

## THE EXPERIMENTAL STUDY OF OSCILLATORY WAVES

by

Martin A. Mason  
Beach Erosion Board  
War Department  
Washington, D. C.

Those of us who occasionally read the Bible find therein many references to waves and storms, indicating that the ancients were familiar with those phenomena. We know also that the Arabian philosopher Avicenna [1]<sup>1</sup> (980-1037), whose work was apparently copied by Leonardo da Vinci [2] (1452-1519), wrote of ocean waves. Since this early beginning the observation of the unceasing movement of the sea has never lost its interest, and it was only a natural step from wonder at the spectacle to study and experimentation with a view to understanding the action. The problem at first glance appears simple, for some features of the movement of the sea may be easily studied even while resting in comfort on a beach; but, oddly enough, less is known of the visible motion of the ocean surface than is known of the invisible motions involved in light and sound. The answer to this paradox probably lies in the complexity of the phenomenon, making even its description a difficult matter. It will be recalled that Jeffreys, in attempting to describe the ocean surface, stated that, "the predominant characteristic of the ocean surface is its irregularity." One is also reminded of the precise but not very helpful axiom of the mathematician to the effect that any natural phenomenon may be expressed as a single equation with an infinite number of variables. This is interesting philosophy; but, in its practical application to ocean wave phenomena, investigators have so far progressed only to the extent of determining that the number of variables involved is unknown and that these unknown variables are probably all dependent.

This confused and, obviously, partially false statement seems, however, to be a reasonably exact expression of our present knowl-

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<sup>1</sup> References appear at the end of the article

edge regarding ocean waves in particular and oscillatory waves in general.

#### EXPERIMENTAL STUDIES

Classically, the experimental study of oscillatory waves has been divided into studies in the ocean, or rather in nature, and studies in the laboratory. This division is natural, since the problem consists of two parts: first, what is the nature of the phenomenon of ocean wave motion, and, second, what are the possible explanations of the phenomenon? With respect to the first, it may be seen that by reason of the large number of uncontrolled variables effective in nature, definition of the character of the ocean surface and its behavior is difficult and may have to be principally statistical. As to the second, the explanation of the natural phenomenon may best be formulated by laboratory studies in which the variables involved are isolated and their relation investigated step by step. Thus the genetic, or causal, features of the problem may be studied.

The first studies of ocean waves date back to Commander Charles Wilkes [3], who attempted to measure accurately the size of ocean waves during an expedition of six United States warships in 1838-39. It may be noted that Wilkes introduced the method of measuring wave height which consists in sighting from the rigging of a ship while the ship is in the trough between two adjacent crests, a method which is still used as a standard.

Following Wilkes, Scoresby [4] in 1850 and Lt. Paris [5] in 1871, continued field measurements in an effort to define the natural phenomenon, using methods similar to those employed by Wilkes, while Abereromby [6] in 1888 and Gerhard Schott [7] in 1893 measured ocean-wave heights employing a sensitive aneroid barometer as a measuring device. A long series of studies was made at Algiers in 1839 by the French investigator Aimé [8], who is believed to have been the first to attempt the observation of the internal movements involved in oscillatory wave motion in nature. Further study and observations of ocean waves were made by Gassenmayr [9] in 1896, Gaillard [10] in 1904, Laas [11] in 1905, Kohlschütter [12] in 1909, Cornish [13] over a long period dating from about 1910, Zimmerman [14] in 1920, and Count von Larisch-Moennich [15] and Meinesz [16] in 1925, and in modern times by Schumacher [17] on the research trip of the *Meteor*, Weinblum

[18] and Schnadel [19] on the trip of the *San Francisco*, and Pabst [20], O'Brien [21], Shepard and LaFond [22], the last three being American observers of shallow-water wave phenomena.

As this long list indicates, many experiments have been made at sea, but most of them have been limited to measurement of the wave velocity, the wave length, the wave height, and the wave period. Not until very recently has information on the shape of waves, or the depth to which the surface disturbance extends, been available, and not even today do we have information on the actual movements of water particles affected by wave motion.

Summarizing our present knowledge of ocean waves derived from actual observation, one cannot improve upon Thorade [23], who wrote in 1931:

No adequate results of observation are available with respect to \* \* \* form, orbital path, and energy, but they are sufficient to shake our confidence in theory. \* \* \* complete agreement between theory and observation is seldom found, and where it is found, it seems suspicious.

#### LABORATORY STUDIES

With respect to laboratory experimentation on oscillatory waves, the situation is somewhat brighter. The mathematical study of waves first received attention from Newton, who in his "Principia" gave an account of tidal motions and some few observations on oscillatory wave motion. Newton was followed in the 18th century by Laplace [24], Lagrange [25], Bernoulli [26], MacLaurin [27], and Euler [28], all of whom worked on both general and particular phases of the wave problem. It was not until 1802, however, that the first complete theory of oscillatory wave motion was developed by Franz von Gerstner [29], basing his study entirely on geometrical considerations suggested by actual observations of ocean waves.

A considerable impetus was thus given to the experimental study of waves by the publication of wave theory, in that a direction of experimentation was provided. The first large-scale laboratory study of wave motion was by the Weber brothers [30], who in 1825 studied waves in a tank, employing water, mercury, and brandy.

Later, de Caligny [31] from 1843 on, and Hagen [32] in 1861, experimented in shallow tanks in an effort to verify and develop

wave theory. They were the first to generate regular systems of waves, previous investigators having contented themselves with study of the disturbance created by a single impulse. Osborne Reynolds [33] reported on the investigation of the action of waves and currents on the beds and foreshores of estuaries by means of working models in 1890. Ahlborn [34] described experiments including a study of the internal movements in a wave in 1922, and was closely followed by Harold Jeffreys [35] who published the results of laboratory experiments and theoretical studies in 1924-25.

The list of investigators is short until about Jefferys' time, when the study of wave action began to be undertaken by hydraulic laboratories at the behest of engineers concerned with harbor construction, shore protection, and marine architecture, as well as oceanographers, geomorphologists, and meteorologists.

In the United States the agencies interested in oscillatory-wave research are the oceanographic institutions, a few of the hydraulic laboratories, and the laboratories of the Beach Erosion Board and Waterways Experiment Station in the Corps of Engineers. The direction of research in this field has been divided between basic research, investigation of generalized problems by model studies, and model studies seeking a solution of some specific problem. As a rule the first two types of study have been limited to two-dimensional studies of simplified problems, in which only a few variables are involved. From such studies it is hoped that a fund of knowledge regarding the interrelation of all the variables entering into the wave problem in nature may be accumulated, which will allow the ultimate solution of the three-dimensional problem in nature. The third type of study has not been successful in those cases in which wave action on a movable bottom is the predominant force. Perhaps twenty models have been built; but only those dealing with the modification of wave characteristics alone, such as the effect of breakwaters on wave heights, have given quantitative results.

Basic research on oscillatory wave motion has been directed entirely to the study of relationships predicted from theory but not verified by experiment or observation in nature. These studies concern the velocity and length of the wave in terms of the wave period, wave height, and water depth; the movement of the water

particles involved in wave motion; the profile of the wave; the wave energy and its dissipation by friction, by obstacles, or by other means; wave pressures, both internal and against obstruction; wave refraction; the movement of materials by waves; and the generation of waves by wind or moving objects, such as ships.

The investigation of generalized problems by model studies and the study of specific problems of wave action at given localities is as yet very limited and the information thus obtained meagre. A few successful studies may be mentioned, however, for example: some studies made at the University of California and described in Masters' theses, including investigation of the equilibrium slopes of sea beaches and verifications of oscillatory wave theories; the work on wave pressures of Larras [36] in France, Bagnold and White [37] in England, and the Waterways Experiment Station in the United States; the researches of Klauck and Hsu at the Massachusetts Institute of Technology, and Stucky and Bonnard at the University of Lausanne, dealing with the effect of sea walls and breakwaters on wave action; and the valuable work of Professor Thyjsse [38] at Delft on the effect of local wind on wave motion.

The experimental study of oscillatory waves has been curtailed considerably because of existing conditions, but some investigation is still in progress in university and governmental laboratories. Reference to the bulletin, "Current Hydraulic Laboratory Research in the United States," published January, 1942, by the National Bureau of Standards, shows that some eleven projects related to oscillatory-wave study are now in progress by the Tennessee Valley Authority, the U. S. Waterways Experiment Station, the Beach Erosion Board, and Lafayette College. In addition, two studies not listed in that bulletin are in progress at the Scripps Institution of Oceanography and the University of California.

Of the studies noted, three are devoted to fundamental research on wave theory or characteristics, two in the field and one in the laboratory; three concern models of specific localities; one is a generalized study of wave forces against breakwaters; three are concerned with the development of experimental equipment; one is a field study of ocean-wave characteristics; and the last two mentioned in the preceding paragraph are field studies of the effect of waves on beaches and beach material.

In general it may be said that laboratory experimentation has progressed almost to a point where further advance is contingent upon more complete information being made available concerning the natural wave phenomena. This information may be supplied by the answers to the following questions:

1. Is there a regular motion of the sea?
2. Does the surface consist of uniform wave trains—i.e., systems of waves of like characteristics?
3. Are ocean waves and wave trains permanent?
4. What is the form of the wind wave; the swell; and the breaking wave?
5. What are the laws of propagation for wind waves; swells; storm seas?
6. What are the characteristics of storm seas?
7. What are the maximum wave characteristics to be expected in a given locality?
8. Do the characteristics of waves in deep water, say greater than 100 fathoms, differ from those of waves in shallower depths?
9. What is the statistical distribution of wave length, height and period, in deep and in shallow water?
10. What is the motion of individual water particles in a wave?
11. Do oscillatory surface waves and translation waves co-exist in the shallower depth regions?
12. What is the mechanism of the movement of bottom and beach materials by surface waves?

#### APPARATUS AND TECHNIQUES

The apparatus, equipment, and techniques employed for the experimental study of oscillatory waves in the laboratory or in the field are relatively simple. The characteristics usually studied are the wave length, height, period, and velocity, the form of the wave profile, and the movement of the water particles affected by the wave motion. Field studies generally include also observations of the direction of travel of the waves, and of the wind characteristics in the locality. Development of apparatus and techniques has been slow, and the field of opportunity for further work in this respect is wide.

The laboratory apparatus for the study of oscillatory waves is relatively simple, consisting essentially of a tank fitted with a

wave generator, and certain special equipment to be described.

Waves are generated in so-called wave tanks, either of the nature of flumes a few feet wide and deep or of shallow basins of large area, by a mechanism which periodically displaces the water from the end or side of the tank. The frequency of displacement governs the period (and thus the length) of the wave, while the volume displaced governs the wave height. The initial shape of the wave is largely a function of the rate of displacement. The Beach Erosion Board, after extensive experimentation with plunger, flapper, and rotating eccentric-body displacement devices, has found that a quarter-circle scoop, operated through a double eccentric system best satisfies the requirements for the formation of a uniform series of waves wherein the water-particle displacements are in phase. This is a fundamental requirement for oscillatory-wave generation. It may be noted that scoop-generated waves reach a stable condition within two wave lengths' travel from the generator, while, for the same result, the much-employed plunger-generated waves require a minimum travel on the order of ten to fifteen wave lengths. This factor is important in determining the length of tank necessary for wave experimentation.

Present evidence indicates that the wave tank should have a minimum width of two feet, and preferably more, to reduce side-wall effects to a minimum. The sides and bottom of the tank should be as smooth as possible, since every irregularity in the solid boundaries affects the water surface.

The dissipation of the wave system in the tank is very important. No thoroughly satisfactory wave absorber is now available, but reasonably acceptable performance may be obtained from a permeable beach set parallel to the wave fronts and on a slope of about 25 degrees.

Wave heights are conveniently measured by a combination point and hook gauge, the point being set to the wave crest and the hook to the trough. However, in many instances it is desirable to obtain a record of wave height, in which case several types of instrument may be used. An electric wave profiler has been developed by the Beach Erosion Board for this purpose, consisting of a pair of wire electrodes connected in series to a recording oscillograph and a 60-cycle alternating-current power supply. A constant potential is

employed and the circuit characteristics are so adjusted that the variation of current flow recorded is practically proportional to the water-level variation. A somewhat similar device employing a contact strip connected to an arrangement of resistors has been developed by the Waterways Experiment Station. A third method employed with considerable success is the photography of the wave profile against a grid system, by which the instantaneous wave height and profile may be obtained. A fourth method sometimes used, but subject to considerable systematic error, is the recording float. Of these methods the photographic is probably the most accurate; the electrical methods are valuable if the wave form is not changing; and the recording float method is least desirable.

The determination of the wave form may also be accomplished by any of the four methods mentioned, with the same comments applicable.

Wave periods may be obtained from records of consecutive wave profiles, or more simply by noting the time required for a given number of waves to pass a point.

Wave lengths may be obtained from photographic records of a series of consecutive waves, or determined by a pair of point gauges, connected in a circuit through the water and a neon bulb. The points are set just to touch the wave crests, and the distance between the points adjusted to obtain synchronous flashing of the neon bulbs. The separation distance may be a single wave length or multiples thereof.

Wave velocities are most conveniently measured by timing wave travel over a known distance.

A successful method for the study of the movement of water particles involved in wave motion has been developed by the University of California and the Beach Erosion Board. Photographs, either single or multiple, of the movement of non-miscible injected drops against a grid background are obtained for complete wave cycles. By including a timing device, such as a seconds clock, in the photograph, particle velocities as well as paths of travel may be determined. The drops employed are a mixture of xylol and butyl-pthalate in the approximate proportion of 1 to 3, colored by the addition of zinc oxide, anthracene, or stamp-pad ink and adjusted to the density of the water at the time of observation.

Equally good results may be obtained from an olive oil-carbon tetrachloride-benzine mixture. Best results are obtained from finely ground aluminum particles about 0.2 millimeters in diameter; but this method requires a very intense light source, such as a cerium arc operated in the Beck-effect range. No adjustment of densities is required in this latter method, the aluminum particles undergoing a selective separation at the time of injection.

The experimental study of oscillatory waves in nature is much more difficult than in the laboratory, largely by reason of the magnitude of the phenomena and the lack of a suitable base of reference. Waves near the shore, where measurements are most easily made, may be as much as 10 feet in height and several hundred feet in length; waves in the open ocean are much larger and can be studied only from boats, which themselves are subject to the wave motion it is desired to investigate.

Since the days of Commander Wilkes in 1838-39, the development of methods employed to measure ocean wave heights and lengths has remained practically static. Wilkes was the first to devise a method for measuring the height of ocean waves which gave reasonably accurate results and was followed more or less closely by subsequent investigators. Wilkes' method [3] consists in viewing the crest of a wave adjacent to a ship from the rigging while the ship is in the trough between two adjacent crests, the observer ascending or descending as necessary in order to align the crest with the horizon. An allowance is sometimes made for the variation in position of the line of flotation of the ship, but in many cases such a correction cannot be made. When it is desired to measure wave heights close to shore, it has been the practice to drive graduated piles offshore in various depths of water and observe the intersection of the water level with such piles by means of a telescope or other viewing device. Obviously the accuracy of this method leaves something to be desired. Another method frequently used for inshore observations involves the use of a recording float mounted on a pier, pile, or other rigid structure. Such a method is generally satisfactory except for the fact that the depth of flotation of the float is not fixed, varying in position with the travel of the wave.

On the open ocean, wave lengths have been measured by com-

parison with the length of the boat on which the observer is stationed. A somewhat more accurate method consists in allowing a calibrated line, to which a float is attached, to pay out astern of the vessel until the float and the observer are in locations such that crests pass below them in synchronism. The number of waves between the observer and the float is noted, as well as the length of line paid out, and the wave length is thus obtained. Inshore observations on the length of waves in the ocean are conveniently made by means of a series of piles driven at known distances apart along the line representing the direction of travel of the wave. The position of the consecutive crests with respect to the spacing of the piles gives a measurement of the wave length. This scheme may also be employed for the case of waves approaching at an angle to the line of the piles, by simply correcting the observed length by the cosine of the angle between the direction of approach of the waves and the line of piles.

Wave periods in the open ocean are determined in the same manner as for inshore or laboratory observations, the simplest and most satisfactory method being to count the time required for a given number of waves to pass a specific point.

Observations of the wave profile in the open ocean are extremely difficult to obtain and up to the present time no completely successful method has been developed for this purpose. The stereophotogrammetric method developed by Laas near the turn of the century offers perhaps the most fruitful field for improvement in this respect. Laas' method [11] involves the use of two cameras taking stereoscopic photographs of the surface of the ocean from the rigging of a ship or some other vantage point above the surface of the sea. A contour map of the ocean surface from which the wave lengths, heights, and profiles or cross sections may be determined is constructed from the photographs by means of a stereocomparator. The accuracy of the method is somewhat limited by the accuracy of reproduction of the contour lines, and the value of the method is reduced because of the relatively small field which may be covered. This method has been used by several German investigators, notably on the *Meteor* and *San Francisco* expeditions. The results obtained were promising, but, because of the limited field of view covered, only one complete wave profile has been obtained although several hundred waves have been observed by this

method. On the *San Francisco* expedition comparisons were obtained between observations of waves made by the stereophotogrammetric method and by the Weiss method [19], which involves the use of a series of contacts placed along the ship's side, operating in much the same manner as the contact-strip wave-height recorder developed by the Waterways Experiment Station and already briefly described. It was found that the Weiss method, while giving a general picture of the state of the sea surface, was subject to considerable error by reason of the variation in the depth of flotation, rolling, and pitching of the ship and the effect of spray in giving fictitious readings.

Insofar as is known, the only observations of internal movements in oscillatory waves made in the open ocean, or rather in nature, were made by Aimé at Algiers in 1839. Aimé's experiments [8] have become classic, but the results he obtained are relatively of little value for the student of ocean waves, in view of the fact that measurements were made in a protected harbor and in the vicinity of an artificial breakwater where the wave motion was influenced to a large extent by the adjacent structure.

Certain advances are now in progress in the development of the equipment and techniques for use in measuring waves in the ocean. Several years ago, in experiments being conducted in the English Channel in the neighborhood of Dover, it was discovered accidentally that a high-frequency echo sounder could apparently be used to measure wave heights. This idea is now being developed in the laboratory of the Beach Erosion Board. The method involves the use of an echo sounder in reverse, that is, the transmitting and receiving units are mounted on a suitable support placed on the sea floor and the signals are projected to the water surface where they are reflected from the water-air interface. Several technical difficulties have been overcome and it is believed that a satisfactory method of employing the echo sounder has been developed.

#### FUTURE EXPERIMENTATION

As was previously noted, our knowledge of oscillatory waves is by no means complete. The progress which has been made to date, while small, has been in general in the right direction. While under present circumstances it is not believed to be reasonable to expect any immediate appreciable advance in the science, it may, however,

be worth while to consider what direction future experimentation should take when it becomes feasible.

It is the opinion of the author that future experimentation should be devoted principally to the development of fundamental knowledge of oscillatory wave motion in the sea. This is the problem which presents itself in nature, not the theoretical problems which have been reported upon by the mathematicians largely responsible for the present-day development of our knowledge of wave motion. We find ourselves, after several thousand years' acquaintance with the problem, in the indefensible position of attempting to study in the laboratory a natural phenomenon which we have not yet defined. It would seem only reasonable, therefore, that our efforts in future experimentation should be devoted principally to a definition of the natural phenomenon, be it on a statistical, causal, or other basis. Once such a definition is available, we may then proceed with our mathematical analyses, our laboratory studies, and our hypotheses concerning the interrelation of the variables involved in the natural phenomenon.

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