (i), it suffices to prove that ϕ is surjective. Since $\phi \circ j_S = j_R$, it follows from Remark 2.7 that we only need to prove that every element in m_R lies in the image of ϕ.

Let now q be any element of m_R. Then there exist elements $r_1, r_2, ..., r_n$ in $j_R(\lambda)$ such that

$$q \equiv \sum_{i=1}^{n} (r_i b_i) \mod m_R^2.$$

Suppose inductively that for $\ell \geq 1$,

$$q \equiv \sum_{i_1} (r^1_{i_1} b_{i_1}) + \sum_{i_1,i_2} (r^2_{i_1 i_2} b_{i_1} b_{i_2}) + ... + \sum_{i_1,i_2,...,i_\ell} (r^\ell_{i_1 i_2 ... i_\ell} b_{i_1} b_{i_2} ... b_{i_\ell}) \mod m_R^{\ell+1}$$

where all the coefficients $r^1_{i_1}, r^2_{i_1 i_2}, ..., r^\ell_{i_1 i_2 ... i_\ell}$ lie in $j_R(\lambda)$. This means that there exists an element x in $m_R^{\ell+1}$ so that

$$q = x + \sum (r^1_{i_1} b_{i_1}) + \sum (r^2_{i_1 i_2} b_{i_1} b_{i_2}) + ... + \sum (r^\ell_{i_1 i_2 ... i_\ell} b_{i_1} b_{i_2} ... b_{i_\ell}).$$

But x is just a finite sum of products of $(\ell + 1)$ elements in m_R, and any such product can be written as $s_1 s_2 ... s_{\ell+1}$ where all the s_i are in m_R. Therefore, for any $1 \leq i \leq \ell + 1$, there exist elements $t^i_1, t^i_2, ..., t^i_n$ in $j_R(\lambda)$ and an element y_i in m_R^2 so that

$$s_i = \sum_{j=1}^{n} (t^i_j b_j) + y_i.$$

But then it follows that each product

$$s_1 s_2 ... s_{\ell+1} \equiv \sum_{i_1,i_2,...,i_{\ell+1}} (w^{\ell+1}_{i_1 i_2 ... i_{\ell+1}} b_{i_1} b_{i_2} ... b_{i_{\ell+1}}) \mod m_R^{\ell+2},$$

where $w^{\ell+1}_{i_1 i_2 ... i_{\ell+1}}$ lies in $j_R(\lambda)$. But this implies

$$q \equiv \sum (r^1_{i_1} b_{i_1}) + \sum (r^2_{i_1 i_2} b_{i_1} b_{i_2}) + ... + \sum (r^{\ell+1}_{i_1 i_2 ... i_\ell} b_{i_1} b_{i_2} ... b_{i_{\ell+1}}) \mod m_R^{\ell+2},$$

where all the coefficients lie in $j_R(\lambda)$, completing the induction. In this way, we form a power series in the elements $b_1, b_2, ..., b_n$ over $j_R(\lambda)$. Since R is assumed to be complete with respect to the m_R-adic topology, this power series converges to q in R. Since $\phi \circ j_S = j_R$, it follows that there exists a power series p in $S = \lambda\langle\langle x_1, ..., x_n\rangle\rangle$ such that $\phi(p) = q$. In other words, ϕ is surjective, which completes the proof of part (i).

(j) Let R be an object in $\hat{\mathcal{C}}$, and let $Z = Z(R)$. If z is any non-unit in Z, then z is a non-unit in R, and hence lies in m_R. It follows that Z is a commutative local ring with unique maximal ideal $m_Z = m_R \cap Z$. Moreover, since $j_R(\lambda) \subseteq Z$ and since $\pi_R(R) = \pi_\lambda(\lambda) = (\pi_R \circ j_R)(\lambda)$, it follows that $Z/(m_R \cap Z) \cong (Z + m_R)/m_R \cong k$. In other words, Z is a local λ-algebra with residue field k.

To show that $Z \cong \varprojlim_\ell Z(R/m_R^\ell)$, let $\pi_\ell : R \to R/m_R^\ell$ be the natural surjection, for all $\ell \geq 1$. Since R is an object in $\hat{\mathcal{C}}$, the natural map $\nu : R \to \varprojlim_\ell (R/m_R^\ell)$ defined by $\nu(r) = (\pi_\ell(r))_\ell$ is an isomorphism. Note that since each π_ℓ is surjective, it follows for any element z in Z that $\pi_\ell(z)$ is an element in $Z(R/m_R^\ell)$. Therefore, the image of ν restricted to Z lies in $\varprojlim_\ell Z(R/m_R^\ell)$. Since ν is injective, so is this restriction. Now suppose $(z_\ell + m_R^\ell)_\ell$ is an element of $\varprojlim_\ell Z(R/m_R^\ell)$. Then $(z_\ell + m_R^\ell)_\ell$ is also an element of $\varprojlim_\ell (R/m_R^\ell)$, and so the surjectivity of ν implies the existence of some z in R such that $z - z_\ell$ lies in m_R^ℓ for all $\ell \geq 1$. Then, if r is any element of R,

$$rz + m_R^\ell = rz_\ell + m_R^\ell = z_\ell r + m_R^\ell = zr + m_R^\ell,$$

for all $\ell \geq 1$, and hence $rz - zr \in \bigcap_\ell m_R^\ell = 0$, which shows that z is in Z. Therefore, letting ν' denote the restriction of ν to Z, we see that ν' is an isomorphism between

Z and $\varprojlim_{\ell} Z(R/m_R^\ell)$.

(k) Let R be an object of $\hat{\mathcal{C}}$ such that m_R is nilpotent. Then there exists some positive integer, ℓ, such that $m_R^\ell = 0$. Let $Z = Z(R)$. We have seen in part (j) that Z is a local λ-algebra with residue field k and unique maximal ideal $m_Z = m_R \cap Z$. For each $i \geq 1$,

$$(Z \cap m_R^i)/(Z \cap m_R^{i+1}) = (Z \cap m_R^i)/((Z \cap m_R^i) \cap m_R^{i+1}) \cong ((Z \cap m_R^i) + m_R^{i+1})/m_R^{i+1}.$$

Since m_R^i/m_R^{i+1} is a finitely generated R-module and since

$$m_R * (m_R^i/m_R^{i+1}) = 0,$$

m_R^i/m_R^{i+1} is a finite dimensional k-vector space. Notice that $((Z \cap m_R^i) + m_R^{i+1})/m_R^{i+1}$ is a Z-module which is annihilated by m_R, which implies that $((Z \cap m_R^i) + m_R^{i+1})/m_R^{i+1}$ is also a k-vector space. Since $((Z \cap m_R^i) + m_R^{i+1})/m_R^{i+1} \subseteq m_R^i/m_R^{i+1}$, and since the action of k on both of these spaces is induced by the action of $j_R(\lambda) \subseteq Z(R)$, it follows that $((Z \cap m_R^i) + m_R^{i+1})/m_R^{i+1}$ must also be a finite dimensional k-vector space. Note that each $Z \cap m_R^i$ is an ideal of Z, and $Z \cap m_R^\ell = 0$. Therefore, the filtration of ideals

$$Z \supseteq Z \cap m_R \supseteq Z \cap m_R^2 \supseteq \ldots \supseteq Z \cap m_R^\ell = 0$$

gives rise to a finite composition series of Z as a Z-module, and hence Z is a commutative Artinian ring. This forces m_Z/m_Z^2 to be a finitely generated left λ-module. Since $m_R^\ell = 0$, it follows that $m_Z^\ell \subseteq Z \cap m_R^\ell = 0$. Therefore, the natural map $Z \to \varprojlim_{i}(Z/m_Z^i)$ is a bijection. In particular, Z satisfies all the conditions necessary to be an object of $\hat{\mathcal{C}}$.

(l) The proof of part (l) follows closely the proof of [10, Lemma 1.1]. Let $f : R \to S$ be a morphism in $\hat{\mathcal{C}}$. If f is an isomorphism in $\hat{\mathcal{C}}$, then f is certainly bijective. Conversely, if f is bijective, then $f(m_R) = m_S$, which implies that f^{-1} is also a morphism in $\hat{\mathcal{C}}$.

Let $\bar{f} : m_R/(\mu_R + m_R^2) \to m_S/(\mu_S + m_S^2)$ be the induced map by f. Notice that $f(\mu_R + m_R^2) \subseteq \mu_S + m_S^2$ and $f(m_R) \subseteq m_S$, so that $\bar{f}$ is well-defined. If f is surjective, then $f(m_R) = m_S$, and hence $\bar{f}$ is surjective.

Now suppose conversely that $\bar{f}$ is surjective. Notice that for each object T of $\hat{\mathcal{C}}$, the map

$$\beta_T : \mu/\mu^2 \to \mu_T/(m_T^2 \cap \mu_T),$$

given by $\beta_T(l + \mu^2) = j_T(l) + (m_T^2 \cap m_T)$, for each l in μ, is surjective. We have a commutative diagram of k-linear maps with exact rows as in Figure 2.2, where $\tilde{f}_1$ and $\tilde{\tilde{f}}_1$ are induced from f.

Figure 2.2: The commutative diagram with $\overline{f}$, $\tilde{f}_1$ and $\tilde{\tilde{f}}_1$.

$$\begin{tikzcd}
0 \arrow[r] & \mu_R/(m_R^2 \cap \mu_R) \arrow[r] \arrow[d, "\tilde{\tilde{f}}_1"] & m_R/m_R^2 \arrow[r] \arrow[d, "\tilde{f}_1"] & m_R/(\mu_R + m_R^2) \arrow[r] \arrow[d, "\bar{f}"] & 0 \\
0 \arrow[r] & \mu_S/(m_S^2 \cap \mu_S) \arrow[r] & m_S/m_S^2 \arrow[r] & m_S/(\mu_S + m_S^2) \arrow[r] & 0
\end{tikzcd}$$

Since β_S is surjective and f is a λ-algebra homomorphism, it follows that $\tilde{\tilde{f}}_1$

is surjective. Since $\bar{f}$ is surjective, it follows that $\tilde{f}_1 : m_R/m_R^2 \to m_S/m_S^2$ is surjective. By Remark 2.7 and since $\pi_S \circ f = \pi_R$, it follows that $\bar{f}_2 : R/m_R^2 \to S/m_S^2$, given by $\bar{f}_2(r + m_R^2) = f(r) + m_S^2$, for all $r \in R$, is surjective. Moreover, if $i \geq 2$ and $y_1, ..., y_i$ are elements in m_S, then there exist elements $x_1, ..., x_i \in m_R$ so that $y_j - f(x_j) \in m_S^2$, for all $1 \leq j \leq i$. But this implies that $y_1 ... y_i - f(x_1 ... x_i) \in m_S^{i+1}$, proving that the map $\tilde{f}_i : m_R^i/m_R^{i+1} \to m_S^i/m_S^{i+1}$ induced from f is surjective for all $i \geq 1$. We obtain, for all $i \geq 1$, a commutative diagram with exact rows as in Figure 2.3, where $\bar{f}_i$ and $\bar{f}_{i+1}$ are induced from f.

Figure 2.3: The commutative diagram with $\tilde{f}_i$, $\overline{f}_{i+1}$ and $\overline{f}_i$.

$$\begin{tikzcd}
0 \arrow[r] & m_R^i/m_R^{i+1} \arrow[r] \arrow[d, "\tilde{f}_i"] & R/m_R^{i+1} \arrow[r] \arrow[d, "\bar{f}_{i+1}"] & R/m_R^i \arrow[r] \arrow[d, "\bar{f}_i"] & 0 \\
0 \arrow[r] & m_S^i/m_S^{i+1} \arrow[r] & S/m_S^{i+1} \arrow[r] & S/m_S^i \arrow[r] & 0
\end{tikzcd}$$

Hence we obtain inductively that for all $\ell \geq 1$, $\bar{f}_\ell : R/m_R^\ell \to S/m_S^\ell$, given by $\bar{f}_\ell(r + m_R^\ell) = f(r) + m_S^\ell$, for all $r \in R$, is surjective. Since R is isomorphic to $\varprojlim_\ell (R/m_R^\ell)$ and S is isomorphic to $\varprojlim_\ell (S/m_S^\ell)$, this implies that f is surjective.

Finally, suppose $R = S$ and f is surjective. Then for all $i \geq 1$, the induced map $\tilde{f}_i : m_R^i/m_R^{i+1} \to m_R^i/m_R^{i+1}$ is surjective. Since m_R^i/m_R^{i+1} is a finite dimensional k-vector space and $\tilde{f}_i$ is k-linear, it follows that $\tilde{f}_i$ is bijective for all $i \geq 1$.

Using induction and the commutative diagram in Figure 2.3, it follows that $\bar{f}_\ell : R/m_R^\ell \to R/m_R^\ell$ is bijective for all $\ell \geq 1$, since $\bar{f}_1$ is the identity morphism of R/m_R. But this means that if r lies in $\ker(f)$ then $r + m_R^\ell$ lies in $\ker(\bar{f}_\ell)$ for all ℓ, which implies that $r \in \bigcap_\ell m_R^\ell$. Since R is m_R-adically complete, it follows that $r = 0$, which means that f is bijective.

□

Remark 2.9. Let R be an object of $\hat{\mathcal{C}}$ and let $Z = Z(R)$ be its center. We have proved in Theorem 2.8, part (j), that Z is a local ring with residue field k and unique maximal ideal $m_Z = m_R \cap Z$. Moreover, $Z \cong \varprojlim_\ell Z(R/m_R^\ell)$. Since by Theorem 2.8, part (k), $Z(R/m_R^\ell)$ is an Artinian ring in $\hat{\mathcal{C}}$ for all $\ell \geq 1$, it follows that Z is isomorphic to the inverse limit of complete local Artinian rings in $\hat{\mathcal{C}}$. However, we do not know whether Z is an object of $\hat{\mathcal{C}}$ in general. The main issue is that $m_Z^2 = (m_R \cap Z)^2$ is usually much smaller than $m_R^2 \cap Z$, which makes it hard to decide whether m_Z/m_Z^2 is a finitely generated λ-module in general.

Definition 2.10. Let $\mathcal{C}$ denote the full subcategory of $\hat{\mathcal{C}}$ consisting of all λ-algebras in $\hat{\mathcal{C}}$ whose maximal ideal is nilpotent. Let $\hat{\mathcal{C}}_{comm}$ and $\mathcal{C}_{comm}$ denote the full subcategories of $\hat{\mathcal{C}}$ and $\mathcal{C}$, respectively, which consist exclusively of commutative rings.

The following result follows from Theorem 2.8.

Lemma 2.11. *The category $\mathcal{C}$ is the same as the full subcategory of $\hat{\mathcal{C}}$ consisting exclusively of Artinian rings. The category $\hat{\mathcal{C}}_{comm}$ (resp. $\mathcal{C}_{comm}$) is the same as the category of all complete local commutative Noetherian (resp. Artinian) λ-algebras* *with residue field k whose morphisms are local λ-algebra homomorphisms inducing the identity on k.*

Proof. We first consider the category $\mathcal{C}$. Let R be an object of $\hat{\mathcal{C}}$. By Theorem 2.8, part (a), we know that R/m_R^ℓ is Artinian for all $\ell \geq 1$. Hence it follows that if R is in $\mathcal{C}$, then R is Artinian. On the other hand, if R is Artinian then the decreasing chain of ideals

$$m_R \supseteq m_R^2 \supseteq ... \supseteq m_R^\ell \supseteq ...$$

must eventually stabilize. Since R is Artinian, R is Noetherian, and hence m_R^ℓ is finitely generated as both a left and a right R-module, for all $\ell \geq 1$. Since these powers eventually stabilize, there exists a positive integer t such that $m_R * (m_R^t) = m_R^t$. We can then use Nakayama's Lemma to conclude that $m_R^t = 0$, and hence m_R is nilpotent. Therefore, an object R of $\hat{\mathcal{C}}$ belongs to $\mathcal{C}$ if and only if R is Artinian.

The statements about $\hat{\mathcal{C}}_{comm}$ and $\mathcal{C}_{comm}$, respectively, follow now from Theorem 2.8, part (e), using that if (R, π_R) and (S, π_S) are commutative Noetherian rings in $\hat{\mathcal{C}}$, then a local λ-algebra homomorphism from R to S inducing the identity on k is, by definition, a λ-algebra homomorphism $f : R \to S$ such that $f(1_R) = 1_S$ and $\pi_S \circ f = \pi_R$. □

We conclude this chapter by discussing the ring of dual numbers, denoted by $k[\epsilon]$, which plays a vital part in our future discussion.

Definition 2.12. The ring of dual numbers, $k[\epsilon]$, is defined to be the ring $k[[x]]/(x^2)$, where we identify ϵ with the image of x in $k[[x]]/(x^2)$. Specifically,

$$k[\epsilon] = \{a + b\epsilon | a, b \in k\},$$

where for all $a_1, a_2, b_1, b_2 \in k$,

$$(a_1 + b_1\epsilon) + (a_2 + b_2\epsilon) = (a_1 + a_2) + (b_1 + b_2)\epsilon$$

and

$$(a_1 + b_1\epsilon)(a_2 + b_2\epsilon) = (a_1a_2) + (a_1b_2 + a_2b_1)\epsilon.$$

Remark 2.13. Notice that $k[\epsilon]$ can be viewed as an object of $\hat{\mathcal{C}}$, where $m_{k[\epsilon]} = \epsilon k[\epsilon] = \{b\epsilon | b \in k\}$ and $\pi_{k[\epsilon]} : k[\epsilon] \to k$ is the natural surjection given by $\pi_{k[\epsilon]}(a + b\epsilon) = a$ for all $a, b \in k$. Additionally, the λ-algebra structure on $k[\epsilon]$ is given by $j_{k[\epsilon]} : \lambda \to k[\epsilon]$ defined by $j_{k[\epsilon]}(l) = \pi_\lambda(l) + 0\epsilon$ for all $l \in \lambda$.

CHAPTER 3
NON-COMMUTATIVE DEFORMATION RINGS

We keep the same definitions and notations that were introduced in Chapter 2. In particular, k is a fixed field, and λ is a fixed complete, local, commutative, Noetherian ring with residue fild k and unique maximal ideal μ. Moreover, we fix the following additional definitions and notations.

Let A be a λ-algebra, and let V be a finite dimensional vector space over k such that V is also a left A-module. Let $n = dim_k V$. After fixing a k-basis of V and recognizing that k is a λ-algebra, there exists a λ-algebra homomorphism, $\rho : A \to Mat_n(k)$, such that the module action of A on V is described as $a \cdot v = \rho(a)(v)$ for all $a \in A$ and $v \in V$. Conversely, if we are given a λ-algebra homomorphism $\rho : A \to Mat_n(k)$, we may associate with it an n-dimensional k-vector space, V, which is simultaneously a left A-module

In this chapter, we define lifts and deformations of V over objects in $\hat{\mathcal{C}}$. Note that we will interchangeably speak of lifts and deformations of V and lifts and deformations of ρ over objects in $\hat{\mathcal{C}}$. We also define versal and universal couples and versal and universal deformation rings. We prove our main result, Theorem 3.16, which shows that if V satisfies a natural finiteness condition, then it has a versal deformation ring in $\hat{\mathcal{C}}$. Many of our definitions, results and proof ideas are based on work by Schlessinger [10] and Mazur [8] and on Vélez-Marulanda's thesis [12]. The main difference is that [10], [8] and [12] worked with $\hat{\mathcal{C}}_{comm}$ instead of $\hat{\mathcal{C}}$. This requires us to modify their definitions, results and proof ideas to fit our situation. We develop this non-commutative deformation theory in Section 3.1, and we give some applications of this theory in Section 3.2. For a related approach to non-commutative deformations, see [6].

3.1 Non-commutative Deformation Theory

We first introduce lifts and deformations of V (or ρ) over objects in $\hat{\mathcal{C}}$.

Definition 3.1. Let (R, π_R) and (S, π_S) be objects of $\hat{\mathcal{C}}$.

(a) We define a lift of V (or ρ) over (R, π_R) to be a λ-algebra homomorphism, $\phi : A \to Mat_n(R)$, such that $\pi_R \circ \phi = \rho$. We will often suppress the residue map and speak of lifts of V (or ρ) over R, instead of lifts of V (or ρ) over (R, π_R).

(b) Two lifts, ϕ and ϕ', of V over R are said to be strictly equivalent if there exists an element P in $GL_n(R)$ such that $\pi_R(P)$ is the identity matrix in $GL_n(k)$, and such that $P\phi(a)P^{-1} = \phi'(a)$ for all $a \in A$.

(c) Strict equivalence forms a partition of the set of all lifts of V over R. We define a deformation of V over R to be an equivalence class of strictly equivalent lifts of V over R. We denote the deformation of V over R corresponding to a lift ϕ by $[\phi]$.

(d) If ϕ is a lift of V over R and if ψ is a lift of V over S, then we say that $f : Mat_n(R) \to Mat_n(S)$ is a morphism from ϕ to ψ if f is the natural extension of a morphism $f : R \to S$ in $\hat{\mathcal{C}}$ such that $f \circ \phi = \psi$.

Lemma 3.2. *Let (R, π_R) and (S, π_S) be objects of $\hat{\mathcal{C}}$.*

(a) If P is a matrix in $Mat_n(R)$ such that P is congruent to the identity matrix modulo $Mat_n(m_R)$, then P is invertible, i.e. P lies in $GL_n(R)$.

(b) If ϕ and ϕ' are strictly equivalent lifts of V over R, and if $f : R \to S$ is a morphism in $\hat{\mathcal{C}}$, then $f \circ \phi$ and $f \circ \phi'$ are strictly equivalent lifts of V over S.

Proof. (a) By Lemma 2.5, $Mat_n(R)$ is $Mat_n(m_R)$-adically complete. Let I denote the $n \times n$ identity matrix in $Mat_n(R)$. It follows that for any matrix X in $Mat_n(m_R)$, the infinite sum $I + X + X^2 + X^3 + ...$ converges in $Mat_n(R)$, and it is clear that this element is the inverse matrix of $I - X$. Therefore, if P is a matrix in $Mat_n(R)$ which is congruent to the identity matrix modulo $Mat_n(m_R)$, then $P = I - X$ for some X in $Mat_n(m_R)$, and hence P is invertible.

(b) Strict equivalence of ϕ and ϕ' means there exists an invertible matrix $P \in Mat_n(R)$ such $P\phi P^{-1} = \phi'$, where $\pi_R(P)$ is the identity matrix in $GL_n(k)$. Since f is a ring homomorphism, we have

$$f \circ \phi' = f \circ (P\phi P^{-1}) = f(P)(f \circ \phi)f(P^{-1}).$$

Further, since $f : R \to S$ is a morphism in $\hat{\mathcal{C}}$, $\pi_S \circ f = \pi_R$, and so f sends P to a matrix in $Mat_n(S)$ which reduces to the identity matrix in $Mat_n(k)$. By (a), this means that $f(P)$ is a unit in $Mat_n(S)$. Therefore, $f(P)$ lies in $GL_n(S)$. Since $I = f(PP^{-1}) = f(P)f(P^{-1})$, we see that $f(P^{-1}) = f(P)^{-1}$, and hence $f \circ \phi$ and

$f \circ \phi'$ are conjugates by the matrix $f(P)$. This proves that $f \circ \phi$ and $f \circ \phi'$ are strictly equivalent lifts of V over S.

□

We next define the deformation functor and couples for V. This leads to the definition of a universal couple and of a universal deformation ring and deformation for V. Note that our definition of couples goes back to [10, Section 2].

Definition 3.3.

(a) The deformation functor $F_V : \hat{\mathcal{C}} \to Sets$ is defined to be the following covariant functor. For any object R of $\hat{\mathcal{C}}$, let $F_V(R)$ be the set consisting of all deformations of V over R. If

$$f : R \to S$$

is a morphism in $\hat{\mathcal{C}}$, then

$$F_V(f) : F_V(R) \to F_V(S)$$

is defined by

$$F_V(f)([\phi]) = [f \circ \phi]$$

for any $[\phi] \in F_V(R)$. We note that we use the notation F_V regardless of whether we speak of deformations of V or of deformations of ρ.

(b) For any object R of $\hat{\mathcal{C}}$, we define the covariant functor $h_R : \hat{\mathcal{C}} \to Sets$ to be $h_R = Hom_{\hat{\mathcal{C}}}(R, -)$. In other words, for any object T of $\hat{\mathcal{C}}$, if

$$f : T \to S$$

is a morphism in $\hat{\mathcal{C}}$, then

$$h_R(f) : h_R(T) \to h_R(S)$$

is given by

$$h_R(f)(\alpha) = f \circ \alpha$$

for any morphism $\alpha : R \to T$ in $\hat{\mathcal{C}}$.

(c) Fix an object R of $\hat{\mathcal{C}}$ and a deformation $[\phi]$ in $F_V(R)$. Then the pair $(R, [\phi])$ is called a couple for V. Define the natural transformation $\tau = \tau(R, [\phi]) : h_R \to F_V$ as follows: For each object T of $\hat{\mathcal{C}}$, let

$$\tau_T : h_R(T) \to F_V(T)$$

be defined by

$$\tau_T(\alpha) = F_V(\alpha)([\phi]).$$

In the case when τ is a natural isomorphism, the functor F_V is said to be representable, and $(R, [\phi])$ is called a universal couple for V. Moreover, R is said to be the universal deformation ring of V, and $[\phi]$ is said to be the universal deformation of V.

Remark 3.4. The definition of F_V in Definition 3.3, part (a), does indeed provide a functor. Namely, if $f : R \to S$ and $g : S \to T$ are morphisms in $\hat{\mathcal{C}}$, then

$$F_V(g \circ f)([\phi]) = [(g \circ f) \circ \phi] = [g \circ (f \circ \phi)] = F_V(g)([f \circ \phi]) = (F_V(g) \circ F_V(f))([\phi])$$

and hence F_V preserves composition. Moreover, for any object R of $\hat{\mathcal{C}}$, we have that

$$F_V(id_R)([\phi]) = [id_R \circ \phi] = [\phi]$$

and so F_V also preserves identity morphisms.

Theorem 3.5. *If $(R, [\phi])$ and $(R', [\phi'])$ are both universal couples for V, then they are isomorphic up to a unique isomorphism. In other words, there exists a morphism $f : R \to R'$ in $\hat{\mathcal{C}}$ such that f is unique with respect to the property that f is an isomorphism in $\hat{\mathcal{C}}$ between R and R' satisfying $[f \circ \phi] = [\phi']$.*

Proof. Let $\tau = \tau(R, [\phi])$ be the natural transformation from Definition 3.3, part (c). Since $(R, [\phi])$ is a universal couple for V, $\tau_{R'} : h_R(R') \to F_V(R')$ is a bijection. Therefore, there exists a unique $f \in h_R(R')$ such that $[f \circ \phi] = [\phi']$. However, since $(R', [\phi'])$ is also a universal couple for V, a completely similar argument provides the existence of a unique $f' \in h_{R'}(R)$ such that $[f' \circ \phi'] = [\phi]$. But then $f' \circ f$ is an element in $h_R(R)$ which gets sent to $[\phi]$ by τ_R. Since id_R also gets sent to $[\phi]$ by τ_R, and since τ_R is a bijection, we must have that $id_R = f' \circ f$. By a completely similar argument, we have that $id_{R'} = f \circ f'$, proving that $f : R \to R'$ is an isomorphism in $\hat{\mathcal{C}}$ which satisfies $[f \circ \phi] = [\phi']$. But since $[f \circ \phi] = \tau_R(f)$, and since τ_R is bijective, f is the unique morphism from R to R' in $\hat{\mathcal{C}}$ which is both an isomorphism and satisfies $[f \circ \phi] = [\phi']$. □

In some cases, however, the functor F_V will fail to be representable, meaning that no universal couple for V will exist. Even so, while this means that there does not exist a couple for V which gives rise to a natural transformation τ that is a natural isomorphism, there may still be a suitable couple for V which gives rise to a natural transformation τ that makes F_V "close to representable." The following definition makes this more precise by defining smoothness, which leads to the definition of a versal couple and of a versal deformation ring and deformation of V. Note that these definitions again go back to [10, Section 2].

Definition 3.6. Fix a couple $(R, [\phi])$ for V. Then the natural transformation $\tau = \tau(R, [\phi]) : h_R \to F_V$ from Definition 3.3, part (c), is said to be smooth if the morphism, q, resulting from the pullback diagram in Figure 3.1 is surjective, whenever $\alpha : B \to B'$ is a surjective morphism in $\mathcal{C}$. If τ is smooth and if $\tau_{k[\epsilon]}$ is bijective, then $(R, [\phi])$ is said to be a versal couple for V. Moreover, R is said to be a versal deformation ring of V, and $[\phi]$ is said to be a versal deformation of V.

Figure 3.1: The pullback diagram needed to define smoothness.

$$\begin{tikzcd}
h_R(B) \arrow[rr, "h_R(\alpha)"] \arrow[dd, "\tau_B"'] \arrow[dr, rightsquigarrow, "q"] & & h_R(B') \arrow[dd, "\tau_{B'}"] \\
& h_R(B') \times_{F_V(B')} F_V(B) \arrow[ur] \arrow[dl] & \\
F_V(B) \arrow[rr, "F_V(\alpha)"] & & F_V(B')
\end{tikzcd}$$

Note that representability implies smoothness, and so in the event that $(R, [\phi])$ is a universal couple for V and $\tau = \tau(R, [\phi])$ is a natural isomorphism, then $(R, [\phi])$

is a versal couple for V and τ is smooth. While we will show in Theorem 3.11 that any two versal couples for V are isomorphic, there may not be a unique isomorphism which relates them, as there is in the universal case. To work our way up to the proof, we begin with a few preliminary results.

Lemma 3.7. *If $(R, [\phi])$ is a versal couple for V, then the smoothness of $\tau = \tau(R, [\phi])$ implies that $\tau_T : h_R(T) \to F_V(T)$ is surjective for all T in $\hat{\mathcal{C}}$.*

Proof. Note that some of the proof ideas are similar to ideas in the proof of [3, Proposition A.1].

Fix an object (T, π_T) of $\hat{\mathcal{C}}$. First notice that $F_V(k)$ consists of only a single element, namely $[\rho]$. Since k is isomorphic to T/m_T, $F_V(T/m_T)$ also contains only a single element. Since R and T share residue fields, there exists some element, f_1, in $h_R(T/m_T)$. Moreover, if $[\gamma]$ is any element of $F_V(T)$, then, considering the natural projections

$$\pi_\ell : T \to T/m_T^\ell,$$

we can define deformations

$$[\gamma_\ell] \in F_V(T/m_T^\ell)$$

by

$$\gamma_\ell = \pi_\ell \circ \gamma.$$

Suppose, inductively, that we have found maps $f_i \in h_R(T/m_T^i)$ for $i \leq \ell$, compatible in the sense that f_i projects onto f_{i-1} under the natural projection $T/m_T^i \to T/m_T^{i-1}$, and such that $\tau_{T/m_T^i}(f_i) = [\gamma_i]$ (see Figure 3.2). Then the smoothness of τ immediately implies the existence of some $f_{\ell+1} \in h_R(T/m_T^{\ell+1})$ which projects onto f_ℓ, and which also satisfies $\tau_{T/m_T^{\ell+1}}(f_{\ell+1}) = [\gamma_{\ell+1}]$ (see Figure 3.3), completing the induction. Therefore, we can construct the inverse limit of the f_ℓ, and by completeness of T we can associate this with an element $f \in h_R(T)$, such that $f(r) = (f_\ell(r))_\ell$ for all r in R, where we identify T with the inverse limit $\varprojlim_\ell (T/m_T^\ell)$. Since $\tau_{T/m_T^\ell}(f_\ell) = [\gamma_\ell]$ for all $\ell \geq 1$, there exists some P_ℓ in $Mat_n(T/m_T^\ell)$ which reduces to the identity in $Mat_n(k)$, such that $P_\ell \gamma_\ell P_\ell^{-1} = f_\ell \circ \phi$. Therefore, we have that

$$f \circ \phi = (f_\ell \circ \phi)_\ell = (P_\ell \gamma_\ell P_\ell^{-1})_\ell.$$

To show that $[f \circ \phi] = [\gamma]$, we need to find some $P \in Mat_n(T)$ which reduces to the identity in $Mat_n(k)$, such that

$$(f_\ell(\phi))_\ell = P(\gamma_\ell)_\ell P^{-1}.$$

Figure 3.2: The induction hypothesis for the construction of $f_{\ell+1}$.

$$T/m_T^\ell \to T/m_T^{\ell-1} \to \ldots \to T/m_T$$

$$\begin{tikzcd}
h_R(T/m_T^\ell) \arrow[r] \arrow[d, "\tau_{T/m_T^\ell}"'] & h_R(T/m_T^{\ell-1}) \arrow[r] \arrow[d, "\tau_{T/m_T^{\ell-1}}"'] & \ldots \arrow[r] & h_R(T/m_T) \arrow[d, "\tau_{T/m_T}"'] \\
F_V(T/m_T^\ell) \arrow[r] & F_V(T/m_T^{\ell-1}) \arrow[r] & \ldots \arrow[r] & F_V(T/m_T)
\end{tikzcd}$$

Figure 3.3: The compatibility conditions for $f_{\ell+1} \in h_R(T/m_T^{\ell+1})$.

$$T/m_T^{\ell+1} \to T/m_T^{\ell}$$

$$\begin{tikzcd}
\exists f_{\ell+1} & f_\ell \\
h_R(T/m_T^{\ell+1}) \arrow[r] \arrow[d, "\tau_{T/m_T^{\ell+1}}"'] & h_R(T/m_T^{\ell}) \arrow[d, "\tau_{T/m_T^{\ell}}"'] \\
F_V(T/m_T^{\ell+1}) \arrow[r] & F_V(T/m_T^{\ell}) \\
{[\gamma_{\ell+1}]} & {[\gamma_\ell]}
\end{tikzcd}$$

To this end, we define, for each $i \geq 1$, the set

$$Z_i = \{X_i \in \mathrm{Mat}_n(T/m_T^i) \mid X_i\gamma_i = \gamma_i X_i \text{ and all entries of } X_i \text{ are in } m_T/m_T^i\}.$$

For all $j \geq i$, let $\sigma_i^j : Mat_n(T/m_T^j) \to Mat_n(T/m_T^i)$ be the natural projection. Note that T/m_T^i has a filtration of left (T/m_T^i)-submodules

$$T/m_T^i \supseteq m_T/m_T^i \supseteq m_T^2/m_T^i \supseteq \ldots \supseteq m_T^i/m_T^i = 0$$

where each of the successive quotients is a finite dimensional k-vector space. Using that this is also a filtration of (left) $j_{T/m_T^i}(\lambda)$-submodules, we see that the successive quotients are Artinian $j_{T/m_T^i}(\lambda)$-modules, showing that T/m_T^i is an Artinian $j_{T/m_T^i}(\lambda)$-module. Therefore, $Mat_n(T/m_T^i)$ is also an Artinian module over $j_{T/m_T^i}(\lambda)$. Since Z_i is a $j_{T/m_T^i}(\lambda)$-submodule of $Mat_n(T/m_T^i)$, Z_i is again an Artinian module over $j_{T/m_T^i}(\lambda)$. Therefore, for each $i \geq 1$, the decreasing sequence of $j_{T/m_T^i}(\lambda)$-modules

$$Z_i \supseteq \sigma_i^{i+1}(Z_{i+1}) \supseteq \sigma_i^{i+2}(Z_{i+2}) \supseteq \ldots$$

stabilizes. For each $i \geq 1$, define the integer $N_i \geq i$ to be the smallest integer such that $\sigma_i^{N_i}(Z_{N_i}) = \sigma_i^j(Z_j)$ for all $j \geq N_i$. Notice that by construction, $N_i \leq N_{i+1}$ for all i.

For each $i \geq 1$, define:

$$\begin{aligned} S_i &= \bigcap_{j \geq i} \sigma_i^j(Z_j) = \sigma_i^{N_i}(Z_{N_i}), \\ Q_i &= \sigma_i^{N_i}(P_{N_i}^{-1})\sigma_i^{N_{i+1}}(P_{N_{i+1}}) - \mathrm{Id}_{T/m_T^i} \in \mathrm{Mat}_n(T/m_T^i), \end{aligned}$$

where Id_{T/m_T^i} denotes the identity matrix in $Mat_n(T/m_T^i)$.

Notice that since each P_ℓ reduces to the identity in $Mat_n(k)$, all the entries of Q_i lie in m_T/m_T^i. Recall that, for all $\ell \geq 1$,

$$P_\ell \gamma_\ell P_\ell^{-1} = f_\ell \circ \phi.$$

From the definition of the γ_ℓ, we have $\sigma_i^j(\gamma_j) = \gamma_i$ for all $j \geq i$, and the compatibility of the f_ℓ means that $\sigma_i^j(f_j) = f_i$ for all $j \geq i$. Therefore, for all $j \geq i$,

$$\begin{aligned} \sigma_i^j(P_j)\gamma_i\sigma_i^j(P_j^{-1}) &= \sigma_i^j(P_j)\sigma_i^j(\gamma_j)\sigma_i^j(P_j^{-1}) \\ &= \sigma_i^j(P_j\gamma_j P_j^{-1}) \\ &= \sigma_i^j(f_j \circ \phi) \\ &= f_i \circ \phi = P_i\gamma_i P_i^{-1}. \end{aligned}$$

Hence, for all $j \geq i$, $P_i^{-1}\sigma_i^j(P_j)$ commutes with γ_i. From this, and from the fact that all the entries of Q_i lie in m_T/m_T^i, it follows that $Q_i \in S_i$ for all $i \geq 1$.

Let $j \geq 3$. For $2 \leq i \leq j$, define

$$\rho_i^j : S_j \to S_i$$

to be the restriction of σ_i^j to S_j. Notice that

$$\rho_i^j(S_j) = \sigma_i^j(\sigma_j^{N_j}(Z_{N_j})) = \sigma_i^{N_j}(Z_{N_j}).$$

By the definition of N_i and the fact that $N_j \geq N_i$, we obtain, for all $j \geq i$,

$$\sigma_i^{N_j}(Z_{N_j}) = \sigma_i^{N_i}(Z_{N_i}).$$

Since $S_i = \sigma_i^{N_i}(Z_{N_i})$, the above equations combine to give, for $j \geq i$,

$$\rho_i^j(S_j) = S_i,$$

showing that each ρ_i^j is surjective.

We now proceed to inductively define a collection of matrices $X_i^{(j)}$ in S_j, for $j \geq 2$ and $2 \leq i \leq j$. For all $j \geq 2$, let $X_j^{(j)} = Q_j$. Fix now $\ell \geq 2$ and suppose inductively that we have defined $X_i^{(j)}$ for all i, j satisfying $3 \leq j \leq \ell$ and $2 \leq i \leq j-1$. We proceed to define $X_i^{(\ell+1)}$, for all $2 \leq i \leq \ell$, as follows. Since $\rho_\ell^{\ell+1}$ is surjective, let $X_i^{(\ell+1)}$ be a matrix in $S_{\ell+1}$ such that $\rho_\ell^{\ell+1}(X_i^{(\ell+1)}) = X_i^{(\ell)}$. Hence, for all $j \geq 3$ and all $2 \leq i \leq j-1$, we have $\rho_{j-1}^j(X_i^{(j)}) = X_i^{(j-1)}$. Since, for all $t \leq j-1$, we have $\rho_t^j = \rho_t^{t+1} \circ \rho_{t+1}^{t+2} \circ \ldots \circ \rho_{j-2}^{j-1} \circ \rho_{j-1}^j$, we obtain inductively, for all $j \geq 3$ and all $2 \leq i \leq t \leq j-1$, that $\rho_t^j(X_i^{(j)}) = X_i^{(t)}$.

Next, define the following matrices L_ℓ, for all $\ell \geq 1$:

$$\begin{aligned}
L_1 &= \sigma_1^{N_1}(P_{N_1}), \\
L_2 &= \sigma_2^{N_2}(P_{N_2}), \\
L_\ell &= \sigma_\ell^{N_\ell}(P_{N_\ell})(\mathrm{Id}_{T/m_T^\ell} + X_{\ell-1}^{(\ell)})^{-1}(\mathrm{Id}_{T/m_T^\ell} + X_{\ell-2}^{(\ell)})^{-1} \cdots (\mathrm{Id}_{T/m_T^\ell} + X_2^{(\ell)})^{-1},
\end{aligned}$$

for $\ell \geq 3$. Notice that the L_ℓ are compatible, in the sense that for $j \geq i$, $\sigma_i^j(L_j) = L_i$. To see this, first notice that

$$\begin{aligned}
\sigma_{j-1}^j(L_j) &= \sigma_{j-1}^j(\sigma_j^{N_j}(P_{N_j})(\mathrm{Id}_{T/m_T^j} + X_{j-1}^{(j)})^{-1}(\mathrm{Id}_{T/m_T^j} + X_{j-2}^{(j)})^{-1} \\
&\quad \cdots (\mathrm{Id}_{T/m_T^j} + X_2^{(j)})^{-1}) \\
&= \sigma_{j-1}^{N_j}(P_{N_j})\sigma_{j-1}^j((\mathrm{Id}_{T/m_T^j} + X_{j-1}^{(j)})^{-1}(\mathrm{Id}_{T/m_T^j} + X_{j-2}^{(j)})^{-1} \\
&\quad \cdots (\mathrm{Id}_{T/m_T^j} + X_2^{(j)})^{-1}) \\
&= \sigma_{j-1}^{N_j}(P_{N_j})(\mathrm{Id}_{T/m_T^{j-1}} + X_{j-1}^{(j-1)})^{-1}(\mathrm{Id}_{T/m_T^{j-1}} + X_{j-2}^{(j-1)})^{-1} \\
&\quad \cdots (\mathrm{Id}_{T/m_T^{j-1}} + X_2^{(j-1)})^{-1} \\
&= \sigma_{j-1}^{N_j}(P_{N_j})(\mathrm{Id}_{T/m_T^{j-1}} + Q_{j-1})^{-1}(\mathrm{Id}_{T/m_T^{j-1}} + X_{j-2}^{(j-1)})^{-1} \\
&\quad \cdots (\mathrm{Id}_{T/m_T^{j-1}} + X_2^{(j-1)})^{-1} \\
&= \sigma_{j-1}^{N_j}(P_{N_j})[\sigma_{j-1}^{N_{j-1}}(P_{N_{j-1}})^{-1}\sigma_{j-1}^{N_j}(P_{N_j})]^{-1}(\mathrm{Id}_{T/m_T^{j-1}} + X_{j-2}^{(j-1)})^{-1} \\
&\quad \cdots (\mathrm{Id}_{T/m_T^{j-1}} + X_2^{(j-1)})^{-1} \\
&= \sigma_{j-1}^{N_{j-1}}(P_{N_{j-1}})(\mathrm{Id}_{T/m_T^{j-1}} + X_{j-2}^{(j-1)})^{-1} \cdots (\mathrm{Id}_{T/m_T^{j-1}} + X_2^{(j-1)})^{-1} \\
&= L_{j-1}
\end{aligned}$$

where we use that σ_i^j coincides with ρ_i^j on S_j. Therefore, we have for all $j \geq i$,

$$L_i = \sigma_i^{i+1}(L_{i+1}) = (\sigma_i^{i+1} \circ \sigma_{i+1}^{i+2})(L_{i+2}) = \ldots = (\sigma_i^{i+1} \circ \sigma_{i+1}^{i+2} \circ \ldots \circ \sigma_{j-1}^{j})(L_j).$$

Since $\sigma_i^j = \sigma_i^{i+1} \circ \sigma_{i+1}^{i+2} \circ \ldots \circ \sigma_{j-1}^{j}$, it follows that $L_i = \sigma_i^j(L_j)$ for all $i \leq j$.

By construction, $X_i^{(\ell)}$ lies in S_ℓ for each $i \leq \ell$. Therefore, $(\mathrm{Id}_{T/m_T^\ell} + X_i^{(\ell)})$ commutes with γ_ℓ, for all $i \leq \ell$. From this it follows, for all $\ell \geq 1$, that

$$\begin{aligned} L_\ell \gamma_\ell L_\ell^{-1} &= \sigma_\ell^{N_\ell}(P_{N_\ell})\gamma_\ell \sigma_\ell^{N_\ell}(P_{N_\ell})^{-1} \\ &= \sigma_\ell^{N_\ell}(P_{N_\ell})\sigma_\ell^{N_\ell}(\gamma_{N_\ell})\sigma_\ell^{N_\ell}(P_{N_\ell})^{-1} \\ &= \sigma_\ell^{N_\ell}(P_{N_\ell}\gamma_{N_\ell}P_{N_\ell}^{-1}) \\ &= \sigma_\ell^{N_\ell}(f_{N_\ell} \circ \phi) = f_\ell \circ \phi. \end{aligned}$$

Since the matrices L_ℓ form an element $(L_\ell)_\ell$ in $\varprojlim_\ell Mat_n(T/m_T^\ell)$, we know from Lemma 2.5 that there exists a matrix L in $Mat_n(T)$ such that, for all $\ell \geq 1$, L projects onto L_ℓ via the natural projection $Mat_n(T) \to Mat_n(T/m_T^\ell)$. Therefore, $L\gamma L^{-1} = f \circ \phi$. Moreover, because L projects onto L_ℓ and because L_ℓ reduces to the identity in $GL_n(k)$, L also reduces to the identity in $GL_n(k)$. Hence

$$\tau_T(f) = [f \circ \phi] = [\gamma],$$

completing the proof that τ_T is surjective. $\square$

As a consequence of the above proof, we get:

Corollary 3.8. *Suppose τ is as in Lemma 3.7. Then τ is a natural isomorphism if and only if τ_B is a bijection for all B in $\mathcal{C}$.*

Proof. Suppose τ_B is a bijection for all B in $\mathcal{C}$. By the definition of completeness, any T in $\hat{\mathcal{C}}$ can be identified with the inverse limit $\varprojlim_{\ell}(T/m_T^{\ell})$. Therefore, if $\tau_T(\alpha) = \tau_T(\alpha')$, then $[\alpha \circ \phi] = [\alpha' \circ \phi]$, which implies $[\pi_\ell \circ \alpha \circ \phi] = [\pi_\ell \circ \alpha' \circ \phi]$, where π_ℓ is again taken to be the natural projection from T onto T/m_T^{ℓ}, for all $\ell \geq 1$. However,

$$[\pi_\ell \circ \alpha \circ \phi] = [(\pi_\ell \circ \alpha) \circ \phi] = \tau_{T/m_T^{\ell}}(\pi_\ell \circ \alpha)$$

proving that we must have $\tau_{T/m_T^{\ell}}(\pi_\ell \circ \alpha) = \tau_{T/m_T^{\ell}}(\pi_\ell \circ \alpha')$ for all ℓ. But since each T/m_T^{ℓ} is an object of $\mathcal{C}$, each $\tau_{T/m_T^{\ell}}$ is a bijection by assumption, and hence $\pi_\ell \circ \alpha = \pi_\ell \circ \alpha'$ for all ℓ. Since T is complete, this proves that $\alpha = \alpha'$, since α and α' are exactly the inverse limits of the maps $\pi_\ell \circ \alpha$ and $\pi_\ell \circ \alpha'$, respectively, in the identification of T with $\varprojlim_{\ell} T/m_T^{\ell}$. This shows that τ_T is injective.

To see surjectivity of τ_T, let $[\eta]$ be any element in $F_V(T)$. Then $[\pi_\ell \circ \eta]$ is an element in $F_V(T/m_T^{\ell})$ for all $\ell \geq 1$. Since $\tau_{T/m_T^{\ell}}$ is assumed to be surjective, there exists some α_ℓ in $h_R(T/m_T^{\ell})$ so that $\tau_{T/m_T^{\ell}}(\alpha_\ell) = [\pi_\ell \circ \eta]$, which is to say that $[\alpha_\ell \circ \phi] = [\pi_\ell \circ \eta]$, for all $\ell \geq 1$. Letting $\nu : T/m_T^{\ell+1} \to T/m_T^{\ell}$ be the natural projection, we have that

$$\begin{aligned}
\tau_{T/m_T^{\ell}}(\alpha_\ell) &= [\alpha_\ell \circ \phi] = [\pi_\ell \circ \eta] = [\nu \circ \pi_{\ell+1} \circ \eta] = [\nu \circ (\pi_{\ell+1} \circ \eta)] \\
&= [\nu \circ (\alpha_{\ell+1} \circ \phi])) = [(\nu \circ \alpha_{\ell+1}) \circ \phi] = \tau_{T/m_T^{\ell}}(\nu \circ \alpha_{\ell+1}).
\end{aligned}$$

However, since $\tau_{T/m_T^{\ell}}$ is assumed to be a bijection, we must have that $\alpha_\ell = \nu \circ \alpha_{\ell+1}$ for all $\ell \geq 1$, which is to say that the α_ℓ are compatible in the sense that they all project onto each other under the natural projections. We then use completeness and mimic the proof of Lemma 3.7, replacing f_ℓ with α_ℓ and γ with η, to find the existence of some $\alpha : R \to T$ in $\hat{\mathcal{C}}$ such that $\pi_\ell \circ \alpha = \alpha_\ell$ for all ℓ, and such that $[\alpha \circ \phi] = [\eta]$, proving that τ_T is surjective, and completing the proof that τ is a natural isomorphism. $\square$

As a final preliminary to the proof that any two versal couples for V are isomorphic, it will be necessary to find certain elements in $h_R(k[\epsilon])$. We first prove the following result:

Lemma 3.9. *For any R in $\hat{\mathcal{C}}$ with $m_R \neq 0$, there exists an integer $m \geq 1$ and an isomorphism in $\hat{\mathcal{C}}$, $f : R/(\mu_R + m_R^2) \to k[\epsilon] \times_k k[\epsilon] \times_k \ldots \times_k k[\epsilon]$, the m-fold pullback of $k[\epsilon]$ over k. More specifically, $m_R/(\mu_R + m_R^2)$ is a vector space over k of k-dimension m. If $b_1, \ldots, b_m$ in m_R generate m_R modulo $(\mu_R + m_R^2)$, then $f(l + \sum_{i=1}^{m} r_i b_i + (\mu_R + m_R^2)) = (\pi_R(l) + \pi_R(r_i)\epsilon)_i$ for all l in $j_R(\lambda)$ and all $r_1, \ldots, r_m$ in R.*

Proof. We know from Theorem 2.8, part (d), that $\mu_R + m_R^2$ is an ideal of R. Since any ring R in $\hat{\mathcal{C}}$ has a maximal ideal m_R such that m_R/m_R^2 is finitely generated as a left R-module, $m_R/(\mu_R + m_R^2)$ is also finitely generated as a left R-module. Moreover, m_R annihilates $m_R/(\mu_R + m_R^2)$, and hence $m_R/(\mu_R + m_R^2)$ is an m-dimensional k-vector space for some finite integer, m. Choose elements $b_1, b_2, \ldots, b_m$ in m_R so that their images form a k-basis of $m_R/(\mu_R + m_R^2)$. Then, letting $\bar{r}$ denote the image of r in $R/(\mu_R + m_R^2)$, we can write any element $\bar{r}$ of $R/(\mu_R + m_R^2)$ as $\overline{l + \sum_{i=1}^{m} r_i b_i}$, where l is some element in $j_R(\lambda)$, and the $r_1, \ldots, r_m$ are certain elements of R. Therefore, we can define a map

$$f : R/(\mu_R + m_R^2) \to k[\epsilon] \times_k k[\epsilon] \times_k \ldots \times_k k[\epsilon]$$

by

$$f(\overline{l + \sum_{i=1}^{m} r_i b_i}) = (\pi_R(l) + \pi_R(r_i)\epsilon)_i.$$

To see this is well-defined, suppose $\overline{l + \sum_{i=1}^{m} r_i b_i} = \overline{l' + \sum_{i=1}^{m} r_i' b_i}$. Then

$$\overline{l - l'} = \overline{\sum_{i=1}^{m} r_i' b_i} - \overline{\sum_{i=1}^{m} r_i' b_i} = \overline{\sum_{i=1}^{m} (r_i' - r_i) b_i}.$$

But since all the b_i lie in m_R and since both l and l' are in $j_R(\lambda)$,

$$l - l' \in j_R(\lambda) \bigcap m_R = \mu_R,$$

and hence

$$\overline{l - l'} = \overline{\sum_{i=1}^{m} (r_i' - r_i) b_i} = \bar{0}.$$

But this shows that $r_i' - r_i \in m_R$ for all i, since the $\overline{b_i}$ form a k-basis of $m_R/(\mu_R + m_R^2)$. Therefore, we have that

$$\pi_R(l) = \pi_R(l')$$

and

$$\pi_R(r_i) = \pi_R(r_i')$$

for all i, and hence f is well-defined. It follows from the definition of f that f is surjective. To see injectivity, suppose $f(\overline{l + \sum_{i=1}^{m} r_i b_i}) = (0)_i$. Then $\pi_R(l) = 0$, and $\pi_R(r_i) = 0$ for all i, and hence $l \in \mu_R$ and $r_i \in m_R$ for all i. But then since $b_i \in m_R$, it is clear that $l + \sum_{i=1}^{m} r_i b_i \in \mu_R + m_R^2$, and hence $\overline{l + \sum_{i=1}^{m} r_i b_i} = \bar{0}$, proving that f is injective. Lastly, we must show that f is a morphism in $\hat{\mathcal{C}}$. To see that f is a ring homomorphism, it is clear that f preserves addition since π_R preserves addition. To see that it preserves the multiplicative structure, notice that

$$\begin{aligned}
f(\overline{(l+\sum_{i=1}^{m} r_i b_i)} * \overline{(l'+\sum_{i=1}^{m} r_i' b_i)}) &= f(\overline{ll'+\sum_{i=1}^{m}(lr_i'+l'r_i)b_i}) \\
&= (\pi_R(ll')+\pi_R(lr_i'+l'r_i)\epsilon)_i \\
&= (\pi_R(l)\pi_R(l')+(\pi_R(l)\pi_R(r_i')+\pi_R(l')\pi_R(r_i))\epsilon)_i \\
&= (\pi_R(l)+\pi_R(r_i)\epsilon)_i * (\pi_R(l')+\pi_R(r_i')\epsilon)_i \\
&= f(\overline{l+\sum_{i=1}^{m} r_i b_i}) * f(\overline{l'+\sum_{i=1}^{m} r_i' b_i}).
\end{aligned}$$

Therefore, f is indeed a ring homomorphism. To see that f is a λ-algebra homomorphism, notice that if x is in λ, then

$$\begin{aligned}
f(x(\overline{l+\sum_{i=1}^{m} r_i b_i})) &= f(\overline{j_R(x)l+\sum_{i=1}^{m} j_R(x)r_i b_i}) \\
&= (\pi_R(j_R(x)l)+\pi_R(j_R(x)r_i)\epsilon)_i \\
&= (\pi_\lambda(x)\pi_R(l)+\pi_\lambda(x)\pi_R(r_i)\epsilon)_i \\
&= x(\pi_R(l)+\pi_R(r_i)\epsilon)_i \\
&= xf(\overline{l+\sum_{i=1}^{m} r_i b_i}).
\end{aligned}$$

Since $\pi_{k[\epsilon]\times_k \ldots \times_k k[\epsilon]}((\pi_R(l)+\pi_R(r_i)\epsilon)_i) = \pi_R(l)$ and since $\pi_{R/(\mu_R+m_R^2)}(\overline{l+\sum_{i=1}^{m} r_i b_i}) = \pi_R(l+\sum_{i=1}^{m} r_i b_i) = \pi_R(l)$, we obtain $\pi_{k[\epsilon]\times_k \ldots \times_k k[\epsilon]} \circ f = \pi_{R/(\mu_R+m_R^2)}$. Since f sends $\bar{1}_R$ to

$1_{k[\epsilon]\times_k ... \times_k k[\epsilon]}$, f is indeed an isomorphism in $\hat{\mathcal{C}}$. $\square$

Corollary 3.10. *Let R be in $\hat{\mathcal{C}}$ with $m_R \neq 0$. Suppose $b_1, b_2, ..., b_m$ are elements in m_R so that their images form a k-basis of $m_R/(\mu_R + m_R^2)$. Then, for $1 \leq i \leq m$, there exists an element $\alpha_{b_i} \in h_R(k[\epsilon])$ such that $\alpha_{b_i}(\mu_R + m_R^2) = 0$, $\alpha_{b_i}(b_i) = \epsilon$ and $\alpha_{b_i}(b_j) = 0$ for all $j \neq i$.*

Proof. For $1 \leq i \leq m$, let $p_i : k[\epsilon] \times_k k[\epsilon] \times_k ... \times_k k[\epsilon] \to k[\epsilon]$ be the i^{th} projection map, where $k[\epsilon] \times_k ... \times_k k[\epsilon]$ is the m-fold pullback of $k[\epsilon]$ over k, and let $\pi : R \to R/(\mu_R + m_R^2)$ be the natural projection. Then we can use the isomorphism f in Lemma 3.9 to see that $\alpha_{b_i} = p_i \circ f \circ \pi : R \to k[\epsilon]$ is a morphism in $\hat{\mathcal{C}}$, for $1 \leq i \leq m$, which satisfies the properties in the statement of the corollary. $\square$

Theorem 3.11. *Any two versal couples for V are isomorphic.*

Proof. The outline of the proof is similar to the proof of [10, Proposition 2.9]. However, we provide many more details.

Suppose $(R, [\phi])$ and $(R', [\phi'])$ are both versal couples for V. Then smoothness of the two associated natural transformations

$$\tau = \tau(R, [\phi]) : h_R \to F_V,$$

$$\tau' = \tau(R', [\phi']) : h_{R'} \to F_V$$

combined with Lemma 3.7 imply the existence of morphisms $f : R \to R'$ and $f' : R' \to R$ in $\hat{\mathcal{C}}$ such that $[f \circ \phi] = [\phi']$ and $[f' \circ \phi'] = [\phi]$. Putting this together we see that

$$[f' \circ f \circ \phi] = [\phi]$$

and

$$[f \circ f' \circ \phi'] = [\phi'].$$

Let $b_1, b_2, ..., b_m$ be elements of m_R such that their images form a k-basis of $m_R/(\mu_R + m_R^2)$. By Corollary 3.10, for each $1 \leq i \leq m$, there exists a morphism $\alpha_{b_i} : R \to k[\epsilon]$ in $\hat{\mathcal{C}}$ such that $\alpha_{b_i}(\mu_R + m_R^2) = 0$, $\alpha_{b_i}(b_i) = \epsilon$ and $\alpha_{b_i}(b_j) = 0$ for all $j \neq i$.

Applying the functors h_R and F_V to these morphisms, we have the commutative diagrams in Figure 3.4. Since

Figure 3.4: The commutative diagrams obtained by applying h_R and F_V to α_{b_i}.

$$\alpha_{b_i} : R \to k[\epsilon]$$

$$\begin{tikzcd}
f' \circ f \arrow[rrr, bend left] \arrow[dddd, bend right] & & & \alpha_{b_i} \circ (f' \circ f) \arrow[dddd, bend left] \\
id_R \arrow[rrr, bend left] \arrow[ddd, bend right] & & & \alpha_{b_i} \arrow[ddd, bend left] \\
& h_R(R) \arrow[r, "h_R(\alpha_{b_i})"] \arrow[d, "\tau_R"'] & h_R(k\,[\epsilon]) \arrow[d, "\tau_{k[\epsilon]}"] & \\
& F_V(R) \arrow[r, "F_V(\alpha_{b_i})"] & F_V(k\,[\epsilon]) & \\
[\phi] \arrow[rrr, bend right] & & & [\alpha_{b_i} \circ \phi]
\end{tikzcd}$$

$$[\phi] = \tau_R(id_R) = \tau_R(f' \circ f)$$

we must have that

$$F_V(\alpha_{b_i})(\tau_R(id_R)) = F_V(\alpha_{b_i})(\tau_R(f' \circ f)).$$

But by the commuting diagram in Figure 3.4,

$$F_V(\alpha_{b_i})(\tau_R(id_R)) = \tau_{k[\epsilon]}(\alpha_{b_i} \circ id_R) = \tau_{k[\epsilon]}(\alpha_{b_i})$$

and

$$F_V(\alpha_{b_i})(\tau_R(f' \circ f)) = \tau_{k[\epsilon]}(\alpha_{b_i} \circ (f' \circ f))$$

which shows that

$$\tau_{k[\epsilon]}(\alpha_{b_i}) = \tau_{k[\epsilon]}(\alpha_{b_i} \circ (f' \circ f)).$$

Since $\tau_{k[\epsilon]}$ is bijective by the assumption that $(R, [\phi])$ is a versal couple for V, it follows that $\alpha_{b_i} = \alpha_{b_i} \circ f' \circ f$, for $1 \leq i \leq m$. Therefore, for $1 \leq i \leq m$,

$$\epsilon = \alpha_{b_i}(b_i) = (\alpha_{b_i} \circ f' \circ f)(b_i).$$

We next show that the images of $(f' \circ f)(b_1), ..., (f' \circ f)(b_m)$ also form a k-basis of $m_R/(\mu_R + m_R^2)$. As before, we let $\bar{r}$ denote the image of r in $R/(\mu_R + m_R^2)$. Suppose there exist elements $l_1, ..., l_m$ in λ such that

$$\overline{\sum_{i=1}^{m} j_R(l_i)(f' \circ f)(b_i)} = \bar{0}.$$

Then since $f' \circ f$ is a λ-algebra homomorphism, we see that

$$\sum_{i=1}^{m} j_R(l_i)(f' \circ f)(b_i) = (f' \circ f)(\sum_{i=1}^{m}(j_R(l_i)b_i)) \in \mu_R + m_R^2.$$

But then for each $1 \leq i \leq m$,

$$\begin{aligned}
0 &= \alpha_{b_i}(0) = (\alpha_{b_i} \circ f' \circ f)(\sum_{j=1}^{m}(j_R(l_j)b_j)) \\
&= \alpha_{b_i}(\sum_{j=1}^{m}(j_R(l_j)b_j)) \\
&= \sum_{j=1}^{m} j_R(l_j)\alpha_{b_i}(b_j) \\
&= j_R(l_i) \cdot \epsilon
\end{aligned}$$

so that $j_R(l_i)$ must be in μ_R, and hence l_i must be in μ, for all $1 \leq i \leq m$. Therefore, the images of

$$(f' \circ f)(b_1), (f' \circ f)(b_2), ..., (f' \circ f)(b_m)$$

in $R/(\mu_R + m_R^2)$ also form a basis for the k-vector space $m_R/(\mu_R + m_R^2)$. Hence, $f' \circ f$ induces a bijection from $m_R/(\mu_R + m_R^2)$ to itself. By Theorem 2.8, part (l), this implies that $f' \circ f$ is an automorphism of R in $\hat{\mathcal{C}}$. By an entirely analagous argument, in which we replace R by R', we see that $f \circ f'$ is an automorphism of R' in $\hat{\mathcal{C}}$. This implies that both f and f' are isomorphisms, and hence $R \cong R'$, completing the proof that any two versal couples for V are isomorphic. □

Definition 3.12. If there exists a versal couple $(R, [\phi])$ for V, we denote the versal deformation ring R by $R(A, V)$ and the versal deformation $[\phi]$ by $[\phi_u]$. Note that we also use this notation when $(R, [\phi])$ is a universal couple for V.

Our goal is to prove that if $F_V(k[\epsilon])$ is a finite dimensional k-vector space, then a versal couple for V exists. To see that this is a natural finiteness condition, we first prove the following result:

Lemma 3.13. *The set $F_V(k[\epsilon])$ is a k-vector space.*

Proof. The proof uses ideas from [9, §15] and [12, Section 3.4].

To define addition in $F_V(k[\epsilon])$, first consider the map $\mathbb{P} : k[\epsilon] \times_k k[\epsilon] \to k[\epsilon]$ defined by

$$\mathbb{P}((x + \epsilon y, x + \epsilon z)) = x + \epsilon(y + z)$$

for all x, y, z in k. This is a k-algebra homomorphism since for all $c, x_1, x_2, y_1, y_2, z_1, z_2$ in k, we have

$$\begin{aligned}
& \mathbb{P}(c(x_1 + \epsilon y_1, x_1 + \epsilon z_1) + (x_2 + \epsilon y_2, x_2 + \epsilon z_2)) \\
= \;& \mathbb{P}((cx_1 + x_2 + \epsilon(cy_1 + y_2), cx_1 + x_2 + \epsilon(cz_1 + z_2)) \\
= \;& cx_1 + x_2 + \epsilon(cy_1 + y_2 + cz_1 + z_2) \\
= \;& c(x_1 + \epsilon(y_1 + z_1)) + (x_2 + \epsilon(y_2 + z_2)) \\
= \;& c\mathbb{P}((x_1 + \epsilon y_1, x_1 + \epsilon z_1)) + \mathbb{P}((x_2 + \epsilon y_2, x_2 + \epsilon z_2))
\end{aligned}$$

and

$$
\begin{aligned}
& \mathbb{P}((x_1+\epsilon y_1, x_1+\epsilon z_1) * (x_2+\epsilon y_2, x_2+\epsilon z_2)) \\
= \;& \mathbb{P}((x_1x_2+\epsilon(x_1y_2+x_2y_1), x_1x_2+\epsilon(x_1z_2+x_2z_1)) \\
= \;& x_1x_2+\epsilon(x_1y_2+x_2y_1+x_1z_2+x_2z_1) = x_1x_2+\epsilon(x_1(y_2+z_2)+x_2(y_1+z_1)) \\
= \;& (x_1+\epsilon(y_1+z_1)) * (x_2+\epsilon(y_2+z_2)) \\
= \;& \mathbb{P}((x_1+\epsilon y_1, x_1+\epsilon z_1)) * \mathbb{P}((x_2+\epsilon y_2, x_2+\epsilon z_2)).
\end{aligned}
$$

Since the λ-algebra structures of $k[\epsilon]$ and $k[\epsilon] \times_k k[\epsilon]$ coincide with their k-algebra structures, it follows that $\mathbb{P}$ is a λ-algebra homomorphism which sends $1_{k[\epsilon]\times_k k[\epsilon]}$ to $1_{k[\epsilon]}$. Moreover, $\pi_{k[\epsilon]} \circ \mathbb{P} = \pi_{k[\epsilon]\times_k k[\epsilon]}$, which implies that $\mathbb{P}$ is a morphism in $\hat{\mathcal{C}}$.

The morphism $\mathbb{P}$ then extends to a ring homomorphism, which we also call $\mathbb{P}$, between the respective $n \times n$ matrix rings. If ϕ and ψ are any two lifts of V over $k[\epsilon]$, then

$$\pi_{k[\epsilon]} \circ \phi = \rho = \pi_{k[\epsilon]} \circ \psi.$$

Hence for all a in A and all $1 \le i, j \le n$, the (i,j)-entries $\phi(a)_{ij}$ and $\psi(a)_{ij}$ of $\phi(a)$ and $\psi(a)$, respectively, define an element

$$(\phi(a)_{ij}, \psi(a)_{ij}) \in k[\epsilon] \times_k k[\epsilon].$$

By abuse of notation, let $(\phi(a), \psi(a))$ denote the matrix in $Mat_n(k[\epsilon] \times_k k[\epsilon])$ with (i,j)-entry given by $(\phi(a)_{ij}, \psi(a)_{ij})$ for all $1 \le i, j, \le n$. Therefore, we can define a lift $\mathbb{P}(\phi, \psi)$ of V over $k[\epsilon] \times_k k[\epsilon]$ by $\mathbb{P}(\phi,\psi)(a) = \mathbb{P}((\phi(a), \psi(a)))$, for all a in A.

Notice that this also extends to a well-defined map on deformations. Namely, using the same name, $\mathbb{P}$, we can construct a map

$$\mathbb{P} : F_V(k[\epsilon]) \times_{F_V(k)} F_V(k[\epsilon]) \to F_V(k[\epsilon])$$

as follows: If $[\phi]$ and $[\psi]$ are two deformations in $F_V(k[\epsilon])$, then $([\phi], [\psi])$ is certainly in $F_V(k[\epsilon]) \times_{F_V(k)} F_V(k[\epsilon])$ due to the fact that $F_V(k)$ contains only a single element, namely $[\rho]$. We now define $\mathbb{P}([\phi], [\psi])$ as $[\mathbb{P}(\phi, \psi)]$. To see that this is well-defined, suppose that $[\phi] = [\phi']$ and $[\psi] = [\psi']$. Then, by definition of strictly equivalent lifts, there exist X and Y in $GL_n(k[\epsilon])$ such that

$$\phi' = X\phi X^{-1},$$

$$\psi' = Y\psi Y^{-1},$$

and both X and Y reduce to the identity matrix in $GL_n(k)$. Notice that if $X = I+\epsilon X'$ and $Y = I + \epsilon Y'$ where I is the $n \times n$ identity matrix in $GL_n(k)$ and X' and Y' are suitable matrices in $Mat_n(k)$, then $X^{-1} = I - \epsilon X'$ and $Y^{-1} = I - \epsilon Y'$. Moreover, $\mathbb{P}(X, Y) = I+\epsilon(X'+Y')$ and $\mathbb{P}(X^{-1}, Y^{-1}) = I-\epsilon(X'+Y') = (\mathbb{P}(X, Y))^{-1}$. Therefore, we have

$$\begin{aligned} \mathbb{P}(\phi', \psi') &= \mathbb{P}(X\phi X^{-1}, Y\psi Y^{-1}) \\ &= \mathbb{P}(X, Y)\mathbb{P}(\phi, \psi)\mathbb{P}(X^{-1}, Y^{-1}) \\ &= \mathbb{P}(X, Y)\mathbb{P}(\phi, \psi)(\mathbb{P}(X, Y))^{-1}, \end{aligned}$$

which proves $[\mathbb{P}(\phi, \psi)] = [\mathbb{P}(\phi', \psi')]$. Hence $\mathbb{P}$ extends to a well-defined map on deformations. We can therefore define addition in $F_V(k[\epsilon])$ as $[\phi] + [\psi] = \mathbb{P}([\phi], [\psi])$.

We can also define scalar multiplication in $F_V(k[\epsilon])$ by considering the map $\mathbb{M} : k \times k[\epsilon] \to k[\epsilon]$, given by $\mathbb{M}(c, a + b\epsilon) = a + cb\epsilon$ for all a, b, c in k. Notice that for any fixed $c \in k$, $\mathbb{M}(c, *) : k[\epsilon] \to k[\epsilon]$ is a morphism in $\hat{\mathcal{C}}$. To see this, note that $\mathbb{M}$ is a λ-algebra homomorphism since for all a_1, a_2, b_1, b_2, c_1 in k we have

$$\begin{aligned}
& \mathbb{M}(c, c_1(a_1 + b_1\epsilon) + (a_2 + b_2\epsilon)) \\
= \;& \mathbb{M}(c, c_1a_1 + a_2 + \epsilon(c_1b_1 + b_2)) \\
= \;& c_1a_1 + a_2 + c\epsilon(c_1b_1 + b_2) \\
= \;& c_1(a_1 + cb_1\epsilon) + (a_2 + cb_2\epsilon) \\
= \;& c_1\mathbb{M}(c, a_1 + b_1\epsilon) + \mathbb{M}(c, a_2 + b_2\epsilon),
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{M}(c, (a_1 + b_1\epsilon) * (a_2 + b_2\epsilon)) \\
= \;& \mathbb{M}(c, a_1a_2 + \epsilon(a_1b_2 + a_2b_1)) \\
= \;& a_1a_2 + c\epsilon(a_1b_2 + a_2b_1) \\
= \;& (a_1 + cb_1\epsilon) * (a_2 + cb_2\epsilon) \\
= \;& \mathbb{M}(c, a_1 + b_1\epsilon) * \mathbb{M}(c, a_2 + b_2\epsilon).
\end{aligned}$$

Since $\mathbb{M}(c, 1_{k[\epsilon]}) = 1_{k[\epsilon]}$ and $\pi_{k[\epsilon]} \circ \mathbb{M} = \pi_{k[\epsilon]}$, this implies that $\mathbb{M}$ is a morphism in $\hat{\mathcal{C}}$. Note that we can extend $\mathbb{M}$ in the natural way to a map $k \times Mat_n(k[\epsilon]) \to Mat_n(k[\epsilon])$, which we also denote by $\mathbb{M}$. If ϕ is a lift of V over $k[\epsilon]$, we define $\mathbb{M}(c, \phi)$ to be the lift of V over $k[\epsilon]$ given by $\mathbb{M}(c,\phi)(a) = \mathbb{M}(c,\phi(a))$, for all a in A. Suppose that ϕ and ϕ' are two strictly equivalent lifts of V over $k[\epsilon]$. Then $\phi' = X\phi X^{-1}$ for some $X \in GL_n(k[\epsilon])$ which reduces to the identity matrix in $GL_n(k)$. Notice that if $X = I + \epsilon X'$, then $\mathbb{M}(c,X) = I + \epsilon cX'$ and $\mathbb{M}(c,X^{-1}) = I - \epsilon cX' = (\mathbb{M}(c,X))^{-1}$. Hence we have

$$\mathbb{M}(c,\phi') = \mathbb{M}(c,X\phi X^{-1}) = \mathbb{M}(c,X)\mathbb{M}(c,\phi)\mathbb{M}(c,X^{-1}) = \mathbb{M}(c,X)\mathbb{M}(c,\phi)\mathbb{M}(c,X)^{-1},$$

which proves $[\mathbb{M}(c,\phi)] = [\mathbb{M}(c,\phi')]$. Therefore, we can define scalar multiplication in $F_V(k[\epsilon])$ as $c \cdot [\phi] = [\mathbb{M}(c,\phi)]$. The vector space axioms for $F_V(k[\epsilon])$ using this addition and scalar multiplication are readily checked, since $\mathbb{P}$ and $\mathbb{M}$ are morphisms in $\hat{\mathcal{C}}$. Hence $F_V(k[\epsilon])$ is a k-vector space. □

We also need to consider ring extensions in $\mathcal{C}$ with "small" kernels, which we now define.

Definition 3.14. We call a morphism $\alpha : B \to B'$ in $\mathcal{C}$ a small extension if α is surjective and $\ker(\alpha)$ has length 1, as a left ideal. Note that there exists then an element $t \in B$ so that $\ker(\alpha) = Bt = tB$ and $m_B t = 0 = t m_B$. In particular, each element in Bt can be written as lt for some l in $j_B(\lambda)$.

Define $k[Bt] = k \oplus Bt = \{a + lt | a \in k, l \in j_B(\lambda)\}$. Then $k[Bt]$ is a ring in $\mathcal{C}$ which is isomorphic to $k[\epsilon]$ by the isomorphism $\omega : k[\epsilon] \to k[Bt]$, defined by

$$\omega(a + b\epsilon) = a + \hat{b}t$$

for all a, b in k, where $\hat{b}$ is any element in $j_B(\lambda)$ satisfying $\pi_B(\hat{b}) = b$.

The next result will be important when showing smoothness of the natural transformation $\tau = \tau(R, [\phi]) : h_R \to F_V$, defined in Definition 3.3, part (c) (see also Definition 3.6).

Lemma 3.15. *Let $\alpha : B \to B'$ be a small extension in $\mathcal{C}$, and let* $\ker(\alpha) = Bt$*, where $t \in B$ satisfies $m_B t = t m_B = 0$.*

(a) If β and γ are two lifts of V over B such that $\alpha \circ \beta = \alpha \circ \gamma$, then $\gamma - \beta$ defines a lift $\psi_{\gamma\beta}$ of V over $k[Bt]$, by defining

$$\psi_{\gamma\beta}(a) = \rho(a) + (\gamma - \beta)(a)$$

for all a in A.

(b) For every lift $\phi : A \to Mat_n(k[Bt])$ of V over $k[Bt]$, define $\tilde{\phi}(a) = \phi(a) - \rho(a) \in Mat_n(Bt)$, for all $a \in A$. If β is a lift of V over B, then for every lift γ of V over B satisfying $[\alpha \circ \beta] = [\alpha \circ \gamma]$, there exists a lift ψ of V over $k[Bt]$ such that γ is strictly equivalent to the lift $\beta_\psi : A \to Mat_n(B)$, defined by

$$\beta_\psi(a) = \beta(a) + \tilde{\psi}(a)$$

for all $a \in A$.

Proof. (a) Since $\pi_{k[Bt]} \circ \psi_{\gamma\beta} = \rho$, it suffices to show that $\psi_{\gamma\beta}$ is a λ-algebra homomorphism. Let $a_1, a_2 \in A$ and $l \in \lambda$. Then

$$\begin{aligned}
\psi_{\gamma\beta}(la_1 + a_2) &= \rho(la_1 + a_2) + (\gamma - \beta)(la_1 + a_2) \\
&= \pi_\lambda(l)\rho(a_1) + \rho(a_2) + j_B(l)(\gamma - \beta)(a_1) + (\gamma - \beta)(a_2) \\
&= l(\rho(a_1) + (\gamma - \beta)(a_1)) + (\rho(a_2) + (\gamma - \beta)(a_2)) \\
&= l\psi_{\gamma\beta}(a_1) + \psi_{\gamma\beta}(a_2)
\end{aligned}$$

and, using the isomorphism ω from Definition 3.14 and using Definition 2.12,

$$\begin{aligned}
\psi_{\gamma\beta}(a_1 a_2) &= \rho(a_1 a_2) + (\gamma - \beta)(a_1 a_2) \\
&= \rho(a_1)\rho(a_2) + \gamma(a_1)\gamma(a_2) - \beta(a_1)\beta(a_2) \\
&= \rho(a_1)\rho(a_2) + \gamma(a_1)(\gamma - \beta)(a_2) + (\gamma - \beta)(a_1)\beta(a_2) \\
&= (\rho(a_1) + (\gamma - \beta)(a_1)) * (\rho(a_2) + (\gamma - \beta)(a_2)) \\
&= \psi_{\gamma\beta}(a_1) * \psi_{\gamma\beta}(a_2).
\end{aligned}$$

Hence $\psi_{\gamma\beta}$ is a lift of V over $k[Bt]$.

(b) Let β and γ be lifts of V over B satisfying $[\alpha \circ \beta] = [\alpha \circ \gamma]$. This means there exists a matrix P' in $GL_n(B')$ which reduces to the identity matrix in $GL_n(k)$ such that

$$\alpha \circ \beta = P'(\alpha \circ \gamma)(P')^{-1}.$$

Let P be any matrix in $Mat_n(B)$ so that $\alpha(P) = P'$. Then P reduces to the identity in $Mat_n(k)$, and hence is a unit in $Mat_n(B)$, and

$$\alpha \circ \beta = \alpha \circ (P\gamma P^{-1}).$$

Define $\gamma' = P\gamma P^{-1}$, so that γ and γ' are strictly equivalent. Define $\psi : A \to Mat_n(k[Bt])$ to be equal to the lift $\psi_{\gamma'\beta}$ defined in part (a). Then it follows that $\tilde{\psi}(a) = (\gamma' - \beta)(a)$, for all a in A. Therefore,

$$\gamma'(a) = \beta(a) + (\gamma' - \beta)(a) = \beta(a) + \tilde{\psi}(a)$$

for all a in A, which means that $\gamma' = \beta_\psi$. Hence, $[\gamma] = [\gamma'] = [\beta_\psi]$.

□

We now prove our main result by constructing a versal couple for V.

Theorem 3.16. *If $F_V(k[\epsilon])$ is finite dimensional as a k-vector space, then there exists a versal couple for V. In particular, V then has a versal deformation ring $R(A,V)$.*

The proof of this theorem consists of several steps and uses similar arguments as in the proof of [10, Theorem 2.11]. We will first construct a certain ring R in $\hat{\mathcal{C}}$, and a certain lift ϕ of V over R. We will then prove the theorem by proving that $(R, [\phi])$ is a versal couple for V.

Construction of $(R, [\phi])$:

If $F_V(k[\epsilon])$ is an m-dimensional k-vector space for some $m \geq 1$, we can select a finite basis $\{[\psi_i]\}_{i=1}^m$ of $F_V(k[\epsilon])$. Due to the fact that each lift ψ_i must satisfy

$\pi_{k[\epsilon]} \circ \psi_i = \rho$, we can then define a lift, ϕ_2', of V over the m-fold pullback of $k[\epsilon]$ over k, $k[\epsilon] \times_k k[\epsilon] \times_k ... \times_k k[\epsilon]$, by

$$\phi_2'(a) = (\psi_1(a), ..., \psi_m(a)),$$

for all a in A. Let R_2' be the m-fold pullback of $k[\epsilon]$ over k, let S be the power series ring over λ in m non-commuting variables $x_1, x_2, ..., x_m$, and let I_2 be the ideal $\mu_S + m_S^2$ in S. Notice that since $j_S : \lambda \to S$ is simply the inclusion map, we commit a slight abuse of notation and write μ to refer to μ_S. Explicitly,

$$R_2' = k[\epsilon] \times_k k[\epsilon] \times_k ... \times_k k[\epsilon],$$

$$S = \lambda\langle\langle x_1, ..., x_m\rangle\rangle,$$

and

$$I_2 = \mu + (x_1, ..., x_m)^2.$$

By Lemma 3.9, we know that $R_2' \cong S/I_2$ via the isomorphism

$$f : S/I_2 \to R_2',$$

$$f(l + \sum_{i=1}^{m}(l_i x_i) + I_2) = (\pi_\lambda(l) + \epsilon\pi_\lambda(l_i))_i,$$

where $l, l_1, ..., l_m$ lie in λ, and we use that $\pi_S \circ j_S = \pi_\lambda$. Note that the inverse of f is the map

$$g : R_2' \to S/I_2$$

given by

$$g((c + \epsilon c_1, ..., c + \epsilon c_m)) = l + \sum_{i=1}^{m}(l_i x_i) + I_2$$

where $c, c_1, ..., c_m$ lie in k and l and $l_1, ..., l_m$ are any elements of λ so that $\pi_\lambda(l) = c$ and $\pi_\lambda(l_i) = c_i$ for $1 \leq i \leq m$. Now define

$$R_2 = S/I_2$$

and

$$\phi_2 = g \circ \phi_2'.$$

The couple $(R_2, [\phi_2])$ for V begins the construction of the versal couple for V. To continue the construction, we proceed by induction: Let $\ell \geq 2$. Define $R_i = S/I_i$, for $2 \leq i \leq \ell$, where we assume that we have found ideals $I_2 \supseteq I_3 \supseteq ... \supseteq I_\ell$ of S, along with lifts ϕ_i of V over R_i for $2 \leq i \leq \ell$, such that $\pi_i \circ \phi_i = \phi_{i-1}$ for $3 \leq i \leq \ell$ when $\pi_i : R_i \to R_{i-1}$ is the natural projection. Assume further that $m_S^i \subseteq I_i$ for $2 \leq i \leq \ell$. We then define $I_{\ell+1}$ to be the minimal ideal among all two-sided ideals, J, of S, such that the following two conditions hold:

(a) $m_S * I_\ell + I_\ell * m_S \subseteq J \subseteq I_\ell$.

(b) There exists a lift ϕ_J of V over S/J so that $\pi_J \circ \phi_J = \phi_\ell$, where $\pi_J : S/J \to R_\ell$ is defined to be the natural projection.

Notice that the set of ideals satisfying conditions (a) and (b) is non-empty, since I_ℓ satisfies both requirements. To see that a unique minimal ideal exists, we first show that the set of all such ideals is closed under pairwise intersection. Supposing both J and J' satisfy the two conditions (a) and (b), then it follows that $m_S * I_\ell + I_\ell * m_S \subseteq J \cap J' \subseteq I_\ell$, and hence $J \cap J'$ satisfies the first condition (a). Since $I_\ell/(m_S * I_\ell + I_\ell * m_S)$

Figure 3.5: The pullback diagram used to show that $J \cap J'$ satisfies condition (b).

$$\begin{tikzcd}
\phi_{J\cap J'} & & \\
S/(J \cap J') \arrow[ddr, "\cong"] & & \\
& \phi_J \times_{\phi_\ell} \phi_{J'} & \phi_{J'} \\
& S/J \times_{S/I_\ell} S/J' \arrow[r] \arrow[d] & S/J' \arrow[d] \\
& S/J \arrow[r] & S/I_\ell \\
& \phi_J & \phi_\ell
\end{tikzcd}$$

is a left/right S-module on which m_S acts trivially, it is a k-vector space. Moreover, since $m_S^\ell \subseteq I_\ell$ by assumption, we know that

$$m_S^{\ell+1} \subseteq m_S * I_\ell + I_\ell * m_S.$$

Because $S/m_S^{\ell+1}$ is Artinian by Theorem 2.8, part (a), it follows that $S/(m_S * I_\ell + I_\ell * m_S)$ is Artinian, and therefore $I_\ell/(m_S * I_\ell + I_\ell * m_S)$ is a finite dimensional k-vector space. Take a k-basis of $(J+J')/(m_S * I_\ell + I_\ell * m_S)$, say $\{d_j + (m_S * I_\ell + I_\ell * m_S)\}_{j=1}^{u}$, and extend it to a k-basis of $I_\ell/(m_S * I_\ell + I_\ell * m_S)$, say $\{d_j + (m_S * I_\ell + I_\ell * m_S)\}_{j=1}^{v}$, where $v \geq u$ and $d_1, ..., d_u$ lie in $J + J'$ and $d_{u+1}, ..., d_v$ lie in $I_\ell - (J + J')$. Consider the two-sided ideal $J'' = J' + (d_{u+1}, ..., d_v)$ in S. Then $J \cap J'' = J \cap J'$ since this equality holds modulo $(m_S * I_\ell + I_\ell * m_S)$. Therefore, we can replace J' by J'' and $\phi_{J'}$ by its projection onto S/J'', to be able to assume that $J + J' = I_\ell$. Hence, we have the pullback diagram in Figure 3.5, where all maps are natural. Note that

$$S/J \times_{S/I_\ell} S/J' = S/J \times_{S/(J+J')} S/J' \cong S/(J \cap J')$$

and that

$$\phi_J \times_{\phi_\ell} \phi_{J'}$$

is a lift of V over $S/J \times_{S/I_\ell} S/J'$ which projects onto ϕ_ℓ. We then compose this lift with the natural isomorphism between $S/J \times_{S/I_\ell} S/J'$ and $S/(J \cap J')$ to arrive at a lift, $\phi_{J\cap J'}$, of V over $S/(J \cap J')$ which projects onto ϕ_ℓ, showing that $J \cap J'$ also satisfies condition (b). This proves that the set of all ideals satisfying conditions (a) and (b) is closed under pairwise intersections. Moreover, since we have already seen that $I_\ell/(m_S * I_\ell + I_\ell * m_S)$ is a finite dimensional k-vector space, it follows that an intersection of any arbitrary collection of two-sided ideals satisfying condition (a) must in fact be able to be written as a finite intersection of such ideals. Since our set of ideals in question is closed under pairwise intersection, we have proved the existence of our minimal ideal, $I_{\ell+1}$. In addition, $m_S^{\ell+1} \subseteq I_{\ell+1}$ since $m_S^{\ell+1} \subseteq m_S * I_\ell + I_\ell * m_S$. Let $\phi_{\ell+1}$ be any lift of V over $R_{\ell+1} = S/I_{\ell+1}$ such that $\pi_{\ell+1} \circ \phi_{\ell+1} = \phi_\ell$, where $\pi_{\ell+1} : R_{\ell+1} \to R_\ell$ is the natural projection. The construction of $I_{\ell+1}$ and $\phi_{\ell+1}$ completes the inductive step. Define I to be the intersection of all the I_ℓ, and define R to be S/I. We claim that R is a versal deformation ring for V. However, before proving that, we first show that R is an object of $\hat{\mathcal{C}}$, and also construct a lift ϕ of V over R.

Proof that R is in $\hat{\mathcal{C}}$:

Since $m_S^\ell \subseteq I_\ell$ for all $\ell \geq 2$, I_ℓ can be written as a union of cosets of the form $b + m_S^\ell$, where the union is taken over all b in I_ℓ. Since the cosets of m_S^ℓ partition S, the complement of I_ℓ can be written as the union of cosets of the form $b' + m_S^\ell$, where the union is taken over all b' in $S - I_\ell$. Hence, the complement of I_ℓ is open, which means that I_ℓ is closed for each $\ell \geq 2$. Therefore, I is the intersection of closed subsets of S, and hence I is closed. Moreover, since each $I_\ell \subseteq m_S$, I is a proper ideal of S. The fact that $R = S/I$ is an object of $\hat{\mathcal{C}}$ now follows from Theorem 2.8, parts (f) and (h).

Construction of a particular ϕ of V over R:

In order to complete the construction of the desired couple, it will be necessary to find a deformation $[\phi]$ of V over R and show it is versal. We now construct a lift ϕ of V over R, as follows. Recall that the ϕ_ℓ were chosen so that $\pi_{\ell+1} \circ \phi_{\ell+1} = \phi_\ell$ for all $\ell \geq 2$, where $\pi_{\ell+1} : R_{\ell+1} \to R_\ell$ is the natural projection. Therefore, we know that for all a in A, the sequence $(\phi_\ell(a))_\ell$ is in the inverse limit $\varprojlim_\ell Mat_n(R_\ell) = \varprojlim_\ell Mat_n(S/I_\ell)$. Consider the following inverse systems $(S/I_\ell, \pi_\ell) = (S/I_\ell)$ and $(S/m_S^\ell, \xi_\ell)$, where $\xi_\ell : S/m_S^\ell \to S/m_S^{\ell-1}$ is the natural projection. Then $(I/m_S^\ell, \tilde{\xi}_\ell) = (I_\ell/m_S^\ell)$ is also an inverse system, where $\tilde{\xi}_\ell : I_\ell/m_S^\ell \to I_{\ell-1}/m_S^{\ell-1}$ is the restriction of ξ_ℓ using that $I_\ell \subseteq I_{\ell-1}$. By extending π_ℓ, ξ_ℓ, and $\tilde{\xi}_\ell$ to matrix rings, we obtain inverse systems $(Mat_n(S/I_\ell), \pi_\ell) = (Mat_n(S/I_\ell))$, $(Mat_n(S/m_S^\ell), \xi_\ell) = (Mat_n(S/m_S^\ell))$ and $(Mat_n(I_\ell/m_S^\ell), \tilde{\xi}_\ell) = (Mat_n(I_\ell/m_S^\ell))$. Since S/m_S^ℓ is an Artinian ring for each ℓ,

$Mat_n(S/m_S^\ell)$ is also an Artinian ring for each ℓ. Therefore, the collection of ideals $\{Mat_n((I_{\ell+i}+m_S^\ell)/m_S^\ell)\}_{i\geq 1}$ stabilizes for each ℓ. In other words, the inverse system $(Mat_n(I_\ell/m_S^\ell))$ satisfies the Mittag-Leffler condition. It follows from [5, Proposition III.10.3] that the following sequence of inverse limits is exact, where all the maps are natural:

$$0 \to \varprojlim_\ell Mat_n(I_\ell/m_S^\ell) \to \varprojlim_\ell Mat_n(S/m_S^\ell) \to \varprojlim_\ell Mat_n(S/I_\ell) \to 0.$$

Therefore, there exists an element $(\bar{B}_\ell)_\ell$ in the inverse limit $\varprojlim_\ell Mat_n(S/m_S^\ell)$ such that $\bar{B}_\ell$ is in $Mat_n(S/m_S^\ell)$ and $\beta_\ell(\bar{B}_\ell) = \phi_\ell(a)$ for all $\ell \geq 2$, where

$$\beta_\ell : Mat_n(S/m_S^\ell) \to Mat_n(S/I_\ell)$$

is the natural projection. By Lemma 2.5, $Mat_n(S)$ is $Mat_n(m_S)$-adically complete. Therefore, we can find a matrix B in $Mat_n(S)$ such that $q_\ell^S(B) = \bar{B}_\ell$ for all ℓ, where

$$q_\ell^S : Mat_n(S) \to Mat_n(S/m_S^\ell)$$

is the natural projection. If $(\bar{B}'_\ell)_\ell$ is any other element in $\varprojlim_\ell Mat_n(S/m_S^\ell)$ such that $\bar{B}'_\ell$ is in $Mat_n(S/m_S^\ell)$ and $\beta_\ell(\bar{B}'_\ell) = \phi_\ell(a)$ for all $\ell \geq 2$, then by a similar argument we can find some matrix B' in $Mat_n(S)$ such that $q_\ell^S(B') = \bar{B}'_\ell$ for all ℓ. But then $\beta_\ell(q_\ell^S(B - B')) = 0$ for all $\ell \geq 2$, and so $B - B'$ lies in $Mat_n(I_\ell)$ for all $\ell \geq 2$, from which it follows that $B - B'$ is in $Mat_n(I)$. Thus, there exists a unique element $B + Mat_n(I)$ in $Mat_n(S/I)$ such that $q_\ell^I(B + Mat_n(I)) = \phi_\ell(a)$ for all $\ell \geq 2$, where

$$q_\ell^I : Mat_n(S/I) \to Mat_n(S/I_\ell)$$

is the natural projection.

Therefore, we obtain a lift ϕ of V over S/I by defining, for all $a \in A$, $\phi(a)$ to be equal to this unique element $B + Mat_n(I)$ in $Mat_n(S/I)$, which for all ℓ projects onto $\phi_\ell(a)$ via the natural projections.

Finally, before proving that the constructed couple $(R, [\phi])$ is a versal couple, we prove two useful relationships between the constructed ideals, I_ℓ, and their intersections, I. Namely, we show

$$I_{\ell+1} = I + m_S * I_\ell + I_\ell * m_S \tag{3.1}$$

for all $\ell \geq 2$, and

$$I_{\ell+i} \subseteq I + m_S^{i+1} \tag{3.2}$$

for all $\ell \geq 2$ and $i \geq 0$.

To prove (3.1), first note that

$$m_S * I_\ell + I_\ell * m_S \subseteq I + m_S * I_\ell + I_\ell * m_S \subseteq I_\ell$$

for all $\ell \geq 2$. Moreover, ϕ is a lift of V over $R = S/I$ which projects onto the lift ϕ_ℓ of V over $R_\ell = S/I_\ell$, for all ℓ. Projecting ϕ onto $S/(I + m_S * I_\ell + I_\ell * m_S)$, we can find a lift of V over $S/(I + m_S * I_\ell + I_\ell * m_S)$ which projects onto ϕ_ℓ. Since $I_{\ell+1}$ is the minimal ideal satisfying both these conditions, it follows that

$$I_{\ell+1} \subseteq I + m_S * I_\ell + I_\ell * m_S.$$

But since both I and $m_S * I_\ell + I_\ell * m_S$ are contained in $I_{\ell+1}$, we obtain (3.1).

To prove (3.2), we fix $\ell \geq 2$ and use induction on $i \geq 0$. Since $I_\ell \subseteq m_S$, it follows that $I_\ell \subseteq I + m_S$, which means that (3.2) holds for $i = 0$. Assume (3.2) is true for $0 \leq i \leq r$. Then, using (3.1), we have

$$\begin{aligned} I_{\ell+r+1} &= I + m_S * I_{\ell+r} + I_{\ell+r} * m_S \\ &\subseteq I + m_S * (I + m_S^{r+1}) + (I + m_S^{r+1}) * m_S = I + m_S^{r+2} \end{aligned}$$

which proves (3.2).

Proof of Theorem 3.16:

It remains to show that $(R, [\phi])$ is a versal couple for V. To see this, we prove the required properties of the natural transformation $\tau = \tau(R, [\phi]) : h_R \to F_V$, recalling that for all B in $\hat{\mathcal{C}}$

$$\tau_B : h_R(B) \to F_V(B)$$

is defined as

$$\tau_B(\alpha) = [\alpha \circ \phi]$$

for all $\alpha : R \to B$ in $\hat{\mathcal{C}}$.

First, we will show that $\tau_{k[\epsilon]}$ is a bijection. Recall that $R = S/I$ and $R_2 = S/I_2$. Then $h_R(k[\epsilon]) = h_{S/I}(k[\epsilon])$ is naturally identified with $h_{R_2}(k[\epsilon]) = h_{S/I_2}(k[\epsilon])$ due to the fact that

$$\mu_R + m_R^2 = (\mu + m_S^2)/I = I_2/I$$

is contained in the kernel of any morphism $R = S/I \to k[\epsilon]$ in $\hat{\mathcal{C}}$. For $1 \leq i \leq m$, let $b_i = x_i + I_2$. Then $b_1, ..., b_m$ are elements in m_{R_2} so that their images form a k-basis of $m_{R_2}/(\mu_{R_2} + m_{R_2}^2)$. Let $\alpha_{b_1}, ..., \alpha_{b_m} \in h_{R_2}(k[\epsilon])$ be the morphisms constructed in Corollary 3.10. Recall that, for $1 \leq i \leq m$, $\alpha_{b_i}(\mu_{R_2} + m_{R_2}^2) = 0$, $\alpha_{b_i}(b_i) = \epsilon$ and $\alpha_{b_i}(b_j) = 0$ for all $j \neq i$. From this it follows that $\{\alpha_{b_i}\}_{i=1}^m$ is a k-basis of $h_{R_2}(k[\epsilon])$. Letting $p_2 : R \to R_2$ be the natural projection, we obtain that $\{\alpha_{b_i} \circ p_2\}_{i=1}^m$ is a k-basis of $h_R(k[\epsilon])$. Recall that our lift $\phi_2 = p_2 \circ \phi$ of V over $R_2 = S/I_2$ has the property that

$$\phi_2 = g \circ \phi_2' = g \circ (\psi_1, ..., \psi_m),$$

where g is the inverse of f and $\{[\psi_i]\}_{i=1}^m$ is a k-basis of $F_V(k[\epsilon])$. From this and the properties of α_{b_i}, $1 \leq i \leq m$, it follows that

$$\tau_{k[\epsilon]}(\alpha_{b_i} \circ p_2) = [\alpha_{b_i} \circ p_2 \circ \phi] = [\alpha_{b_i} \circ \phi_2] = [\psi_i]$$

for $1 \leq i \leq m$. As $\tau_{k[\epsilon]}$ maps basis elements to basis elements, it follows that $\tau_{k[\epsilon]}$ is a bijection.

In order to show that τ is smooth, consider any surjective morphism $\alpha : B \to B'$ in the category $\mathcal{C}$. We must show that the natural map q in Figure 3.1 is surjective. This is equivalent to the following statement, outlined in Figure 3.6: If $[\beta]$ is in $F_V(B)$ and u is in $h_R(B')$ such that $[u \circ \phi] = [\alpha \circ \beta]$, we need to find an element δ in $h_R(B)$ such that $\alpha \circ \delta = u$ and $[\delta \circ \phi] = [\beta]$. Since $[u \circ \phi] = [\alpha \circ \beta]$ there exists a matrix P' in $GL_n(B')$ which reduces to the identity matrix in $GL_n(k)$ such that

$$u \circ \phi = P'(\alpha \circ \beta)(P')^{-1}.$$

Let P in $Mat_n(B)$ be any matrix such that $\alpha(P) = P'$. Then P reduces to the

Figure 3.6: The smoothness conditions on δ.

$$\begin{tikzcd} & {[\phi]} \\ & R \arrow[d, "u"] \arrow[dl, dashed, "\delta"'] \\ B \arrow[r, "\alpha"'] & B' \\ {[\beta]} & {[\beta']} \end{tikzcd}$$

$$[\alpha \circ \beta] = [u \circ \phi]$$

identity in $Mat_n(k)$ and

$$u \circ \phi = \alpha \circ (P\beta P^{-1}).$$

Replacing β by the strict equivalent $P\beta P^{-1}$, we can assume that

$$u \circ \phi = \alpha \circ \beta.$$

Define $\beta' = u \circ \phi = \alpha \circ \beta$.

Since B is Artinian and α is surjective, the kernel of α has finite length as a left ideal. Hence we can factor α into finitely many compositions of surjections, each of which has a kernel of length 1, as left ideals. By a simple induction argument, suggested by Figure 3.7, we can assume that the kernel of α is of length 1 as a left ideal. Note that in Figure 3.7, each morphism $B_i \xrightarrow{\alpha_i} B_{i-1}$ has a kernel of length 1 as a left ideal, and $\beta_{i-1} = \alpha_i \circ \beta_i$, for $1 \leq i \leq v$.

Hence, we assume now that α is a small extension, as defined in Definition

3.14. In particular, $m_B * \ker(\alpha) = 0 = \ker(\alpha) * m_B$. By (3.2), $I_{2+i} \subseteq I + m_S^{i+1}$ for any $i \geq 0$, which we can rewrite as $I_i \subseteq I + m_S^{i-1}$ for any $i \geq 2$.

Figure 3.7: The induction argument used to show that the kernel of α can be assumed to have length 1.

$$
\begin{tikzcd}
 & & & & \phi \\
 & & & & R \arrow[dllll, dashed, "\delta = \delta_v"'] \arrow[dlll, dashed, "\delta_{v-1}"] \arrow[dl, dashed, "\delta_1"'] \arrow[d, "u"] \\
B = B_v \arrow[r, "\alpha_v"] & B_{v-1} \arrow[r, "\alpha_{v-1}"] & \dots \arrow[r] & B_1 \arrow[r, "\alpha_1"] & B_0 = B' \\
\beta = \beta_v & \beta_{v-1} & \dots & \beta_1 & \beta_0 = \beta'
\end{tikzcd}
$$

Since B' is Artinian, there exists an integer $\ell \geq 2$ such that $m_{B'}^{\ell-1} = 0$. From $I_\ell \subseteq I + m_S^{\ell-1}$ it follows that $u : S/I = R \to B'$ factors through a morphism $\bar{u} : S/I_\ell \to B'$ in $\hat{\mathcal{C}}$. Let $w : S \to B'$ be the composition of u with the natural projection $S \to S/I$, as indicated in the vertical right paths of Figure 3.8. For $1 \leq i \leq m$, let y_i be an element of B belonging to $\alpha^{-1}(w(x_i))$. Since α and w are morphisms in $\hat{\mathcal{C}}$, y_i is in m_B for all i. We can then define $\eta(x_i) = y_i$ for all $1 \leq i \leq m$. Since $S = \lambda\langle\langle x_1, ..., x_m\rangle\rangle$, we can extend η naturally to a morphism $\eta : S \to B$ in $\hat{\mathcal{C}}$ such that $\alpha \circ \eta = w$. Since $w(I_\ell) = 0$, we see that $\eta(I_\ell) \subseteq ker(\alpha)$. Recalling that

$$m_B * \ker(\alpha) = 0 = \ker(\alpha) * m_B$$

we then have that

$$\eta(m_S * I_\ell + I_\ell * m_S) \subseteq m_B * \eta(I_\ell) + \eta(I_\ell) * m_B \subseteq m_B * \ker(\alpha) + \ker(\alpha) * m_B = 0$$

and hence $m_S * I_\ell + I_\ell * m_S \subseteq \ker(\eta)$.

Figure 3.8: The diagram used to find a morphism $\delta' : R = S/I \to B$ with $\alpha \circ \delta' = u$.

$$\begin{tikzcd}
 & S \arrow[d] \arrow[dddl, dashed, "\psi"'] \arrow[ddddl, bend right=60, "\eta"'] \arrow[dddd, bend left=70, "w"] \\
 & S/I \arrow[d] \arrow[ddd, bend left=40, "u"] \\
 & S/I_{\ell+1} \arrow[d] \\
\mathcal{H} \arrow[r, "h_2"] \arrow[d, "h_1"] & S/I_\ell \arrow[d, "\bar{u}"'] \\
B \arrow[r, "\alpha"'] & B'
\end{tikzcd}$$

Let $\mathcal{H}$ be the pullback of α and $\bar{u}$, as indicated in Figure 3.8. Then there exists a morphism $\psi : S \to \mathcal{H}$ in $\hat{\mathcal{C}}$ such that $\psi(s) = (\eta(s), s + I_\ell)$ for all $s \in S$. It follows from above that $m_S * I_\ell + I_\ell * m_S \subseteq \ker(\psi)$, and it is clear from the definition of ψ that $\ker(\psi) \subseteq I_\ell$. Notice that $h_2 : \mathcal{H} \to S/I_\ell$ is surjective, since $h_2 \circ \psi : S \to S/I_\ell$ is surjective. Furthermore, the kernel of h_2 is the set of all elements of $\mathcal{H}$ of the form $(x, 0 + I_\ell)$, where x is an element of the kernel of α. Since we assumed that the kernel of α has length 1, we see that the kernel of h_2 must also be of length 1. Therefore,

$$length(S/I_\ell) \leq length(Im(\psi)) \leq length(\mathcal{H}) = 1 + length(S/I_\ell).$$

If ψ is not surjective, then $Im(\psi)$, which is in $\hat{\mathcal{C}}$ by Theorem 2.8, part (g), is a proper subalgebra of $\mathcal{H}$. Hence, by a length argument, $Im(\psi)$ must be isomorphic to S/I_ℓ via the restriction of h_2. Letting $\sigma : S/I_\ell \to Im(\psi)$ be the morphism in $\hat{\mathcal{C}}$ with $h_2 \circ \sigma = id_{S/I_\ell}$ and letting $\pi_\ell^I : R = S/I \to S/I_\ell$ be the natural projection, we set δ' equal to $h_1 \circ \sigma \circ \pi_\ell^I$. However, if ψ is surjective, then $\mathcal{H}$ is isomorphic to $S/\ker(\psi)$. Recall that we assume $u \circ \phi = \alpha \circ \beta = \beta'$. Since $u \circ \phi = \bar{u} \circ \phi_\ell$, we obtain that the pullback of lifts $\beta \times_{\beta'} \phi_\ell$ is a lift over $\mathcal{H}$ which projects onto ϕ_ℓ via h_2. Hence there is also a lift over $S/\ker(\psi)$ that projects onto ϕ_ℓ. Moreover, we have already established that $m_S * I_\ell + I_\ell * m_S \subseteq \ker(\psi) \subseteq I_\ell$. But by the minimality of $I_{\ell+1}$, we therefore have that $I_{\ell+1} \subseteq \ker(\psi)$, and hence ψ factors through a morphism $\bar{\psi} : S/I_{\ell+1} \to \mathcal{H}$. Letting $\pi_{\ell+1}^I : R = S/I \to S/I_{\ell+1}$ be the natural projection, we set δ' equal to $h_1 \circ \bar{\psi} \circ \pi_{\ell+1}^I$, in this case.

In both cases, it follows that $\alpha \circ \delta' = u$. Define $\gamma = \delta' \circ \phi$, so that γ is a lift of V over B. Then we have

$$\alpha \circ \gamma = \alpha \circ \delta' \circ \phi = u \circ \phi = \alpha \circ \beta.$$

By Lemma 3.15, part (a), $\beta - \gamma$ defines a lift $\psi_{\beta\gamma}$ of V over $k[Bt]$, where

$$\psi_{\beta\gamma}(a) = \rho(a) + (\beta - \gamma)(a)$$

for all a in A. Since we have shown that $\tau_{k[\epsilon]} : h_R(k[\epsilon]) \to F_V(k[\epsilon])$ is a bijection, we can use the isomorphism $\omega : k[\epsilon] \to k[Bt]$ in $\hat{\mathcal{C}}$, from Definition 3.14, to see that there exists a unique morphism $\delta_{\beta\gamma} : R \to k[Bt]$ in $\hat{\mathcal{C}}$ such that $[\delta_{\beta\gamma} \circ \phi] = [\psi_{\beta\gamma}]$. Hence there exists $X \in Mat_n(B)$ so that

$$\delta_{\beta\gamma} \circ \phi = (1 + Xt)\psi_{\beta\gamma}(1 - Xt).$$

Define the map $\tilde{\delta}_{\beta\gamma} : R \to Bt$ by

$$\tilde{\delta}_{\beta\gamma}(r) = \delta_{\beta\gamma}(r) - \pi_R(r)$$

for all r in R, and define $\delta : R \to B$ by

$$\delta(r) = \delta'(r) + \tilde{\delta}_{\beta\gamma}(r)$$

for all r in R. Then δ is a λ-algebra homomorphism, since δ', $\delta_{\beta\gamma}$ and π_R are λ-algebra homomorphisms, and $\tilde{\delta}_{\beta\gamma}(1_R) = 0t$. Moreover, we have for all $r_1, r_2 \in R$, $\delta_{\beta\gamma}(r_1 r_2) = \delta_{\beta\gamma}(r_1)\delta_{\beta\gamma}(r_2)$. Using the isomorphism ω and Definition 2.12, this implies

$$\tilde{\delta}_{\beta\gamma}(r_1 r_2) = \widehat{\pi_R(r_1)}\tilde{\delta}_{\beta\gamma}(r_2) + \tilde{\delta}_{\beta\gamma}(r_1)\widehat{\pi_R(r_2)},$$

where, for $i \in \{1, 2\}$, $\widehat{\pi_R(r_i)}$ denotes any element q_i in B with $\pi_B(q_i) = \pi_R(r_i)$. Note that since $m_B t = 0 = t m_B$, it does not matter which elements q_1, q_2 we choose. Therefore, we obtain for all r_1, r_2 in R,

$$\begin{aligned}
\delta(r_1 r_2) &= \delta'(r_1 r_2) + \tilde{\delta}_{\beta\gamma}(r_1 r_2) \\
&= \delta'(r_1)\delta'(r_2) + \delta'(r_1)\tilde{\delta}_{\beta\gamma}(r_2) + \tilde{\delta}_{\beta\gamma}(r_1)\delta'(r_2) \\
&= (\delta'(r_1) + \tilde{\delta}_{\beta\gamma}(r_1)) * (\delta'(r_2) + \tilde{\delta}_{\beta\gamma}(r_2))
\end{aligned}$$

where the last equality follows since $\tilde{\delta}_{\beta\gamma}(r_1)\tilde{\delta}_{\beta\gamma}(r_2) = 0$ because $(Bt)^2 = 0$. Additionally, we have $\pi_B \circ \delta = \pi_B \circ \delta' = \pi_R$, which implies that δ is a morphism in $\hat{\mathcal{C}}$. Moreover, using that $(Bt)^2 = 0$ and that $m_B t = 0 = t m_B$, we have for all $a \in A$,

$$\begin{aligned}
(\delta \circ \phi)(a) &= \delta'(\phi(a)) + \tilde{\delta}_{\beta\gamma}(\phi(a)) \\
&= \gamma(a) + \delta_{\beta\gamma}(\phi(a)) - \pi_R(\phi(a)) \\
&= \gamma(a) + (1 + Xt)\psi_{\beta\gamma}(a)(1 - Xt) - \rho(a) \\
&= \gamma(a) + (\psi_{\beta\gamma}(a) - \rho(a)) + Xt\psi_{\beta\gamma}(a) - \psi_{\beta\gamma}(a)Xt \\
&= \gamma(a) + (\beta - \gamma)(a) + Xt\beta(a) - \beta(a)Xt \\
&= \beta(a) + Xt\beta(a) - \beta(a)Xt \\
&= (1 + Xt)\beta(a)(1 - Xt).
\end{aligned}$$

Since $t \in m_B$, $1 + Xt$ is a matrix in $GL_n(B)$ which reduces to the identity in $GL_n(k)$. Therefore, $[\delta \circ \phi] = [\beta]$. Since $\tilde{\delta}_{\beta\gamma}(r) \in Bt = \ker(\alpha)$ for all $r \in R$, we also obtain that

$$\alpha \circ \delta = \alpha \circ \delta' = u.$$

This completes the proof that $(R, [\phi])$ is a versal couple for V, and hence the proof of Theorem 3.16.

3.2 Applications

We will now compute an example of a versal deformation ring of a one-dimensional representation. Let $A = S/I$, where $S = \lambda\langle\langle x_1, ..., x_m\rangle\rangle$ and where I is any proper, closed ideal of S. In particular, A is in $\hat{\mathcal{C}}$ by Theorem 2.8, parts (f)

and (h). Let $\pi_I : S \to S/I = A$ be the natural surjection. Then $j_A : \lambda \to A$ is given as $j_A = \pi_I \circ j_S$. Moreover, $\pi_A : A \to k$ satisfies $\pi_A \circ \pi_I = \pi_S$.

Let V be a one-dimensional k-vector space, and give V a left A-module structure by defining $a \cdot v = \pi_A(a)(v)$ for all $v \in V$. In other words, $\pi_A = \rho$.

Notice that $\pi_A \circ id_A = \pi_A$, and since $\pi_A = \rho$, we also have $\pi_A \circ id_A = \rho$. This means that $id_A : A \to A$ is both a morphism in $\hat{\mathcal{C}}$ as well as a lift of V over A.

The claim is that $(A, [id_A])$ is a versal couple for V. To see this, let T be any object of $\hat{\mathcal{C}}$ and let $[\psi]$ be any element of $F_V(T)$. Then by virtue of ψ being a lift of V over A, $\psi(1_A) = 1_T$ and $\pi_T \circ \psi = \rho$. But since $\rho = \pi_A$, this gives $\pi_T \circ \psi = \pi_A$, and hence $\psi : A \to T$ is also a morphism in $\hat{\mathcal{C}}$.

Since $[\psi] = [\psi \circ id_A]$, $\tau_T : h_A(T) \to F_V(T)$ is surjective, where $\tau_T(\psi) = [\psi \circ id_A]$ for all ψ in $h_A(T)$. Suppose α and α' are elements in $h_A(T)$ such that $\tau_T(\alpha) = \tau_T(\alpha')$. Then $[\alpha \circ id_A] = [\alpha' \circ id_A]$, which means $[\alpha] = [\alpha']$. Therefore, there exists some t in $Mat_1(T) = T$ which reduces to the identity in $Mat_1(k) = k$, such that $t \cdot \alpha \cdot t^{-1} = \alpha'$. Hence, if T is commutative, in particular if $T = k[\epsilon]$, τ_T is bijective.

It remains to show τ is smooth. Let $\alpha : B \to B'$ be a surjective morphism in $\mathcal{C}$, let $[\beta] \in F_V(B)$ and let $u : A \to B'$ be a morphism in $\hat{\mathcal{C}}$ such that $[u \circ id_A] = [\alpha \circ \beta]$. Hence there exists some b' in B' with $\pi_{B'}(b') = 1$ such that

$$u = b'(\alpha \circ \beta)(b')^{-1}.$$

Letting $b \in B$ be any element with $\beta(b) = b'$, we obtain that $\pi_B(b) = 1$. Hence

$\tilde{\beta} = b\beta b^{-1}$ is strictly equivalent to β, and

$$u = \alpha \circ \tilde{\beta}.$$

Since $\tilde{\beta} : A \to B$ is a morphism in $\hat{\mathcal{C}}$, we can take $\delta = \tilde{\beta}$ to see that $\alpha \circ \delta = u$ and $[\delta \circ id_A] = [\tilde{\beta}] = [\beta]$. This proves that $(A, [id_A])$ is a versal couple for V.

Remark 3.17. This example shows that it is easier to obtain universal couples, and hence universal deformation rings, when restricting the functor F_V to $\hat{\mathcal{C}}_{comm}$, i.e., to the full subcategory of $\hat{\mathcal{C}}$ consisting of the commutative objects of $\hat{\mathcal{C}}$. Moreover, we see that every object of $\hat{\mathcal{C}}$ is a versal deformation ring of a one-dimensional representation, and every object of $\hat{\mathcal{C}}_{comm}$ is a universal deformation ring of a one-dimensional representation.

Let now $F_{V,c} : \hat{\mathcal{C}}_{comm} \to Sets$ be the restriction of F_V to $\hat{\mathcal{C}}_{comm}$. For any object R of $\hat{\mathcal{C}}$, let $[R, R] = \{ab - ba | a, b \in R\}$, and define $R_c = R/[R, R]$. Let $\pi_c : R \to R_c$ be the natural projection.

Theorem 3.18. *Let $(R, [\phi])$ be a versal couple for V with respect to the category $\hat{\mathcal{C}}$. Then $(R_c, [\pi_c \circ \phi])$ is a versal couple of V with respect to the category $\hat{\mathcal{C}}_{comm}$.*

Proof. Since $k[\epsilon]$ is a commutative object in $\hat{\mathcal{C}}$, the sets $h_{R_c}(k[\epsilon])$ and $h_R(k[\epsilon])$ are in bijection via the map that sends $\alpha : R_c \to k[\epsilon]$ to $\alpha \circ \pi_c : R \to k[\epsilon]$. Since $(R, [\phi])$ is a versal couple for V, the sets $h_R(k[\epsilon])$ and $F_V(k[\epsilon])$ are in bijection via the map that sends $f : R \to k[\epsilon]$ to $[f \circ \phi]$. Therefore, the sets $h_{R_c}(k[\epsilon])$ and $F_V(k[\epsilon])$ are also in bijection via the map that sends $\alpha : R_c \to k[\epsilon]$ to $[\alpha \circ (\pi_c \circ \phi)]$. Therefore, all that remains is to prove a smoothness condition. Let B and B' be objects of $\mathcal{C}_{comm}$, and let $\alpha : B \to B'$ be a morphism in $\mathcal{C}_{comm}$. We must show that the map q' resulting from the pullback diagram in Figure 3.9 is surjective.

Figure 3.9: The pullback diagram for q'.

$$
\begin{tikzcd}
h_{R_c}(B) \arrow[rr, "h_{R_c}(\alpha)"] \arrow[dd] \arrow[dr, rightsquigarrow, "q'"] & & h_{R_c}(B') \arrow[dd] \\
& h_{R_c}(B') \times_{F_{V,c}(B')} F_{V,c}(B) \arrow[ur] \arrow[dl] & \\
F_{V,c}(B) \arrow[rr, "F_{V,c}(\alpha)"] & & F_{V,c}(B')
\end{tikzcd}
$$

However, since $(R, [\phi])$ is a versal couple for V with respect to the category $\hat{\mathcal{C}}$, we know that the map q resulting from the pullback diagram in Figure 3.10 is surjective.

Any morphism in $\hat{\mathcal{C}}$ from R to a commutative ring in $\hat{\mathcal{C}}_{comm}$ must factor through $\pi_c : R \to R_c$. Specifically, since B and B' are both commutative, $h_R(B) \cong h_{R_c}(B)$ and $h_R(B') \cong h_{R_c}(B')$. From this, it follows that the surjectivity of q implies the surjectivity of q', completing the proof that $(R_c, [\pi_c \circ \phi])$ is a versal couple for V with respect to the category $\hat{\mathcal{C}}_{comm}$. $\square$

Figure 3.10: The pullback diagram for q.

$$\begin{tikzcd}
h_R(B) \arrow[rr, "h_R(\alpha)"] \arrow[dd] \arrow[dr, rightsquigarrow, "q"] & & h_R(B') \arrow[dd] \\
& h_R(B') \times_{F_V(B')} F_V(B) \arrow[ur] \arrow[dl] & \\
F_V(B) \arrow[rr, "F_V(\alpha)"] & & F_V(B')
\end{tikzcd}$$

Theorem 3.19. *Let $(R, [\phi])$ be a universal couple for V with respect to the category $\hat{\mathcal{C}}$. Then $(R_c, [\pi_c \circ \phi])$ is a universal couple for V with respect to the category $\hat{\mathcal{C}}_{comm}$.*

Proof. Let B be an object of $\mathcal{C}_{comm}$. As a consequence of Theorem 3.18 and Corollary 3.8, it suffices to prove that the map from $h_{R_c}(B)$ to $F_{V,c}(B)$ which sends $\alpha : R_c \to B$ in $\hat{\mathcal{C}}_{comm}$ to $[\alpha \circ (\pi_c \circ \phi)]$ is injective. Suppose that $f : R_c \to B$ and $g : R_c \to B$ are two elements of $h_{R_c}(B)$ such that $[f \circ (\pi_c \circ \phi)] = [g \circ (\pi_c \circ \phi)]$, which we can rewrite as $[(f \circ \pi_c) \circ \phi] = [(g \circ \pi_c) \circ \phi]$. Since $(R, [\phi])$ is a universal couple for V with respect to the category $\hat{\mathcal{C}}$, the map from $h_R(B)$ to $F_V(B)$ which sends $h : R \to B$ in $\hat{\mathcal{C}}$ to $[h \circ \phi]$ must be injective, implying that $f \circ \pi_c = g \circ \pi_c$. In other words,

$$0 = f \circ \pi_c - g \circ \pi_c = (f - g) \circ \pi_c.$$

Since $h_{R_c}(B)$ and $h_R(B)$ are in bijection via the map which sends $\alpha : R_c \to B$ to $\alpha \circ \pi_c : R \to B$, it follows that $f - g = 0$, and hence $(R_c, [\pi_c \circ \phi])$ is a universal couple for V with respect to the category $\hat{\mathcal{C}}_{comm}$. $\square$

CHAPTER 4
TWO PARTICULAR EXAMPLES

In this chapter, we consider two particular examples of λ, A, and V and determine the versal deformation ring $R(A,V)$ and a corresponding versal couple $(R(A,V),[\phi_u])$.

Let k be any field, let $r \geq 2$ be a fixed integer, and define

$$\Omega_r = k\langle\langle x,y\rangle\rangle/(x^2, y^2, (xy)^r - (yx)^r).$$

We consider a particular representation

$$\rho_r : \Omega_r \to Mat_{2r+1}(k).$$

Since r can be arbitrarily large, it becomes necessary to find an efficient way to write down and multiply arbitrarily large matrices. The following two remarks introduce our notation for such large matrices.

Remark 4.1. Let S be an object of $\hat{\mathcal{C}}$. Let c_i denote the $n \times 1$ column vector with 1 in the i^{th} row, and zeros everywhere else. Using this notation, we can define matrices in $Mat_n(S)$ by writing each column as an S-linear combination of the c_i. For example, if $n = 3$ and

$$B = \begin{bmatrix} 0 & 0 & a \\ d & f & 0 \\ g & 0 & s \end{bmatrix} \in Mat_3(S),$$

then, using the above notation, we would write B as

$$B = \left[\begin{array}{c|c|c} dc_2 & fc_2 & ac_1 \\ & & \\ +gc_3 & & +sc_3 \end{array} \right].$$

Additionally, if B represents a matrix with elements in the ring S/J for some ideal J of S, the coefficients of the c_i in the above notation are understood to be taken modulo J.

Remark 4.2. Let S be again an object of $\hat{\mathcal{C}}$, and let B and D be two matrices in $Mat_n(S)$. Let $B_{i,*}$ denote the i^{th} row of B, and let $D_{*,j}$ denote the j^{th} column of D. Then the j^{th} column of BD is $(BD)_{*,j} = B \cdot D_{*,j}$. If $D_{*,j} = s_1c_1 + s_2c_2 + ... + s_nc_n$ with each s_i in S, then $(BD)_{*,j} = B_{*,1} \cdot s_1 + B_{*,2} \cdot s_2 + ... + B_{*,n} \cdot s_n$.

Similarly, the i^{th} row of BD is $(BD)_{i,*} = B_{i,*} \cdot D$. Notice that we can write row vectors as the transpose, ^{T}, of column vectors, and vice versa. If $B_{i,*} = (r_1c_1 + r_2c_2 + ... + r_nc_n)^T$ with each r_i in S, then

$$(BD)_{i,*} = r_1 \cdot D_{1,*} + r_2 \cdot D_{2,*} + ... + r_n \cdot D_{n,*} = (r_1 \cdot D_{1,*}^T + r_2 \cdot D_{2,*}^T + ... + r_n \cdot D_{n,*}^T)^T.$$

To define our representation $\rho_r : \Omega_r \to Mat_{2r+1}(k)$, we need the following two matrices in $Mat_{2r+1}(k)$:

$$\begin{aligned} X &= \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & c_1 & c_4 & 0 & c_6 & 0 & \dots & c_{2r} & 0 & 0 \end{array} \right] \\ &= \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} c_2 & 0 & 0 & c_3 & 0 & c_5 & \dots & 0 & c_{2r-1} & 0 \end{array} \right]^T \end{aligned}$$

and

$$
\begin{aligned}
Y &= \left[\begin{array}{c|c|c|c|c|c|c|c|c|c}
0 & c_3 & 0 & c_5 & 0 & c_7 & \dots & 0 & c_{2r+1} & 0
\end{array}\right] \\
&= \left[\begin{array}{c|c|c|c|c|c|c|c|c|c}
0 & 0 & c_2 & 0 & c_4 & 0 & \dots & c_{2r-2} & 0 & c_{2r}
\end{array}\right]^T.
\end{aligned}
$$

Let

$$\bar{x} = x + (x^2, y^2, (xy)^r - (yx)^r) \in \Omega_r,$$

and

$$\bar{y} = y + (x^2, y^2, (xy)^r - (yx)^r) \in \Omega_r.$$

Since $X^2 = Y^2 = (XY)^r - (YX)^r = 0$, we may define a k-algebra homomorphism

$$\rho_r : \Omega_r \to Mat_{2r+1}(k)$$

by letting

$$\rho_r(\bar{x}) = X,$$

$$\rho_r(\bar{y}) = Y,$$

and extending this definition k-linearly. Let V_r be the associated left Ω_r-module which is a $(2r+1)$-dimensional k-vector space and on which Ω_r acts by the representation ρ_r.

In the following two sections, we consider the following two situations. In Section 4.1, we let $\lambda = k$, $A = \Omega_r$, and $V = V_r$ and compute $R(\Omega_r, V_r)$ for any $r \geq 2$. In Section 4.2, we take k to be a perfect field of characteristic 2, and we let W be the ring of infinite Witt vectors over k. We take $\lambda = W$ and we take A to be the group ring $W[D_8]$ of a dihedral group D_8 of order 8. In this case, it turns out that $A/2A \cong \Omega_2$, which means that we can view V_2 as a 5-dimensional representation of D_8 over k. We determine $R(W[D_8], V_2)$.

4.1 The case $\lambda = k$, $A = \Omega_r$, and $V = V_r$

In this section, let $r \geq 2$ be arbitrary, let $\lambda = k$, $A = \Omega_r$, $V = V_r$, and $\rho = \rho_r$. Our goal is to prove the following result.

Theorem 4.3. *The versal deformation ring $R(\Omega_r, V_r)$ is isomorphic to Ω_r for all $r \geq 2$.*

We first prove the following lemma.

Lemma 4.4. *The k-vector space $F_V(k[\epsilon])$ is 2-dimensional.*

Proof. Any k-algebra homomorphism $\psi : \Omega_r \to Mat_{2r+1}(k[\epsilon])$ is completely determined by its values on $\bar{x}$ and $\bar{y}$. Therefore, may view a lift ψ of V over $k[\epsilon]$ as a pair of matrices $(X_\psi, Y_\psi) \in Mat_{2r+1}(k[\epsilon]) \times Mat_{2r+1}(k[\epsilon])$ such that

$$X_\psi^2 = Y_\psi^2 = (X_\psi Y_\psi)^r - (Y_\psi X_\psi)^r = 0 \in Mat_{2r+1}(k[\epsilon]).$$

Conversely, if we have any two matrices, B and D, in $Mat_{2r+1}(k[\epsilon])$ which satisfy the above equations, then we may define a lift of V over $k[\epsilon]$ by sending $\bar{x}$ to B, $\bar{y}$ to D, and extending this k-linearly. Since ψ must reduce to ρ via the natural projection $Mat_{2r+1}(k[\epsilon]) \to Mat_{2r+1}(k)$, it follows that there exist matrices $L = [l_{ij}]$

and $L' = [l'_{ij}]$ in $Mat_{2r+1}(m_{k[\epsilon]})$ such that

$$X_\psi = X + L,$$

and

$$Y_\psi = Y + L'.$$

Because X_ψ^2 and Y_ψ^2 must equal 0, we see that $(X+L)^2 = (Y+L')^2 = 0$. But since $m_{k[\epsilon]}^2 = 0$, it follows that L^2 and L'^2 are both 0. Since $X^2 = Y^2 = 0$, it then follows that

$$X_\psi^2 = XL + LX = 0$$

and

$$Y_\psi^2 = YL' + L'Y = 0.$$

Since

$$\begin{aligned} X &= \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & c_1 & c_4 & 0 & c_6 & 0 & \dots & c_{2r} & 0 & 0 \end{array} \right] \\ &= \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} c_2 & 0 & 0 & c_3 & 0 & c_5 & \dots & 0 & c_{2r-1} & 0 \end{array} \right]^T \end{aligned}$$

and

$$\begin{aligned} Y &= \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & c_3 & 0 & c_5 & 0 & c_7 & \dots & 0 & c_{2r+1} & 0 \end{array} \right] \\ &= \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & 0 & c_2 & 0 & c_4 & 0 & \dots & c_{2r-2} & 0 & c_{2r} \end{array} \right]^T, \end{aligned}$$

it follows that

$$0 = X_\psi^2 = XL + LX \quad = \quad \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} L_{2,*}^T & 0 & 0 & L_{3,*}^T & 0 & L_{5,*}^T & \cdots & 0 & L_{2r-1,*}^T & 0 \end{array}\right]^T$$
$$+ \quad \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & L_{*,1} & L_{*,4} & 0 & L_{*,6} & 0 & \cdots & L_{*,2r} & 0 & 0 \end{array}\right], \quad (4.1)$$

and

$$0 = Y_\psi^2 = YL' + L'Y \quad = \quad \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & 0 & {L'}_{2,*}^T & 0 & {L'}_{4,*}^T & 0 & \cdots & {L'}_{2r-2,*}^T & 0 & {L'}_{2r,*}^T \end{array}\right]^T$$
$$+ \quad \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & L'_{*,3} & 0 & L'_{*,5} & 0 & L'_{*,7} & \cdots & 0 & L'_{*,2r+1} & 0 \end{array}\right]. \quad (4.2)$$

These equations, in addition to the equation

$$(X_\psi Y_\psi)^r - (Y_\psi X_\psi)^r = 0,$$

serve as restrictions as to what our lifts may actually look like. Before proving that $F_V(k[\epsilon])$ is 2-dimensional for all r, we first look explicitly at the case $r = 2$, to help guide the reader in the general case.

Let $r = 2$. Then

$$\begin{aligned} 0 = X_\psi^2 &= XL + LX \\ &= \left[\begin{array}{c|c|c|c|c} L_{2,*}^T & 0 & 0 & L_{3,*}^T & 0 \end{array}\right]^T + \left[\begin{array}{c|c|c|c|c} 0 & L_{*,1} & L_{*,4} & 0 & 0 \end{array}\right] \end{aligned}$$

and

$$
\begin{aligned}
0 = Y_\psi^2 &= YL' + L'Y \\
&= \left[\begin{array}{c|c|c|c|c} 0 & 0 & L'^T_{2,*} & 0 & L'^T_{4,*} \end{array}\right]^T + \left[\begin{array}{c|c|c|c|c} 0 & L'_{*,3} & 0 & L'_{*,5} & 0 \end{array}\right],
\end{aligned}
$$

which we may rewrite as

$$
0 = XL + LX = \begin{bmatrix} l_{21} & l_{22} + l_{11} & l_{23} + l_{14} & l_{24} & l_{25} \\ 0 & l_{21} & l_{24} & 0 & 0 \\ 0 & l_{31} & l_{34} & 0 & 0 \\ l_{31} & l_{32} + l_{41} & l_{33} + l_{44} & l_{34} & l_{35} \\ 0 & l_{51} & l_{54} & 0 & 0 \end{bmatrix} \tag{4.3}
$$

and

$$
0 = YL' + L'Y = \begin{bmatrix} 0 & l'_{13} & 0 & l'_{15} & 0 \\ 0 & l'_{23} & 0 & l'_{25} & 0 \\ l'_{21} & l'_{22} + l'_{33} & l'_{23} & l'_{24} + l'_{35} & l'_{25} \\ 0 & l'_{43} & 0 & l'_{45} & 0 \\ l'_{41} & l'_{42} + l'_{53} & l'_{43} & l'_{44} + l'_{55} & l'_{45} \end{bmatrix}. \tag{4.4}
$$

Let E be any matrix in $Mat_5(m_{k[\epsilon]})$. Then we obtain

$$XE - EX = \begin{bmatrix} e_{21} & e_{22} - e_{11} & e_{23} - e_{14} & e_{24} & e_{25} \\ 0 & -e_{21} & -e_{24} & 0 & 0 \\ 0 & -e_{31} & -e_{34} & 0 & 0 \\ e_{31} & e_{32} - e_{41} & e_{33} - e_{44} & e_{34} & e_{35} \\ 0 & -e_{51} & -e_{54} & 0 & 0 \end{bmatrix} \tag{4.5}$$

and

$$YE - EY = \begin{bmatrix} 0 & -e_{13} & 0 & -e_{15} & 0 \\ 0 & -e_{23} & 0 & -e_{25} & 0 \\ e_{21} & e_{22} - e_{33} & e_{23} & e_{24} - e_{35} & e_{25} \\ 0 & -e_{43} & 0 & -e_{45} & 0 \\ e_{41} & e_{42} - e_{53} & e_{43} & e_{44} - e_{55} & e_{45} \end{bmatrix}. \tag{4.6}$$

We note that any lift $\zeta = (X_\zeta, Y_\zeta)$ of V over $k[\epsilon]$ which is strictly equivalent to the original lift $\rho = (X, Y)$ must be of the form

$$X_\zeta = (1 - E)X(1 + E) = X + XE - EX,$$

$$Y_\zeta = (1 - E)Y(1 + E) = Y + YE - EY,$$

for some matrix $E = [e_{ij}]$ in $Mat_5(m_{k[\epsilon]})$. The conditions imposed on L by the equation $XL + LX = 0$ are not sufficient to force L to be of the form $L = XE - EX$, for some matrix $E = [e_{ij}]$ in $Mat_5(m_{k[\epsilon]})$. Similarly, the conditions imposed on L' by the equation $YL' + L'Y = 0$ are not sufficient to force L' to be of the form $L' = YE - EY$, for some matrix $E = [e_{ij}]$ in $Mat_5(m_{k[\epsilon]})$. However, we will now investigate what is forced by those equations.

We begin by examining the implications of equations (4.3) and (4.5). Notice that the requirement that $XL + LX = 0$ forces 8 elements of L to be 0, namely $l_{21}, l_{31}, l_{51}, l_{24}, l_{34}, l_{54}, l_{25}, l_{35}$. Moreover, if we look at these respective entries in the matrix $XE - EX$ we find that they are all 0. The only entry in $XE - EX$ which is 0 but is not "referred to" by the requirement that $XL + LX = 0$ is the entry $(XE - EX)_{55}$. In other words, $(XE - EX)_{55} = 0$, but the element l_{55} does not appear in the matrix $XL + LX$. Additionally, there are 4 duplicate entries in the matrix $XL+LX$, namely $l_{21}, l_{24}, l_{31}, l_{34}$ appear twice. Notice that the sums appearing as entries in the matrix $XL + LX$ exactly prescribe the locations of the duplicate entries in $XL + LX$. For example, $l_{22} + l_{11}$ appears as an entry in $XL + LX$, and we note that

$$(XL + LX)_{11} = (XL + LX)_{22}.$$

Moreover, the conditions imposed on L by the sums found in $XL + LX = 0$ are satisfied in the matrix $XE - EX$. As an example, we take the condition $l_{22} + l_{11} = 0$ and note that $(XE - EX)_{22} + (XE - EX)_{11} = 0$. Finally, with the single exception that $(XE - EX)_{55} = 0$, there are no other dependencies found in $XE - EX$ which are not prescribed by the conditions found in $XL + LX = 0$. Therefore, any L satisfying $XL + LX = 0$ will be of the form $XE - EX$, with the possible exception of differing by an element in $m_{k[\epsilon]}$ in entry L_{55}.

Next, we continue by examining the implications of equations (4.4) and (4.6). By looking at the conditions imposed on L' in equation (4.4), in addition to observing the form of equation (4.6), it follows that any L' satisfying $YL' + L'Y = 0$ will be of the form $YE - EY$, with the possible exception of differing by an element in $m_{k[\epsilon]}$ in entry L'_{11}.

Let Q_{ij} be the matrix in $Mat_5(m_{k[\epsilon]})$ with ϵ in the (i,j)-entry and zeros everywhere else. What we know now is that there exist matrices F and F' in $Mat_5(m_{k[\epsilon]})$ such that any lift $(X+L, Y+L')$ of V over $k[\epsilon]$ is of the form

$$X + L = (1-F)X(1+F) + cQ_{55}, \tag{4.7}$$

$$Y + L' = (1-F')Y(1+F') + c'Q_{11}, \tag{4.8}$$

for some c and c' in k.

We now show that $F_V(k[\epsilon])$ is 2-dimensional. This means that we need to find two lifts β_1 and β_2 of ρ over $k[\epsilon]$ such that any other such lift, ψ, satisfies $[\psi] = b_1[\beta_1] + b_2[\beta_2]$ for some b_1 and b_2 in k. Note that we use here the addition and multiplication in $F_V(k[\epsilon])$ introduced in Lemma 3.13. As lifts, this means that there exists a matrix E in $Mat_5(m_{k[\epsilon]})$, depending on ψ, such that $\psi = (1-E)\mathbb{P}(\mathbb{M}(b_1,\beta_1),\mathbb{M}(b_2,\beta_2))(1+E)$, where $\mathbb{P}$ and $\mathbb{M}$ are the maps introduced in the proof of Lemma 3.13. Translating this into matrices, we must find matrices M_1, M_2, M_3, M_4 in $Mat_5(m_{k[\epsilon]})$ such that

$$\beta_1 = (X + M_1, Y + M_3),$$

$$\beta_2 = (X + M_2, Y + M_4),$$

and so that for any other lift $\psi = (X+L, Y+L')$, we can find some E in $Mat_5(m_{k[\epsilon]})$, depending on ψ, such that

$$\begin{aligned} X+L &= (1-E)(X+b_1M_1+b_2M_2)(1+E) = (1-E)X(1+E)+b_1M_1+b_2M_2, \\ Y+L &= (1-E)(Y+b_1M_3+b_2M_4)(1+E) = (1-E)Y(1+E)+b_1M_3+b_2M_4. \end{aligned}$$

Moreover, we must show that the deformations $[\beta_1]$ and $[\beta_2]$ are k-linearly independent. We begin by defining our lifts, β_1 and β_2, and showing that the resulting deformations are k-linearly independent.

Define

$$\begin{aligned} \beta_1 &= (\beta_1(\bar{x}), \beta_1(\bar{y})) = \left(\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \epsilon & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \right), \\ \beta_2 &= (\beta_2(\bar{x}), \beta_2(\bar{y})) = \left(\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \right). \end{aligned}$$

In other words,

$$\beta_1 = (X + Q_{55}, Y + Q_{31})$$

and

$$\beta_2 = (X, Y + Q_{11}).$$

It is a straightforward calculation to see that, for $i \in \{1, 2\}$,

$$\begin{aligned} \beta_i(\bar{x})^2 &= 0, \\ \beta_i(\bar{y})^2 &= 0, \\ (\beta_i(\bar{x})\beta_i(\bar{y}))^2 - (\beta_i(\bar{y})\beta_i(\bar{x}))^2 &= 0, \end{aligned}$$

and so both β_1 and β_2 give lifts of ρ over $k[\epsilon]$. We now show that $[\beta_1]$ and $[\beta_2]$ are k-linearly independent. If they are not, then there exists some non-zero element c in k such that $c[\beta_1] = [\beta_2]$. In other words, there exists some matrix E in $Mat_5(m_{k[\epsilon]})$ such that $\mathbb{M}(c, \beta_1) = (1 - E)\beta_2(1 + E)$. But notice that

$$(1 - E)\beta_2(\bar{x})(1 + E) = \beta_2(\bar{x}) + \beta_2(\bar{x})E - E\beta_2(\bar{x}).$$

Since the 5^{th} row and 5^{th} column of $\beta_2(\bar{x})$ is 0, it follows that the $(5, 5)$-entry

$$(\beta_2(\bar{x}) + \beta_2(\bar{x})E - E\beta_2(\bar{x}))_{55} = (\beta_2(\bar{x}))_{55} = 0.$$

Since $(\beta_1(\bar{x}))_{55} = \epsilon$, it is impossible to find a non-zero element c in k such that

$$\mathbb{M}(c, \beta_1)(\bar{x}) = (1 - E)\beta_2(\bar{x})(1 + E).$$

Therefore, $[\beta_1]$ and $[\beta_2]$ are k-linearly independent deformations.

Now that we have established that the dimension of $F_V(k[\epsilon])$ is greater than or equal to 2, we proceed to show its dimension is exactly 2. Let

$$\psi = (\psi(\bar{x}), \psi(\bar{y})) = (X + L, Y + L')$$

be any lift of ρ over $k[\epsilon]$. We recall the equations (4.3) and (4.5), and begin to define E so that $XE - EX$ is as close to L as possible. Looking at the 6 negative single entries in $XE - EX$, as well as the 2 single entries in the 5^{th} column, we begin to define the matrix E:

$$E = \begin{bmatrix} * & * & * & * & * \\ -l_{22} & * & * & -l_{23} & l_{15} \\ -l_{32} & * & * & -l_{33} & l_{45} \\ * & * & * & * & * \\ -l_{52} & * & * & -l_{53} & * \end{bmatrix}.$$

Next, we recall the equations (4.4) and (4.6). We continue to define the matrix E by looking at the 5 negative single entries in $YE - EY$, excluding the $(2,4)$-entry, as well as all of the single entries in the 1^{st} column, except for the $(3,1)$-entry. We exclude these two particular entries, because we note that both e_{25} and e_{21} have already been defined. Setting $e_{13} = -l'_{12}, e_{15} = -l'_{14}, e_{23} = -l'_{22}, e_{43} = -l'_{42}, e_{45} = -l'_{44}, e_{41} = l'_{51}$, we continue to fill in the matrix E as:

$$E = \begin{bmatrix} * & * & -l'_{12} & * & -l'_{14} \\ -l_{22} & * & -l'_{22} & -l_{23} & l_{15} \\ -l_{32} & * & * & -l_{33} & l_{45} \\ l'_{51} & * & -l'_{42} & * & -l'_{44} \\ -l_{52} & * & * & -l_{53} & * \end{bmatrix}.$$

Next, we define the remaining 11 undetermined entries of E by examining the entries which contain a difference expression in $XE - EX$ and $YE - EY$. Noting that e_{22} has yet to be defined, we choose to set $e_{22} = 0$. Then we define e_{11} as $e_{11} = e_{22} - l_{12} = -l_{12}$, which we obtain by noticing that $e_{22} - e_{11} = (XE - EX)_{12}$. Similarly, we set $e_{33} = e_{22} - l'_{32} = -l'_{32}$, which we obtain by noticing that the expression $e_{22} - e_{33} = (YE - EY)_{32}$. We may then continue in this manner, defining $e_{14} = e_{23} - l_{14} = -l'_{22} - l_{13}, e_{32} = e_{41} + l_{42} = l'_{51} + l_{42}, e_{44} = e_{33} - l_{43} = -l'_{32} - l_{43}$, $e_{55} = e_{44} - l'_{54} = -l'_{32} - l_{43} - l'_{54}$. Also, since neither e_{53} nor e_{42} have yet been defined, we choose to set $e_{53} = 0$, which results in $e_{42} = e_{53} + l'_{52} = l'_{52}$. The only difference expression we have yet to address is the expression $e_{24} - e_{35} = (YE - EY)_{34}$, due to the fact that both e_{24} and e_{35} have previously been defined.

To summarize, all but 3 of the expressions found in $XE - EX$ and $YE - EY$ have been utilized to define elements of E. In particular, we were unable to use expressions found in $(YE - EY)_{24}, (YE - EY)_{31}, (YE - EY)_{34}$. For all the elements of E that we have yet to define, we choose to set them all equal to 0, which results in the matrix:

$$E = \begin{bmatrix} -l_{12} & 0 & -l'_{12} & -l'_{22} - l_{13} & -l'_{14} \\ -l_{22} & 0 & -l'_{22} & -l_{23} & l_{15} \\ -l_{32} & l'_{51} + l_{42} & -l'_{32} & -l_{33} & l_{45} \\ l'_{51} & l'_{52} & -l'_{42} & -l'_{32} - l_{43} & -l'_{44} \\ -l_{52} & 0 & 0 & -l_{53} & -l'_{32} - l_{43} - l'_{54} \end{bmatrix}.$$

Recall that Q_{ij} represents the matrix in $Mat_5(m_{k[\epsilon]})$ with ϵ in the (i, j)-entry and zeros everywhere else. Using equations (4.7) and (4.8), we now know the following: If $L = [l_{ij}]$ and $L' = [l'_{ij}]$ are any matrices in $Mat_5(m_{k[\epsilon]})$ such that $(X+L, Y+L')$

is a lift of V over $k[\epsilon]$, then there exist elements a_1, a_2, a_3, a_4, a_5 in k such that

$$\begin{aligned} X + L &= (1 - E)X(1 + E) + a_1 Q_{55}, \\ Y + L' &= (1 - E)Y(1 + E) + a_2 Q_{11} + a_3 Q_{24} + a_4 Q_{31} + a_5 Q_{34}. \end{aligned}$$

However, we have yet to consider the conditions imposed on L and L' by the equation

$$((X + L)(Y + L'))^2 - ((Y + L')(X + L))^2 = 0.$$

Notice that because L and L' have entries lying in $m_{k[\epsilon]}$,

$$((X + L)(Y + L'))^2 - ((Y + L')(X + L))^2 =$$

$$XYXY + LYXY + XL'XY + XYLY + XYXL'$$

$$-(YXYX + L'XYX + YLYX + YXL'X + YXYL).$$

We then compute the matrices $X, Y, XY, YX, XYX, YXY, XYXY, YXYX$ to examine the second column of $((X + L)(Y + L'))^2 - ((Y + L')(X + L))^2$. Since $YX = \left[\begin{array}{c|c|c|c|c} 0 & 0 & c_5 & 0 & 0 \end{array}\right]$, the matrices $YXYX, L'XYX$ and $YLYX$ have a zero second column. Moreover, XYX is the zero matrix so that $XYXY$ and $XYXL'$ also have a zero second column. On the other hand,

$$\begin{aligned}
(LYXY)_{*,2} &= L_{*,5}, \\
(XL'XY)_{*,2} &= (l'_{24}, 0, 0, l'_{34}, 0)^T, \\
(XYLY)_{*,2} &= (0, 0, 0, l_{23}, 0)^T, \\
-(YXL'X)_{*,2} &= (0, 0, 0, 0, -l'_{31})^T, \\
-(YXYL)_{*,2} &= (0, 0, 0, 0, -l_{22}).
\end{aligned}$$

Therefore, we obtain

$$(((X+L)(Y+L'))^2-((Y+L')(X+L))^2)_{*,2} = (l_{15}+l'_{24}, l_{25}, l_{35}, l_{45}+l_{23}+l'_{34}, l_{55}-l_{22}-l'_{31})^T.$$

Since $((X+L)(Y+L'))^2 - ((Y+L')(X+L))^2$ must equal 0, it follows that all the elements in its second column are zero. Notice that this gives the 3 following dependencies between elements of L and L':

$$\begin{aligned}
l'_{24} &= -l_{15} \\
l'_{34} &= -l_{45} - l_{23} \\
l'_{31} &= l_{55} - l_{22}.
\end{aligned}$$

Therefore, since these three elements of L' are determined by L, we must set $a_4 = a_1$ and $a_3 = a_5 = 0$ in the expression for L' to arrive at the following expressions for L and L':

$$\begin{aligned} X + L &= (1-E)X(1+E) + a_1 Q_{55}, \\ Y + L' &= (1-E)Y(1+E) + a_2 Q_{11} + a_1 Q_{31}. \end{aligned}$$

Recalling our expressions for β_1 and β_2, we may now write

$$[\psi] = [(\psi(\bar{x}), \psi(\bar{y}))] = [(X+L, Y+L')] = a_1[\beta_1] + a_2[\beta_2].$$

Therefore, due to the fact that $[\beta_1]$ and $[\beta_2]$ have already been shown to be k-linearly independent, we conclude that $F_V(k[\epsilon])$ is a 2-dimensional k-vector space when $r = 2$.

In the general case, for arbitrary $r \geq 2$, we proceed in a similar manner. Let Q_{ij} be the matrix in $Mat_{2r+1}(m_{k[\epsilon]})$ with ϵ in the (i,j)-entry and zeros everywhere else. We define the following two lifts of V over $k[\epsilon]$:

$$\beta_1 = (\beta_1(\bar{x}), \beta_1(\bar{y})) = (X + Q_{(2r+1)(2r+1)}, Y + Q_{31}) = \tag{4.9}$$
$$\left(\left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & c_1 & c_4 & 0 & c_6 & 0 & \dots & c_{2r} & 0 & \epsilon c_{2r+1} \end{array} \right], \right.$$
$$\left. \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} \epsilon c_3 & c_3 & 0 & c_5 & 0 & c_7 & \dots & 0 & c_{2r+1} & 0 \end{array} \right] \right)$$

and

$$\beta_2 = (\beta_2(\bar{x}), \beta_2(\bar{y})) = (X, Y + Q_{11}) = \tag{4.10}$$
$$(\left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & c_1 & c_4 & 0 & c_6 & 0 & \dots & c_{2r} & 0 & 0 \end{array}\right],$$
$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c} \epsilon c_1 & c_3 & 0 & c_5 & 0 & c_7 & \dots & 0 & c_{2r+1} & 0 \end{array}\right]).$$

As in the case when $r = 2$, we see that the deformations $[\beta_1]$ and $[\beta_2]$ are k-linearly independent. Let

$$\psi = (\psi(\bar{x}), \psi(\bar{y})) = (X + L, Y + L')$$

be any lift of V over $k[\epsilon]$. We seek elements b_1 and b_2 in k such that we can write

$$[\psi] = b_1[\beta_1] + b_2[\beta_2].$$

This will be accomplished if we can find a matrix E in $Mat_{2r+1}(m_{k[\epsilon]})$ such that

$$X + L = (1 - E)X(1 + E) + b_1 Q_{(2r+1)(2r+1)}, \tag{4.11}$$

$$Y + L' = (1 - E)Y(1 + E) + b_2 Q_{11} + b_1 Q_{31}. \tag{4.12}$$

Note that, as in the equations (4.7) and (4.8) when $r = 2$, we see that there exist matrices F and F' in $Mat_{2r+1}(m_{k[\epsilon]})$ such that

$$X + L = (1 - F)X(1 + F) + cQ_{(2r+1)(2r+1)}, \tag{4.13}$$

$$Y + L' = (1 - F')Y(1 + F') + c'Q_{11} \tag{4.14}$$

for certain c and c' in k. To find E in (4.11) and (4.12), we use the structure of the equations (4.1) and (4.2) to obtain

$$\begin{aligned} XE - EX &= \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} E_{2,*}^T & 0 & 0 & E_{3,*}^T & 0 & E_{5,*}^T & \dots & 0 & E_{2r-1,*}^T & 0 \end{array} \right]^T \\ &- \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & E_{*,1} & E_{*,4} & 0 & E_{*,6} & 0 & \dots & E_{*,2r} & 0 & 0 \end{array} \right] \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} YE - EY &= \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & 0 & E_{2,*}^T & 0 & E_{4,*}^T & 0 & \dots & E_{2r-2,*}^T & 0 & E_{2r,*}^T \end{array} \right]^T \\ &- \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & E_{*,3} & 0 & E_{*,5} & 0 & E_{*,7} & \dots & 0 & E_{*,2r+1} & 0 \end{array} \right]. \end{aligned} \tag{4.16}$$

We then begin to define entries in E in terms of entries from L by first examining all of the negative, single-element entries in the matrix $XE - EX$, along with all of the single-element entries in the $(2r+1)^{th}$ column of $XE - EX$. We then continue to define entries of E in terms of entries from L' by looking at all of the negative, single-element entries in the matrix $YE - EY$, in addition to all of the single-element entries in the 1^{st} column of $YE - EY$, but excluding any of those entries which contain entries in E that have already been defined. Finally, we continue to define the remaining entries of E in terms of elements from L and L' by examining the entries consisting of differences in the matrices $XE - EX$ and $YE - EY$. This can be done in a manner so that, excluding element $L_{(2r+1)(2r+1)}$ by (4.13), every element of L is either 0, as determined by equation (4.1), or is defined in terms of an entry found in E. While this method does not result in knowing every non-zero element of L' in terms of E, it does result in the following: With the exception of L'_{11} by equation (4.14), those entries of L' that are currently not determined by E are in fact determined by the equation

$$((X+L)(Y+L'))^r - ((Y+L')(X+L))^r = 0.$$

Notice that since L and L' are matrices in $Mat_{2r+1}(m_{k[\epsilon]})$, and because $m_{k[\epsilon]}^2 = 0$, it follows that

$$((X+L)(Y+L'))^r - ((Y+L')(X+L))^r =$$

$$((XY)^r + LY(XY)^{r-1} + XL'(XY)^{r-1} + \dots (XY)^{r-1}XL')$$

$$-((YX)^r + L'X(YX)^{r-1} + YL(YX)^{r-1} + \dots (YX)^{r-1}YL).$$

Setting the above equation equal to zero allows us to solve for the currently undetermined entries of L' in terms of entries of L, and hence in terms of entries of E. From this, we are able to find elements b_1 and b_2 in k such that

$$\begin{aligned} X+L &= (1-E)X(1+E) + b_1 Q_{(2r+1)(2r+1)}, \\ Y+L' &= (1-E)Y(1+E) + b_2 Q_{11} + b_1 Q_{31}, \end{aligned}$$

and so we can write

$$[\psi] = b_1[\beta_1] + b_2[\beta_2].$$

Since $[\beta_1]$ and $[\beta_2]$ are k-linearly independent, this concludes the proof of Lemma 4.4. $\square$

Proof of Theorem 4.3:

We prove this by constructing a versal couple for V, using the construction in the proof of Theorem 3.16. Define

$$S = k\langle\langle s,t\rangle\rangle.$$

Since $F_V(k[\epsilon])$ is a 2-dimensional vector space, we know from Theorem 3.16 that a versal couple for V exists, and we further know that the versal deformation ring, $R = R(A,V)$, will be a quotient of S by some proper closed ideal, I.

In the process of constructing the versal couple, we know we can find ideals I_ℓ of S, for all integers $\ell \geq 2$, which satisfy

$$m_S * I_\ell + I_\ell * m_S \subseteq I_{\ell+1}.$$

Moreover, we can find compatible lifts ϕ_ℓ of V over S/I_ℓ for all ℓ. Notice that any lift of V over S/I_ℓ is completely determined by its values on $\bar{x}$ and $\bar{y}$. Namely, these lifts correspond to pairs of matrices,

$$(\phi_\ell(\bar{x}), \phi_\ell(\bar{y})) = (X_\ell, Y_\ell) \in Mat_{2r+1}(S/I_\ell) \times Mat_{2r+1}(S/I_\ell)$$

such that

$$(X_\ell)^2 = (Y_\ell)^2 = (X_\ell Y_\ell)^r - (Y_\ell X_\ell)^r = 0 \in Mat_{2r+1}(S/I_\ell). \tag{4.17}$$

Let

$$\pi_\ell : S \to S/I_\ell$$

be the natural surjections, and let $\mathscr{X}_\ell$ and $\mathscr{Y}_\ell$ be any elements in $\pi_\ell^{-1}(X_\ell)$ and $\pi_\ell^{-1}(Y_\ell)$, respectively, where we have extended π_ℓ naturally to a surjection of matrix rings.

The relations in equation (4.17) involving X_ℓ and Y_ℓ then translate into the following relations regarding $\mathscr{X}_\ell$ and $\mathscr{Y}_\ell$:

$$(\mathscr{X}_\ell)^2, (\mathscr{Y}_\ell)^2, (\mathscr{X}_\ell\mathscr{Y}_\ell)^r - (\mathscr{Y}_\ell\mathscr{X}_\ell)^r \in I_\ell. \tag{4.18}$$

Since we know the lifts ϕ_ℓ will be compatible, each $X_{\ell+1}$ will project onto X_ℓ, and $Y_{\ell+1}$ will project onto Y_ℓ via the natural projection of $Mat_{2r+1}(S/I_{\ell+1})$ onto $Mat_{2r+1}(S/I_\ell)$. Notice that this precisely means that $\pi_{\ell+1}^{-1}(X_{\ell+1}) \subseteq \pi_\ell^{-1}(X_\ell)$ and $\pi_{\ell+1}^{-1}(Y_{\ell+1}) \subseteq \pi_\ell^{-1}(Y_\ell)$, for all ℓ. These set containments then immediately yield:

$$\mathscr{X}_{\ell+1} - \mathscr{X}_\ell \in I_\ell,$$

$$\mathscr{Y}_{\ell+1} - \mathscr{Y}_\ell \in I_\ell.$$

We proceed in the construction of the versal couple of V by finding the ideals I_ℓ and appropriate matrices $\mathscr{X}_\ell$ and $\mathscr{Y}_\ell$, for each ℓ.

Following the constructive proof of the existence of a versal couple in the proof of Theorem 3.16, we take

$$I_2 = (s^2, t^2, st, ts).$$

To find our matrices $\mathscr{X}_2$ and $\mathscr{Y}_2$, we recall equations (4.9) and (4.10) from Lemma 4.4. Since $[\beta_1]$ and $[\beta_2]$ are k-linearly independent elements of the 2-dimensional k-vector space $F_V(k[\epsilon])$, they form a basis of $F_V(k[\epsilon])$. Therefore, as in the proof of Theorem 3.16, we take the pullback of β_1 and β_2 over $Mat_{2r+1}(k)$ to arrive at our desired lift $\phi_2 = (X_2, Y_2)$ of V over S/I_2:

$$X_2 = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & c_1 & c_4 & 0 & c_6 & 0 & \dots & c_{2r} & 0 & sc_{2r+1} \end{array}\right]$$

and

$$Y_2 = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} tc_1 & c_3 & 0 & c_5 & 0 & c_7 & \dots & 0 & c_{2r+1} & 0 \\ +sc_3 & & & & & & & & & \end{array}\right].$$

Define an element b in S by

$$b = t(-st)^{r-1}, \tag{4.19}$$

and define two matrices $\mathscr{X}$ and $\mathscr{Y}$ in $Mat_{2r+1}(S)$ by

$$\mathscr{X} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & c_1 & c_4 & 0 & c_6 & 0 & \dots & c_{2r} & 0 & bc_1 \\ & & & & & & & & & -bsc_2 \\ & & & & & & & & & +sc_{2r+1} \end{array}\right]$$

and by

$$\mathscr{Y} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} tc_1 & c_3 & 0 & c_5 & 0 & c_7 & \dots & 0 & c_{2r+1} & 0 \\ -stc_2 & & & & & & & & & \\ +sc_3 & & & & & & & & & \end{array}\right].$$

Notice that $\mathscr{X}$ mod $I_2 = X_2$, and $\mathscr{Y}$ mod $I_2 = Y_2$. Now let

$$(\mathscr{X}_2, \mathscr{Y}_2) \in Mat_{2r+1}(S) \times Mat_{2r+1}(S)$$

be a pair of matrices such that

$$\pi_2(\mathscr{X}_2, \mathscr{Y}_2) = (X_2, Y_2)$$

where we have extended π_2 to the natural map between Cartesian products.

Suppose inductively that we have found ideals $I_2, I_3, ..., I_\ell$ of S and pairs of matrices

$$(\mathscr{X}_2, \mathscr{Y}_2), (\mathscr{X}_3, \mathscr{Y}_3), ..., (\mathscr{X}_\ell, \mathscr{Y}_\ell) \in Mat_{2r+1}(S) \times Mat_{2r+1}(S)$$

such that

$$\mathscr{X}_i^2, \mathscr{Y}_i^2, (\mathscr{X}_i\mathscr{Y}_i)^r - (\mathscr{Y}_i\mathscr{X}_i)^r \in Mat_{2r+1}(I_i)$$

for all $2 \leq i \leq \ell$. Further assume that it is permissible to take $\mathscr{X}_\ell = \mathscr{X}$ and $\mathscr{Y}_\ell = \mathscr{Y}$. We seek an ideal $I_{\ell+1}$ minimal among all ideals J of S that satisfy the following two conditions:

(a) $m_S * I_\ell + I_\ell * m_S \subseteq J \subseteq I_\ell$, and

(b) there exists a pair of matrices $(\mathscr{X}_J, \mathscr{Y}_J)$ in $Mat_{2r+1}(S) \times Mat_{2r+1}(S)$ such that $\pi_J((\mathscr{X}_J, \mathscr{Y}_J))$ defines a lift of V over S/J, where $\pi_J : S \to S/J$ is the natural projection extended to the Cartesian product of matrix rings, and so that $\mathscr{X}_J - \mathscr{X}_\ell$ and $\mathscr{Y}_J - \mathscr{Y}_\ell$ both lie in $Mat_{2r+1}(I_\ell)$.

Since $\mathscr{X}_J - \mathscr{X}_\ell$ and $\mathscr{Y}_J - \mathscr{Y}_\ell$ must both lie in $Mat_{2r+1}(I_\ell)$, there are matrices H_X and H_Y in $Mat_{2r+1}(I_\ell)$ such that $\mathscr{X}_J = \mathscr{X}_\ell + H_X$ and $\mathscr{Y}_J = \mathscr{Y}_\ell + H_Y$. Moreover, since we are assuming $\mathscr{X}_\ell = \mathscr{X}$ and $\mathscr{Y}_\ell = \mathscr{Y}$, we must have $\mathscr{X}_J = \mathscr{X} + H_X$ and $\mathscr{Y}_J = \mathscr{Y} + H_Y$. Letting $(X)_{i,*}$ and $(X)_{*,i}$ denote the i^{th} row and column, respectively, of $\mathscr{X}$, we notice the following:

$$(\mathscr{X})_{2r+1,*} = (0, 0, ..., 0, s),$$

$$(\mathscr{X})_{*,2r+1} = (b, -bs, 0, 0, ..., 0, s)^T.$$

Furthermore,

$$\begin{aligned}(X_J^2)_{2r+1,2r+1} &= ((\mathscr{X})_{2r+1,*} + (H_X)_{2r+1,*}) \cdot ((\mathscr{X})_{*,2r+1} + (H_X)_{*,2r+1}) \\ &= (\mathscr{X})_{2r+1,*} \cdot (\mathscr{X})_{*,2r+1} + (\mathscr{X})_{2r+1,*} \cdot (H_X)_{*,2r+1} \\ &+ (H_X)_{2r+1,*} \cdot (\mathscr{X})_{*,2r+1} + (H_X)_{2r+1,*} \cdot (H_X)_{*,2r+1}.\end{aligned}$$

However, recall that

$$m_S * I_\ell + I_\ell * m_S \subseteq J \subseteq I_\ell.$$

Additionally, notice that every entry of the matrix H_X lies in I_ℓ and every entry of $(\mathscr{X})_{2r+1,*}$ and $(\mathscr{X})_{*,2r+1}$ lies in m_S. Therefore, letting

$$a = (\mathscr{X})_{2r+1,*} \cdot (H_X)_{*,2r+1} + (H_X)_{2r+1,*} \cdot (\mathscr{X})_{*,2r+1} + (H_X)_{2r+1,*} \cdot (H_X)_{*,2r+1}$$

we see that

$$(\mathscr{X}_J^2)_{2r+1,2r+1} = (\mathscr{X})_{2r+1,*} \cdot (\mathscr{X})_{*,2r+1} + a = (\mathscr{X}^2)_{2r+1,2r+1} + a$$

where a lies in J. Notice that since $(\mathscr{X}_J, \mathscr{Y}_J)$ constitutes a lift of V over S/J, $\mathscr{X}_J^2$ must be a matrix in $Mat_{2r+1}(J)$. The $(2r+1, 2r+1)$-entry of this matrix is exactly $(\mathscr{X}_J^2)_{2r+1,2r+1}$, which means that

$$(\mathscr{X}_J^2)_{2r+1,2r+1} = (\mathscr{X}^2)_{2r+1,2r+1} + a$$

must lie in J. Therefore, $(\mathscr{X}^2)_{2r+1,2r+1}$ must lie in J. Since $(\mathscr{X}^2)_{2r+1,2r+1} = s^2$, we see that s^2 must lie in J. Since this is true for all J satisfying the two conditions

(a) and (b) above, it follows that this must also be true for the minimal J satisfying these conditions. Therefore, s^2 must lie in $I_{\ell+1}$.

Next, notice that

$$(\mathscr{Y})_{1,*} = (t, 0, 0, ..., 0),$$

$$(\mathscr{Y})_{*,1} = (t, -st, s, 0, 0, ..., 0),$$

where every entry lies in m_S. Therefore, when looking at $(\mathscr{Y}_J^2)_{1,1}$, a completely analogous argument can be used to show that t^2 must also lie in $I_{\ell+1}$. Specifically, because all the elements in $(\mathscr{Y})_{1,*}$ and $(\mathscr{Y})_{*,1}$ lie in m_S, there exists an a' in J such that $(\mathscr{Y}_J^2)_{1,1} = (\mathscr{Y})_{1,1}^2 + a' = t^2 + a'$. Therefore, since $(\mathscr{Y}_J)^2$ is a matrix in $Mat_{2r+1}(J)$, it must be the case that t^2 is in J for all J satisfying the two conditions (a) and (b). Hence t^2 must lie in $I_{\ell+1}$.

Finally, we focus on the requirement that

$$(\mathscr{X}_J\mathscr{Y}_J)^r - (\mathscr{Y}_J\mathscr{X}_J)^r \in Mat_{2r+1}(J).$$

Using Remark 4.2, we see that

$$\mathscr{X}^2 = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \begin{matrix} -bs^2c_2 \\ +s^2c_{2r+1} \end{matrix} \end{array}\right],$$

$$\mathscr{Y}^2 = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} \begin{matrix} t^2c_1 \\ -st^2c_2 \end{matrix} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{array}\right],$$

$$\mathscr{X}\mathscr{Y} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} -stc_1 & c_4 & 0 & c_6 & 0 & \dots & c_{2r} & 0 & bc_1 & 0 \\ +sc_4 & & & & & & & & -bsc_2 & \\ & & & & & & & & +sc_{2r+1} & \end{array}\right],$$

and

$$\mathscr{Y}\mathscr{X} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & tc_1 & c_5 & 0 & c_7 & 0 & \dots & c_{2r+1} & 0 & tbc_1 \\ & -stc_2 & & & & & & & & -stbc_2 \\ & +sc_3 & & & & & & & & +(sb-bs)c_3 \end{array}\right].$$

Let $X_J = \mathscr{X} \bmod J$ and $Y_J = \mathscr{Y} \bmod J$. Furthermore, let us recall that when we write

$$X_J = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & c_1 & c_4 & 0 & c_6 & 0 & \dots & c_{2r} & 0 & bc_1 \\ & & & & & & & & & -bsc_2 \\ & & & & & & & & & +sc_{2r+1} \end{array}\right],$$

we implicitly mean that the coefficients of the c_i are to be taken modulo J. Since s^2 and t^2 are in J, we see that

$$X_J^2 = Y_J^2 = 0$$

in $Mat_{2r+1}(S/J)$. Moreover, we have

$$X_J Y_J = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} -stc_1 & c_4 & 0 & c_6 & 0 & \dots & c_{2r} & 0 & bc_1 & 0 \\ +sc_4 & & & & & & & & -bsc_2 & \\ & & & & & & & & +sc_{2r+1} & \end{array}\right],$$

$$Y_J X_J = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & \begin{array}{c} tc_1 \\ -stc_2 \\ +sc_3 \\ \end{array} & c_5 & 0 & c_7 & 0 & \dots & c_{2r+1} & 0 & (sb-bs)c_3 \end{array}\right],$$

since $b = t(-st)^{r-1}$ by equation (4.19), and hence tb and $-stb$ are zero in S/J.

Let

$$w = sb - bs. \tag{4.20}$$

Looking at small powers of $X_J Y_J$, we get

$$(X_J Y_J)^2 = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} \begin{array}{c} (-st)^2 c_1 \\ +sc_6 \\ \end{array} & c_6 & 0 & c_8 & 0 & \dots & c_{2r} & 0 & \begin{array}{c} bc_1 \\ -bsc_2 \\ +sc_{2r+1} \end{array} & 0 & wc_4 & 0 \end{array}\right],$$

$$(X_J Y_J)^3 = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c} \begin{array}{c} (-st)^3 c_1 \\ +sc_8 \\ \end{array} & c_8 & 0 & c_{10} & 0 & \dots & c_{2r} & 0 & \begin{array}{c} bc_1 \\ -bsc_2 \\ +sc_{2r+1} \end{array} & 0 & wc_4 & 0 & wc_6 & 0 \end{array}\right].$$

We see inductively that

$$(X_J Y_J)^{r-1} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} \begin{array}{c} (-st)^{r-1} c_1 \\ +sc_{2r} \\ \end{array} & c_{2r} & 0 & \begin{array}{c} bc_1 \\ -bsc_2 \\ +sc_{2r+1} \end{array} & 0 & wc_4 & 0 & wc_6 & \dots & 0 & wc_{2(r-1)} & 0 \end{array}\right],$$

and

$$(X_JY_J)^r = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} ((-st)^r+bs)c_1 & bc_1 & 0 & wc_4 & 0 & wc_6 & \dots & 0 & wc_{2r} & 0 \\ & -bsc_2 & & & & & & & & \\ & +sc_{2r+1} & & & & & & & & \end{array}\right].$$

Rewriting $(-st)^r = (-s)(t)(-st)^{r-1} = -sb$, we have $(-st)^r + bs = -sb + bs = -w$.

Therefore we can write

$$(X_JY_J)^r = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} -wc_1 & bc_1 & 0 & wc_4 & 0 & wc_6 & \dots & 0 & wc_{2r} & 0 \\ & -bsc_2 & & & & & & & & \\ & +sc_{2r+1} & & & & & & & & \end{array}\right].$$

We similarly look at powers of Y_JX_J. We have

$$(Y_JX_J)^2 = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} 0 & t(-st)c_1 & c_7 & 0 & c_9 & \dots & 0 & c_{2r+1} & 0 & wc_3 & 0 & wc_5 \\ & +(-st)^2c_2 & & & & & & & & & & \\ & +sc_5 & & & & & & & & & & \end{array}\right].$$

Inductively, we see that

$$(Y_JX_J)^{r-1} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & t(-st)^{r-2}c_1 & c_{2r+1} & 0 & wc_3 & 0 & wc_5 & \dots & 0 & wc_{2(r-1)+1} \\ & +(-st)^{r-1}c_2 & & & & & & & & \\ & +sc_{2(r-1)+1} & & & & & & & & \end{array}\right],$$

and

$$(Y_JX_J)^r = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & t(-st)^{r-1}c_1 & wc_3 & 0 & wc_5 & \dots & 0 & wc_{2(r-1)+1} & 0 & wc_{2r+1} \\ & +(-st)^rc_2 & & & & & & & & \\ & +sc_{2r+1} & & & & & & & & \end{array}\right].$$

Recalling from equation (4.19) that $b = t(-st)^{r-1}$ and hence $(-st)^r = -sb$, we see that

$$(Y_J X_J)^r = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & \begin{array}{c} bc_1 \\ -sbc_2 \\ +sc_{2r+1} \end{array} & wc_3 & 0 & wc_5 & \dots & 0 & wc_{2(r-1)+1} & 0 & wc_{2r+1} \end{array}\right].$$

Therefore, using that $w = sb - bs$ by equation (4.20), we obtain

$$(X_J Y_J)^r - (Y_J X_J)^r = \left[\begin{array}{c|c|c|c|c|c} -wc_1 & wc_2 & -wc_3 & \dots & wc_{2r} & -wc_{2r+1} \end{array}\right]. \tag{4.21}$$

Recall that $X_J = \mathscr{X}$ mod J and $Y_J = \mathscr{Y}$ mod J. Equation (4.21), combined with our calculation of $\mathscr{X}^2$ and $\mathscr{Y}^2$ results in the following conclusion: The pair of matrices $(\mathscr{X}, \mathscr{Y})$ defines a lift of V over $S/(s^2, t^2, w)$.

Since the first row of $(X_J Y_J)^{r-1}$ is $((-st)^{r-1}, 0, 0, b, 0, ..., 0)$, it follows that the first row of $(\mathscr{X}\mathscr{Y})^{r-1}$ is

$$((-st)^{r-1}, 0, 0, b, 0, ..., 0) + (j_1, j_2, ..., j_{2r+1}),$$

where each j_i is in $J \subseteq I_\ell$. However, recall that $\mathscr{X}_{\ell+1}$ must be of the form $\mathscr{X} + H_X$, where H_X is a matrix in $Mat_{2r+1}(I_\ell)$, and that $\mathscr{Y}_{\ell+1}$ must be of the form $\mathscr{Y} + H_Y$, where H_Y is a matrix in $Mat_{2r+1}(I_\ell)$. Since $J \subseteq I_\ell$, it follows that the first row of $(\mathscr{X}_{\ell+1}\mathscr{Y}_{\ell+1})^{r-1}$ is

$$((-st)^{r-1}, 0, 0, b, 0, ..., 0) + (a_1, a_2, ..., a_{2r+1}),$$

where each a_i is in I_ℓ. Similarly, since the first column of $(X_J Y_J)$ is $(-st, 0, 0, s, 0, 0, ..., 0)^T$, it follows that the first column of $(\mathscr{X}_{\ell+1}\mathscr{Y}_{\ell+1})$ is

$$(-st, 0, 0, s, 0, 0, ..., 0)^T + (a'_1, a'_2, ..., a'_{2r+1})^T,$$

where each of the a_i' lies in I_ℓ. Since $m_S * I_\ell + I_\ell * m_S \subseteq I_{\ell+1}$, it follows that there exists an element d in $I_{\ell+1}$ so that

$$(\mathscr{X}_{\ell+1}\mathscr{Y}_{\ell+1})_{11}^r = (-st)^r + bs + d = -w + d.$$

If we can show that there exists an element d' in $I_{\ell+1}$ so that $(\mathscr{Y}_{\ell+1}\mathscr{X}_{\ell+1})_{11}^r = d'$, then $(\mathscr{X}_{\ell+1}\mathscr{Y}_{\ell+1})_{11}^r - (\mathscr{Y}_{\ell+1}\mathscr{X}_{\ell+1})_{11}^r = w + d - d'$. Since this entry is required to be in $I_{\ell+1}$, it then follows that w must be in $I_{\ell+1}$.

To this end, notice that the first row of $(Y_J X_J)$ is $(0, t, 0, ..., 0)$, and the first column of $(Y_J X_J)^{r-1}$ is $(0, 0, ..., 0)^T$. This implies that there exist elements a_i'' and a_i''' in I_ℓ for $1 \leq i \leq 2r+1$ so that the first row of $\mathscr{Y}_{\ell+1}\mathscr{X}_{\ell+1}$ is $(0, t, 0, ..., 0) + (a_1'', ..., a_{2r+1}'')$, and the first column of $(\mathscr{Y}_{\ell+1}\mathscr{X}_{\ell+1})^{r-1}$ is $(0, 0, ..., 0)^T + (a_1''', ..., a_{2r+1}''')^T$. Once again recalling that $m_S * I_\ell + I_\ell * m_S \subseteq I_{\ell+1}$, it follows that there exists an element d' in $I_{\ell+1}$ so that $(\mathscr{Y}_{\ell+1}\mathscr{X}_{\ell+1})_{11}^r = d'$, completing the proof that w must lie in $I_{\ell+1}$.

Therefore, we obtain that $(\mathscr{X}, \mathscr{Y})$ defines a lift of V over $S/(s^2, t^2, w)$ and that $(s^2, t^2, w) + m_S * I_\ell + I_\ell * m_S$ is the minimal ideal satisfying conditions (a) and (b). Therefore,

$$I_{\ell+1} = (s^2, t^2, w) + m_S * I_\ell + I_\ell * m_S \subseteq I_\ell, \tag{4.22}$$

and we may take $\mathscr{X}_{\ell+1} = \mathscr{X}$ and $\mathscr{Y}_{\ell+1} = \mathscr{Y}$, completing the induction step.

Notice that $I_2 = m_S^2 = (s^2, t^2, w) + m_S^2$. If we suppose inductively that $I_i = (s^2, t^2, w) + m_S^i$ for $2 \leq i \leq \ell$, then applying equation (4.22) yields

$$\begin{aligned}
I_{\ell+1} &= (s^2,t^2,w) + m_S * I_\ell + I_\ell * m_S \\
&= (s^2,t^2,w) + m_S * ((s^2,t^2,w) + m_S^\ell) + ((s^2,t^2,w) + m_S^\ell) * m_S \\
&= (s^2,t^2,w) + m_S^{\ell+1},
\end{aligned}$$

completing the induction. Therefore,

$$I_\ell = (s^2,t^2,w) + m_S^\ell \tag{4.23}$$

for all $\ell \geq 2$. We know from the proof of Theorem 3.16 that the versal deformation ring $R(A,V)$ of V is isomorphic to $R = S/I$, where $I = \bigcap I_\ell$. Since $m_S^\ell \subseteq (s^2,t^2,w)$ for all $\ell \geq 2r+1$, applying equation (4.23) yields

$$I = \bigcap_\ell I_\ell = \bigcap_\ell ((s^2,t^2,w) + m_S^\ell) = (s^2,t^2,w).$$

In particular, $R(A,V) = R(\Omega_r, V_r) \cong \Omega_r$, proving Theorem 4.3.

Additionally, we have

$$\begin{aligned}
(X_\ell, Y_\ell) &= (\mathscr{X}_\ell \mod I_\ell,\ \mathscr{Y}_\ell \mod I_\ell) \\
&= (\mathscr{X} \mod I_\ell,\ \mathscr{Y} \mod I_\ell)
\end{aligned}$$

for all ℓ. Therefore, it follows from the construction of a versal deformation in the proof of Theorem 3.16 that the versal deformation $[\phi_u]$ of V over S/I corresponds to the lift ϕ_u with

$$(\phi_u(\bar{x}), \phi_u(\bar{y})) = (\mathscr{X} \mod I,\ \mathscr{Y} \mod I).$$

4.2 The case $\lambda = W$, $A = W[D_8]$ and $V = V_2$

In this section, suppose that k is a perfect field of characteristic 2, and let $\lambda = W$ be the ring of infinite Witt vectors over k. Note that W is a complete, local, commutative, Noetherian ring with residue field isomorphic to k, whose unique maximal ideal is the principal ideal $\mu = (2)$. We let $A = W[D_8]$ be the group ring over W of a dihedral group of order 8. We first prove the following connection between group rings over W of dihedral groups of order 2^{m+2} and Ω_{2^m}, for $m \geq 1$. Recall that

$$\Omega_{2^m} = k\langle\langle x, y\rangle\rangle/(x^2, y^2, (xy)^{2^m} - (yx)^{2^m})$$

and that $\bar{x}$ and $\bar{y}$ denote the images in Ω_{2^m} of x and y, respectively

Lemma 4.5. *For $m \geq 1$, let $D_{2^{m+2}}$ be a dihedral group of order 2^{m+2} with presentation $D_{2^{m+2}} = \langle u, v | u^2 = v^2 = 1, (uv)^{2^m} = (vu)^{2^m}\rangle$. Define*

$$\widehat{\Omega}_{2^m} = W\langle\langle x, y\rangle\rangle/(x^2 - 2x, y^2 - 2y, ((x-1)(y-1))^{2^m} - ((y-1)(x-1))^{2^m})$$

and let $\hat{x}$ and $\hat{y}$ denote the images in $\widehat{\Omega}_{2^m}$ of x and y, respectively. Then the map $f : W[D_{2^{m+2}}] \to \widehat{\Omega}_{2^m}$, defined by $f(u) = \hat{x}-1$, $f(v) = \hat{y}-1$, and extending W-linearly, is a W-algebra isomorphism. Furthermore, f induces a k-algebra isomorphism

$$\bar{f} : k[D_{2^{m+2}}] \cong W[D_{2^{m+2}}]/2W[D_{2^{m+2}}] \to \Omega_{2^m}$$

given by $\bar{f}(u) = \bar{x} - 1, \bar{f}(v) = \bar{y} - 1$, and extending k-linearly.

Proof. Since $u^2 = v^2 = 1$ and $(uv)^{2^m} = (vu)^{2^m}$, it follows that f is a bijective W-algebra homomorphism. Since $W/2W \cong k$, we have $k[D_{2^{m+2}}] \cong W[D_{2^{m+2}}]/2W[D_{2^{m+2}}]$.

To complete the proof of the lemma, it suffices to show that the ideals

$$(x^2-2x, y^2-2y, ((x-1)(y-1))^{2^m} - ((y-1)(x-1))^{2^m})$$

and

$$(x^2, y^2, (xy)^{2^m} - (yx)^{2^m})$$

are equal in $k\langle\langle x, y\rangle\rangle$.

Note that since the characteristic of k is 2, $2x = 2y = 0$ in $k\langle\langle x, y\rangle\rangle$, and so we can already write

$$\begin{gathered}(x^2-2x, y^2-2y, ((x-1)(y-1))^{2^m} - ((y-1)(x-1))^{2^m})\\ = (x^2, y^2, ((x-1)(y-1))^{2^m} - ((y-1)(x-1))^{2^m}).\end{gathered}$$

Let ℓ be a positive integer, and define the ideal L of $k\langle\langle x, y\rangle\rangle$ as

$$L = (x^2, y^2).$$

Notice that since the characteristic of k is 2,

$$(x-1)^2 + L = (y-1)^2 + L = 1 + L = -1 + L.$$

We also note that

$$(x-1)(y-1) - (y-1)(x-1) = (xy - x - y + 1) - (yx - x - y + 1) = xy - yx.$$

Suppose inductively that for $i \leq \ell - 1$,

$$((x-1)(y-1))^i - ((y-1)(x-1))^i + L = (xy)^i - (yx)^i + L.$$

Letting $a = ((x-1)(y-1))^{\ell-1} - ((y-1)(x-1))^{\ell-1}$, we have the following equations:

$$\begin{aligned}
&(xy)^{\ell} - (yx)^{\ell} + L \\
&= (xy - x - y + 1)((xy)^{\ell-1} - (yx)^{\ell-1}) - ((xy)^{\ell-1} - (yx)^{\ell-1})(yx - x - y + 1) + L \\
&= (x-1)(y-1)((xy)^{\ell-1} - (yx)^{\ell-1}) - ((xy)^{\ell-1} - (yx)^{\ell-1})(y-1)(x-1) + L \\
&= (x-1)(y-1)a - a(y-1)(x-1) + L \\
&= ((x-1)(y-1))^{\ell} - ((y-1)(x-1))^{\ell} + L.
\end{aligned}$$

Therefore, the ideals

$$(x^2, y^2, ((x-1)(y-1))^{2^m} - ((y-1)(x-1))^{2^m})$$

and

$$(x^2, y^2, (xy)^{2^m} - (yx)^{2^m})$$

of $k\langle\langle x, y\rangle\rangle$ are equal, which proves the lemma. □

By Lemma 4.5, we can identify $A = W[D_8]$ with the W-algebra

$$\widehat{\Omega}_2 = W\langle\langle x, y\rangle\rangle/(x^2 - 2x, y^2 - 2y, ((x-1)(y-1))^2 - ((y-1)(x-1))^2),$$

and $A/2A \cong k[D_8]$ with

$$\Omega_2 = k\langle\langle x, y\rangle\rangle/(x^2, y^2, (xy)^2 - (yx)^2).$$

In this way, we can view V_2 as a 5-dimensional k-vector space which is also a left A-module. More precisely, using the representation ρ_2 of Ω_2 from the beginning of Chapter 4, we define the following matrices in $Mat_5(k)$:

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c|c|c|c} 0 & c_1 & c_4 & 0 & 0 \end{array}\right]$$

and

$$Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \left[\begin{array}{c|c|c|c|c} 0 & c_3 & 0 & c_5 & 0 \end{array}\right].$$

We then obtain a W-algebra homomorphism

$$\widehat{\rho}_2 : \widehat{\Omega}_2 \to Mat_5(k)$$

by defining $\widehat{\rho}_2(\hat{x}) = X$, $\widehat{\rho}_2(\hat{y}) = Y$, and extending W-linearly. In other words, if $\pi : \widehat{\Omega}_2 \to \Omega_2$ is the natural projection sending $\hat{x}$ to $\bar{x}$ and $\hat{y}$ to $\bar{y}$, then $\widehat{\rho}_2 = \rho_2 \circ \pi$. Let $\widehat{V}_2$ denote the corresponding $\widehat{\Omega}_2$-module.

Our goal is to prove the following result.

Theorem 4.6. *The versal deformation ring $R(\widehat{\Omega}_2, \widehat{V}_2)$ is isomorphic to $\widehat{\Omega}_2$.*

Proof. Let $V = \widehat{V}_2$ and $\rho = \widehat{\rho}_2$. Since $\widehat{\rho}_2 = \rho_2 \circ \pi$, it follows from Lemma 4.4 that $F_V(k[\epsilon])$ is a two-dimensional k-vector space. Therefore, a versal couple for V exists, and the versal deformation ring $R = R(\widehat{\Omega}_2, V)$ will be a quotient of $W\langle\langle s,t\rangle\rangle$. In finding this versal couple we closely follow the methods and notations of Section 4.1.

Define $\widehat{S} = W\langle\langle s,t\rangle\rangle$, and define the following matrices in $\widehat{S}/(2, s^2, t) \cong k[\epsilon]$:

$$X_s = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & s \end{bmatrix} = \left[\begin{array}{c|c|c|c|c} 0 & c_1 & c_4 & 0 & sc_5 \end{array}\right]$$

and

$$Y_s = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ s & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \left[\begin{array}{c|c|c|c|c} sc_3 & c_3 & 0 & c_5 & 0 \end{array}\right].$$

Also, define the following matrices in $\widehat{S}/(2, s, t^2) \cong k[\epsilon]$:

$$X_t = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c|c|c|c} 0 & c_1 & c_4 & 0 & 0 \end{array}\right]$$

and

$$Y_t = \begin{bmatrix} t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \left[\begin{array}{c|c|c|c|c} tc_1 & c_3 & 0 & c_5 & 0 \end{array} \right].$$

It follows from the arguments in Section 4.1 that the pairs of matrices (X_s, Y_s) and (X_t, Y_t) define non-isomorphic lifts of V over $k[\epsilon]$. Let

$$\widehat{I_2} = \mu + m_{\widehat{S}}^2 = (2, s^2, t^2, st, ts).$$

Taking the pullback over k of these pairs of matrices, we arrive at the following matrices in $Mat_5(\widehat{S}/\widehat{I_2})$:

$$X_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & s \end{bmatrix} = \left[\begin{array}{c|c|c|c|c} 0 & c_1 & c_4 & 0 & sc_5 \end{array} \right]$$

and

$$Y_2 = \begin{bmatrix} t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ s & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \left[\begin{array}{c|c|c|c|c} tc_1 + sc_3 & c_3 & 0 & c_5 & 0 \end{array} \right].$$

Then (X_2, Y_2) defines a lift of V over $\widehat{S}/\widehat{I}_2$. We define the following elements of $\widehat{S}$:

$$\begin{aligned} b &= -tst, \\ \alpha &= b - 2t - 4s + 2st + 2ts, \\ q &= \alpha s - s\alpha, \\ w &= (st)^2 - (ts)^2. \end{aligned}$$

We also define the following matrices in $Mat_5(\widehat{S})$:

$$\mathcal{X} = \begin{bmatrix} 2 & 1 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & -\alpha s \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & s \end{bmatrix},$$

$$\mathcal{Y} = \begin{bmatrix} t & 0 & 0 & 0 & 0 \\ 2s - st & 2 & 0 & 4 & -8 \\ s & 1 & 0 & 2 & -4 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Let $\pi_2 : \widehat{S} \to \widehat{S}/\widehat{I}_2$ be the natural projection. Then $\pi_2((\mathcal{X}, \mathcal{Y})) = (X_2, Y_2)$.

To proceed with the construction of the versal couple for V, suppose inductively that we have found ideals of $\widehat{I}_2, \widehat{I}_3, ..., \widehat{I}_\ell$ of $\widehat{S}$ and pairs of matrices

$$(\mathcal{X}_2, \mathcal{Y}_2), (\mathcal{X}_3, \mathcal{Y}_3), ..., (\mathcal{X}_\ell, \mathcal{Y}_\ell) \in Mat_5(\widehat{S}) \times Mat_5(\widehat{S})$$

such that

$$\mathcal{X}_i^2 - 2\mathcal{X}_i, \mathcal{Y}_i^2 - 2\mathcal{Y}_i, ((\mathcal{X}_i - 1)(\mathcal{Y}_i - 1))^2 - ((\mathcal{Y}_i - 1)(\mathcal{X}_i - 1))^2 \in Mat_{2r+1}(\widehat{I}_i)$$

for all $2 \leq i \leq \ell$. Further assume that it is permissible to take $\mathcal{X}_\ell = \mathcal{X}$, and $\mathcal{Y}_\ell = \mathcal{Y}$. We seek an ideal $\widehat{I}_{\ell+1}$, minimal among all ideals J of $\widehat{S}$ that satisfy the following two conditions:

(a) $m_{\widehat{S}} * \widehat{I}_\ell + \widehat{I}_\ell * m_{\widehat{S}} \subseteq J \subseteq \widehat{I}_\ell$, and

(b) there exists a pair of matrices $(\mathcal{X}_J, \mathcal{Y}_J)$ in $Mat_5(\widehat{S}) \times Mat_5(\widehat{S})$ such that $\pi_J((\mathcal{X}_J, \mathcal{Y}_J))$ defines a lift of V over $\widehat{S}/J$, where $\pi_J : \widehat{S} \to \widehat{S}/J$ is the natural projection extended to the Cartesian product of matrix rings, and so that $\mathcal{X}_J - \mathcal{X}_\ell$ and $\mathcal{Y}_J - \mathcal{Y}_\ell$ both lie in $Mat_5(\widehat{I}_\ell)$.

Since $\mathcal{X}_J - \mathcal{X}_\ell$ and $\mathcal{Y}_J - \mathcal{Y}_\ell$ must both lie in $Mat_5(\widehat{I}_\ell)$, there are matrices G_X and G_Y in $Mat_5(\widehat{I}_\ell)$ such that $\mathcal{X}_J = \mathcal{X}_\ell + G_X$ and $\mathcal{Y}_J = \mathcal{Y}_\ell + G_Y$. Moreover, since we are assuming $\mathcal{X}_\ell = \mathcal{X}$ and $\mathcal{Y}_\ell = \mathcal{Y}$, we must have $\mathcal{X}_J = \mathcal{X} + G_X$ and $\mathcal{Y}_J = \mathcal{Y} + G_Y$. Letting $\mathcal{X}_{i,*}$ and $\mathcal{X}_{*,i}$ denote the i^{th} row and column, respectively, of $\mathcal{X}$, we notice the following:

$$(\mathcal{X})_{5,*} = (0, 0, 0, 0, s),$$

$$(\mathcal{X})_{*,5} = (\alpha, -\alpha s, 0, 0, s)^T.$$

Furthermore,

$$\begin{aligned} (\mathcal{X}_J^2)_{5,5} &= ((\mathcal{X})_{5,*} + (G_X)_{5,*}) \cdot ((\mathcal{X})_{*,5} + (G_X)_{*,5}) \\ &= (\mathcal{X})_{5,*} \cdot (\mathcal{X})_{*,5} + (\mathcal{X})_{5,*} \cdot (G_X)_{*,5} + (G_X)_{5,*} \cdot (\mathcal{X})_{*,5} + (G_X)_{5,*} \cdot (G_X)_{*,5}. \end{aligned}$$

However, recall that

$$m_{\widehat{S}} * \widehat{I}_\ell + \widehat{I}_\ell * m_{\widehat{S}} \subseteq J \subseteq \widehat{I}_\ell.$$

Additionally, notice that every entry of the matrix G_X lies in $\widehat{I}_\ell$ and every entry of $(\mathcal{X})_{5,*}$ and $(\mathcal{X})_{*,5}$ lies in $m_{\widehat{S}}$. Therefore, letting

$$a = (\mathcal{X})_{5,*} \cdot (G_X)_{*,5} + (G_X)_{5,*} \cdot (\mathcal{X})_{*,5} + (G_X)_{5,*} \cdot (G_X)_{*,5}$$

we see that

$$(\mathcal{X}_J^2)_{5,5} = (\mathcal{X})_{5,*} \cdot (\mathcal{X})_{*,5} + a = (\mathcal{X}^2)_{5,5} + a$$

where a lies in J. Additionally, $2\mathcal{X}_J = 2\mathcal{X} + 2G_X$. Since $2 \in m_{\widehat{S}}$ and

$$m_{\widehat{S}} * \widehat{I}_\ell + \widehat{I}_\ell * m_{\widehat{S}} \subseteq J \subseteq \widehat{I}_\ell,$$

it follows that there exists some h in J such that

$$(2\mathcal{X}_J)_{5,5} = (2\mathcal{X})_{5,5} + h.$$

Therefore,

$$(\mathcal{X}_J^2 - 2\mathcal{X}_J)_{5,5} - (\mathcal{X}^2 - 2\mathcal{X})_{5,5} \in Mat_5(J).$$

Notice that since $(\mathcal{X}_J, \mathcal{Y}_J)$ defines a lift of V over $\widehat{S}/J$, it follows that $\mathcal{X}_J^2 - 2X_J$ is a matrix in $Mat_5(J)$. Therefore, it follows that

$$(\mathcal{X}^2 - 2\mathcal{X})_{5,5} = s^2 - 2s \in J.$$

Since this is true for all J satisfying the conditions (a) and (b) above, it follows that it is in particular true for the minimal J satisfying these conditions. Therefore, $s^2 - 2s$ must lie in $\widehat{I}_{\ell+1}$.

Next, notice that

$$(\mathcal{Y})_{1,*} = (t, 0, 0, 0, 0),$$

$$(\mathcal{Y})_{*,1} = (t, 2s - st, s, 0, 0)^T,$$

where every entry lies in $m_{\widehat{S}}$. Therefore, when looking at $(\mathcal{Y}_J^2)_{1,1}$, an analogous argument can be used to show that $t^2 - 2t$ must also lie in $\widehat{I}_{\ell+1}$. Specifically, because all the entries of $(\mathcal{Y})_{1,*}$ and $(\mathcal{Y})_{*,1}$ lie in $m_{\widehat{S}}$, there exists an a' in J such that

$$(\mathcal{Y}_J^2)_{1,1} = (\mathcal{Y})_{1,1}^2 + a' = t^2 + a'.$$

Additionally, since $2\widehat{I}_\ell \subseteq m_{\widehat{S}} * \widehat{I}_\ell \subseteq J$, there exists an h' in J such that

$$(2\mathcal{Y}_J)_{1,1} = (2\mathcal{Y})_{1,1} + h' = 2t + h'.$$

Therefore, since $\mathcal{Y}_J^2 - 2Y_J$ is a matrix in $Mat_5(J)$, it must be the case that $t^2 - 2t$ is in J for all J satisfying the conditions (a) and (b). Hence, $t^2 - 2t$ must lie in $\widehat{I}_{\ell+1}$.

Let N be the ideal of $\widehat{S}$ generated by $s^2 - 2s$ and $t^2 - 2t$. Since $N \subseteq J$, it will be useful to compute

$$((\mathcal{X} - 1)(\mathcal{Y} - 1))^2 - ((\mathcal{Y} - 1)(\mathcal{X} - 1))^2$$

modulo N. To do so, we first note that

$$\mathcal{X}^2 = \begin{bmatrix} 4 & 2 & 0 & 0 & 2\alpha \\ 0 & 0 & 0 & 0 & -\alpha s^2 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & s^2 \end{bmatrix},$$

$$2\mathcal{X} = \begin{bmatrix} 4 & 2 & 0 & 0 & 2\alpha \\ 0 & 0 & 0 & 0 & -2\alpha s \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2s \end{bmatrix},$$

$$\mathcal{Y}^2 = \begin{bmatrix} t^2 & 0 & 0 & 0 & 0 \\ 2st - st^2 + 2(2s - st) & 4 & 0 & 8 & -16 \\ 2s & 2 & 0 & 4 & -8 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix},$$

$$2\mathcal{Y} = \begin{bmatrix} 2t & 0 & 0 & 0 & 0 \\ 2(2s - st) & 4 & 0 & 8 & -16 \\ 2s & 2 & 0 & 4 & -8 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix},$$

and so $\mathcal{X}^2 - 2\mathcal{X}$ and $\mathcal{Y}^2 - 2\mathcal{Y}$ are each 0 modulo N. Therefore,

$$((\mathcal{X}-1)(\mathcal{Y}-1))^2 - ((\mathcal{Y}-1)(\mathcal{X}-1))^2 \equiv (\mathcal{X}\mathcal{Y})^2 - (\mathcal{Y}\mathcal{X})^2 - 2(\mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X}) \mod N.$$

To compute

$$(\mathcal{X}\mathcal{Y})^2 - (\mathcal{Y}\mathcal{X})^2 - 2(\mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X}) \mod N,$$

we first perform the following initial calculations in $\widehat{S}/N$, using the fact that $s^2 - 2s$ and $t^2 - 2t$ are in N:

$$\begin{aligned} t\alpha + N &= -4t + N, \\ \alpha s + N &= bs + 2ts - 8s + 2sts + N, \\ q + N &= w - (2st - 2ts) + N = ((s-1)(t-1))^2 - ((t-1)(s-1))^2 + N. \end{aligned}$$

Additionally, we compute

$$\mathcal{XY} = \begin{bmatrix} 2t+2s-st & 2 & 0 & \alpha+4 & -8 \\ 0 & 0 & 0 & -\alpha s & 0 \\ 2s & 2 & 0 & 4 & -8 \\ s & 1 & 0 & 2 & -4 \\ 0 & 0 & 0 & s & 0 \end{bmatrix},$$

$$\mathcal{YX} = \begin{bmatrix} 2t & t & 0 & 0 & t\alpha \\ 4s-2st & 2s-st & 4 & 0 & -2q+4st-8s \\ 2s & s & 2 & 0 & -q-4s \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Using these relations, we see that

$$((\mathcal{X}-1)(\mathcal{Y}-1))^2 - ((\mathcal{Y}-1)(\mathcal{X}-1))^2 \equiv \begin{bmatrix} q & 0 & 0 & -2q & 2tq \\ -2q & -q & 2q & 0 & (4s-2st+4)q \\ 0 & 0 & q & -2q & 2sq \\ 0 & 0 & 0 & -q & 2q \\ 0 & 0 & 0 & 0 & q \end{bmatrix}$$

modulo N. Therefore, the pair $(\mathcal{X}, \mathcal{Y})$ defines a lift of V over

$$\widehat{S}/(N+(q)) = \widehat{S}/(s^2-2s, t^2-2t, ((s-1)(t-1))^2 - ((t-1)(s-1))^2).$$

We now examine the first rows and columns of the matrices $\mathcal{XY}$ and $\mathcal{YX}$, to see that q must also be in $\widehat{I}_{\ell+1}$. Recall that

$$((\mathcal{X}_J - 1)(\mathcal{Y}_J - 1))^2 - ((\mathcal{Y}_J - 1)(\mathcal{X}_J - 1))^2$$

must be a matrix in $Mat_5(J)$. However, since $\mathcal{X}_J^2 - 2\mathcal{X}_J$ and $\mathcal{Y}_J^2 - 2\mathcal{Y}_J$ are also matrices in $Mat_5(J)$, we get

$$((\mathcal{X}_J - 1)(\mathcal{Y}_J - 1))^2 - ((\mathcal{Y}_J - 1)(\mathcal{X}_J - 1))^2 \equiv (\mathcal{X}_J\mathcal{Y}_J)^2 - (\mathcal{Y}_J\mathcal{X}_J)^2 - 2(\mathcal{X}_J\mathcal{Y}_J - \mathcal{Y}_J\mathcal{X}_J)$$

modulo J. Recall that $\mathcal{X} - \mathcal{X}_J$ and $\mathcal{Y} - \mathcal{Y}_J$ must both be matrices in $Mat_5(\widehat{I}_\ell)$, and that $m_{\widehat{S}} * \widehat{I}_\ell + \widehat{I}_\ell * m_{\widehat{S}} \subseteq J$. Since 2 is in $m_{\widehat{S}}$,

$$2(\mathcal{X}_J\mathcal{Y}_J - \mathcal{Y}_J\mathcal{X}_J) \equiv 2(\mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X}) \mod J.$$

Looking at the $(1,1)$-entries of these matrices yields

$$2(\mathcal{X}_J\mathcal{Y}_J - \mathcal{Y}_J\mathcal{X}_J)_{1,1} \equiv 2(\mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X})_{1,1} \mod J.$$

We next look at the $(1,1)$-entry of $(\mathcal{X}_J\mathcal{Y}_J)^2 - (\mathcal{Y}_J\mathcal{X}_J)^2$. Notice that our earlier computations show that all entries in the first row and first column of both $\mathcal{X}\mathcal{Y}$ and $\mathcal{Y}\mathcal{X}$ lie in $m_{\widehat{S}}$. Since $m_{\widehat{S}} * \widehat{I}_\ell + \widehat{I}_\ell * m_{\widehat{S}} \subseteq J$, and because each entry of $\mathcal{X} - \mathcal{X}_J$ and $\mathcal{Y} - \mathcal{Y}_J$ must lie in $\widehat{I}_\ell$, it follows that the $(1,1)$-entries of the matrices

$$(\mathcal{X}_J\mathcal{Y}_J)^2 - (\mathcal{Y}_J\mathcal{X}_J)^2$$

and

$$(\mathcal{X}\mathcal{Y})^2 - (\mathcal{Y}\mathcal{X})^2$$

only differ by an element of J.

Therefore, the $(1,1)$-entries of the matrices

$$(\mathcal{X}_J\mathcal{Y}_J)^2 - (\mathcal{Y}_J\mathcal{X}_J)^2 - 2(\mathcal{X}_J\mathcal{Y}_J - \mathcal{Y}_J\mathcal{X}_J)$$

and

$$(\mathcal{X}\mathcal{Y})^2 - (\mathcal{Y}\mathcal{X})^2 - 2(\mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X})$$

are congruent to each other modulo J. Since

$$(\mathcal{X}_J\mathcal{Y}_J)^2 - (\mathcal{Y}_J\mathcal{X}_J)^2 - 2(\mathcal{X}_J\mathcal{Y}_J - \mathcal{Y}_J\mathcal{X}_J)$$

must be a matrix in $Mat_5(J)$, and because the $(1,1)$-entry of

$$(\mathcal{X}\mathcal{Y})^2 - (\mathcal{Y}\mathcal{X})^2 - 2(\mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X})$$

is congruent to q modulo N and $N \subseteq J$, it follows that q must lie in J for all J satisfying the conditions (a) and (b). Hence q must lie in $\widehat{I}_{\ell+1}$.

We obtain that $(\mathcal{X},\mathcal{Y})$ defines a lift of V over $\widehat{S}/(s^2-2s, t^2-2t, q)$. Moreover, we know that $(s^2-2s, t^2-2t, q) + m_{\widehat{S}} * \widehat{I}_\ell + \widehat{I}_\ell * m_{\widehat{S}}$ is the minimal ideal satisfying the conditions (a) and (b). Therefore,

$$\widehat{I}_{\ell+1} = (s^2-2s, t^2-2t, q) + m_{\widehat{S}} * \widehat{I}_\ell + \widehat{I}_\ell * m_{\widehat{S}}, \tag{4.24}$$

and we may take $\mathcal{X}_{\ell+1} = \mathcal{X}$ and $\mathcal{Y}_{\ell+1} = \mathcal{Y}$, completing the induction step.

Notice that $\widehat{I}_2 = \mu + m_{\widehat{S}}^2 = (s^2-2s, t^2-2t, q) + \mu + m_{\widehat{S}}^2$. If we suppose inductively that $\widehat{I}_i = (s^2-2s, t^2-2t, q) + \mu m_{\widehat{S}}^{i-2} + m_{\widehat{S}}^i$ for $2 \le i \le \ell$, then applying equation (4.24) yields

$$\begin{aligned}
\widehat{I}_{\ell+1} &= (s^2-2s, t^2-2t, q) + m_{\widehat{S}} * \widehat{I}_\ell + \widehat{I}_\ell * m_{\widehat{S}} \\
&= (s^2-2s, t^2-2t, q) + m_{\widehat{S}} * ((s^2-2s, t^2-2t, q) + \mu m_{\widehat{S}}^{\ell-2} + m_{\widehat{S}}^{\ell}) \\
&+ ((s^2-2s, t^2-2t, q) + \mu m_{\widehat{S}}^{\ell-2} + m_{\widehat{S}}^{\ell})) * m_{\widehat{S}} \\
&= (s^2-2s, t^2-2t, q) + \mu m_{\widehat{S}}^{\ell-1} + m_{\widehat{S}}^{\ell+1},
\end{aligned}$$

completing the induction. Therefore,

$$\widehat{I}_\ell = (s^2-2s, t^2-2t, q) + \mu m_{\widehat{S}}^{\ell-2} + m_{\widehat{S}}^{\ell} \tag{4.25}$$

for all $\ell \geq 2$. We know from Theorem 3.16 that the versal deformation ring $R(\widehat{\Omega}_2, V)$ of V is isomorphic to $\widehat{R} = \widehat{S}/\widehat{I}$, where $\widehat{I} = \bigcap_{\ell\geq 2} \widehat{I}_\ell$. Let $\widehat{L} = (s^2-2s, t^2-2t, q)$, $\widehat{T} = \widehat{S}/\widehat{L}$, and consider the natural projection $\widehat{p}: \widehat{S} \to \widehat{S}/\widehat{L} = \widehat{T}$. Then, for all $\ell \geq 2$, $\widehat{p}$ maps $\widehat{L} + \mu m_{\widehat{S}}^{\ell-2} + m_{\widehat{S}}^{\ell}$ into $\mu m_{\widehat{T}}^{\ell-2} + m_{\widehat{T}}^{\ell}$. Therefore, $\bigcap_{\ell\geq 2}(\widehat{L} + \mu m_{\widehat{S}}^{\ell-2} + m_{\widehat{S}}^{\ell})$ is sent by $\widehat{p}$ into the set

$$\bigcap_{\ell\geq 2}(\mu m_{\widehat{T}}^{\ell-2} + m_{\widehat{T}}^{\ell}) \subseteq \bigcap_{\ell\geq 2} m_{\widehat{T}}^{\ell-1} = \left\{0_{\widehat{T}}\right\}.$$

Applying equation (4.25), this implies that

$$\widehat{I} = \bigcap_{\ell\geq 2} \widehat{I}_\ell = \bigcap_{\ell\geq 2}(\widehat{L} + \mu m_{\widehat{S}}^{\ell-2} + m_{\widehat{S}}^{\ell}) = \widehat{L} = (s^2-2s, t^2-2t, q).$$

In particular, $R(\widehat{\Omega}_2, V) = R(\widehat{\Omega}_2, \widehat{V}_2) \cong \widehat{\Omega}_2$, proving Theorem 4.6. □

Additionally, we have

$$\begin{aligned}(X_\ell, Y_\ell) &= (\mathcal{X}_\ell \mod \widehat{I}_\ell,\ \mathcal{Y}_\ell \mod \widehat{I}_\ell) \\ &= (\mathcal{X} \mod \widehat{I}_\ell,\ \mathcal{Y} \mod \widehat{I}_\ell)\end{aligned}$$

for all ℓ. Therefore, it follows from the construction of a versal deformation in the proof of Theorem 3.16 that the versal deformation $[\phi_u]$ of $V = \widehat{V}_2$ over $\widehat{S}/\widehat{I}$ corresponds to the lift ϕ_u with

$$(\phi_u(\hat{x}), \phi_u(\hat{y})) = (\mathcal{X} \mod \widehat{I},\ \mathcal{Y} \mod \widehat{I}).$$

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