GAS-WAVE ANALOGIES IN OPEN-CHANNEL FLOW

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In connection with power, irrigation, and flood-control projects, numerous hydraulic structures have been erected in recent years, in which rather high velocities of flow were employed for the efficient handling of large discharges in channels with a free surface. In many instances, the first design of such structures was approached with such concepts of flow as had been formed through contact with flow phenomena in natural streams and in artificial channels of moderate slopes. Consequently, the performance of these high-velocity structures very often did not come up to expectations. Sometimes the chief reasons were surprising standing-wave patterns, which affected the flow in curved sections of steep flood channels, in channel contractions and expansions, in log chutes, storm-drain intakes, and other types of hydraulic structures. The elimination of the undesirable features of these structures was often achieved by the experimental method, applied in most cases rather hurriedly and resulting in empirical corrections without general benefit to the knowledge of high-velocity flow in open channels. Some research projects in this field, however, were carried out in such a way as to permit quite a detailed analysis of the nature of this so-called "shooting" or "rapid" flow in certain cases. It is the purpose of this paper to summarize the general results of these theoretical and experimental investigations and to bring them to wider attention among hydraulic engineers as useful tools in the design of high-velocity structures.

Analogy Between Supersonic Flow of Gases and Supercritical Flow of Water

Since much of the progress in the analysis of high-velocity flow of water was due to the existence of extensively-developed solutions

in the supersonic flow of gases, it is only fitting to start out with a brief summary of this analogy. The fundamental relations common to both flows are included here, while readers interested in a more extensive discussion of this problem must be referred to the papers listed in the bibliography. It is believed that the existing theory of high-velocity gas dynamics has been exploited to the limit by the hydraulic engineers who worked in this field in recent years. The types of problems encountered in both fields are quite different, and practical applications of the fundamental relations in either field will therefore follow their individual courses more and more. The mathematical analogy was discussed and treated by Jouguet, Riabouchinsky [3],[1] Prandtl, and von Kármán [7]. The theory was extended and confirmed experimentally for certain cases in hydraulics by Preiswerk [11], who will be referred to frequently on account of the extent and detail of his work, by R. T. Knapp and the author in experimental studies at the California Institute of Technology [4, 5, 6, 8], and by the latter at Lehigh University.

That physical relationships must exist between high-velocity gas and liquid flow is immediately recognized from pictures, showing, for example, the pressure-wave pattern created by a bullet in air and the wave disturbances of a similarly-shaped bridge pier in water. The difference in the two cases is only apparent from the fact that the shock waves originating at the sharp tip of the bullet are visible as thin lines typical of shock waves in gases, while shock waves in water, better known as standing wave fronts or hydraulic jumps, extend over a larger width.

The basic features of the analogy are obtained from the continuity, energy, and momentum relationships as commonly applied in hydrodynamics. The following assumptions are made: the flow in either medium is considered two-dimensional in the xy plane, the flow is steady and frictionless, and the gas-flow is considered adiabatic—i.e., no heat is added to or taken from the fluid. The energy equation then reads:

$$h + \frac{V^2}{2g} = H_0 = \text{const.} \tag{1a}$$

wherein V is the velocity of flow at any point and h or H_0

[1] References appear at the end of the article.

represents enthalpy or total heat content in foot-pounds per pound. The difference in enthalpy $(H_0 - h)$ can be expressed in accordance with the assumption of adiabatic conditions in terms of H_0 by the following expression:

$$\frac{V^2}{2gH_0} = \frac{H_0 - h}{H_0} = \frac{T_0 - T}{T_0} \tag{1b}$$

with T representing total absolute temperature. Considering the flow of water with a free surface on a horizontal xy plane and assuming hydrostatic pressure distribution and no friction losses, Bernoulli's theorem is written in exactly the same form as Eqs. (1a) and (1b). The term H_0 becomes the total head H_0 with respect to the bottom, commonly known as "specific energy;" h is simply depth of flow.

Since the enthalpy for gases under the conditions stated depends only on certain constants and the temperature T, the conclusion from Eq. (1b) is that

$$\left(\frac{h}{H_0}\right)_{\text{water}} = \left(\frac{T}{T_0}\right)_{\text{gas}}$$

The continuity equations are written for gas and for water in the two-dimensional case as follows:

$$\text{For gas:}\quad \frac{\partial(\rho V_x)}{\partial x} + \frac{\partial(\rho V_y)}{\partial y} = 0 \tag{2}$$

$$\text{For water:}\quad \frac{\partial(hV_x)}{\partial x} + \frac{\partial(hV_y)}{\partial y} = 0 \tag{3}$$

and result in the analogy stated in Eq. (4):

$$\left(\frac{\rho}{\rho_0}\right)_{\text{gas}} = \left(\frac{T}{T_0}\right)^{1/k-1} = \left(\frac{h}{H_0}\right)_{\text{water}} \tag{4}$$

The ratio of the densities, ρ/ρ_0, is found to correspond to the ratio of the water depths. Since the latter is also analogous to the ratio of temperatures in gas flow, as the result of Eq. (1b), a perfect correspondence of the related quantities is only possible for a hypothetical gas with $k = c_p/c_v = 2$. However, many flow phenomena do not depend very strongly on the value of k, so that

actual gas flow (for air, $k = 1.4$) shows satisfactory agreement with the corresponding flow of water in numerous problems.

The pressure p in gas flow is not proportional to the depth h as one might be inclined to expect. In accordance with the thermodynamic principle $p/p_0 = (\rho/\rho_0)^k$, and using $k = 2$ as derived before, the comparison gives

$$\left(\frac{p}{p_0}\right)_{\text{gas}} = \left(\frac{\rho}{\rho_0}\right)^k_{\text{gas}} = \left(\frac{h}{H_0}\right)^2_{\text{water}} \tag{5}$$

This latter analogy may be given a physical meaning, however, by relating total hydrostatic pressure $P = wh^2/2$ to the pressure intensity in gas flow. This results in

$$\left(\frac{p}{p_0}\right)_{\text{gas}} = \left(\frac{P}{P_0}\right)_{\text{water}}$$

It remains to relate the velocity of sound or of pressure disturbances in gases to the velocity of equivalent disturbances in water flowing in a horizontal channel (the channel may also be inclined so as to compensate for friction forces by a gravity component). The velocity of sound in gases is given by $a = \sqrt{kp/\rho}$, and the ratio of a to the velocity of sound a_0 for an initial state H_0 would be given by

$$\frac{a}{a_0} = \sqrt{\frac{p}{p_0} \cdot \frac{\rho_0}{\rho}} \tag{6}$$

Using the analogies found so far from Eqs. (4) and (5), the equivalent ratio for water flow is $\sqrt{h/H_0}$, or, written in the final form,

$$\frac{a}{a_0} = \sqrt{\frac{gh}{gH_0}} \tag{7}$$

The quantity $\sqrt{gh}$ is recognized as the speed of propagation of small waves or surges in shallow water, which is also often called the "celerity" (C). Therefore, the quantity analogous to the velocity of sound in gases is the wave velocity in open channels, provided the height of the wave front remains small as compared to the depth of flow.

In gas dynamics as well as in hydraulics, use is made of the ratio of the velocity of flow to the velocity of a pressure disturbance to indicate the state and character of flow. The ratio of the velocity of gas flow to the velocity of sound, V/a, is commonly known as the

Mach number, which has its equivalent in hydraulics in the Froude number $V/\sqrt{gh} = V/C$. Equal values of these numbers therefore indicate comparable conditions of flow. The following table presents a summary of the elements of the gas-water flow analogy as discussed in these pages.

THE STANDING WAVE IN HYDRAULICS

The hydraulic engineer encounters quite frequently in his practical problems a sudden change in water-surface elevation, which is known as a standing wave; if it is at right angles to the direction of flow it becomes the familiar hydraulic jump. The latter has been thoroughly treated and its theory and experimental confirmation can easily be learned from most standard texts. The following treatment will take up the slanting hydraulic jump or the standing wave of the shock type; the word "shock" is borrowed from the terminology in gas dynamics and indicates a rather sudden change of flow conditions [1, 11, 13].

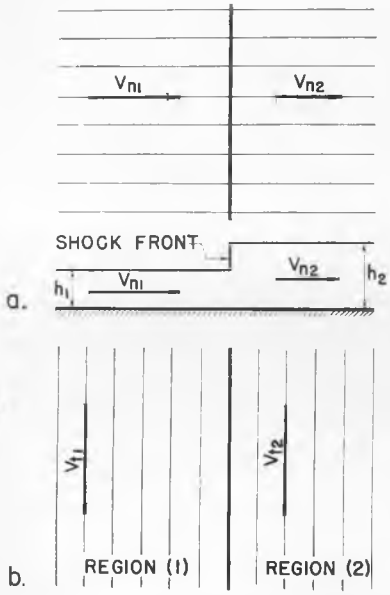

FIG. 1.—STANDING JUMP AT RIGHT ANGLES TO FLOW.

Let Fig. 1a be the plan and elevation of such a standing jump at right angles to the flow and let the jump front be represented by a single line. The velocity of approach, for reasons apparent later, may be called V_{n1} and the exit velocity V_{n2}; the corresponding depths are h_1 and h_2. The streamlines are normal to the jump front.

In Fig. (1b) below, a second field of flow is shown—or, rather, two fields of flow—with streamlines parallel to the vertical line, indicating a change in depth from h_1 to h_2 but with velocities V_{t1} and V_{t2} equal and parallel to the front. Since there is no flow across the front,

TABLE I

SUMMARY OF GAS-WATER FLOW ANALOGY

Analogous Quantities	Gas Flow	Liquid Flow
Flow Conditions	Adiabatic Process	No Friction (or slope to compensate for friction loss)
	Enthalpy—Total Head H_0	Total Head H_0
Properties of Fluid	Hypothetical Compressible	Incompressible
	(Gas with $k = \frac{c_p}{c_v} = 2$)	(Water)
Boundaries of Channel	Sides Geometrically Similar to Sides of Water Flow	Sides Vertical
	Bottom Parallel to Top	Bottom so that H_0 Remains Constant Top—Free Surface
Wave Velocity	Acoustic Velocity $a = \sqrt{k\ p/\rho}$	Wave Celerity $C = \sqrt{gh}$
Shock Wave (Normal and at Angle)	Shock Wave (Compression Wave)	Standing Wave (Hydraulic Jump)
Flow Velocity	Velocity V Components V_x, V_y	Velocity V Components V_x, V_y
Characteristic Ratios	Acoustic-Velocity Ratio: Mach Number $\frac{V}{a} = \mathbf{M}$	Wave-Celerity Ratio: Froude Number $\frac{V}{C} = \mathbf{F}$
	Density Ratio $\frac{\rho}{\rho_0}$	Depth Ratio $\frac{h}{H_0}$
	Temperature Ratio $\frac{T}{T_0}$	Depth Ratio $\frac{h}{H_0}$
	Pressure Ratio $\frac{p}{p_0}$	Total-Pressure Ratio $\frac{P}{P_0} = \frac{h^2}{H_0^2}$

it may best be thought of as a definite boundary such as a thin plate separating the two streams.

In Fig. 2 the flows of (1a) and (1b) are superposed; the definition of the original quantities should be well remembered. The velocities V_1 and V_2 obtained by this superposition are composed of components normal and parallel to the wave front, and the place of the boundary in Fig. 1b is now taken by the jump front separating the regions (1) and (2). The normal component V_{n1} by definition is also the velocity of the jump or wave front, so that V_{n1}/V_1 is the sine of the angle β, properly termed the wave angle. The fact that V_{n1} changes to V_{n2} for flow across the jump line, while V_{t1} and V_{t2} remain the same, means, in effect, a change in direction by an angle θ for the velocity of flow from V_1 to V_2.

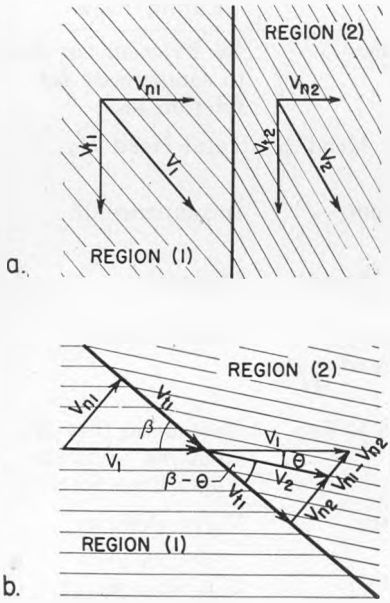

FIG. 2.—FLOWS OF FIG. 1 SUPERPOSED.

The application of continuity and momentum relationships, combined with the geometry of the velocity vectors, results in Eqs. (8) to (14). Thus,

$$h_1 V_{n1} = h_2 V_{n2} \tag{8a}$$

$$h_1 V_1 \sin \beta_1 = h_2 V_2 \sin (\beta_1 - \theta) \tag{8b}$$

$$\frac{h_2^2 - h_1^2}{2h_1} = \frac{V_{n1}(V_{n1} - V_{n2})}{g} \tag{9}$$

From Eq. (9), with the aid of Eq. (8a), the jump celerity V_{n1} can be expressed solely in terms of h_1 and h_2:

$$V_{n1} = V_1 \sin \beta_1 = C = \sqrt{g h_1 \frac{1}{2} \frac{h_2}{h_1} \left(1 + \frac{h_2}{h_1}\right)} \tag{10}$$

The celerity becomes equal to the critical velocity only if the ratio h_2/h_1 approaches unity. Replacing $V_1/\sqrt{gh_1}$ by the familiar Froude number $\mathbf{F}_1$, Eq. (10) may also be written as

$$\sin \beta_1 = \frac{1}{\mathbf{F}_1} \sqrt{\frac{1}{2} \frac{h_1}{h_2} \left(1 + \frac{h_2}{h_1} \right)} \tag{11}$$

For a given relative height of jump the angle β_1 will depend only on the value of $\mathbf{F}_1$—i.e., on the depth and velocity of approach. The minimum value of sin β for any given $\mathbf{F}$ can only be sin $\beta = 1/\mathbf{F}$, since h_2 would have to become smaller than h_1 to reduce sin β below this value. A hydraulic jump is always accompanied by a certain energy loss ΔH, however small, which is given by Eq. (12) in terms of initial depth h_1:

$$\frac{\Delta H}{h_1} = \frac{(h_2/h_1 - 1)^3}{4\ h_2/h_1} \tag{12}$$

A hydraulic drop of the shock type is physically impossible under the assumptions made. A sudden surface drop such as may occur at a corner or a sharp bend or curve might be considered as a negative shock wave; however, here the assumption of hydrostatic pressure distribution over the depth will no longer be valid. Such cases were treated by Warren E. Wilson [12] at the Institute of Hydraulic Research in Iowa City by experiments in which the pressure distribution along walls of rapidly expanding channels was measured. Such cases, however, must remain beyond the scope of this paper.

Returning to Eq. (12), it is easily seen that for all ratios h_2/h_1 smaller than 2, the energy loss is only a small fraction of the initial depth h_1, which in itself is small compared to the total head or energy in most supercritical flows of a practical nature. We may therefore treat shock waves even as high as the initial depth as jumps without energy dissipation. This is borne out by the physical appearance of such waves, which do not show the extremely turbulent surface-roller characteristic of higher jumps.

Eq. (11) should be discussed further to bring out two particular points. First, the maximum value of β cannot exceed 90°; if it approaches that value, the jump will detach itself from the source of disturbance and travel upstream, the limiting values of $\mathbf{F}$ and

h_2/h_1 easily being found from the equation for this condition. The second point to note is the fact that for the slanting jump there is no restriction placed on the values of h_2 with respect to V_2. Depending upon the relative magnitudes of V_t and V_{n2}, $\sqrt{gh_2}$ may be smaller or larger than V_2—i.e., the flow for a slanting jump may be either tranquil or rapid, streaming or shooting. The critical flow, where $V_2 = \sqrt{gh_2}$, can be obtained from the equations by substituting this condition.

Missing so far is an expression for the angle of deflection θ. The geometry of the vector diagram and Eq. (8b) result in

$$\frac{h_1}{h_2} \tan \beta = \tan (\beta - \theta)$$

so that finally

$$\tan \theta = \frac{(1 - h_1/h_2) \tan \beta}{1 + h_1/h_2 \tan^2 \beta} \tag{13}$$

This last equation may be simplified considerably if it is applied to values of h_1/h_2 and β occurring in practice. If h_1/h_2 is assumed to vary only between $2/3$ and $1/3$ and if β ranges between values of $10°$ and $35°$, only a small error results from replacing $\tan \beta / (1 + h_1 / h_2 \tan^2 \beta)$ simply by $\sin \beta$. Eq. (14) is the result:

$$\tan \theta = \left(1 - \frac{h_1}{h_2}\right) \sin \beta \tag{14}$$

This equation can be combined with Eq. (11) to give the deflection θ in terms of **F** and h_1/h_2 alone.

The equations are all written in the given form with a view toward practical application. In designing a channel contraction for super-critical flow, for example, one will probably start out with definite assumptions of **F** and permissible ratios h_1/h_2; the angles β and θ are then easily obtained from the equations.

WAVE FRONTS OF SMALL HEIGHT—ASSUMPTION OF CONSTANT ENERGY

All previous equations are valid for the assumptions made, regardless of the energy dissipation, ΔH. If for the present the case of zero energy dissipation is considered, these equations can be greatly simplified. Zero energy dissipation requires a ratio h_2/h_1 approaching unity, which obviously is only possible for a very small deflec-

tion angle $d\theta$, while sin β approaches the limiting value 1/**F**. Eq. (13) then results in the differential expression

$$d\theta = \frac{dh}{\frac{V^2}{g} \tan \beta} \tag{15}$$

The small angular change $d\theta$ may be negative or positive, causing a decrease or increase of the depth along a disturbance line crossing the flow at an angle $\beta = \sin^{-1} 1/\mathbf{F}$. Such disturbance lines may be drawn along a curved wall, as shown in Fig. 3, approximating the curvature by a series of short tangents or chords for definite small angles—say $d\theta = 2°$ or even 4°. Fig. 3 shows these disturbance fronts and their effect on the stream lines. Positive disturbances converge, whereas negative disturbances diverge. Velocities will increase for flow passing under negative disturbance lines, and the deflection $d\theta$ will be away from the wave front; for positive disturbances the opposite will take place. The convergence of positive wave fronts must ultimately cause a finite jump and lead to the breakdown of the assumption of constant energy made in this chapter, if the lengths involved are sufficient. For negative disturbance lines no such danger exists and no energy dissipation is possible.

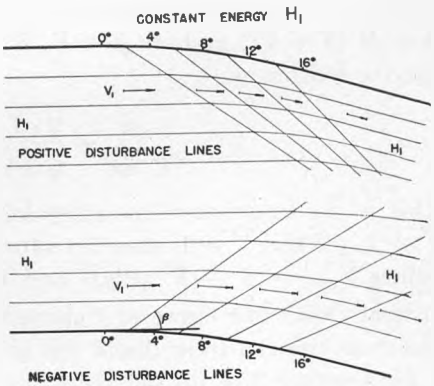

FIG. 3.—FLOW ALONG CURVED WALLS.

Eq. (15) for $d\theta$ may be integrated along a curved wall for certain basic assumptions. The resulting equations have been given in various publications [5, 6, 7, 8] and need not be stated here. The two assumptions made are:

1. Constant energy throughout the fluid field in the xy plane.
2. Constant velocity throughout the xy plane.

At this point however, the attention of the practical engineer may be called to a graphical solution for the basic equation, which

was first developed by Busemann for applications in gas dynamics [1, 2] and which, at the suggestion of Professor von Kármán, the writer in 1935-1936 adapted to the hydraulics problem of flow in curves. Preiswerk has since applied it to flow in channel expansions and to other problems described in detail in his paper [9, 11].

THE "METHOD OF CHARACTERISTICS" FOR SUPERCRITICAL FLOW WITH CONSTANT ENERGY

The basis for the graphical method known as the "method of characteristics" in gas dynamics is given by Eq. (15). This equation can be transformed into a general relationship for a dimensionless velocity defined as $\overline{V} = V/\sqrt{2gH}$, wherein the total head $H = h + V^2/2g$ in accordance with the former notation. Since then $h = H\,(1 - \overline{V}^2)$ and $\tan\beta = V_n/V_t = \sqrt{gh}/\sqrt{V^2 - gh}$, Eq. (15) may be written as

$$\frac{1}{\overline{V}}\frac{d\overline{V}}{d\theta} = \frac{\sqrt{1 - \overline{V}^2}}{\sqrt{3\,\overline{V}^2 - 1}} \tag{16}$$

This is the equation of an epicycloid between the circles of radii $\overline{V} = 1/\sqrt{3}$ and $\overline{V} = 1$, since for supercritical flow $\overline{V}$ has as limiting values $\overline{V}_{max} = 1 = \overline{V}\,\sqrt{2gH}$ and $\overline{V}_{min} = \sqrt{H/3H} = 1/\sqrt{3}$ for critical flow. The curve or "characteristic" plotted for the above equation between these limits will give all possible values of $\overline{V}$ and θ for shooting flow for the assumption of constant energy and for a wave velocity $V_n = \sqrt{gh}$. In agreement with these conditions the angular intervals $\Delta\theta$ have to be small—say 2° or 4°.

In Fig. 4 such characteristic curves are drawn at angular intervals of 4° to correspond to angular changes of streamlines along curved walls or under wave fronts of equal influence. Curves are drawn upward and downward, giving positive and negative direction changes with respect to any initial direction. Moving outward along any curve will give increasing values of $\overline{V}$ corresponding to the crossing of negative wave fronts, and moving inward will result in decreasing values of $\overline{V}$ due to positive wave fronts. In order to obtain not only the changes of the velocities but also the directions of the wave fronts or boundaries along which these changes take place, it remains to construct an additional curve. This curve must have the property of giving the magnitudes and the directions of $\overline{V}_n$ and $\overline{V}_t$ for all possible values of $\overline{V}$ and can be

shown to be an ellipse with the major axis $\overline{V}_{max} = 1$ and the minor axis $\overline{V}_{min} = 1/\sqrt{3}$. The following development gives the equation for $\overline{V}_n$ and $\overline{V}_t$:

$$V_n^2 = gh = gH - \frac{V^2}{2}$$

$$V_t^2 = V^2 - V_n^2 = V^2 - gH + \frac{V^2}{2}$$

On eliminating V^2 from these equations, there is obtained the general equation

$$3\,V_n^2 + V_t^2 = 2gH$$

or

$$\frac{\overline{V}_n^2}{1/3} + \frac{\overline{V}_t^2}{1} = 1 \tag{17}$$

This ellipse, for purposes of graphical construction, is best engraved on a piece of celluloid which can be rotated freely over the

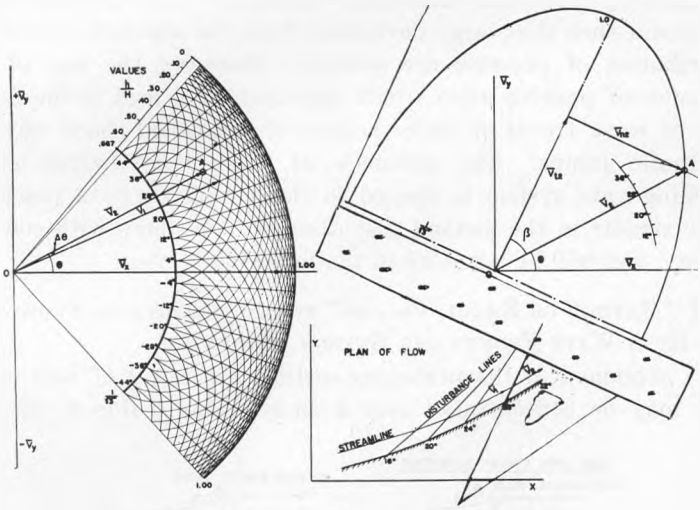

FIG. 4.—CHARACTERISTICS DIAGRAM.

characteristics diagram. The two diagrams are shown separately in Fig. 4 only to avoid confusion. The example showing the principle of application considers the flow along a wall curved toward the stream in the xy plane. For a given value of $\overline{V}_1$ at an angle θ_1 to the x axis the epicycloid is found in the characteristics diagram. A further change of angle $\triangle\theta = 4^\circ$ along the wall will

cause $\overline{V}_1$ to decrease to a value $\overline{V}_2$ along this epicycloid. The ellipse can then be rotated until it intersects this epicycloid at A. The direction of the major axis is the direction of $\overline{V}_t$ and therefore of the wave front separating zone (1) from zone (2) in the plan. The value of V_2 determines the ratio h_2/H, which can be obtained on the diagram directly from the scale indicated.

It is possible to build up by this method a relatively complicated pattern of wave fronts of positive and negative sign, which in the end permits the construction of a contour map of the entire fluid surface. The method described here only in principle and in its barest essentials can be adapted to almost automatic use by certain technical and graphical procedures, which are not within the scope of this paper. It will give good results for all flow phenomena which take place within a relatively short length with gradual changes in direction, or in longer stretches of channel if friction and gravity forces are reasonably balanced. The curvatures have to remain such that large deviations from the assumed hydrostatic distribution of pressure are avoided. However, the fact of convergence of positive wave fronts may even then lead to the formation of wave fronts of finite height—the so-called shock waves or hydraulic jumps. The influence of finite wave height on the standing-wave system is opened to closer analysis by a procedure quite similar to the method just discussed for flows with constant energy, and will be taken up in the following section.

THE "METHOD OF SHOCK POLARS" FOR SUPERCRITICAL FLOW WITH HIGH WAVE FRONTS AND ENERGY DISSIPATION

As explained in the preceding section, the effect of wall curvature may be concentrated over a short distance due to the con-

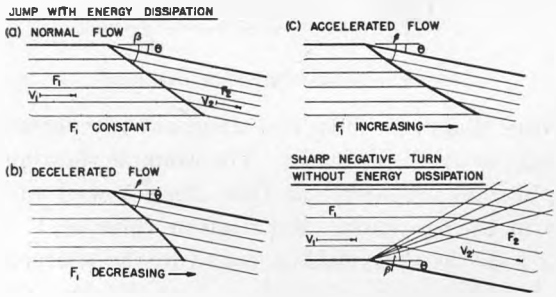

FIG. 5.—FLOW PAST SHARP CORNER.

vergence of the positive wave fronts, and a steep wave front or shock wave may be created. This would be equivalent to the effect of a sudden angular change along the wall, as shown in Fig. 5, with the wall turning toward the flow. A shock wave or jump with energy dissipation originates at any such sharp corner crossing the flow at an angle β, as given by Eq. (11). If, however, the wall turns away from the flow (Fig. 5), no shock-wave is created, since there can be no energy dissipation. The sudden drop can be imagined to disintegrate into a number of diverging wave fronts of small height. The assumption of hydrostatic pressure distribution naturally can no longer be maintained in the immediate neighborhood of such a corner. This latter problem is therefore quite a distinct one and is not included in the further treatment.

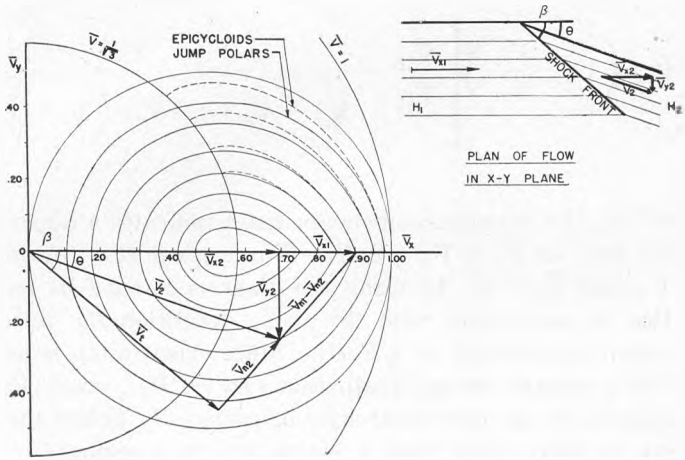

FIG. 6.—POLAR DIAGRAM FOR SLANTING HYDRAULIC JUMP.

The "method of shock polars" for supercritical flow is based on the fundamental relationships contained in Eqs. (8) to (14) and takes into account the influence of energy dissipation and wave height. The graphical analogy to the epicycloids in the preceding chapter results in other characteristic curves, the so-called shock polars, which as before give the direction and magnitude of the velocity $\overline{V}_2$ for all possible deflection angles θ if the initial conditions $\overline{V}_1$ and h_1 are given. These curves are shown in Fig. 6. The portions of the curves lying within the limiting circles of radius

$\overline{V} = 1$ and $\overline{V} = 1/\sqrt{3}$ give very nearly the end condition $\overline{V}_2$ in the supercritical range of flow and the parts within the inner circle give the subcritical velocity $\overline{V}_2$. Within the latter are therefore located the values of $\overline{V}_2$ for $\theta = 0$ — i.e., for the normal hydraulic jump. The plotting of these curves is materially aided if the equations are transformed to contain the x and y components of $\overline{V}_2$ instead of $\overline{V}_{n2}$ and $\overline{V}_t$. The geometric relationships of Eqs. (18) and (19) introduced into Eqs. (8) and (9) together with the energy equation yield the following expressions for the "jump-polar" or "shock-polar curves:"

$$\overline{V}_{n1}\,(\overline{V}_{n1} - \overline{V}_{n2}) = \overline{V}_{x1}\,(\overline{V}_{x1} - \overline{V}_{x2}) \tag{18}$$

$$(\overline{V}_{n1} - \overline{V}_{n2})^2 = \overline{V}_y^2 + (\overline{V}_{x1} - \overline{V}_{x2})^2 \tag{19}$$

$$\overline{V}_y^2 = \overline{V}_{x1}^2\left(1 - \frac{\overline{V}_{x2}}{\overline{V}_{x1}}\right)\left[\frac{\overline{V}_{x2}}{\overline{V}_{x1}} - \frac{\sqrt{1 - \overline{V}_{x1}^2}}{\sqrt{1 - \overline{V}_{x1}^2 + 4\,\overline{V}_{x1}^2\left(1 - \frac{\overline{V}_{x2}}{\overline{V}_{x1}}\right)}}\right] \tag{20}$$

The bar over the velocity components again indicates a dimensionless ratio such as $\overline{V}_1 = V_1/\sqrt{2gH_1}$. For critical flow, obviously, $\overline{V}_{x1} = \overline{V}_{x2}$ and $\overline{V}_y = 0$. In using the diagram it must be remembered that in accordance with the above definition the velocities $\overline{V}_2$ represent percentages of $\sqrt{2gH_1}$. Since every jump is accompanied by a certain energy dissipation $(H_1 - H_2)$, every $\overline{V}_2$ has to be adjusted to the new lower state of energy H_2, before the diagram can be used again from a region (2) to a region (3). In other words, the quantity $\overline{V}_2$ is to be multiplied by $\sqrt{H_1/H_2}$, which is always larger than unity. The diagram is then oriented anew with the x axis in the direction of the corrected value of $\overline{V}_2 = V_2/\sqrt{2gH_2}$. The relationship between plan of flow and polar diagram is easily seen from the figure by means of the vector diagrams shown. In case standing waves or jumps have to be analyzed rather frequently, it is obvious that the work can be reduced considerably by plotting also the ratios H_2/H_1 and h_2/h_1 once and for all in the $\overline{V}_x$, $\overline{V}_y$ plane for all possible values of θ and $\overline{V}_{x1}$. The use of these curves is similar to that of V_2 curves and eliminates practically all further calculations.

APPLICATIONS OF THE THEORY

As mentioned before, the theory outlined and the method of characteristics have been applied to various problems of supercritical flow with promising results. The author applied it to the case of flow in curves and Preiswerk investigated the case of Laval expansions for water, a problem directly analogous to the supersonic gas flow through Laval nozzles. The results of experimental investigations confirm the solutions obtained by the method of characteristics not only in a qualitative manner, but also quantitatively to a surprising degree, for certain ranges of supercritical flow for which the fundamental assumptions are reasonably well fulfilled in practice. The method of shock polars needs considerably more experimental verification; however, measurements mentioned in Preiswerk's treatise for a Froude number $\mathbf{F} = 2$ show in general the expected variation of deflection angle θ with wave angle β. Considerable interest in further investigations along this line is therefore justified. The investigation carried on at present at the Hydraulic Laboratory of Lehigh University is concerned with applications to channel contractions. Fig. 7 shows the apparatus used, looking downstream, the problem encountered being quite evident. The more-or-less conventional design of S-shaped side walls produces standing waves of considerable magnitude, which, due to the symmetry of the contraction, results in the very regular diamond pat-

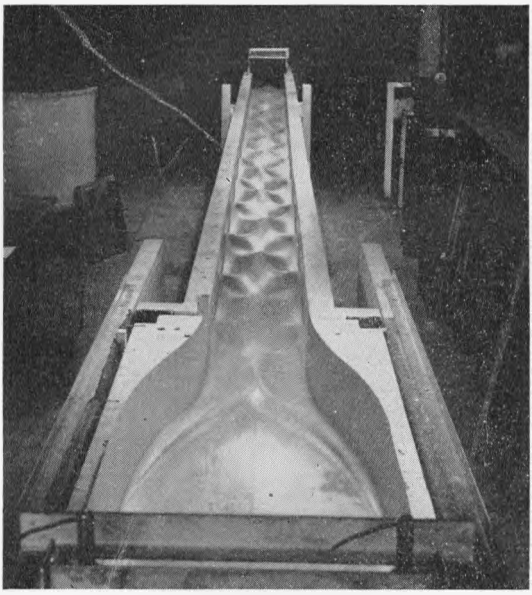

FIG. 7.—STANDING WAVES IN LABORATORY APPARATUS.

tern extending throughout the length of the flume. This phenomenon is entirely in agreement with observations in large channels, of which a few have been published here and there. The theoretical analysis gives satisfactorily the location and the height of the disturbances and therefore may well be relied upon to point the way toward forms of contractions which will not show the undesirable standing-wave features of Fig. 7. Such a form of contraction based in its design on a correct evaluation of the supercritical-flow phenomenon is shown in Fig. 8. Since the standing waves or jumps cannot be avoided in a contraction with a level channel bottom, they may as well be produced by a definite wall deflection of angle θ. This angle is determined in such a way as to give the proper contracted channel width at the point where the standing wave reflected from the center again meets the wall. The original and final directions of flow must be parallel, since both original and reflected waves must cause the same deflection of the flow through the angle θ. Because principles of wave interference have full application in these problems, it is easily seen why the two designs in Fig. 8 are labeled as correct and incorrect, respectively. The latter design, as shown by the stream lines and wave fronts, gives the undesirable diamond waves, while the former results in undisturbed flow downstream from the contraction. It may be well to point out again in conclusion that Fig. 8 remains as yet a theoretical solution which is to be confirmed or modified by experiment. In presenting it, the intention is to show the value of the theory in producing rather promising possibilities for investigations and practical applications in this field. The line of attack seems very definitely established.

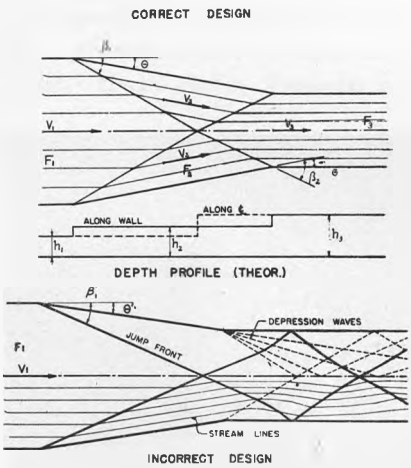

FIG. 8.—DESIGNS OF SHARP CONTRACTION.

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