

POTENTIAL-FLOW ANALOGS AND COMPUTERS

by

S. C. LING

INTRODUCTION

An analogy is often used to explain an unfamiliar phenomenon in terms of a better known counterpart. In particular, if two physical systems are described by the same mathematical relationships, quantitative results for one system can be obtained by studying the other system as an analog. Consequently, the physical system which performs the numerical calculations using analogous variables of the prototype system is known as an analog computer.

For inviscid irrotational fluid motion [1], in which the tangential stress on a fluid element due to viscous shear is absent, the velocity component in any direction can be expressed as the corresponding space derivative of a velocity potential ϕ . This type of flow, generally known as potential flow, is described mathematically by the equation

$$\nabla^2 \phi = 0 \quad (1)$$

where ∇^2 represents the Laplacian operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

More generally the potential satisfies

$$\text{div } (k \text{ grad } \phi) = 0 \quad (2)$$

where k is the permeability coefficient of the flow field, and is taken as unity for fluid motion in free space.

Although water, the fluid which is of primary concern to hydraulic engineers, cannot be considered as inviscid, for many flow conditions where the thickness of the laminar or turbulent boundary layer due to viscous shear between the fluid and the flow boundary is thin as compared to the geometrical size of the flow boundary under investigation, it is generally safe to assume that the flow velocity at the outer edge of this boundary layer is essentially that due to potential flow, and the pressure on the flow boundary is the same as that outside the boundary layer [2]. This therefore forms the basis upon which potential flow theory can be applied to some problems which involve the study of pressure distribution along solid boundaries. Examples of fundamental importance in hydraulic

engineering are: designing boundary profiles that will avoid or minimize the destructive phenomena of cavitation, such as the investigation of boundary transitions for flow inlets and contractions; and testing profiles for head forms and hydrofoils. A second type of problem that can be treated as potential motion is the seepage type of flow in which the fluid acceleration is negligible so that the velocity potential ϕ can be considered as equivalent to the force potential; that is, ϕ is equal to pressure head plus elevation head

$$\phi = \left[\frac{P}{\gamma} + h \right]$$

The gradient of this potential times the permeability coefficient k of the soil bed is the flow velocity in the corresponding direction

$$U_n = -k \frac{\partial \phi}{\partial n} \quad (3)$$

For a number of potential flow problems which involve simple boundary values the exact mathematical solutions can be determined, but for most problems that are of practical interest the boundary values are too complicated for obtaining solutions in the orthodox manner. Consequently, the integration of Eq. (2) is generally carried out by the analog method because of the ease and simplicity with which useful solutions can be obtained.

PRINCIPLE OF ANALOG

The mathematical expression of Eq. (2) not only describes the potential field of fluid flow but also a number of physical fields such as the electric potential field, the magnetic potential field, and the thermal potential field in various transmitting media. Any of these fields may be used as an analog to study another through proper scaling of the physical constants. For example, the differential equation for an electric potential field is

$$\text{div } (g \text{ grad } E) = 0 \quad (4)$$

where E is the electric potential and g is the conductivity of the electrical field. If one lets

$$E = A\phi \text{ and } g = Bk$$

and substitutes them into Eq. (4), the resultant expression is identical to Eq. (2) for the fluid counterpart with A and B as the scaling constants. In most flow problems, the solutions sought can be expressed in dimensionless ratios so that an exact knowledge of these

scaling factors is unnecessary. For instance, the solution can be stated in the form of velocity ratio U/U_0 , where U is the fluid velocity at any point of the field and U_0 is the reference velocity, usually taken as the uniform flow velocity that exists in the flow field. If n is the distance normal to the equipotential surface, then

$$\frac{U}{U_0} = \frac{\frac{\partial \phi}{\partial n}}{\frac{\partial \phi}{\partial n} \Big|_0} = \frac{\frac{\partial E}{\partial n}}{\frac{\partial E}{\partial n} \Big|_0} \quad (5)$$

Hence the velocity ratio U/U_0 of the prototype system is simply the ratio of the electric potential gradients at the respective points in the model system.

The corresponding pressure p of the flow field may be obtained from the Bernoulli relationship

$$\frac{p - p_0}{\rho U_0^2 / 2} = 1 - \left[\frac{U}{U_0} \right]^2 = 1 - \left[\frac{\frac{\partial E}{\partial n}}{\frac{\partial E}{\partial n} \Big|_0} \right]^2 \quad (6)$$

where p_0 is the pressure at the point where the reference velocity is U_0 .

Unlike mathematical solutions which operate upon numerical values, the analog method works on physical quantities. Consequently, the accuracy of the solutions obtained depends on the precision with which these physical quantities can be set up and measured. In consideration of this fact, the electric potential analog is used almost exclusively because quantities such as the electrical potential, the electrical current and the electrical resistance can all be measured with great precision by simple instruments that are either available commercially or found in most laboratories. This paper will therefore be confined to the discussion of various electrical analogs which are designed to solve steady-state potential flow problems.

The difference in various electrical analogs lies primarily in the use of different media for the conducting field. Those most commonly used are the conducting paper, the electrolytic tank, and the resistance network. The application of each of these techniques will be discussed.

CONDUCTING PAPER AND ELECTROLYTIC TANK METHODS

Perhaps the oldest known potential analog is the work of Kirchhoff [3] who used thin copper sheet as the electrically conducting medium. His work was published in 1845, but only in recent years has his technique received increasing attention [4] partly due to intensive interest in the application of potential theory in many fields of engineering and partly due to the availability of more uniform conducting materials. Examples of the latter are the paper which is used in making tape resistors and the paper which is made for use in teleprinters. In fact laboratory units complete with accessories for solving simple two-dimensional flow fields are available commercially from the General Electric Company.

Since a paper conductor is essentially a two-dimensional medium, it can solve only flow problems of a similar nature; i.e., two-dimensional flows, which satisfy the equations

$$\nabla^2 \phi = 0 \text{ and } \nabla^2 \psi = 0 \quad (7)$$

where ψ is the stream function and ∇^2 is the Laplacian operator

$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Since the differential equations for ϕ and ψ are similar

they can be solved separately by the electrical potential analog. As an illustration, consider the solution of a two-dimensional flow-inlet problem shown in Fig. 1.

The dark line A-B-C-D-E-F-G represents the outline of the conducting paper; the sides A-B and F-G represent the two dimensional conduit; B-C the inlet transition; C-D is the wall of reservoir and E-F is the floor of reservoir or the centerline of a pipe heading from a semi-infinite reservoir. A-G represents an equipotential line which should be straight and normal to the velocity vector of the uniform flow in the conduit, while D-E represents an equipotential line at a distance R from the inlet. The latter line is an arc of a circle with radius R centering at F , if R is large compared with the inlet opening C-F. A conducting paint applied to these equipotential boundaries may serve as electrodes between which an ac voltage is impressed to set up the flow field. The four-dial decade potentiometer which is calibrated to read down to 1/10000 part of the total applied potential, the sensitive null detector such as the Ballantine Voltmeter with a sensitivity of .0001 volts and the probe P , as indicated in Fig. 1, are typical accessories required to trace the equipotential lines $\phi_0, \phi_1, \dots, \phi_m$ or to measure the potential at any position in the field. From these readings, velocity

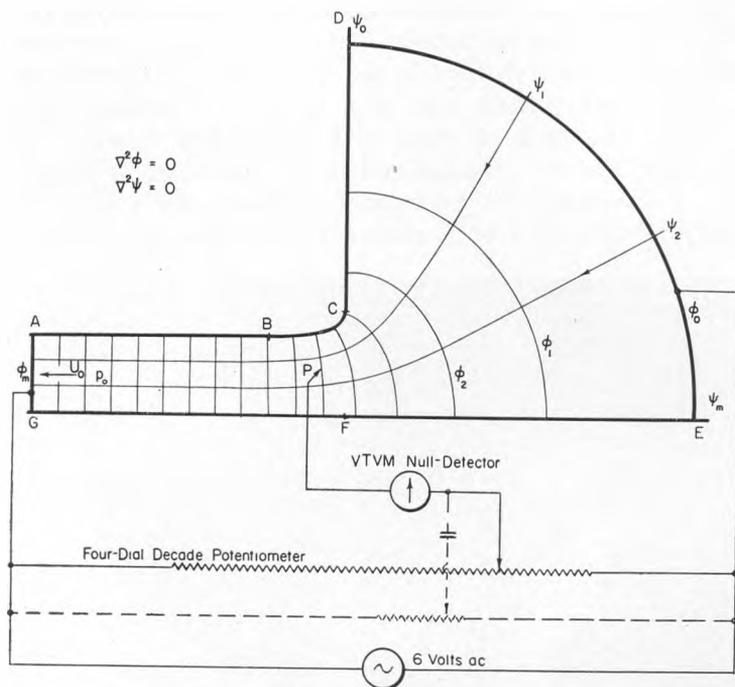


FIG. 1

ratios and pressure ratios of the corresponding fluid systems can be obtained through the use of Eqs. (5) and (6). From Eq. (7) it is further possible to trace the stream lines $\psi_1, \psi_2, \dots, \psi_m$ by impressing the ac potential along ψ_0 and ψ_m instead of along ϕ_0 and ϕ_m .

The main advantage of using the conducting paper technique is its simplicity and low cost. However, due to the fact that perfectly uniform conducting paper is still not readily available, and the accuracy of cutting out or setting the boundary profiles is somewhat dependent on human factors, the paper technique is not considered ideal for precision work.

The other type of electrical analog commonly used [5] is the so-called electrolytic tank method, in which an electrolytic solution serves as the conducting field. The solution is usually held in a glass or plastic tank whose shape and form are dictated by the boundary values of the flow problem. Figure 2 shows the picture of one such tank built for the study of inlet transitions from an infinite reservoir wall to a square or to a circular conduit. The tank for holding the electrolyte is made of 5/16" lucite plates glued together with chloroform. The transition boundary between the

reservoir wall and the conduit is made detachable so that different transition forms can be tested. This plastic tank is placed on a $\frac{3}{4}$ " plate glass which in turn is set on a rigid wooden frame work in a manner that permits the tank to lie in a precise horizontal plane. When the tank is filled with electrolyte it will represent the flow field of a two-dimensional inlet transition, or when tilted at a 15° angle from the horizontal position, the electrolyte will represent the flow field of a symmetrical sector of a circular-con-

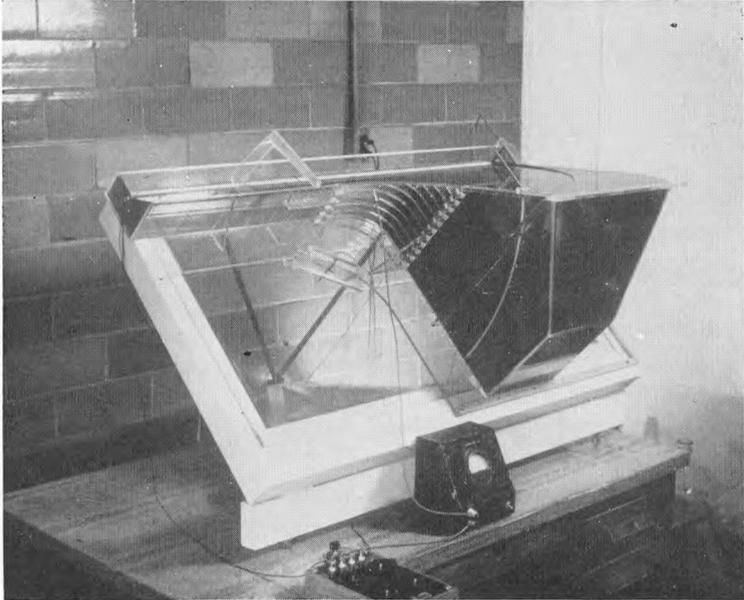


FIG. 2

duit and its inlet transition. With the tank tilted at 45° from the horizontal position, as shown in Fig. 2, the electrolyte represents the flow field which is a segment between planes of symmetry of a square conduit and its inlet transition. Also, varying the liquid level in the tank permits various contraction ratios (i.e. the size of inlet to the size of conduit) to be studied. Six volts ac at 60 cycles was impressed between one electrode at the left end of the tank, representing an equipotential plane of uniform flow in the conduit, and the other electrode made of wire mesh at the right end of the tank, representing part of an equipotential sphere some distance from the inlet opening. The resultant electrical potential distribution along the flow boundary is detected by electrodes made of No. 36 gage copper wires embedded normal to and flush with the boundary surfaces. These electrodes are placed at precisely

$\frac{1}{2}$ -inch center-to-center spacing in an orthogonal pattern, thus permitting the total velocity vector U_n at any point along the flow boundary to be evaluated; that is

$$U_n = \frac{\partial \phi}{\partial n} = \sqrt{\left(\frac{\Delta \phi}{\Delta x}\right)^2 + \left(\frac{\Delta \phi}{\Delta y}\right)^2} \quad (8)$$

where $\Delta \phi / \Delta x$ and $\Delta \phi / \Delta y$ are the potential gradients measured between the corresponding pairs of orthogonal electrodes.

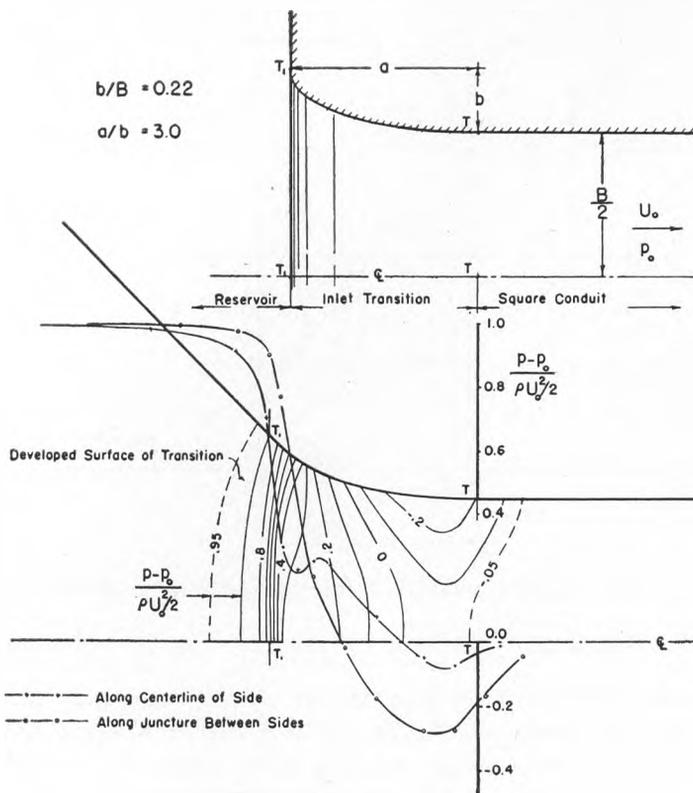


FIG. 3

A typical experimental result is shown in Fig. 3 in which the dimensionless pressure distribution along the boundary of a square conduit with a 3:1 elliptic transition curve is plotted in terms of dimensionless pressure parameter $\frac{P - P_0}{\rho U_0^2 / 2}$. It is interesting to note that the lowest pressure always occurs at the corners of the conduit; it is no coincidence that the cavitation erosion found in

prototype inlet structures of high dams was also located at the position predicted by the electric analog.

From the viewpoint of structural economy, the most efficient transition curve is of course one that gives the least negative pressure with a shortest length of transition, provided that a complicated curve does not lead to excessive form costs. The elliptic transition curves were chosen for the study because of their form and functional simplicity. The results obtained by electrical analog are shown in Figs. 4, 5 and 6 for two-dimensional,

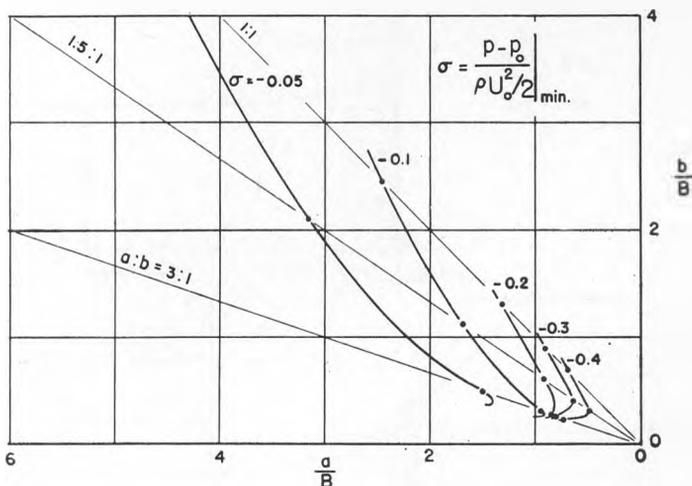


FIG. 4

circular-, and square-conduit inlet transitions respectively. The

loci of cavitation parameters $\sigma = \frac{p - p_0}{\rho U_0^2 / 2} \Big|_{\min.}$, or the minimum pres-

sure ratios that exist in a particular elliptic transition geometry, are plotted as functions of a/B and b/B ratios, where a and b are lengths of the semi-major and the semi-minor axes respectively of the ellipses, and B is the width or the diameter of the conduit. From these figures one can obtain the required elliptic form and size for a given cavitation parameter σ . Also one notes that a 3:1 to 4:1 elliptic transition is the most efficient form to use. Elliptic forms with slenderness ratio greater than 3:1 have not been investigated, but from the trend of the above figures one can see that for higher slenderness ratios longer transition curves would be required to provide for the same cavitation number σ .

The difficulties that arise from the use of an electrolytic solution as a conducting medium are many. Among the major problems

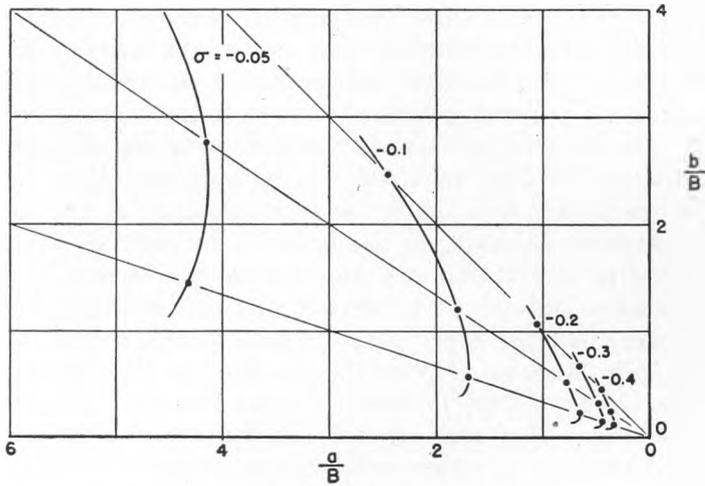


FIG. 5

encountered are 1) the necessity of a water-tight tank to hold the liquid, 2) the requirement of precision leveling of the tank in case the free-liquid surface is used as one of the flow boundaries, and 3) the presence of errors due to the polarization layer formed on the electrodes. However, with due care in designing the equipment, inaccuracies due to the first two problems can be effectively eliminated. The third problem can be taken care of by the use of concentrated copper-sulfate solution, in the order of 100 gr. of CuSO_4 -

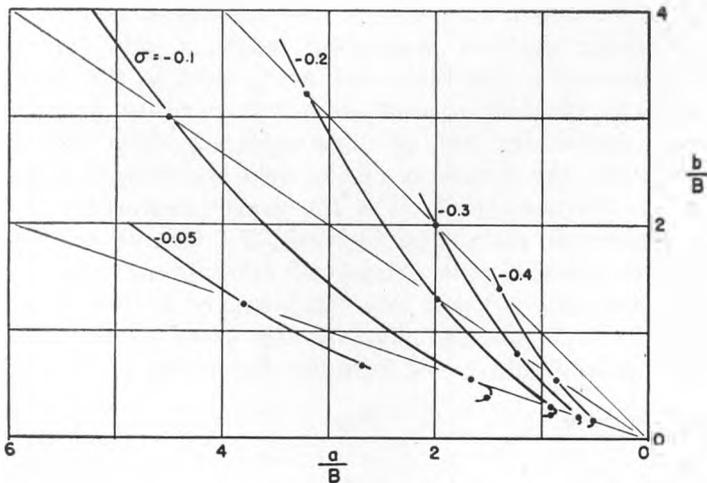


FIG. 6

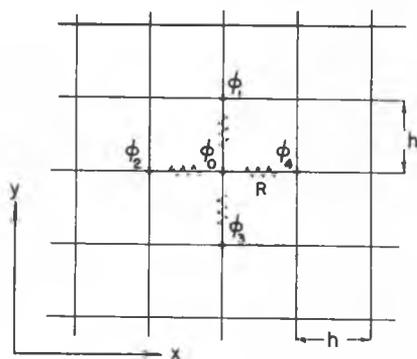
$\cdot 5\text{H}_2\text{O}$ per liter of water with few drops of concentrated sulfuric acid added to stabilize the solution. An electrolyte made in this proportion will not only keep the copper electrodes clean and increase the conductivity of the solution with resultant better impedance matching (or more sensitivity) to the measuring circuits, but, above all, it reduces the time required for the polarized film to reach a stabilized condition from hours or even days, as in the case of a weak electrolytic solution, to the order of a fraction of a second. The equivalent circuit for this film can be represented by an unwanted resistor plus an extra capacitor combined in parallel. This extra resistor not only takes time to reach an equilibrium value but causes a false increase of potential gradient in the vicinity of the electrodes. If ac voltage is used to excite the analog, the electrolytic condenser effect will cause a quadrature current to flow in the field. However, by using an impedance-bridge system as shown by the dotted line in Fig. 1, one can balance out the quadrature voltages due to this quadrature current and thus permit the desired potential distribution in the analog to be measured.

With proper care and selection of model scale the electrolytic tank method can be considered as a precision analog for the study of 3-dimensional flow problems. Accuracy better than ± 1.0 percent in the measurement of velocity ratios and ± 2.0 percent in the measurement of pressure ratios can be readily achieved.

RESISTANCE NETWORK COMPUTER

A still more versatile potential-flow analog, which may be considered as an analog computer, is the resistance network [6]. Instead of using uniform conducting media, a grid system of resistors connected in the form of a net is used as the field. This is equivalent to dividing a continuum into a finite grid system of flow paths, and if the sizes of these grids are made infinitely small compared with the geometry of the field one obtains essentially a continuum. Mathematically it is the exact counterpart of the well known numerical relaxation method [7] (better known as the Hardy Cross Method or the Southwell relaxation method) in which the field is usually divided into small square lattices, Fig. 7. The basic Eq. (7) of potential flow for the two-dimensional field can be written in finite difference form for this lattice as

$$\nabla^2 \phi \Big|_o = \frac{\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_o}{h^2} + \frac{h^2}{12} \left[\frac{d^4 \phi}{dx^4} + \frac{d^4 \phi}{dy^4} \right] = 0 \quad (9)$$



$$\nabla^2 \phi|_0 = \frac{\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_0}{h^2} + \frac{h^2}{12} \left(\frac{\delta^4 \phi}{\delta x^4} + \frac{\delta^4 \phi}{\delta y^4} \right) + \dots = 0$$

$$\nabla^2 \phi|_0 \doteq \frac{\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_0}{h^2} = 0$$

$$\frac{\phi_1 - \phi_0}{R} + \frac{\phi_2 - \phi_0}{R} + \frac{\phi_3 - \phi_0}{R} + \frac{\phi_4 - \phi_0}{R} = \frac{\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_0}{R} = 0$$

FIG. 7

neglecting the higher order terms one has

$$\nabla^2 \phi|_0 = \frac{\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_0}{h^2} = 0 \quad (10)$$

where h is the lattice size, and $\phi_0, \phi_1, \phi_2, \phi_3, \phi_4$ are the potentials at the node points 0, 1, 2, 3, 4 respectively. Through a tedious mathematical accounting system the ϕ function at each node point is relaxed step by step until it satisfies both the boundary values as well as Eq. (10), throughout the field. Suppose that the grids of Fig. 7 were to be replaced by resistors whose value R is made proportional to h . Then from the law of continuity the total electric currents flowing into the node 0 must be equal to zero or

$$\begin{aligned} \frac{\phi_1 - \phi_0}{R} + \frac{\phi_2 - \phi_0}{R} + \frac{\phi_3 - \phi_0}{R} + \frac{\phi_4 - \phi_0}{R} \\ = \frac{\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_0}{R} = 0 \end{aligned} \quad (11)$$

One notes the identity of Eq. (11) to Eq. (10). Furthermore the resistance network represented by Eq. (11) is self-consistent; that

is once the values of $\phi_1, \phi_2, \phi_3, \phi_4$ are given ϕ_0 will adjust itself to satisfy the equation. In the case of a complete network, the potential ϕ at each node point will adjust itself to the correct value once the boundary values are fixed. Consequently the solution obtained by the resistance network is equivalent to that obtained by the numerical integration, but without the tedious relaxation processes.

Neither the relaxation method nor the network analog is absolutely correct, because both Eqs. (10) and (11) omit the term

$$\frac{h^2}{12} \left[\frac{\partial^4 \phi}{\partial x^4} + \frac{\partial^4 \phi}{\partial y^4} \right] \text{ and other higher order terms of Eq. (9). How-}$$

ever, if due care is taken to provide for sufficiently small nets such that the potential distribution is nearly linear within each net, then the error due to neglecting these higher order terms will be insignificant.

Against this approximation of finite difference operation, the advantages gained by using the resistance network are many. First of all, the boundary geometry can be represented by simple resistance values, and consequently the human elements involved in the construction of models, as in the case of the paper analog and the electrolytic tank analog can be eliminated. This ease of setting up the flow boundaries permits the solution of many flow problems involving previously unknown flow boundaries that will satisfy certain preset flow conditions. The free flow surface of an overfall under gravitational action, profiles of constant velocity-inlet transitions, and the water-table surfaces of seepage flow are some of the potential-flow problems that can be ideally handled. The process involves successive cut and try steps which quickly converge to the correct solution [8, 9]. Second, the structure of the network permits any flow fields whose permeability k is either a constant or a function of space to be accurately set up; that is, the conductivity $g(x, y)$ of the resistance network can be distributed in accordance with the permeability $k(x, y)$. For the special case of 3-dimensional flow where the flow contains an axis of symmetry, a sector of such a field can be represented by a two-dimensional network whose conductivity $g(r)$ is made proportional to the radial distance r from the axis. Third, the accuracy of the resistance network analog is inherently high. By using precision resistors with a consistency of ± 0.5 percent, one can easily estimate the corresponding space distance of each unit net with the same order of accuracy. With the same precision provided in the measurement of potentials between each node of the network, one can obtain an

accuracy of ± 1.0 percent in the calculation of potential gradients. Consequently the solutions obtained by the resistance network are highly reproducible and free from human factors.

In order to set up a network with proper resistance values for various boundary forms and permeabilities of flow media, some rules or equations are required. The most complete and general equations known to the writer are due to Tschiasny [10] who derived a basic equation by replacing a triangular element of continuum with an equivalent resistance net. This elementary triangle is then the basis upon which a flow field with complicated boundary geometry can be constructed.

Figure 8 shows an elementary two-dimensional triangular continuum $A_1A_2A_3$ which can, in the sense of finite differences, be replaced by a triangular network of resistors with conductances g_1 , g_2 and g_3 .

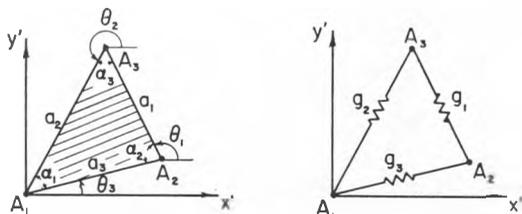


FIG. 8

Let $k_{x'x'}$ and $k_{y'y'}$ be the principal conductivities of the continuum, where x' and y' are the arbitrary Cartesian coordinates which coincide with the principal axes of the conductivity tensor. Assuming only that the continuum considered is so small that the potential distribution is linear within that region, a general expression for the conductance of the resistor element is

$$g_3 = \frac{1}{2} \left[\frac{k_{x'x'} + k_{y'y'}}{2} \cot a_3 + \frac{k_{x'x'} - k_{y'y'}}{2} \frac{\cos (\theta_1 + \theta_2)}{\sin a_3} \right] \tag{12}$$

The same equation applies to g_1 and g_2 after proper rotation of the indices.

One notes three important characteristics of Eq. (12): First, if the field is isotropic, that is $k_{x'x'} = k_{y'y'} = k$, then the equation reduces to

$$g_3 = \frac{k}{2} \cot a_3 \tag{13}$$

Second, if angle a is acute, right, and obtuse then the conductance of the resistor on the side opposite the angle is positive, zero and

negative respectively. In order to prevent negative conductances, obtuse angles should be avoided. Third, the conductance is independent of the absolute size of the triangle.

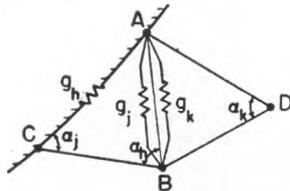
To combine these triangular elements into a complete flow field, one notes that there are always two conductances to be added up between two nodes unless both nodes are on the boundary.

$$k_{x'x'} \quad , \quad k_{y'y'}$$

$$g_s = \frac{1}{2} \left[\frac{k_{x'x'} + k_{y'y'}}{2} \cot \alpha_s + \frac{k_{x'x'} - k_{y'y'}}{2} \frac{\cos(\theta_1 + \theta_2)}{\sin \alpha_s} \right]$$

$$k_{x'x'} = k_{y'y'} = k$$

$$g = \frac{k}{2} \cot \alpha_s$$



$$g = g_j + g_k$$

$$R = \frac{1}{g}$$

FIG. 9

Figure 9 shows two adjacent elements with common nodes A and B from which it may be seen that the total conductance between A and B is $g = g_j + g_k$, or the total resistance value is $R = 1/g$.

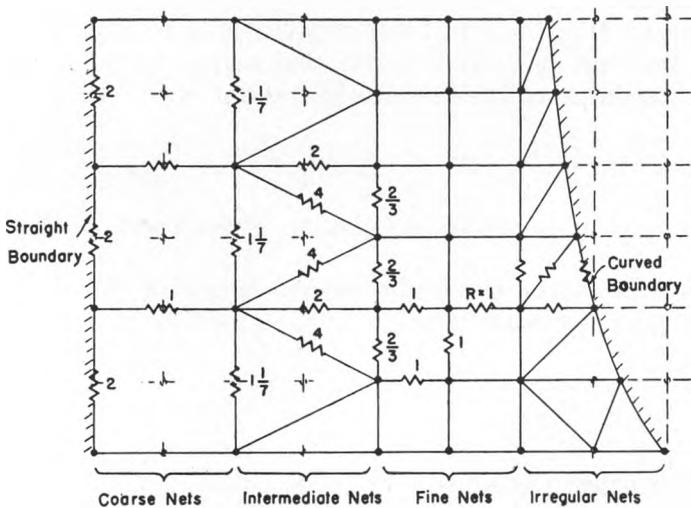


FIG. 10

It may be further shown that if the angles a_i and a_k are right angles, the conductance for the diagonal $A-B$ of a square element will be zero. Figure 10 depicts a typical square-net system representing a flow field. A coarse network may be joined to a finer one by an intermediate network, and the elements in the vicinity of a curved boundary may be represented by irregular nets. The fineness of network required, of course depends on the linearity of potential distribution within each element of net considered. As a rule, finer nets are required where the potential distribution is found to be nonlinear.

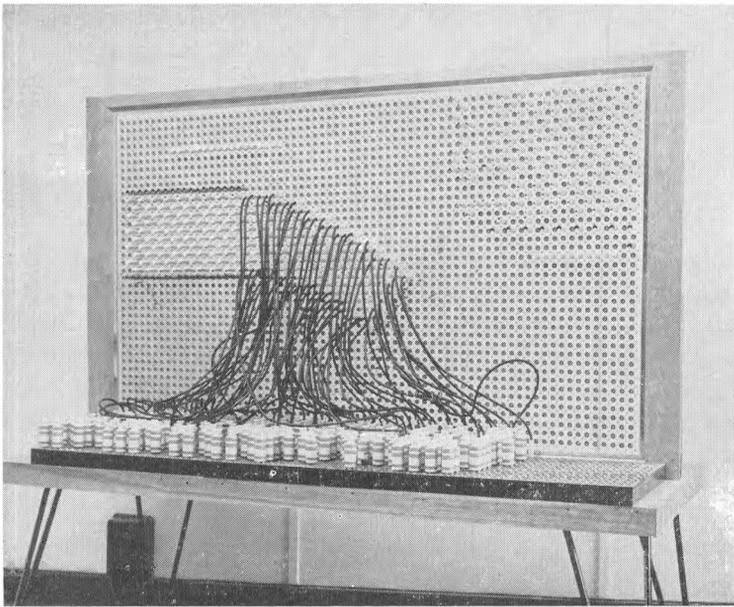


FIG. 11

A computer based on this network principle is shown in Fig. 11. The vertical panel contains a grid of sockets into which resistor elements may be plugged to form the flow field. The board is divided into 70 squares wide and 40 squares high which gives a total of 2,800 square nets. A four-pin socket is provided at the center of each net, and they are interconnected as shown in Fig. 12. A one-pin socket is also provided at each node point for the measurement of potentials or for the feeding in of boundary values. The panel is made of $\frac{1}{4}$ inch clear Lucite plate, so a to-scale graph can be hung on the opposite side of the panel to provide the operator with a visual reference to the flow field. For regular square elements,

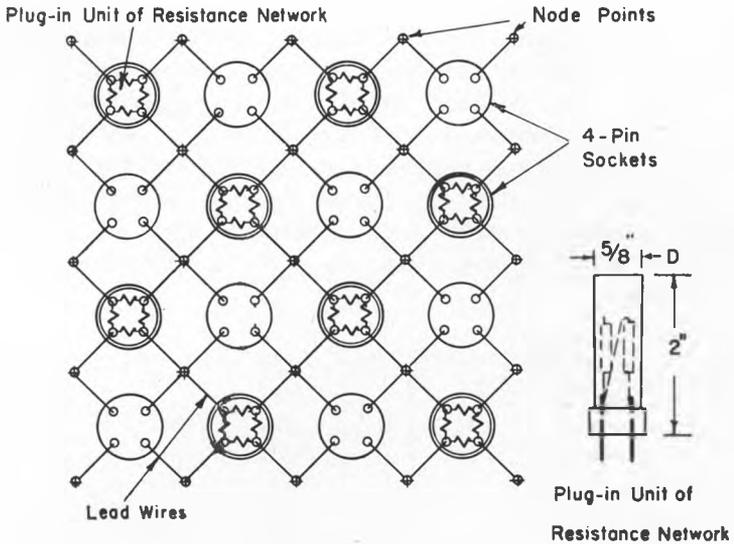


FIG. 12

four 1000-ohm precision resistors are mounted in a ring on a four-pin plug which forms the basic plug-in unit. These units are then plugged into alternate sockets on the panel, as shown in Fig. 12, to form a network of square nets. For irregular-net elements a special board is provided on the table of the computer (see Fig. 11). On this board 78 units of irregular nets can be set up. Each irregular net may consist of from one to five units of adjustable resistors which are special 10,000-ohms 3-dial decades made by the Telex Company. Each decade unit is only $1\frac{1}{2}$ inches diameter by 3 inches high, and is provided with a convenient plug-in base. The necessary number of these decade units is plugged into the board on the table to form a particular irregular net element, and the precise resistance values are then simply dialed out on the decades. A plug-in cable connects this irregular unit to the main network on the vertical panel.

This liberal use of plug-in units permits a maximum of flexibility in the setting up of a problem, as well as a maximum utilization of the costly precision resistors and decade elements. Future modifications can also be performed with a minimum of alterations because all major parts are separated into special units.

CONCLUSIONS

Three of the most commonly used analogs for the solution of potential-flow problems have been presented. Emphasis is placed upon the precision and on the methods of setting up the fields of these analogs. Much of the distrust of all forms of analogs in the past has been based on inability of providing an absolute check on the accuracy of results obtained. It is therefore a primary object of this paper to show that if proper care and attention are used in setting up the analog, and its tolerance and characteristics are understood, there is no reason why the results obtained cannot be reproducible and accepted with confidence.

The main characteristics of the three analogs are summarized in the following table:

Conducting Paper

Advantages

Simple and cheap; applicable to complicated boundary geometries, especially to cut and try solution for correct boundary forms.

Disadvantages

Limited to two-dimensional problems; accuracy is limited by the uniformity of conducting material and the human factors.

Electrolytic Tank

Capable of handling three-dimensional flow problems in a simple manner; reasonably accurate if handled properly.

Needs complicated watertight tanks; construction of precision boundary forms is time consuming; accuracy involves human factors.

Resistance Network

High accuracy, and free from human factors; extremely flexible in the setting up of boundary values and fields.

More expensive; finite difference manner of representing a continuum needs special attention.

A literature survey has revealed an intensive progress in the art of electrical analogs, and the published papers on this subject are numerous; most of these papers deal with very specialized problems in other fields of engineering. Also there are many other forms of analogs and applications which are considered beyond the scope of this paper and consequently have had to be omitted.

ACKNOWLEDGMENTS

The writer wishes to express his gratitude to Mr. Michel Hug for his many brilliant ideas and his work on the resistance network analog, and to the staff members of the Iowa Institute of Hydraulic Research, who have contributed to the improvements of these analogs. This work was supported by the Office of Naval Research under Contract N8onr-500, T. O. III., and the David Taylor Model Basin, U.S. Navy.

DISCUSSION

Mr. McNown initiated the discussion by emphasizing the need for the solutions of a great many potential-flow problems. He stated that the analogs probably offered considerably more power in the attack on these problems than theoretical methods which, in part at least, have been exhausted. The relaxation approach is numerical by nature but it offers serious limitations in the time and patience required. However, he warned anyone who wished to use the electrolytic tank first to take steps to have a man with the dexterity and skill of Mr. Ling to set up the models.

Mr. Murray then inquired if circulation can be superposed in the analog. Mr. Ling replied in the affirmative. For the case in point the flow would be two-dimensional for which the paper-conducting technique would be suitable. Circulation is superposed upon the field by modifying the values of the input potentials on the boundaries in accordance with a desired or trial circulation.

Baines pointed out that free-streamline problems are the ones for which potential flow solutions are most commonly needed in practical work, and that they are more difficult than problems with fixed boundaries to handle by relaxation, and asked which of the three techniques is best for this purpose. Mr. Ling thought that the paper technique is easier for approximate solutions, but that the network analog is superior because it also does not require the construction of special models and it is more precise.

Mr. Baines inquired further about application to problems involving gravitational effects. Mr. Ling asserted that this could also be done very simply by measuring the current flowing through the net, instead of the potential gradient, and determining the velocities along a trial boundary from the current readings.

An unidentified discussor commented that the network is not necessarily limited to the analysis of potential flows and inquired whether analogs have been used to study other problems. He also

called attention to the fact that M. Germain at the University of Brussels had made an electrolytic tank for two-dimensional problems, equipped with an automatic sensing probe and a pantograph system, which automatically plotted the field on a sheet of paper. Mr. Ling replied that there are many differential equations which can be solved by networks, as has been shown by Mr. Kron of G.E. who has contributed most to this art. By using network principles, time-dependent and other resistance elements can be used to solve various types of differential equations, such as for the fluttering of a wing. The network described here is a very simple type because it is designed to solve only one equation. In response to the second remark he stated that there are many papers describing automatic analog systems, but that these are of little interest in connection with the solution of hydraulics problems where interest is focussed upon the velocity and pressure distributions on solid boundaries rather than upon the flow characteristics in the entire field.

Mr. Bauer wished to know the order of magnitude of the time required to set up and solve a typical flow problem, such as for a two-dimensional inlet, or an analog computer. Mr. Ling indicated that problems with a definite boundary form could be solved in less than a day, but that flow problems for which a trial and error procedure is used would take more time; although he estimated the analog method to be 20 times faster than the relaxation process.

Mr. DeHaven inquired about the adaptability of the analog to 3-dimensional problems. Mr. Ling assured him that it was applicable, but that the set-up became much more complicated since many more boards are required. Nevertheless such systems have been built.

Mr. McPherson called attention to the fact that there is a paper analog manufactured by G.E. available on the market for about a hundred dollars, which includes a rectifier, a voltmeter, a pantograph, a mounting board, and about a five years' supply of paper. Mr. Cahuff contributed the additional information that the G.E. computer is now available from Sunshine Electric in Philadelphia.

Finally, Mr. McPherson asked whether, in order to duplicate seepage flow studies, it would be necessary to have resistances with a squared characteristic. Mr. Ling stated that that was not necessary, that in Darcy's law, the equation for the pressure drop, one could operate on the force potential instead of on the velocity potential. Mr. McPherson also mentioned that there are many

electric-power network analyzers throughout the country in all metropolitan areas which might be available for the solution of potential flow problems.

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