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REPRODUCING KERNEL HILBERT SPACES AND COMPLEX DYNAMICS

by

James Edward Tipton

A thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

December 2016

Thesis Supervisor: Professor Palle Jorgensen

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Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

James Edward Tipton

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the December 2016 graduation.

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This thesis is dedicated to the love of my life, Saypon Sayasane. She put her own education on hold not just so that I could complete my own, but also so that we could start a family. It is because of her that I have two beautiful daughters, and it is also because of her that I was able to finish this thesis. Saypon Sayasane, this work belongs to you even more so than it does to me.

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ABSTRACT

Both complex dynamics and the theory of reproducing kernel Hilbert spaces have found widespread application over the last few decades. Although complex dynamics started over a century ago, the gravity of its importance was only recently realized due to B. Mandelbrot's work in the 1980's. B. Mandelbrot demonstrated to the world that fractals, which are chaotic patterns containing a high degree of self-similarity, often times serve as better models of natural phenomena than conventional smooth models. The theory of reproducing kernel Hilbert spaces, also having started over a century ago, didn't gain popularity until N. Aronszajn's classic was written in 1950. Since then, the theory has found widespread application to many fields including machine learning, quantum mechanics, and harmonic analysis.

In the paper, *Infinite Product Representations for Kernels and Iterations of Functions*, the authors, D. Alpay, P. Jorgensen, I. Lewkowicz, and I. Martiziano, show how a kernel function can be constructed on an attracting set of an iterated function system. Furthermore, they show that when certain conditions are met, one can construct an orthonormal basis of the associated Hilbert space via certain pull-back and multiplier operators.

In this thesis we take for our iterated function system, the family of iterates of a given rational map. Following the approach of the above paper, we investigate for which

rational maps their kernel and orthonormal basis constructions hold. We are able to show that the kernel construction applies to any rational map conjugate to a polynomial with an attracting fixed point at 0. Within such rational maps, we are able to find a family of polynomials for which the orthonormal basis construction holds. It is then natural to ask how the orthonormal basis changes as the polynomial within a given family varies. We are able to determine, for certain families of polynomials, that the dynamics of the corresponding orthonormal bases are well-behaved. Finally, we conclude with some possible avenues of future investigation.

PUBLIC ABSTRACT

A fractal is an object which exhibits a property called self-similarity; no matter how far one zooms into the image, one will find slightly perturbed copies of the original image. Their are many natural phenomena which are currently best modeled by fractals. These include lightning, plants, animals, the coast line, and brownian motion, to name a few.

A Hilbert space is a collection of objects which satisfy two special properties. The first property is the existence of a two-variable function called an inner product. An inner product allows one to assign a notion of length and angle for the objects in the space. The second property is that the space is complete; intuitively this means there aren't any "naturally obtained" objects missing from the space. A reproducing kernel Hilbert space is a Hilbert space consisting of functions, which satisfy a third property. The third property, called the reproducing property, is the existence of a function, called the kernel function, which allows one to compute values of other functions in the space using the inner product. Hilbert spaces have the nice feature that any object in the space can be obtained as a sum involving a collection of orthogonal normal objects within the set. The term normal means each object has length one, and the term orthogonal means that each pair of objects meet at a 90° angle. The collection of these orthogonal normal objects is called an orthonormal basis.

In this thesis we consider fractals which are obtained from rational maps. A rational map is a fraction in which both the numerator and denominator are polynomials. We investigate a technique that associates to each fractal a reproducing kernel Hilbert space. We ask how does the orthonormal basis associated to the reproducing kernel Hilbert space change as we let the fractal vary? We are able to find a class of fractals for which the dynamics of the associated orthonormal bases is well-behaved in a certain sense.

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CHAPTER 1
HISTORICAL INTRODUCTION

In what follows we present a brief historical introduction to the fields relevant to this thesis. The information presented below is by no means complete; we have chosen to emphasize those works and results which are closely related to our interests. Furthermore, this information has been gleaned from a variety of books and papers from which a more detailed history has been given. Detailed accounts of the history of complex dynamics can be found in [1] and [2]. For the history of reproducing kernel Hilbert space theory, the references [3], [24], and [5] go into greater detail than presented here. Analysis on fractals, having earnestly started in the late 1980's, is a relatively new field. The book of the same name, [17], provides greater details of the history of analysis on fractals.

1.1 Complex Dynamics and Fractals

Complex dynamics is concerned with the dynamics of iterated function systems of complex number spaces. Newton's method is an early example of iteration, but examples can be found even earlier, such as the ancient Babylonian iterative method of approximating the square root of some number a:

$$z_{n+1} = \frac{1}{2}\left(z_n + \frac{a}{z_n}\right)$$

Concern for the actual dynamics of such iterations didn't make an appearance until the 1870's, with E. Schröder's only two papers in the field [25], [26]. E. Schröder was the first to discover the characteristic property of fixed points whose evaluation in the derivative has absolute value less than one: successive iterates of nearby points tend to the attracting fixed point. E. Schröder was also the first to study iteration in general. His search for a general formula for the iterates of a given function, although eventually abandoned, had led him to independently discover what are now called the Abel-Schröder functional equations:

$$\phi(G(z)) = G(z+h) \quad \text{and} \quad \phi(G(z)) = G(hz)$$

Following E. Schöder, advances were made by A. Cayley, A. Korkine, J. Farkas, G. Koenigs, A. Grévy, and L. Leau. Using the Abel-Schröder functional equations, G. Koenigs was able to build a local theory on the dynamics of iteration near attracting fixed points. Although P. Montel wasn't directly interested in iteration, his theory of normal families played an important role in complex dynamics. One only needs to look at the work of G. Julia and P. Fatou to understand the impact P. Montel's work had. Both G. Julia and P. Fatou made significant contributions to complex dynamics. In fact, the famous complex dynamical sets, the Fatou set and the Julia set, were named after them. Using P. Montel's theory, both G. Julia and P. Fatou were able to prove many properties of the Julia and Fatou sets.

An important conjecture made by G. Julia was the non-existence of certain hypothetical domains, now called Siegel discs. The discs were named after C. Siegel, who demonstrated that iteration in a Siegel disc is conjugate to an irrational rotation. Unfortunately, interest in complex dynamics began to dwindle after C. Siegel's result. Two important contributors to the field during this period were P. Rosenbloom and I. Baker; their work would receive little attention until interest in complex dynamics was rekindled.

This rekindling began in the early 80's with the work of B. Mandelbrot, who had access to some of the most powerful supercomputers at the time. B. Mandelbrot began investigating the Julia and Fatou sets graphically, by printing approximations of the sets. Due to the images self-similar nature, B. Mandelbrot coined the term fractal to describe the general class of objects exhibiting self-similarity. A particular class of fractals that B. Mandelbrot studied was the Julia sets for quadratics of the form $R(z) = z^2 + c$. Intricately related to the Julia sets for this class of quadratics is the Mandelbrot set, named in honor of B. Mandelbrot. An important open conjecture on the Mandelbrot set is that its boundary is locally connected. The proof of this conjecture would have far reaching consequences. After B. Mandelbrot had made the convincing argument that fractals serve as better models to natural phenomena than current conventional models, the interest in complex dynamics exploded. One of the most important advances made since was D. Sullivan's no wandering domains theorem. This theorem asserts that when R is a rational map, every Fatou component must be eventually periodic. A more recent advance came from J. Yoccoz, who was able to obtain a result in the direction of the local connectedness conjecture. J. Yoccoz was able to give certain conditions for when components of the Mandelbrot set are locally connected.

1.2 Reproducing Kernel Hilbert Spaces

The essence of a reproducing kernel, K, is captured by the following inner product property it satisfies:

$$f(y) = < f(x), K(x, y) >$$

This property is aptly named the reproducing property. Examples of such two-variable functions can be found as far back as 1828, as Green's functions of self-adjoint ordinary differential equations [10]. The reproducing property itself was first noted by S. Zaremba in 1907 [31]. S. Zaremba used the reproducing property to study boundary value problems of harmonic and biharmonic functions. The positive definite property of reproducing kernels was introduced by J. Mercer in 1909. J. Mercer had interest in the theory of integral equations and in [20], he pointed out a connection to reproducing kernels. It wasn't until the early 1920's that reproducing kernels resurfaced. The mathematicians S. Bergmann, S. Bochner, and G. Szegő, had each written their dissertation on reproducing kernels within a year of each other. The general theory of reproducing kernels was then developed by N. Aronszajn [5].

The fact that every positive definite kernel is a reproducing kernel for a unique class of functions can be found in [5], although N. Aronszajn attributes this to E. H. Moore. This class of functions is called a reproducing kernel Hilbert space. Combined with the fact that every reproducing kernel is a positive definite function, one stumbles across the fundamental result of the theory: there is a one-to-one correspondence between the collection of all positive definite functions and the collection of all reproducing kernel Hilbert spaces. This result serves as a link between operator theory and the theory of functions.

The classical reference for the history and theory of reproducing kernels is of course [5]. Another great historical reference can be found in [24], which is itself an excellent reference for the general theory and applications of reproducing kernels. A great modern introduction to the theory of reproducing kernels can be found in [22].

The theory of reproducing kernel Hilbert spaces has found widespread use in a variety of fields, including partial differential equations, harmonic analysis, operator theory, interpolation, integral transformations, and probability theory, to name a few. A larger list of current applications may be found in [3]. In particular, the theory has found use in dynamical systems, which is the general subject matter of this thesis.

1.3 Analysis on Fractals

Prior to B. Mandelbrot's publications, interest in the analysis of Fractals was quite limited. Even so, some important breakthroughs came about in the measure theory of fractals. Measure theory on fractals can be traced back to the year 1919; Hausdorff showed that Caratheodory's outer measure construction applies to non-integral dimensional measures. One of the many examples he gave was the middle thirds Cantor set, which has positive finite $\frac{\log 2}{\log 3}$-dimensional measure. Measures of this type are now called Hausdorff measures. Although Hausdorff measures play an important role in the measure theory of fractals, there seems to have been little work combining the two prior to 1975. Excellent references on the general theory of Hausdorff measures can be found in [23] and [8]

After B. Mandelbrot made the argument that many natural systems were better modeled by fractals, research on the analysis of fractals grew rapidly. Rekindled by B. Mandelbrot, the measure theory of fractals was developed extensively, notable references include [11] and [7]. In 1987, the first steps towards a calculus on fractals were taken. The mathematicians S. Goldstein and S. Kusuoka independently constructed an analog of

Brownian motion on the Sierpinski gasket in [9] and [18], respectively. This was achieved by considering a sequence of random walks on approximations of the Sierpinski gasket which could be made to converge to a diffusion process on the Sierpinski gasket. In the following year, J. Kigame constructed an analog of the Laplacian on the Sierpinski spaces [15]. This Laplacian was found as the limit of a sequence of discrete Laplacians on approximations of the Sierpinski gasket. Following this approach, both J. Kigame and S. Kusuoka were able to extend the construction to a larger class of finitely ramified fractals in [16] and [19], respectively. Details on the Laplacian of this particular class of finitely ramified fractals and its applications can be found in [17].

As the branches of Analysis grew, so did their use in the theory of fractals. In particular, interest grew in the harmonic analysis of fractals. Among the first papers to discuss this topic were [27], [28], [29], by R. S. Strichartz, and [12], [13], by P. E. T. Jorgensen and S. Pedersen. The results contained in these papers suggested asymptotic estimates for Fourier analysis of fractal measures. A surprising result then appeared a few years after, in 1998. In [14], P. E. T. Jorgensen and S. Pederson gave explicit formulas for the Fourier series of certain Cantor $L^2(\mu)$ spaces. It is then natural to investigate the harmonic analysis of other fractals. In the case of Julia and Fatou sets, two approaches have been offered so far and both are related to the theory of reproducing kernels. In 2007, K. Thirulogasanthar, A. Krzyżak, and G. Honnouvo constructed reproducing kernels on certain Julia sets using coherent states [30]. The other approach, developed in 2013 by D. Alpay, P. E. T. Jorgensen, I. Lewkowicz, and I. Martziano, focused instead on Fatou components containing an attracting fixed point [4]. Surprisingly, their reproducing kernel was constructed as an infinite product of other reproducing kernels; such factorizations of reproducing kernels seem to be rather uncommon. Furthermore, they were able to obtain an orthonormal basis of the associated Hilbert space, provided that a certain class of operators satisfied the Cuntz relations [6]. It is this approach that this thesis investigates.

CHAPTER 2
COMBINING THE TWO THEORIES

2.1 A Natural Interplay

The theories of complex dynamics and reproducing kernel Hilbert spaces compliment each other quite well. Combining the two offers many ways to construct kernel functions on components of the Fatou and Julia sets of a rational map. We begin by noting that Proposition B.14 gives us a natural way to pull-back kernel functions on sets common to complex dynamics. Combining the conjugacy theory with the pull-back property we can easily obtain kernel functions for a variety of conjugacy classes. In such cases, the pull-back map is just the Möbius transformation conjugating two rational maps. For the entirety of this chapter, we will assume that R and S are non-constant rational maps satisfying

$$S = g \circ R \circ g^{-1}$$

for some Möbius transformation g.

Proposition 2.1. *If K is a kernel function on $J(S)$ (resp. $F(S)$), then $K \circ g(z, w) = K(g(z), g(w))$ is a kernel function on $J(R)$ (resp. $F(R)$).*

Proof. Since $F(S) = g(F(R))$, we may restrict g to $F(R)$, giving a map, $g|_{F(R)}$, from $F(R)$ to $F(S)$. Thus $K \circ g|_{F(R)}$ is a kernel function on $F(R)$. Of course $K \circ g|_{F(R)}$, is just the map $K \circ g$ restricted to $F(R)$. The reasoning is similar with $J(R)$. $\square$

As noted earlier, the above proposition allows one to focus their attention on a single member of a conjugacy class. For example, Proposition A.8 tells us that given any quadratic polynomial R we can find a kernel function on $F(R)$ by investigating kernel functions of $F(z^2+c)$ or $F(\lambda z(1-z))$. Thus instead of searching for a kernel function of a general quadratic, one can instead focus on quadratics of the form z^2+c or $\lambda z(1-z)$. A similar result is achieved with basins of attraction by combining Proposition B.14 and Proposition A.7:

Proposition 2.2. *Let $B_{R,z}$ and $B_{S,g(z)}$ be basins of attraction of R and S respectively. If K is a kernel function on $B_{S,g(z)}$, then $K \circ g$ is a kernel function on $B_{R,z}$.*

Proof. Restricting g to $B_{R,z}$ yields a map from one basin of attraction to the other. Therefore we may pull-back to a kernel function just as was done in the previous proposition. □

We will see in Chapter 4 that the theory laid out in Chapter 3 works particularly well for polynomials with an attracting fixed point at 0. The previous proposition will allow us to extend the theory to any rational map conjugate to a polynomial with an attracting fixed point at 0. Now Proposition A.4 tells us that

Proposition 2.3. *If K is a kernel function on $F(R)$ (resp. $J(R)$), then K is a kernel function on $F(R^{\circ n})$ (resp. $J(R^{\circ n})$).*

Proof. This is actually quite trivial since families of iterates share the same Julia and Fatou set. □

Remark. In spite of the triviality of the above proposition, it hints at a rather novel idea. The rational map R is itself a map between its Fatou components. Thus we may use R

as a pull-back map. In particular, we can find a kernel function on a Fatou component of R and then use R to transfer the kernel function to other Fatou components of R.

Example 2.1. Consider the polynomial $R(z) = z^2 - 1$. It has $\alpha = \{-1, 0\}$ as an attracting cycle. Therefore $S(z) = R^{\circ 2}(z)$ has both -1 and 0 as attracting fixed points. Furthermore $R(B_{S,-1}) = B_{S,0}$. Thus if we can find a kernel function on $B_{S,0}$, we can pull it back along R to a kernel function on $B_{S,-1}$. In fact, since $B_{R,\alpha} = B_{S,0} \cup B_{S,-1}$, we can actually pull back along a piecewise defined function to a kernel function on $B_{R,\alpha}$.

CHAPTER 3
INFINITE PRODUCT REPRESENTATIONS OF KERNEL FUNCTIONS AND ITERATED FUNCTION SYSTEMS

The results presented in this chapter will serve as the foundation and motivation for the work presented in the following chapters. These results were developed and proven in [4].

3.1 Infinite Product Representations

Proposition 3.1. *Let k be a kernel function on U, a topological space, and let K be a continuous map, not identically zero, which satisfies*

$$K(z,w) = k(z,w)K(R(z),R(w)) \tag{1}$$

for all $z,w \in U$. Suppose there exists $l \in U$ such that

$$\lim_{n\to\infty} R^{\circ n}(z) = l \quad \forall z \in U \tag{2}$$

If $K(l,l) > 0$ then K is a kernel function on U and

$$K(z,w) = \left(\prod_{n=0}^{\infty} k(R^{\circ n}(z), R^{\circ n}(w))\right) K(l,l)$$

Proof. By (2), we have that $R(l) = l$. For each natural number N, we have by (1)

$$K(z,w) = \left(\prod_{n=0}^{N} k(R^{\circ n}(z), R^{\circ n}(w))\right) K(R^{\circ n+1}(z), R^{\circ n+1}(w))$$

for all $z,w \in U$. As $n \to \infty$, we have that $K(R^{\circ n+1}(z), R^{\circ n+1}(w)) \to K(l,l) > 0$. Therefore the infinite product, obtained by letting N approach ∞, converges to the quotient $\dfrac{K(z,w)}{K(l,l)}$ on U, and in particular, $K(z,w)$ is a kernel function on U. $\square$

The above proposition provides sufficient conditions for which a continuous map is a kernel function on its domain and can be represented as an infinite product involving some other kernel function.

Proposition 3.2. *Let U be a set, k a kernel function on U such that $k = 1 + t$ where t is a kernel function on U. Consider a map $R : U \to U$. If the set*

$$\Omega = \left\{ z \in U : \sum_{n=0}^{\infty} |t(R^{\circ n}(z), R^{\circ n}(z))| < \infty \right\}$$

is non-empty then

$$K(z,w) = \prod_{n=0}^{\infty} (1 + t(R^{\circ n}(z), R^{\circ n}(w)))$$

converges, and satisfies

$$K(z,w) = k(z,w)K(R(z), R(w)) \qquad \forall z, w \in \Omega \tag{1}$$

Proof. For $z, w \in \Omega$, we have by the Cauchy-Schwarz inequality

$$\begin{aligned} |t(R^{\circ n}(z), R^{\circ n}(w))| = |\langle t_{R^{\circ n}(z)}, t_{R^{\circ n}(w)} \rangle| &\leq \|t_{R^{\circ n}(z)}\| \|t_{R^{\circ n}(w)}\| \\ &= \sqrt{t(R^{\circ n}(z), R^{\circ n}(z)) t(R^{\circ n}(w), R^{\circ n}(w))} \end{aligned}$$

Thus applying Cauchy-Schwarz inequality again we have

$$\begin{aligned} \sum_{n=0}^{N} |t(R^{\circ n}(z), R^{\circ n}(w))| &\leq \sum_{n=0}^{N} \sqrt{t(R^{\circ n}(z), R^{\circ n}(z)) t(R^{\circ n}(w), R^{\circ n}(w))} \\ &\leq \sum_{n=0}^{N} \sqrt{t(R^{\circ n}(z), R^{\circ n}(z))} \sum_{n=0}^{N} \sqrt{t(R^{\circ n}(w), R^{\circ n}(w))} \end{aligned}$$

Therefore as $N \to \infty$, we obtain

$$\sum_{n=0}^{\infty} |t(R^{\circ n}(z), R^{\circ n}(w))| \leq \sum_{n=0}^{\infty} \sqrt{t(R^{\circ n}(z), R^{\circ n}(z))} \sum_{n=0}^{\infty} \sqrt{t(R^{\circ n}(w), R^{\circ n}(w))} < \infty$$

and so the infinite product converges as well. That $K(z,w)$ satisfies (1), we simply compute

$$\begin{aligned} K(z,w) &= \prod_{n=0}^{\infty}(1+t(R^{\circ n}(z),R^{\circ n}(w))) = (1+t(R(z),R(w)))\prod_{n=1}^{\infty}(1+t(R^{\circ n}(z),R^{\circ n}(w))) \\ &= k(z,w)\prod_{n=0}^{\infty}(1+t(R^{\circ n+1}(z),R^{\circ n+1}(w))) = k(z,w)K(R(z),R(w)) \end{aligned}$$

□

Thus the map K as given in the above proposition is a reproducing kernel whenever K satisfies the sufficient conditions given in proposition 3.1. For ease of exposition such a kernel function will be referred to as a product kernel with respect to the kernel function k and map R.

3.2 An Orthonormal Basis

Fundamental in the construction of an orthonormal basis will be a set of relations called the Cuntz relations [6].

Definition 3.1. Let $\mathcal{H}$ be a Hilbert space. A collection of operators, $\{S_i\}_{i\in I}$, is said to satisfy the Cuntz relations if

$$S_i^*S_j = \delta_{ij}I \text{ and } \sum_{i\in I} S_iS_i^* = I$$

holds for all $i,j\in I$.

Under the setting of the previous section, an orthonormal basis can be built for a product kernel K with respect to a kernel function k and a map R, provided R satisfies certain conditions. The presence of these conditions allows one to show that a particular class of pull-back multiplier operators dependent on k and R satisfy the Cuntz relations. From these operators, one is able to then construct an orthonormal basis for the associated Hilbert space of K. In what follows, let $\mathcal{H}_k$ be the reproducing kernel Hilbert space for k with orthonormal basis $\{e_i\}_{i\in I}$, let $\mathcal{H}$ be the reproducing kernel Hilbert space for K, and suppose that for every $z \in \Omega$, the map R satisfies the cardinality condition

$$M(z) = \mathrm{Card}\{\zeta \in \Omega : \ R(\zeta) = z\} < \infty$$

and either

$$\frac{1}{M(z)} \sum_{R(\zeta)=z} e_i(\zeta)\overline{e_j(\zeta)} = \delta_{ij}, \qquad \forall i, j \in I \tag{†}$$

or

$$\frac{1}{M(z)} \sum_{R(\zeta)=z} e_i(\zeta)e_j(\zeta) = \delta_{ij}, \qquad \forall i, j \in I \tag{‡}$$

Under these assumptions we can show that the class of operators consisting of e_i-f pull-back multipliers satisfy the Cuntz relations. We begin by deriving an explicit formula for the adjoints of the e_i-f pull-back multipliers.

Lemma 3.3. *Denote by S_i the operator defined on $\mathcal{H}$ by $S_i(f)(z) = e_i(z)f(R(z))$. The adjoint of S_i is given by*

$$S_i^*(f)(z) = \frac{1}{M(z)} \sum_{R(\zeta)=z} \overline{e_i(\zeta)}f(\zeta)$$

if † holds, and

$$S_i^*(f)(z) = \frac{1}{M(z)} \sum_{R(\zeta)=z} e_i(\zeta)f(\zeta)$$

if ‡ holds.

Proof. If † holds, then for $z, w \in \Omega$ we have

$$\begin{aligned}
\frac{1}{M(z)} \sum_{R(\zeta)=z} \left(\overline{e_i(\zeta)} K(\zeta, w)\right) &= \frac{1}{M(z)} \sum_{R(\zeta)=z} \left(\overline{e_i(\zeta)} k(\zeta, w) K(R(\zeta), R(w))\right) \\
&= \frac{1}{M(z)} \sum_{R(\zeta)=z} \left(\overline{e_i(\zeta)} \sum_{j \in I} e_j(\zeta) \overline{e_j(w)}\right) K(z, R(w)) \\
&= \sum_{j \in I} \left(\overline{e_j(w)} \cdot \frac{1}{M(z)} \sum_{R(\zeta)=z} \left(\overline{e_i(\zeta)} e_j(\zeta)\right)\right) K(z, R(w)) \\
&= \sum_{j \in I} \left(\overline{e_j(w)} \cdot \delta_{ij}\right) K(z, R(w)) = \overline{e_i(w)} K(z, R(w)) \\
&= S_i^*(K_w)(z)
\end{aligned}$$

Since K_w is dense in $\mathcal{H}$ and S_i^* is continuous, the equality holds for all $f \in \mathcal{H}$. On the other hand if ‡ holds, we find similarly:

$$\begin{aligned}
\frac{1}{M(z)} \sum_{R(\zeta)=z} \left(e_i(\zeta) K(\zeta, w)\right) &= \frac{1}{M(z)} \sum_{R(\zeta)=z} \left(e_i(\zeta) k(\zeta, w) K(R(\zeta), R(w))\right) \\
&= \frac{1}{M(z)} \sum_{R(\zeta)=z} \left(e_i(\zeta) \sum_{j \in I} e_j(\zeta) \overline{e_j(w)}\right) K(z, R(w)) \\
&= \sum_{j \in I} \left(\overline{e_j(w)} \cdot \frac{1}{M(z)} \sum_{R(\zeta)=z} \left(e_i(\zeta) e_j(\zeta)\right)\right) K(z, R(w)) \\
&= \sum_{j \in I} \left(\overline{e_j(w)} \cdot \delta_{ij}\right) K(z, R(w)) = \overline{e_i(w)} K(z, R(w)) \\
&= S_i^*(K_w)(z)
\end{aligned}$$

□

Theorem 3.4. *Consider the operators $\{S_i\}_{i \in I}$ defined on $\mathcal{H}$ by $S_i(f)(z) = e_i(z) f(R(z))$. If either † or ‡ holds, then the operators $\{S_i\}_{i \in I}$ satisfy the Cuntz relations.*

Proof. Let $w \in \Omega$. By Theorem B.19 we have that the adjoint of S_i satisfies

$$S_i^*(k_w) = \overline{e_i(w)} k_{R(w)}$$

and therefore

$$S_iS_i^*(k_w)(z) = S_i(\overline{e_i(w)}k_{R(w)})(z) = e_i(z)\overline{e_i(w)}k_{R(w)}(R(z)) = e_i(z)\overline{e_i(w)}K(R(z), R(w))$$

Now summing over $i \in I$ we have by Proposition 3.2

$$\sum_{i\in I} S_iS_i^*(K_w)(z) = \sum_{i\in I} e_i(z)\overline{e_i(w)}K(R(z), R(w)) = k(z,w)K(R(z), R(w)) = K(z,w)$$

Thus we have $\sum_{i\in I} S_iS_i^* = I$ as desired. Now we show that $S_i^*S_j = \delta_{ij}I$. If † holds then by Lemma 3.3

$$\begin{aligned} S_i^*S_j(f)(z) &= \frac{1}{M(z)} \sum_{R(\zeta)=z} \overline{e_i(\zeta)}S_j(f)(\zeta) = \frac{1}{M(z)} \sum_{R(\zeta)=z} \overline{e_i(\zeta)}e_j(\zeta)f(R(\zeta)) \\ &= \left(\frac{1}{M(z)} \sum_{R(\zeta)=z} \overline{e_i(\zeta)}e_j(\zeta)\right) f(z) = \delta_{ij}f(z) \end{aligned}$$

If on the other hand, ‡ holds, we have by Lemma 3.3

$$\begin{aligned} S_i^*S_j(f)(z) &= \frac{1}{M(z)} \sum_{R(\zeta)=z} e_i(\zeta)S_j(f)(\zeta) = \frac{1}{M(z)} \sum_{R(\zeta)=z} e_i(\zeta)e_j(\zeta)f(R(\zeta)) \\ &= \left(\frac{1}{M(z)} \sum_{R(\zeta)=z} e_i(\zeta)e_j(\zeta)\right) f(z) = \delta_{ij}f(z) \end{aligned}$$

Thus the operators S_i satisfy the Cuntz relations. □

We are now in position to construct an orthonormal basis of $\mathcal{H}$. First observe that the function $\mathbf{1}(z) = 1$ is in $\mathcal{H}$. Let $\rho = \dim \mathcal{H}_k$ and let J be an index set with cardinality equal to ρ. For each natural number N, let $S_{\iota_N} = S_{j_1} \cdots S_{j_N}$, where $\iota_N = (j_1, \ldots, j_N) \in J^N$. Next define an index set J^∞ by

$$J^\infty = \bigcup_{N=1}^{\infty} J^N$$

Finally, for each $v \in J^\infty$ we have the function

$$b_v(z) = S_v\mathbf{1}(z)$$

We show that these functions form an orthonormal basis on $\mathcal{H}$.

Theorem 3.5. *If the operators $\{S_i\}_{i\in I}$ satisfy the Cuntz relations then the collection of functions $\{b_v\}_{v\in J^\infty}$ form an orthonormal basis for $\mathcal{H}$ and thus*

$$K(z,w) = \sum_{v\in J^\infty} b_v(z)\overline{b_v(w)}$$

Proof. First we show that the functions b_v form an orthonormal system. Let

$$b_v = S_{v_1}\cdots S_{v_a}\mathbf{1} \text{ and } b_u = S_{u_1}\cdots S_{u_b}\mathbf{1}$$

such that $b_v \neq b_u$. Suppose that $a = b$, then there must be some $i \leq a$ such that $S_{v_i} \neq S_{u_i}$. To illustrate the plan of attack, suppose this is the case for $i = 1$. Then we have

$$\begin{aligned}
\langle b_v, b_u\rangle &= \langle S_{v_1}\cdots S_{v_a}\mathbf{1}, S_{u_1}\cdots S_{u_b}\mathbf{1}\rangle = \langle S_{v_2}\cdots S_{v_a}\mathbf{1}, S_{v_1}^* S_{u_1}\cdots S_{u_b}\mathbf{1}\rangle \\
&= \langle S_{v_2}\cdots S_{v_a}\mathbf{1}, 0\rangle = 0
\end{aligned}$$

Now suppose i is chosen to be the smallest such index and is strictly larger than 1. Then for all $j < i$, $S_{v_j} = S_{u_j}$. Thus we may compute

$$\begin{aligned}
\langle b_v, b_u\rangle &= \langle S_{v_1}\cdots S_{v_a}\mathbf{1}, S_{u_1}\cdots S_{u_a}\mathbf{1}\rangle = \langle S_{v_i}\cdots S_{v_a}\mathbf{1}, S_{v_{i-1}}^*\cdots S_{v_1}^* S_{u_1}\cdots S_{u_a}\mathbf{1}\rangle \\
&= \langle S_{v_i}\cdots S_{v_a}\mathbf{1}, S_{u_i}\cdots S_{u_a}\mathbf{1}\rangle = \langle S_{v_{i-1}}\cdots S_{v_a}\mathbf{1}, S_{v_i}^* S_{u_i}\cdots S_{u_a}\mathbf{1}\rangle \\
&= \langle S_{v_{i-1}}\cdots S_{v_a}\mathbf{1}, 0\rangle = 0
\end{aligned}$$

Now suppose that $a \neq b$. We may assume without loss of generality that $a < b$. If for some index $i \leq a$ we have that $S_{v_i} \neq S_{u_i}$, then a computation similar to the one above shows that $\langle b_v, b_u \rangle = 0$. Suppose instead that for all $i \leq a$ we have that $S_{v_i} = S_{u_i}$. Choose p so that $S_{u_{a+1}} \neq S_p$. Then we have

$$\langle b_v, b_u \rangle = \langle S_{v_1} \cdots S_{v_a} \mathbf{1}, S_{u_1} \cdots S_{u_b} \mathbf{1} \rangle = \langle S_{v_a} \mathbf{1}, S^*_{v_{a-1}} \cdots S^*_{v_1} S_{u_1} \cdots S_{u_b} \mathbf{1} \rangle = 0$$

Now we verify that $\langle b_v, b_v \rangle = 1$.

$$\begin{aligned} \langle b_v, b_v \rangle &= \langle S_{v_1} \cdots S_{v_a} \mathbf{1}, S_{v_1} \cdots S_{v_a} \mathbf{1} \rangle = \langle S_{v_a} \mathbf{1}, S^*_{v_{a-1}} \cdots S^*_{v_1} S_{v_1} \cdots S_{v_a} \mathbf{1} \rangle \\ &= \langle e_{v_a}, e_{v_a} \rangle = 1 \end{aligned}$$

Thus the b_v form an orthonormal system, and now we only need to establish completeness. To do this recall that

$$K(z,w) = \lim_{N \to \infty} \prod_{n=0}^{N} \left(\sum_{i \in I} e_i(R^{\circ n}(z)) \overline{e_i(R^{\circ n}(w))} \right)$$

In particular, letting $i = (i_0, ..., i_N) \in I^N$, we have for each $N \geq 0$

$$\prod_{n=0}^{N} \left(\sum_{i \in I} e_i(R^{\circ n}(z)) \overline{e_i(R^{\circ n}(w))} \right) = \sum_{i \in I^N} \left(\prod_{n=0}^{N} e_{i_n}(R^{\circ n}(z)) \overline{e_{i_n}(R^{\circ n}(w))} \right)$$

Now observe that

$$\begin{aligned} \prod_{n=0}^{N} e_{i_n}(R^{\circ n}(z)) &= S_{i_0} \left(\prod_{n=1}^{N} e_{i_n}(R^{\circ n-1}(\cdot)) \right)(z) = S_{i_0} S_{i_1} \left(\prod_{n=2}^{N} e_{i_n}(R^{\circ n-2}(\cdot)) \right)(z) \\ &= \cdots = S_{i_0} S_{i_1} \cdots S_{i_{N-1}} (e_{i_N}(\cdot)) = S_{i_0} S_{i_1} \cdots S_{i_N} \mathbf{1} = b_v \end{aligned}$$

thus we have that

$$K(z,w) = \lim_{N \to \infty} \sum_{i \in I^N} \left(\prod_{n=0}^{N} e_{i_n}(R^{\circ n}(z)) \overline{e_{i_n}(R^{\circ n}(w))} \right) = \sum_{v \in V} b_v(z) \overline{b_v(w)}$$

and so completeness follows. $\square$

3.3 Concluding Remarks

As noted earlier, this thesis is based almost entirely upon the ideas presented in this chapter. In the following chapters, we start by investigating for which rational maps the product kernel construction presented above will work. Such maps will have an associated Hilbert space: the reproducing kernel Hilbert space of the obtained product kernel. It is then natural to determine an orthonormal basis for these Hilbert spaces. Not every rational map with an associated Hilbert space is compatible with the construction given above, but it might be possible to suitably modify the construction in such cases. Lastly, we study the dynamics of orthonormal bases associated to certain families of parameter dependent rational maps.

CHAPTER 4
KERNEL FUNCTIONS FOR BASINS OF ATTRACTION

4.1 Ω is a Basin of Attraction

We begin by demonstrating that when R is a polynomial with an attracting fixed point at 0, the set Ω from Theorem 3.2 is in fact the basin of attraction of $R(z)$ containing 0.

Lemma 4.1. *If*

$$\Omega = \left\{ z \in \mathbb{C} : \sum_{n=0}^{\infty} R^{\circ n}(z)\overline{R^{\circ n}(z)} < \infty \right\} \neq \emptyset$$

then $\Omega = B_{R,0}$

Proof. Since $R^{\circ n}(z) \to 0$ as $n \to \infty$ we have immediately $\Omega \subseteq B_{R,0}$. Now suppose $z \in B_{R,0}$. Since $R^{\circ n}(z)\overline{R^{\circ n}(z)} = |R^{\circ n}(z)|^2$ we have that the tail of the series

$$\sum_{n=0}^{\infty} R^{\circ n}(z)\overline{R^{\circ n}(z)}$$

is bounded by the tail of the series

$$\sum_{n=0}^{\infty} |R^{\circ n}(z)|$$

the latter was shown to converge in Proposition A.9, and thus the former series must converge as well. Thus $B_{R,0} \subseteq \Omega$. $\square$

4.2 $\mathbf{K(z, w)}$ when $\mathbf{k(z, w) = 1 + z\overline{w}}$

Theorem 4.2. *Let* $R(z) = \sum_{i=1}^{n} c_i z^i$ *such that* $|c_1| < 1$. *Then*

$$K(z, w) = \prod_{n=0}^{\infty} \left(1 + R^{\circ n}(z)\overline{R^{\circ n}(w)}\right)$$

is a kernel function on the basin of attraction at 0.

Proof. The functions $t(z, w) = z\overline{w}$ and $k(z, w) = 1 + t(z, w)$ are kernel functions on $\mathbb{C}$. Define

$$\Omega = \left\{ z \in U : \sum_{n=0}^{\infty} |t(R^{\circ n}(z), R^{\circ n}(z))| < \infty \right\}$$

Since $0 \in \Omega$ we may conclude that Ω is non-empty. Since $\Omega = B_{R,0}$ we have for all $z \in \Omega$ that $R^{\circ n}(\zeta) \to 0$ as $n \to \infty$. Since $K(0, 0) = 1 > 0$, both Propositions 3.2 and 3.1 may be applied to obtain the result.

□

Example 4.1. For the polynomial $R(z) = z^2$, we have that 0 is an attracting fixed point, and $R^{\circ n}(z) = z^{2^n}$. Thus we have as our kernel function

$$K(z, w) = \prod_{n=0}^{\infty} \left(1 + (z\overline{w})^{2^n}\right)$$

It is worth noting that the basin of attraction in this case is just interior of the unit circle. In fact, one can show that this is the well known Szegö kernel.

Example 4.2. Suppose $R(z) = \lambda z(1 - z)$ with $|\lambda| < 1$. It is clear that 0 is an attracting fixed point, thus Theorem 4.2 applies. Combining Proposition A.8 with the following corollary, we obtain a product kernel for any quadratic with a finite attracting fixed point.

Combining the previous theorem with proposition 3.3 we achieve

Corollary 4.3. *Suppose R and S are rational maps satisfying*

$$S(z) = \sum_{i=1}^{n} c_i z^i = g \circ R \circ g^{-1}(z)$$

where $|c_1| < 1$ and g is a Möbius map. Then

$$K(z,w) = \prod_{n=0}^{\infty} (1 + S^{\circ n}(g(z))\overline{S^{\circ n}(g(w))})$$

is a kernel function on $B_{R,g^{-1}(0)}$.

Proof. This follows from applying Proposition 2.2 to the previous theorem. □

Example 4.3. Suppose in the above Corollary, $R(z)$ is itself a polynomial with a finite attracting fixed point ζ. If we conjugate $R(z)$ with the Möbius transformation $g(z) = z - \zeta$ we obtain a polynomial S with an attracting fixed point at $g(\zeta) = 0$. Applying Corollary 4.3 we have that

$$K(z,w) = \prod_{n=0}^{\infty} (1 + S^{\circ n}(z-\zeta)\overline{S^{\circ n}(w-\zeta)})$$

is a kernel function on $B_{R,\zeta}$. A particular case worth mentioning is when $R(z) = z^2 + c$ where $c \in \mathbb{C}$ is chosen so that R has a finite attracting fixed point. If ζ is the attracting fixed point of R then we conjugate R by $g(z) = z - \zeta$, giving the map $S(z) = g \circ R \circ g^{-1}(z) = z^2 + 2\zeta z$. Since $S(z)$ has an attracting fixed point at 0 we may apply Corollary 4.3 to obtain a kernel function on $B_{R,\zeta}$.

Example 4.4. Suppose $R(z)$ is a polynomial with an attracting n-cycle $\zeta = \{\zeta_1, ..., \zeta_n\}$. Then $R^{\circ n}$ has an attracting fixed point at each ζ_i. We can conjugate $R^{\circ n}$ by $g(z) = z - \zeta_i$ to obtain a polynomial $S(z)$ with an attracting fixed point at $g(\zeta_i) = 0$. Thus we may apply Corollary 4.3 to obtain a kernel function on $B_{R^{\circ n},\zeta_i}$. In the next section we will see how we can piece these kernel functions together to get a kernel function on $B_{R,\zeta}$.

4.3 Pulling Back with R

We now explore how one can use powers of R to pull-back to a kernel function on the basin of attraction of R containing an attracting cycle.

Theorem 4.4. *Suppose R is a polynomial with an attracting N-cycle $\zeta = \{\zeta_1, ..., \zeta_N\}$ with $\zeta_N = 0$, and let $S = R^{\circ N}$. If $R(B_{S,\zeta_i}) = B_{S,\zeta_{i+1}}$ for $i \neq N$ and $B_{R,\zeta} = \bigcup_{i=1}^{N} B_{S,\zeta_i}$, then*

$$K(z,w) = \prod_{n=0}^{\infty} \left(1 + S^{\circ n}(M(z))\overline{S^{\circ n}(M(w))}\right)$$

is a kernel function on $B_{R,\zeta}$, where $M(z)$ is given in the proof below.

Proof. Since 0 is an attracting fixed point of S, we have that

$$K_0(z,w) = \prod_{n=0}^{\infty} \left(1 + S^{\circ n}(z)\overline{S^{\circ n}(w)}\right)$$

is a kernel function on $B_{S,0}$. Now define $M : B_{R,\zeta} \to B_{S,0}$ by

$$M(z) = \begin{cases} z & \text{for } z \in B_{S,0} \\ R(z) & \text{for } z \in B_{S,\zeta_{N-1}} \\ R^{\circ 2}(z) & \text{for } z \in B_{S,\zeta_{N-2}} \\ \vdots & \vdots \\ R^{\circ N-1}(z) & \text{for } z \in B_{S,\zeta_1} \end{cases}$$

Then we may pull-back along M to a kernel function on $B_{R,\zeta}$ given by

$$K(z,w) = K_0(M(z), M(w))$$

□

Example 4.5. Consider the polynomial $R(z) = z^2 - 1$. It has an attracting cycle $\zeta = \{0, -1\}$. Thus $S(z) = R^{\circ 2}(z)$ has an attracting fixed point at both 0 and -1. Furthermore $R(B_{S,-1}) = B_{S,0}$ and $B_{R,\zeta} = B_{S,-1} \cup B_{S,0}$. Thus we have that

$$K(z,w) = \begin{cases} \prod_{n=0}^{\infty} \left(1 + R^{\circ 2n}(z)\overline{R^{\circ 2n}(w)}\right) & \text{for } z \in B_{S,0}\ w \in B_{S,0} \\ \prod_{n=0}^{\infty} \left(1 + R^{\circ 2n}(z)\overline{R^{\circ 2n+1}(w)}\right) & \text{for } z \in B_{S,0}\ w \in B_{S,-1} \\ \prod_{n=0}^{\infty} \left(1 + R^{\circ 2n+1}(z)\overline{R^{\circ 2n}(w)}\right) & \text{for } z \in B_{S,-1}\ w \in B_{S,0} \\ \prod_{n=0}^{\infty} \left(1 + R^{\circ 2n+1}(z)\overline{R^{\circ 2n+1}(w)}\right) & \text{for } z \in B_{S,-1}\ w \in B_{S,-1} \end{cases}$$

is a kernel function on $B_{R,\zeta}$. This kernel function has the desirable property that when restricted to $B_{S,0}$ we recover the original kernel function on $B_{S,0}$.

4.4 $\mathbf{K(z,w)}$ when $\mathbf{k(z,w) = 1 + (z\overline{w})^{\alpha}}$

Ultimately we will want to find an orthonormal basis for the associated Hilbert space of K. For some polynomials we can make the construction from chapter 3 work if we change the initial kernel function k. What we present now is a generalization of Theorem 4.2.

Theorem 4.5. *Let $R(z) = \sum_{i=1}^{n} c_i z^i$ such that $|c_1| < 1$ and suppose $1 < \alpha \in \mathbb{N}$. Then*

$$K(z,w) = \prod_{n=0}^{\infty} \left(1 + \left[R^{\circ n}(z)\overline{R^{\circ n}(w)}\right]^{\alpha}\right)$$

is a kernel function on the basin of attraction at 0.

Proof. This is essentially the same argument given in Theorem 4.1 with a few modifications to our initial kernel functions. Let $t(z,w) = (z\overline{w})^{\alpha}$ and $k(z,w) = 1 + t(z,w)$.

Defining

$$\Omega = \left\{ z \in U : \sum_{n=0}^{\infty} |t(R^{\circ n}(z), R^{\circ n}(z))| < \infty \right\}$$

we have that $0 \in \Omega$ and $K(0,0) = 1 > 0$ so that $K(z,w)$ converges to a kernel function on Ω. Clearly $\Omega \subseteq B_{R,0}$. Lastly we can show that the tail of the above series is bounded by the tail of the series in Proposition A.9 just as we did in Theorem 4.1. Hence $B_{R,0} \subseteq \Omega$. $\square$

Applying Proposition 2.2 we obtain a generalization of Corollary 4.3.

Corollary 4.6. *Let $1 < \alpha \in \mathbb{N}$, and suppose that R and S are rational maps satisfying*

$$S(z) = \sum_{i=1}^{n} c_i z^i = g \circ R \circ g^{-1}(z)$$

where $|c_1| < 1$ and g is a Möbius map. Then

$$K(z,w) = \prod_{n=0}^{\infty} \left(1 + \left[S^{\circ n}(g(z))\overline{S^{\circ n}(g(w))}\right]^{\alpha}\right)$$

is a kernel function on $B_{R,g^{-1}(0)}$.

CHAPTER 5
ORTHONORMAL BASES

5.1 An Example Satisfying ‡

Finding an orthonormal basis for the associated reproducing kernel Hilbert space seems to be a rather daunting task. The construction in Chapter 3 seems to work for very few rational functions. In [1], the construction was demonstrated with the polynomial $R(z) = z^4 - 2z^2$, which satisfies the condition ‡. It is worth outlining this particular case as we will use the same approach for a class of similar examples. Taking $k(z, w) = 1 + z\overline{w}$ we have that $e_1(z) = 1$ and $e_2(z) = z$ form an orthonormal basis for the reproducing kernel Hilbert space of k. Thus the condition ‡ that R needs to satisfy in order to apply Theorem 2.17 becomes: for all $z \in B_{R,0}$,

$$\frac{1}{M(z)} \sum_{R(\zeta)=z} 1 = 1, \qquad \frac{1}{M(z)} \sum_{R(\zeta)=z} \zeta^2 = 1, \qquad \frac{1}{M(z)} \sum_{R(\zeta)=z} \zeta = 0$$

where $M(z) = \mathrm{Card}\{\zeta \in \Omega : R(\zeta) = z\} < \infty$. Since $B_{R,0}$ is completely invariant, the fundamental theorem of algebra tells us that $M(z) = 4$. Thus the first equality holds. To see that the third equality holds, simply apply Vieta's formula for the sum of the roots of a polynomial. To verify the second equality, consider the roots of $u^2 - 2u - z$. If α is a root, then $\alpha = 1 \pm \sqrt{1+z}$. Thus $\zeta^2 = \alpha$, which implies the second equality. Therefore by Theorem 3.4 we may conclude that the operators S_0 and S_1 satisfy the Cuntz relations. Hence, Theorem 3.5 gives us an orthonormal basis for the reproducing kernel Hilbert space associated to R through Theorem 4.2.

5.2 Quadratics Satisfying Neither † Nor ‡

Observe that the conditions † and ‡ depend on the kernel function $k(z, w)$. Thus for those rational maps which satisfy neither condition under the kernel function $1 + z\overline{w}$, one may try using some other kernel function. The issue with changing the kernel function arbitrarily is that we would need to verify whether Ω is still $B_{R,0}$. We now explore which polynomials satisfy conditions † or ‡ under the kernel function $1 + z\overline{w}$. By the design of Ω it is clear that we should restrict ourselves to polynomials with an attracting fixed point at 0.

Proposition 5.1. *Suppose $R(z) = az^2 + bz$ with $|b| < 1$. Then R satisfies neither † nor ‡*

Proof. Suppose instead that R satisfies either † or ‡. Then in particular, for any $z \in \Omega$ we have

$$\frac{1}{2} \sum_{R(\zeta)=z} \zeta = 0$$

Let $w \in \Omega$ and consider the equation $a\zeta^2 + b\zeta = w$. Vieta's formula tells us immediately that $b = 0$. Therefore $\zeta^2 = \dfrac{w}{a}$. Since R has only two roots, we see that the sum of their squares is $\dfrac{2(w)}{a} = 2$. Since a is fixed, Ω must be the singleton $\{a\}$. But Ω being a basin of attraction cannot be a singleton, so ‡ cannot be satisfied. Similarly, since $|\zeta|^2 = \left|\dfrac{w}{a}\right|^2$, condition † cannot be satisfied either; a contradiction. □

5.3 Classes of Polynomials Satisfying ‡

The situation does seem to improve if we increase the degree of our polynomial.

Proposition 5.2. *The only quartics of the form $az^4+bz^3+cz^2$ that satisfy ‡ are $R(z) = az^4 - 2az^2$ where $a \neq 0$.*

Proof. In order for the roots to sum to zero, Vieta's formula requires that $b = 0$. Let $w \in \Omega$ and consider the equation $a\zeta^4 + c\zeta^2 = w$. Solving for ζ^2 we get

$$\zeta^2 = \frac{-c \pm \sqrt{c^2 + 4aw}}{2a}$$

Therefore

$$\frac{1}{4} \sum_{R(\zeta)=w} \zeta^2 = \frac{-c}{2a}$$

Thus the sum will be equal to 1 whenever $c = -2a$. So $R(z) = az^4 - 2az^2$ as desired. Finally, that

$$\frac{1}{4} \sum_{R(\zeta)=w} 1 = 1$$

follows directly from the fact that Ω is completely invariant and the fundamental theorem of algebra. $\square$

Since we can construct an orthonormal basis for the Hilbert spaces associated to the family of polynomials $\{az^4 - 2az^2 : a \neq 0\}$ we will be able to study the dynamics of the basis vectors as the parameter a varies. We can actually do this for higher degree polynomials of a similar form, provided we change the kernel function k suitably, as the next proposition shows.

Proposition 5.3. *Let $k(z,w) = 1 + (z\overline{w})^{2^n}$ where n is a non-negative integer. Then $R(z) = az^{2^{n+2}} - 2az^{2^{n+1}}$ satisfies ‡ for all $a \neq 0$.*

Proof. We need to verify that the following equalities hold for any $w \in \Omega$

$$\frac{1}{M(w)} \sum_{R(\zeta)=w} 1 = 1, \qquad \frac{1}{M(w)} \sum_{R(\zeta)=w} \zeta^{2^{n+1}} = 1, \qquad \frac{1}{M(w)} \sum_{R(\zeta)=w} \zeta^{2^n} = 0$$

where $M(w) = \text{Card}\{w \in \Omega \ : R(\zeta) = w\} < \infty$. That $M(w) = 2^{n+2}$ follows from the fundamental theorem of algebra and that Ω is completely invariant with respect to R. Thus the first equality follows. Recall that if $au^2 - 2au - w = 0$ then

$$u = 1 \pm \frac{\sqrt{a^2 + aw}}{a}$$

So we have that

$$\zeta^{2^{n+1}} = 1 \pm \frac{\sqrt{a^2 + aw}}{a}$$

which implies the second equality. Taking the square root of both sides of the previous equality gives

$$\zeta^{2^n} = \pm\sqrt{1 \pm \frac{\sqrt{a^2 + aw}}{a}}$$

which implies the final equality. $\square$

CHAPTER 6
DYNAMICS OF BASIS VECTORS

6.1 A Family With Well-Behaved Dynamics

Consider any family of polynomials with the property that each member of the family has a finite attracting fixed point. Assume that for each polynomial in this family, we are able to obtain an orthonormal basis for the associated reproducing kernel Hilbert space. It is natural then to investigate the dynamics of the basis vectors as the members of the family varies. As our first example we will consider the family

$$\mathcal{F} = \{R_a(z) = a(z^4 - 2z^2) : a \in \mathbb{C}/\{0\}\}$$

We have already demonstrated that every polynomial, R_a, of the family is associated to a kernel function of the form

$$K_a(z, w) = \prod_{i=0}^{\infty} \left(1 + R_a^{\circ i}(z)\overline{R_a^{\circ i}(w)}\right)$$

Furthermore this family was shown to have satisfied the conditions necessary to construct an orthonormal basis for the associated reproducing kernel Hilbert space. In order to study the dynamics of the basis vectors, it will help to understand how they look as functions. To do this we note the setting we are in. We have that $e_0(z) = 1$, $e_1(z) = z$, and $S_i(f)(z) = e_i(z)f(R_a(z))$ where $i \in \{0, 1\}$. Thus the basis vectors b_v are of the form $S_{i_0} \cdots S_{i_N}\mathbf{1}$ where $N \in \mathbb{N}$ and $i_j \in \{0, 1\}$. Observe that in an exhaustive list of such compositions, each distinct basis vector will be repeated infinitely many times. We can actually say much more about how the b_v look.

Proposition 6.1. *Suppose v is chosen so that $b_v(z) \neq \mathbf{1}(z)$ and $b_v(z) \neq z$. If $R_a(z) = az^4 - 2az^2$ then the remaining associated basis vectors b_v are polynomials with non-zero coefficients in non-constant polynomials of a*

Proof. Observe that $S_0\mathbf{1}(z) = 1$ and $S_1\mathbf{1}(z) = z$. The next pair of operators one obtains which are distinct from the previous two are given by $R_a(z)$ and $zR_a(z)$. The shortest composition for each is $S_0S_1\mathbf{1}$ and $S_1S_1\mathbf{1}$, both of which are polynomials with non-zero coefficients in non-constant polynomials of a. If f is a polynomial with non-zero coefficients in non-constant polynomials of a, then so too is $S_0f = f(R_a(z))$ and $S_1f = zf(R_a(z))$. Therefore by induction, the b_v are polynomials with non-zero coefficients in non-constant polynomials of a. That is each b_v not identically 1 nor z is of the form

$$\sum_{i=0}^{n} \alpha_i z^i$$

where α_i is either 0 or a non-constant polynomial in a. □

Remark. If $\alpha_0 \neq 0$, then $b_v = \mathbf{1}$. This is because R_a has constant term 0, and the operator S_1 increases the power of each non-zero term by at least 1. Although α_i are polynomials in a, it is difficult to precisely determine each α_i. To do this, one could try solving the corresponding Böttcher functional equation. Fortunately, knowing that the α_i are polynomials in a will suffice.

Because of how the b_v are computed, we obtain the following corollary.

Corollary 6.2. *Suppose that the polynomials R_{a_1} and R_{a_2} are in $\mathcal{F}$ and let*

$$\beta_i(x) = \sum_{l=0}^{k_i} c_{i_l} x^l$$

$$\textit{Then } b_{v,a_1}(z) = \sum_{i=0}^{n} \beta_i(a_1) z^i \textit{ if and only if } b_{v,a_2}(z) = \sum_{i=0}^{n} \beta_i(a_2) z^i$$

Theorem 6.3. *For each $v \in J^{\infty}$, the map Γ_v on $\mathbb{C}/\{0\}$ given by $a \mapsto b_{v,a}$ is continuous.*

Proof. Let a_n be a sequence in $\mathbb{C}/\{0\}$ converging to $a \in \mathbb{C}/\{0\}$. Then we have

$$\lim_{n\to\infty} \Gamma_v(a_n) = \lim_{n\to\infty} b_{v,a_n} = b_{v,a} = \Gamma(a)$$

Since the coefficients of b_v is given by the polynomial $\beta_i(x)$ which is continuous, we have that Γ_v is continuous as well. $\square$

Remark. What this theorem tells us is that the basis elements associated to the polynomial R_a depend continuously on the parameter a. That is we can approximate the basis vectors associated to R_a with the basis vectors associated to R_b by choosing b suitably close to a.

With this result one might conjecture that a similar result holds for polynomials of the form $az^{2^{n+2}} - 2az^{2^{n+1}}$ provided we use the kernel function $k(z, w) = 1 + (z\overline{w})^{2^n}$. Although a bit more clunky, the same arguments should prevail. Examining the dynamics of basis vectors for other families of rational maps is a more daunting task. As shown in Proposition 5.1, there are no quadratics for which the orthonormal basis construction developed in Chapter 3 will work. Therefore, commonly studied families of quadratics such as the logistic maps $\lambda z(1 - z)$ remain elusive to the methods developed in Chapter 3.

CHAPTER 7
OPEN QUESTIONS

7.1 Extending Both Constructions

Here we discuss a few problems which might merit further investigation.

- There exist rational maps with attracting fixed points which are not conjugate to any polynomial. Consider for example $R(z) = 1 + \frac{1}{z^2}$, which has an attracting fixed point at $\zeta \approx 1.47$. We can conjugate $R(z)$ to a rational map with an attracting fix point at 0. Thus the conditions of Theorem 3.2 are satisfied for this new rational map. Is it the case that Ω is the basin of attraction? What about for other rational maps, with a finite attracting fixed point, not conjugate to any polynomial? Another issue one needs to keep in mind is that ∞ can be in the basin of attraction containing a finite attracting fixed point for such rational maps.

- How can one construct an orthonormal basis for the Hilbert spaces obtained via Theorem 3.2 for rational maps satisfying neither of the conditions † nor ‡ under the kernel $k(z, w) = 1 + z\overline{w}$? Is it possible in general to alter $k(z, w)$ so that one of the conditions † or † hold? We have seen this to be the case with rational maps of the form $az^{2^{n+2}} - 2z^{2^{n+1}}$.

- Given a rational map with a finite attracting fixed point, we can conjugate it to a rational map with an attracting fixed point at 0. Suppose we are able to find an orthonormal basis associated to the new rational map. How is this orthonormal basis related to the orthonormal basis on the original rational map? We know how the kernel functions are related. In fact, knowing one kernel function allows us to pull-back to the other kernel function. Thus in particular, if we know the orthonormal basis associated to a given rational map, can we determine the orthonormal basis associated to one of its conjugates?

- Given a family of rational maps for which Theorem 3.2 applies. Assuming we are able to construct the associated orthonormal bases, what are the dynamics of the orthonormal bases as the family member varies? A particular family which would be interesting to study is the family $\{z^2 + c : c \in \mathbb{C}\}$. Other families of interest would be the polynomials with an attracting fixed point at 0, the rational maps, with an attracting fixed point at 0, not conjugate to any polynomial, and of course the rational maps in general.

APPENDIX A
COMPLEX DYNAMICS

We present here the basic definitions and theory of complex dynamics which we will require through out this thesis. Our focus in particular will be on attracting fixed points and their basins of attraction. Unless stated otherwise, no claim to originality is made in this appendix, nor is it to be considered a complete introduction to complex dynamics.

A.1 Partitioning the Complex Sphere: The Fatou and Julia Sets

By the Uniformization Theorem of Poincaré and Koebe, every simply connected Riemann surface is conformally isomorphic to either the complex plane, the open unit disk, or the Riemann sphere. For each case one may opt to study the dynamics of iterated holomorphic maps. Since this thesis pays special attention to polynomials, we will outline the theory surrounding the dynamics of iterated holomorphic mappings on the Riemann sphere. We choose the Riemann sphere over the complex plane because a polynomial viewed as a holomorphic map on the complex plane can be uniquely extended to a holomorphic map on the Riemann sphere. Furthermore, the analytic maps on the Riemann sphere coincide with the rational maps. Thus the dynamics of iterated polynomials on the complex plane can be viewed as a special case of the dynamics of iterated rational maps on the Riemann sphere. Lastly, many of the results for polynomials can be easily extended to rational maps.

Definition A.1. Suppose (X, d_x) and (Y, d_y) are metric spaces. A family of maps $\{f_i :$

$X \to Y\}_{i \in I}$ is said to be *equicontinuous* at $x_0 \in X$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that $d_x(x, x_0) < \delta$ implies $d_y(f_i(x), f_i(x_0)) < \epsilon$ for all $i \in I$. A family of maps is said to be *equicontinuous on a set* if it is equicontinuous at each point of the set.

Theorem A.1. *Given a family of maps F on a set X, there exists a unique maximal open subset of X for which the family is equicontinuous.*

Proof. The family of maps F is vacuously equicontinuous on the empty set. Let M consist of every open subset of X for which F is equicontinuous, and set

$$U = \bigcup_{V \in M} V$$

By construction, U is open and F is equicontinuous on U. Furthermore, U contains any open set for which F is equicontinuous. Thus U is the unique maximal set for which F is equicontinuous. Any set contradicting this proposition would have to belong to M, implying containment by U. $\square$

Our metric of choice for the Riemann sphere will be the *chordal metric*:

Definition A.2. If z and w are complex numbers in the Riemann Sphere, then the chordal metric is given by

$$d_c(z, w) = \frac{|z - w|}{(|z|^2 + 1)^{\frac{1}{2}}(|w|^2 + 1)^{\frac{1}{2}}}$$

When z and w are finite complex numbers, we have $d_c(z, w) < |z - w|$. Thus convergence properties in $\mathbb{C}$ can be used to determine convergence properties in $\mathbb{C}_\infty/\{\infty\}$. It will be useful to note the following Lipschitz property of the chordal metric:

Proposition A.2. *Suppose R is a rational map on the Riemann sphere. For some non-negative real number M, the rational map R satisfies the Lipschitz condition*

$$d_c(R(z), R(w)) \leq M d_c(z, w)$$

for all complex numbers, z and w, on the Riemann sphere.

With the Riemann sphere as our metric space, Theorem A.1 allows us to partition the Riemann sphere in a way that depends on the choice of a rational map.

Definition A.3. Let R be a rational map on the complex sphere. Consider the family of iterates $\{R^{\circ n}\}_{n=1}^{\infty}$, where $R^{\circ n}$ denotes R composed with itself n times. The *Fatou set of R*, denoted $F(R)$ is the maximal subset of the complex sphere for which the family of iterates is equicontinuous. The *Julia set of R*, denoted $J(R)$, is the complement of the Fatou set of R.

Intuitively, the Julia set is comprised of the points whose neighbors' behavior under the repeated iteration of R is chaotic while the Fatou set is comprised of the points whose neighbors' behavior under the repeated iteration of R is predictable. It is quite convenient when the study of a class of objects can be reduced to the study of a subclass of those objects. Fortunately, this is true when it comes to the study of the Julia and Fatou sets. This can be achieved through the notion of conjugacy of rational maps.

A.2 Conjugacy and Fixed Points

Definition A.4. A *Möbius transformation* is a rational map of the form $\dfrac{az+b}{cz+d}$. Two rational maps R and S are said to be *conjugate* if there exists a Möbius transformation

g such that $R = g \circ S \circ g^{-1}$.

Proposition A.3. *Let R and S be non-constant conjugate rational maps; that is there exists a Möbius transformation g such that $S = g \circ R \circ g^{-1}$. Then $F(S) = g(F(R))$ and $J(S) = g(J(R))$.*

Proof. Let $\epsilon > 0$ and suppose $x_0 \in g(F(R))$, then $g^{-1}(x_0) \in F(R)$. Thus by definition of the Fatou set, there exists $\delta > 0$ such that $|g^{-1}(x_0) - g^{-1}(x)| < \delta$ implies

$$\left|R^{\circ n} \circ g^{-1}(x_0) - R^{\circ n} \circ g^{-1}(x)\right| < \frac{\epsilon}{\|g\|} \qquad \forall n \in \mathbb{N}$$

By conjugation and the Lipschitz condition on Möbius transformations,

$$|S^{\circ n}(x_0) - S^{\circ n}(x)| \leq \|g\| \left|R^{\circ n} \circ g^{-1}(x_0) - R^{\circ n} \circ g^{-1}(x)\right| = \epsilon \qquad \forall n \in \mathbb{N}$$

Thus $x_0 \in F(S)$, giving that $g(F(R)) \subseteq F(S)$. Since $R = g^{-1} \circ S \circ g$, following the same line of reasoning above yields $g^{-1}(F(S)) \subseteq F(R)$. The Möbius transformations g and g^{-1} are bijective, so we have that $F(S) \subseteq g(F(R))$; from which we conclude $F(S) = g(F(R))$. Finally, since the Julia set is the complement of the Fatou set, we also obtain $J(S) = g(J(R))$. □

A similar proposition states that the iterates of a rational map share the same Julia and Fatou sets.

Proposition A.4. *Suppose R is a non-constant rational map. Then $F(R) = F(R^{\circ n})$ and $J(R) = J(R^{\circ n})$ for any choice of $n \in \mathbb{N}$.*

Proof. Letting $S = R^{\circ n}$ for some $n \in \mathbb{N}$, it is clear that the family of maps $\{S^{\circ k}\}_{k=0}^{\infty}$ is a subset of the family of maps $\{R^{\circ k}\}_{k=0}^{\infty}$. Thus $F(S) \subseteq F(R)$ since the former family is equicontinuous anywhere the latter family is. For the reverse inclusion we make use of the fact that rational maps satisfy a Lipschitz condition.

□

Definition A.5. Suppose R is a rational map. A fixed point of R is a complex number ζ satisfying $R(\zeta) = \zeta$. The multiplier of a finite fixed point, ζ, is given by $|R'(z)|$. A fixed point is called

- attracting if the multiplier is less than 1
- repelling if the multiplier is greater than 1
- indifferent if the multiplier is equal to 1

Remark. The notion of fixed point can be generalized to an n-cycle. An n-cycle $\{\zeta_0, ..., \zeta_{n-1}\}$ satisfies the property that $\zeta_i = R^i(\zeta_0)$ and $\zeta_0 = R^n(\zeta_0)$. The multiplier of the cycle is given by $R^{n'}(\zeta_i)$ which is independent of the choice of ζ_i. Each result presented here about fixed points correspond to an analogous result about cycles.

Our focus will be on those rational maps with an attracting fixed point. As the name suggests, if z is a complex number close enough to the fixed point, then $R^{\circ n}(z)$ moves progressively closer to the fixed point as the number of iterates n increases. In particular, we will study those rational maps which are conjugate to a polynomial with an attracting fixed point at 0. As such it is useful to note the form of such a polynomial:

Lemma A.5. *If $P(z)$ is a degree n polynomial with an attracting fixed point at 0, then*

$$P(z) = \sum_{i=1}^{n} c_i z^i$$

where $|c_1| < 1$.

Proof. Since $P(0) = 0$, we see there can be no constant term. That $|c_1| < 1$ follows from the fact that 0 is an attracting fixed point. □

To each attracting fixed point corresponds a basin of attraction consisting of all complex numbers "attracted" to the fixed point.

Definition A.6. Suppose R is a rational map. The basin of attraction of R containing the attracting fixed point ζ, denoted by $B_{R,\zeta}$, is given by

$$B_{R,\zeta} = \{z \in \mathbb{C}_\infty : \lim_{n\to\infty} R^{\circ n}(z) = \zeta\}$$

Basins of attraction can be described in terms of conjugates just as Fatou and Julia sets were.

Lemma A.6. *Suppose $S = g \circ R \circ h$ where g is a Möbius transformation and $h = g^{-1}$; R and S rational. If c is a fixed point of R then $d = g(c)$ is a fixed point of S, and $S'(d) = R'(c)$*

Proof. Clealry, $S(d) = d$ by conjugacy. Now by the chain rule and inverse function theorem

$$\begin{aligned} S'(d) &= g'(R \circ h(d)) \cdot R'(h(d)) \cdot h'(d) \\ &= g'(R(c)) \cdot R'(c) \cdot \frac{1}{g'(c)} \\ &= R'(c) \end{aligned}$$

□

Remark. Since the multiplier is preserved under conjugacy, we can define the multiplier of ∞ to be the multiplier of any of its finite conjugates.

Under the previous hypotheses we have:

Proposition A.7. *If $R'(c) < 1$ then $B_{S,d} = g(B_{R,c})$*

Proof. Suppose $w \in g(B_{R,c})$. Then there exists $z \in B_{R,c}$ such that $w = g(z)$. We have

$$S^n(w) = g \circ R^n \circ h(w) = g \circ R^n(z)$$

Since $\lim_{n\to\infty} R^n(c) = c$, we have that $\lim_{n\to\infty} S^n(w) = d$. So $w \in B_{S,d}$ implies that $g(B_{R,c}) \subseteq F_d$.

Now suppose $w \in B_{S,d}$. Set $z = h(w)$. Using a similar argument we find that $z \in B_{R,c}$. Thus $g(z) = w$, which implies that $w \in g(B_{R,c})$, so $B_{S,d} \subseteq g(B_{R,c})$. We conclude that $B_{S,d} = g(B_{R,c})$. □

Given a rational map with a finite fixed point, we will often want to consider instead a conjugate rational map with the corresponding fixed point at 0. This is easily done by conjugating the rational map with a Möbius transformation of the form $z - \zeta$, where ζ is the finite fixed point of the original rational map. If R is our original rational map, then the conjugate map we desire is $Q(z) = R(z + \zeta) - \zeta$. In the case of the quadratic polynomials, there are two particularly useful conjugates one may obtain:

Proposition A.8. *Suppose $R(z) = az^2 + 2bz + c$. Then R is conjugate to a map of the form $z^2 + d$ and also a map of the form $\lambda z(1 - z)$.*

Proof. Let $g(z) = az + b$, $\gamma = b^2 - b + c$, and $\beta = \sqrt{1 - 4\gamma}$; it is easy to verify that $R(z)$ is conjugate to $z^2 + \gamma$ with respect to g. Similarly, one can verify with enough ambition that $R(z)$ is conjugate to $\lambda z(1 - z)$, where

$$\lambda = \frac{2(1 - \beta - 2(\gamma))}{1 + \beta}$$

and with respect to the Möbius transformation

$$h(z) = \left(\frac{a}{1 + \beta}\right) z - \frac{2b - 1 - \beta}{2 + 2\beta}$$

□

We conclude with an alternative description of basins of attraction of polynomials containing an attracting fixed point at 0. Although readily derived from classic results within the literature, see for example [21], the description itself seems absent. We feel the need to mention it here since it will be of great use to us.

Proposition A.9. *Let P be a polynomial with an attracting fixed point at 0. The basin of attraction of P containing 0 is given by*

$$B_{P,0} = \Omega := \{z \in \mathbb{C} : \sum_{j=1}^{\infty} |P^{\circ j}(z)| < \infty\}$$

Proof. If $z \in \Omega$, then the convergence of the sum requires that $\lim_{j\to\infty} P^{\circ j}(z) = 0$. Thus by definition, $z \in B_{P,0}$. Now suppose $z \in B_{P,0}$ and let $z_j = P^{\circ j}(z)$. We demonstrate convergence by the ratio test:

$$\lim_{j\to\infty} \left| \frac{P^{\circ j+1}(z)}{P^{\circ j}(z)} \right| = \lim_{j\to\infty} \left| \frac{P(z_j)}{z_j} \right| = \lim_{j\to\infty} \left| \sum_{i=2}^{n} c_i z_n^i + c_1 \right| = |c_1| < 1$$

The second equality follows from Lemma A.5, while last equality follows from the fact that $z_n \to 0$ as $n \to \infty$. □

Remark. Although the above result seems quite restrictive, one must remember that we can use it to obtain information on the conjugacy classes of each such polynomial.

APPENDIX B
REPRODUCING KERNEL HILBERT SPACES

Here we present a brief outline to the theory of reproducing kernel Hilbert spaces. No claim to originality is made, and important details can be found in the classic reference [5]. We begin by briefly surveying some prequisites from the theory of Hilbert spaces and linear functionals.

B.1 Hilbert Spaces and Linear Functionals

Definition B.1. An inner product space $\mathcal{S}$ is a vector space over the complex numbers with a map $G : \mathcal{S} \times \mathcal{S} \to \mathbb{C}$ satisfying for all vectors $x, y, z \in \mathcal{S}$ and all complex numbers $a \in \mathbb{C}$:

1. $G(x, y) = \overline{G(y, x)}$
2. $G(ax, y) = aG(x, y)$
3. $G(x + y, z) = G(x, z) + G(y, z)$
4. $G(x, x) \geq 0$ where equality holds if and only if $x = 0$

Such a map is called an inner product and is typically written $\langle x, y\rangle := G(x, y)$.

Definition B.2. A Hilbert space is a complete inner product space.

It is well known that every inner product space can be completed to form a Hilbert space.

Theorem B.1. *Suppose $\mathcal{S}$ is an inner product space. There exists a Hilbert space $\mathcal{H}$ and an injective linear map $\Phi : \mathcal{S} \to \mathcal{H}$ satisfying*

1. $\langle \Phi(x), \Phi(y) \rangle_{\mathcal{H}} = \langle x, y \rangle_{\mathcal{S}}$ *for all vectors* $x, y \in \mathcal{S}$.

2. $\Phi(\mathcal{S})$ *is dense in* $\mathcal{H}$.

Remark. Since $\mathcal{S}$ is in a one-to-one correspondence with the subset $\Phi(\mathcal{S})$ of $\mathcal{H}$, it will be convenient to identify $\mathcal{S}$ with $\Phi(\mathcal{S})$.

Theorem B.2. *If* $\mathcal{H}$ *is a Hilbert space, then* $\mathcal{H}$ *has an orthonormal basis. Furthermore, if* $\{e_i\}_{i \in I}$ *is an orthonormal basis of* $\mathcal{H}$*, then every vector* $h \in \mathcal{H}$ *satisfies the Parseval identity:*

$$h = \sum_{i \in I} \langle h, e_i \rangle e_i$$

We now shift our attention to linear functionals.

Definition B.3. Let X be a vector space over $\mathbb{C}$. A *linear functional* is a map $l : X \to \mathbb{C}$ satisfying for all $x, y \in X$ and for all complex numbers c.

$$\begin{aligned} l(x+y) &= l(x) + l(y) \\ l(cx) &= cl(x) \end{aligned}$$

A linear functional is *continuous* if $\lim_{n\to\infty} |x_n - x| = 0$ implies $\lim_{n\to\infty} l(x_n) = l(x)$; and is *bounded* if there exists a positive constant c such that $|l(x)| < c|x|$ holds for all $x \in \mathbb{C}$.

It turns out that when X is a Hilbert space, every bounded (or equivalently continuous) linear functional is in fact a restriction of the corresponding inner product.

Theorem B.3 (Riesz-Frechet Representation Theorem)**.** *If* $\mathcal{H}$ *is a Hilbert space and* $l : \mathcal{H} \to \mathbb{C}$ *is a bounded linear functional, then there exists a unique* $h \in \mathcal{H}$ *such that*

$$l(g) = \langle g, h \rangle$$

for all $g \in \mathcal{H}$.

B.2 The Reproducing Property

We now introduce a particular type of Hilbert space whose linear evaluation functionals are bounded. Such Hilbert spaces satisfy a reproducing property which has found widespread application in many disciplines.

Definition B.4. Let X be a set. A *reproducing kernel Hilbert space*, $\mathcal{H}$, on X is a Hilbert space consisting of complex functions on X such that all linear evaluation functionals, $L_x : \mathcal{H} \to \mathbb{C}$ defined by $L_x(f) = f(x)$, are bounded.

Proposition B.4. *Let X be a set. If $\mathcal{H}$ is a reproducing kernel Hilbert space on X, then there exists a map $K : X \times X \to \mathbb{C}$ satisfying $f(y) = \langle f(x), K(x,y)\rangle$ for all $y \in X$. This map is called the kernel function of $\mathcal{H}$.*

Proof. Consider the linear evaluation functional $L_y : \mathcal{H} \to \mathbb{C}$, where $y \in X$, given by $L_y(f) = f(y)$. By definition of reproducing kernel Hilbet space, L_y is bounded. Thus the Riesz-Frechet representation theorem guarantees the existence of a unique vector $k_y \in \mathcal{H}$ such that

$$f(y) = L_y(f) = \langle f, k_y\rangle$$

Hence we may define a function $K : X \times X \to \mathbb{C}$ by $K(x,y) = k_y(x)$, giving us the desired equality. $\square$

Example B.1. Consider the unit disk $\mathbb{D}$ and consider functions on $\mathbb{D}$ of the form

$$f(z) = \sum_{i=0}^{\infty} a_i z^i$$

this space, denoted $H^2(\mathbb{D})$, is called the Hardy space on the unit disk. It can be made into a Hilbert space by endowing it with the inner product

$$\left\langle \sum_{i=0}^{\infty} a_i z^i, \sum_{i=0}^{\infty} b_i z^i \right\rangle = \sum_{i=0}^{\infty} a_i \overline{b_i}$$

By the Cauchy-Schwarz inequality, every linear evaluation functional is bounded, and so $H^2(\mathbb{D})$ is a reproducing kernel Hilbert space. It's reproducing kernel, which is called the Szegő kernel, is given by

$$K(z, w) = \sum_{i=0}^{\infty} z^i \overline{w}^i = \frac{1}{1 - z\overline{w}}$$

Example B.2. The Sobelov space on $[0, 1]$ is the set

$$\mathcal{H} = \{f : [0, 1] \to \mathbb{R} : f \text{ is absolutely continuous, } f(0) = f(1) = 0, f^{'} \in L^2[0, 1]\}$$

It can be given the inner product

$$\langle f, g\rangle = \int_0^1 f'(t)g'(t)dt$$

$\mathcal{H}$ is in fact a reproducing kernel Hilbert space, every linear evaluation functional can be shown to be bounded with the Cauchy-Schwarz inequality. The associated reproducing kernel is found by solving the boundary-value problem $-k_y^{"}(t) = \delta_{y,t}$, $k_y(0) = 0 = k_y(1)$. This solution is called the Green's function, and it is given by

$$K(x, y) = \begin{cases} (1 - y)x & x \leq y \\ \\ (1 - x)y & x \geq y \end{cases}$$

We now list some of the properties shared by kernel functions, the most important being their uniqueness.

Lemma B.5. *If K is a kernel function for some Hilbert space, then K satisfies $K(x, y) = \langle k_y, k_x \rangle$ and $K(y, x) = \overline{K(x, y)}$.*

Proof. The first equality follows directly from the reproducing property:

$$\langle k_y, k_x \rangle = \langle k_y(x), k_x(x) \rangle = k_y(x) = K(x, y)$$

The second equality follows from conjugate symmetry of the inner product:

$$K(y, x) = \langle k_x, k_y \rangle = \overline{\langle k_y, k_x \rangle} = \overline{K(x, y)}$$

□

We establish now the fact that every reproducing kernel Hilbert space has a unique kernel function.

Proposition B.6. *Suppose K_1 and K_2 both satisfy the reproducing property for some reproducing kernel Hilbert space $\mathcal{H}$, then $K_1 = K_2$.*

Proof. For any $f \in \mathcal{H}$, we have

$$\langle f, K_1 - K_2 \rangle = \langle f, K_1 \rangle - \langle f, K_2 \rangle = f(y) - f(y) = 0$$

Thus if we take $f = K_1 - K_2$, we have that

$$\langle K_1 - K_2, K_1 - K_2 \rangle = 0$$

which holds if and only if $K_1 - K_2 = 0$ which is equivalent to $K_1 = K_2$. □

Remark. Therefore the only kernel function a reproducing kernel Hilbert space has is the one we constructed in the existence proof.

To illustrate the convenience of the reproducing property, we will now show how one can compute the kernel function in terms of a given orthonormal basis for the as-soicated Hilbert space. We begin with a result concerning density and convergence.

Proposition B.7. *Suppose K is a kernel function for a reproducing kernel Hilbert space $\mathcal{H}$ on a set X.*

1. *The span of the set $S = \{k_y : y \in X\}$ is dense in $\mathcal{H}$.*

2. *If $\|f_n - f\| \to 0$ as $n \to \infty$ for $f_n, f \in \mathcal{H}$, then $f_n(x) \to f(x)$ as $n \to \infty$.*

Proof.

1. If f belongs to the orthogonal complement of the span of S, then for any $y \in X$, we have $0 = \langle f, k_y \rangle = f(y)$. Thus f is the 0 vector and so every non-zero vector in $\mathcal{H}$ belongs to the span of S.

2. Observe that $|f_n(y) - f(y)| = |\langle f_n - f, k_y \rangle| \leq \|f_n - f\| \|k_x\|$. Since $\|f_n - f\| \to 0$ as $n \to \infty$, we may conclude that

$$\lim_{n \to \infty} |f_n(y) - f(y)| = 0$$

□

Theorem B.8. *Suppose $\{e_i\}_{i \in I}$ is an orthonormal basis for the reproducing kernel Hilbert space $\mathcal{H}$. Let K be the kernel function of $\mathcal{H}$. Then*

$$K(x, y) = \sum_{i \in I} e_i(x) \overline{e_i(y)}$$

Proof. For each $y \in X$, we have that $\langle k_y, e_i \rangle = \overline{\langle e_i, k_y \rangle} = \overline{e_i(y)}$. Thus by the Parseval Identity we have

$$k_y(x) = \sum_{i \in I} \langle k_y, e_i \rangle e_i(x) = \sum_{i \in I} \overline{e_i(y)} e_i(x)$$

where convergence occurs under the norm in $\mathcal{H}$. But since norm convergence implies point wise convergence we may conclude that

$$K(x, y) = \sum_{i \in I} e_i(x) \overline{e_i(y)}$$

□

We end this section by showing that two distinct reproducing kernel Hilbert spaces can not share a kernel function. This will help us in exposing the fundamental result of the theory which ties operator theory to the theory of functions.

Proposition B.9. *Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are reproducing kernel Hilbert spaces on a set X sharing a kernel function K, then $\mathcal{H}_1 = \mathcal{H}_2$.*

Proof. Let W be the span of the set $\{k_y : y \in X\}$. By Proposition B.7, the set W is dense in both $\mathcal{H}_1$ and $\mathcal{H}_2$. If $f \in W$ then

$$f(x) = \sum_{i \in I} c_i k_{y_i}(x)$$

which in turn implies

$$\langle f, f \rangle_1 = \sum_{i,j \in I} c_i \overline{c_j} \langle k_{y_i}, k_{y_j} \rangle_1 = \sum_{i,j \in I} c_i \overline{c_j} K(y_i, y_j)$$

but the same computation can be made for $\langle f, f \rangle_2$, and thus $\langle f, f \rangle_1 = \langle f, f \rangle_2$ whenever $f \in W$.

Now suppose $i, j \in \{1, 2\}$ and $i \neq j$. By the density property, given $h \in \mathcal{H}_i$, we may find a sequence of functions f_n which converge to h in the norm of $\mathcal{H}_i$. But the computation above shows that this sequence of functions also converges to some $g \in \mathcal{H}_j$ in the norm of $\mathcal{H}_j$. Since norm convergence implies pointwise convergence, we have for every $x \in X$,

$$h(x) = \lim_{n\to\infty} f_n(x) = g(x)$$

Thus $\mathcal{H}_i \subseteq \mathcal{H}_j$. By making this argument for $i = 1$ and then for $i = 2$ we obtain $\mathcal{H}_1 = \mathcal{H}_2$. It is worth noting as well that density also implies the equivalence of the two norms. □

B.3 Positive Definite Functions

Definition B.5. Suppose X is a set with a complex-valued function $K : X \times X \to \mathbb{C}$. We call K positive definite if for every natural number N,

$$\sum_{i,j=1}^{N} c_j \overline{c_i} K(x_i, x_j) \geq 0$$

holds for any choice of N complex numbers c_i and any choice of N elements x_i belonging to X.

Proposition B.10. *If K is a kernel function on a set X, then K is positive definite.*

Proof. Let N be a natural number; consider N complex numbers c_i and N elements x_i belonging X. Thus we have

$$\begin{aligned}\sum_{i,j=1}^{N} c_j \overline{c_i} K(x_i, x_j) &= \sum_{i,j=1}^{N} c_j \overline{c_i} \langle K_{x_j}, K_{x_i} \rangle = \sum_{i,j=1}^{N} \langle c_j K_{x_j}, c_i K_{x_i} \rangle \\ &= \left\langle \sum_{i=1}^{N} c_j K_{x_j}, \sum_{j=1}^{N} c_i K_{x_i} \right\rangle \geq 0\end{aligned}$$

□

Thus one can associate a positive definite function to each reproducing kernel Hilbert space on a set X. The remarkable converse, attributed to E. H. Moore, states that every positive definite function is the kernel function for a unique reproducing kernel Hilbert space.

Theorem B.11. *Suppose f is a positive definite function on a set X. Then there exists a reproducing kernel Hilbert space, $\mathcal{H}$, such that f is the kernel function for $\mathcal{H}$.*

Outline of Proof: Let $\mathcal{S}$ be the span of the set $\{k_y(x) := f(x,y) : y \in X\}$. Define a map $G : \mathcal{S} \times \mathcal{S} \to \mathbb{C}$ by

$$G\left(\sum_{i=1}^{N} a_i k_{y_i}, \sum_{j=1}^{M} b_j k_{y_j}\right) = \sum_{i=1}^{N}\sum_{j=1}^{M} a_i \overline{b_j} f(y_j, y_i)$$

where $a_i, b_j \in \mathbb{C}$. The pair $(\mathcal{S}, G)$ form an inner product space. Furthermore, given any $s \in \mathcal{S}$ and $x \in X$, we have

$$G(s, k_x) = \sum_{i=1}^{N} a_i f(x, y_i) = s(x)$$

Let $\mathcal{H}$ denote the Hilbert space obtained by completing this inner product space. The Cauchy-Schwartz inequality shows that every $h \in \mathcal{H}$ is a well-defined function on X. Let $L_y : \mathcal{H} \to \mathbb{C}$ be the linear evaluation functional given by $L_y(h) = h(y)$. If f_n is a cauchy sequence converging to h, then

$$\begin{aligned} |L_y(h)| &= |h(y)| = \lim_{n\to\infty} |f_n(y)| = \lim_{n\to\infty} |\langle f_n, k_y\rangle| \\ &\leq \lim_{n\to\infty} \|f_n\|^2 \|k_y\|^2 = f(y,y)\|h\|^2 \end{aligned}$$

So every linear evaluation functional is bounded, making $\mathcal{H}$ a reproducing kernel Hilbert space. $\square$

Combining this result with Proposition B.10 and Proposition B.9, we obtain:

Theorem B.12. *There is a one-to-one correspondence between positive definite functions and reproducing kernel Hilbert spaces.*

Remark. Because of this one-to-one correspondence it is convenient to refer to the reproducing kernel Hilbert space $\mathcal{H}$ and the corresponding positive definite function K as a *reproducing kernel pair*, which we will denote $(\mathcal{H}, K)$.

B.4 Pullbacks and Multipliers

We begin by citing a theorem due to N. Aronszajn which characterizes when the difference of two kernel functions is again a kernel function.

Theorem B.13. *Suppose $(\mathcal{H}_i, K_i)$ is a reproducing kernel pair on a set X for $i \in \{1, 2\}$. The following are equivalent:*

1. $\mathcal{H}_1 \subseteq \mathcal{H}_2$

2. *There exists $c > 0$ such that $c^2 K_2 - K_1$ is positive definite on X.*

Furthermore, if either condition holds, then $\|f\|_2 \leq c\|f\|_1$

With this characterization we can proceed to characterize the boundedness of a variety of linear operators.

Proposition B.14. *Let S be a set and suppose $(\mathcal{H}, K)$ is a reproducing kernel pair on some other set X. Every map $\phi : S \to X$ induces a reproducing kernel pair $(\mathcal{H}_\phi, K \circ \phi)$ on S where $\mathcal{H}_\phi = \{h \circ \phi : h \in \mathcal{H}\}$ and $K \circ \phi(x, y) = K(\phi(x), \phi(y))$. Furthermore, if*

$h \in \mathcal{H}_\phi$, then

$$\|h\|_{\mathcal{H}_\phi} = inf\{\|f\|_{\mathcal{H}} : h = f \circ \phi\}$$

Outline of Proof: We start by defining two index sets $I = \{i_1, \ldots, i_N\}$ and $J = \{j_1, \ldots, j_M\}$, where $M \leq N$. For each $i \in I$ let $s_i \in S$, $c_i \in \mathbb{C}$, and $x_i = \phi(s_i)$. Since ϕ is not necessarily injective, the x_i are not necessarily distinct. We assume there are M distinct x_i, and for each $j \in J$ let x_j be a distinct x_i. Now define for each $j \in J$ the set $I_j = \{i \in I : x_j = \phi(s_i)\}$. We now compute

$$\sum_{a,b \in I} c_a \overline{c_b} K(\phi(s_a), \phi(s_b)) = \sum_{a,b \in I} c_a \overline{c_b} K(x_a, x_b) = \sum_{k,l \in J} \left(\sum_{a \in I_k} c_a \right) \left(\sum_{b \in I_l} \overline{c_b} \right) K(x_k, x_l) \geq 0$$

Thus $(\mathcal{H}_\phi, K \circ \phi)$ is a reproducing kernel pair on X. Applying Theorem B.13 we see that $\{h \circ \phi : h \in \mathcal{H}\} \subseteq \mathcal{H}_\phi$. This gives a contractive linear map $L_\phi : \mathcal{H} \to \mathcal{H}_\phi$ given by $h \mapsto h \circ \phi$. There is a well-defined isometry $\mathcal{I} : \mathcal{H}_\phi \to \mathcal{H}$ satisfying $\mathcal{I}(k \circ \phi_s) = k_{\phi(s)}$. Lastly, we note that $L_\phi \circ \mathcal{I}$ is the identity on $\mathcal{H}_\phi$.

$\square$

Definition B.6. The Hilbert space $\mathcal{H}_\phi$ is called the pull-back of $\mathcal{H}$ along ϕ. The linear map $L_\phi : \mathcal{H} \to \mathcal{H}_\phi$ defined by $L_\phi(h) = h \circ \phi$ is called the ϕ-pull-back map.

Theorem B.15. *Suppose $(\mathcal{H}_i, K_i)$ is a reproducing kernel pair on the set X_i for $i \in \{1, 2\}$. Given a map $\phi : X_1 \to X_2$, the following are equivalent:*

1. *$L_\phi : \mathcal{H}_2 \to \mathcal{H}_1$ defined by $L_\phi(h) = h \circ \phi$ is a bounded linear operator*

2. *There exists $c > 0$ such that the map $c^2 K_1 - K_2 \circ \phi$ is positive definite*

Proof. 1 implies $\mathcal{H}_\phi \subseteq \mathcal{H}$ which is equivalent to 2 by Theorem B.13. Assuming 2, we have by Theorem B.13 that $L_\phi(f) \in \mathcal{H}_1$ for all $f \in \mathcal{H}_2$. Furthermore, by Theorem B.13 and Proposition B.14 we have that $\|L_\phi(f)\|_{\mathcal{H}_1} \leq c\|f\|_{\mathcal{H}_\phi} \leq c\|f\|_{\mathcal{H}_2}$. Thus 2 implies 1. $\square$

Definition B.7. Suppose $(\mathcal{H}_i, K_i)$ is a reproducing kernel pair on a set X for $i \in \{1, 2\}$. A function $f : X \to \mathbb{C}$ is called a multiplier of $\mathcal{H}_1$ into $\mathcal{H}_2$ if

$$f\mathcal{H}_1 := \{fh : h \in \mathcal{H}_1\} \subseteq \mathcal{H}_2$$

The set of all such multipliers will be denoted by $M(\mathcal{H}_1, \mathcal{H}_2)$. When $\mathcal{H}_1 = \mathcal{H}_2$, we refer to f as a multiplier of $\mathcal{H}$. Each multiplier induces a linear map $M_f : \mathcal{H}_1 \to \mathcal{H}_2$ given by $M_f(h) = fh$, which we will call the f-multiplier map.

Proposition B.16. *Suppose $(\mathcal{H}, K)$ is a reproducing kernel pair on a set X. If f is a complex valued function of X, then $(\mathcal{H}_f, K_f)$ is a reproducing kernel pair on X where $\mathcal{H}_f = f\mathcal{H}$ and $K_f(x, y) = f(x)\overline{f(y)}K(x, y)$*

Proof. Firstly, $\mathcal{H}_f$ becomes a Hilbert space if we assign it the inner product

$$\langle fh_1, fh_2\rangle_{\mathcal{H}_f} = \langle h_1, h_2\rangle_{\mathcal{H}}$$

since the map $h \mapsto fh$ is a surjective linear isometry. Now set $\mathcal{H}_0 = \{h \in \mathcal{H} : fh = 0\}$, $\mathcal{H}_1 = \mathcal{H}_0^\perp$, and decompose $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. For each $y \in X$ we have through our decomposition $k_y = k_{y_0} + k_{y_1}$, where each k_{y_i} is a kernel function on $\mathcal{H}_i$. Now for $h \in \mathcal{H}$ we have

$$f(y)h(y) = f(y)\langle h, k_y\rangle_{\mathcal{H}} = f(y)\langle fh, fk_y\rangle_{\mathcal{H}_f} = \langle fh, \overline{f(y)}fk_{y_1}\rangle_{\mathcal{H}_f}$$

demonstrating that $\overline{f(y)}f(x)k_{y_1}(x,y)$ satisfies the reproducing property. But now $K_{f_y} = \overline{f(y)}fk_y = \overline{f(y)}f(k_{y_1}+k_{y_0}) = \overline{f(y)}fk_{y_1}$, and so $(\mathcal{H}, K_f)$ is a reproducing kernel pair on X. □

Theorem B.17. *Suppose $(\mathcal{H}_i, K_i)$ is a reproducing kernel pair on a set X for $i \in \{1,2\}$. Given a function $f : X \to \mathbb{C}$, the following are equivalent:*

1. *$f \in M(\mathcal{H}_1, \mathcal{H}_2)$ and M_f is a bounded linear operator.*

2. *There exists $c > 0$ such that $c^2K_2 - K_{1_f}$ is positive definite.*

Moreover, the adjoint of M_f satisfies $M_f^*(k_{1_y}) = \overline{f(y)}k_{2_y}$ for all $y \in X$.

Proof. 1 implies 2 by Theorem B.13 and the definition of a multiplier. Assuming 2, Theorem B.13 shows that $f \in M(\mathcal{H}_1, \mathcal{H}_2)$. Furthermore Theorem B.13 asserts that $\|M_f(h)\|_{\mathcal{H}_2} \le c\|fh\|_{\mathcal{H}_f}$. By the definition of the inner product on $\mathcal{H}_f$ we have that $c\|fh\|_{\mathcal{H}_f} = c\|h\|_{\mathcal{H}_1}$. Thus M_f is a bounded linear operator. Finally the adjoint of M_f is given by

$$\langle h, \overline{f(y)}k_{1_y}\rangle_{\mathcal{H}_1} = f(y)\langle h, k_{1_y}\rangle_{\mathcal{H}_1} = f(y)h(y) = \langle M_f(h), k_{2_y}\rangle_{\mathcal{H}_2} = \langle h, M_f^*k_{2_y}\rangle_{\mathcal{H}_2}$$

□

Fundamental to our work is an operator which is a combination of a pull-back map and a multiplier map.

Proposition B.18. *Suppose $(\mathcal{H}, K)$ is a reproducing kernel pair on a set X, and let $\phi : X \to X$ and $f : X \to \mathbb{C}$ be maps. If $f\mathcal{H}_\phi = \{fh \circ \phi : h \in \mathcal{H}\}$ and $K_f \circ \phi(x,y) = f(x)\overline{f(y)}K(\phi(x), \phi(y))$ then $(f\mathcal{H}\phi, K_f \circ \phi)$ is a reproducing kernel pair on X.*

Proof. Applying Proposition B.14 to the reproducing kernel pair $(\mathcal{H}, K)$ yields the reproducing kernel pair $(\mathcal{H}_\phi, K \circ \phi)$. Now apply Proposition B.16 to this new pair to obtain the reproducing kernel pair $(f\mathcal{H}\phi, K_f \circ \phi)$ on X. $\square$

Definition B.8. Given a reproducing kernel pair $(\mathcal{H}, K)$ on a set X and two maps $\phi : X \to X$ and $f : X \to \mathbb{C}$, the ϕ-f pull-back multiplier, $S_{\phi,f} : \mathcal{H} \to f\mathcal{H}_\phi$, is given by $S_{\phi,f}(h) = fh \circ \phi$.

Remark. Such operators do not appear to have been named in the literature, so for ease of exposition, I have selected a name accordingly.

Theorem B.19. *Suppose $(\mathcal{H}, K)$ is a reproducing kernel pair on a set X. Given two maps $\phi : X \to X$ and $f : X \to \mathbb{C}$, the following are equivalent:*

1. *$S_{\phi,f} : \mathcal{H} \to f\mathcal{H}_\phi$ is a bounded operator and $f\mathcal{H}_\phi \subseteq \mathcal{H}$.*

2. *There exists $c > 0$ such that $c^2 K - K_f \circ \phi$ is positive definite on X.*

Furthermore, when either condition holds, the adjoint of $S_{\phi,f}$ satisfies

$$S^*_{\phi,f}(k_y) = \overline{f(y)} k_{\phi(y)}$$

Proof. 1 implies 2 by Theorem B.13. Now assume that 2 is true. By Theorem B.13 we know that $f\mathcal{H}_\phi \subseteq \mathcal{H}$. Thus we need to demonstrate that $S_{\phi,f}$ is bounded. Let $\mathcal{G} = \{h \in \mathcal{H} : fh \circ \phi = 0\}$, and decompose $\mathcal{H} = \mathcal{G} \oplus \mathcal{G}^\perp$. For $h = h_0 + h_1$ where $h_0 \in \mathcal{G}$ and $h_1 \in \mathcal{H}^\perp$, we have

$$\|S_{\phi,f}(h)\|_{\mathcal{H}} = \|fh \circ \phi\|_{\mathcal{H}} = \|fh_1 \circ \phi\|_{\mathcal{H}} \leq c\|fh_1 \circ \phi\|_{\mathcal{H}_{\phi,f}} = c\|h_1\|_{\mathcal{G}^\perp} \leq c\|h\|_{\mathcal{H}}$$

So $S_{\phi,f}$ is bounded. Now compute

$$\langle S_{\phi,f}(h), K_y\rangle = \langle fh \circ \phi, K_y\rangle, = f(y)h \circ \phi(y) = \langle h, \overline{f(y)}K_{\phi(y)}\rangle$$

giving the above adjoint property. $\square$

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