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Pick interpolation, displacement equations, and W*-correspondences

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PICK INTERPOLATION, DISPLACEMENT EQUATIONS, AND W^*-CORRESPONDENCES

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CERTIFICATE OF APPROVAL

PH.D. THESIS

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To my parents, Mike and Nancy

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ABSTRACT

The classical Nevanlinna-Pick interpolation theorem, proved in 1915 by Pick and in 1919 by Nevanlinna, gives a condition for when there exists an interpolating function in $H^\infty(\mathbb{D})$ for a specified set of data in the complex plane. In 1967, Sarason proved his commutant lifting theorem for $H^\infty(\mathbb{D})$, from which an operator theoretic proof of the classical Nevanlinna-Pick theorem followed. Several competing noncommutative generalizations arose as a consequence of Sarason's result, and two strategies emerged for proving generalized Nevanlinna-Pick theorems: via a commutant lifting theorem or via a resolvent, or *displacement*, equation.

We explore the difference between these two approaches. Specifically, we compare two theorems: one by Constantinescu-Johnson from 2003 and one by Muhly-Solel from 2004. Muhly-Solel's theorem is stated in the highly general context of W^*-correspondences and is proved via commutant lifting. Constantinescu-Johnson's theorem, while stated in a less general context, has the advantage of an elegant proof via a displacement equation. In order to make the comparison, we first generalize Constantinescu-Johnson's theorem to the setting of W^*-correspondences in Theorem 3.0.1. Our proof, modeled after Constantinescu-Johnson's, hinges on a modified version of their displacement equation. Then we show that Theorem 3.0.1 is fundamentally different from Muhly-Solel's. More specifically, interpolation in the sense of Muhly-Solel's theorem implies interpolation in the sense of Theorem 3.0.1, but the converse is not true. Nevertheless, we identify a commutativity assumption under

which the two theorems yield the same result.

In addition to the two main theorems, we include smaller results that clarify the connections between the notation, space of interpolating maps, and point evaluation employed by Constantinescu-Johnson and those employed by Muhly-Solel. We conclude with an investigation of the relationship between Theorem 3.0.1 and Popescu's generalized Nevanlinna-Pick theorem proved in 2003.

PUBLIC ABSTRACT

Suppose you are given a finite number of points $(x_1, y_1), \ldots, (x_n, y_n)$ in the xy-plane. The problem of finding a function that goes through the given points is called an *interpolation problem*. In the early 1900s, mathematicians were interested in an interpolation problem in which the desired function was required to satisfy certain conditions. While not overly restrictive, the conditions on the function made it hard to find in some cases and impossible in others. Georg Pick and Rolf Nevanlinna were the first to solve this problem. Instead of finding a formula for the function, they gave an easy way to check if it exists. Since then, theorems about interpolation problems of this form have been called Nevanlinna-Pick theorems, even if the setting is much more general than the original setting.

In this thesis, we compare two very general Nevanlinna-Pick theorems, one due to Constantinescu and Johnson and the other due to Muhly and Solel. Though the theorems seem similar, their connection is obfuscated by the authors' notation and context. First we reformulate Constantinescu-Johnson's theorem in the setting of Muhly-Solel's result. Our proof, modeled after Constantinescu-Johnson's, employs the so-called displacement equation. When we compare our new theorem to Muhly-Solel's, we find fundamental differences. Nevertheless, we identify certain additional assumptions on the original data points which guarantee that the two theorems will yield the same result.

TABLE OF CONTENTS

CHAPTER 1
INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Consider the following interpolation problem: Given N distinct points $z_1, \ldots, z_N$ in the open unit disc $\mathbb{D}$ in the complex plane and N complex numbers $w_1, \ldots, w_N$, when does there exist a function f in the Hardy space $H^\infty(\mathbb{D})$ of norm at most 1 that interpolates the data, i.e., $f(z_1) = w_1, \ldots, f(z_N) = w_N$? In 1915 in [Pic15], Georg Pick was the first to prove a necessary and sufficient condition for the existence of the interpolating function. Rolf Nevanlinna independently answered the same question in 1919, and the following theorem became known as the Nevanlinna-Pick theorem.

Theorem 1.1.1 (Nevanlinna-Pick theorem)**.** *Given N distinct points $z_1, \ldots, z_N \in \mathbb{D}$ and N points $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$, there exists $f \in H^\infty(\mathbb{D})$ such that $||f||_\infty \leq 1$ and*

$$f(z_i) = \lambda_i, \quad i = 1, \ldots, N,$$

if and only if the Pick matrix

$$\left[\frac{1-\overline{\lambda_i}\lambda_j}{1-\overline{z_i}z_j}\right]_{i,j=1}^{N}$$

is positive semidefinite.

In 1967, Donald Sarason gave an operator theoretic proof of the Nevanlinna-Pick theorem in [Sar67]. Sarason's proof hinged on his commutant lifting theorem for $H^\infty(\mathbb{D})$, which was the first of its kind. The following year, Bela Sz.-Nagy and

Ciprian Foiaş extended Sarason's commutant lifting theorem and proved their own generalized Nevanlinna-Pick theorem.

Several competing noncommutative generalizations arose as a consequence of Sarason and Nagy-Foiaş's work, including Muhly-Solel's Theorem 5.3 in [MS04]. Stated in the highly general context of W^*-correspondences, Muhly-Solel's theorem encompasses a variety of Nevanlinna-Pick theorems as special cases. There is one result, however, which does not appear to fall under the umbrella of [MS04, Theorem 5.3]: Constantinescu-Johnson's Theorem 3.4 in [CJ03]. Constantinescu-Johnson's theorem was proved via a so-called displacement equation, an equation of the form $A - \theta(A) = B$, where A and B are bounded operators on Hilbert space and θ is a completely positive map of norm less than 1 on the algebra of bounded operators. While Constantinescu-Johnson's theorem is stated in a relatively less general setting, the simplicity of its proof makes it of interest.

The goal of this thesis is to improve our understanding of the relationship between the commutant lifting approach and the displacement equation approach to proving generalized Nevanlinna-Pick theorems. To do this, we focus on the relationship between Muhly-Solel's theorem and Constantinescu-Johnson's theorem. In Chapter 2, we compare the Hardy algebras where the interpolating functions live and the point evaluations defined on those spaces. While Muhly-Solel's point evaluation defined in [MS04, Proposition 5.1] is multiplicative, Constantinescu-Johnson's is not. However, in Theorem 2.3.4, we prove that it gives rise to an antihomomorphism on the Hardy algebra of the dual correspondence.

In Chapter 3, we generalize Constantinescu-Johnson's theorem to the setting of W^*-correspondences where Muhly-Solel work. This theorem, Theorem 3.0.1, is one of our main results. Its proof is modeled after Constantinescu-Johnson's proof and depends on a modified version of their displacement equation. It also relies on results about time varying linear systems and a corollary of Douglas's lemma, all of which we state and prove in Chapter 3. The importance of the new Nevanlinna-Pick theorem is that, despite its generality, its proof does not rely on commutant lifting. To conclude Chapter 3, we prove Constantinescu-Johnson's theorem and the classical Nevanlinna-Pick theorem as corollaries, and we give a restatement of Theorem 3.0.1 with no reference to the dual correspondence.

In Chapter 4, we compare Theorem 3.0.1 with Muhly-Solel's theorem. A close study of the Pick matrices shows that interpolation in the sense of Muhly-Solel's theorem implies interpolation in the sense of Theorem 3.0.1. However, a simple example, brought to our attention by the referee of [Nor17], illustrates that the converse is not true in general. Nevertheless, in Theorem 4.1.6 we identify commutativity assumptions under which the two theorems yield the same result. To prove Theorem 4.1.6, we extend Muhly-Solel's isomorphism between central correspondences in [MS08] to an isomorphism of the associated Hardy algebras. We also require a generalization of the Schur Product theorem for matrices with operator entries.

We conclude with an investigation of the connections between our generalized Nevanlinna-Pick theorem and Popescu's Theorem 7.4 in [Pop03]. Though Popescu's setting is more general than Constantinescu-Johnson's and his proof utilizes both commutant lifting and a displacement equation, Theorem 7.4 is strikingly similar to Constantinescu-Johnson's theorem. In Lemma 5.3.4 we develop a lexicon for going between our notation and that of Popescu. Then we provide an alternate proof of a nontangential version of Popescu's theorem based on a generalization of Constantinescu-Johnson's theorem to Popescu's setting and Lemma 5.3.4.

1.2 Preliminaries

In this section, we establish notation and conventions that will be used throughout this document. We define the objects of interest, assuming that the reader is familiar with the basic theory of C^*-algebras.

Definition 1.2.1. *A* W^*-algebra M *is a* C^**-algebra for which there is a Banach space* M_* *such that its dual is* M*.*

Following Sakai's convention, we define W^*-algebras abstractly, without reference to a preferred representation on Hilbert space, while we think of von Neumann algebras concretely as weakly closed unital $*$-subalgebras of $B(H)$, for some Hilbert space H. Though every von Neumann algebra is a W^*-algebra, and every W^*-algebra is $*$-isomorphic to a von Neumann algebra [Sak98, Theorem 1.16.7], the focus of our study will be W^*-algebras. Examples of W^*-algebras include $\mathbb{C}, M_n(\mathbb{C})$, and $B(H)$. An example of a C^*-algebra that fails to be a W^*-algebra is $C(X)$, the set of all continuous functions from a locally compact connected Hausdorff space X into the complex numbers.

Before stating the definition of a W^*-correspondence, we recall the definition of a Hilbert C^*-module from [Lan95, Chapter 1].

Definition 1.2.2. *We say that E is a* right Hilbert C^*-module *over a C^*-algebra M if it satisfies the following conditions:*

1. *E is a vector space with a right M-module structure.*

2. *E is equipped with an inner product $\langle\cdot,\cdot\rangle : E \times E \to M$ that is full in M and such that for all $x, y, z \in E, a \in M,$ and $\lambda, \gamma \in \mathbb{C}$,*

   - $\langle x, \lambda y + \gamma z\rangle = \lambda\langle x, y\rangle + \gamma\langle x, z\rangle$

   - $\langle x, ya\rangle = \langle x, y\rangle a$

   - $\langle y, x\rangle = \langle x, y\rangle^*$

   - $\langle x, x\rangle \geq 0,$ *and* $\langle x, x\rangle = 0 \implies x = 0$

3. *E is complete with respect to the norm defined by $\|x\| := \|\langle x, x\rangle\|^{1/2}$ for all $x \in E$.*

Definition 1.2.3. *A* W^*-correspondence *E over a W^*-algebra M is a self-dual right Hilbert C^*-module over M equipped with a faithful, normal $*$-homomorphism $\varphi : M \to \mathcal{L}(E)$ that gives the left action of M on E, where $\mathcal{L}(E)$ denotes the W^*-algebra of adjointable operators on E.*

The requirement that the W^*-correspondence be *self-dual* means that every M-linear map $T : E \to M$ is given by an inner product, i.e., there exists $x \in E$ such that for all $y \in E$, $T(y) = \langle x, y\rangle$. Lance shows in [Lan95, Chapter 1] that every adjointable operator on E is bounded and M-linear. The assumption of self-duality guarantees that all bounded linear operators on E are adjointable. By *faithful* we mean injective, and by *normal* we mean that the representation is continuous with respect to the ultraweak topology. For convenience, sometimes we will denote the left action by $\cdot$ instead of φ.

For a fixed W^*-correspondence E over a W^*-algebra M, we form $E^{\otimes 2} = E \otimes_\varphi E$ by taking the self-dual completion of a quotient of the algebraic tensor product $E \otimes E$ balanced over M. The quotient is determined by the semi inner product defined by $\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle := \langle \eta_1, \varphi(\langle \xi_1, \xi_2 \rangle)\eta_2 \rangle$ for all $\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \in E^{\otimes 2}$. For $k \in \mathbb{N}$, define the tensor power $E^{\otimes k}$ recursively by $E^{\otimes k} = E \otimes_\varphi E^{\otimes k-1}$. $E^{\otimes k}$ is a W^*-correspondence over M with the left action given by $\varphi_k : M \to \mathcal{L}(E^{\otimes k})$, where

$$\varphi_k(a)(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_k) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots \otimes \xi_k \tag{1.1}$$

for all $\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_k \in E^{\otimes k}$.

Definition 1.2.4. *The* Fock space *of E is the ultraweak direct sum*

$$\mathcal{F}(E) := \bigoplus_{k=0}^{\infty} E^{\otimes k},$$

where $E^{\otimes 0} = M$ is viewed as a bimodule over itself.

Two types of operators on the Fock space are of special interest because they generate the Hardy algebra: the left action operators and the left creation operators. The Fock space of E is a W^*-correspondence over M, and we let $\varphi_\infty : M \to \mathcal{L}(\mathcal{F}(E))$

denote the left action of M on $\mathcal{F}(E)$ given by

$$\varphi_\infty(a) = \begin{bmatrix} a & & & \\ & \varphi(a) & & \\ & & \varphi_2(a) & \\ & & & \ddots \end{bmatrix},$$

where $\varphi_k(a)$ is defined in equation (1.1).

For $\xi \in E$, the *left creation operator* T_ξ is the map $T_\xi : \mathcal{F}(E) \to \mathcal{F}(E)$ that is given by $T_\xi(\eta) = \xi \otimes \eta, \eta \in \mathcal{F}(E)$. Matricially,

$$T_\xi = \begin{bmatrix} 0 & & & \\ T_\xi^{(1)} & 0 & & \\ & T_\xi^{(2)} & 0 & \\ & & \ddots & \ddots \end{bmatrix},$$

where $T_\xi^{(k)} : E^{\otimes k-1} \to E^{\otimes k}$ is defined by $T_\xi^{(k)}(\eta_1 \otimes \cdots \otimes \eta_{k-1}) := \xi \otimes \eta_1 \otimes \cdots \otimes \eta_{k-1}$.

Definition 1.2.5. *The* tensor algebra *over E, denoted $\mathcal{T}_+(E)$, is the norm-closed subalgebra of $\mathcal{L}(\mathcal{F}(E))$ generated by the left action operators $\{\varphi_\infty(a) \mid a \in M\}$ and the left creation operators $\{T_\xi \mid \xi \in E\}$. The* Hardy algebra *of E is the ultraweak closure of $\mathcal{T}_+(E)$ in $\mathcal{L}(\mathcal{F}(E))$ and is denoted by $H^\infty(E)$.*

Having defined the Hardy algebra of a W^*-correspondence, we turn our attention to intertwining spaces.

Definition 1.2.6. *Given two representations $\sigma_1 : M \to B(H_1)$ and $\sigma_2 : M \to B(H_2)$ of M on Hilbert space, we define the* intertwining space

$$\mathfrak{I}(\sigma_1, \sigma_2) := \{\eta \in B(H_1, H_2) \mid \eta\sigma_1(a) = \sigma_2(a)\eta \quad \forall a \in M\}.$$

In the W^*-correspondence setting, we fix a W^*-correspondence E over a W^*-algebra M and a faithful, normal representation $\sigma : M \to B(H)$, for some Hilbert space H. We form $E \otimes_\sigma H$ by taking the Hausdorff completion of the algebraic tensor product $E \otimes H$ in the positive semidefinite sesquilinear form defined by the formula $\langle \xi \otimes h, \eta \otimes k \rangle := \langle h, \sigma(\langle \xi, \eta \rangle) k \rangle$, for $\xi \otimes h, \eta \otimes k \in E \otimes H$. Note that $E \otimes_\sigma H$ is a Hilbert space with respect to the norm defined by this inner product. Moreover, σ induces the representation $\sigma^E : \mathcal{L}(E) \to B(E \otimes_\sigma H)$ given by $\sigma^E(T) = T \otimes I_H, T \in \mathcal{L}(E)$. Consequently, both σ and $\sigma^E \circ \varphi$ are representations of M on Hilbert space. The intertwining space $\mathfrak{I}(\sigma, \sigma^E \circ \varphi)$ is of particular importance to us.

Definition 1.2.7. *We define the* σ-dual of E *or the* dual correspondence*, denoted* E^σ*, to be the intertwining space* $\mathfrak{I}(\sigma, \sigma^E \circ \varphi)$*. That is,*

$$E^\sigma := \{\eta \in B(H, E \otimes_\sigma H) \mid \eta\sigma(a) = \sigma^E \circ \varphi(a)\eta \quad \forall a \in M\}.$$

The following result about elements in E^σ will be useful in Chapter 2.

Lemma 1.2.8. *For* $a \in M$*,* $n \in \mathbb{N}$*, and* $\eta_1, \ldots, \eta_n \in E^\sigma$*,*

$$(\varphi_n(a) \otimes I_H)(I_{E^{\otimes n-1}} \otimes \eta_1) \cdots (I_E \otimes \eta_{n-1})\eta_n = (I_{E^{\otimes n-1}} \otimes \eta_1) \cdots (I_E \otimes \eta_{n-1})\eta_n \sigma(a).$$

Proof. First note that $(\varphi(a) \otimes I_H)\eta_n = \eta_n \sigma(a)$ since $\eta_n \in E^\sigma$. It remains to show that for $k \geq 1$,

$$(\varphi_{k+1}(a) \otimes I_H)(I_{E^{\otimes k}} \otimes \eta_{n-k}) = (I_{E^{\otimes k}} \otimes \eta_{n-k})(\varphi_k(a) \otimes I_H).$$

For $\xi_1 \otimes \cdots \otimes \xi_k \otimes h \in E^{\otimes k} \otimes H$, the left hand side gives

$$(\varphi_{k+1}(a) \otimes I_H)(I_{E^{\otimes k}} \otimes \eta_{n-k})(\xi_1 \otimes \cdots \otimes \xi_k \otimes h)$$

$$= (\varphi_{k+1}(a) \otimes I_H)(\xi_1 \otimes \cdots \otimes \xi_k \otimes \eta_{n-k} h) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots \otimes \xi_k \otimes \eta_{n-k} h.$$

The right hand side gives

$$(I_{E^{\otimes k}} \otimes \eta_{n-k})(\varphi_k(a) \otimes I_H)(\xi_1 \otimes \cdots \otimes \xi_k \otimes h)$$

$$= (I_{E^{\otimes k}} \otimes \eta_{n-k})((\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots \otimes \xi_k \otimes h) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots \otimes \xi_k \otimes \eta_{n-k} h.$$

□

Though an element in E^σ cannot be composed with itself, we define a notion of powers for elements in E^σ in the following definition. We will see in Chapter 2 that this notion is closely related to the tensor power of an element in E^σ.

Definition 1.2.9. *For $\eta \in E^\sigma$, define the* Cauchy Kernel $C(\eta) \in B(H, \mathfrak{F}(E) \otimes_\sigma H)$ *by* $C(\eta) = \begin{bmatrix} I_H & \eta^{(1)} & \eta^{(2)} & \eta^{(3)} & \cdots \end{bmatrix}^T$, *where $\eta^{(1)} = \eta$ and $\eta^{(k)} \in \mathfrak{I}(\sigma, \sigma^{E^{\otimes k}} \circ \varphi_k)$ is defined recursively by the formula $\eta^{(k)} = (I_{E^{\otimes k-1}} \otimes \eta)\eta^{(k-1)}$.*

We note that for $\xi, \eta \in E^\sigma$, the inner product $\langle C(\xi), C(\eta) \rangle$ is given by the formula $\langle C(\xi), C(\eta) \rangle = C(\xi)^* C(\eta)$.

E^σ is a W^*-correspondence over $\sigma(M)'$, the commutant of $\sigma(M)$ in $B(H)$, under the following actions and inner product: For $a, b \in \sigma(M)'$ and $\eta, \xi \in E^\sigma$, $a \cdot \eta \cdot b := (I_E \otimes a)\eta b$ and $\langle \eta, \xi \rangle := \eta^* \xi$. We will denote the left action of $\sigma(M)'$ on E^σ by φ^σ. As above, form the tensor powers $(E^\sigma)^{\otimes k}, k \in \mathbb{N}$, and the Fock space $\mathfrak{F}(E^\sigma)$. The left action maps of $\sigma(M)'$ on $(E^\sigma)^{\otimes k}$ and $\mathfrak{F}(E^\sigma)$ are denoted by φ_k^σ and

φ_∞^σ, respectively. Finally, construct the tensor algebra $\mathcal{T}_+(E^\sigma)$ and the Hardy algebra $H^\infty(E^\sigma)$ as described in Definition 1.2.5. The Hardy algebras $H^\infty(E)$ and $H^\infty(E^\sigma)$ will be objects of central interest throughout this document. Chapters 2 and 4 will give us more insight into how they are related.

We conclude this section with a few examples of W^*-correspondences. First note that every Hilbert space is a W^*-correspondence over $\mathbb{C}$. Furthermore, every W^*-algebra is a W^*-correspondence over itself. Therefore, the notion of the W^*-correspondence generalizes the notions of Hilbert space and the W^*-algebra. Now we turn our attention to a few concrete examples.

Example 1.2.10 ([MS04, Example 4.1])**.** *The most basic example is $M = E = \mathbb{C}$. In this case, for $a, b, \in M$ and $c, d \in E$, we define the left and right actions by $a \cdot c \cdot b := acb$ and the inner product by $\langle c, d \rangle := \overline{c}d$. The Fock space $\mathcal{F}(E)$ is $\ell^2(\mathbb{Z}_+)$, which may be identified with the Hardy space $H^2(\mathbb{T})$ on the unit circle via the Fourier transform. The Hardy algebra $H^\infty(E)$ is the algebra of lower triangular Toeplitz matrices, which may be identified with $H^\infty(\mathbb{T})$ via the Fourier transform.*

For an arbitrary Hilbert space H, there is only one possible representation σ of M on H, $\sigma(a) = aI_H$. Thus $\sigma(M)' = B(H)$ and the dual correspondence

$$E^\sigma = \{\eta : H \to \mathbb{C} \otimes_\sigma H \mid \eta\sigma(a) = (\sigma(a) \otimes I_H)\eta \quad \forall a \in M\}$$

can be identified with $B(H)$ as well, since $\mathbb{C} \otimes_\sigma H \approx H$ and σ is given by scalar multiples of the identity on H.

Example 1.2.11 ([MS04, Example 4.2])**.** *Let $M = \mathbb{C}$ and $E = \mathbb{C}^n$ for $n \in \mathbb{N}$. Define* *the left and right actions and inner product as follows:*

$$a \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \cdot b := \begin{bmatrix} ac_1b \\ \vdots \\ ac_nb \end{bmatrix} \quad \text{and} \quad \left\langle \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \right\rangle := \sum_{k=1}^{n} \overline{c_k} d_k.$$

The Fock space $\mathcal{F}(E)$ is $\mathcal{F}(\mathbb{C}^n)$. The Hardy algebra $H^\infty(E)$ is the ultraweak closure of Popescu's noncommutative disc algebra $\mathcal{A}_n$ as defined in [Pop96, Section 1]. We note that Popescu's $\mathcal{A}_n$ is the tensor algebra $\mathcal{T}_+(\mathbb{C}^n)$ in the notation of [MS04].

As in the previous example, there is only one representation $\sigma : M \to H$ for any Hilbert space H. That is, $\sigma : M \to H$ is defined by $\sigma(a) = aI_H$ for all $a \in M$. Thus $\sigma(M)' = B(H)$ and the dual correspondence

$$E^\sigma = \{\eta : H \to \mathbb{C}^n \otimes_\sigma H \mid \eta\sigma(a) = (\sigma(a) \otimes I_H)\eta \quad \forall a \in M\}$$

is isomorphic to $C_n(B(H))$, column n-space over $B(H)$, since $\mathbb{C}^n \otimes_\sigma H \approx H^{\oplus n}$ and σ is given by scalar multiples of the identity on H.

Example 1.2.12 ([MS04, Example 4.3])**.** *W^*-correspondences can also be generated from directed graphs, or quivers. Let G be a directed graph with n vertices, $v_1, \ldots, v_n$, and let c_{ij} denote the number of edges from v_j to v_i. We will assume $0 \le c_{ij} < \infty$ for all $1 \le i, j \le n$. Let C denote the $n \times n$ incidence matrix $[c_{ij}]_{1 \le i,j \le n}$.*

Define M to be the algebra of $n \times n$ diagonal matrices, denoted D_n, over $\mathbb{C}$. Define E, or $E(C)$, to be the direct sum of Hilbert spaces $\bigoplus_{1 \le i,j \le n} H_{ij}$, where the dimension of H_{ij} is c_{ij}. For each H_{ij}, fix an orthonormal basis $\{e_{ij}^{(k)}\}_{1 \le k \le c_{ij}}$. Let e_{ll} denote the matrix unit in D_n with 1 in the ll^{th} entry and zeros elsewhere. Define the *right and left actions of M on $E(C)$ by*

$$\begin{aligned} e_{ij}^{(k)} \cdot e_{ll} &:= \delta_{jl} e_{ij}^{(k)} \\ e_{ll} \cdot e_{ij}^{(k)} &:= \delta_{il} e_{ij}^{(k)}, \end{aligned}$$

where δ_{kl} denotes the Kronecker delta function that is 1 when $k = l$ and 0 otherwise. Define the M-valued inner product on $E(C)$ by

$$\langle e_{ij}^{(k)}, e_{lm}^{(p)} \rangle := \delta_{kp} \delta_{il} \delta_{jm} e_{jj}.$$

Equipped with these actions and inner product, $E(C)$ is a W^*-correspondence over M. In [MS99, Corollary 5.2], Muhly and Solel showed that $E(C)^{\otimes n}$ and $E(C^n)$ are naturally isomorphic as W^*-correspondences, and therefore the Fock space $\mathcal{F}(E(C)) = \bigoplus_{n=0}^{\infty} E(C^n)$.

Let $\sigma : M \to B(H)$ be a faithful, normal representation of M on a Hilbert space H. Since every element $a \in M$ is of the form $a = \sum_{k=1}^{n} a_k e_{kk}$, σ is completely determined by its value at e_{kk}, for $1 \leq k \leq n$. In fact, it is completely determined up to unitary equivalence by the dimension m_k of $H_k := \sigma(e_{kk})H$. Note that $0 < m_k \leq \infty$ for all k since σ is faithful. Thus we may write $H = \bigoplus_{1 \leq k \leq n} H_k = \bigoplus_{1 \leq k \leq n} \mathbb{C}^{m_k}$, where $\mathbb{C}^{\infty}$ is interpreted as $\ell^2(\mathbb{Z}_+)$. Then $\sigma(M)' = \bigoplus_{1 \leq k \leq n} B(\mathbb{C}^{m_k})$. By definition, the dual correspondence

$$\begin{aligned} (E(C))^{\sigma} &= \{\eta : H \to E(C) \otimes_{\sigma} H \mid \eta \sigma(a) = (\varphi(a) \otimes I_H)\eta \quad \forall a \in M\} \\ &= \{\eta = [\eta_{ij}]_{1 \leq i,j \leq n} \mid \eta_{ij} \in B(\mathbb{C}^{m_j}, H_{ji} \otimes \mathbb{C}^{m_i}) \quad \forall 1 \leq i, j \leq n\}, \end{aligned}$$

since $\sigma(e_{kk})H = \mathbb{C}^{m_k}$ and $(\varphi(e_{kk}) \otimes I_H)(E(C) \otimes_{\sigma} H) = \bigoplus_{1 \leq i \leq n} H_{ki} \otimes \mathbb{C}^{m_i}$.

CHAPTER 2
THE HARDY ALGEBRA

We examine the Hardy algebras $H^\infty(E)$ and $H^\infty(E^\sigma)$ in this chapter. They play a major role in our study since Muhly-Solel's interpolant lives in $H^\infty(E)$ and our interpolant lives in $H^\infty(E^\sigma)$. It is natural, therefore, to consider Constantinescu-Johnson's Schur class simultaneously since it is the space where their interpolating map lies. In this chapter, we explore the relationships between these spaces, and we study the point evaluations defined for their elements.

2.1 Definitions

Recall that the tensor algebra over a W^*-correspondence E, denoted $\mathcal{T}_+(E)$, is the norm-closed subalgebra of $\mathcal{L}(\mathcal{F}(E))$ generated by the left action operators $\{\varphi_\infty(a) \mid a \in M\}$ and the left creation operators $\{T_\xi \mid \xi \in E\}$. The Hardy algebra of E, denoted $H^\infty(E)$, is the ultraweak closure of $\mathcal{T}_+(E)$ in $\mathcal{L}(\mathcal{F}(E))$. For a fixed faithful, normal representation σ of M on a Hilbert space H, recall that E^σ is a W^*-correspondence over $\sigma(M)'$. Thus we can construct $H^\infty(E^\sigma)$, the Hardy algebra of E^σ, as in Definition 1.2.5.

In Constantinescu-Johnson's Theorem 3.4 in [CJ03], the interpolating map is a contraction that belongs to an algebra of upper triangular operators (see equation (3.1) in [CJ03]). Translated to the W^*-correspondence setting, this space is defined as follows.

Definition 2.1.1. *Let E be a W^*-correspondence over a W^*-algebra M and σ be a* *faithful, normal representation of M on a Hilbert space H. Define $\mathcal{U}_{\mathcal{T}}(E,H,\sigma)$ to be the algebra of upper triangular operators $T = \begin{bmatrix} T_{ij} \end{bmatrix}_{i,j=0}^{\infty} \in B(\mathcal{F}(E) \otimes_\sigma H)$ such that $T_{0j} \in \mathfrak{I}(\sigma^{E^{\otimes j}} \circ \varphi_j, \sigma)$, for $j \geq 0$, and $T_{ij} = I_E \otimes T_{i-1,j-1}$, for $1 \leq i \leq j$. The collection of contractions in $\mathcal{U}_{\mathcal{T}}(E,H,\sigma)$ is called the* Schur class *and is denoted by $\mathcal{S}(E,H,\sigma)$.*

That is, $T \in \mathcal{U}_{\mathcal{T}}(E,H,\sigma)$ is a bounded, linear operator on $\mathcal{F}(E) \otimes_\sigma H$ of the form

$$T = \begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} & \cdots \\ 0 & I_E \otimes T_{00} & I_E \otimes T_{01} & I_E \otimes T_{02} & \cdots \\ 0 & 0 & I_{E^{\otimes 2}} \otimes T_{00} & I_{E^{\otimes 2}} \otimes T_{01} & \cdots \\ 0 & 0 & 0 & \vdots & \ddots \end{bmatrix}, \tag{2.1}$$

such that $T_{0j}(\sigma^{E^{\otimes j}} \circ \varphi_j(a)) = \sigma(a) T_{0j}$ for all $a \in M$ and $j \geq 0$.

2.2 Comparing $H^\infty(E), H^\infty(E^\sigma)$, and $\mathcal{U}_{\mathcal{T}}(E,H,\sigma)$

In order to understand the relationship between the Hardy algebras, $H^\infty(E)$ and $H^\infty(E^\sigma)$, and the algebra of upper triangular opertators, $\mathcal{U}_{\mathcal{T}}(E,H,\sigma)$, we must first familiarize ourselves with a couple of maps. Let $\iota : \sigma(M)' \to B(H)$ be the identity representation of $\sigma(M)'$ on H, and let $E^\sigma \otimes_\iota H$ denote the Hausdorff completion of the algebraic tensor product $E^\sigma \otimes H$ with respect to the positive semidefinite sesquilinear form $\langle \xi \otimes h, \eta \otimes k \rangle := \langle h, \iota(\langle \xi, \eta \rangle) k \rangle$ for $\xi \otimes h, \eta \otimes k \in E^\sigma \otimes H$. We construct $(E^\sigma)^{\otimes k} \otimes_\iota H$, for $k \in \mathbb{N}$, and $\mathcal{F}(E^\sigma) \otimes_\iota H$ analogously.

Definition 2.2.1. *Define $U : \mathcal{F}(E^\sigma) \otimes_\iota H \to \mathcal{F}(E) \otimes_\sigma H$ to be the Hilbert space direct sum $U = \bigoplus_{k=0}^\infty U_k$, where $U_k : (E^\sigma)^{\otimes k} \otimes_\iota H \to E^{\otimes k} \otimes_\sigma H$ is given by the formula $U_k(\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_k \otimes h) := (I_{E^{\otimes k-1}} \otimes \eta_1)(I_{E^{\otimes k-2}} \otimes \eta_2) \cdots (I_E \otimes \eta_{k-1}) \eta_k h.$*

In [MS04, Lemma 3.8], Muhly and Solel showed that U isometrically maps $\mathcal{F}(E^\sigma) \otimes_\iota H$ onto $\mathcal{F}(E) \otimes_\sigma H$.

Definition 2.2.2. *Define* $\rho : H^\infty(E^\sigma) \to B(\mathcal{F}(E) \otimes_\sigma H)$ *by the formula*

$$\rho(X) = U(X \otimes I_H)U^*, \quad X \in H^\infty(E^\sigma). \tag{2.2}$$

In [MS04, Theorem 3.9], Muhly and Solel showed that ρ is an ultraweakly continuous, completely isometric representation of $H^\infty(E^\sigma)$ on $\mathcal{F}(E) \otimes_\sigma H$ and that $\rho(H^\infty(E^\sigma)) = \sigma^{\mathcal{F}(E)}(H^\infty(E))'$, thus clarifying the relationship between the Hardy algebras.

Jenni Good drew our attention to the relationship between $H^\infty(E^\sigma)$ and $\mathcal{U}_{\mathcal{T}}(E, H, \sigma)$, which is made precise by the following lemma and whose proof follows easily thanks to Muhly-Solel's Theorem 3.9.

Lemma 2.2.3. *Define* $\rho : H^\infty(E^\sigma) \to B(\mathcal{F}(E) \otimes_\sigma H)$ *as in equation* (2.2)*. Then* $\mathcal{U}_{\mathcal{T}}(E, H, \sigma)^* = \rho(H^\infty(E^\sigma))$.

Proof. According to Theorem 3.9 in [MS04], to show that $\mathcal{U}_{\mathcal{T}}(E, H, \sigma)^* \subseteq \rho(H^\infty(E^\sigma))$, it suffices to show that every element of $\mathcal{U}_{\mathcal{T}}(E, H, \sigma)^*$ commutes with $\sigma^{\mathcal{F}(E)}(\varphi_\infty(a)), a \in M$, and with $\sigma^{\mathcal{F}(E)}(T_\xi), \xi \in E$.

For $a \in M$ and $T \in \mathcal{U}_{\mathcal{T}}(E, H, \sigma)$,

$$T^* \circ \sigma^{\mathcal{F}(E)}(\varphi_\infty(a))$$

$$= \begin{bmatrix} T_{00}^* & 0 & 0 & \cdots \\ T_{01}^* & I_E \otimes T_{00}^* & 0 & \cdots \\ T_{02}^* & I_E \otimes T_{01}^* & I_{E^{\otimes 2}} \otimes T_{00}^* & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} a \otimes I_H & & & \\ & \varphi(a) \otimes I_H & & \\ & & \varphi_2(a) \otimes I_H & \\ & & & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} T_{00}^*\sigma(a) & 0 & 0 & \cdots \\ T_{01}^*\sigma(a) & \varphi(a) \otimes T_{00}^* & 0 & \cdots \\ T_{02}^*\sigma(a) & \varphi(a) \otimes T_{01}^* & \varphi_2(a) \otimes T_{00}^* & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \sigma^{\mathcal{F}(E)}(\varphi_\infty(a)) \circ T^*$$

since $T_{0j} \in \mathfrak{I}(\sigma^{E^{\otimes j}} \circ \varphi_j, \sigma)$ and $a \otimes I_H \approx \sigma(a)$.

For $\xi \in E$ and $T \in \mathcal{U}_{\mathcal{T}}(E, H, \sigma)$,

$$T^* \circ \sigma^{\mathcal{F}(E)}(T_\xi)$$

$$= \begin{bmatrix} T_{00}^* & 0 & 0 & \cdots \\ T_{01}^* & I_E \otimes T_{00}^* & 0 & \cdots \\ T_{02}^* & I_E \otimes T_{01}^* & I_{E^{\otimes 2}} \otimes T_{00}^* & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \cdots \\ T_\xi^{(1)} \otimes I_H & 0 & 0 & \cdots \\ 0 & T_\xi^{(2)} \otimes I_H & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \cdots \\ (I_E \otimes T_{00}^*)(T_\xi^{(1)} \otimes I_H) & 0 & 0 & \cdots \\ (I_E \otimes T_{01}^*)(T_\xi^{(1)} \otimes I_H) & (I_{E^{\otimes 2}} \otimes T_{00}^*)(T_\xi^{(2)} \otimes I_H) & 0 & \cdots \\ (I_E \otimes T_{02}^*)(T_\xi^{(1)} \otimes I_H) & (I_{E^{\otimes 2}} \otimes T_{01}^*)(T_\xi^{(2)} \otimes I_H) & (I_{E^{\otimes 3}} \otimes T_{00}^*)(T_\xi^{(3)} \otimes I_H) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= \sigma^{\mathcal{F}(E)}(T_\xi) \circ T^*$$

because for $i, k \in \mathbb{N}$ with $i \geq k$,

$$(I_{E^{\otimes k}} \otimes T_{0,i-k}^*)(T_\xi^{(k)} \otimes I_H)(\eta_1 \otimes \cdots \otimes \eta_{k-1} \otimes h) = (I_{E^{\otimes k}} \otimes T_{0,i-k}^*)(\xi \otimes \eta_1 \otimes \cdots \otimes \eta_{k-1} \otimes h)$$

$$= \xi \otimes \eta_1 \otimes \cdots \otimes \eta_{k-1} \otimes T_{0,i-k}^*(h) = (T_\xi^{(i)} \otimes I_H)(I_{E^{\otimes k-1}} \otimes T_{0,i-k}^*)(\eta_1 \otimes \cdots \eta_{k-1} \otimes h),$$

where $T_\xi^{(i)}$ denotes the i^{th} component of the left creation operator T_ξ.

For the other inclusion, note that it is a consequence of [MS04, Theorem 3.9] that $\rho(\varphi_\infty^\sigma(a)) = I_{\mathcal{F}(E)} \otimes a, a \in \sigma(M)'$, and

$$\rho(T_\eta) = \begin{bmatrix} 0 & & & & \\ \eta & 0 & & & \\ & I_E \otimes \eta & 0 & & \\ & & I_{E^{\otimes 2}} \otimes \eta & \ddots & \\ & & & \ddots & \end{bmatrix}, \quad \eta \in E^\sigma.$$

Now it is easy to see that $\rho(\varphi_\infty^\sigma(a))$ and $\rho(T_\eta)$ are elements of $\mathfrak{U}_{\mathfrak{T}}(E, H, \sigma)^*$. □

We will rely on Lemma 2.2.3 to help us define a point evaluation on the elements of $H^\infty(E^\sigma)$ in the following section. In addition, the lemma will play a role in the proof of our generalized Nevanlinna-Pick theorem, Theorem 3.0.1.

2.3 Point Evaluations

2.3.1 Point Evaluation for $\mathfrak{U}_{\mathfrak{T}}(E, H, \sigma)$

In accordance with Constantinescu-Johnson's definition ([CJ03, Equation 3.2]), we define the point evaluation of $T \in \mathfrak{U}_{\mathfrak{T}}(E, H, \sigma)$ at $\eta \in E^\sigma$ with $\|\eta\| < 1$ by

$$T(\eta) := \langle C(0), TC(\eta) \rangle, \tag{2.3}$$

where $C(0) = \begin{bmatrix} I_H & 0 & 0 & \cdots \end{bmatrix}^T$ denotes the Cauchy kernel at 0 and $C(\eta)$ denotes the Cauchy kernel at η.

The following result about the point evaluation is a generalization of [CJ03, Lemma 3.1] to the setting of W^*-correspondences. It will be useful in the proof of Theorem 3.0.1.

Lemma 2.3.1. *If $T \in \mathfrak{U}_{\mathfrak{I}}(E, H, \sigma)$ and $\eta \in E^\sigma$ with $\|\eta\| < 1$, then*

$$TC(\eta) = (I_{\mathcal{F}(E)} \otimes T(\eta))C(\eta).$$

Proof. Expand the left hand side:

$$TC(\eta) = \begin{bmatrix} T_{00} & T_{01} & T_{02} & \cdots \\ 0 & I_E \otimes T_{00} & I_E \otimes T_{01} & \cdots \\ 0 & 0 & I_{E^{\otimes 2}} \otimes T_{00} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} I_H \\ \eta \\ \eta^{(2)} \\ \vdots \end{bmatrix} = \begin{bmatrix} \sum_{r=0}^{\infty} T_{0r}\eta^{(r)} \\ \sum_{r=0}^{\infty}(I_E \otimes T_{0r})\eta^{(r+1)} \\ \sum_{r=0}^{\infty}(I_{E^{\otimes 2}} \otimes T_{0r})\eta^{(r+2)} \\ \vdots \end{bmatrix}.$$

Expand the right hand side:

$$(I_{\mathcal{F}(E)} \otimes T(\eta))C(\eta) = \begin{bmatrix} T(\eta) & & & \\ & I_E \otimes T(\eta) & & \\ & & I_{E^{\otimes 2}} \otimes T(\eta) & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} I_H \\ \eta \\ \eta^{(2)} \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} T(\eta) \\ (I_E \otimes T(\eta))\eta \\ (I_{E^{\otimes 2}} \otimes T(\eta))\eta^{(2)} \\ \vdots \end{bmatrix}.$$

The k^{th} entry of the right hand side is

$$(I_{E^{\otimes k}} \otimes T(\eta))\eta^{(k)} = (I_{E^{\otimes k}} \otimes C(0)^* TC(\eta))\eta^{(k)} = (I_{E^{\otimes k}} \otimes \sum_{r=0}^{\infty} T_{0r}\eta^{(r)})\eta^{(k)}$$

$$= \sum_{r=0}^{\infty} \big(I_{E^{\otimes k}} \otimes (T_{0r}(I_{E^{\otimes r-1}} \otimes \eta) \cdots (I_E \otimes \eta)\eta)\big)\eta^{(k)}$$

$$= \sum_{r=0}^{\infty}(I_{E^{\otimes k}} \otimes T_{0r})(I_{E^{\otimes k}} \otimes I_{E^{\otimes r-1}} \otimes \eta) \cdots (I_{E^{\otimes k}} \otimes I_E \otimes \eta)(I_{E^{\otimes k}} \otimes \eta)\eta^{(k)}$$

$$= \sum_{r=0}^{\infty}(I_{E^{\otimes k}} \otimes T_{0r})\eta^{(r+k)},$$

which agrees with the k^{th} entry of the left hand side. □

2.3.2 Point Evaluation for $H^\infty(E^\sigma)$

Since $\mathcal{U}_{\mathfrak{I}}(E, H, \sigma)^* = \rho(H^\infty(E^\sigma))$, we define the point evaluation of an element in $H^\infty(E^\sigma)$ at a point in E^σ to agree with equation (2.3).

Definition 2.3.2. *For $X \in H^\infty(E^\sigma)$ and $\eta \in E^\sigma$ with $\|\eta\| < 1$, we define the point evaluation by the formula*

$$\hat{X}(\eta) := \langle \rho(X)C(0), C(\eta)\rangle, \quad \eta \in E^\sigma, \tag{2.4}$$

where $C(0) = \begin{bmatrix} I_H & 0 & 0 & \dots \end{bmatrix}^T$ denotes the Cauchy kernel at 0 and $C(\eta)$ denotes the Cauchy kernel at η.

The following proposition shows that the point evaluation takes values in $\sigma(M)'$ and is linear.

Proposition 2.3.3. *For $X, Y \in H^\infty(E^\sigma), \eta \in E^\sigma$ with $\|\eta\| < 1$, and $\lambda \in \mathbb{C}$,*

1. *$\hat{X}(\eta) \in \sigma(M)'$.*

2. $\widehat{X+\lambda Y} = \hat{X} + \overline{\lambda}\hat{Y}$.

Proof. First note that for all $h \in H$,

$$\|C(\eta)h\|^2 = \left\| \begin{bmatrix} h \\ \eta h \\ \eta^{(2)} h \\ \vdots \end{bmatrix} \right\|^2 = \sum_{k=0}^{\infty} \|\eta^{(k)} h\|^2 \leq \sum_{k=0}^{\infty} \|\eta^{(k)}\|^2 \|h\|^2 \leq \sum_{k=0}^{\infty} \|\eta\|^{2k} \|h\|^2.$$

Thus $\|C(\eta)\|^2 \leq \sum_{k=0}^{\infty} \|\eta\|^{2k} < \infty$, since $\|\eta\| < 1$. Thus $\hat{X}(\eta)$ is a product of bounded, linear operators on H, hence $\hat{X}(\eta) \in B(H)$. Now we show that $\hat{X}(\eta) \in \sigma(M)'$. Since $\eta \in E^\sigma$, then $\eta\sigma(a) = (\varphi(a) \otimes I_H)\eta, a \in M$. Therefore,

$$C(\eta)\sigma(a) = \begin{bmatrix} I_H \\ \eta \\ \eta^{(2)} \\ \vdots \end{bmatrix} \sigma(a) = \begin{bmatrix} \sigma(a) \\ \eta\sigma(a) \\ (I_E \otimes \eta)\eta\sigma(a) \\ \vdots \end{bmatrix} = \begin{bmatrix} \sigma(a) \\ (\varphi(a) \otimes I_H)\eta \\ (\varphi_2(a) \otimes I_H)(I_E \otimes \eta)\eta \\ \vdots \end{bmatrix}$$

$$= \left(\begin{bmatrix} a & & & \\ & \varphi(a) & & \\ & & \varphi_2(a) & \\ & & & \ddots \end{bmatrix} \otimes I_H \right) \begin{bmatrix} I_H \\ \eta \\ \eta^{(2)} \\ \vdots \end{bmatrix} = (\varphi_\infty(a) \otimes I_H)C(\eta),$$

where the third equality holds by Lemma 1.2.8. Thus we have

$$\hat{X}(\eta)\sigma(a) = C(0)^*\rho(X)^*C(\eta)\sigma(a) = C(0)^*\rho(X^*)(\varphi_\infty(a) \otimes I_H)C(\eta)$$

$$= C(0)^*(\varphi_\infty(a) \otimes I_H)\rho(X^*)C(\eta) = \sigma(a)C(0)^*\rho(X)^*C(\eta) = \sigma(a)\hat{X}(\eta),$$

where the third equality is due to Theorem 3.9 in [MS04]. Therefore $\hat{X}(\eta) \in \sigma(M)'$.

It is easy to see that the second claim holds because ρ is linear, and the inner product is conjugate linear in the first variable. □

Though the point evaluation is not multiplicative, i.e., $\widehat{XY}(\eta) \neq \hat{X}(\eta)\hat{Y}(\eta)$, we may view it as giving rise to antihomomorphisms from $H^\infty(E^\sigma)$ into the completely bounded maps on $\sigma(M)'$ thanks to [MS09, Theorem 19]. The proof of this assertion is simply a matter of shifting emphasis. We let E^σ play the role of E in [MS09, Theorem 19]. Here are the details.

First recall that for a pair of points $\mathfrak{z}$ and $\mathfrak{w}$ in E^σ with norm less than 1, the inner product $\langle C(\mathfrak{z}), C(\mathfrak{w})\rangle$ is given by the formula $\langle C(\mathfrak{z}), C(\mathfrak{w})\rangle = C(\mathfrak{z})^* C(\mathfrak{w})$. Consequently, the map $a \mapsto \langle C(\mathfrak{z}), \rho(\varphi_\infty^\sigma(a))C(\mathfrak{w})\rangle = \langle C(\mathfrak{z}), (I_{\mathcal{F}(E)} \otimes a)C(\mathfrak{w})\rangle$ is a completely bounded map from $\sigma(M)'$ into itself. So if $\mathfrak{z} \in E^\sigma$ with $\|\mathfrak{z}\| < 1$ and $X \in H^\infty(E^\sigma)$, we may define the map $\Phi_X^{\mathfrak{z}}$ on $\sigma(M)'$ by the formula

$$\Phi_X^{\mathfrak{z}}(a) := \langle C(\mathfrak{z}), \rho(\varphi_\infty^\sigma(a))\rho(X)C(0)\rangle, \quad a \in \sigma(M)'. \tag{2.5}$$

Note that $C(0) = \begin{bmatrix} I_H & 0 & 0 & \cdots \end{bmatrix}^T$, so our definition is precisely that in Theorem 19 of [MS09] with E^σ replacing E, $\rho(\varphi_\infty^\sigma(a))$ replacing $\varphi_\infty(a)$, $\rho(X)$ replacing X, and $\mathfrak{z}$ replacing ξ in the notation of that theorem.

Now note that formula (2) of Theorem 19 of [MS09] translated to our context allows us to write

$$\rho(X)^*\rho(\varphi_\infty^\sigma(a^*))C(\mathfrak{z}) = \rho(\varphi_\infty^\sigma(\Phi_X^{\mathfrak{z}}(a)^*))C(\mathfrak{z}).$$

Consequently, for $X, Z \in H^\infty(E^\sigma), a \in \sigma(M)'$, and $\mathfrak{z} \in E^\sigma$ with $\|\mathfrak{z}\| < 1$,

$$\begin{aligned}\Phi^{\mathfrak{z}}_{XZ}(a) &= \langle C(\mathfrak{z}), \rho(\varphi^\sigma_\infty(a))\rho(XZ)C(0)\rangle = \langle \rho(X)^*\rho(\varphi^\sigma_\infty(a^*))C(\mathfrak{z}), \rho(Z)C(0)\rangle \\ &= \langle \rho(\varphi^\sigma_\infty(\Phi^{\mathfrak{z}}_X(a)^*))C(\mathfrak{z}), \rho(Z)C(0)\rangle = \langle C(\mathfrak{z}), \rho(\varphi^\sigma_\infty(\Phi^{\mathfrak{z}}_X(a)))\rho(Z)C(0)\rangle \\ &= \Phi^{\mathfrak{z}}_Z(\Phi^{\mathfrak{z}}_X(a)),\end{aligned}$$

which shows that $\Phi^{\mathfrak{z}}_{XZ} = \Phi^{\mathfrak{z}}_Z \circ \Phi^{\mathfrak{z}}_X$. Thus we arrive at the following theorem.

Theorem 2.3.4. *Let $\mathfrak{z} \in E^\sigma$ with $\|\mathfrak{z}\| < 1$ and $X \in H^\infty(E^\sigma)$. Define $\Phi^{\mathfrak{z}}_X : \sigma(M)' \to \sigma(M)'$ by formula (2.5). Then the map $X \mapsto \Phi^{\mathfrak{z}}_X$ is an algebra antihomomorphism from $H^\infty(E^\sigma)$ into the completely bounded maps on $\sigma(M)'$.*

2.3.3 Point Evaluation for $H^\infty(E)$

Muhly-Solel's point evaluation for elements in $H^\infty(E)$ is defined in [MS04], [MS09], and elsewhere. In this subsection, we explore the different notations used and make explicit the connections between various definitions. We begin with the point evaluation defined on the page preceding Proposition 5.1 in [MS04].

Definition 2.3.5. *For $X \in H^\infty(E)$ and $\eta \in E^\sigma$ with $\|\eta\| < 1$, define the point evaluation $\hat{X}(\eta^*)$ by the formula*

$$\hat{X}(\eta^*) := \sigma \times \hat{\eta^*}(X),$$

where $\sigma \times \hat{\eta^*}$ denotes the ultraweakly continuous, completely contractive representation of $H^\infty(E)$ on H defined on the generators of $H^\infty(E)$ by

$$\begin{aligned}\varphi_\infty(a) &\mapsto \sigma(a) \\ T_\xi &\mapsto \hat{\eta^*}(\xi),\end{aligned}$$

where $\hat{\eta^*} : E \to B(H)$ *is given by* $\hat{\eta^*}(\xi)(h) = \eta^*(\xi \otimes h)$*. The resulting map* $\hat{X} : \mathbb{D}(E^\sigma)^* \to B(H)$*, where* $\mathbb{D}(E^\sigma)^* := \{\eta^* \in (E^\sigma)^* \mid \|\eta\| < 1\}$*, is called the* Fourier transform *of* X*.*

In Proposition 5.1 in [MS04], the authors prove that the point evaluation in Definition 2.3.5 can be rewritten in the form

$$\hat{X}(\eta^*) = L_\eta^* \rho_1(X) \iota_H, \tag{2.6}$$

where

- the insertion operator $L_\eta : H \to \mathcal{F}(E^\sigma) \otimes_\iota H$ is given by

$$L_\eta = \begin{bmatrix} I_H \\ L_\eta^{(1)} \\ L_\eta^{(2)} \\ \vdots \end{bmatrix},$$

  where $L_\eta^{(k)} : H \to (E^\sigma)^{\otimes k} \otimes_\iota H$ is defined by $L_\eta^{(k)}(h) = \eta^{\otimes k} \otimes h$. In other words, $L_\eta = U^* C(\eta)$, for U as in Definition 2.2.1, and this is how we will prefer to think of it.

- $\rho_1 : H^\infty(E) \to B(\mathcal{F}(E^\sigma) \otimes_\iota H)$ is defined on generators by

$$\begin{aligned} \rho_1(\varphi_\infty(a)) &= I_{\mathcal{F}(E^\sigma)} \otimes \sigma(a) \\ \rho_1(T_\xi) &= I_{\mathcal{F}(E^\sigma)} \otimes W(\xi), \end{aligned} \tag{2.7}$$

  where $W(\xi) : H \to E^\sigma \otimes_\iota H$ is defined in the paragraphs before Section 4 in [MS04] by the formula $W(\xi) = U_1^* L_\xi^{(1)}$, with U_1 denoting the restriction of U to $E^\sigma \otimes_\iota H$ and $L_\xi^{(1)} : H \to E \otimes_\sigma H$ given by $L_\xi^{(1)}(h) = \xi \otimes h$.

Thus we can rewrite the point evaluation in equation (2.6) in terms of the Cauchy kernel as follows:

$$\hat{X}(\eta^*) = C(\eta)^* U \rho_1(X) \iota_H. \tag{2.8}$$

Furthermore, in [MS04, Proposition 5.1] the authors prove that this point evaluation is multiplicative, i.e.,

$$\widehat{XY}(\eta^*) = \hat{X}(\eta^*)\hat{Y}(\eta^*)$$

using the facts that

$$L_\eta^* \rho_1(X) = L_\eta^* \rho_1(X) \iota_H L_\eta^*$$

and

$$C(\eta)^* U \rho_1(X) = C(\eta)^* U \rho_1(X) \iota_H C(\eta)^* U,$$

which are observed in [MS04, Remark 5.2].

In [MS09], Muhly and Solel initially define the point evaluation as in Definition 2.3.5, but in [MS09, Proposition 7], they rewrite the point evaluation as

$$\hat{X}(\eta^*) = C(\eta)^* U \rho_2(X) \iota_H, \tag{2.9}$$

where $\rho_2 : H^\infty(E) \to (\iota^{\mathcal{F}(E^\sigma)}(H^\infty(E^\sigma)))'$ is the ultraweakly continuous, completely isometric isomorphism given by

$$\rho_2(X) = U^*(X \otimes I_H)U. \tag{2.10}$$

They rewrite the point evaluation, but they do not provide the details that show it is equivalent to either Definition 2.3.5 or equation (2.8). We will prove that $\rho_1 = \rho_2$, thereby justifying the rewrite in equation (2.9).

Proposition 2.3.6. *Let* ρ_1 *be defined as in equation* (2.7) *and* ρ_2 *be defined as in equation* (2.10)*. Then* $\rho_1(X) = \rho_2(X)$ *for all* $X \in H^\infty(E)$*.*

The proof of Proposition 2.3.6 will require the following lemma.

Lemma 2.3.7. *For* $n \in \mathbb{N}, \eta_1, \ldots, \eta_n \in E^\sigma, \xi \in E,$ *and* $h \in H$*,*

$$U_n^*((I_{E^{\otimes n}} \otimes \eta_1) \cdots (I_E \otimes \eta_n) L_\xi^{(1)} h) = \eta_1 \otimes \cdots \otimes \eta_n \otimes U_1^*(L_\xi^{(1)} h),$$

where U_k *is defined in Definition 2.2.1 and* $L_\xi^{(1)} : H \to E \otimes_\sigma H$ *is given by* $L_\xi^{(1)}(h) := \xi \otimes h$ *for all* $h \in H$*.*

Proof. Since U is surjective, for fixed $\xi \in E$ and $h \in H$, there exist $\eta \in E^\sigma$ and $k \in H$ such that

$$U_1(\eta \otimes k) = \eta k = \xi \otimes h = L_\xi^{(1)} h.$$

Thus we have

$$\begin{aligned} U_n^*((I_{E^{\otimes n}} \otimes \eta_1) \cdots (I_E \otimes \eta_n) L_\xi^{(1)} h) &= U_n^*((I_{E^{\otimes n}} \otimes \eta_1) \cdots (I_E \otimes \eta_n) \eta k) \\ &= \eta_1 \otimes \cdots \otimes \eta_n \otimes \eta \otimes k = \eta_1 \otimes \cdots \otimes \eta_n \otimes U_1^*(L_\xi^{(1)} h). \end{aligned}$$

□

Now we are ready to show $\rho_1 = \rho_2$.

Proof of Proposition 2.3.6. Since both ρ_1 and ρ_2 are $*$-homomorphisms, it suffices to show that they agree on the generators of $H^\infty(E)$.

Let $n \in \mathbb{N}$ and let $\eta_1 \otimes \cdots \otimes \eta_n \otimes h \in (E^\sigma)^{\otimes n} \otimes H$. Then

$$\rho_2(\varphi_\infty(a)) \left(\begin{bmatrix} 0 \\ \vdots \\ \eta_1 \otimes \cdots \otimes \eta_n \otimes h \\ 0 \\ \vdots \end{bmatrix} \right) = U^*(\varphi_\infty(a) \otimes I_H) U \begin{bmatrix} 0 \\ \vdots \\ \eta_1 \otimes \cdots \otimes \eta_n \otimes h \\ 0 \\ \vdots \end{bmatrix}$$

$$= U^* \begin{bmatrix} a & & & \\ & \varphi(a) \otimes I_H & & \\ & & \varphi_2(a) \otimes I_H & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ (I_{E^{\otimes n-1}} \otimes \eta_1) \cdots (I_E \otimes \eta_{n-1}) \eta_n h \\ 0 \\ \vdots \end{bmatrix}$$

$$= U^* \begin{bmatrix} 0 \\ \vdots \\ (\varphi_n(a) \otimes I_H)(I_{E^{\otimes n-1}} \otimes \eta_1) \cdots (I_E \otimes \eta_{n-1}) \eta_n h \\ 0 \\ \vdots \end{bmatrix}$$

$$= U^* \begin{bmatrix} 0 \\ \vdots \\ (I_{E^{\otimes n-1}} \otimes \eta_1) \cdots (I_E \otimes \eta_{n-1}) \eta_n \sigma(a) h \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \eta_1 \otimes \cdots \otimes \eta_n \otimes \sigma(a) h \\ 0 \\ \vdots \end{bmatrix},$$

where the penultimate equality holds because of Lemma 1.2.8. Thus $\rho_2(\varphi_\infty(a)) = I_{\mathcal{F}(E^\sigma)} \otimes \sigma(a)$. Moreover,

$$\rho_2(T_\xi) \left(\begin{bmatrix} 0 \\ \vdots \\ \eta_1 \otimes \cdots \otimes \eta_n \otimes h \\ 0 \\ \vdots \end{bmatrix} \right) = U^*(T_\xi \otimes I_H) U \begin{bmatrix} 0 \\ \vdots \\ \eta_1 \otimes \cdots \otimes \eta_n \otimes h \\ 0 \\ \vdots \end{bmatrix}$$

$$= U^*(T_\xi \otimes I_H) \begin{bmatrix} 0 \\ \vdots \\ (I_{E^{\otimes n-1}} \otimes \eta_1) \cdots (I_E \otimes \eta_{n-1}) \eta_n h \\ 0 \\ \vdots \end{bmatrix}$$

$$= U^* \begin{bmatrix} 0 \\ \vdots \\ \xi \otimes ((I_{E^{\otimes n-1}} \otimes \eta_1) \cdots (I_E \otimes \eta_{n-1})\eta_n h) \\ 0 \\ \vdots \end{bmatrix}$$

$$= U^* \begin{bmatrix} 0 \\ \vdots \\ (I_{E^{\otimes n}} \otimes \eta_1) \cdots (I_{E^{\otimes 2}} \otimes \eta_{n-1})(I_E \otimes \eta_n)(\xi \otimes h) \\ 0 \\ \vdots \end{bmatrix}$$

$$= U^* \begin{bmatrix} 0 \\ \vdots \\ (I_{E^{\otimes n}} \otimes \eta_1) \cdots (I_{E^{\otimes 2}} \otimes \eta_{n-1})(I_E \otimes \eta_n)L_\xi^{(1)} h \\ 0 \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \vdots \\ U_n^*((I_{E^{\otimes n}} \otimes \eta_1) \cdots (I_{E^{\otimes 2}} \otimes \eta_{n-1})(I_E \otimes \eta_n)L_\xi^{(1)} h) \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \eta_1 \otimes \cdots \otimes \eta_n \otimes U_1^*(L_\xi^{(1)} h) \\ 0 \\ \vdots \end{bmatrix},$$

where the final equality is due to Lemma 2.3.7. Now note

$$(I_{\mathcal{F}(E^\sigma)} \otimes W(\xi)) \left(\begin{bmatrix} 0 \\ \vdots \\ \eta_1 \otimes \cdots \otimes \eta_n \otimes h \\ 0 \\ \vdots \end{bmatrix} \right) = (I_{\mathcal{F}(E^\sigma)} \otimes U_1^* L_\xi^{(1)}) \left(\begin{bmatrix} 0 \\ \vdots \\ \eta_1 \otimes \cdots \otimes \eta_n \otimes h \\ 0 \\ \vdots \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 0 \\ \vdots \\ \eta_1 \otimes \cdots \otimes \eta_n \otimes U_1^* L_\xi^{(1)}(h) \\ 0 \\ \vdots \end{bmatrix} \right).$$

Thus $\rho_2(T_\xi) = I_{\mathcal{F}(E^\sigma)} \otimes W(\xi)$. Since ρ_1 and ρ_2 agree on the generators of $H^\infty(E)$, they are equal. □

We recapitulate our results regarding the Muhly-Solel point evaluation in the following remark.

Remark 2.3.8. *Since $\rho_1 = \rho_2$, the point evaluations in Definition 2.3.5 and equations* (2.6), (2.8), *and* (2.9) *are all the same. The point evaluation is multiplicative. Finally, for $X \in H^\infty(E)$ and $\eta \in E^\sigma$ with $\|\eta\| < 1$, we can rewrite the point evaluation one last time as*

$$\begin{aligned} \hat{X}(\eta^*) = C(\eta)^* U \rho_2(X) \iota_H = C(\eta)^* U U^* (X \otimes I_H) U \iota_H = C(\eta)^* (X \otimes I_H) C(0) \\ = \langle C(\eta), (X \otimes I_H) C(0) \rangle = \langle (X \otimes I_H) C(0), C(\eta) \rangle^*, \end{aligned}$$

and this will be our preferred way to write it.

CHAPTER 3
A NEW NEVANLINNA-PICK THEOREM

In this chapter, we prove our generalized noncommutative Nevanlinna-Pick theorem. Originally constructed as a generalization of [CJ03, Theorem 3.4] to the W^*-correspondence setting, the theorem is as follows.

Theorem 3.0.1. *Let E be a W^*-correspondence over a W^*-algebra M, with the left action of M on E given by a faithful, normal $*$-homomorphism $\varphi : M \to \mathcal{L}(E)$. Let σ be a faithful, normal representation of M on a Hilbert space H. Let $\mathfrak{z}_1, \ldots, \mathfrak{z}_N$ be N distinct elements of E^σ with $\|\mathfrak{z}_i\| < 1$ for all $i = 1, \ldots, N$, and let $\Lambda_1, \ldots, \Lambda_N \in \sigma(M)'$. There exists $X \in H^\infty(E^\sigma)$ with $\|X\| \leq 1$ such that*

$$\hat{X}(\mathfrak{z}_i) = \Lambda_i, \quad i = 1, \ldots, N,$$

if and only if the Pick matrix

$$\mathcal{A}_N = \left[\langle C(\mathfrak{z}_i), C(\mathfrak{z}_j) \rangle - \langle (I_{\mathcal{F}(E)} \otimes \Lambda_i) C(\mathfrak{z}_i), (I_{\mathcal{F}(E)} \otimes \Lambda_j) C(\mathfrak{z}_j) \rangle \right]_{i,j=1}^{N} \tag{3.1}$$

is positive semidefinite.

We remark that Theorem 3.0.1, like [CJ03, Theorem 3.4], is a nontangential interpolation theorem, in contrast to [MS04, Theorem 5.3]. Due to the nature of our proof, the necessary conditions on the left-tangential operators would be somewhat restrictive. Therefore, we choose to focus on the nontangential version stated above, and to compare it to nontangential versions of Theorem 5.3 in [MS04] and Theorem

7.4 in [Pop03]. We do, however, acknowledge that Theorem 3.0.1 can be generalized to a left-tangential version.

The proof of Theorem 3.0.1 is modeled after Constantinescu-Johnson's proof of Theorem 3.4 in [CJ03]. Our main tool is the displacement equation, which will be discussed in detail in Section 3.1. We also rely on results from the theory of time varying linear systems as well as a corollary of Douglas's lemma, all of which are stated and proved in Section 3.2.

3.1 The Displacement Equation

The displacement equation is a resolvent equation. In the context of this subject, the equation was given the special name *displacement equation* by Kailath, Kung, and Morf in [KKM79] for the purpose of measuring the extent to which a matrix is Toeplitz (see also [KS95]). We are interested in a displacement equation of the form

$$A - \theta(A) = B,$$

where A and B are bounded operators on Hilbert space and θ is a completely positive map of norm less than 1 on the algebra of bounded operators. Given B and θ, one can solve for the unique solution A by computing the resolvent, $A = (I - \theta)^{-1}(B) = \sum_{k=0}^{\infty} \theta^k(B)$. Note that $(I - \theta)^{-1}$ is completely positive since it is the sum of powers of a completely positive map, and it is bounded since $\|\theta\| < 1$.

In order to apply the displacement theory to our context, fix $\mathfrak{z}_1, \mathfrak{z}_2, \ldots, \mathfrak{z}_N \in E^\sigma$

with $\|\mathfrak{z}_i\| < 1$ for all $i = 1, \ldots, N$, and $\Lambda_1, \ldots, \Lambda_N \in \sigma(M)'$. Then form the matrices

$$U := \begin{bmatrix} I_H \\ \vdots \\ I_H \end{bmatrix}, V := \begin{bmatrix} \Lambda_1^* \\ \vdots \\ \Lambda_N^* \end{bmatrix}, \text{ and } \mathfrak{z} := \begin{bmatrix} \mathfrak{z}_1 & & \\ & \ddots & \\ & & \mathfrak{z}_N \end{bmatrix}. \tag{3.2}$$

For the remainder of this chapter, we reserve the above notation for these matrices. We emphasize that U defined in equation (3.2) is in accordance with [CJ03] and should not be confused with the isomorphism from Definition 2.2.1.

Let $H^{(N)}$ denote $H \otimes \mathbb{C}^N \approx H^{\oplus N}$, and let $\sigma^{(N)} : M \to B(H^{(N)})$ be the representation of M on $H^{(N)}$ defined at $a \in M$ to be the $N \times N$ diagonal matrix

$$\sigma^{(N)}(a) = \begin{bmatrix} \sigma(a) & & \\ & \ddots & \\ & & \sigma(a) \end{bmatrix}.$$

As in Definition 1.2.6, define the intertwining space $\mathfrak{I}(\sigma^{(N)}, (\sigma^{(N)})^E \circ \varphi) := \{\eta \in B(H^{(N)}, E \otimes_{\sigma^{(N)}} H^{(N)}) \mid \eta\sigma^{(N)}(a) = (\sigma^{(N)})^E \circ \varphi(a)\eta \quad \forall a \in M\}$. Also define $\eta^{(k)}$ and the Cauchy Kernel $C(\eta)$ for $\eta \in \mathfrak{I}(\sigma^{(N)}, (\sigma^{(N)})^E \circ \varphi)$ as in Definition 1.2.9. Observe that $\mathfrak{z}$ from equation (3.2) belongs to $\mathfrak{I}(\sigma^{(N)}, (\sigma^{(N)})^E \circ \varphi)$ and $\|\mathfrak{z}\| < 1$.

Consider the displacement equation

$$A - \theta_{\mathfrak{z}}(A) = UU^* - VV^*, \tag{3.3}$$

where U, V, and $\mathfrak{z}$ are as in equation (3.2) and $\theta_{\mathfrak{z}} : \sigma^{(N)}(M)' \to \sigma^{(N)}(M)'$ is defined by the formula $\theta_{\mathfrak{z}}(A) = \mathfrak{z}^*(I_E \otimes A)\mathfrak{z}$. Equation (3.3) admits a unique solution $A \in \sigma^{(N)}(M)'$. To see this, let $k \in \mathbb{N}$, and note that $\theta_{\mathfrak{z}}^k(B) = (\mathfrak{z}^{(k)})^*(I_{E^{\otimes k}} \otimes B)\mathfrak{z}^{(k)}$.

Since $\|\theta_{\mathfrak{z}}\| \leq \|\mathfrak{z}\|^2 < 1$, $(I_{B(H^{(N)})} - \theta_{\mathfrak{z}})^{-1}$ is a completely bounded map on $\sigma^{(N)}(M)'$. Consequently, we can solve equation (3.3) for A:

$$\begin{aligned} A = (I_{B(H^{(N)})} - \theta_{\mathfrak{z}})^{-1}(UU^* - VV^*) &= \sum_{k=0}^{\infty} \theta_{\mathfrak{z}}^k (UU^* - VV^*) \\ &= \sum_{k=0}^{\infty} (\mathfrak{z}^{(k)})^* (I_{E^{\otimes k}} \otimes (UU^* - VV^*))\mathfrak{z}^{(k)} \\ &= \sum_{k=0}^{\infty} (\mathfrak{z}^{(k)})^* (I_{E^{\otimes k}} \otimes UU^*)\mathfrak{z}^{(k)} - \sum_{k=0}^{\infty} (\mathfrak{z}^{(k)})^* ((I_{E^{\otimes k}} \otimes VV^*)\mathfrak{z}^{(k)} \\ &= C(\mathfrak{z})^*(I_{\mathcal{F}(E)} \otimes UU^*)C(\mathfrak{z}) - C(\mathfrak{z})^*(I_{\mathcal{F}(E)} \otimes VV^*)C(\mathfrak{z}) \\ &= \left[\langle C(\mathfrak{z}_i), C(\mathfrak{z}_j)\rangle - \langle (I_{\mathcal{F}(E)} \otimes \Lambda_i)C(\mathfrak{z}_i), (I_{\mathcal{F}(E)} \otimes \Lambda_j)C(\mathfrak{z}_j)\rangle \right], \end{aligned}$$

which is the Pick matrix $\mathcal{A}_{\mathcal{N}}$ in Theorem 3.0.1.

In the proof of Theorem 3.0.1, it will be convenient to write $\mathcal{A}_{\mathcal{N}}$ in terms of different notation. Thus define two maps U_∞^* and V_∞^* both from $\mathcal{F}(E) \otimes H$ to $H^{(N)}$ by

$$\begin{aligned} U_\infty^* &:= \begin{bmatrix} U & \mathfrak{z}^*(I_E \otimes U) & (\mathfrak{z}^{(2)})^*(I_{E^{\otimes 2}} \otimes U) & \ldots \end{bmatrix} = C(\mathfrak{z})^*(I_{\mathcal{F}(E)} \otimes U) \\ V_\infty^* &:= \begin{bmatrix} V & \mathfrak{z}^*(I_E \otimes V) & (\mathfrak{z}^{(2)})^*(I_{E^{\otimes 2}} \otimes V) & \ldots \end{bmatrix} = C(\mathfrak{z})^*(I_{\mathcal{F}(E)} \otimes V). \end{aligned}$$

A quick calculation shows that the Pick matrix $\mathcal{A}_{\mathcal{N}} = U_\infty^* U_\infty - V_\infty^* V_\infty$. Keeping the definitions of U, V, and $\mathfrak{z}$ from equation (3.2) in mind, we rewrite U_∞ and V_∞ in terms of the Cauchy kernels as follows:

$$U_\infty = \begin{bmatrix} C(\mathfrak{z}_1) & \cdots & C(\mathfrak{z}_N) \end{bmatrix}$$

and

$$V_\infty = \begin{bmatrix} (I_{\mathcal{F}(E)} \otimes \Lambda_1)C(\mathfrak{z}_1) & \cdots & (I_{\mathcal{F}(E)} \otimes \Lambda_N)C(\mathfrak{z}_N) \end{bmatrix}.$$

These observations will be useful in the proof of Theorem 3.0.1, so we summarize them in the following remark.

Remark 3.1.1. *The Pick matrix $\mathcal{A}_{\mathcal{N}}$ in equation* (3.1) *is the unique solution to the displacement equation* (3.3)*, and it may be written in the form $\mathcal{A}_{\mathcal{N}} = U_\infty^* U_\infty - V_\infty^* V_\infty$, where $U_\infty = \begin{bmatrix} C(\mathfrak{z}_1) & \cdots & C(\mathfrak{z}_N) \end{bmatrix}$ and*
$V_\infty = \begin{bmatrix} (I_{\mathcal{F}(E)} \otimes \Lambda_1) C(\mathfrak{z}_1) & \cdots & (I_{\mathcal{F}(E)} \otimes \Lambda_N) C(\mathfrak{z}_N) \end{bmatrix}$.

The following lemma is the crux of the proof of Theorem 3.0.1. It relates the positivity of the Pick matrix to the existence of a special element in the Schur class, $\mathcal{S}(E, H, \sigma)$. Recall that $\mathcal{S}(E, H, \sigma)$ is defined in Definition 2.1.1 to be the collection of contractive upper triangular operators $T = [T_{ij}]_{i,j=0}^\infty$ such that $T_{ij} = I_E \otimes T_{i-1,j-1}$ for $1 \le i \le j$ and $T_{0j}(\sigma^{E^{\otimes j}} \circ \varphi_j(a)) = \sigma(a) T_{0j}$ for all $a \in M$ and $j \ge 0$.

Lemma 3.1.2. *The solution to the displacement equation* (3.3) *is positive semidefinite if and only if there exists $T \in \mathcal{S}(E, H, \sigma)$ such that $T U_\infty = V_\infty$.*

The proof of Lemma 3.1.2 will require some results about time varying linear systems and a corollary of Douglas's lemma. We pause to discuss this background information in detail before proceeding with the proof.

3.2 Background Results

3.2.1 The Transfer Map

We begin with the definition of a state-space model of a discrete time varying linear system. For more information, see [Con96, Section 2.3].

Definition 3.2.1. *The* state-space model *of a discrete time varying linear system is an equation of the form*

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \begin{bmatrix} x(t+1) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{Z}, \tag{3.4}$$

where $\{\mathcal{U}(t)\}_{t\in\mathbb{Z}}, \{\mathcal{Y}(t)\}_{t\in\mathbb{Z}}$*, and* $\{\mathcal{H}(t)\}_{t\in\mathbb{Z}}$ *are given families of Hilbert spaces called the input, output, and state spaces, respectively;* $A(t) \in B(\mathcal{H}(t+1), \mathcal{H}(t)), B(t) \in B(\mathcal{U}(t), \mathcal{H}(t)), C(t) \in B(\mathcal{H}(t+1), \mathcal{Y}(t))$*, and* $D(t) \in B(\mathcal{U}(t), \mathcal{Y}(t))$ *are given operators; and* $u(t) \in \mathcal{U}(t), y(t) \in \mathcal{Y}(t)$*, and* $x(t) \in \mathcal{H}(t)$ *for all* $t \in \mathbb{Z}$*. The system is said to be* contractive *if* $\left\| \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \right\| \leq 1$ *for all* $t \in \mathbb{Z}$ *and* unitary *if* $\begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix}$ *is unitary for all* $t \in \mathbb{Z}$*.*

Given a state-space model, the operators $A(t), B(t), C(t)$, and $D(t)$ uniquely determine the so-called transfer map of the system.

Definition 3.2.2. *A* transfer map *of a time varying linear system is an operator* $T : \bigoplus_{t\in\mathbb{Z}} \mathcal{U}(t) \to \bigoplus_{t\in\mathbb{Z}} \mathcal{Y}(t)$ *that satisfies* $T(u(t))_{t\in\mathbb{Z}} = (y(t))_{t\in\mathbb{Z}}$*.*

In [Con96, Section 2.3, Lemma 3.1], the author proves that the transfer map of a unitary time varying linear system is a contraction. We use the same line of reasoning to prove a similar result about contractive time varying linear systems in the following lemma.

Lemma 3.2.3. *The transfer map of a contractive time varying linear system is a contraction.*

Proof. Fix $t_0 \in \mathbb{Z}$. Suppose $x(t_0) = 0$, and let $\{y(t)\}_{t<t_0}$ be the output generated from the input $\{u(t)\}_{t<t_0}$ by the contractive time varying linear system

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \begin{bmatrix} x(t+1) \\ u(t) \end{bmatrix}.$$

Since $\left\| \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \right\| \leq 1$, we have

$$\|x(t)\|^2 + \|y(t)\|^2 \leq \|x(t+1)\|^2 + \|u(t)\|^2.$$

By induction,

$$\|x(t)\|^2 \leq \sum_{k=t}^{t_0-1} \|u(k)\|^2 - \sum_{k=t}^{t_0-1} \|y(k)\|^2, \quad t < t_0.$$

In particular,

$$\sum_{k=t}^{t_0-1} \|y(k)\|^2 \leq \sum_{k=t}^{t_0-1} \|u(k)\|^2.$$

Since this holds for arbitrary $t_0 \in \mathbb{Z}$ and arbitrary $t < t_0$, it follows that the transfer map T is a contraction. □

3.2.2 Douglas's Lemma

The connection between majorization, factorization, and range inclusion of operators on Hilbert space has been well-studied. Perhaps the most well-known result in this area is Douglas's lemma ([Dou66, Theorem 1]), though similar theorems have been proved elsewhere, including [Dav79, Theorem 3.2]. Douglas's original statement follows.

Lemma 3.2.4. *[Dou66, Theorem 1] Let A and B be bounded, linear operators on a Hilbert space H. The following are equivalent:*

1. *Range*$(A) \subseteq$ *Range*(B).
2. $AA^* \leq \lambda^2 BB^*$ *for some* $\lambda \geq 0$.
3. *There exists a bounded, linear operator C on H such that $A = BC$.*

Moreover, if (1), (2), and (3) are valid, then there exists a unique operator C such that

1. $\|C\|^2 = \inf\{\mu \mid AA^* \leq \mu BB^*\}$.
2. $ker(A) = ker(C)$.
3. *Range*$(C) \subseteq \overline{Range(B^*)}$.

A corollary of Douglas's lemma will enable us to show that the operator T in Lemma 3.1.2 satisfies the necessary intertwining conditions to lie in $\mathcal{S}(E, H, \sigma)$. While Douglas's lemma was originally stated for operators on one Hilbert space, at the end of [Dou66] Douglas remarks that his proof holds for operators A and B with domains H_1 and H_2, respectively, and common range H. The only modification in this case is that C is an operator from H_1 to H_2. In the corollary that follows, $\hat{A}$ and $\hat{B}$ map from a common Hilbert space into different Hilbert spaces, and $\hat{A}^*\hat{A} = \hat{B}^*\hat{B}$. Therefore, the adjoints $\hat{A}^*$ and $\hat{B}^*$ satisfy the modified hypotheses of Douglas's lemma.

Before we continue, recall that if $\sigma : M \to H$ is a faithful, normal representation of M on a Hilbert space H, then for $N \in \mathbb{N}$ we define $H^{(N)} := \mathbb{C}^N \otimes H$, and we define $\sigma^{(N)} : M \to B(H^{(N)})$ by the formula

$$\sigma^{(N)}(a) := \begin{bmatrix} \sigma(a) & & \\ & \ddots & \\ & & \sigma(a) \end{bmatrix}.$$

Corollary 3.2.5. *Let $N \in \mathbb{N}$. If $\hat{A} \in \mathfrak{I}(\sigma^{(N)}, \sigma^{(N+1)})$, $\hat{B} \in \mathfrak{I}(\sigma^{(N)}, (\sigma^{(N)})^E \circ \varphi \oplus \sigma)$, and $\hat{A}^*\hat{A} = \hat{B}^*\hat{B}$, then there exists a unique partial isometry $\Omega : (E \otimes H^{(N)}) \oplus H \to H^{(N+1)}$ such that $\hat{A} = \Omega\hat{B}$ and $ker(\Omega)^\perp \subseteq \overline{Range(\hat{B})}$. Moreover, for all $a \in M$,*

$$\sigma^{(N+1)}(a)\Omega = \Omega \begin{bmatrix} (\sigma^{(N)})^E \circ \varphi(a) & 0 \\ 0 & \sigma(a) \end{bmatrix}. \tag{3.5}$$

Proof. Since $\hat{A}^*\hat{A} = \hat{B}^*\hat{B}$ and $\hat{A}$ and $\hat{B}$ are bounded, linear operators on Hilbert space, Douglas's lemma implies that there exists a unique partial isometry $\Omega : (E \otimes H^{(N)}) \oplus H \to H^{(N+1)}$ such that $\hat{A} = \Omega\hat{B}$ and $ker(\Omega)^\perp \subseteq \overline{Range(\hat{B})}$. All that remains to show is that equation (3.5) holds. Recall that since M is a W^*-algebra, it is generated by its unitaries. Thus it suffices to prove equation (3.5) for all unitary elements of M. Let $u \in M$ be unitary, and define the partial isometry $\hat{\Omega} = \sigma^{(N+1)}(u^*)\Omega \begin{bmatrix} (\sigma^{(N)})^E \circ \varphi(u) & 0 \\ 0 & \sigma(u) \end{bmatrix}$. We will show $\hat{\Omega} = \Omega$.

Note that the intertwining relations satisfied by $\hat{A}$ and $\hat{B}$ imply

$$\sigma^{(N+1)}(a)\hat{A} = \hat{A}\sigma^{(N)}(a) \text{ and } \begin{bmatrix} (\sigma^{(N)})^E \circ \varphi(a) & 0 \\ 0 & \sigma(a) \end{bmatrix} \hat{B} = \hat{B}\sigma^{(N)}(a) \tag{3.6}$$

for all $a \in M$. Then $\hat{A} = \hat{\Omega}\hat{B}$ since $\hat{A} = \Omega\hat{B}$ and the relations in (3.6) hold. By the uniqueness of Ω, it remains to show that $ker(\hat{\Omega})^\perp \subseteq \overline{Range(\hat{B})}$. That is, we must show $P \leq Q$, where $P = \hat{\Omega}^*\hat{\Omega}$ is projection onto $ker(\hat{\Omega})^\perp$ and $Q = \hat{B}\hat{B}^*$ is projection onto $\overline{Range(\hat{B})}$.

Observe that Q commutes with $\begin{bmatrix} (\sigma^{(N)})^E \circ \varphi(a) & 0 \\ 0 & \sigma(a) \end{bmatrix}$ for all $a \in M$ since (3.6) holds. Thus

$$P = \hat{\Omega}^*\hat{\Omega} = \begin{bmatrix} (\sigma^{(N)})^E \circ \varphi(u^*) & 0 \\ 0 & \sigma(u^*) \end{bmatrix} \Omega^*\Omega \begin{bmatrix} (\sigma^{(N)})^E \circ \varphi(u) & 0 \\ 0 & \sigma(u) \end{bmatrix}$$
$$\leq \begin{bmatrix} (\sigma^{(N)})^E \circ \varphi(u^*) & 0 \\ 0 & \sigma(u^*) \end{bmatrix} Q \begin{bmatrix} (\sigma^{(N)})^E \circ \varphi(u) & 0 \\ 0 & \sigma(u) \end{bmatrix} = Q,$$

where the inequality follows from the fact that $ker(\Omega)^\perp \subseteq \overline{Range(\hat{B})}$. □

Now we are equipped to prove Lemma 3.1.2.

Proof of Lemma 3.1.2. If the solution $\mathcal{A}_{\mathcal{N}}$ to the displacement equation (3.3) is positive semidefinite, then there exists $L \in \sigma^{(N)}(M)'$ such that $\mathcal{A}_{\mathcal{N}} = LL^*$. Rewrite the displacement equation in terms of L:

$$LL^* - \mathfrak{z}^*(I_E \otimes LL^*)\mathfrak{z} = UU^* - VV^*.$$

Let $\hat{A} = \begin{bmatrix} L^* \\ V^* \end{bmatrix}$ and $\hat{B} = \begin{bmatrix} (I_E \otimes L^*)\mathfrak{z} \\ U^* \end{bmatrix}$. Then the previous equation may be rewritten as

$$\hat{A}^*\hat{A} = \hat{B}^*\hat{B}.$$

Since $L^* \in \sigma^{(N)}(M)', V^* \in \mathfrak{I}(\sigma^{(N)}, \sigma), \mathfrak{z} \in \mathfrak{I}(\sigma^{(N)}, (\sigma^{(N)})^E \circ \varphi)$, and $U^* \in \mathfrak{I}(\sigma^{(N)}, \sigma)$, then $\hat{A}$ and $\hat{B}$ satisfy the necessary intertwining conditions to apply Corollary 3.2.5.

Let Ω be as in Corollary 3.2.5. Then we may write $\Omega = \begin{bmatrix} X & Z \\ Y & W \end{bmatrix}$, for some $X \in B(E \otimes H^{(N)}, H^{(N)}), Z \in B(H, H^{(N)}), Y \in B(E \otimes H^{(N)}, H)$, and $W \in B(H)$. The following intertwining relations are a consequence of equation (3.5):

$$X \in \mathfrak{I}((\sigma^{(N)})^E \circ \varphi, \sigma^{(N)}), \quad Z \in \mathfrak{I}(\sigma, \sigma^{(N)}), \quad Y \in \mathfrak{I}((\sigma^{(N)})^E \circ \varphi, \sigma), \quad W \in \sigma(M)'. \tag{3.7}$$

Writing $\hat{A} = \Omega \hat{B}$ in terms of the entries of $\hat{A}, \Omega$, and $\hat{B}$, we get the system of equations

$$\begin{aligned} L^* &= X(I_E \otimes L^*)\mathfrak{z} + ZU^* \\ V^* &= Y(I_E \otimes L^*)\mathfrak{z} + WU^*. \end{aligned} \tag{3.8}$$

After substituting the first equation into the second K times, we get

$$\begin{aligned} V^* = WU^* + \sum_{k=0}^{K-1} Y(I_E \otimes ((X^*)^{(k)})^*)(I_{E^{\otimes k+1}} \otimes ZU^*)\mathfrak{z}^{(k+1)} \\ + Y(I_E \otimes ((X^*)^{(K)})^*)(I_{E^{\otimes K+1}} \otimes L^*)\mathfrak{z}^{(K+1)}. \end{aligned} \tag{3.9}$$

We can bound the last term in equation (3.9) by

$$\begin{aligned} \|Y(I_E \otimes ((X^*)^{(K)})^*)(I_{E^{\otimes K+1}} \otimes L^*)\mathfrak{z}^{(K+1)}\| &\leq \|Y\|\|((X^*)^{(K)})^*\|\|L^*\|\|\mathfrak{z}^{(K+1)}\| \\ &\leq \|Y\|\|X\|^K\|L^*\|\|\mathfrak{z}\|^{K+1}. \end{aligned}$$

Since $\|\mathfrak{z}\| < 1$ and $\|X\| \leq 1$, the last term goes to 0 as K goes to infinity, which shows

$$V^* = WU^* + \sum_{k=0}^{\infty} Y(I_E \otimes ((X^*)^{(k)})^*)(I_{E^{\otimes k+1}} \otimes ZU^*)\mathfrak{z}^{(k+1)}.$$

Form the infinite upper triangular matrix $T = [T_{ij}]_{i,j=0}^{\infty}$ defined as follows in terms of the entries of Ω:

$$T_{ij} = \begin{cases} 0 & j < i \\ I_{E^{\otimes i}} \otimes W & j = i \\ I_{E^{\otimes i}} \otimes Y(I_E \otimes Z) & j = i+1 \\ I_{E^{\otimes i}} \otimes Y(I_E \otimes ((X^*)^{(j-i-1)})^*)(I_{E^{\otimes j-i}} \otimes Z) & j > i+1 \end{cases}$$

That is,

$$T = \begin{bmatrix} W & Y(I_E \otimes Z) & Y(I_E \otimes X)(I_{E^{\otimes 2}} \otimes Z) & Y(I_E \otimes ((X^*)^{(2)})^*)(I_{E^{\otimes 3}} \otimes Z) & \cdots \\ 0 & I_E \otimes W & I_E \otimes Y(I_E \otimes Z) & I_E \otimes Y(I_E \otimes X)(I_{E^{\otimes 2}} \otimes Z) & \cdots \\ 0 & 0 & I_{E^{\otimes 2}} \otimes W & I_{E^{\otimes 2}} \otimes Y(I_E \otimes Z) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is easy to check that $TU_\infty = V_\infty$. We want to prove that T extends to an element of $\mathcal{S}(E, H, \sigma)$. A quick calculation shows that $T_{0j} \in \mathfrak{I}(\sigma^{E^{\otimes j}} \circ \varphi_j, \sigma), j \geq 0$, because of the intertwining relations (3.7) satisfied by X, Z, Y, and W. To show that $\|T\| \leq 1$, we prove that T is the transfer map of a contractive time varying linear system.

From the system of equations (3.8) we have the following system of equations for all $t \in \mathbb{N}$ and for all $h \in H^{(N)}$:

$$\begin{aligned} (I_{E^{\otimes t}} \otimes L^*)\mathfrak{z}^{(t)}h &= (I_{E^{\otimes t}} \otimes X)(I_{E^{\otimes t+1}} \otimes L^*)\mathfrak{z}^{(t+1)}h + (I_{E^{\otimes t}} \otimes Z)(I_{E^{\otimes t}} \otimes U^*)\mathfrak{z}^{(t)}h \\ (I_{E^{\otimes t}} \otimes V^*)\mathfrak{z}^{(t)}h &= (I_{E^{\otimes t}} \otimes Y)(I_{E^{\otimes t+1}} \otimes L^*)\mathfrak{z}^{(t+1)}h + (I_{E^{\otimes t}} \otimes W)(I_{E^{\otimes t}} \otimes U^*)\mathfrak{z}^{(t)}h. \end{aligned} \tag{3.10}$$

Fix $h \in H^{(N)}$. For $t \in \mathbb{N}$, define $x(t) := (I_{E^{\otimes t}} \otimes L^*)\mathfrak{z}^{(t)}h, u(t) := (I_{E^{\otimes t}} \otimes U^*)\mathfrak{z}^{(t)}h$, and $y(t) := (I_{E^{\otimes t}} \otimes V^*)\mathfrak{z}^{(t)}h$. Also define $A(t) := I_{E^{\otimes t}} \otimes X, B(t) := I_{E^{\otimes t}} \otimes Z, C(t) :=$

$I_{E^{\otimes t}} \otimes Y$, and $D(t) := I_{E^{\otimes t}} \otimes W$. Then

$$U_\infty h := \begin{bmatrix} U^* \\ (I_E \otimes U^*)\mathfrak{z} \\ (I_{E^{\otimes 2}} \otimes U^*)\mathfrak{z}^{(2)} \\ \vdots \end{bmatrix} h = \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \end{bmatrix}$$

and

$$V_\infty h := \begin{bmatrix} V^* \\ (I_E \otimes V^*)\mathfrak{z} \\ (I_{E^{\otimes 2}} \otimes V^*)\mathfrak{z}^{(2)} \\ \vdots \end{bmatrix} h = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \end{bmatrix},$$

and the system (3.10) may be rewritten as

$$\begin{aligned} x(t) &= A(t)x(t+1) + B(t)u(t) \\ y(t) &= C(t)x(t+1) + D(t)u(t), \quad t \in \mathbb{N}, \end{aligned}$$

where $\mathcal{U}(t) = E^{\otimes t} \otimes H$, $\mathcal{Y}(t) = E^{\otimes t} \otimes H$, and $\mathcal{H}(t) = E^{\otimes t} \otimes H^{(N)}$. Since $TU_\infty h = V_\infty h$, T is the transfer map of the system. The matrix

$$\begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} = \begin{bmatrix} I_{E^{\otimes t}} \otimes X & I_{E^{\otimes t}} \otimes Z \\ I_{E^{\otimes t}} \otimes Y & I_{E^{\otimes t}} \otimes W \end{bmatrix} = I_{E^{\otimes t}} \otimes \begin{bmatrix} X & Z \\ Y & W \end{bmatrix}$$

has norm equal to 1 for all t, since $\Omega = \begin{bmatrix} X & Z \\ Y & W \end{bmatrix}$ is of norm 1. Thus T is the transfer map of a contractive system. Lemma 3.2.3 implies that $T \in \mathcal{S}(E, H, \sigma)$.

Conversely, if there exists $T \in \mathcal{S}(E, H, \sigma)$ such that $TU_\infty = V_\infty$, then the solution to the displacement equation may be written as follows:

$$\mathcal{A}_{\mathcal{N}} = U_\infty^* U_\infty - V_\infty^* V_\infty = U_\infty^* U_\infty - U_\infty^* T^* T U_\infty = U_\infty^*(I - T^*T)U_\infty \geq 0,$$

since $\|T\| \leq 1$. □

3.3 Proof of the New Nevanlinna-Pick Theorem

Now that Lemma 3.1.2 has been proved, we are ready to prove our generalized Nevanlinna-Pick theorem.

Proof of Theorem 3.0.1. We noted in Remark 3.1.1 that the Pick matrix $\mathcal{A}_{\mathcal{N}}$ in equation (3.1) is the unique solution to the displacement equation, and we may write $\mathcal{A}_{\mathcal{N}} = U_\infty^* U_\infty - V_\infty^* V_\infty$. If $\mathcal{A}_{\mathcal{N}} \geq 0$, then by Lemma 3.1.2, there exists $T \in \mathcal{S}(E, H, \sigma)$ such that $TU_\infty = V_\infty$. By Remark 3.1.1, we rewrite U_∞ and V_∞ in terms of the Cauchy kernels to get

$$T \begin{bmatrix} C(\mathfrak{z}_1) & \cdots & C(\mathfrak{z}_N) \end{bmatrix} = \begin{bmatrix} (I_{\mathcal{F}(E)} \otimes \Lambda_1)C(\mathfrak{z}_1) & \cdots & (I_{\mathcal{F}(E)} \otimes \Lambda_N)C(\mathfrak{z}_N) \end{bmatrix}$$

Comparing the matrices entrywise, we see that

$$TC(\mathfrak{z}_i) = (I_{\mathcal{F}(E)} \otimes \Lambda_i)C(\mathfrak{z}_i), \quad i = 1, \ldots, N. \tag{3.11}$$

Using Lemma 2.3.1, we rewrite the left hand side of equation (3.11) to get

$$(I_{\mathcal{F}(E)} \otimes T(\mathfrak{z}_i))C(\mathfrak{z}_i) = (I_{\mathcal{F}(E)} \otimes \Lambda_i)C(\mathfrak{z}_i), \quad i = 1, \ldots, N.$$

It follows that $T(\mathfrak{z}_i) = \Lambda_i$ for all $i = 1, \ldots, N$. Together with Lemma 2.2.3, this implies that there exists $X \in H^\infty(E^\sigma)$ with $\|X\| \leq 1$ such that $\hat{X}(\mathfrak{z}_i) = \Lambda_i$ for all $i = 1, \ldots, N$.

Conversely, suppose there exists $X \in H^\infty(E^\sigma)$ with $\|X\| \leq 1$ such that $\hat{X}(\mathfrak{z}_i) = \Lambda_i$ for all $i = 1, \ldots, N$. Then by Lemma 2.2.3, there exists $T \in \mathcal{S}(E, H, \sigma)$ such that $T(\mathfrak{z}_i) = \Lambda_i$ for all $i = 1, \ldots, N$. By the above calculations, $TU_\infty = V_\infty$, and by Lemma 3.1.2, $\mathcal{A}_N \geq 0$. □

3.4 Corollaries

We obtain the classical Nevanlinna-Pick theorem and Constantinescu-Johnson's Nevanlinna-Pick theorem as special cases of Theorem 3.0.1.

Recall that $\mathbb{D}$ denotes the open unit disc in the complex plane, and $H^\infty(\mathbb{D})$ denotes the algebra of bounded, analytic functions from $\mathbb{D}$ into $\mathbb{C}$.

Corollary 3.4.1 (Nevanlinna-Pick theorem)**.** *Given N distinct points $z_1, \ldots, z_N \in \mathbb{D}$ and N points $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$, there exists $f \in H^\infty(\mathbb{D})$ such that $||f||_\infty \leq 1$ and*

$$f(z_i) = \lambda_i, \quad i = 1, \ldots, N,$$

if and only if the Pick matrix

$$\left[\tfrac{1-\overline{\lambda_i}\lambda_j}{1-\overline{z_i}z_j}\right]_{i,j=1}^N$$

is positive semidefinite.

Proof. In Theorem 3.0.1, let $M = E = H = \mathbb{C}$, and let $\sigma : M \to B(H)$ be given by $\sigma(a) = a$. Now we compute

$$E^\sigma := \{\eta \in B(H, E \otimes_\sigma H) \mid \eta\sigma(a) = \sigma^E \circ \varphi(a)\eta \,\forall a \in M\}$$
$$= \{\eta \in B(\mathbb{C}, \mathbb{C} \otimes \mathbb{C}) \mid \eta a = (a \otimes I_H)\eta \,\forall a \in \mathbb{C}\} = B(\mathbb{C}) \approx \mathbb{C}.$$

Moreover, in Theorem 3.0.1, the initial data are required to have norm less than one. Therefore, in the current setting, the initial data must lie in $\mathbb{D}$. In addition, $\sigma(M) = B(\mathbb{C}) \approx \mathbb{C}$, so $\sigma(M)' = B(\mathbb{C}) \approx \mathbb{C}$. Therefore the target data lie in $\mathbb{C}$.

Now we consider the interpolating map and point evaluation. Note that the maps U and ρ defined in Chapter 2 are the identity in this case since $\mathcal{F}(E^\sigma) \otimes_\iota H \approx \mathcal{F}(E) \otimes_\sigma H$ and $X \otimes I_H \approx X$. Thus the point evaluation of $X \in H^\infty(\mathbb{C})$ at $z_i \in \mathbb{C}$ is given by

$$\hat{X}(z_i) := \langle \rho(X)C(0), C(z_i)\rangle = \langle XC(0), C(z_i)\rangle = C(0)^* X^* C(z_i) = \sum_{k=0}^{\infty} X_{0k}^* z_i^k,$$

where X_{k0} is the k^{th} entry in the left-most column of X written as a matrix. The requirement that the interpolating map lie in $H^\infty(E^\sigma)$ and have norm at most one guarantees that X is a bounded, analytic map with sup norm at most one. Therefore, $X \in H^\infty(\mathbb{D})$.

Finally, we note that the Pick matrix in Theorem 3.0.1 can be simplified as follows:

$$\begin{aligned}
\mathcal{A}_{\mathcal{N}} &= [\langle C(z_i), C(z_j)\rangle - \langle (I_{\mathcal{F}(E)} \otimes \lambda_i)C(z_i), (I_{\mathcal{F}(E)} \otimes \lambda_j)C(z_j)\rangle]_{i,j=1}^N \\
&= [\sum_{k=0}^{\infty} z_i^{k*} z_j^k - \sum_{k=0}^{\infty} z_i^{k*} \lambda_i^* \lambda_j z_j^k]_{i,j=1}^N = [\sum_{k=0}^{\infty} (z_i^{k*} z_j^k - z_i^{k*} \lambda_i^* \lambda_j z_j^k)]_{i,j=1}^N \\
&= [\sum_{k=0}^{\infty} z_i^{k*} (1 - \lambda_i^* \lambda_j) z_j^k]_{i,j=1}^N = [(1 - z_i^* z_j)^{-1} (1 - \lambda_i^* \lambda_j)]_{i,j=1}^N \\
&= \left[\frac{1-\overline{\lambda_i}\lambda_j}{1-\overline{z_i}z_j}\right]_{i,j=1}^N
\end{aligned}$$

Thus in the setting $M = E = H = \mathbb{C}$, we recover the classical Nevanlinna-Pick theorem from Theorem 3.0.1. □

Before we state and prove Constantinescu-Johnson's Theorem 3.4 in [CJ03] as a corollary, we make the following definition in accordance with the authors' notation in [CJ03].

Definition 3.4.2. *Let H be a Hilbert space, fix $n \in \mathbb{N}$, and define*

$$B_n(H) := \{\mathfrak{z} = \begin{bmatrix} Z_1 & \dots & Z_n \end{bmatrix} \mid Z_k \in B(H)\, \forall k = 1, \dots, n \text{ and } \sum_{k=1}^{n} Z_k^* Z_k < I_H\}.$$

We also introduce the following notation. Let F_n^+ denote the free semigroup on n generators. For $\mathfrak{z} = \begin{bmatrix} Z_1 & \dots & Z_n \end{bmatrix} \in B_n(H)$ and $\alpha = g_{i_1} g_{i_2} \cdots g_{i_k} \in F_n^+$, let Z_α denote the operator $Z_{i_1} Z_{i_2} \cdots Z_{i_k}$. Following Constantinescu-Johnson's convention, we distinguish between $(Z_\alpha)^* = Z_{i_k}^* \cdots Z_{i_2}^* Z_{i_1}^*$ and $Z_\alpha^* = Z_{i_1}^* Z_{i_2}^* \cdots Z_{i_k}^*$.

Corollary 3.4.3 ([CJ03, Theorem 3.4])**.** *Fix a Hilbert space H, and let $\sigma : \mathbb{C} \to B(H)$ be given by $\sigma(a) = aI_H$. Let $\left\{\mathfrak{z}_i = \begin{bmatrix} Z_{i1} & \dots & Z_{in} \end{bmatrix}\right\}_{i=1}^{N}$ be N distinct elements of $B_n(H)$, and let $\Lambda_1, \dots, \Lambda_N \in B(H)$. There exists $T \in \mathcal{S}(\mathbb{C}^n, H, \sigma)$ such that*

$$T(\mathfrak{z}_i) = \Lambda_i, \quad i = 1, \dots, N,$$

if and only if the Pick matrix

$$\begin{bmatrix} L(\mathfrak{z}_i) diag[I - \Lambda_i^* \Lambda_j] L(\mathfrak{z}_j)^* \end{bmatrix}_{i,j=1}^{N}$$

is positive semidefinite, where $L(\mathfrak{z}_i) := [Z_\alpha^*]_{|\alpha|=0}^{\infty}$ is the infinite row vector whose entries are listed according to the lexicographical order on F_n^+.

Proof. In Theorem 3.0.1, let $M = \mathbb{C}$ and $E = \mathbb{C}^n$. By assumption, H is a Hilbert space and $\sigma : M \to B(H)$ is given by $\sigma(a) = aI_H$. Thus $\sigma(M)' = B(H)$. The dual correspondence

$$E^\sigma := \{\eta \in B(H, E \otimes_\sigma H) \mid \eta\sigma(a) = \sigma^E \circ \varphi(a)\eta \, \forall a \in M\}$$
$$= \{\eta \in B(H, H^{(n)}) \mid \eta a I_H = (\varphi(a) \otimes I_H)\eta \, \forall a \in \mathbb{C}\} \approx C_n(B(H)),$$

where $C_n(B(H))$ denotes column n-space over $B(H)$. Note that the points in $B_n(H)$ are row vectors while the points in E^σ are column vectors. However, these spaces can be identified since an element $\mathfrak{z} = \begin{bmatrix} Z_1 & \cdots & Z_n \end{bmatrix}$ is in $B_n(H)$ if and only if its transpose is in E^σ. To see this, we show that $\sum_{k=1}^n Z_k^* Z_k < I_H$ if and only if $\|\mathfrak{z}\|_{E^\sigma} < 1$. For all $h \in H \setminus \{0\}$, we have

$$\sum_{k=1}^n Z_k^* Z_k < I_H \iff I_H - \sum_{k=1}^n Z_k^* Z_k > 0 \iff \langle (I_H - \sum_{k=1}^n Z_k^* Z_k)h, h\rangle > 0$$
$$\iff \langle h, h\rangle - \sum_{k=1}^n \langle Z_k^* Z_k h, h\rangle > 0 \iff \langle h, h\rangle - \sum_{k=1}^n \langle Z_k h, Z_k h\rangle > 0$$
$$\iff \|h\|_H^2 - \sum_{k=1}^n \|Z_k h\|_H^2 > 0 \iff \|h\|_H^2 > \sum_{k=1}^n \|Z_k h\|_H^2$$
$$\iff \|h\|_H^2 > \|\mathfrak{z}^T h\|_{H^{\oplus n}}^2 \iff \|h\|_H > \|\mathfrak{z}^T h\|_{H^{\oplus n}} \iff \|\mathfrak{z}^T\|_{E^\sigma} < 1.$$

Thus we identify E^σ with $B_n(H)$, and we will switch between the column vector form of an element in E^σ and its row vector form in $B_n(H)$ as necessary.

By Lemma 2.2.3 and the point evaluation defined in equation (2.3), there exists $X \in H^\infty(E^\sigma)$ such that $\hat{X}(\mathfrak{z}_i) = \Lambda_i$ for all $i = 1, \ldots, N$, if and only if there exists $T = \rho(X)^* \in \mathfrak{U}_{\mathfrak{I}}(E, H, \sigma)$ such that $T(\mathfrak{z}_i) = \Lambda_i$ for all $i = 1, \ldots, N$. Moreover, $\|X\| \leq 1$ if and only if $\|T\| \leq 1$ since ρ is an isometry.

Lastly, we focus on the Pick matrices. Observe that for $\mathfrak{z}^* = \begin{bmatrix} Z_1^* & \cdots & Z_n^* \end{bmatrix} \in$

$(E^\sigma)^*$,

$$(\mathfrak{z}^{(2)})^* = \mathfrak{z}^*(I_E \otimes \mathfrak{z}^*)$$
$$= \begin{bmatrix} Z_1^* & \cdots & Z_n^* \end{bmatrix} \begin{bmatrix} \begin{bmatrix} Z_1^* & \cdots & Z_n^* \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} Z_1^* & \cdots & Z_n^* \end{bmatrix} \end{bmatrix}$$
$$= \begin{bmatrix} Z_1^*Z_1^* & Z_1^*Z_2^* & \cdots & Z_1^*Z_n^* & \cdots\cdots & Z_n^*Z_1^* & Z_n^*Z_2^* & \cdots & Z_n^*Z_n^* \end{bmatrix}.$$

By induction, since $(\mathfrak{z}^{(k)})^* = (\mathfrak{z}^{(k-1)})^*(I_{E^{\otimes k-1}} \otimes \mathfrak{z}^*)$, it can be shown that

$$\mathfrak{z}^{(k)*} = [Z_\alpha^*]_{|\alpha|=k},$$

where $Z_\alpha^* := Z_{i_1}^* \cdots Z_{i_k}^*$ if $\alpha = g_{i_1} \cdots g_{i_k}$, and the words in F_n^+ are listed in lexicographical order. Therefore we have the following connection between Muhly-Solel's and Constantinescu-Johnson's notation:

$$C(\mathfrak{z})^* = \begin{bmatrix} I_H & \mathfrak{z}^* & (\mathfrak{z}^{(2)})^* & \cdots \end{bmatrix} = [Z_\alpha^*]_{|\alpha|=0}^\infty = L(\mathfrak{z}).$$

Thus we may rewrite the Pick matrix as follows:

$$\mathcal{A}_\mathcal{N} = [\langle C(\mathfrak{z}_i), C(\mathfrak{z}_j)\rangle - \langle (I_{\mathcal{F}(E)} \otimes \Lambda_i)C(\mathfrak{z}_i), (I_{\mathcal{F}(E)} \otimes \Lambda_j)C(\mathfrak{z}_j)\rangle]$$
$$= [C(\mathfrak{z}_i)^*C(\mathfrak{z}_j) - C(\mathfrak{z}_i)^*(I_{\mathcal{F}(E)} \otimes \Lambda_i^*)(I_{\mathcal{F}(E)} \otimes \Lambda_j)C(\mathfrak{z}_j)]$$
$$= [L(\mathfrak{z}_i)L(\mathfrak{z}_j)^* - L(\mathfrak{z}_i)(I_{\mathcal{F}(E)} \otimes \Lambda_i^*\Lambda_j)L(\mathfrak{z}_j)^*]$$
$$= [L(\mathfrak{z}_i)diag[I - \Lambda_i^*\Lambda_j]L(\mathfrak{z}_j)^*].$$

Putting all the pieces together, we see that we recover [CJ03, Theorem 3.4] by restating Theorem 3.0.1 in this context. □

3.5 A Restatement

We arrived at Theorem 3.0.1 by generalizing [CJ03, Theorem 3.4], and it lends itself most naturally to a comparison with [MS04, Theorem 5.3]. Nevertheless, a statement that avoids E^σ may be preferable in some cases. We can state Theorem 3.0.1 without reference to the σ-dual as follows. Let F be a W^*-correspondence over a W^*-algebra P. Let $\tau : P \to B(H)$ be a faithful, normal representation of P on a Hilbert space H. For $\eta \in F$ and $X \in H^\infty(F)$, define

$$\tilde{X}(\eta) := \langle (X \otimes I_H)L_0, L_\eta \rangle$$

to be the P-valued point evaluation of X at η, where the insertion operator $L_\eta = \begin{bmatrix} I_H & L_\eta^{(1)} & L_\eta^{(2)} & \dots \end{bmatrix}^T$ and $L_\eta^{(k)} : H \to F^{\otimes k} \otimes_\tau H$ is given by $L_\eta^{(k)}(h) = \eta^{\otimes k} \otimes h$. By Theorem 3.6 in [MS04], there exists a W^*-correspondence E over a W^*-algebra M and a faithful, normal representation $\sigma : M \to B(H)$ such that $F = E^\sigma, P = \sigma(M)'$, and τ is the identity map. Let $X \in H^\infty(F) = H^\infty(E^\sigma), \eta \in F = E^\sigma$, and $\Lambda \in P = \sigma(M)'$. A simple calculation shows $L_\eta = U^*C(\eta)$, and it immediately follows that $\tilde{X}(\eta) = \hat{X}(\eta)$, as defined in equation (2.4). Furthermore, Lemma 3.8 in [MS04] implies that $(I_{\mathcal{F}(E)} \otimes \Lambda)C(\eta) = U(\varphi_\infty^F(\Lambda) \otimes I_H)L_\eta$. Thus we arrive at the following theorem.

Theorem 3.5.1. *Let F be a W^*-correspondence over a W^*-algebra P. Let $\mathfrak{z}_1, \dots, \mathfrak{z}_N$ be N distinct elements of F with $\|\mathfrak{z}_i\| < 1$ for all $i = 1, \dots, N$, and let $\Lambda_1, \dots, \Lambda_N \in P$. There exists $X \in H^\infty(F)$ with $\|X\| \leq 1$ such that*

$$\tilde{X}(\mathfrak{z}_i) = \Lambda_i, \quad i = 1, \dots, N,$$

if and only if the Pick matrix

$$\left[\langle L_{\mathfrak{z}_i}, L_{\mathfrak{z}_j}\rangle - \langle(\varphi_\infty^F(\Lambda_i)\otimes I_H)L_{\mathfrak{z}_i}, (\varphi_\infty^F(\Lambda_j)\otimes I_H)L_{\mathfrak{z}_j}\rangle\right]_{i,j=1}^N$$

is positive semidefinite.

CHAPTER 4
COMPARISON WITH MUHLY-SOLEL'S GENERALIZED NEVANLINNA-PICK THEOREM

Since Theorem 3.0.1 is a nontangential Nevanlinna-Pick thereom, we compare it to the following nontangential version of Muhly-Solel's generalized noncommutative Nevanlinna-Pick theorem, [MS04, Theorem 5.3]. The following version of [MS04, Theorem 5.3] (in which the adjoints of the target points are interpolated) lends itself most naturally to the comparison.

Theorem 4.0.1. *Let E be a W^*-correspondence over a W^*-algebra M, and let σ be a faithful, normal representation of M on a Hilbert space H. Given N distinct points $\mathfrak{z}_1, \ldots, \mathfrak{z}_N \in E^\sigma$ with $\|\mathfrak{z}_i\| < 1$ for all $i = 1, \ldots, N$, and $\Lambda_1, \ldots, \Lambda_N \in B(H)$, there exists $Y \in H^\infty(E)$ with $\|Y\| \leq 1$ such that*

$$\hat{Y}(\mathfrak{z}_i^*) = \Lambda_i^*, \quad i = 1, \ldots, N,$$

if and only if the map from $M_N(\sigma(M)')$ to $M_N(B(H))$ defined by

$$[B_{ij}]_{i,j=1}^N \overset{\Phi_{\mathcal{MS}}}{\longmapsto} [\langle C(\mathfrak{z}_i), (I_{\mathcal{F}(E)} \otimes B_{ij})C(\mathfrak{z}_j)\rangle - \langle C(\mathfrak{z}_i)\Lambda_i, (I_{\mathcal{F}(E)} \otimes B_{ij})C(\mathfrak{z}_j)\Lambda_j\rangle]_{i,j=1}^N$$

is completely positive.

In Section 4.1 we compare Theorems 4.0.1 and 3.0.1, and we give a condition for when the two theorems yield the same result. First, we investigate the Pick matrices to reveal a significant difference between the theorems.

Recall that the Pick matrix in Theorem 3.0.1 is given by

$$\mathcal{A}_{\mathcal{N}} = (I_{B(H^{(N)})} - \theta_{\mathfrak{z}})^{-1}(UU^* - VV^*) = (I_{B(H^{(N)})} - \theta_{\mathfrak{z}})^{-1}([I_H - \Lambda_i^*\Lambda_j]_{i,j=1}^N),$$

where $\theta_{\mathfrak{z}} : \sigma^{(N)}(M)' \to \sigma^{(N)}(M)'$ is defined by $\theta_{\mathfrak{z}}(B) = \mathfrak{z}^*(I_E \otimes B)\mathfrak{z}$. Note that $\|\theta_{\mathfrak{z}}\| < 1$ and $\theta_{\mathfrak{z}}$ is completely positive, so $(I_{B(H^{(N)})} - \theta_{\mathfrak{z}})^{-1} = \sum_{k=0}^{\infty} \theta_{\mathfrak{z}}^k$ is well-defined and completely positive. Given $\Lambda_1, \ldots, \Lambda_N \in B(H)$, define $\Lambda := diag[\Lambda_i]$. Then define $\Psi_\Lambda : M_N(B(H)) \to M_N(B(H))$ by $\Psi_\Lambda(B) = \Lambda^* B \Lambda$. We can rewrite the Pick matrix in Theorem 3.0.1 as

$$\mathcal{A}_{\mathcal{N}} = (I_{B(H^{(N)})} - \theta_{\mathfrak{z}})^{-1} \circ (I_{B(H^{(N)})} - \Psi_\Lambda) \left(\begin{bmatrix} I_H & \cdots & I_H \\ \vdots & & \vdots \\ I_H & \cdots & I_H \end{bmatrix} \right). \tag{4.1}$$

On the other hand, the Pick matrix map $\Phi_{\mathcal{MS}}$ in Theorem 4.0.1 can be rewritten in the form

$$B \overset{\Phi_{\mathcal{MS}}}{\longmapsto} (I_{B(H^{(N)})} - \Psi_\Lambda) \circ (I_{B(H^{(N)})} - \theta_{\mathfrak{z}})^{-1}(B), \quad B \in M_N(\sigma(M)').$$

Consider the map $\Phi_{\mathcal{MS}} = (I - \Psi_\Lambda) \circ (I - \theta_{\mathfrak{z}})^{-1}$, where I denotes the identity operator on $B(H^{(N)})$. Suppose $\Phi_{\mathcal{MS}}$ is a completely positive map. Then $(I - \theta_{\mathfrak{z}})^{-1} \circ (I - \Psi_\Lambda) = (I - \theta_{\mathfrak{z}})^{-1} \circ \Phi_{\mathcal{MS}} \circ (I - \theta_{\mathfrak{z}})$ is a composition of completely positive maps, hence completely positive. Conversely, if $(I - \theta_{\mathfrak{z}})^{-1} \circ (I - \Psi_\Lambda)$ is completely positive, then $\Phi_{\mathcal{MS}} = (I - \theta_{\mathfrak{z}}) \circ [(I - \theta_{\mathfrak{z}})^{-1} \circ (I - \Psi_\Lambda)] \circ (I - \theta_{\mathfrak{z}})^{-1}$ is completely positive as well. Thus $(I - \theta_{\mathfrak{z}})^{-1} \circ (I - \Psi_\Lambda)$ is completely positive if and only if $\Phi_{\mathcal{MS}} = (I - \Psi_\Lambda) \circ (I - \theta_{\mathfrak{z}})^{-1}$ is completely positive.

Therefore, if $\Phi_{\mathcal{MS}}$ is completely positive, then equation (4.1) implies that the Pick matrix $\mathcal{A}_{\mathcal{N}}$ in Theorem 3.0.1 is positive semidefinite. However, the converse is not true, i.e., if $\mathcal{A}_{\mathcal{N}} \geq 0$, then the Pick matrix map $\Phi_{\mathcal{MS}}$ may not be completely positive. It might be tempting to apply a generalization of [Cho75, Theorem 2] to prove the converse, but it would not be prudent since $A_{\mathcal{N}}$ is the evaluation of $(I-\theta_{\mathfrak{z}})^{-1}\circ(I-\Psi_\Lambda)$ at a matrix of identities on H instead of the evaluation at Choi's matrix units. We summarize these observations in the following remark.

Remark 4.0.2. *The complete positivity of Muhly-Solel's Pick matrix map*

$$\Phi_{\mathcal{MS}} = (I-\Psi_\Lambda)\circ(I-\theta_{\mathfrak{z}})^{-1}$$

implies the positive semidefiniteness of the Pick matrix $\mathcal{A}_{\mathcal{N}}$ in Theorem 3.0.1, but the converse is not true. Stated differently, if there exists $Y \in H^\infty(E)$ such that $\hat{Y}(\mathfrak{z}_i^*) = \Lambda_i^*, i = 1,\ldots,N$, then there exists $X \in H^\infty(E^\sigma)$ such that $\hat{X}(\mathfrak{z}_i) = \Lambda_i, i = 1,\ldots,N$, but the converse does not hold.

The following simple example, brought to our attention by the referee of [Nor17], illustrates a case in which the converse fails.

Example 4.0.3. *Let Z and Λ be 2×2 matrices given by*

$$Z = \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}, 0<r<1, \text{ and } \Lambda = \begin{bmatrix} \epsilon & 0 \\ 0 & 0 \end{bmatrix}, 0<\epsilon\leq 1.$$

Now consider two problems:

1. *Find $F(z) = \sum_{k=0}^\infty A_k z^k$ in the unit ball of $H^\infty(\mathbb{D})\otimes\mathbb{C}^2$ such that*

$$F(Z) = \sum_{k=0}^\infty A_k Z^k = \Lambda,$$

where $A_k Z^k$ is given by multiplication of 2×2 matrices.

2. Find $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *in in the unit ball of* $H^\infty(\mathbb{D})$ *such that*

$$f(Z^*) = \sum_{k=0}^{\infty} a_k (Z^*)^k = \Lambda^*,$$

where $a_k(Z^*)^k$ *is given by scalar multiplication of a matrix.*

We start by showing that the first problem is a specific case of [CJ03, Theorem 3.4]. First we focus on $B_n(H)$*, as defined in Definition 3.4.2. Since* Z *is a single matrix, then* $n = 1$*. Moreover,* $Z : \mathbb{C}^2 \to \mathbb{C}^2$*. Therefore* $H = \mathbb{C}^2$ *and* $B_n(H) = B_1(\mathbb{C}^2) = \{W \in M_2(\mathbb{C}) \mid W^*W < I_2\}$*. In addition,* E *is defined to be* $\mathbb{C}^n$ *in [CJ03]. Thus* $E = \mathbb{C}$ *in this example.*

For Z *to belong to* $B_1(\mathbb{C}^2)$*, it must satisfy* $Z^*Z < I_2$*. Note that* $Z^*Z = \begin{bmatrix} 0 & 0 \\ 0 & r^2 \end{bmatrix}$*, so* $Z^*Z < I_2$ *since* $0 < r < 1$ *by assumption. Also,* $\Lambda \in B(\mathbb{C}^2)$*, as required.*

Now we turn our attention to the interpolating map. Define $\sigma : \mathbb{C} \to M_2(\mathbb{C})$ *by* $\sigma(a) = aI_2$*. By definition,* $\mathcal{S}(E, H, \sigma) \subset \mathcal{L}(\mathcal{F}(E) \otimes H)$*. In particular, if* $T = [T_{ij}] \in \mathcal{S}(E, H, \sigma)$*, then* $T_{0k} : \mathbb{C}^2 \to \mathbb{C}^2$ *for all* $k \geq 0$*. Thus for any* $T \in \mathcal{S}(E, H, \sigma)$*,*

$$T = \begin{bmatrix} T_{00} & T_{01} & T_{02} & \cdots \\ 0 & T_{00} & T_{01} & \cdots \\ 0 & 0 & T_{00} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $T_{0k} \in M_2(\mathbb{C})$ *for all* $k \geq 0$*. According to equation* (2.3)*, for any* $T \in \mathcal{S}(E, H, \sigma)$*,*

$$T(Z) := \langle C(0), TC(Z)\rangle = \begin{bmatrix} I_2 & 0 & 0 & \cdots \end{bmatrix} \begin{bmatrix} T_{00} & T_{01} & T_{02} & \cdots \\ 0 & T_{00} & T_{01} & \cdots \\ 0 & 0 & T_{00} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} I_2 \\ Z \\ Z^2 \\ \vdots \end{bmatrix}$$

$$= \sum_{k=0} T_{0k} Z^k = T_{00} + T_{01} Z,$$

since $Z^2 = 0$. *Therefore, the first problem is a matter of finding an operator* $T \in \mathcal{S}(E, H, \sigma)$ *that interpolates the given data. [CJ03, Theorem 3.4] tells us that the interpolant exists if and only if the Pick matrix is positive semidefinite. That is, the interpolant exists if and only if*

$$L(Z)diag(I_2 - \Lambda^*\Lambda)L(Z)^* = \begin{bmatrix} I_2 & Z^* & Z^{*2} & \cdots \end{bmatrix} \begin{bmatrix} I_2 - \Lambda^*\Lambda & & \\ & I_2 - \Lambda^*\Lambda & \\ & & \ddots \end{bmatrix} \begin{bmatrix} I_2 \\ Z \\ Z^2 \\ \vdots \end{bmatrix}$$

$$= I_2 - \Lambda^*\Lambda + Z^*(I_2 - \Lambda^*\Lambda)Z = \begin{bmatrix} 1 - \epsilon^2 & 0 \\ 0 & 1 + r^2(1 - \epsilon^2) \end{bmatrix}$$

is positive semidefinite, which it is since $0 < \epsilon \leq 1$ *by assumption. Therefore, the interpolant exists in the context of [CJ03, Theorem 3.4], and the first problem has a solution. Since [CJ03, Theorem 3.4] is a special case of Theorem 3.0.1, the interpolant exists in the context of Theorem 3.0.1 as well.*

For the second problem, note that in the context of Theorem 4.0.1, the initial data are assumed to lie in E^σ *and map* H *into* $E \otimes_\sigma H$. *Since* $Z : \mathbb{C}^2 \to \mathbb{C}^2$, *this* *implies* $H = \mathbb{C}^2$ *and* $E = \mathbb{C}$*. Moreover, the interpolating map* f *must lie in* $H^\infty(\mathbb{D})$*, i.e.,* f *has the form*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

where $a_k \in \mathbb{C}$ *for all* $k \geq 0$*. Thus for any* $f \in H^\infty(\mathbb{D})$*,*

$$f(Z^*) = \sum_{k=0}^{\infty} a_k (Z^*)^k = a_0 I_2 + a_1 Z^*,$$

since $(Z^*)^2 = 0$*. It is clear that there does not a exist a map* $f \in H^\infty(\mathbb{D})$ *such that* $f(Z^*) = \Lambda^*$ *due to the forms of* Z^* *and* Λ^**.*

We confirm that no solution exists by studying Muhly-Solel's Pick matrix map $\Phi_{\mathcal{MS}} : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ *defined by* $\Phi_{\mathcal{MS}}(B) = (I_{\mathbb{C}^2} - \Psi_\Lambda) \circ (I_{\mathbb{C}^2} - \theta_Z)^{-1}(B) = B + Z^* B Z - \Lambda^* B \Lambda$*. By [Cho75, Theorem 2],* $\Phi_{\mathcal{MS}}$ *is completely positive if and only if the matrix*

$$[\Phi_{\mathcal{MS}}(E_{ij})]_{1 \leq i,j \leq 2}$$

is positive semidefinite, where E_{ij} *denotes the* 2×2 *matrix with* 1 *in the* ij^{th} *entry and zeros elsewhere. Since*

$$[\Phi_{\mathcal{MS}}(E_{ij})]_{1 \leq i,j \leq 2} = \begin{bmatrix} \Phi_{\mathcal{MS}}(E_{11}) & \Phi_{\mathcal{MS}}(E_{12}) \\ \Phi_{\mathcal{MS}}(E_{21}) & \Phi_{\mathcal{MS}}(E_{22}) \end{bmatrix} = \begin{bmatrix} 1-\epsilon^2 & 0 & 0 & 1 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \not\geq 0,$$

then Muhly-Solel's Pick matrix map is not completely positive. Therefore, it is confirmed that no interpolating function exists in the context of Theorem 4.0.1.

4.1 The Comparison Theorem

Despite the differences between Theorems 3.0.1 and 4.0.1, the theorems yield the same result under certain additional commutativity assumptions. We begin this section by defining the center of a W^*-correspondence and exploring the connection between the centers of E and E^σ. We conclude with a theorem that makes the connection between Theorems 3.0.1 and 4.0.1 explicit, and gives a new characterization for interpolation in terms of completely bounded maps.

Definition 4.1.1 ([MS08, Definition 4.11]). *If E is a W^*-correspondence over a W^*-algebra M, then the* center *of E, denoted $\mathfrak{Z}(E)$, is the collection of points $\xi \in E$ such that $a \cdot \xi = \xi \cdot a$ for all $a \in M$.*

In [MS08, Lemma 4.12], the authors proved that if E is a W^*-correspondence over a W^*-algebra M, then $\mathfrak{Z}(E)$ is a W^*-correspondence over the commutative W^*-algebra $\mathfrak{Z}(M)$. In general, we will say that a W^*-correspondence E over a commutative W^*-algebra M is *central* if E equals its center. Note that in Theorem 3.0.1 (respectively, Theorem 4.0.1), the Pick matrix, and thus its positivity (resp., complete positivity), does not change if the correspondence $(\sigma(M)', E^\sigma)$ (resp., (M, E)) is replaced by the correspondence $(\mathfrak{Z}(\sigma(M)'), \mathfrak{Z}(E^\sigma))$ (resp., $(\mathfrak{Z}(M), \mathfrak{Z}(E))$). Thus there exists an interpolating map in $H^\infty(E^\sigma)$ (resp., $H^\infty(E)$) if and only if there exists an interpolating map in $H^\infty(\mathfrak{Z}(E^\sigma))$ (resp., $H^\infty(\mathfrak{Z}(E))$). Consequently, for the final theorem we restrict our attention to the correspondences $(\mathfrak{Z}(\sigma(M)'), \mathfrak{Z}(E^\sigma))$ and $(\mathfrak{Z}(M), \mathfrak{Z}(E))$. We choose to do this because the centers are isomorphic as correspondences in the following sense.

Definition 4.1.2 ([MS08, Definition 2.2]). *An* isomorphism of W^*-correspondences (M_1, E_1) *and* (M_2, E_2) *is a pair* (σ, Ψ) *where* $\sigma : M_1 \to M_2$ *is an isomorphism of* W^**-algebras,* $\Psi : E_1 \to E_2$ *is a vector space isomorphism, and for* $e, f \in E_1$ *and* $a, b \in M_1$*, we have* $\Psi(a \cdot e \cdot b) = \sigma(a) \cdot \Psi(e) \cdot \sigma(b)$ *and* $\langle \Psi(e), \Psi(f) \rangle = \sigma(\langle e, f \rangle)$*.*

Define $\gamma : \mathfrak{Z}(E) \to \mathfrak{Z}(E^\sigma)$ by $\gamma(\xi) = L_\xi^{(1)}$, where $L_\xi^{(1)} : H \to E \otimes_\sigma H$ is given by $L_\xi^{(1)}(h) = \xi \otimes h$. For convenience, we will write L_ξ instead of $L_\xi^{(1)}$ for the remainder of the chapter. In [MS08, Lemma 4.12], Muhly and Solel proved that the pair (σ, γ) is an isomorphism of the correspondences $(\mathfrak{Z}(M), \mathfrak{Z}(E))$ and $(\mathfrak{Z}(\sigma(M)'), \mathfrak{Z}(E^\sigma))$.

Proposition 4.1.3. *For* $k \in \mathbb{N}$*, define the map* $\gamma_k : \mathfrak{Z}(E)^{\otimes k} \to \mathfrak{Z}(E^\sigma)^{\otimes k}$ *by*

$$\gamma_k(\xi_1 \otimes \cdots \otimes \xi_k) := L_{\xi_1} \otimes \cdots \otimes L_{\xi_k}.$$

The pair (σ, γ_k) *is an isomorphism of* $(\mathfrak{Z}(M), \mathfrak{Z}(E)^{\otimes k})$ *onto* $(\mathfrak{Z}(\sigma(M)'), \mathfrak{Z}(E^\sigma)^{\otimes k})$*.*

Proof. Let $\xi_1 \otimes \cdots \otimes \xi_k, \eta_1 \otimes \cdots \otimes \eta_k \in \mathfrak{Z}(E)^{\otimes k}$. Since $\xi_i, \eta_i \in \mathfrak{Z}(E)$, $L_{\xi_j}, L_{\eta_i} \in \mathfrak{Z}(E^\sigma)$, and (σ, γ) is an isomorphism of correspondences, we have

$$\begin{aligned}
\langle \gamma_k(\xi_1 \otimes \cdots \otimes \xi_k), \gamma_k(\eta_1 \otimes \cdots \otimes \eta_k) \rangle &= \langle L_{\xi_1} \otimes \cdots \otimes L_{\xi_k}, L_{\eta_1} \otimes \cdots \otimes L_{\eta_k} \rangle \\
&= \langle L_{\xi_2} \otimes \cdots \otimes L_{\xi_k}, \langle L_{\xi_1}, L_{\eta_1} \rangle \cdot L_{\eta_2} \otimes \cdots \otimes L_{\eta_k} \rangle \\
= \langle L_{\xi_2} \otimes \cdots \otimes L_{\xi_k}, L_{\eta_2} \otimes \cdots \otimes L_{\eta_k} \rangle \langle L_{\xi_1}, L_{\eta_1} \rangle &= \ldots = \langle L_{\xi_k}, L_{\eta_k} \rangle \cdots \langle L_{\xi_1}, L_{\eta_1} \rangle \\
&= \sigma(\langle \xi_k, \eta_k \rangle) \cdots \sigma(\langle \xi_1, \eta_1 \rangle) = \sigma(\langle \xi_1 \otimes \cdots \otimes \xi_k, \eta_1 \otimes \cdots \otimes \eta_k \rangle).
\end{aligned}$$

Thus $\langle \gamma_k(\xi), \gamma_k(\eta) \rangle = \sigma(\langle \xi, \eta \rangle)$ for all $\xi, \eta \in \mathfrak{Z}(E)^{\otimes k}$. Furthermore, since σ is an isomorphism, $\|\sigma(\langle \xi, \xi \rangle)\| = \|\langle \xi, \xi \rangle\|$ for all $\xi \in \mathfrak{Z}(E)^{\otimes k}$. Thus γ_k is an isometry.

For $\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_k \in \mathfrak{Z}(E)^{\otimes k}$ and $a, b \in \mathfrak{Z}(M)$ we have

$$\gamma_k(a \cdot (\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_k) \cdot b) = \gamma_k((a \cdot \xi_1) \otimes \xi_2 \otimes \cdots \otimes (\xi_k \cdot b))$$

$$= (\sigma(a) \cdot L_{\xi_1}) \otimes L_{\xi_2} \otimes \cdots \otimes (L_{\xi_k} \cdot \sigma(b)) = \sigma(a) \cdot (L_{\xi_1} \otimes L_{\xi_2} \otimes \cdots \otimes L_{\xi_k}) \cdot \sigma(b)$$

$$= \sigma(a) \cdot \gamma_k(\xi_1 \otimes \cdots \xi_k) \cdot \sigma(b).$$

□

Define $\gamma_\infty : \mathcal{F}(\mathfrak{Z}(E)) \to \mathcal{F}(\mathfrak{Z}(E^\sigma))$ by $\gamma_\infty = diag[\sigma, \gamma, \gamma_2, \ldots]$. Since (σ, γ_k) is an isomorphism of correspondences for each $k \in \mathbb{N}$, it follows that (σ, γ_∞) is an isomorphism of correspondences as well.

Proposition 4.1.4. *For $\xi \in \mathfrak{Z}(E)$, $\gamma_\infty T_\xi \gamma_\infty^{-1} = T_{\gamma(\xi)}$, where T_ξ is the left creation operator in $H^\infty(\mathfrak{Z}(E))$ determined by ξ, and $T_{\gamma(\xi)}$ is the left creation operator in $H^\infty(\mathfrak{Z}(E^\sigma))$ determined by $\gamma(\xi)$. For $a \in \mathfrak{Z}(M)$, $\gamma_\infty \varphi_\infty(a) \gamma_\infty^{-1} = \varphi_\infty^\sigma(\sigma(a))$, where $\varphi_\infty(a)$ is the left action operator in $H^\infty(\mathfrak{Z}(E))$ determined by a, and $\varphi_\infty^\sigma(\sigma(a))$ is the left action operator in $H^\infty(\mathfrak{Z}(E^\sigma))$ determined by $\sigma(a)$.*

Proof. Let $\begin{bmatrix} \eta_0 & \eta_1 & \eta_2 & \cdots \end{bmatrix}^T \in \mathcal{F}(\mathfrak{Z}(E^\sigma))$. Since $\mathcal{F}(\mathfrak{Z}(E^\sigma))$ is isomorphic to $\mathcal{F}(\mathfrak{Z}(E))$ via γ_∞, there exist $\alpha_{ij} \in \mathfrak{Z}(E)$ such that $\eta_i = L_{\alpha_{i1}} \otimes \cdots \otimes L_{\alpha_{ii}}$, for $i \geq 1$, and $\alpha_0 \in \mathfrak{Z}(M)$ such that $\sigma(\alpha_0) = \eta_0$. Thus for $\xi \in \mathfrak{Z}(E)$, we have

$$\gamma_\infty T_\xi \gamma_\infty^{-1} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \end{bmatrix} = \gamma_\infty T_\xi \begin{bmatrix} \alpha_0 \\ \alpha_{11} \\ \alpha_{21} \otimes \alpha_{22} \\ \vdots \end{bmatrix} = \gamma_\infty \begin{bmatrix} 0 \\ \xi \cdot \alpha_0 \\ \xi \otimes \alpha_{11} \\ \xi \otimes \alpha_{21} \otimes \alpha_{22} \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ L_\xi \sigma(\alpha_0) \\ L_\xi \otimes L_{\alpha_{11}} \\ L_\xi \otimes L_{\alpha_{21}} \otimes L_{\alpha_{22}} \\ \vdots \end{bmatrix} = T_{\gamma(\xi)} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \end{bmatrix}.$$

For $a \in \mathfrak{Z}(M)$, we have

$$\gamma_\infty \varphi_\infty(a) \gamma_\infty^{-1} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \end{bmatrix} = \gamma_\infty \varphi_\infty(a) \begin{bmatrix} \alpha_0 \\ \alpha_{11} \\ \alpha_{21} \otimes \alpha_{22} \\ \vdots \end{bmatrix} = \gamma_\infty \begin{bmatrix} a\alpha_0 \\ \varphi(a)(\alpha_{11}) \\ \varphi_2(a)(\alpha_{21} \otimes \alpha_{22}) \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \sigma(a\alpha_0) \\ L_{a\cdot\alpha_{11}} \\ L_{a\cdot\alpha_{21}} \otimes L_{\alpha_{22}} \\ \vdots \end{bmatrix} = \begin{bmatrix} \sigma(a)\sigma(\alpha_0) \\ \sigma(a) \cdot L_{\alpha_{11}} \\ \sigma(a) \cdot L_{\alpha_{21}} \otimes L_{\alpha_{22}} \\ \vdots \end{bmatrix} = \varphi_\infty^\sigma(\sigma(a)) \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \end{bmatrix}.$$

□

Thus we arrive at the following isomorphism from $H^\infty(\mathfrak{Z}(E))$ onto $H^\infty(\mathfrak{Z}(E^\sigma))$.

Proposition 4.1.5. *The map defined on the generators of* $H^\infty(\mathfrak{Z}(E))$ *by* $T_\xi \mapsto \gamma_\infty T_\xi \gamma_\infty^{-1}, \xi \in \mathfrak{Z}(E)$, *and* $\varphi_\infty(a) \mapsto \gamma_\infty \varphi_\infty(a) \gamma_\infty^{-1}, a \in \mathfrak{Z}(M)$, *extends to an isomorphism* Γ *from* $H^\infty(\mathfrak{Z}(E))$ *onto* $H^\infty(\mathfrak{Z}(E^\sigma))$.

With Γ in hand, we are able to prove that Theorems 3.0.1 and 4.0.1 yield the same result when we restrict to the centers of the correspondences.

Theorem 4.1.6. *Given N distinct elements $\mathfrak{z}_1, \ldots, \mathfrak{z}_N \in \mathfrak{Z}(E^\sigma)$ with $\|\mathfrak{z}_i\| < 1$ for all $i = 1, \ldots, N$, and $\Lambda_1, \ldots, \Lambda_N \in \mathfrak{Z}(\sigma(M)')$, define $\Psi^{\mathfrak{z}_i}_{\Lambda_i} : \sigma(M)' \to \sigma(M)'$ by*

$$\Psi^{\mathfrak{z}_i}_{\Lambda_i}(a) = \langle C(\mathfrak{z}_i), (I_{\mathcal{F}(E)} \otimes a\Lambda_i^*)C(0)\rangle.$$

The following are equivalent:

1. *there exists $X \in H^\infty(\mathfrak{Z}(E^\sigma))$ with $\|X\| \leq 1$ such that $\hat{X}(\mathfrak{z}_i) = \Lambda_i, i = 1, \ldots, N$, in the sense of Theorem 3.0.1.*

2. *there exists $Y \in H^\infty(\mathfrak{Z}(E))$ with $\|Y\| \leq 1$ such that $\hat{Y}(\mathfrak{z}_i^*) = \Lambda_i^*, i = 1, \ldots, N$, in the sense of Theorem 4.0.1.*

3. *$\Phi^{\mathfrak{z}_i}_X = \Psi^{\mathfrak{z}_i}_{\Lambda_i}, i = 1, \ldots, N$, where $\Phi^{\mathfrak{z}_i}_X$ is defined in equation (2.5).*

In order to prove this theorem, we will need the following generalization of the Schur product theorem to matrices with operator entries.

Lemma 4.1.7. *Let $\mathcal{A}$ be a W^*-algebra represented on a Hilbert space H. Let $\mathcal{A}'$ be the commutant of $\mathcal{A}$ in $B(H)$. If $A \in M_N(\mathcal{A}')$ is positive semidefinite and $B \in M_N(\mathcal{A})$ is positive semidefinite, then the Schur product of A and B, $A \circ B$, is a positive semidefinite element of $M_N(B(H))$.*

Proof. By Theorem 7.4 in [HR13], there exists a unitary matrix $U \in M_N(\mathcal{A}')$ such that UAU^* is a diagonal matrix. Since A is positive semidefinite, the diagonal entries of UAU^* are nonnegative elements of $\mathcal{A}'$. Therefore there exist $a_i, i = 1, \ldots, N$, in $\mathcal{A}'$ such that $UAU^* = \operatorname{diag}[a_i a_i^*]_{i=1}^N$. For $i = 1, \ldots, N$, define $\tilde{V}_i$ to be the column vector with a_i in the i^{th} entry and zeros elsewhere so that $UAU^* = \sum_{i=1}^N \tilde{V}_i \tilde{V}_i^*$. Then we may write $A = \sum_{i=1}^{N} V_i V_i^*$, where each $V_i = U^* \tilde{V}_i$ is a column vector with entries in $\mathcal{A}'$. Similarly, we may write $B = \sum_{i=1}^{N} W_i W_i^*$, where each W_i is a column vector with entries in $\mathcal{A}$. Then

$$A \circ B = \left(\sum_{i=1}^{N} V_i V_i^* \right) \circ \left(\sum_{i=1}^{N} W_i W_i^* \right) = \sum_{i,j=1}^{N} (V_i V_i^*) \circ (W_j W_j^*)$$

$$= \sum_{i,j=1}^{N} \left(\begin{bmatrix} V_{1i} \\ \vdots \\ V_{Ni} \end{bmatrix} \begin{bmatrix} V_{1i}^* & \cdots & V_{Ni}^* \end{bmatrix} \right) \circ \left(\begin{bmatrix} W_{1j} \\ \vdots \\ W_{Nj} \end{bmatrix} \begin{bmatrix} W_{1j}^* & \cdots & W_{Nj}^* \end{bmatrix} \right)$$

$$= \sum_{i,j=1}^{N} \begin{bmatrix} V_{1i} V_{1i}^* W_{1j} W_{1j}^* & \cdots & V_{1i} V_{Ni}^* W_{1j} W_{Nj}^* \\ \vdots & & \vdots \\ V_{Ni} V_{1i}^* W_{Nj} W_{1j}^* & \cdots & V_{Ni} V_{Ni}^* W_{Nj} W_{Nj}^* \end{bmatrix} = \sum_{i,j=1}^{N} (V_i \circ W_j)(V_i \circ W_j)^* \geq 0.$$

□

We are now ready to prove our comparison theorem.

Proof of Theorem 4.1.6. To see that (1) is equivalent to (2), we apply Lemma 4.1.7 to show that the Pick matrix $\mathcal{A}_{\mathcal{N}}$ from equation (3.1) is positive semidefinite if and only if the Pick matrix map

$$[B_{ij}]_{i,j=1}^{N} \overset{\Phi_{\mathcal{M}\mathfrak{S}}}{\longmapsto} [\langle C(\mathfrak{z}_i), (I_{\mathcal{F}(E)} \otimes B_{ij}) C(\mathfrak{z}_j) \rangle - \langle C(\mathfrak{z}_i) \Lambda_i, (I_{\mathcal{F}(E)} \otimes B_{ij}) C(\mathfrak{z}_j) \Lambda_j \rangle]_{i,j=1}^{N}$$

from Theorem 4.0.1 is a completely positive map from $M_N(\sigma(M)')$ into $M_N(B(H))$.

Let $B = [B_{ij}]_{i,j=1}^N \in M_N(\sigma(M)')$, and consider

$$\Phi_{\mathcal{MS}}([B_{ij}]_{i,j=1}^N) = [\langle C(\mathfrak{z}_i), (I_{\mathcal{F}(E)} \otimes B_{ij})C(\mathfrak{z}_j)\rangle - \langle C(\mathfrak{z}_i)\Lambda_i, (I_{\mathcal{F}(E)} \otimes B_{ij})C(\mathfrak{z}_j)\Lambda_j\rangle]_{i,j=1}^N$$

$$= [C(\mathfrak{z}_i)^*(I_{\mathcal{F}(E)} \otimes B_{ij})C(\mathfrak{z}_j) - \Lambda_i^* C(\mathfrak{z}_i)^*(I_{\mathcal{F}(E)} \otimes B_{ij})C(\mathfrak{z}_j)\Lambda_j]_{i,j=1}^N$$

$$= [C(\mathfrak{z}_i)^* C(\mathfrak{z}_j)B_{ij} - \Lambda_i^* C(\mathfrak{z}_i)^* C(\mathfrak{z}_j)\Lambda_j B_{ij}]_{i,j=1}^N$$

$$= [(\langle C(\mathfrak{z}_i), C(\mathfrak{z}_j)\rangle - \langle (I_{\mathcal{F}(E)} \otimes \Lambda_i)C(\mathfrak{z}_i), (I_{\mathcal{F}(E)} \otimes \Lambda_j)C(\mathfrak{z}_j)\rangle)B_{ij}]_{i,j=1}^N$$

$$= [A_{ij} \cdot B_{ij}]_{i,j=1}^N,$$

where $\mathcal{A}_{\mathcal{N}} = [A_{ij}]_{i,j=1}^N$ is the Pick matrix from equation (3.1). Thus $\Phi_{\mathcal{MS}}(B) = \mathcal{A}_{\mathcal{N}} \circ B$. Since $\mathcal{A}_{\mathcal{N}} \in M_N(\mathfrak{Z}(\sigma(M)'))$ and $B \in M_N(\sigma(M)')$, Lemma 4.1.7 implies that $\Phi_{\mathcal{MS}}(B) \geq 0$ if $\mathcal{A}_{\mathcal{N}} \geq 0$ and $B \geq 0$. Thus $\Phi_{\mathcal{MS}}$ is a positive map if $\mathcal{A}_{\mathcal{N}} \geq 0$. For complete positivity, let $k \in \mathbb{N}$ and consider the map $(\Phi_{\mathcal{MS}})_k : M_k(M_N(\sigma(M)')) \to M_k(M_N(B(H)))$ given by $(\Phi_{\mathcal{MS}})_k([B_{ij}]_{ij}) = [\Phi_{\mathcal{MS}}(B_{ij})]_{ij}$. If $B = [B_{ij}]_{i,j=1}^k \in M_k(M_N(\sigma(M)'))$, then $\Phi_k(B) = [\mathcal{A}_{\mathcal{N}} \circ B_{ij}]_{i,j=1}^k$. Let $\overline{\mathcal{A}}_{\mathcal{N}}$ denote the $k \times k$ block matrix with $\mathcal{A}_{\mathcal{N}}$ at each entry. Thinking of $\overline{\mathcal{A}}_{\mathcal{N}}$ as a $kN \times kN$ matrix with entries in $\mathfrak{Z}(\sigma(M)')$ and B as a $kN \times kN$ matrix with entries in $\sigma(M)'$, we may write $(\Phi_{\mathcal{MS}})_k(B) = \overline{\mathcal{A}}_{\mathcal{N}} \circ B$. Since $\overline{\mathcal{A}}_{\mathcal{N}} \geq 0$ if $\mathcal{A}_{\mathcal{N}} \geq 0$, by Lemma 4.1.7 $\Phi_{\mathcal{MS}}$ is completely positive if $\mathcal{A}_{\mathcal{N}} \geq 0$.

Conversely, if $\Phi_{\mathcal{MS}}$ is completely positive, then $\Phi_{\mathcal{MS}}(B) = \mathcal{A}_{\mathcal{N}} \circ B \geq 0$ for all $B \geq 0$. In particular, $\Phi_{\mathcal{MS}}([I_H]_{ij}) = \mathcal{A}_{\mathcal{N}} \circ [I_H]_{ij} = \mathcal{A}_{\mathcal{N}} \geq 0$.

Hence we have the following string of equivalences. By Theorem 3.0.1, there exists $X \in H^\infty(\mathfrak{Z}(E^\sigma))$ with $\|X\| \leq 1$ satisfying $\hat{X}(\mathfrak{z}_i) = \Lambda_i, i = 1, \ldots, N$, if and only if $\mathcal{A}_{\mathcal{N}} \geq 0$ if and only if $\Phi_{\mathcal{MS}}$ is completely positive if and only if there exists

$Y \in H^\infty(\mathfrak{Z}(E))$ with $\|Y\| \leq 1$ satisfying $Y(\mathfrak{z}_i^*) = \Lambda_i^*, i = 1, \ldots, N$, by Theorem 4.0.1.

The following calculation shows that (1) and (3) are equivalent. Note that the proof of Lemma 2.3.1 shows that for all $X \in H^\infty(E^\sigma)$, $\mathfrak{z} \in \mathfrak{Z}(E^\sigma)$ with $\|\mathfrak{z}\| < 1$, and $a \in \sigma(M)'$,

$$\rho(X)^*\rho(\varphi_\infty^\sigma(a^*))C(\mathfrak{z}) = \rho(X)^*(I_{\mathcal{F}(E)} \otimes a^*)C(\mathfrak{z}) = (I_{\mathcal{F}(E)} \otimes \hat{X}(\mathfrak{z}))(I_{\mathcal{F}(E)} \otimes a^*)C(\mathfrak{z}).$$

Thus we have

$$\begin{aligned}\Phi_X^{\mathfrak{z}_i}(a) &= \langle C(\mathfrak{z}_i), \rho(\varphi_\infty^\sigma(a))\rho(X)C(0)\rangle = \langle \rho(X)^*\rho(\varphi_\infty^\sigma(a^*))C(\mathfrak{z}_i), C(0)\rangle \\ &= \langle (I_{\mathcal{F}(E)} \otimes \hat{X}(\mathfrak{z}_i))(I_{\mathcal{F}(E)} \otimes a^*)C(\mathfrak{z}_i), C(0)\rangle = \langle C(\mathfrak{z}_i), (I_{\mathcal{F}(E)} \otimes a\hat{X}(\mathfrak{z}_i)^*C(0)\rangle\end{aligned}$$

which is equal to $\Psi_{\Lambda_i}^{\mathfrak{z}_i}(a)$ if and only if $\hat{X}(\mathfrak{z}_i) = \Lambda_i$ for $i = 1, \ldots, N$. $\square$

Therefore, in the setting of central correspondences, the proof of Theorem 3.0.1 provides a proof of Theorem 4.0.1 which does not rely on commutant lifting. We expect that the proof of Theorem 3.0.1 will generalize to the setting of weighted Nevanlinna-Pick interpolation which was studied by Good in [Goo17]. If so, our proof via the displacement equation will serve as a new proof of the nontangential version of [Goo17, Theorem 6.5] in the commutative setting.

CHAPTER 5
CONNECTIONS TO POPESCU'S GENERALIZED NEVANLINNA-PICK THEOREM

Popescu's generalized Nevanlinna-Pick theorem, [Pop03, Theorem 7.4], is strikingly similar to [CJ03, Theorem 3.4]. However, Popescu's "unit disc" strictly contains Constantinescu-Johnson's unit disc $B_n(H)$, and Popescu uses a combination of intertwining lifting and a displacement equation to prove his theorem. In this chapter, we state Theorem 3.0.1 in the more general context of Popescu's Theorem 7.4 in [Pop03] and give a lexicon for switching between Popescu's notation and our own. Then we use a corollary of Theorem 3.0.1 to eliminate intertwining lifting from the proof of a nontangential version of [Pop03, Theorem 7.4].

5.1 Definitions and Notation

What follows is a brief introduction to Popescu's notation from [Pop03]. See the original paper for more detail.

Let H_n denote an n-dimensional complex Hilbert space with orthonormal basis $e_1, \ldots, e_n$, where $n \in \mathbb{N}$ or $n = \infty$. $F^2(H_n) := \oplus_{k=0}^{\infty} H_n^{\otimes k}$ denotes the *full Fock space* over H_n. The *left creation operators* $S_i : F^2(H_n) \to F^2(H_n), i = 1, \ldots, n$, are defined by

$$S_i \psi := e_i \otimes \psi, \quad \psi \in F^2(H_n).$$

The *right creation operators* $R_i : F^2(H_n) \to F^2(H_n), i = 1, \ldots, n$, are defined by

$$R_i \psi := \psi \otimes e_i, \quad \psi \in F^2(H_n).$$

For $\alpha = g_{i_1}g_{i_2}\cdots g_{i_k} \in F_n^+$, the free semigroup on n generators, let S_α denote the operator $S_{i_1}S_{i_2}\cdots S_{i_k}$ and R_α denote the operator $R_{i_1}R_{i_2}\cdots R_{i_k}$. Then for $\psi \in F^2(H_n), S_\alpha(\psi) = e_\alpha \otimes \psi$ and $R_\alpha(\psi) = \psi \otimes e_{\widetilde{\alpha}}$, where $e_\alpha = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ and $\widetilde{\alpha}$ denotes the reverse of α, i.e., $\widetilde{\alpha} = g_{i_k}\cdots g_{i_2}g_{i_1}$.

The *noncommutative analytic Toeplitz algebra* F_n^∞ is defined to be the weakly-closed subalgebra of $\mathcal{L}(F^2(H_n))$ generated by the left creation operators $\{S_i\}_{i=1}^n$ and the identity operator on $F^2(H_n)$. For a Hilbert space H, $F_n^\infty \overline{\otimes} B(H)$ is the weakly-closed operator space generated by the spacial tensor product. Popescu notes that any element $f \in F_n^\infty \overline{\otimes} B(H)$ has a unique Fourier representation

$$f \sim \sum_{\alpha \in F_n^+} S_\alpha \otimes A_{(\alpha)}$$

for some operators $A_{(\alpha)} \in B(H)$ such that $\sum_{\alpha \in F_n^+} A_{(\alpha)}^* A_{(\alpha)} \leq \|f\|^2 I$. For $\mathfrak{z} = \begin{bmatrix} Z_1 & \dots & Z_n \end{bmatrix}$ with $Z_j \in B(H), j = 1, \dots, n$, the *spectral radius* of $\mathfrak{z}$ is defined by

$$r(\mathfrak{z}) := \inf_k \left\| \sum_{|\alpha|=k} Z_\alpha (Z_\alpha)^* \right\|^{\frac{1}{2k}} = \lim_{k\to\infty} \left\| \sum_{|\alpha|=k} Z_\alpha (Z_\alpha)^* \right\|^{\frac{1}{2k}},$$

where the second equality is due to [Pop89, Lemma 2.2].

Definition 5.1.1 ([Pop03, equation (7.5)])**.** *If* $\mathfrak{z} = \begin{bmatrix} Z_1 & \dots & Z_n \end{bmatrix}$ *with* $r(\mathfrak{z}) < 1$ *and* $f \sim \sum_{\alpha \in F_n^+} S_\alpha \otimes A_{(\alpha)} \in F_n^\infty \overline{\otimes} B(H)$, *then the* point evaluation *of* f *at* $\mathfrak{z}$ *is given by*

$$f(\mathfrak{z}) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} Z_{\widetilde{\alpha}} A_{(\alpha)},$$

where $\widetilde{\alpha}$ *denotes the reverse of* α.

Through correspondence with Popescu, we know that equation (7.5) in [Pop03] was inadvertently written without the tilde, but the typo has been corrected in Definition 5.1.1.

Lastly, for $\mathfrak{z} = \begin{bmatrix} Z_1 & \cdots & Z_n \end{bmatrix}$ with $r(\mathfrak{z}) < 1$ and $C \in B(H)$, the *controllability operator* $W_{\{\mathfrak{z},C\}} : F^2(H_n) \otimes H \to H$ associated with $\{\mathfrak{z}, C\}$ is defined by the formula

$$W_{\{\mathfrak{z},C\}} \left(\sum_{\alpha \in F_n^+} e_\alpha \otimes h_\alpha \right) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} Z_{\widetilde{\alpha}} C h_\alpha.$$

Note that the point evaluation and controllability operator are well-defined for $\mathfrak{z}$ with spectral radius less than 1 [Pop03, Section 7].

5.2 Spectral Radius in the W^*-Correspondence Setting

Let $\mathfrak{z} = \begin{bmatrix} Z_1 & \cdots & Z_n \end{bmatrix}$, where $Z_j \in B(H)$ for all $j = 1, \ldots, n$. Then $\mathfrak{z}^* = \begin{bmatrix} Z_1^* & \cdots & Z_n^* \end{bmatrix}^T \in E^\sigma$ for $E = \mathbb{C}^n$, and

$$(\mathfrak{z}^*)^{(k)} := (I_{(\mathbb{C}^n)^{\otimes k-1}} \otimes \mathfrak{z}^*) \cdots (I_{\mathbb{C}^n} \otimes \mathfrak{z}^*)\mathfrak{z}^* = [Z_\alpha]^*_{|\alpha|=k}$$

is an $n^k \times 1$ column vector, where $\{\alpha \in F_n^+ \mid |\alpha| = k\}$ are listed in lexicographical order. Thus, according to Popescu's definition of spectral radius,

$$r(\mathfrak{z}) = \inf_k \|((\mathfrak{z}^*)^{(k)})^*(\mathfrak{z}^*)^{(k)}\|^{\frac{1}{2k}} = \inf_k \|(\mathfrak{z}^*)^{(k)}\|^{\frac{1}{k}}.$$

Inspired by Popescu's definition of spectral radius, we make the following definition for spectral radius in the W^*-correspondence setting.

Definition 5.2.1. *For $\mathfrak{z} \in E^\sigma$, define the* spectral radius *of $\mathfrak{z}$ by*

$$r(\mathfrak{z}) := \inf_k \|\mathfrak{z}^{(k)}\|^{\frac{1}{k}}.$$

Lemma 5.2.2. *For* $\mathfrak{z} \in E^\sigma$, $r(\mathfrak{z}) = \lim_{k\to\infty} \|\mathfrak{z}^{(k)}\|^{\frac{1}{k}}$.

Our proof follows the line of reasoning employed to prove a similar statement, Proposition 8, in [BD73].

Proof. Let $\mathfrak{z} \in E^\sigma$ and fix $\epsilon > 0$. By Definition 5.2.1, there exists $N \in \mathbb{N}$ such that $\|\mathfrak{z}^{(N)}\|^{\frac{1}{N}} < r(\mathfrak{z}) + \epsilon$.

For all $n \in \mathbb{N}$, there exist $p(n), q(n) \in \mathbb{N}$ such that $n = p(n)N + q(n)$ and $q(n) < N$. Therefore, $\lim_{n\to\infty} \frac{1}{n} q(n) = 0$ and $\lim_{n\to\infty} \frac{1}{n} p(n)N = 1$. Consequently, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|\mathfrak{z}^{(n)}\|^{\frac{1}{n}} &= \|\mathfrak{z}^{(p(n)N+q(n))}\|^{\frac{1}{n}} = \|(I_{E^{\otimes p(n)N}} \otimes \mathfrak{z}^{(q(n))})\mathfrak{z}^{(p(n)N)}\|^{\frac{1}{n}} \\ &\le (\|\mathfrak{z}^{(q(n))}\| \|\mathfrak{z}^{(p(n)N)}\|)^{\frac{1}{n}} \le \|\mathfrak{z}\|^{\frac{q(n)}{n}} \|\mathfrak{z}^{(N)}\|^{\frac{p(n)}{n}} \overset{n\to\infty}{\longrightarrow} 1 \cdot \|\mathfrak{z}^{(N)}\|^{\frac{1}{N}} < r(\mathfrak{z}) + \epsilon.\end{aligned}$$

Thus for large n, $\|\mathfrak{z}^{(n)}\|^{\frac{1}{n}} \le r(\mathfrak{z}) + \epsilon$. Also, $r(\mathfrak{z}) \le \|\mathfrak{z}^{(n)}\|^{\frac{1}{n}}$ for all $n \in \mathbb{N}$, by definition of spectral radius. Therefore,

$$r(\mathfrak{z}) := \inf_k \|\mathfrak{z}^{(k)}\|^{\frac{1}{k}} = \lim_{k\to\infty} \|\mathfrak{z}^{(k)}\|^{\frac{1}{k}}.$$

□

In analogy to Proposition 2.11 in [Pop89] and Proposition 6 in [Bun84], we prove the following lemma.

Lemma 5.2.3. *For* $\mathfrak{z} \in E^\sigma$, *the following are equivalent:*

1. *There exists a positve operator* $P \in B(H)$ *such that* $\mathfrak{z}^*(I_E \otimes P)\mathfrak{z} + I_H = P$.

2. *The infinite series* $\sum_{k=1}^\infty (\mathfrak{z}^{(k)})^*\mathfrak{z}^{(k)}$ *converges in the strong operator topology.*

3. $r(\mathfrak{z}) < 1$.

4. $C(\mathfrak{z})$ *is a bounded operator.*

Proof. $(1 \implies 2)$ Suppose there exists a positive operator $P \in B(H)$ such that $\mathfrak{z}^*(I_E \otimes P)\mathfrak{z} + I_H = P$. Since $\mathfrak{z}^*(I_E \otimes P)\mathfrak{z} \geq 0$, then $P - I_H \geq 0$, i.e., $I_H \leq P$. Therefore,

$$\mathfrak{z}^*\mathfrak{z} \leq \mathfrak{z}^*(I_E \otimes P)\mathfrak{z} = P - I_H.$$

Thus $I_H + \mathfrak{z}^*\mathfrak{z} \leq P$. Now assume, as inductive hypothesis, that

$$I_H + \sum_{k=1}^{m} (\mathfrak{z}^{(k)})^*\mathfrak{z}^{(k)} \leq P.$$

We have already proved the base case $(m = 1)$. Assuming the equation is true for $m \in \mathbb{N}$, then

$$\begin{aligned} I_E \otimes (I_H + \sum_{k=1}^{m} (\mathfrak{z}^{(k)})^*\mathfrak{z}^{(k)}) &\leq I_E \otimes P \\ \implies \mathfrak{z}^*(I_E \otimes (I_H + \sum_{k=1}^{m} (\mathfrak{z}^{(k)})^*\mathfrak{z}^{(k)}))\mathfrak{z} &\leq \mathfrak{z}^*(I_E \otimes P)\mathfrak{z} \\ \implies \mathfrak{z}^*\mathfrak{z} + \sum_{k=1}^{m} \mathfrak{z}^*(I_E \otimes (\mathfrak{z}^{(k)})^*)(I_E \otimes \mathfrak{z}^{(k)})\mathfrak{z} &\leq \mathfrak{z}^*(I_E \otimes P)\mathfrak{z} \\ \implies \mathfrak{z}^*\mathfrak{z} + \sum_{k=1}^{m} (\mathfrak{z}^{(k+1)})^*\mathfrak{z}^{(k+1)} &\leq \mathfrak{z}^*(I_E \otimes P)\mathfrak{z} = P - I_H \\ \implies I_H + \sum_{k=1}^{m+1} (\mathfrak{z}^{(k)})^*\mathfrak{z}^{(k)} &\leq P. \end{aligned}$$

Therefore the infinite series $\sum_{k=1}^{\infty} (\mathfrak{z}^{(k)})^*\mathfrak{z}^{(k)}$ converges in the strong operator topology.

$(2 \implies 1)$ Assume the infinite series $\sum_{k=1}^{\infty} (\mathfrak{z}^{(k)})^*\mathfrak{z}^{(k)}$ converges in the strong operator topology, and define

$$P := I_H + \sum_{k=1}^{\infty} (\mathfrak{z}^{(k)})^*\mathfrak{z}^{(k)}.$$

Then

$$
\begin{aligned}
I_E \otimes P = I_E \otimes I_H + \sum_{k=1}^{\infty} (I_E \otimes (\mathfrak{z}^{(k)})^*)(I_E \otimes \mathfrak{z}^{(k)}) \\
\implies \mathfrak{z}^*(I_E \otimes P)\mathfrak{z} = \mathfrak{z}^*\mathfrak{z} + \sum_{k=1}^{\infty} \mathfrak{z}^*(I_E \otimes (\mathfrak{z}^{(k)})^*)(I_E \otimes \mathfrak{z}^{(k)})\mathfrak{z} \\
\implies \mathfrak{z}^*(I_E \otimes P)\mathfrak{z} = \sum_{k=1}^{\infty} (\mathfrak{z}^{(k)})^*\mathfrak{z}^{(k)} = P - I_H \\
\implies \mathfrak{z}^*(I_E \otimes P)\mathfrak{z} + I_H = P.
\end{aligned}
$$

($2 \implies 3$) Assuming 2 holds, then for $x \in H$ we have

$$
\langle Px, x\rangle = \langle (I_H + \sum_{k=1}^{\infty} (\mathfrak{z}^{(k)})^*\mathfrak{z}^{(k)})x, x\rangle = \|x\|^2 + \sum_{k=1}^{\infty} \|\mathfrak{z}^{(k)}x\|^2.
$$

Therefore $\sup_k \|\mathfrak{z}^{(k)}x\|^2 < \infty$, so by the Uniform Boundedness Principle there exists $M > 0$ such that $\|\mathfrak{z}^{(k)}\|^2 = \|(\mathfrak{z}^{(k)})^*\mathfrak{z}^{(k)}\| \le M$ for all $k \in \mathbb{N}$.

For any r such that $0 < r < M^{-\frac{1}{2}}$, define $m : H \to \{0, 1, 2, \ldots\}$ by

$$
m(x) = \inf\{k \ge 0 \mid \|\mathfrak{z}^{(k)}x\| \le r\|x\|\}.
$$

If $m(x) \ge 1$, then

$$
\begin{aligned}
r\|x\| &< \|\mathfrak{z}^{(k)}x\|, \quad \text{for } k = 0, \ldots, m(x) - 1 \\
\implies r^2\|x\|^2 &< \|\mathfrak{z}^{(k)}x\|^2, \quad \text{for } k = 0, \ldots, m(x) - 1 \\
\implies m(x)r^2\|x\|^2 &\le \sum_{k=0}^{m(x)-1} \|\mathfrak{z}^{(k)}x\|^2.
\end{aligned}
$$

Consequently,

$$
m(x)r^2\|x\|^2 \le \sum_{k=0}^{\infty} \|\mathfrak{z}^{(k)}x\|^2 = \langle Px, x\rangle \le \|P\|\|x\|^2.
$$

Therefore, for all $x \in H$, $m(x) \leq \frac{\|P\|}{r^2}$. Fix $k > \frac{\|P\|}{r^2}$ and consider

$$\|\mathfrak{z}^{(k)}x\| = \|(I_{E^{\otimes m(x)}} \otimes \mathfrak{z}^{(k-m(x))})\mathfrak{z}^{(m(x))}x\| \leq \|\mathfrak{z}^{(k-m(x))}\|\|\mathfrak{z}^{(m(x))}x\| \leq M^{\frac{1}{2}}r\|x\|,$$

where the last inequality holds by definition of M and $m(x)$. Thus for $k > \frac{\|P\|}{r^2}$, we have $\|\mathfrak{z}^{(k)}\| \leq M^{\frac{1}{2}}r < 1$. By Definition 5.2.1, this implies $r(\mathfrak{z}) < 1$.

($3 \implies 4$) Suppose $r(\mathfrak{z}) < 1$. Then for any r' such that $r(\mathfrak{z}) < r' < 1$, there exists N such that for all $k \geq N$, $\|\mathfrak{z}^{(k)}\|^{\frac{1}{k}} < r'$. Thus, for all $k \geq N$, $\|\mathfrak{z}^{(k)}\| < (r')^k$. Consequently,

$$\begin{aligned}\|C(\mathfrak{z})^*C(\mathfrak{z})\| = \|\sum_{k=0}^{\infty}(\mathfrak{z}^{(k)})^*\mathfrak{z}^{(k)}\| &\leq \sum_{k=0}^{\infty}\|(\mathfrak{z}^{(k)})^*\mathfrak{z}^{(k)}\| \\ &\leq \sum_{k=0}^{N-1}\|(\mathfrak{z}^{(k)})^*\mathfrak{z}^{(k)}\| + \sum_{k=N}^{\infty}(r')^{2k} < \infty.\end{aligned}$$

Thus $\|C(\mathfrak{z})\| = \|C(\mathfrak{z})^*C(\mathfrak{z})\|^{\frac{1}{2}} < \infty$.

($4 \implies 2$) If $C(\mathfrak{z})$ is bounded, then

$$\sum_{k=1}^{\infty}(\mathfrak{z}^{(k)})^*\mathfrak{z}^{(k)} = C(\mathfrak{z})^*C(\mathfrak{z}) - I_H$$

converges in the strong operator topology. □

Note that for $\mathfrak{z} \in E^\sigma$, $\|\mathfrak{z}\| < 1$ implies $r(\mathfrak{z}) < 1$ because for all $k \in \mathbb{N}$,

$$\|\mathfrak{z}^{(k)}\|^{\frac{1}{k}} \leq (\|\mathfrak{z}\|^k)^{\frac{1}{k}} = \|\mathfrak{z}\| < 1.$$

However, the converse is not true. For example, consider $\mathfrak{z} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \in E^\sigma$, where $M = \mathbb{C}, E = \mathbb{C}, H = \mathbb{C}^2$, and $\sigma : M \to B(H)$ is given by $\sigma(a) = aI_2$. Then $\|\mathfrak{z}\| = 2$ but $r(\mathfrak{z}) = 0$ since $\|\mathfrak{z}^{(k)}\| = \|\mathfrak{z}^k\| = 0$ for all $k \geq 2$. We summarize our observations in a remark.

Remark 5.2.4. *For* $\mathfrak{z} \in E^\sigma, \|\mathfrak{z}\| < 1$ *implies* $r(\mathfrak{z}) < 1$*, but the converse is not true.*

In fact, nilpotent elements in E^σ with arbitarily large norm have spectral radius 0. Therefore, Popescu's requirement that the spectral radius be less than 1 is far less restrictive than the requirement that the initial data have norm less than 1 in Theorem 3.0.1.

5.3 Connections between Popescu's Theorem and Theorem 3.0.1

Since Theorem 3.0.1 is nontangential, we work with the nontangential version of [Pop03, Theorem 7.4] that follows.

Theorem 5.3.1. *Let* H *be a Hilbert space. For* $i = 1, \ldots, N$*, let* $\mathfrak{z}_i = \begin{bmatrix} Z_{i1} & \ldots & Z_{in} \end{bmatrix} :$ $H^{(n)} \to H$ *be distinct points such that* $r(\mathfrak{z}_i) < 1$ *for all* i*, and let* $\Lambda_i \in B(H)$*. There exists* $\Phi \in F_n^\infty \overline{\otimes} B(H)$ *such that* $\|\Phi\| \leq 1$ *and*

$$\Phi(\mathfrak{z}_i) = \Lambda_i, \quad i = 1, \ldots, N,$$

if and only if the operator matrix

$$\mathcal{A}_{\mathcal{P}} = \left[\sum_{k=0}^{\infty} \sum_{|\alpha|=k} Z_{i\alpha}(I_H - \Lambda_i \Lambda_j^*)(Z_{j\alpha})^* \right]_{i,j=1}^{N}$$

is positive semidefinite.

By Remark 5.2.4, Popescu's hypothesis that $r(\mathfrak{z}_i) < 1$, for all $i = 1, \ldots, N$, is a relaxation of the hypotheses that $\mathfrak{z}_i \in B_n(H)$ in [CJ03, Theorem 3.4] and $\mathfrak{z}_i \in E^\sigma$ with $\|\mathfrak{z}_i\| < 1$ in Theorem 3.0.1 and [MS04, Theorem 5.3]. Fortunately, Theorem 3.0.1 holds under the broader assumption that $r(\mathfrak{z}_i) < 1$ for all $i = 1, \ldots, N$. In fact, the proof of Theorem 3.0.1 generalizes to this context with only one minor modification. Note that since $C(\mathfrak{z}_i)$ are bounded operators as long as $r(\mathfrak{z}_i) < 1$, then $(I - \theta_{\mathfrak{z}})^{-1} : \sigma^{(N)}(M)' \to \sigma^{(N)}(M)'$ defined by

$$(I - \theta_{\mathfrak{z}})^{-1}(B) = C(\mathfrak{z})^*(I_{\mathcal{F}(E)} \otimes B)C(\mathfrak{z})$$

is a bounded operator. Moreover,

$$U_\infty = \begin{bmatrix} C(\mathfrak{z}_1) & \cdots & C(\mathfrak{z}_N) \end{bmatrix}$$

and

$$V_\infty = \begin{bmatrix} (I_{\mathcal{F}(E)} \otimes \Lambda_1)C(\mathfrak{z}_1) & \cdots & (I_{\mathcal{F}(E)} \otimes \Lambda_N)C(\mathfrak{z}_N) \end{bmatrix}$$

are bounded operators. Thus the Pick matrix

$$\mathcal{A}_{\mathcal{N}} = (I - \theta_{\mathfrak{z}})^{-1}(UU^* - VV^*) = U_\infty U_\infty^* - V_\infty V_\infty^*$$

is bounded. In addition, the point evaluation of $X \in H^\infty(E^\sigma)$ at $\mathfrak{z} \in E^\sigma$ with $r(\mathfrak{z}) < 1$ given by $\hat{X}(\mathfrak{z}) = \langle \rho(X)C(0), C(\mathfrak{z})\rangle$ is well-defined. The only necessary modification to the proof of Theorem 3.0.1 is in showing that the last term in equation (3.9) goes to 0 as $K \to \infty$. To show this, note that if $r(\mathfrak{z}) < r' < 1$, then for large $K \in \mathbb{N}$, $\|\mathfrak{z}^{(K+1)}\| < (r')^{K+1}$. Thus $\|\mathfrak{z}^{(K+1)}\| \to 0$ as $K \to \infty$. Therefore, we have the following more general statement of Theorem 3.0.1.

Theorem 5.3.2. *Let E be a W^*-correspondence over a W^*-algebra M, with the left action of M on E given by a faithful, normal $*$-homomorphism $\varphi : M \to \mathcal{L}(E)$. Let σ be a faithful, normal representation of M on a Hilbert space H. Let $\mathfrak{z}_1, \ldots, \mathfrak{z}_N$ be*

N distinct elements of E^σ with $r(\mathfrak{z}_i) < 1$ for all $i = 1, \ldots, N$, and let $\Lambda_1, \ldots, \Lambda_N \in \sigma(M)'$. There exists $X \in H^\infty(E^\sigma)$ with $\|X\| \leq 1$ such that

$$\hat{X}(\mathfrak{z}_i) = \Lambda_i, \quad i = 1, \ldots, N,$$

if and only if the Pick matrix

$$\mathcal{A}_{\mathcal{N}} = \Big[\langle C(\mathfrak{z}_i), C(\mathfrak{z}_j)\rangle - \langle (I_{\mathcal{F}(E)} \otimes \Lambda_i)C(\mathfrak{z}_i), (I_{\mathcal{F}(E)} \otimes \Lambda_j)C(\mathfrak{z}_j)\rangle\Big]_{i,j=1}^{N} \tag{5.1}$$

is positive semidefinite.

For the given data in Theorem 5.3.1, $\mathfrak{z}_i^* = \begin{bmatrix} Z_{i1}^* & \cdots & Z_{in}^* \end{bmatrix}^T$ is an element of E^σ with $r(\mathfrak{z}_i^*) < 1$ for all $i = 1, \ldots, N$, where $M = \mathbb{C}, E = \mathbb{C}^n$, and $\sigma : M \to B(H)$ is given by $\sigma(a) = aI_H$. We have the following corollary of Theorem 5.3.2 for this data.

Corollary 5.3.3. *There exists $X \in H^\infty(E^\sigma)$ such that $\hat{X}(\mathfrak{z}_i^*) = \Lambda_i^*$ for all $i = 1, \ldots, N$ if and only if*

$$\mathcal{A}_{\mathcal{N}} = [C(\mathfrak{z}_i^*)^*(I_{\mathcal{F}(E)} \otimes (I_H - \Lambda_i\Lambda_j^*))C(\mathfrak{z}_j^*)]_{i,j=1}^{N}$$

is positive semidefinite.

We note that Corollary 5.3.3 is almost identical to [CJ03, Theorem 3.4], except that the hypotheses have been weakened to include all $\mathfrak{z}_i^*$ such that $r(\mathfrak{z}_i^*) < 1$. We will use Corollary 5.3.3 to prove Theorem 5.3.1, but first we state and prove the following lemma which will serve as a lexicon for going between Popescu's notation in Theorem 5.3.1 and our notation in Corollary 5.3.3 and in the proof of Theorem 3.0.1.

Lemma 5.3.4. *Given the interpolation data in Theorem 5.3.1, define*

$$U := \begin{bmatrix} I_H \\ \vdots \\ I_H \end{bmatrix} \text{ and } V := \begin{bmatrix} \Lambda_1 \\ \vdots \\ \Lambda_N \end{bmatrix}.$$

Define $Y := \begin{bmatrix} Y_1 & \cdots & Y_n \end{bmatrix}$, *where*

$$Y_j := \begin{bmatrix} Z_{1j} & & \\ & \ddots & \\ & & Z_{Nj} \end{bmatrix}, \quad j = 1, \ldots, n.$$

Finally, define $J : F^2(H_n) \to F^2(H_n)$ *by* $J(e_\alpha) = e_{\widetilde{\alpha}}$. *We have the following relationships between Popescu's notation and our own:*

1. *The Pick matrices* $\mathcal{A}_{\mathcal{P}}$ *in Theorem 5.3.1 and* $\mathcal{A}_{\mathcal{N}}$ *in Corollary 5.3.3 are equal.*

2. $W_{\{Y,U\}} = U_\infty^*(J \otimes I_H)$, *where* $W_{\{Y,U\}}$ *denotes the controllability operator associated with* $\{Y, U\}$ *and* $U_\infty = \begin{bmatrix} C(\mathfrak{z}_1^*) & \cdots & C(\mathfrak{z}_N^*) \end{bmatrix}$.

3. $W_{\{Y,V\}} = V_\infty^*(J \otimes I_H)$. *where* $W_{\{Y,V\}}$ *denotes the controllability operator associated with* $\{Y, V\}$ *and* $V_\infty = \begin{bmatrix} (I_{\mathcal{F}(\mathbb{C}^n)} \otimes \Lambda_1^*)C(\mathfrak{z}_1^*) & \cdots & (I_{\mathcal{F}(\mathbb{C}^n)} \otimes \Lambda_N^*)C(\mathfrak{z}_N^*) \end{bmatrix}$.

4. $W_{\{Y,U\}}W_{\{Y,U\}}^* - W_{\{Y,V\}}W_{\{Y,V\}}^* = U_\infty^* U_\infty - V_\infty^* V_\infty$.

5. $\Phi \in F_n^\infty \overline{\otimes} B(H)$ *with* $\|\Phi\| \leq 1$ *if and only if* $T := (J \otimes I_H)\Phi^*(J \otimes I_H) \in \mathcal{S}(\mathbb{C}^n, H, \sigma)$. *In this case,* $W_{\{Y,U\}}\Phi = W_{\{Y,V\}}$ *if and only if* $TU_\infty = V_\infty$.

Proof. 1. Consider the ij^{th} entry of $\mathcal{A}_{\mathcal{N}}$:

$$C(\mathfrak{z}_i^*)^*(I_{\mathcal{F}(E)} \otimes (I_H - \Lambda_i\Lambda_j^*))C(\mathfrak{z}_j^*)$$

$$= \begin{bmatrix} I_H & Z_{i1} & \cdots & Z_{in} & Z_{i1}Z_{i1} & \cdots \end{bmatrix} \begin{bmatrix} I_H - \Lambda_i\Lambda_j^* & & \\ & I_H - \Lambda_i\Lambda_j^* & \\ & & \ddots \end{bmatrix} \begin{bmatrix} I_H \\ Z_{j1}^* \\ \vdots \\ Z_{jn}^* \\ Z_{j1}^*Z_{j1}^* \\ Z_{j2}^*Z_{j1}^* \\ \vdots \end{bmatrix}$$

$$= \sum_{k=0}^{\infty} \sum_{|\alpha|=k} Z_{i\alpha}(I_H - \Lambda_i\Lambda_j^*)(Z_{j\alpha})^*,$$

which equals the ij^{th} entry of $\mathcal{A}_{\mathcal{P}}$.

2. Let $\sum_{\alpha\in F_n^+} e_\alpha \otimes h_\alpha \in F^2(H_n)\otimes H$. By definition of the controllability operator,

$$W_{\{Y,U\}}\left(\sum_{\alpha\in F_n^+} e_\alpha\otimes h_\alpha\right) = \sum_{\alpha\in F_n^+} Y_{\tilde{\alpha}} U h_\alpha = \sum_{\alpha\in F_n^+} \begin{bmatrix} Z_{1\tilde{\alpha}} & & \\ & \ddots & \\ & & Z_{N\tilde{\alpha}} \end{bmatrix} U h_\alpha$$

$$= \begin{bmatrix} \sum_{\alpha\in F_n^+} Z_{1\tilde{\alpha}} h_\alpha \\ \vdots \\ \sum_{\alpha\in F_n^+} Z_{N\tilde{\alpha}} h_\alpha \end{bmatrix}.$$

On the other hand,

$$U_\infty^*(J\otimes I_H)\left(\sum_{\alpha\in F_n^+} e_\alpha\otimes h_\alpha\right) = U_\infty^*\left(\sum_{\alpha\in F_n^+} e_{\widetilde{\alpha}}\otimes h_\alpha\right) = \begin{bmatrix} C(\mathfrak{z}_1^*)^* \\ \vdots \\ C(\mathfrak{z}_N^*)^* \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_n \\ h_{11} \\ h_{21} \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{\alpha\in F_n^+} Z_{1\alpha} h_{\widetilde{\alpha}} \\ \vdots \\ \sum_{\alpha\in F_n^+} Z_{N\alpha} h_{\widetilde{\alpha}} \end{bmatrix}.$$

Since the sums are absolutely convergent, this proves the claim.

3. Let $\sum_{\alpha\in F_n^+} e_\alpha\otimes h_\alpha \in F^2(H_n)\otimes H$. By definition of the controllability operator,

$$W_{\{Y,V\}}\left(\sum_{\alpha\in F_n^+} e_\alpha\otimes h_\alpha\right) = \sum_{\alpha\in F_n^+} Y_{\widetilde{\alpha}} V h_\alpha = \sum_{\alpha\in F_n^+} \begin{bmatrix} Z_{1\widetilde{\alpha}} & & \\ & \ddots & \\ & & Z_{N\widetilde{\alpha}} \end{bmatrix} V h_\alpha$$

$$= \begin{bmatrix} \sum_{\alpha\in F_n^+} Z_{1\widetilde{\alpha}} \Lambda_1 h_\alpha \\ \vdots \\ \sum_{\alpha\in F_n^+} Z_{N\widetilde{\alpha}} \Lambda_N h_\alpha \end{bmatrix}.$$

On the other hand,

$$V_\infty^*(J\otimes I_H)\left(\sum_{\alpha\in F_n^+} e_\alpha\otimes h_\alpha\right) = V_\infty^*\left(\sum_{\alpha\in F_n^+} e_{\widetilde{\alpha}}\otimes h_\alpha\right)$$

$$= \begin{bmatrix} C(\mathfrak{z}_1^*)^*(I_{\mathcal{F}(\mathbb{C}^n)} \otimes \Lambda_1) \\ \vdots \\ C(\mathfrak{z}_N^*)^*(I_{\mathcal{F}(\mathbb{C}^n)} \otimes \Lambda_N) \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_n \\ h_{11} \\ h_{21} \\ \vdots \end{bmatrix} = \begin{bmatrix} \sum_{\alpha \in F_n^+} Z_{1\alpha} \Lambda_1 h_{\widetilde{\alpha}} \\ \vdots \\ \sum_{\alpha \in F_n^+} Z_{N\alpha} \Lambda_N h_{\widetilde{\alpha}} \end{bmatrix}.$$

Since the sums are absolutely convergent, this proves the claim.

4. This claim is an easy consequence of parts 2 and 3 since J is unitary.

5. Suppose $\Phi \sim \sum_{\alpha \in F_n^+} S_\alpha \otimes A_{(\alpha)} \in F_n^\infty \overline{\otimes} B(H)$ with $\|\Phi\| \leq 1$. Recall that if $T \in \mathcal{S}(\mathbb{C}^n, H, \sigma)$, then $T = [T_{ij}]_{i,j=0}^\infty : \mathcal{F}(\mathbb{C}^n) \otimes H \to \mathcal{F}(\mathbb{C}^n) \otimes H$ is upper triangular, obeys the rule $T_{ij} = I_{\mathbb{C}^n} \otimes T_{i-1,j-1}$ for $i, j \geq 1$, and $T_{0k} : \mathbb{C}^{n \otimes k} \otimes H \to H$ is an element of $\mathfrak{I}(\sigma^{\mathbb{C}^{n \otimes k}} \circ \varphi_k, \sigma)$ for all $k \geq 0$. In particular, T^* is of the form

$$T^* := \begin{bmatrix} T_{00}^* & 0 & 0 & \cdots \\ T_{01}^* & I_{\mathbb{C}^n} \otimes T_{00}^* & 0 & \cdots \\ T_{02}^* & I_{\mathbb{C}^n} \otimes T_{01}^* & I_{(\mathbb{C}^n)^{\otimes 2}} \otimes T_{00}^* & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since T^* is completely determined by its first column, we need only define T_{0k}^* for all $k \geq 0$. Fix $k \geq 0$ and let $\alpha_1, \ldots, \alpha_{n^k}$ denote all the words in F_n^+ of length

k, listed in lexicographical order. Given $\Phi \sim \sum_{\alpha \in F_n^+} S_\alpha \otimes A_{(\alpha)}$, define

$$T_{0k}^* := \begin{bmatrix} A_{(\widetilde{\alpha}_1)} \\ A_{(\widetilde{\alpha}_2)} \\ \vdots \\ A_{(\widetilde{\alpha}_{n^k})} \end{bmatrix}.$$

Form T^* as above and consider the k^{th} subdiagonal of T^*:

$$\begin{bmatrix} 0 & & \\ T_{0k}^* & 0 & \\ 0 & I_{\mathbb{C}^n} \otimes T_{0k}^* & \ddots \\ & 0 & \ddots \end{bmatrix} = \begin{bmatrix} 0 & & \\ \begin{bmatrix} A_{(\widetilde{\alpha}_1)} \\ \vdots \\ A_{(\widetilde{\alpha}_{n^k})} \end{bmatrix} & 0 & \\ 0 & I_{\mathbb{C}^n} \otimes \begin{bmatrix} A_{(\widetilde{\alpha}_1)} \\ \vdots \\ A_{(\widetilde{\alpha}_{n^k})} \end{bmatrix} & \ddots \\ & 0 & \ddots \end{bmatrix} = \sum_{|\alpha|=k} R_{\widetilde{\alpha}} \otimes A_{(\widetilde{\alpha})},$$

where $R_{\widetilde{\alpha}}$ denotes the right creation operator associated with $\widetilde{\alpha} \in F_n^+$.

Thus

$$T^* = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} R_\alpha \otimes A_{(\alpha)}.$$

Then for $e_\beta \otimes h \in F^2(H_n) \otimes H$, we have

$$(J \otimes I_H) T^* (J \otimes I_H)(e_\beta \otimes h) = (J \otimes I_H)\Big(\sum_{\alpha \in F_n^+} R_\alpha \otimes A_{(\alpha)}\Big)(e_{\widetilde{\beta}} \otimes h)$$

$$= (J \otimes I_H)\Big(\sum_{\alpha \in F_n^+} e_{\widetilde{\beta}\widetilde{\alpha}} \otimes A_{(\alpha)} h\Big) = \sum_{\alpha \in F_n^+} e_{\alpha\beta} \otimes A_{(\alpha)} h.$$

Therefore $(J \otimes I_H)T^*(J \otimes I_H) = \sum_{\alpha \in F_n^+} S_\alpha \otimes A_{(\alpha)} = \Phi$. Since J is unitary and $\|\Phi\| \leq 1$, then $\|T\| = \|T^*\| \leq 1$ as well. It remains to show that $T_{0k} \in \mathfrak{I}(\sigma^{\mathbb{C}^{n \otimes k}} \circ \varphi_k, \sigma)$ for all $k \geq 0$.

Since $\sigma : \mathbb{C} \to B(H)$ is defined by $\sigma(a) = aI_H$,

$$\begin{aligned} \mathfrak{I}(\sigma^{\mathbb{C}^{n \otimes k}} \circ \varphi_k, \sigma) &= \{\eta \in B((\mathbb{C}^n)^{\otimes k} \otimes H, H) \mid \eta(\sigma^{\mathbb{C}^{n \otimes k}} \circ \varphi_k)(a) = \sigma(a)\eta \quad \forall a \in \mathbb{C}\} \\ &= B((\mathbb{C}^n)^{\otimes k} \otimes H, H). \end{aligned}$$

Therefore $T \in \mathcal{S}(\mathbb{C}^n, H, \sigma)$, as desired.

For the converse, form Φ from the entries of T^* as above. The same calculations show that $T \in \mathcal{S}(\mathbb{C}^n, H, \sigma)$ implies that $\Phi = (J \otimes I_H)T^*(J \otimes I_H) \in F_n^\infty \overline{\otimes} B(H)$ and $\|\Phi\| \leq 1$.

The proof that $W_{\{Y,U\}}\Phi = W_{\{Y,V\}}$ if and only if $\rho(X)^* U_\infty = V_\infty$ now follows from parts 2 and 3.

□

We can now prove Theorem 5.3.1. Our proof relies solely on Corollary 5.3.3 and the calculations in Lemma 5.3.4. In essence, it shows that Theorem 5.3.1 and Corollary 5.3.3 are two different ways of writing the same theorem.

Proof of Theorem 5.3.1. ($\Longrightarrow$) Suppose there exists $\Phi \sim \sum_{\alpha \in F_n^+} S_\alpha \otimes A_{(\alpha)}$ in $F_n^\infty \overline{\otimes} B(H)$ such that $\|\Phi\| \leq 1$ and

$$\Phi(\mathfrak{z}_i) = \Lambda_i, \quad i = 1, \ldots, N.$$

By Definition 5.1.1,

$$\Phi(\mathfrak{z}_i) = \sum_{\alpha \in F_n^+} Z_{i\alpha} A_{(\widetilde{\alpha})} = \Lambda_i, \quad i = 1, \ldots, N.$$

By the proof of part 5 of Lemma 5.3.4, the operator $T := (J \otimes I_H)\Phi^*(J \otimes I_H)$ is an element of $\mathcal{S}(\mathbb{C}^n, H, \sigma)$. Moreover, by the definition of T and the definition of the point evaluation in equation (2.3), we have

$$\begin{aligned}
(T(\mathfrak{z}_i^*))^* &= \langle C(0), TC(\mathfrak{z}_i^*)\rangle^* = C(\mathfrak{z}_i^*)^* T^* C(0) \\
&= \begin{bmatrix} I_H & Z_{i1} & \cdots & Z_{in} & Z_{i1}Z_{i1} & \cdots \end{bmatrix} \begin{bmatrix} T_{00}^* & 0 & 0 & \cdots \\ T_{01}^* & I_{\mathbb{C}^n} \otimes T_{00}^* & 0 & \cdots \\ T_{02}^* & I_{\mathbb{C}^n} \otimes T_{01}^* & I_{\mathbb{C}^{n \otimes 2}} \otimes T_{00}^* & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} I_H \\ 0 \\ 0 \\ \vdots \end{bmatrix} \\
&= \begin{bmatrix} I_H & Z_{i1} & \cdots & Z_{in} & Z_{i1}Z_{i1} & Z_{i1}Z_{i2} & \cdots \end{bmatrix} \begin{bmatrix} T_{00}^* \\ T_{01}^* \\ T_{02}^* \\ \vdots \end{bmatrix} = \sum_{k=0}^{\infty} \begin{bmatrix} Z_{i\alpha_1} & \cdots & Z_{i\alpha_{n^k}} \end{bmatrix} T_{0k}^* \\
&= \sum_{k=0}^{\infty} \sum_{|\alpha|=k} Z_{i\alpha} A_{(\widetilde{\alpha})}.
\end{aligned}$$

Therefore $T(\mathfrak{z}_i^*) = \Lambda_i^*$ for $i = 1, \ldots, N$. By Lemma 2.2.3 and the definitions of the point evaluations in Chapter 2, there exists $X \in H^\infty(E^\sigma)$ such that $\|X\| \leq 1$ and $\hat{X}(\mathfrak{z}_i^*) = \Lambda_i^*$ for all $i = 1, \ldots, N$. Therefore $\mathcal{A}_{\mathcal{N}}$ is positive semidefinite by Corollary 5.3.3. Finally, $\mathcal{A}_{\mathcal{N}} = \mathcal{A}_{\mathcal{P}}$ by part 1 of Lemma 5.3.4, so $\mathcal{A}_{\mathcal{P}}$ is positive semidefinite, as desired.

For the converse, we can follow the argument backward to show that $\mathcal{A}_{\mathcal{P}} \geq 0$ implies that there exists $\Phi \in F_n^\infty \overline{\otimes} B(H)$ such that $\|\Phi\| \leq 1$ and $\Phi(\mathfrak{z}_i) = \Lambda_i$ for all $i = 1, \ldots, N$. □

As mentioned at the beginning of the chapter, Popescu originally proved Theorem 7.4 in [Pop03] using a combination of intertwining lifting ([Pop03, Corollary 7.3]) and a displacement equation. Since Corollary 5.3.3 was proved solely via the displacement equation, the above proof shows that intertwining lifting is not necessary in the proof of the nontangential version of [Pop03, Theorem 7.4].

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