THE EFFECT OF TURBULENCE IN RETARDING SETTLING

by

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This paper is a further development of studies completed by Dr. William E. Dobbins [1][1] at the Massachusetts Institute of Technology in September, 1941. Dobbins' work includes the derivation of the basic mathematical theory and an experimental verification of certain special cases, all of which has been reported upon in the paper cited. The studies of Dobbins will be described very briefly in this paper, but it will be necessary to refer to the original paper for the steps in the development of the theory and for the details of the experiment. The purpose of the present paper is to describe an approximate application of the results of Dobbins' work to open-channel flow in order to predict the effect of turbulence in retarding settling.

The turbulent mixing process works in a manner similar to diffusion. Suspended material is transported in the direction of decreasing concentration, and the rate of transport by turbulence in any direction is proportional to the concentration gradient in that direction. If the suspended material has the same density as the fluid, there will be no separation by settling and the concentration will approach uniformity throughout. If the density of the suspended particles differs from that of the fluid, there will also be transportation in a vertical direction due to settling (or rising) velocities. Settling tends to increase the magnitude of the concentration gradient by clearing the upper region of the fluid, and thus settling promotes turbulent transfer of suspended matter in the opposite direction.

If the rate of settling is at equilibrium with the turbulent trans-

[1] References appear at the end of the article.

port in the vertical direction, then in any horizontal plane at height y above the bed:

$$\varepsilon \frac{dc}{dy} = -w\,c \qquad (1)$$

in which c in the concentration, w is the settling velocity of the particles, $-dc/dy$ is the concentration gradient and ε is the mixing coefficient. Integration of this equation results in an expression for the relative concentration at height y with respect to any other height a:

$$\ln \frac{c}{c_a} = -w \int_a^y \frac{dy}{\varepsilon} \qquad (2)$$

The evaluation of the integral requires that ε be expressed as a function of y.

Eq. (2) has been verified experimentally by Rouse [2] and later by Dobbins [1] for constant values of ε in vertical cylindrical tanks equipped with mechanical agitators designed to produce a constant intensity of mixing throughout the depth. It has also been verified by Vanoni [3] for two-dimensional open-channel flow with the value of ε as given by the logarithmic velocity distribution.

It is pertinent to note here that the vertical distribution of sediment characterized by Eq. (2) depends upon the equilibrium of Eq. (1) at all horizontal planes including the bottom. That is, the rate of settling out of suspension must equal the rate of scour from the bed. To obtain the absolute concentration it is necessary to know the rate of scour from the bed. There is no such thing as transportation in suspension *per se*. If particles which settle onto the bed are permitted to remain there, the stream must clarify itself. Transportation in suspension is a continuous process of settling out and scour. Theoretically the equilibrium postulated by Eqs. (1) and (2) is reached only after infinite time. It is approached very rapidly, however. Dobbins [1] has shown experimentally that the equilibrium distribution of suspended matter is independent of the manner in which the sediment is introduced to the fluid.

In a settling tank the rate of scour from the bottom is always less than the rate of settling out. The process is therefore not at equilibrium, and the vertical distribution of suspended matter depends also upon the time of settling. In a river the non-equilibrium condition is also to be expected, because of expansions and contractions in cross section and changes in the rate of scour from station to station.

The non-equilibrium condition may be investigated for discrete particles with two-dimensional flow in an open channel by studying the changes in concentration at some point m in the channel (see Fig. 1).

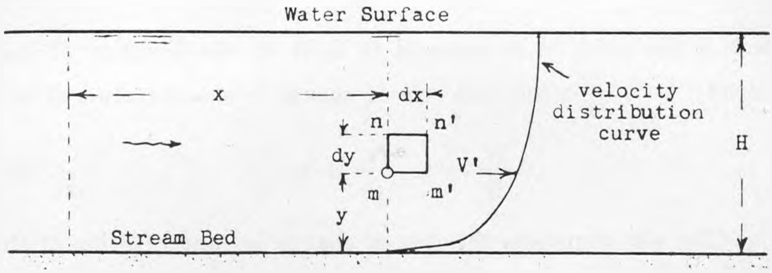

FIG. 1.—TWO-DIMENSIONAL FLOW IN AN OPEN CHANNEL.

Let c be the concentration at m at any time t of particles with settling velocity w. Let V' be the temporal mean velocity of the liquid at m, and ε_y and ε_x be the mixing coefficients at m in the direction of y and x, respectively. Consider the elementary volume $mm'n'n$ with dimensions dx, dy, and unity. By means of the equation of continuity, the amount of sediment transported out of the elementary volume in time dt across the face nn' by turbulence and settling and across the face $m'n'$ by turbulence and the liquid velocity may be subtracted from the amount carried in across the faces mm' and mn in the same time. It may thus be shown for the steady state, in which the concentration at the fixed point m remains unchanged, that

$$V' \frac{\partial c}{\partial x} = \varepsilon_y \frac{\partial^2 c}{\partial y^2} + \left(w + \frac{\partial \varepsilon_y}{\partial y} \right) \frac{\partial c}{\partial y} + \varepsilon_x \frac{\partial^2 c}{\partial x^2} \qquad (3)$$

This is the general differential equation for the concentration changes in the two-dimensional case. An equation similar to Eq. (3) but which does not include the term $\frac{\partial \varepsilon_y}{\partial y} \frac{\partial c}{\partial y}$ has been presented by Kalinske [4].

The integration of Eq. (3) requires that V', ε_y and ε_x be expressed as functions of y. No solution has been found for the equation in its general form because of the difficulty of separating the variables. If the assumptions are made that $\varepsilon_x \frac{\partial^2 c}{\partial x^2}$ is zero and that both V' and ε_y are constant throughout the depth H and the length of the channel, Eq. (3) is reduced to

$$V \frac{\partial c}{\partial x} = \varepsilon \frac{\partial^2 c}{\partial y^2} + w \frac{\partial c}{\partial y} \tag{3a}$$

Now if the point m is assumed to move in the direction of x at velocity $V = \frac{\partial x}{\partial t}$, the time rate of change in concentration at m is

$$\frac{\partial c}{\partial t} = \varepsilon \frac{\partial^2 c}{\partial y^2} + w \frac{\partial c}{\partial y} \tag{3b}$$

Eq. (3b) also expresses the rate of change in concentration at any point m in a settling container in which turbulence is imposed of uniform intensity throughout.

A solution for Eq. (3b) has been obtained by Dobbins [1] for the following boundary conditions: (1) the rate of transport across the free liquid surface is zero; (2) the rate of transport across the bottom is equal to the rate of settling out less a constant rate of pickup from the bottom; (3) at the start, the concentration of suspended matter is some function of y; and (4) as t approaches infinity the concentration of suspended matter approaches the equilibrium distribution $A\, e^{-\frac{wy}{\epsilon}}$ as given by Eq. (2) where A is the concentration at the bottom. If the initial concentration is represented by the function $A_0 e^{-\frac{wy}{\epsilon_0}}$, where A_0 is the initial concentration at the bottom and ε_0 is the initial value of the mixing coefficient, ε_0 being suddenly reduced to ε at time $t = 0$, the solution for Eq. (3b) is

$$c = A\, e^{-2\left(\frac{wH}{2\epsilon}\right)\frac{y}{H}} + e^{-\left(\frac{wH}{2\epsilon}\right)\frac{y}{H}} \sum_{n=1}^{n=\infty} e^{-\left[\left(\frac{wH}{2\epsilon}\right)^2 + a_n^2\right]\frac{\epsilon t}{H^2}} C_n Y_n \tag{4}$$

in which

$$C_n = \frac{4\,\alpha_n^2}{\left(\frac{wH}{2\varepsilon}\right)^2 + \alpha_n^2 + 2\,\frac{wH}{2\varepsilon}}$$

$$\left\{ \frac{A_0\left[\frac{wH}{2\varepsilon_0} + H_n\left(\frac{wH}{2\varepsilon} - \frac{wH}{2\varepsilon_0}\right) e^{\left(\frac{wH}{2\epsilon} - \frac{2wH}{2\epsilon_0}\right)}\right]}{\left(\frac{wH}{2\varepsilon} - \frac{2wH}{2\varepsilon_0}\right)^2 + \alpha_n^2} - \frac{A\,\frac{wH}{2\varepsilon}}{\left(\frac{wH}{2\varepsilon}\right)^2 + \alpha_n^2} \right\} \tag{5}$$

$$Y_n = \cos\frac{y}{H}\alpha_n + \frac{wH/2\varepsilon}{\alpha_n}\sin\frac{y}{H}\alpha_n \tag{6}$$

$\alpha_1, \alpha_2, \alpha_3, \ldots\ldots \alpha_n$ are the successive real positive roots of the transcendental equation

$$2\cot\alpha = \frac{\alpha}{wH/2\varepsilon} - \frac{wH/2\varepsilon}{\alpha} \tag{7}$$

and H_n is $+1$ when α is in the first and second quadrants and -1 when α is in the third and fourth quadrants.

If A_0 is taken as zero in Eq. (5), the solution is for the case of scour starting with a clear liquid. If A is taken as zero in Eq. (5), the solution is for the case of settling out with no scour. The complete solution is the sum of the two cases.

Dobbins [1] has made an experimental verification of the above equations with a unigranular suspension of lucite powder in water. The experiments were conducted in a vertical cylinder equipped with a reciprocating mixer similar to that used by Rouse [2]. Two cases were verified, in both of which ε_0 was made equal to ε. In the first case, the scour was suddenly reduced to zero—that is, $A = 0$. In the second case, the scour was suddenly changed to a lower magnitude—that is, $A < A_0$. Good verification was obtained for both cases.

The application of this theory to studies of settling and scour in open channels cannot be made with confidence in the results until an integration is obtained for the general differential equation. With certain bold assumptions, however, some idea may be obtained of the magnitude of the effect of turbulence in retarding settling. The solution obtained by Dobbins [1] for Eq. (3b) is equally applicable to Eq. (3a), since these equations are similar mathematically. Therefore, in order to use the only integration so far available, Eq. (4) must be adapted to open channels through the following assumptions:

1. The fluid velocity is the same at every point in the channel,
2. The mixing coefficient is the same at every point in the channel.

In order to isolate the effect of turbulence in retarding settling, the effect of scour must be eliminated. That is, $A = 0$ in Eqs. (4) and (5). To simplify the equations further, a uniform concentration throughout the depth H was assumed for zero time, which requires a value of $\varepsilon_0 = \infty$. With these assumptions, Eqs. (4) and (5) may be combined in the following dimensionless form:

$$\frac{c}{c_0} = \frac{wH}{2\varepsilon} e^{\frac{wH}{2\epsilon}\left(1-\frac{y}{H}\right)} \sum_{n=1}^{n=\infty} \frac{a_n{}^2 H_n Y_n e^{-\left[\left(\frac{wH}{2\epsilon}\right)^2 + a_n{}^2\right]\frac{\epsilon t}{H^2}}}{\left[\left(\frac{wH}{2\varepsilon}\right)^2 + a_n{}^2 + 2\frac{wH}{2\varepsilon}\right]\left[\left(\frac{wH}{2\varepsilon}\right)^2 + a_n{}^2\right]} \qquad (8)$$

This equation gives the relative concentration of suspended matter at any point in the channel a distance $x = Vt$ from the starting point.

The average relative concentration throughout the depth H may be computed from the integral $\int_0^1 \frac{c}{c_0} d\left(\frac{y}{H}\right)$, wherein the value of $\frac{c}{c_0}$ is taken from Eq. (8) This integral is readily evaluated. The removal from suspension by settling in the distance x is the difference between this integral and the initial concentration. Since the relative initial concentration is unity throughout the depth H, the relative removal is

$$r = 1 - 8\left(\frac{wH}{2\varepsilon}\right)^2 e^{\frac{wH}{2\epsilon}} \sum_{n=1}^{n=\infty} \frac{\alpha_n^2 H_n e^{-\left[\left(\frac{wH}{2\epsilon}\right)^2 + \alpha_n^2\right]\frac{w}{w_0}\frac{1}{2wH/2\epsilon}}}{\left[\left(\frac{wH}{2\varepsilon}\right)^2 + \alpha_n^2 + 2\frac{wH}{2\varepsilon}\right]\left[\left(\frac{wH}{2\varepsilon}\right) + \alpha_n^2\right]^2} \tag{9}$$

It will be noted that the term Y_n in Eq. (8) has been eliminated by integration over the depth H and does not appear in Eq. (9).

The quantity w_0 in Eq. (9) is the "overflow rate." In the sanitary-engineering field, the overflow rate is usually defined as the discharge per unit of surface area. It is also the settling velocity required for a particle to settle from the surface to the bottom in time t. Therefore $t = \frac{H}{w_0}$. This value for t has been used in Eq. (9) to facilitate comparison with the removal obtained in a stream without turbulence. In a stream without turbulence, in which the velocity is the same throughout and the concentration is uniform from top to bottom at the start, the removal is

$$r = \frac{w}{w_0} \tag{10}$$

The removal in Eq. (9) is expressed as a function of only three variables, $wH/2\varepsilon$, α_n, and w/w_0, all of which are dimensionless. Only two of these variables, $wH/2\varepsilon$ and w/w_0, are independent, since α_n is a function of $wH/2\varepsilon$ as defined by Eq. (7). The use of Eq. (9) for numerical computations of removal is too tedious and cumbersome. Some problems of practical significance require 15 terms or more for convergence of the series. It is convenient, therefore, to have a graph for the solution of Eq. (9), and Fig. 2 is presented for this purpose.

The successive values of α_n required for each computed point on this dimensionless graph were determined to six decimal places by trial-and-error solutions of Eq. (7). Convergence of the series is rapid for low values of $wH/2\varepsilon$, only one term being required for a value of 0.1. However, 14 terms were required for $wH/2\varepsilon = 30$, and no solution was practical for $wH/2\varepsilon = 100$.

The effect of turbulence in retarding settling is apparent from

Fig. 2. When turbulence is relatively great with a correspondingly high value of ε, the value of $wH/2\varepsilon$ is low and the removal is reduced. For example, for particles which would be just 100 percent settled out in the stretch of stream considered without turbulence (i.e., $w/w_0 = 1.0$), the removal is only 64 percent if $wH/2\varepsilon =$

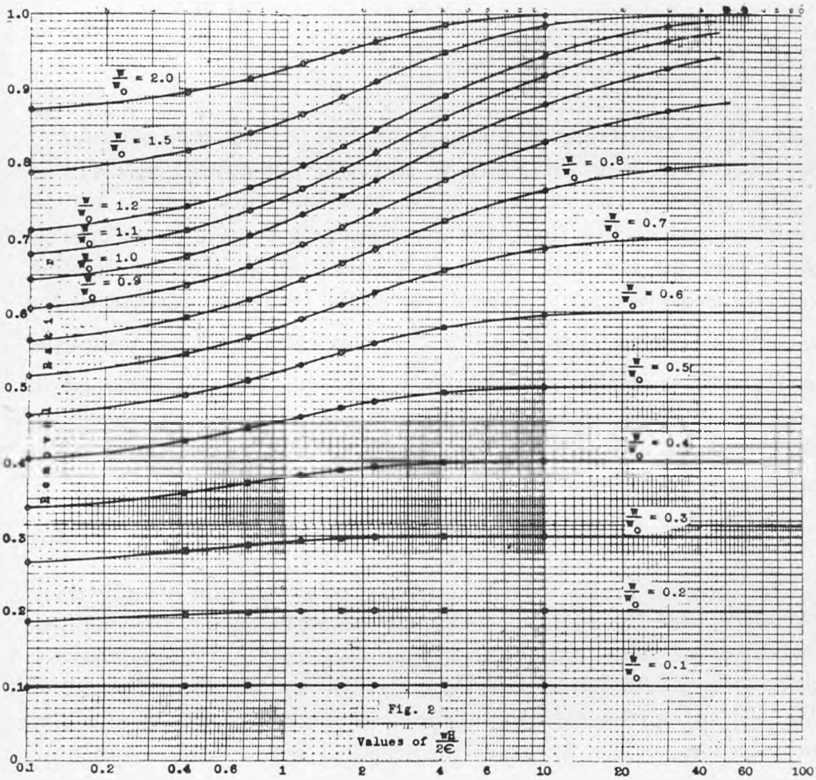

FIG. 2.—GRAPHICAL SOLUTION OF EQ. (9).

0.1, but is 72 percent if $wH/2\varepsilon$ is 1.0 and is 94 percent if $wH/2\varepsilon$ is 40. The effect of turbulence on settling out of particles is much less if the removal is less, that is, for low values of w/w_0.

In order to use this graph to study the effect of turbulence in an actual stream, a mean value for ε must be estimated. The value of ε is a function of the mean velocity and the velocity gradient as follows:

$$\varepsilon = \frac{f}{8} V^2 \frac{1 - Y/H}{\dfrac{dV'}{dy}} \tag{11}$$

The logarithmic velocity distribution in terms of the velocity defect is as follows:

$$\frac{V_{max} - V'}{V\sqrt{f/8}} = \frac{1}{\kappa} \ln \frac{H}{y} = 5.75 \log \frac{H}{y} \tag{12}$$

in which f is the friction factor for the channel and κ is the von Kármán universal constant, taken as 0.4 in this study. If the value of $\frac{dV'}{dy}$ as determined from Eq. (12) is substituted in Eq. (11), the mixing coefficient becomes

$$\frac{\varepsilon}{HV\sqrt{f/8}} = \kappa \frac{y}{H}\left(1 - \frac{y}{H}\right) \tag{13}$$

The mean value of the coefficient over the depth H is

$$\frac{\varepsilon}{HV\sqrt{f/8}} = \frac{\kappa}{6} = 0.0667 \tag{14}$$

If the value of ε is assumed to be constant, the corresponding velocity distribution required by Eq. (11) is parabolic with a bottom velocity greater than zero. This parabolic velocity distribution with the value of ε as given by Eq. (14) is defined in terms of the velocity defect as follows:

$$\frac{V_{max} - V'}{V\sqrt{f/8}} = \frac{3}{\kappa}\left(1 - \frac{y}{H}\right) = 7.5\left(1 - \frac{y}{H}\right)^2 \tag{15}$$

The values of ε as defined by Eqs. (13) and (14) and the corresponding velocity distribution curves are shown in Fig. 3.

The values of the abscissas $wH/2\varepsilon$ in Fig. 2 may now be expressed in terms of the mean velocity of the stream by means of Eq. (14) as follows:

$$\frac{wH}{2\varepsilon} = \frac{3w}{\kappa V\sqrt{f/8}} = 7.5 \frac{w}{V\sqrt{f/8}} \tag{16}$$

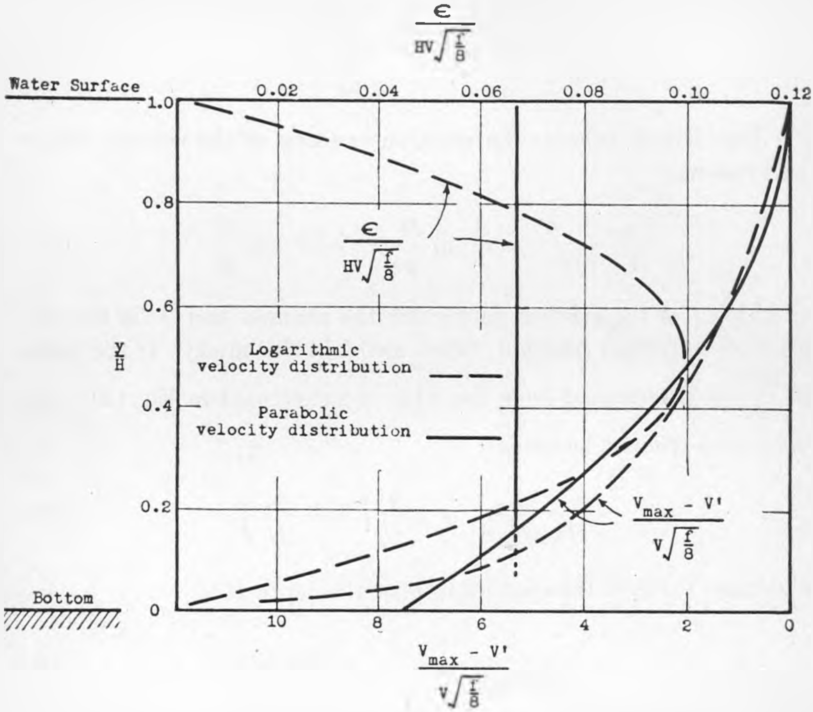

FIG. 3.—VALUES OF ϵ AS DEFINED BY EQS. (13) AND (14).

Also, since by similar triangles

$$\frac{wH}{2\varepsilon} = \frac{3}{\kappa\sqrt{f/8}}\frac{H}{L}\frac{w}{w_0} = \frac{7.5}{\sqrt{f/8}}\frac{H}{L}\frac{w}{w_0} \tag{17}$$

As an example, let it be required to find the effect of turbulence on the settling of particles which would be just 70 percent removed $\left(\frac{w}{w_0} = 0.7\right)$ without turbulence in a 1000-ft. stretch of a stream 10 ft. deep $\left(\frac{H}{L} = 0.01\right)$ if $f = 0.024$. From Eq. (17), $\frac{wH}{2\varepsilon} = 0.96$, and from the graph in Fig. 2 the removal $r = 0.58$ or 58 percent. The effect of turbulence is thus to reduce the removal by about 17 percent.

It should be emphasized in conclusion that the method proposed in this paper for estimating the effect of turbulence in retarding settling is only a rough approximation. For a more precise method we must await the solution of the general differential equation.

REFERENCES

[1] Dobbins, W. E., "Effect of Turbulence on Sedimentation." *Proc. Am. Soc. Civil Engineers,* Vol. 69, No. 2, Feb., 1943, pp. 235-262.

[2] Rouse, H., "Experiments on the Mechanics of Sediment Suspension," *Proc. Fifth Intern. Congress Appl. Mech.,* 1938.

[3] Vanoni, V. A., "Some Experiments on the Transportation of Suspended Load," *Trans. Am. Geophysical Union,* 1941.

[4] Kalinske, A. A., "Suspended-Material Transportation under Non-equilibrium Conditions," *Trans. Am. Geophysical Union,* 1940.