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Computing spectral data for Maass cusp forms using resonance

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COMPUTING SPECTRAL DATA FOR MAASS CUSP FORMS USING RESONANCE

by

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CERTIFICATE OF APPROVAL

PH.D. THESIS

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ABSTRACT

The primary arithmetic information attached to a Maass cusp form is its Laplace eigenvalue. However, in the case of cuspidal Maass forms, the range that these eigenvalues can take is not well-understood. In particular it is unknown if, given a real number r, one can prove that there exists a primitive Maass cusp form with Laplace eigenvalue $1/4 + r^2$. Conversely, given the Fourier coefficients of a primitive Maass cusp form f on $\Gamma_0(D)$, it is not clear whether or not one can determine its Laplace eigenvalue. In this paper we show that given only a finite number of Fourier coefficients one can first determine the level D, and then compute the Laplace eigenvalue to arbitrarily high precision.

The key to our results will be understanding the resonance and rapid decay properties of Maass cusp forms. Let f be a primitive Maass cusp form with Fourier coefficients $\lambda_f(n)$. The resonance sum for f is given by

$$S_X(f;\alpha,\beta) = \sum_{n\geq 1} \lambda_f(n)\phi\left(\frac{n}{X}\right) e(\alpha n^\beta)$$

where $\phi \in C_c^\infty((1,2))$ is a Schwartz function and $\alpha \in \mathbb{R}$ and β, $X > 0$ are real numbers.

Sums of this form have been studied for many different classes of functions f, including holomorphic modular forms for SL(2, $\mathbb{Z}$), and Maass cusp forms for SL(n,$\mathbb{Z}$). In this paper we take f to be a primitive Maass cusp form for a congruence subgroup $\Gamma_0(D) \subset \mathrm{SL}(2,\mathbb{Z})$. Thus our result extends the family of automorphic forms for which their resonance properties are understood. Similar analysis and algorithms

can be easily implemented for holomorphic cusp forms for $\Gamma_0(D)$. Our techniques include Voronoi summation, weighted exponential sums, and asymptotics expansions of Bessel functions.

We then use these estimates in a new application of resonance sums. In particular we show that given only limited information about a Maass cusp form f (in particular a finite list of high Fourier coefficients), one can determine its level and estimate its spectral parameter, and thus its Laplace eigenvalue. This is done using a large parallel computing cluster running MATLAB and Mathematica.

PUBLIC ABSTRACT

Prime numbers are numbers greater than 1 that are divisible only by 1 and themselves, such as 2, 3, 5, 7 and 11. Because any whole number can be written uniquely as the product of prime numbers, prime numbers are the fundamental building blocks in mathematics. Therefore it is of great importance to understand the prime numbers. In particular, how are the prime numbers distributed? Are they separated uniformly, or at random?

The goal of number theory is to answer questions like these. In this paper we investigate a function which exhibits many of the same properties as the prime numbers. We study when such functions can occur, and what properties they must have. Finally, using computational techniques and given only a small amount of information, we are able to determine key properties of these functions.

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CHAPTER 1
INTRODUCTION AND BACKGROUND

1.1 Automorphic Forms and the Riemann Zeta Function

In 1859 Bernhard Riemann introduced the complex-valued function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}; \qquad\qquad s \in \mathbb{C}, Re(s) > 1$$

This function is now referred to as the Riemann zeta function. In the same paper he also showed that the zeta function has the following "Euler product"

$$\zeta(s) = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1}$$

for s in the same range as above. Thus we can see that the zeta function carries information about the prime numbers. Riemann also showed that knowing the first few zeros of the zeta function would allow one to get very good approximations to the prime counting function

$$\pi(X) = |\text{primes } p : p \leq X|$$

Moreover, the more zeros one knew, the better the approximation could be made. All of this and more was discovered by Riemann in [Rie59]. Riemann's analysis spurred others into researching the zeros of the zeta function. In order to study the zeta function it is natural to form functions that are not only superficially similar to the zeta function, but also satisfy the same key properties. These functions are known as "automorphic forms."

1.2 Modular Forms

The classical automorphic forms are described below. These are the most thoroughly studied functions in the family of automorphic forms.

There are two families of functions which are called classical automorphic forms for $SL(2,\mathbb{Z})$. These are the holomorphic modular forms, and the nonholomorphic Maass forms.

Holomorphic modular forms are well understood. One can construct many examples of holomorphic modular forms, such as Dirichlet series, holomorphic Eisenstein series, and modular forms arising from artihmetic functions. Moreover, modular forms can be grouped according to their "weight", which is a non-negative integer. All modular forms with a given weight form a complex vector space. Dimensions and spanning sets are known for all such vector spaces. For a discussion of all of these topics see [Ser12] Chapter 7.

1.3 Maass Forms

As their name suggests, Maass forms were first considered by Hans Maass in 1949 in [Maa49]. Unlike holomorphic forms, Maass forms are not well understood. The first (and only) family of Maass forms one can easily construct are the so-called nonholomorphic Eisenstein series, which we now construct.

Let

$$\mathcal{H} = \{z \in \mathbb{C} : z = x + iy, y > 0\}$$

be the upper-half plane. If $z \in \mathcal{H}$ and $s \in \mathbb{C}$, then we may form the conditionally convergent series

$$E(z,s) = \frac{1}{2} \sum_{(m,n)=1} \frac{y^s}{|mz+n|^{2s}}$$

where $y = Re(z)$. It can be shown that this sum is absolutely convergent for the right-half plane $Re(s) > 1$. In addition it has an analytic continuation to the entire s-plane (see [Bum98] pg. 65).

Now, define a **Maass form of type** ν (see [Bum98] pg. 104) to be a function $f : \mathcal{H} \to \mathbb{C}$ satisfying the following properties:

1. $f(\gamma z) = f(z)$ for all $\gamma \in SL(2,\mathbb{Z})$
2. $\Delta f = \nu(1-\nu)f$ where $\Delta = y^2(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y})$ and $\nu \in \mathbb{C}$
3. $|f(z)| \leq Cy^B$ for some constants $C > 0$ and $B \geq 1$

Then through direct computation one can show that the function

$$E_s(z) = E(z,s)$$

is a Maass form of type s.

The collection of all Maass forms of a given type form a $\mathbb{C}$ vector space under pointwise addition. As we have just seen, Eisenstein series are one such family of Maass forms. If we take the orthogonal complement of the subspace of Eisenstein series, it can be shown that that the orthogonal complement consists of Maass forms with one additional quality, the so-called "cuspidality" condition (see [Gol06] Ch. 3 Section 16).

Let f be a Maass form of type ν. Then if f satisfies the additional condition

4. $\int_0^1 f(z)dx = 0$ for all $y > 0$

then f is called a **Maass cusp form of type** ν. This range of Maass forms is concisely expressed in Selberg's spectral decomposition. Symbolically it states that

$$\mathcal{L}^2(SL(2,\mathbb{Z})\backslash\mathcal{H}) = \mathbb{C} \oplus \mathcal{L}^2_{Eis} \oplus \mathcal{L}^2_{cusp}$$

where the $\mathcal{L}^2$ spaces have the measure

$$\frac{dxdy}{y^2}$$

It can be easily verified that this is indeed a measure on $\mathcal{H}$ which is invariant under the action of SL(2,$\mathbb{Z}$).

Given a Maass form f with Fourier coefficients $\lambda_f(n)$ we may form the associated L-function

$$L(s,f) = \sum_{n\in\mathbb{Z}} \frac{\lambda_f(n)}{n^s}$$

Using property 1 for special choices of γ one can show that the "completed L-function" Λ defined by

$$\Lambda(s,f) = \pi^{-s}\Gamma\Big(\frac{s+i\nu}{2}\Big)\Gamma\Big(\frac{s-i\nu}{2}\Big)L(s,f)$$

is an entire function. In addition, this completed L-function satisifes the functional equation

$$\Lambda(s,f) = (-1)^{\epsilon}\Lambda(1-s,f) \tag{1.1}$$

where $\epsilon = \pm 1$ comes from the properties of the Maass form.

1.3.1 The Spectrum of the Laplacian

Sometimes the Maass forms coming from Eisenstein series are referred to as the "continuous part" of the spectrum, and the cusp forms as the "discrete part" of the spectrum. This is because

$$\Delta E(z,s) = s(1-s)E(z,s)$$

Thus for any complex number s, $E(z,s)$ has eigenvalue $s(1-s)$. Therefore we have a one-to-one correspondence between Eisenstein series and $\mathbb{C}$. On the other hand, it is not at all clear for which complex numbers ν does there exist a corresponding Maass *cusp* form with eigenvalue $\nu(1-\nu)$. Largely for this reason, Eisenstein series are well understood, however comparatively little is known about Maass cusp forms. What is known (see [Gol06] pgs. 70-73) is that the for any given ν the space of Maass cusp forms with Laplace eigenvalue $\nu(1-\nu)$ is finite dimensional. Moreover, there are no Maass forms of type ν with $\nu(1-\nu) < 3\pi^2/2$. It has been conjectured by Selberg ([Sel65]) that the correct lower bound should be 1/4.

The first condition of a Maass form f guarantees that f has a Fourier expansion of the form

$$f(z) = \sum_{n\in\mathbb{Z}} \lambda_f(n)e(nz); \qquad \lambda_f \in \mathbb{C}$$

The fourth condition (cuspidality) is equivalent to saying that $\lambda_f(0) = 0$. By solving the differential equation in condition 2, and applying the bounded growth condition 3, one can further show that any Maass cusp form has a Fourier expansion given by

$$f(z) = \sum_{n\in\mathbb{Z}} a_f(n)\sqrt{y}K_{ir}(2\pi|n|y)e(nx); \qquad a_f \in \mathbb{C}$$

where $1/4 + r^2 = \nu(1-\nu)$ and K_{ir} is the K-Bessel function of type ir. Therefore, understanding Maass cusp forms amounts to understanding their Fourier coefficients.

1.4 Maass Cusp Forms for SL(n,Z)

In addition to the classical Maass cusp forms for SL(2,$\mathbb{Z}$), Maass cusp forms also exist for SL(n,$\mathbb{Z}$) for any $n \geq 2$. We now outline the main properties of these higher dimensional Maass cusp forms.

In order to define Maass forms for SL(n,$\mathbb{Z}$), the first step is to define the analog of the upper half plane. In the classical case

$$\mathcal{H} = \{x + iy : x \in \mathbb{R}, y > 0\}$$

However the Iwasawa decomposition ([Gol06] pg. 8) for GL(2,$\mathbb{R}$) states that if $g \in GL(2, \mathbb{R})$ then

$$g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$$

where $y > 0$, $x, d \in \mathbb{R}$ with $d \neq 0$ and

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in O(2, \mathbb{R})$$

The decomposition is not uniquely determined, but x and y *are* uniquely determined. Thus there is a natural bijection

$$GL(2, \mathbb{R})/(O(2, \mathbb{R}) \cdot \mathbb{R}^\times) \leftrightarrow \mathcal{H}$$

given by $g \to z = x + iy \in \mathcal{H}$ where x and y are as described in the Iwasawa decomposition. The reverse map is defined by sending an element of the upper half plane to the corresponding coset.

It is clear that this correspondence can be extended in a natural way. Define

$$\mathcal{H}^n = GL(n,\mathbb{R})/(O(n,\mathbb{R})\cdot\mathbb{R}^\times)$$

for any integer $n \geq 2$. We call $\mathcal{H}^n$ the generalized upper-half plane. Explicitly, an element $z \in \mathcal{H}$ has a uniquely defined coset representative given by

$$z = x \cdot y$$

where

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix}$$

$$y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix}$$

with $x_{i,j} \in \mathbb{R}$ for $1 \leq i < j \leq n$ and $y_i > 0$ for $1 \leq i \leq n-1$.

Next one must define the analog of the eigenvalue. In the classical case there is a single eigenvalue because the center of the universal enveloping algebra of GL(2,$\mathbb{R}$) is a polynomial algebra over $\mathbb{C}$ of rank 1 (see [Bum98] Section 2.2). However this is no longer the case for GL(n,ℝ) (see [Gol06] pg. 46). Let $\nu = (\nu_1, \nu_2, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$, and define the generalized imaginary part function I_ν by

$$I_\nu(z) = \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{i,j}\nu_j}, \qquad z \in \mathcal{H}^n$$

where

$$b_{i,j} = \begin{cases} ij & \text{if } i + j \leq n \\ (n-i)(n-j) & \text{if } i + j \geq n \end{cases}$$

Let $\mathfrak{D}^n$ be the center of the universal enveloping algebra for GL(n,ℝ). It can be shown ([Gol06] pg. 51) that for every $D \in \mathfrak{D}^n$

$$DI_\nu(z) = \lambda_D \cdot I_\nu(z)$$

for some $\lambda_D \in \mathbb{C}$. That is, the generalized imaginary part function is an eigenfunction of $\mathfrak{D}^n$. We are now in a position to define Maass cusp forms for SL(n,ℤ).

Def *Let $n \geq 2$, $\nu = (\nu_1, \nu_2, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$. Then a Maass cusp form of type ν for SL(n,ℤ) is a smooth function $f : \mathcal{H}^n \to \mathbb{C}$ satisfying*

1) $f(\gamma z) = f(z)$ for all $\gamma \in SL(n, \mathbb{Z})$, $z \in \mathcal{H}^n$,

2) $Df(z) = \lambda_D f(z)$ for all $D \in \mathfrak{D}^n$ with λ_D as above,

3) $\int_{(SL(n,\mathbb{Z}) \cap U)\backslash U} f(z) = 0$ where U is the set of all block upper triangular matrices,

4) $|f(z)|$ is of at most polynomial growth as $|z| \to \infty$.

Just as in the classical case, Maass cusp forms for SL(n,ℤ) have Fourier expansions.

Theorem *For $n \geq 2$, let U_n denote the group of $n \times n$ upper triangular matrices with 1s on the diagonal. Let ϕ be a Maass form for SL(n,$\mathbb{Z}$). Then for all $z \in SL(n,\mathbb{Z})\backslash\mathcal{H}^n$*

$$\phi(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z})\backslash SL(n-1,\mathbb{Z})} \sum_{m_1 \neq 0} \sum_{m_2=1}^{\infty} \cdots \sum_{m_{n-1}=1}^{\infty} \tilde{\phi}_{(m_1,\ldots,m_{n-1})}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z\right)$$

([Gol06] pg. 118) where the sum is independent of the choice of the coset representative γ and

$$\tilde{\phi}_{(m_1,\ldots,m_{n-1})}(z) := \int_0^\infty \cdots \int_0^\infty \phi(u \cdot z) e^{-2\pi i(m_1 u_{1,2} + m_2 u_{2,3} + \cdots + m_{n-1} u_{n-1,n})} d^*u$$

with $u \in U_n(\mathbb{R})$ given by

$$u = \begin{pmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ & 1 & u_{2,3} & \cdots & u_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & u_{n-1,n} \\ & & & & 1 \end{pmatrix}$$

and $d^*u = \prod_{1 \leq i < j \leq n} du_{i,j}$.

The function $\tilde{\phi}$ is called a Whittaker function. In the special case $n = 2$ this can be explicitly computed and is a scaling of the K-Bessel function. In [Jac67] Jacquet gave an explicit construction of a Whittaker function, called the Jacquet-Whittaker function, for any $n \geq 2$. However, for computational purposes this function is still difficult to work with. As discussed in the subsection "Hejhal's Algorithm," the fact that we have limited computational understanding of the Jacquet-Whittaker function for $n > 2$ poses significant problems to adapting Hejhal's algorithm to higher dimensions. Therefore, to continue computational work to SL(3,$\mathbb{Z}$) and higher, a different approach was needed.

CHAPTER 2
COMPUTATION OF AUTOMORPHIC FORMS

As we have seen, explicit examples can be given of Maass forms, in particular Eisenstein series. However as explained above, Eisenstein series are not cuspidal. In fact, writing down examples of Maass cusp forms is extremely difficult, and very few are known.

There are indeed infinitely many Maass cusp forms (see [Hej76], or [Gol06] Ch. 4). Therefore there is great interest in finding examples of such forms in order to study them more concretely. Because of the interest in these forms and the difficulty in explicitly writing down examples, one may choose to instead numerically approximate Maass cusp forms. As we explained in the previous chapter, this amounts to finding a Laplace eigenvalue for which a form exists, and then finding the form's Fourier coefficients.

In the sections that follow we will discuss major work that has been done in this area. All of these techniques leverage the Fourier expansion of Maass cusp form and the functional equation. These properties allow one to "work backwards" from what is expected of a cusp form to locate legitimate forms.

2.1 Hejhal's Algorithm

In 1999 David Hejhal in [Hej99] presented an algorithm to find eigenvalues for which Maass cusp forms exist, and then to compute the first several hundred Fourier coefficients of the corresponding form. His approach uses automorphy, and the fact that the asymptotics of the K-Bessel function are well understood. In particular the K-Bessel function satisfies the asymptotic

$$K_\alpha(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) \tag{2.1}$$

as $|z| \to \infty$ and $|arg(z)| < \frac{3\pi}{2}$ ([GIJZ07] pg. 920). In addition, [Kim03] showed that the Fourier coefficients of Maass cusp forms satisfy the asymptotic

$$a(n) = \mathcal{O}\left(n^{7/64+\epsilon}\right) \tag{2.2}$$

Thus the rapid decay of the Bessel function dominates any growth from the Fourier coefficients for large n. Therefore if one wishes to compute a Maass cusp form to high accuracy, it is sufficient to restrict the Fourier expansion to $|n| < N$ for some large positive integer N.

In addition, Hejhal used the automorphy condition to form a system of linear equations. In particular he chose various "test points" $z_i \in \mathcal{H}$, and wrote $z_i^* = \gamma z_i$ where $\gamma \in SL(2, \mathbb{Z})$ is a carefully chosen element in the modular group. For each z_i we can explicitly write

$$f(z_i) = \sum_{|n|<N} a_f(n) \sqrt{y_i} K_{ir}(2\pi |n| y_i) e(n x_i)$$

Then by the automorphy condition, any Maass form will agree at z_i and z_i^*. By choosing many test points he formed a system of linear equations (with variables being the Fourier coefficients of a Maass cusp form).

$$\begin{aligned}
\sum_{|n|<N} a_f(n)\sqrt{y_i}K_{ir}(2\pi|n|y_1)e(nx_1) &= \sum_{|n|<N} a_f(n)\sqrt{y_1^*}K_{ir}(2\pi|n|y_1^*)e(nx_1^*) \\
&\vdots \\
\sum_{|n|<N} a_f(n)\sqrt{y_M}K_{ir}(2\pi|n|y_M)e(nx_M) &= \sum_{|n|<N} a_f(n)\sqrt{y_M^*}K_{ir}(2\pi|n|y_M^*)e(nx_M^*)
\end{aligned}$$

He then solved this system by substituting the asymptotic expansion of the K-Bessel function.

Note that this assumes that one has singled out a legitimate eigenvalue $\frac{1}{4}+r^2$ for which such a cusp form exists. Up to this point we have stressed that computing automorphic forms is equivalent to computing their Fourier coefficients. However the primary difficulty one encounters is finding eigenvalues for which automorphic forms exist. Once such an eigenvalue has been located then solving for the associated form(s) is generally very straightforward. This is because solving linear equations is easy theoretically and computationally. Thus the central issue is in fact locating legitimate eigenvalues. Therefore we will take a moment to cover in detail Hejahl's approach.

In order to find eigenvalues for which automorphic forms exist, Hejhal varied his choice of test points z_i. In particular he choses the 2Q points z_j where

$$z_j = \frac{1}{2Q}\left(j - \frac{1}{2}\right) + iY, \quad 1 - Q \le j \le Q$$

for two values of Y, say Y_1 and Y_2.

Next for each Y_i he solves the above system. This gives two sets of Fourier coefficients, say $\{a_n^{'}\}$ and $\{a_n^{''}\}$ coming from Y_1 and Y_2 respectively. However, the

Fourier coefficients of a legitimate Maass form should not depend on the choice of Y because they are only a function of n. Therefore if the chosen "eigenvalue" $\frac{1}{4}+r^2$ corresponds to a legitimate eigenvalue, then $a_n^{'} \approx a_n^{''}$ for all n. However if an r is chosen for which no Maass form with eigenvalue $\frac{1}{4}+r^2$ exists, then there is no reason to believe that the solutions $\{a_n^{'}\}$ and $\{a_n^{''}\}$ will agree for different values of Y.

In their paper "Effective Computation of Maass Cusp Forms", Booker, Strombergsson, and Venkatesh [BSV06] took Hejhal's algorithm and adapted it slightly to work well with modern computers. Using a 1.5GHZ PC with computation time of between one and three weeks, they computed the first ten eigenvalues on PSL(2,$\mathbb{Z}$)$\backslash\mathcal{H}$ to a precision of more than 1000 decimal places, and the associated first 455 Fourier coefficients to at 900 decimal places. They also used a slight modification of his algorithm to compute the first eigenvalues on $\Gamma_0(5)\setminus\mathcal{H}$ and $\Gamma_0(6)\setminus\mathcal{H}$. Their results show the power of Hejhal's algorithm in obtaining numerical data about Maass forms to high precision. On a network of today's computers one could expect greater results in significantly less time.

These results show that Hejhal's algorithm is a good choice for finding SL(2,$\mathbb{Z}$) Maass forms. However Maass forms also exist for higher dimension groups SL(n,$\mathbb{Z}$). Before discussing the techniques being used to find these forms we give a brief overview of the properties of these higher dimensional forms.

2.2 Bian's SL(3,$\mathbb{Z}$) Computations

One may wonder if Hejhal's algorithm could be used to compute Maass cusp forms for higher dimensions $n \geq 3$. Recall that Hejhal's algorithm relies on a thorough understanding of the K-Bessel function, including its asymptotics. In the case of Maass cusp forms for higher dimensions (i.e. cusp forms for SL(n,$\mathbb{Z}$), the functions which play the role of the K-Bessel function when $n = 2$ are significantly more complicated, and are not well understood. Therefore Hejhal's algorithm does not scale to SL(3,$\mathbb{Z}$) because the Whittaker function for SL(3,$\mathbb{Z}$) is not easily computed, so another approach is needed.

In 2010 Ce Bian [Bia10] provided the first numerical evidence of Maass cusp forms for SL(3,$\mathbb{Z}$) not arising from a Gelbart-Jacquet lift (see [GJ78]). That is, Maass cusp forms that are not constructed using a known form from SL(2,$\mathbb{Z}$). In this subsection we outline his approach.

In the case of $n = 3$ one can simplify the Fourier expansion given in the previous expression. Although the Whittaker function still cannot be expressed in terms of a well-understood function, one can rewrite the Fourier expansion to better highlight the Fourier coefficients.

SL(3) Fourier Expansion *Suppose ϕ is a Maass cusp form of type $\nu = (\nu_1, \nu_2)$ for SL(3,$\mathbb{Z}$), as described in the previous subsection. Then the Fourier expansion of*

ϕ can be written as

$$\phi(z) = \sum_{\gamma\in\Gamma^2_\infty\backslash\Gamma^2}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{A(m_1,m_2)}{m_1m_2}W_{\nu_1,\nu_2}\left(\begin{pmatrix} m_1m_2 & & \\ & m_1 & \\ & & 1\end{pmatrix}\gamma z\right) \tag{2.3}$$

where Γ^2 is all matrices of the form

$$\begin{pmatrix} a & b & \\ c & d & \\ & & 1\end{pmatrix}$$

with integer entries and determinant ± 1. The set $\Gamma^2_\infty \subset \Gamma^2$ is the set

$$\begin{pmatrix} 1 & b & \\ & 1 & \\ & & 1\end{pmatrix}$$

Here W_{ν_1,ν_2} is the Jacquet Whittaker function

$$W_{\nu_1,\nu_2}(z) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} I_{\nu_1,\nu_2}(w\theta z)\bar{\psi}_m(\theta)d\theta_{1,2}d\theta_{1,3}d\theta_{2,3}$$

In addition, the L-function for ϕ and any twisted L-function has an Euler product. If χ is a Dirichlet character of conductor q, then

$$L(s,\phi\times\chi) = \prod_{p\ prime}\frac{1}{1-A(1,p)\chi(p)p^{-s}+A(p,1)\chi^2(p)p^{-2s}-\chi^3(p)p^{-3s}} \tag{2.4}$$

and satisfies the functional equation

$$q^{(3/4)s}\Lambda(s,\phi\times\chi) = \epsilon_\chi^3 q^{(3/2)(1-s)}\Lambda(1-s,\tilde{\phi}\times\bar{\chi}) \tag{2.5}$$

where

$$\Lambda(s, \phi \times \chi) = L(s, \phi \times \chi)\Gamma_{\mathbb{R}}(s + k + i\nu_1)\Gamma_{\mathbb{R}}(s + k + i\nu_2)\Gamma_{\mathbb{R}}(s + k + i\nu_3)$$
$$\Lambda(s, \tilde{\phi} \times \overline{\chi}) = L(s, \tilde{\phi} \times \overline{\chi})\Gamma_{\mathbb{R}}(s + k + i\nu_1)\Gamma_{\mathbb{R}}(s + k + i\nu_2)\Gamma_{\mathbb{R}}(s + k + i\nu_3)$$

and

$$\Gamma_{\mathbb{R}}(z) = \pi^{-z/2}\Gamma(z/2)$$

The key ingredient of Bian's approach was the converse theorem of Jacquet, Piatetski-Shapiro, and Shalika [JPSS79]. Their result essentially states that one can test if a function Π is an automorphic form for SL(n,$\mathbb{Z}$) by looking at the properties of L-function L(s,$\Pi \times \pi$) for various known automorphic forms π for SL(n-2,$\mathbb{Z}$). In the case of $n = 3$ this reduces to twisting Π by Dirichlet characters. This simpler SL(3,$\mathbb{Z}$) converse theorem is as follows:

SL(3) Converse Theorem *Suppose ϕ is a Fourier series as in (2.3), such that the associated L-function L(s, $\phi \times \chi$) continues to an entire function of finite order, has an Euler product of the form (2.4), and satisfies the functional equation (2.5), for all primitive characters. Then ϕ is a cusp form for SL(3,$\mathbb{Z}$).*

Bian uses the converse theorem by starting with complex variables $\{a_n\}$ such that the associated L-function with $A(1, n) := a_n$ is assumed to have the properties described in the converse theorem. He then leverages these properties to form a system of equations which one can use to solve for the variables $\{a_n\}$. We outline this process below.

Given the eigenvalues $\nu = (\nu_1, \nu_2)$ one can write

$$\nu_1 = \frac{1}{3} + i\frac{u}{3}, \qquad \nu_2 = \frac{1}{3} + i\frac{v}{3}$$

Then define

$$g_{u,v}(s) := \Gamma_{\mathbb{R}}\left(s - i\frac{2u+v}{3}\right)\Gamma_{\mathbb{R}}\left(s + i\frac{u-v}{3}\right)\Gamma_{\mathbb{R}}\left(s + i\frac{u+2v}{3}\right)$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. One should note that $g_{u,v}(s)$ is precisely the gamma factors appearing in the completed L-function $\Lambda(s, \phi)$.

Bian then defines a family of complex-valued functions $F_{\phi\times\chi}$ where χ is a Dirichlet character of conductor q such that

$$k := \begin{cases} 1 & \text{if } \chi(-1) = -1 \\ 0 & \text{otherwise} \end{cases}$$

given by

$$F_{\phi}(x) := \frac{e^{(\pi/4)(u+v)}}{2\pi i}\int_{\Re(s)=c>0} g_{u,v}(s)x^{-s}ds$$

and

$$F_{\phi\times\chi}(x) := \frac{e^{(\pi/4)(u+v)}}{2\pi i}\int_{\Re(s)=c} g_{u,v}(s+k)x^{-s}ds$$

Using the fact that F_{ϕ} is the inverse Mellin transform of the gamma factors of Λ one can show that the inverse Mellin transform of Λ at a real variable x is given by

$$\frac{e^{(\pi/4)(u+v)}}{2\pi i}\int_{\Re(s)=c>0} \Lambda(s, \phi\times\chi)x^{s-1/2}ds = x^{-1/2}\sum_{n=1}^{\infty} a_n\chi(n)F_{\phi\times\chi}\left(\frac{n}{x}\right)$$

where χ is a Dirichlet character of our choosing (including the trivial character).

Define

$$S(x, \phi \times \chi) := x^{-1/2} \sum_{n=1}^{\infty} a_n \chi(n) F_{\phi\times\chi}\left(\frac{n}{x}\right)$$

Using the functional equation and the properties of the Mellin transform, Bian shows the functional equation

$$S(x, \phi \times \chi) = \epsilon_\chi^3 \overline{S(q^3/x, \phi \times \chi)} \tag{2.6}$$

As x and χ vary this gives a system of linear equations in the complex variables $\{a_n\}$. Due to the decay of $F_{\phi\times\chi}$ the infinite sums can in fact be truncated to allow arbitrary precision.

Testing for $u, v \in [10, 20]$ they found what appeared to be a Maass cusp form at $u \approx 18.9024126, v \approx 11.7612471$. If this were a Maass form coming from a Gelbart-Jacquet lift, then it would follow that $u = v$. Thus this form does not arise in that manner, and so is quite interesting.

One may wonder if the techniques that Bian uses to search for forms can be applied to Maass forms for higher dimensions. After all, the converse theorem applies to SL(n,$\mathbb{Z}$) for all $n \geq 2$. However, the converse theorem says that we must understand the twisted L-functions, where the object one twists by are forms of dimension $n - 2$. Thus in the case $n = 4$ one is twisting by SL(2,$\mathbb{Z}$) automorphic forms. This is significantly more complicated and presents serious computational and theoretical obstacles. Thus it is not expected that these techniques will directly apply to higher dimensions.

2.3 SL(3) and SL(4) L-functions from a Functional Equation

In their 2012 paper "Maass Forms on GL(3) and GL(4)" Farmer, Koutsoliotas, and Lemurell [FKL14] introduced a new approach to find all L-functions satisfying a given functional equation. In their approach they also solve a system of equations for the Fourier coefficients, however they use significantly less equations at the cost of having a non-linear system.

A key ingredient of their work is using the approximate functional equation. Suppose that $\Lambda(s)$ is the completed functional equation for some L-function $L(s)$. Next, one introduces an entire function $g : \mathbb{C} \to \mathbb{C}$ such that for a fixed s, one has $|\Lambda(z+s)g(z+s)z^{-1}| \to 0$ as $|Im(z)| \to \infty$ in vertical strips. Then by the Mellin inversion formula and by shifting contours, one gets an equation relating $\Lambda(s)g(s)$ to integral transforms of g. The value in this is two-fold: first, the spectral parameters appear on the right hand side, allowing one to manipulate them directly. Second, after some manipulations and use of the Hardy Z-function (see [THB86] Ch. 15), one can arrive at an equation that the left hand side is independent of the choice of test-function $g(s)$. This allows one to choose various test functions to test for consistency of the system, as in Hejhal's algorithm. Finally, in order to reduce the number of "false positives," the authors use the multiplicativity relations arising from the Euler product. It is interesting to note that they seem to be the first to actually solve for Fourier coefficients of forms using nonlinear equations. While Bian used the multiplicativity relations to test the validity of his results, he did not involve them in the initial computations.

Using these techniques, the authors found more than 2000 spectral parameters for SL(3,$\mathbb{Z}$) Maass forms, and more than 200 spectral parameters for SL(4,$\mathbb{Z}$). Recall from previous discussions that the primary difficulty in computing Maass forms of any dimension is in first locating their spectral parameters. Once these have been located, computing the Fourier coefficients is generally just a matter of solving a system of linear and nonlinear equations. Thus their techniques were successful, and allowed insight into a family of forms not seen before.

2.4 Difficulties of Above Approaches

We have seen that by manipulating the known properties of an L-function of an automorphic form one can locate previously unknown forms. In the case of Hejhal's algorithm this involved knowing precisely what form the Fourier expansion of an SL(2,$\mathbb{Z}$) Maass form takes. In Bian's case this involved using the converse theorem to twist by GL(1,$\mathbb{Z}$) automorphic forms, or more explicitly by Dirichlet characters. Neither of these techniques extend naturally to higher dimensions. In the case of Hejhal's algorithm, one simply does not know a nice form for the Fourier series of automorphic forms of dimension 3 or higher. In Bian's case the converse theorem requires one to understand the properties of your form twisted by forms of dimension two smaller. Thus for SL(4,$\mathbb{Z}$) one would need to twist by SL(2,$\mathbb{Z}$) automorphic forms. Due to their complexity this approach seems unfeasible.

The approach of Farmer and Lemurell holds the most promise for extending to higher dimensions. As one moves to higher dimensions the computational complexity necessarily increases. However in their approach the approximate functional equation, which is the primary object they deal with, is not drastically altered. Unfortunately the authors do not discuss the computational setup or difficulty of their approach, though one can assume this is the primary inhibitor in finding forms of arbitrarily high dimensions.

2.5 Our Goal

Our goal is to work in the reverse direction of the above approaches. That is, given only limited information about a Maass cusp form f (in particular a finite list of high Fourier coefficients of a primitive Maass cusp form), we will determine its level and estimate its spectral parameter, and thus its Laplace eigenvalue.

Since our approach allows one to ascertain properties of a given Maass form, one may wonder where Maass forms show up in the larger theory. A Maass form can be lifted to an automorphic cuspidal representation $\pi = \otimes_{\nu \leq \infty} \pi_\nu$ of GL(2) over the adelic ring $\mathbb{A}_\mathbb{Q}$ of $\mathbb{Q}$ (see [Bum98] Section 3.2). Our analysis and algorithms show that the non-Archimedean local representations π_p, $p < \infty$, or a finite list of them, can be used to uniquely determine the Archimedean local representation π_∞ and the global conductor. This can be regarded as a new type of strong multiplicity one theorem. The Langlands program (see [KKDS$^+$13]) predicts that all L-functions can be expressed as products of automorphic L-functions for cuspidal representations of GL(n,$\mathbb{A}_\mathbb{Q}$). Our results offer a possible new approach to this conjecture when an otherwise defined L-function is only known to match finitely many local components and L-factors of an automorphic L-function.

The fact that resonance and rapid decay of sums of Fourier coefficients of f can be used to determine the level D and Laplace eigenvalue $1/4 + r^2$ supports the belief that these resonance and rapid decay properties can be used to characterize the underlying Maass form. This valuable insight allows us to understand more about the oscillatory nature of Maass forms.

CHAPTER 3
PROOF OF RESULTS

Resonance is a general phenomenon relating to automorphic forms, either holomorphic or Maass. Given a form of dimension n with Fourier coefficients $A(m_1, \ldots, m_{n-1})$ one can form the sum

$$S_X(f) = \sum_{m \geq 1} A(m, 1, \ldots, 1) e(\alpha m^\beta) \phi\left(\frac{m}{X}\right)$$

where $\alpha \neq 0$ and $\beta > 0$ are real parameters and $\phi \in C_c^\infty([1,2])$. In the case of SL(2,$\mathbb{Z}$) this simplifies to

$$S_X(f) = \sum_{n \geq 1} \lambda_f(n) \phi\Big(\frac{n}{X}\Big) e\big(\alpha n^\beta\big) \tag{3.1}$$

By varying the choice of parameters one can get different estimates for the sum.

The simplest case is for f a holomorphic cusp form and ϕ is replaced with an indicator function. In [RY10] Ren and Ye showed the following:

Theorem [Ren-Ye]: *Let f be a holomorphic form for SL(2,$\mathbb{Z}$) with Fourier coefficients $\lambda_f(n)$, $0 < \beta < 1$ and $\alpha \neq 0$. Then*

1. *For $\beta \neq 1/2$ and $\alpha \neq 0$, we have*

$$\sum_{X < n \leq 2X} \lambda_f(n) e(\alpha n^\beta) \ll |2\beta - 1|^{-1/2} (|\alpha| \beta X^\beta)^{1+\epsilon} + X^{1/2+\epsilon} (|\alpha| \beta X^\beta)^{-1/4}$$

2. *If $\alpha = \pm 2\sqrt{q}$ with any integer $1 \leq q < X/4$, then for some absolute constant $\overline{c_0}$*

$$\sum_{X < n \leq 2X} \lambda_f(n) e(\pm 2\sqrt{qn}) = \overline{c_0} \lambda_f(q) q^{-1/4} X^{3/4} + \mathcal{O}((qX)^{1/4+\epsilon} + X^{3/8+\epsilon} q^{-1/8})$$

In particular we see that the particular choice $\beta = 1/2$ gives a precise main term, whereas choosing $\beta \neq 1/2$ only allows one an asymptotic bound. We say then that an SL(2,$\mathbb{Z}$) holomorphic cusp form has resonance at $\beta = 1/2$. More generally, Ren and Ye proved in ([RY14]) that a Maass cusp form for SL(m,$\mathbb{Z}$) has resonance at $\beta = 1/m$.

Our idea is to search for automorphic forms by understanding the properties of their resonance sums. This has the advantage that it allows one to form many linear equations simply by varying the parameters α and β. In addition, resonance results can theoretically be computed for high dimensions, and do not require twisting by forms of lower dimension.

For this to work we need a version of the above results for Maass forms, and also with sharper error terms. That is our primary result, and is given and proved below.

3.1 Statement of Results

Theorem 1: *Let f be a fixed primitive Maass cusp form with Laplace eigenvalue $1/4 + r^2$ for a Hecke congruence subgroup $\Gamma_0(D)$ of $SL(2,\mathbb{Z})$, and $\lambda_f(n)$ its n-th Fourier coefficient. Let $\phi \in C_c^\infty((1,2))$ be a Schwartz function and $\alpha \in \mathbb{R}$, $\beta > 0$, $X > 0$ be real numbers such that for some $\epsilon > 0$ we have $r^4 D \ll X^{1-\epsilon}$. Then for any* *positive integer* N*,*

$$\begin{aligned}\sum_{n\geq 1}\lambda_f(n)e(\alpha n^\beta)\phi\Big(\frac{n}{X}\Big) &= \frac{i-1}{D\lambda_f(D)}\sum_{n<4b^*}\lambda_{f_D}(n)\sum_{k=0}^{N-1}C_{r,k}X^{3/4-k}\\ &\times \Big(\frac{n}{D}\Big)^{-1/4-k}P^+_{\alpha,\beta,X}\left(-sgn(\alpha)2\sqrt{\frac{nX}{D}},k\right)\\ &- \frac{1+i}{4\pi D\lambda_f(D)}\sum_{n<4b^*}\lambda_{f_D}(n)\sum_{k=0}^{N-1}C_{r,k}d_{r,k}X^{1/4-k}\\ &\times \Big(\frac{n}{D}\Big)^{-3/4-k}P^-_{\alpha,\beta,X}\left(-sgn(\alpha)2\sqrt{\frac{nX}{D}},k\right)\\ &+ \mathcal{E}(X)\end{aligned}$$

where

$$\begin{aligned}b^* &:= (|\alpha|\beta)^2X^{2\beta-1}D\min\{1,2^{1-2\beta}\}\\ C_{r,k} &:= \frac{(-1)^k\Gamma(2ir+2k+1/2)}{(2k)!\Gamma(2ir-2k+1/2)}(8\pi)^{-2k}\\ P^\pm_{\alpha,\beta,X}(w,k) &:= \int_1^{\sqrt{2}}t^{\pm 1/2-2k}\phi(t^2)e(\alpha X^\beta t^{2\beta}+wt)dt\\ d_{r,k} &:= -\frac{4r^2+(2k+1/2)^2}{2(2k+1)}\\ \lambda_{f_D}(n) &:= \begin{cases}\lambda_f(n) & if\ (n,D)=1\\ \overline{\lambda_f(n)} & if\ (n,D)>1\end{cases}\\ \mathcal{E}(X) &= \mathcal{O}_{\beta,N,\phi}\left((1+r^3N)\Big(\frac{X}{D}\Big)^{\frac{3}{4}-N}\right)\end{aligned}$$

Set $Q(w)=\min_{t\in[1,\sqrt{2}]}|2\alpha\beta X^\beta t^{2\beta-1}+w|$*. If* $Q(w)\neq 0$ *then*

$$P^\pm_{\alpha,\beta,X}(w,k)=\mathcal{O}_{\beta,k,\phi}\left(\frac{\alpha X^\beta}{Q(w)^3}+\frac{1}{Q(w)^2}\right).$$

In particular, by trivial estimation $P^{\pm}_{\alpha,\beta,X}(w,k) \ll 1$.

Corollaries 1 and 2 simplify Theorem 1 by preserving only the largest terms. Corollary 1 gives conditions for rapid decay, while Corollary 2 gives conditions for a main term. Comparing Corollary 2 to the result of Sun and Wu [SW14] one can see that our result shows same the main term of size $X^{3/4}$, however our corollary also shows the role that the level D plays.

Corollary 1 *With notations as in Theorem 1, if*

$$\alpha\beta X^{\beta}\min\{1, 2^{\frac{1}{2}-\beta}\} < \frac{1}{2}\sqrt{\frac{X}{D}}$$

then

$$\sum_{n\geq 1}\lambda_f(n)\phi\left(\frac{n}{X}\right)e(\alpha n^{\beta}) \ll X^{-M}$$

for all $M>0$. *The implied constants may depend on* α, β, r, D, M *and* ϕ, *but not on* X.

Corollary 2 *With notations as in Theorem 1, let* $q < X/D$ *be a positive integer and set* $\alpha = 2\sqrt{q/D}$ *and* $\beta = 1/2$. *Then*

$$\begin{aligned}\sum_{n\geq 1}\lambda_f(n)\phi\left(\frac{n}{X}\right)e\left(2\sqrt{\frac{qn}{D}}\right) &= \frac{c_q^{+}}{q^{\frac{1}{4}}\lambda_f(D)}\left(\frac{X}{D}\right)^{\frac{3}{4}}\lambda_{f_D}(q) \\ &+ \frac{c_q^{-}d_{r,0}}{q^{\frac{3}{4}}\lambda_f(D)}\left(\frac{X}{D}\right)^{\frac{1}{4}}\lambda_{f_D}(q) \\ &+ \mathcal{E}'(X)\end{aligned}$$

where $c_q^{\pm}$ *are nonzero constants depending on* q *given in (3.17) and*

$$\mathcal{E}'(X) \ll_{\phi} \left(\frac{X}{D}\right)^{-\frac{1}{4}} (1+r^3)$$

is given in (3.15).

In Corollary 3 we see that by carefully keeping track of all constants one can use the resonance properties of f to solve for the spectral parameter r. Doing so requires knowing the level D of f, which is handled in Corollary 4.

Corollary 3 *Let* f *be as in Theorem 1, and recall that* f *has Laplace eigenvalue* $\frac{1}{4}+r^2$. *Then*

$$\begin{aligned} r &= \left| \frac{\lambda_f(D)}{2c^-}\left(\frac{X}{D}\right)^{-1/4}\left(\sum_{X\le n\le 2X} \lambda_f(n)\phi\left(\frac{n}{X}\right)e\left(2\sqrt{\frac{n}{D}}\right) - \frac{c^+}{\lambda_f(D)}\left(\frac{X}{D}\right)^{3/4}\right) - \frac{1}{16}\right|^{\frac{1}{2}} \\ &+ \mathcal{O}_{N,\phi}\left(\lambda_f(D)^{\frac{1}{2}}\left(\frac{X}{D}\right)^{-\frac{1}{4}}\right) \end{aligned}$$

where

$$c^+ = (i-1)\int_1^{\sqrt{2}} t^{\frac{1}{2}}\phi(t^2)dt, \qquad c^- = -\frac{i+1}{4\pi}\int_1^{\sqrt{2}} t^{-\frac{1}{2}}\phi(t^2)dt$$

In Corollary 4 the parameter c plays the role of a guess at the level D. Indeed, if $c = D$ then α_ϵ will satisfy the rapid decay conditions of Corollary 1, and α_q will satisfy the resonance conditions on Corollary 2. Thus Corollary 4 shows that if the c behaves sufficiently like the level D, then in fact the two are close. Numerical examples demonstrating the ideas in Corollaries 3 and 4 are given in Section 4.

Corollary 4 *With notation as in Theorem 1, let* X, $q \in \mathbb{Z}_{>0}$, $c < X$ *and* $0 < \epsilon < 1$. *Define*

$$\alpha_\epsilon(c) = \frac{\epsilon}{\sqrt{c}}, \qquad \alpha_q(c) = 2\sqrt{\frac{q}{c}}.$$

If

$$\sum_{n \geq 1} \lambda_f(n) \phi\left(\frac{n}{X}\right) e\big(\alpha_\epsilon(c)\sqrt{n}\big) \ll X^{-M}$$

for all $M > 0$ *as* $X \to \infty$, *and*

$$\sum_{n \geq 1} \lambda_f(n) \phi\left(\frac{n}{X}\right) e\big(\alpha_q(c)\sqrt{n}\big) = \mathcal{O}\left(X^\delta\right)$$

for some $\delta > 0$ *as* $X \to \infty$, *then for* $X \gg 1$

$$\frac{c}{\left(\sqrt{q} + \sqrt{\frac{c}{4X}}\right)^2} < D < \frac{c}{\epsilon^2}.$$

Since D is an integer, if one can choose ϵ and q to make this range small enough that it only contains a single integer, then one has solved for D. Note that as $X \to \infty$ the range for D becomes $c/q \leq D \leq c/\epsilon^2$. Thus unless some computational reason prohibits it, choosing $q = 1$ is optimal.

3.2 Proof of Theorem 1

Let f be a primitive Maass cusp form for $\Gamma_0(D)$ with Laplace eigenvalue $1/4 + r^2$. Then f has Fourier expansion (see [Bum98] Section 1.9)

$$f(z) = \sum_{n \neq 0} \lambda_f(n) \sqrt{y} K_{ir}(2\pi |n| y) e(nx).$$

If $\Phi \in C^\infty(\mathbb{R}_{>0})$ vanishes in a neighborhood of zero and is rapidly decreasing, then we have the Voronoi summation formula (Kowalski-Michel-VanderKam [KMV02])

$$\sum_{n\geq 1}\lambda_f(n)\Phi(n) = \frac{1}{D\lambda_f(D)}\sum_{n\geq 1}\lambda_{f_D}(n)\int_0^\infty \Phi(x)J_f\left(4\pi\sqrt{\frac{nx}{D}}\right)dx + \frac{\epsilon_f}{D\lambda_f(D)}\sum_{n\geq 1}\lambda_{f_D}(n)\int_0^\infty \Phi(x)K_f\left(4\pi\sqrt{\frac{nx}{D}}\right)dx, \tag{3.2}$$

where

$$J_f(z) := \frac{-\pi}{\sin(\pi i r)}\Big(J_{2ir}(z) - J_{-2ir}(z)\Big), \qquad K_f(z) := 4\epsilon_f\cosh(\pi r)K_{2ir}(z),$$

$$\lambda_{f_D}(n) = \begin{cases} \lambda_f(n) & \text{if } (n,D) = 1; \\ \overline{\lambda_f(n)} & \text{if } (n,D) > 1. \end{cases}$$

Here $J_{\pm 2ir}$ is the Bessel function of the first kind, and K_{2ir} is the modified Bessel function of rapid decay (see [Wat95] p. 181), and $\epsilon_f = \pm 1$ depending on whether f is an even or odd Maass form respectively. In our case we set

$$\Phi(n) = \phi\left(\frac{n}{X}\right)e(\alpha n^\beta).$$

Asymptotics for $J_v(z)$ and $K_v(z)$ are given (see [GIJZ07] p. 920) by

$$K_v(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\left(1 + \mathcal{O}\left(\frac{\nu^2 - \frac{1}{4}}{z}\right)\right)$$

for any $|nu| < |z|$ and

$$J_{\pm\nu}(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right)\left[\sum_{k=0}^{N-1}\frac{(-1)^k\Gamma(\nu+2k+1/2)}{(2z)^{2k}(2k)!\Gamma(\nu-2k+1/2)} + R_1\right] - \sqrt{\frac{2}{\pi z}}\sin\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right)\left[\sum_{k=0}^{N-1}\frac{(-1)^k\Gamma(\nu+2k+3/2)}{(2z)^{2k+1}(2k+1)!\Gamma(\nu-2k-1/2)} + R_2\right]$$

where

$$\begin{aligned} |R_1(N)| &< \left|\frac{\Gamma(\nu+2N+1/2)}{(2z)^{2N}(2N)!\Gamma(\nu-2N+1/2)}\right| \\ |R_2(N)| &< \left|\frac{\Gamma(\nu+2N+3/2)}{(2z)^{2N+1}(2N+1)!\Gamma(\nu-2N-1/2)}\right| \end{aligned}$$

for any $\nu \in \mathbb{C}$ and $|z| \gg 1$. After rearranging we have

$$\begin{aligned} J_f(z) &= e^{iz}\frac{(i+1)}{\sqrt{\pi z}}\sum_{k=0}^{N-1}\frac{C_{r,k}}{z^{2k}}\left(1-\frac{d_{r,k}}{iz}\right) - e^{-iz}\frac{(i-1)}{\sqrt{\pi z}}\sum_{k=0}^{N-1}\frac{C_{r,k}}{z^{2k}}\left(1+\frac{d_{r,k}}{iz}\right) \\ &+ \mathcal{O}\left(\frac{E_{r,N}}{z^{2N+1/2}}\right) \end{aligned} \tag{3.3}$$

where the implied constant is absolute and

$$\begin{aligned} d_{r,k} &= -\frac{4r^2+(2k+1/2)^2}{2(2k+1)} = \mathcal{O}(r^2k), \\ C_{r,k} &= \frac{(-1)^k\Gamma(2ir+2k+1/2)}{(8\pi)^{2k}(2k)!\Gamma(2ir-2k+1/2)} = \mathcal{O}\left(\frac{r^{4k}}{(8\pi)^{2k}(2k)!}\right), \\ E_{r,N} &= \frac{1}{(2N)!}\prod_{\ell=1}^{4N}(2ir+\frac{1}{2}-\ell) = \mathcal{O}\left(\frac{r^{4N}}{(2N)!}\right). \end{aligned} \tag{3.4}$$

We first apply the asymptotics of K_{2ir} to $K_f := 4\epsilon_f\cosh(2\pi r)K_{2ir}$ appearing in (3.2) to arrive at

$$\begin{aligned} \int_0^\infty \Phi(x)K_f\left(4\pi\sqrt{\frac{nx}{D}}\right)dx &\ll \cosh(\pi r)\left(\frac{D}{16\pi^2 n}\right)^{\frac14}\int_0^\infty x^{-\frac14}e^{-4\pi\sqrt{nx/D}}\Phi(x) \\ &\times \left\{1+\mathcal{O}\left(\left(4r^2+\frac14\right)\left(\frac{D}{nx}\right)^{\frac12}\right)\right\}dx \\ &\ll_\Phi \cosh(\pi r)\left(\frac{D}{n}\right)^{\frac14}\left\{1+\left(4r^2+\frac14\right)\left(\frac{D}{nX}\right)^{\frac12}\right\} \\ &\times \int_X^{2X} x^{-\frac14}e^{-4\pi\sqrt{nx/D}}dx \\ &\ll X^{\frac34}\left(\frac{D}{n}\right)^{\frac14}e^{-4\pi\sqrt{nX/D}}\cosh(\pi r) \\ &\times \left\{1+\left(4r^2+\frac14\right)\left(\frac{D}{nX}\right)^{\frac12}\right\}. \end{aligned}$$

Using the known bound $\lambda_f(n) \ll n^\theta$ for $\theta = \frac{7}{64} + \epsilon$ (see [Kim03]) we see that

$$\begin{aligned}
E_1(X) \quad &:= \quad \frac{1}{D\lambda_f(D)} \sum_{n\geq 1} \lambda_{f_D}(n) \int_0^\infty \Phi(x) K_f\left(4\pi\sqrt{\frac{nx}{D}}\right) dx \qquad (3.5)\\
&\ll_\Phi \quad \frac{1}{\lambda_f(D)}\left(\frac{X}{D}\right)^{\frac{3}{4}} \sum_{n\geq 1} n^{\theta-\frac{1}{4}} e^{-4\pi\sqrt{nX/D}} \left(1+\left(4r^2+\frac{1}{4}\right)\left(\frac{D}{nX}\right)^{\frac{1}{2}}\right)\\
&\ll \quad \frac{1}{\lambda_f(D)}\left(\frac{X}{D}\right)^{\frac{3}{4}} e^{-4\pi\sqrt{X/D}} \left(1+r^2\left(\frac{D}{X}\right)^{\frac{1}{2}}\right)\\
&\ll \quad \frac{1}{\lambda_f(D)} e^{-4\pi\sqrt{X/D}} \left(\frac{X}{D}\right)^{\frac{3}{4}},
\end{aligned}$$

where the last inequality comes from applying the assumed bound $r^4/D \ll X^{1-\epsilon}$.

Next we use the asymptotics for J_f from (3.3). To simplify the presentation we write

$$\begin{aligned}
&\frac{1}{D\lambda_f(D)} \sum_{n\geq 1} \lambda_{f_D}(n) \int_0^\infty \Phi(x) J_f\left(4\pi\sqrt{\frac{nx}{D}}\right) dx\\
= \quad &\frac{1}{D\lambda_f(D)} \sum_{n\geq 1} \lambda_{f_D}(n) G_N^-(n) + \frac{1}{D\lambda_f(D)} \sum_{n\geq 1} \lambda_{f_D}(n) G_N^+(n) + E_2(X),
\end{aligned}$$

where $G_N^-(n)$ comes substituting the first sum in (3.3), $G_N^+(n)$ from substituting the second, and

$$E_2(X) = \mathcal{O}\left(\frac{E_{r,N}}{D\lambda_f(D)} \sum_{n\geq 1} \lambda_{f_D}(n) n^{-2N-1/2} \int_0^\infty \Phi(x) \left(\frac{D}{X}\right)^{2N+1/2} dx\right)$$

comes from the error term in (3.3). Recall that $N \geq 1$, and thus the sum in the error term is absolutely convergent. In addition, the function $\Phi(y) := \phi(y/X)e(\alpha y^\beta)$ has compact support in $(X, 2X) \subset \mathbb{R}$, and thus the integral in the error term is also absolutely convergent. Once again using the bound $\lambda_f(n) \ll n^\theta$ for $\theta = 7/64 + \epsilon$ the sum in $E_2(X)$ is $\ll_N 1$ for $N \geq 1$. In addition the integral is $\ll_\Phi X(D/X)^{2N+1/2}$.

Thus

$$E_2(X) \ll_{\Phi,N} \frac{r^{4N}}{\lambda_f(D)}\left(\frac{X}{D}\right)^{-2N+\frac{1}{2}} \ll \frac{X^{\frac{1}{2}-N-\epsilon}}{D^{\frac{1}{2}}\lambda_f(D)}. \tag{3.6}$$

We now return to estimating the sums involving $G_N^{\pm}$. After making the change of variables $x = Xt^2$ we arrive at

$$\begin{aligned} G_N^{-}(n) &= (1+i)\sum_{k=0}^{N-1} C_{r,k} X^{\frac{3}{4}-k}\left(\frac{n}{D}\right)^{-\frac{1}{4}-k} P^{+}_{\alpha,\beta,X}\left(2\sqrt{\frac{nX}{D}},k\right) \\ &+ \frac{i-1}{4\pi}\sum_{k=0}^{N-1} C_{r,k}\ d_{r,k} X^{\frac{1}{4}-k}\left(\frac{n}{D}\right)^{-\frac{3}{4}-k} P^{-}_{\alpha,\beta,X}\left(2\sqrt{\frac{nX}{D}},k\right) \end{aligned}$$

and

$$\begin{aligned} G_N^{+}(n) &= (i-1)\sum_{k=0}^{N-1} C_{r,k} X^{\frac{3}{4}-k}\left(\frac{n}{D}\right)^{-\frac{1}{4}-k} P^{+}_{\alpha,\beta,X,D}\left(-2\sqrt{\frac{nX}{D}},k\right) \\ &- \frac{1+i}{4\pi}\sum_{k=0}^{N-1} C_{r,k}\ d_{r,k} X^{\frac{1}{4}-k}\left(\frac{n}{D}\right)^{-\frac{3}{4}-k} P^{-}_{\alpha,\beta,X,D}\left(-2\sqrt{\frac{nX}{D}},k\right) \end{aligned}$$

where

$$P^{\pm}_{\alpha,\beta,X}(w,k) = \int_0^\infty t^{\pm 1/2-2k}\phi(t^2)e(\alpha X^\beta t^{2\beta} + wt)dt.$$

A similar situation arises in [RY15] in the proof of Theorem 4, however with $N = 1$ and with the terms appearing in the SL(3,$\mathbb{Z}$) case. Nonetheless the techniques are the same, and so we use the analogous techniques for our situation. We will now summarize that approach.

By repeated integration by parts we have

$$P^{\pm}_{\alpha,\beta,X}(w,k) = \int_1^{\sqrt{2}} g_s^{\pm}(t;k)e(\psi(t))dt,$$

where

$$g_0^{\pm}(t;k) = t^{\pm 1/2-2k}\phi(t^2), \qquad g_s^{\pm}(t;k) = \left(\frac{g_{s-1}^{\pm}(t;k)}{2\pi i\psi'(t)}\right)' \text{ for } s \geq 1.$$

Suppose that $|\psi'(t)| \gg Q(w) > 0$. Then by the arguments in [RY15] p. 13 we have

$$P^{\pm}_{\alpha,\beta,X}(w,k) \ll_{\phi,\beta,s} \sum_{0\le m\le s} \frac{(|\alpha|\beta X^{\beta})^m}{Q(w)^{m+s}}. \tag{3.7}$$

If $sgn(\alpha) = sgn(w)$ then the phase function

$$\psi(t) := \alpha X^{\beta} t^{2\beta} + wt = sgn(\alpha)\left(|\alpha| X^{\beta} t^{2\beta} + |w|t\right) \tag{3.8}$$

has no critical points since

$$\psi'(t) \gg |\alpha| X^{\beta} + |w|.$$

Thus in this case we may choose $Q(w) = |\alpha|X^{\beta} + |w|$ and by (3.7) we obtain

$$P^{\pm}_{\alpha,\beta,X}(sgn(\alpha)|w|,k) \ll |w|^{-s}$$

On the other hand, if $sgn(\alpha) = -sgn(w)$ then we set

$$b^* = (|\alpha|\beta)^2 X^{2\beta-1} D \min\{1, 2^{1-2\beta}\}$$

For $n \geq 4b^*$ one has $\psi'(t) \gg Q = \sqrt{nX/D} \gg \sqrt{b^* X} \gg \alpha\beta X^{\beta}$. Thus using (3.7) we also have

$$P^{\pm}_{\alpha,\beta,X}(w,k) \ll_{\phi,\beta,s} \left(\frac{nX}{D}\right)^{-s/2}$$

for all $s \geq 0$, and therefore in either of these cases we have

$$\begin{aligned} G_N^{sgn(\alpha)}(n) &\ll_{\phi,\beta,N,s} X \sum_{k=0}^{N-1} C_{r,k} \left(\frac{nX}{D}\right)^{-k-\frac{s}{2}-\frac{1}{4}} (1+d_{r,k}) \\ &\ll X\left(\frac{nX}{D}\right)^{-\frac{1}{4}-\frac{s}{2}} \sum_{k=0}^{N-1} \left(\frac{r^4 D}{nX}\right)^k (1+r^2 k) \\ &\ll X\left(\frac{nX}{D}\right)^{-\frac{1}{4}-\frac{s}{2}} \sum_{k=0}^{N-1} \frac{1+r^2k}{(nDX^{\epsilon})^k} \\ &\ll_N X\left(\frac{nX}{D}\right)^{-\frac{1}{4}-\frac{s}{2}} (1+r^3 N) \end{aligned}$$

Recall that $|w| = 2\sqrt{nX/D}$ and $n \geq 1$. Set $s = 2N$ and define

$$E_3(X) := \frac{1}{D\lambda_f(D)} \sum_{n\geq 1} \lambda_{f_D}(n) G_N^{sgn(-\alpha)}(n) + \frac{1}{D\lambda_f(D)} \sum_{n\geq 4b^*} \lambda_{f_D}(n) G_N^{sgn(\alpha)}(n)$$

Then by the above analysis and using $\lambda_f(n) \ll n^\theta$ with $\theta = 7/64 + \epsilon$ we have the bound

$$E_3(X) \ll_{\phi,\beta,N} \frac{1}{\lambda_f(D)} \left(\frac{X}{D}\right)^{\frac{3}{4}-N} (1 + r^3 N) \tag{3.9}$$

for all $N \geq 1$.

Combining the above estimates we have

$$\sum_{n\geq 1} \lambda_f(n) e(\alpha n^\beta) \phi\left(\frac{n}{X}\right) = \frac{1}{D\lambda_f(D)} \sum_{n<4b^*} \lambda_{f_D}(n) G_N^{sgn(\alpha)}(n) + \mathcal{E}(X) \tag{3.10}$$

for all $N \geq 1$, where $\mathcal{E}(X) = E_1(X) + E_2(X) + E_3(X)$. By combining the estimates for $E_i(X)$ given in (3.5), (3.6) and (3.9) we have

$$\begin{aligned} \mathcal{E}(X) &\ll_{\phi,\beta,N} \frac{1}{\lambda_f(D)} \left[e^{-4\pi\sqrt{X/D}} \left(\frac{X}{D}\right)^{\frac{3}{4}} + \frac{X^{\frac{1}{2}-N-\epsilon}}{D^{\frac{1}{2}}} + \left(\frac{X}{D}\right)^{\frac{3}{4}-N} (1 + r^3 N) \right] \\ &= \mathcal{O}_{\phi,\beta,N}\left(\frac{1 + r^3 N}{\lambda_f(D)} \left(\frac{X}{D}\right)^{\frac{3}{4}-N} \right) \end{aligned} \tag{3.11}$$

This gives the equality in Theorem 1.

Next we estimate the integrals

$$P^\pm(w,k) := P^\pm_{\alpha,\beta,X}(w,k) := \int_1^{\sqrt{2}} t^{\pm\frac{1}{2}-2k} \phi(t^2) e(\alpha X^\beta t^{2\beta} + wt) dt$$

appearing in the Theorem 1 and consider several special cases which will be of interest in the numerical example in Section 4. We use the weighted first derivative test from Huxley [Hux96], Lemma 5.5.5.

Recall that in the theorem $w = -sgn(\alpha)2\sqrt{nX/D}$. Setting

$$\begin{aligned} g_{\pm}(t) &= t^{\pm 1/2-2k}\phi(t^2) \\ f(t) &= \alpha X^{\beta} t^{2\beta} + wt \end{aligned}$$

then

$$P^{\pm}(w,k) = \int_1^{\sqrt{2}} g_{\pm}(t)e(f(t))dt$$

Following the notation of [Hux96] the integral will be estimated in terms of the parameters satisfying

$$\begin{aligned} f^{(r)}(t) &\leq C_r \frac{T}{M^r} \\ g^{(s)}(t) &\leq C_s \frac{U}{N^r} \end{aligned}$$

for $r = 2,3$ and $s = 0,1,2$. Since $\phi(t)$ is a Schwartz function and $t \in [1, \sqrt{2}]$ we have

$$g^{(s)}(t) \leq C_s$$

for some constant C_s depending only on ϕ. In addition,

$$f^{(r)}(t) = 2\beta(2\beta-1)\cdots(2\beta-r+1)\frac{\alpha X^{\beta}}{t^{2\beta-r}}$$

for all $r \geq 2$. Set

$$\begin{aligned} C_r &= |2\beta(2\beta-1)\cdots(2\beta-r+1)| \max\{1, 2^{\beta-\frac{r}{2}}\} \\ T &= |\alpha X^{\beta}| \\ M &= 1 \end{aligned}$$

and similarly for g we set $U = N = 1$. Finally, we consider

$$Q := \min_{t\in[1,\sqrt{2}]} |f'(t)|$$

Note that this value can be arbitrarily small. Indeed, in the case of primary interest where $\beta = 1/2$ and $\alpha = 2\sqrt{q/D}$ for any $q \in \mathbb{Z}_{>0}$ then $f'(t) \equiv 0$. However, in this case the integral can be easily evaluated for specific choices of ϕ, since if $f'(t) \equiv 0$ we have

$$P^{\pm}\left(-sgn(\alpha)2\sqrt{\frac{nX}{D}}, k\right) = \int_1^{\sqrt{2}} t^{\frac{1}{2}-2k}\phi(t^2)dt$$

On the other and, by the weighted first derivative test if $f'(t)$ is not identically zero then we have

$$P^{\pm}\left(-sgn(\alpha)2\sqrt{\frac{nX}{D}}, k\right) = \mathcal{O}_{\beta,k,\phi}\left(\frac{\alpha X^{\beta}}{Q^3} + \frac{1}{Q^2}\right)$$

In particular we note that when $\beta = 1/2$ and $\alpha = 2\sqrt{q/D}$ we have

$$|f'(t)| = \sqrt{\frac{X}{D}}\left|\sqrt{q} - \sqrt{n}\right|$$

Thus provided $q \neq n$ we have

$$P^{\pm}_{2\sqrt{\frac{q}{D}},\frac{1}{2},X}\left(2\sqrt{\frac{nX}{D}}, k\right) = \mathcal{O}_{k,\phi}\left(\frac{D\sqrt{q}}{X|\sqrt{n} - \sqrt{q}|^3}\right) \tag{3.12}$$

These estimates conclude the proof of Theorem 1. □

3.3 Proof of Corollaries

Corollary 1 is the case of rapid decay. There will be no main terms precisely when the sum in the right-hand side of (3.10) vanishes, which is when $4b^* < 1$. Rearranging this we see that there will be no main terms if

$$\alpha\beta X^{\beta}\min\{1, 2^{\frac{1}{2}-\beta}\} < \frac{1}{2}\sqrt{\frac{X}{D}}$$

This gives Corollary 1.

Corollary 2 covers the case of a single main term. Theorem 1 gives an estimate of the resonance sum with error term having rapid decay in X. However we also wish to give an estimate which only has a single Fourier coefficient on the right, at the expense of an error term which is not of rapid decay in X. Since our case of primary interest is when $\beta = 1/2$, we now make this assumption.

Since the sum in the right-hand side of (3.10) vanishes when $4b^* < 1$, we suppose $4b^* \geq 1$. Then for $n \neq q$ we can use the asymptotic for $P^{\pm}(w,k)$ given in (3.12) to calculate the contribution for $n \neq q$ in (3.10) as

$$\frac{\sqrt{q}}{X\lambda_f(D)} \sum_{k=0}^{N-1} r^{4k} \left(\frac{X}{D}\right)^{-k} \sum_{\substack{1 \leq n < 4b^* \\ n \neq q}} n^{\theta} \frac{n^{-\frac{1}{4}-k}}{|\sqrt{n}-\sqrt{q}|^3} \left(1 + d_{r,k}\left(\frac{nX}{D}\right)^{-\frac{1}{2}}\right) \tag{3.13}$$

$$\begin{aligned} &\ll_N \frac{\sqrt{q}}{X\lambda_f(D)} \sum_{k=0}^{N-1} \left(\frac{Dr^4}{X}\right)^k \left(1 + r^2 k \left(\frac{nX}{D}\right)^{-\frac{1}{2}}\right) \\ &\ll_N \frac{\sqrt{q}}{X\lambda_f(D)} \sum_{k=0}^{N-1} X^{-\epsilon k} \left(1 + r^2 k \left(\frac{nX}{D}\right)^{-\frac{1}{2}}\right) \\ &\ll_N \frac{\sqrt{q}N}{X\lambda_f(D)} \left(1 + r^2 N \left(\frac{X}{D}\right)^{-\frac{1}{2}}\right) \end{aligned}$$

Combining this estimate with those in (3.11) we arrive at

$$\sum_{n \geq 1} \lambda_f(n) e\left(\sqrt{\frac{qn}{D}}\right) \phi\left(\frac{n}{X}\right) = \frac{\lambda_{f_D}(q)}{D\lambda_f(D)} G_N^{sgn(\alpha)}(q) + \mathcal{E}'(X) \tag{3.14}$$

for any integer $N \geq 1$, where $\mathcal{E}'(X)$ is $\mathcal{E}(X)$ plus the contribution for $n \neq q$ calculated in (3.13). Thus

$$\begin{aligned} \mathcal{E}'(X) \ &\ll_{\phi,\beta,N} \ \frac{1}{\lambda_f(D)} \Bigg[e^{-4\pi\sqrt{X/D}} \left(\frac{X}{D}\right)^{\frac{3}{4}} + \frac{X^{\frac{1}{2}-N-\epsilon}}{D^{\frac{1}{2}}} \\ &+ \left(\frac{X}{D}\right)^{\frac{3}{4}-N} (1 + r^3 N) + \frac{\sqrt{q}N}{X\lambda_f(D)} \left(1 + r^2 N \left(\frac{X}{D}\right)^{-\frac{1}{2}}\right) \Bigg] \end{aligned} \tag{3.15}$$

To allow easier comparison to similar results for holomorphic cusp forms and Maass forms for the full modular group we set $N = 1$ and substitute the definition of $G_N^-(q)$ to arrive at

$$\begin{aligned} \sum_{n\geq 1} \lambda_f(n)\phi\left(\frac{n}{X}\right) e\left(2\sqrt{\frac{qn}{D}}\right) &= \frac{i-1}{D\lambda_f(D)}\lambda_{f_D}(q)X^{\frac{3}{4}}\left(\frac{q}{D}\right)^{-\frac{1}{4}} \\ &\times P^+_{\alpha,\frac{1}{2},X}\left(-sgn(\alpha)2\sqrt{\frac{qX}{D}},0\right) \\ &- \frac{1+i}{4\pi D\lambda_f(D)}\lambda_{f_D}(q)d_{r,0}X^{\frac{1}{4}}\left(\frac{q}{D}\right)^{-\frac{3}{4}} \\ &\times P^-_{\alpha,\frac{1}{2},X}\left(-sgn(\alpha)2\sqrt{\frac{qX}{D}},0\right) \\ &+ \mathcal{E}'(X) \end{aligned} \tag{3.16}$$

Set

$$\begin{aligned} c_q^+ &:= (i-1)P^+_{\alpha,\frac{1}{2},X}\left(-sgn(\alpha)2\sqrt{\frac{qX}{D}},0\right) \\ c_q^- &:= -\frac{(1+i)}{4\pi}P^-_{\alpha,\frac{1}{2},X}\left(-sgn(\alpha)2\sqrt{\frac{qX}{D}},0\right) \end{aligned} \tag{3.17}$$

Then (3.16) becomes

$$\begin{aligned} \sum_{n\geq 1} \lambda_f(n)\phi\left(\frac{n}{X}\right) e\left(2\sqrt{\frac{qn}{D}}\right) &= \frac{c_q^+}{q^{\frac{1}{4}}\lambda_f(D)}\left(\frac{X}{D}\right)^{\frac{3}{4}}\lambda_{f_D}(q) \\ &+ \frac{c_q^- d_{r,0}}{q^{\frac{3}{4}}\lambda_f(D)}\left(\frac{X}{D}\right)^{\frac{1}{4}}\lambda_{f_D}(q) \\ &+ \mathcal{E}'(X) \end{aligned} \tag{3.18}$$

This gives Corollary 2.

From Corollary 2 we see that it is simple to solve for $d_{r,0} = -2r^2 - \frac{1}{8}$, and thus for r. Indeed, one can rearrange (3.18) to solve for r for any value of q. However, it is desirable to simultaneously maximize the main term and minimize the error term in (3.18). This is accomplished when $q = 1$, and thus this is the case we use. Numerical computations show that once q gets larger one needs to choose significantly larger X to achieve similar accuracy. Finally, note that all constants above are nonzero.

It is interesting to ask whether one can improve the error term in Corollary 3. The obvious way to do this is to use $N \geq 2$ in Theorem 1, since the error term is of size

$$\mathcal{O}\left(D^{-N-\frac{1}{4}}X^{-N+\frac{3}{4}}\right)$$

However when $N = 2$, rather than having a quadratic polynomial in r (as in the case for $N = 1$), one has a degree 6 polynomial. While this cannot be solved by hand, it can be numerically solved. If one first estimates the eigenvalue with the equation in Corollary 3, then it is feasible to improve the precision of r (without needing to know more Fourier coefficients) by using $N = 2$ and throwing away the extraneous solutions. Indeed, if one only has very limited knowledge of the Fourier coefficients then this approach may be useful.

To prove Corollary 4 we first consider α_ϵ. From Corollary 1 we see that the resonance sum will be of rapid decay if and only if

$$\alpha\beta X^\beta \min\{1, 2^{\frac{1}{2}-\beta}\} < \frac{1}{2}\sqrt{\frac{X}{D}}$$

Setting $\beta = 1/2$ and $\alpha = \alpha_\epsilon$ the assumption of rapid decay means that

$$\frac{\epsilon}{\sqrt{c}} < \frac{1}{\sqrt{D}}$$

Solving for D this becomes

$$D < \frac{c}{\epsilon^2} \tag{3.19}$$

Using Corollary 2 we see that the resonance sum will *not* be of rapid decay when $\alpha = \alpha_q$. Then setting $\beta = 1/2$ and $\alpha = \alpha_q$ the assumption of a main term at some q means that

$$\left| \sqrt{\frac{q}{c}} - \sqrt{\frac{q}{D}} \right| < \frac{1}{2} X^{-\frac{1}{2}}$$

Solving this for D yields

$$\frac{4cqX}{\left(2\sqrt{qX} + \sqrt{c}\right)^2} < D < \frac{4cqX}{\left(2\sqrt{qX} - \sqrt{c}\right)^2} \tag{3.20}$$

Using $q \geq 1$ and combining the left-hand side of (3.20) with (3.19) we arrive at

$$\frac{c}{\left(1 + \sqrt{\frac{c}{4qX}}\right)^2} < D < \frac{c}{\epsilon^2}$$

Note that as $X \to \infty$ this bound on D becomes

$$c \leq D \leq \frac{c}{\epsilon^2}$$

CHAPTER 4
NUMERICAL RESULTS

4.1 Numerical Examples

In this section we illustrate the above ideas with a concrete example. We take a specific primitive self-dual Maass cusp form f (see [Col13] for details of this form, and [LMF13] for many other examples) and estimate its level and spectral parameter r, and then compare these to the known values.

We begin by estimating the level of f. This involves evaluating the sums given in Corollary 4 for various choices of $c \geq 1$. We first evaluate the sum involving $\alpha_q(c)$. To make the range for D given in Corollary 4 as small as possible we choose $q = 1$. Unless some computational purposes prohibits it, this choice is optimal. In Figure 1 below we show four graphs illustrating the size of

$$\left| \sum_{n \geq 1} \lambda_f(n) \phi\left(\frac{n}{X}\right) e(\alpha\sqrt{n}) \right|$$

for $\alpha = \alpha_q(c)$ as a function of X. We see that at $c = 5$ the graph grows as a positive power of X. Thus we evaluate the sum with $\alpha = \alpha_\epsilon(c)$ for $c = 5$ and $\epsilon = 0.95$. In Figure 2 we see that this graph shows rapid decay in X. Thus from Figures 1 and 2 we deduce that $D \approx 5$. Using Corollary 4 we can guarantee that $4.77 < D < 5.54$ and thus $D = 5$. Indeed, f is a Maass cusp form on $\Gamma_0(5)$.

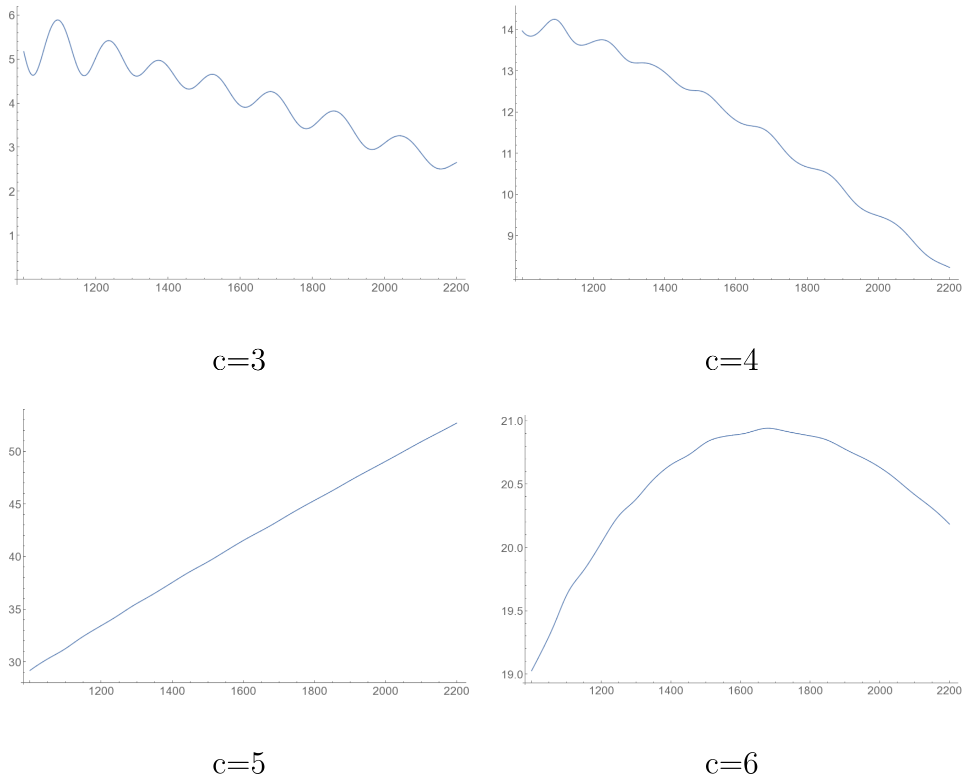

Figure 4.1: $|S_X(f;q,\eta)|$ for $\alpha = \alpha_1(c)$ for X from 1000 to 2200

It is important to note that neither graph alone can determine the level. In Figure 3 we see that for $c = 1$ and $q = 1$ the resonance sum with $\alpha = \alpha_q(c)$ has a main term, and thus would suggest $D \approx 1$. However, the sum with $\alpha = \alpha_\epsilon(1)$ does not show rapid decay, and thus D is in fact *not* near 1.

Now that we have located the level, we will use this knowledge to compute the eigenvalue. All terms in Corollary 3 are easily computed. Recall that the constants $c^\pm$ both involve integrals, however the integrals are of the form

$$\int_1^{\sqrt{2}} t^{\pm\frac{1}{2}}\phi(t^2)dt$$

and are easily handled by any modern mathematical software. In Figure 4 we see

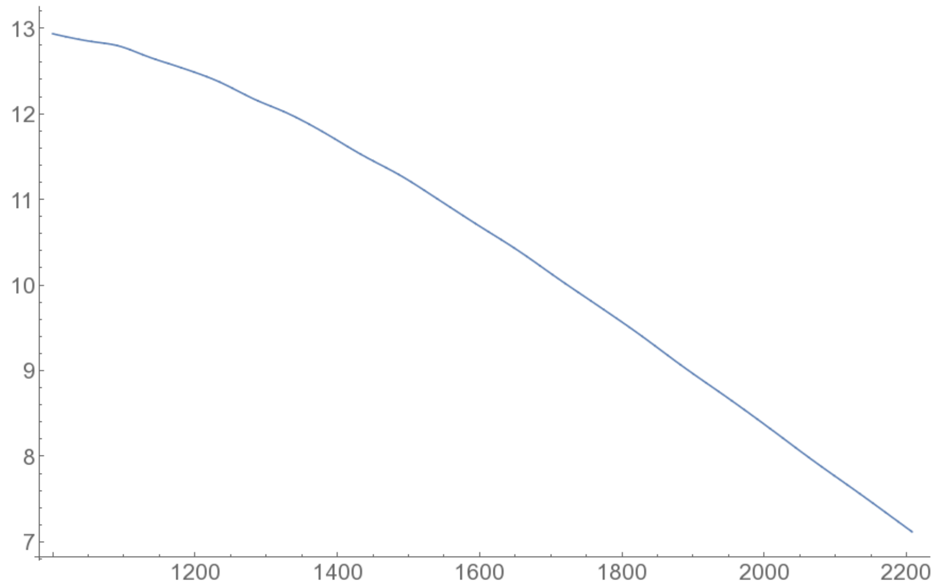

Figure 4.2: $|S_X(f;q,\eta)|$ for $\alpha = \alpha_{0.95}(5)$ for X from 1000 to 2200

that as $X \to \infty$ the graph converges to a value near 8. Indeed, the true spectral parameter is $r \approx 8.01848237839$. For $X = 2200$ the calculated value is off by only 0.02. In cases where one can easily compute tens of thousands of Fourier coefficients one could achieve arbitrarily high accuracy.

All computations were carried out on the Neon High Performance Computing Cluster at the University of Iowa, and run in Mathematica 10.

4.2 An Algorithm for Locating Spectral Parameters

In [RY15] Ren and Ye suggested that resonance sums could be used in an algorithm to locate the spectral parameters of Maass cusp forms. As discussed in Section 2.1, an algorithm to locate spectral parameters for SL(2,$\mathbb{Z}$) has been known since the early 1990's. However Ren and Ye's suggestion was unique in that it suggested an algorithm to locate spectral parameters for SL(n, $\mathbb{Z}$) for any $n \geq 2$. This suggestion

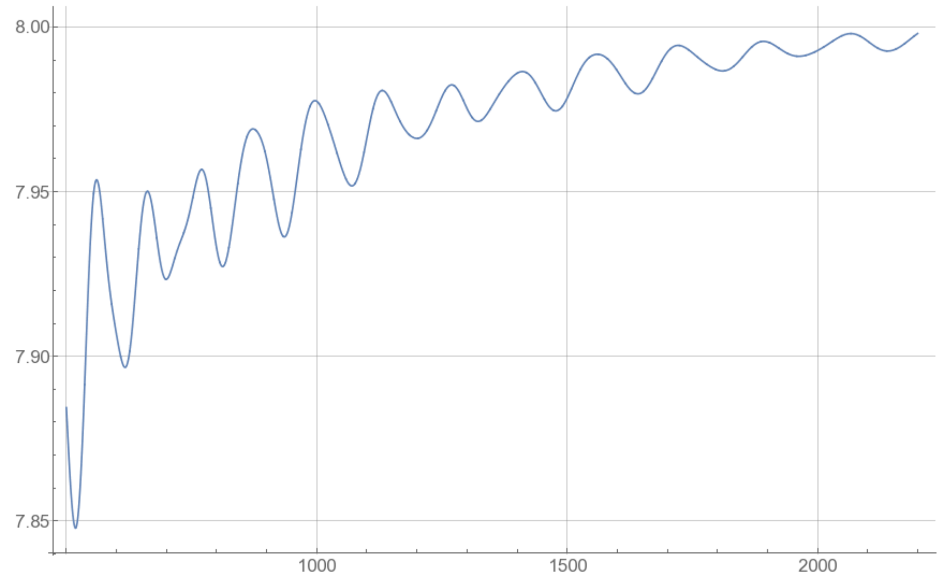

Figure 4.3: Calculated spectral parameter r using Corollary 3 for X from 1000 to 2200. The true spectral parameter is approximately 8.01848237839

was the initial motivation into our research into estimating resonance sums of Maass cusp forms. In this section we will discuss our work in this area, including limitations we encountered.

The suggested algorithm from [RY15] is as follows: Let $r > 0$ be a fixed real number, and for each $q \in \{1, 2, \ldots, \lfloor X^{1-\epsilon} \rfloor\}$ set $\alpha_q^{\pm} = \pm 2\sqrt{\frac{q}{D}}$. Then from Corollary 2 in Section 3.2 we have

$$\begin{aligned} \sum_{n \geq 0} \lambda_f(n) e(\alpha_q^{\pm} \sqrt{n}) \phi\left(\frac{n}{X}\right) &= \frac{c_q^{+}}{q^{\frac{1}{4}} \lambda_f(D)} \left(\frac{X}{D}\right)^{\frac{3}{4}} \lambda_{f_D}(q) \\ &+ \frac{c_q^{-} d_{r,0}}{q^{\frac{3}{4}} \lambda_f(D)} \left(\frac{X}{D}\right)^{\frac{1}{4}} \lambda_{f_D}(q) \\ &+ \mathcal{E}'(X) \end{aligned}$$

where all terms are as in the corollary. If we replace $\lambda_f(n)$ with complex variables a_n

then this gives a system of equations with $X+1$ variables and $2\lfloor X^{1-\epsilon}\rfloor+1$ equations. Thus we have an overdetermined system of equations. This system will have a solution when the a_n are Fourier coefficients (normalized with $\lambda_f(1) = 1$) for some form f. Therefore, if our real number r coincides with a spectral parameter attached to some Maass cusp form then this system of equations has a solution. Therefore the idea is to vary r and look for the existence of a solution to the system.

This is the algorithm outlined for SL(2, $\mathbb{Z}$). However, Maass cusp forms for SL(n, $\mathbb{Z}$) also have spectral parameters $\mu_1, \ldots, \mu_{n-1}$. By estimating the resonance sums for these forms one could use the same idea to search for spectral parameters in $\mathbb{C}^{n-1}$.

As discussed in Section 2.2, Bian (see [Bia10]) implemented a philosophically similar approach to search spectral parameters attached to SL(3, $\mathbb{Z}$). However his approach relied on twisting arithmetic sums by Dirichlet characters. In dimensions $n > 3$ his approach would require twisting by automorphic forms on SL(n-1, $\mathbb{Z}$). This means that his approach does not easily generalize to higher dimensions.

We attempted to implement Ren and Ye's approach in SL(3,$\mathbb{Z}$). In doing so we saw both the promise and limitations of this approach. In order to further discuss our research in this area we must introduce some results from [RY15].

4.2.1 GL(3) Background

Suppose f is a $GL(3)$ Maass form with Fourier coefficients $A_f(m,n), m \geq 1, n \neq 0$. Consider the smooth cut sum

$$\sum_{n\geq 0} A_f(m,n)\phi\left(\frac{n}{X}\right) e(\alpha n^{\beta})$$

where ϕ is a smooth function of $\mathbb{R}$ supported on $[1,2]$ with bounded derivatives, $\beta > 0$ fixed, $\alpha > 0$ and $X > 1$ main parameters, with $m < X$. It is known that a Maass form f can be written as a finite sum of Hecke eigenforms, i.e.

$$f = \sum_{j=1}^{e_f} f_j,$$

where $f_j, j = 1, 2, ..., e_f$ are Hecke eigenforms. Then from ([RY15]) we have

Theorem 4.1. *Suppose* $\max\{2^{\max\{\beta,1/3\}}\alpha\beta, 1\} \cdot m^{1/3} \leq X^{\frac{1}{3}-\beta}$. *Then the estimate*

$$\sum_{n\geq 0} A_f(m,n)\phi\left(\frac{n}{X}\right) e(\alpha n^{\beta}) \ll X^{-M}$$

holds for any $M > 0$*, where the implied constant may depend on* M, β *and* f *only.*

Theorem 4.2. *Let* f *be a* $GL(3)$ *Maass form and* ℓ *a positive integer. Assume toward the Ramanujan conjecture that* $|A_f(mn)| \ll |mn|^{\theta+\epsilon}$ *for* $\theta < 1/3$*. Then*

$$\begin{aligned}\sum_{X\leq n\leq 2X} & A_f(m,n)e(3(\ell n)^{1/3}) \\ = -i\tfrac{\sqrt{3}}{2}(2^{2/3}-1)\ell^{-1/3}X^{2/3} & \sum_{d|m}\frac{\mu(d)}{d}\sum_{j=1}^{e_f} A_{f_j}(\frac{m}{d},1)A_{f_j}(1,d\ell) \\ & + \mathcal{O}_{f,\epsilon}(m^{\theta+\epsilon}(X^{\frac{1+\theta}{2}+\epsilon} + \ell^{\frac{1}{3}+\theta+\epsilon}X^{1/3}))\end{aligned}$$

In particular when $m = 1$,

$$\sum_{X \le n \le 2X} A_f(1,n) e(3(\ell n)^{1/3}) = -i\frac{\sqrt{3}}{2}(2^{2/3}-1)\ell^{-1/3}X^{2/3}A_f(1,\ell)$$
$$+ \ \mathcal{O}_{f,\epsilon}(X^{\frac{1+\theta}{2}+\epsilon} + \ell^{\frac{1}{3}+\theta+\epsilon}X^{1/3})$$

4.2.2 Lagrange Multipliers

We recall the method of Lagrange multipliers (see [Bel56] for an in-depth discussion of the topic). Suppose $f, g : \mathbb{C}^n \to \mathbb{R}$ and we wish to maximize f subject to the constraint $g(z_1, ..., z_n) = c$. Then we define a new function

$$\Lambda(z_1, ..., z_n, \lambda) = f(z_1, ..., z_n) + \lambda(g(z_1, ..., z_n) - c).$$

If $f(w_1, ..., w_n)$ is a maximum of f, then there exists a λ_0 such that $(w_1, ..., w_n, \lambda_0)$ satisfies $\nabla\Lambda(w_1, ..., w_n, \lambda_0) = 0$. Note that in addition, $\nabla\Lambda(w_1, ..., w_n, \lambda_0) = 0$ implies $g(z_1, ..., z_n) = c$. Thus in the method of Lagrange multipliers, one wishes to find solutions to

$$\nabla\Lambda(z_1, ..., z_n, \lambda) = 0$$

Choose some finite collection of α and β satisfying the conditions in Theorem 1, and ℓ a positive integer. We define the following functions:

$$g_{\alpha,\beta}(a_X, ..., a_{2X}) = \sum_{n=X}^{2X} a_n e(\alpha n^{\beta})\phi\left(\frac{n}{X}\right)$$
$$f_{\ell}(a_X, ..., a_{2X}) = \sum_{n=X}^{2X} a_n e(3(\ell n)^{1/3})\phi\left(\frac{n}{X}\right)$$
$$g(a_X, ..., a_{2X}) = \sum_{n=X}^{2X} a_n^2 - 1$$

Then $g_{\alpha,\beta}$ is from Theorem 1, f_ℓ from Theorem 6, and g is used as a normalization factor. We will use the method of Lagrange multipliers to maximize f_ℓ subject to $g_{\alpha,\beta} = 0 = g$.

Define

$$\Lambda(a_X, ..., a_{2X}, (\lambda_{\alpha,\beta}), \lambda) = f_\ell + \sum_{\alpha,\beta} \lambda_{\alpha,\beta} g_{\alpha,\beta} + \lambda g$$

Taking $\frac{\partial \Lambda}{\partial a_m} = 0$ and solving for a_m we get

$$a_m = -\frac{\phi(\frac{m}{X})}{2\lambda}[e(3(\ell m)^{1/3}) + \sum_{\alpha,\beta} \lambda_{\alpha,\beta} e(\alpha m^\beta)]$$

Then substituting into g_{α_0,β_0} the value of a_m from above, we get

$$g_{\alpha_0,\beta_0}(a_X, ..., a_{2X}) = -\sum_{n=X}^{2X} \frac{\phi(\frac{n}{X})}{2\lambda}[e(3(\ell n)^{1/3}) + \sum_{\alpha,\beta} \lambda_{\alpha,\beta} e(\alpha n^\beta)] e(\alpha_0 n^{\beta_0}) \phi\left(\frac{n}{X}\right)$$

Note that

$$\frac{\partial \Lambda}{\partial \lambda_{\alpha_0,\beta_0}} = g_{\alpha_0,\beta_0}$$

Thus in the method of Lagrange multipliers, this value is zero. Setting this equal to zero and rewriting the result as a linear system in the variables $\lambda_{\alpha,\beta}$ gives

$$\sum_{\alpha,\beta} \lambda_{\alpha,\beta} \sum_{n=X}^{2X} e(\alpha n^\beta) e(\alpha_0 n^{\beta_0}) \phi\left(\frac{n}{X}\right)^2 = -\sum_{n=X}^{2X} e(\alpha_0 n^{\beta_0}) e(3(\ell n)^{1/3}) \phi\left(\frac{n}{X}\right)^2 \tag{4.1}$$

This is solved numerically in MATLAB. The $\lambda_{\alpha,\beta}$ are solved for directly as complex numbers, without needing to consider real and imaginary parts as separate variables.

4.2.3 Numerical Results

First we will consider how closely the potential solutions produced by the method of Lagrange multipliers mirror the behavior predicted in Theorem 6. The parameters we can vary to get different solutions are X, the number of (α, β) pairs, and the integer ℓ. A note here should be made. Even though Theorem 6 requires certain growth properties for all positive integers ℓ, due to how the method of Lagrange multipliers is set up, when we vary ℓ we get (potentially) different solutions. Therefore, once a possible solution corresponding to a specific ℓ value has been found, it must then be tested again in Theorem 6 against many other ℓ values.

To test the potential Fourier coefficients computed in MATLAB, we compute the sum on the LHS of Theorem 6, compute the main term on the RHS of Theorem 6 and subtract these two values. Then we would expect to remainder to be close to the error term predicted by Theorem 6, which is

$$X^{\frac{1+\theta}{2}+\epsilon} + \ell^{1/3+\theta+\epsilon} X^{1/3}$$

So for a solution $\{a_X, ..., a_{2X}\}$ computed in MATLAB, we then compute

$$Error := \frac{\sum_{n=X}^{2X} a_n e(3(\ell n)^{1/3}) + i\frac{\sqrt{3}}{2}(2^{2/3}-1)\ell^{-1/3}X^{2/3}a_\ell}{X^{\frac{1+\theta}{2}+\epsilon} + \ell^{1/3+\theta+\epsilon} X^{1/3}}$$

In absolute value we would expect this to be localized about 1. In particular, in order to preserve the error term (and not have it dominate the main term) we would expect that the absolute value of the above computation would be less than $X^{1/6}$. The table below gives the value of $Error$ to four decimal places. Note that $100^{1/6} \approx 2.14$ and $800^{1/6} \approx 3.05$.

Size of $Error$ for values of ℓ, X and $\#(\alpha, \beta)$

	$X = 100$	$X = 200$	$X = 400$	$X = 800$	$X = 800$	$X = 800$
$\#(\alpha, \beta) =$	6	6	6	6	9	12
ℓ	Error	Error	Error	Error	Error	Error
1	0.7767	1.2971	1.1378	1.1173	0.9089	4.8100
2	0.3631	0.5609	0.4690	0.8910	1.8283	32.9244
3	1.5121	0.4146	0.1946	1.1415	3.6874	10.2940
4	0.8862	0.8363	0.3786	1.4838	5.0193	9.7609
5	0.7275	0.2500	0.4382	0.4884	3.5031	15.9221
6	0.3573	0.5427	0.4374	0.6970	1.7466	59.4012
7	0.4438	0.0530	0.4002	0.7667	3.9581	56.3663
8	1.0647	0.7104	0.0807	1.2613	4.9970	10.7853
9	0.4653	1.1129	0.6238	0.6803	0.9480	76.6143
10	1.0104	0.7656	0.7136	0.6706	5.6328	5.4461
11	0.3040	0.6867	0.6435	0.4133	0.7970	73.6106
12	1.1566	1.0080	0.3605	1.0328	5.8116	5.7850
13	0.3567	0.6054	0.6984	0.5081	1.0233	78.7533
14	0.7674	0.5609	0.2778	0.7363	4.2183	11.6942
15	0.7400	0.5869	0.0714	0.3251	3.0189	59.2335

Thus even though the method of Lagrange multipliers simply maximizes the sum in Theorem 6, in fact the "Fourier coefficients" it generates adhere to those predicted, provided the number of α, β pairs is small.

We now choose a specific solutions corresponding to $\ell = 1$ in the above chart, and test those against the growth in Theorem 6 for many values of ℓ in Theorem 6.

A similar idea seems to be true in general. That is, we found a set of "Fourier coefficients" numerically, which were computed by picking a specific ℓ, (α, β) pairs, and X. Then when that solution is tested against the expected growth in Theorem 6 for many different ℓ values, the growth was still close to what is predicted. Therefore we can conclude that computed results hold some significance.

4.2.4 Graphs

We have chosen the case of $\ell = 6$ with six (α, β) pairs. In order to produce a more interesting graph, we have used a smooth cutoff function with support in [1,10]. This does not affect the results in [Ren-Ye]. In this, and just about all cases, the absolute value of the solutions is in the shape of a vibrating arc.

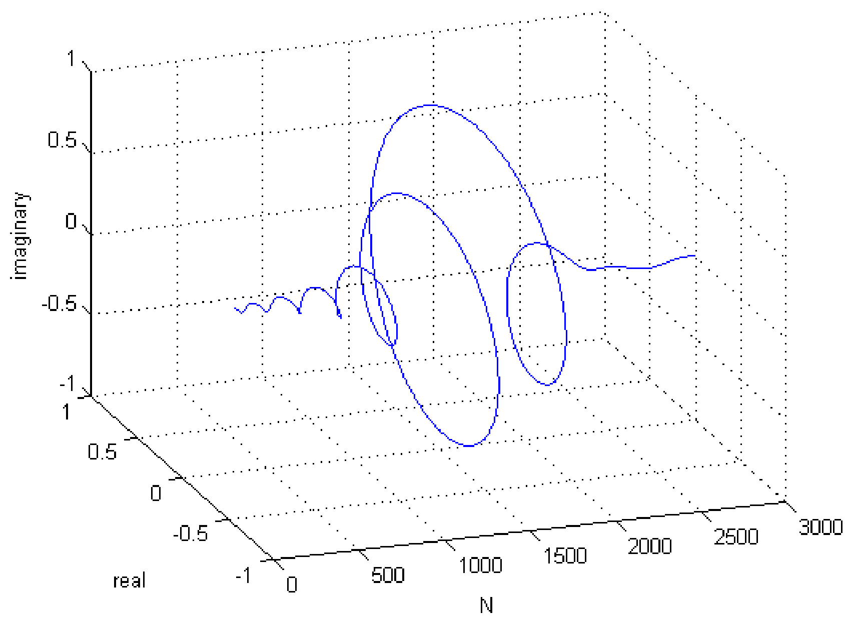

Figure 4.4: Computed Fourier coefficients with $300 \leq X \leq 3000, \#(\alpha, \beta) = 6, \ell = 6$

4.2.5 Rank of the System

The method of Lagrange multipliers initially seems promising in terms of minimizing the Theorem 1 sum, while matching the expected growth in Theorem 6. Moreover, the general decay seen in the graphs looks correct. However, when one begins to investigate the system of equations we see some problems. In particular, while the system is consistent, it is very close to being singular. The simplest way to demonstrate this to compute the singular values of the system.

Recall that given an $m \times n$ matrix M over $\mathbb{C}$, we can decompose M as

$$M = U \Sigma V^*$$

where U is an $m \times m$ unitary matrix, V is an $n \times n$ unitary matrix, V^* denotes the conjugate transpose of V, and Σ is a diagonal matrix which is unique up to ordering. The columns of U give an orthonormal basis for $\mathbb{C}^n$, and the columns of V^* give an orthonormal basis for $\mathbb{C}^m$. Furthermore, if $\vec{v}_i$ is the i^{th} column vector of V^*, and σ_i the i^{th} singular value (i.e. i^{th} nonzero entry of Σ), then

$$M^* M \vec{v}_i = \sigma_i^2 \vec{v}_i.$$

Therefore, small singular values indicate vectors which are "nearly" in the null-space of M^*M. That is, fix a value $\epsilon > 0$. Then

$$r_\epsilon := \#\{\sigma_i > \epsilon \ : \ \sigma_i \text{ is a singular value of } M\}$$

is an approximation to the rank of M (citation goes here). This definition takes into the account measurement error in a system. In the figure below we show the singular values for the system in (4.1) with $X = 200$

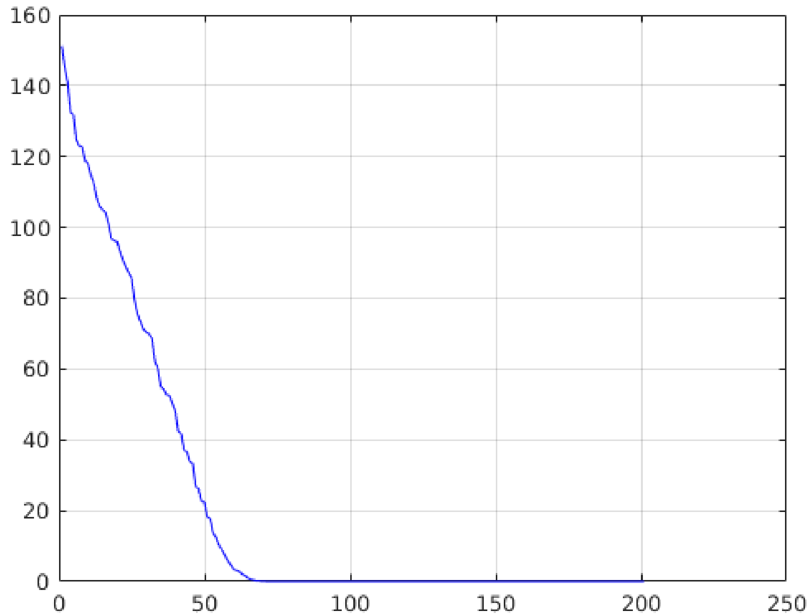

Figure 4.5: Singular values for (4.1) with $X = 200, \ell = 1$

We see that the although M is nonsingular, it's effective rank is nearer to 60. Thus M cannot be used reliably to solve for Fourier coefficients.

4.2.6 Conclusions

While the theory of the algorithm suggested by Ren and Ye is sound, some numerical problems get in the way of actual implementation. The primary difficulty is that the system M has very small determinant (or determinant of M^*M in the case where M is not square), and thus error terms are greatly magnified. We are continuing research into ways to avoid this problem, perhaps by introducing alternate versions of the parameters α and β.

CHAPTER 5
MATHEMATICA CODE

In this section we include the Mathematica code needed to estimate the eigenvalue of a Maass cusp form f, given its level D, and Fourier coefficients $\{\lambda_{f_D}(n)\}_{n=X}^{2X}$.

```
    // Define constants
ClearAll[q, c, level, eig, t, X];
b = 1/2; // beta
a[q_, c_] := 2*Sqrt[q/c]; // alpha
level = 5; // Level of the form
eig = 8.01848237839; // Eigenvalue of the form

// Functions appearing in Corollary 3
// Bump is a particular choice of a bump function
Bump[t_] := Piecewise[Exp[(4*(2*(t-3/2))^2)/(((2*(t-3/2))^2-1))], t > 1 && t < 2];
Pplus = NIntegrate[t^(1/2)*Bump[t^2], {t, 1, Sqrt[2]}];
Pminus = NIntegrate[t^(-1/2)*Bump[t^2], {t, 1, Sqrt[2]}];
LhsPhi[q_, c_, X_] := Table[Bump[n/X]*Exp[2*Pi]*a[q, c]*n^b], n, X, 2*X];


// Functions comprising the left and right hand sides of Corollary 3
Lhs1[q_, c_, X_] := Ar8N5[[Range[X, 2*X]]] . LhsPhi[q, c, X];
Cor2EstPt1[q_, X_] := -((((1 - I)*Pplus*X^(3/4))*Ar8N5[[q]])/
(level^(3/4)*q^(1/4)*Ar8N5[[level]]));
```

```
Cor2EstPt2[q_, X_] := (((1 + I)*(-2*eig^2 - 1/8)*Pminus*X^(1/4))*Ar8N5[[q]])/
(4*Pi*level^(1/4)*q^(3/4)*Ar8N5[[level]]);
Cor2Est[q_, X_] := Cor2EstPt1[q, X] - Cor2EstPt2[q, X];


// EigSolve combines all these functions to solve for the eigenvalue
EigSolve[q_, X_] := Sqrt[Abs[(-(1/2))*(1/8 - (Lhs1[q, level, X] -
Cor2EstPt1[q, X])/((((I + 1)*Pminus*X^(1/4))*Ar8N5[[q]])/(4*Pi*q^(3/4)*
Ar8N5[[level]]* level^(1/4))))]];
```

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